PHASE TRANSITION IN THE EM SCHEME OF AN SDE DRIVEN BY α-STABLE NOISES WITH $\alpha \in (0,2]$

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Abstract. We study in this paper the EM scheme for a family of well-posed critical SDEs with the drift $-x\log(1+|x|)$ and α-stable noises. Specifically, we find that when the SDE is driven by a rotationally symmetric α-stable processes with $\alpha = 2$ (i.e. Brownian motion), the EM scheme is bounded in the L^2 sense uniformly w.r.t. the time. In contrast, if the SDE is driven by a rotationally symmetric α-stable process with $\alpha \in (0,2)$, all the β-th moments, with $\beta \in (0,\alpha)$, of the EM scheme blow up. This demonstrates a phase transition phenomenon as $\alpha \uparrow 2$. We verify our results by simulations.

Contents

1. Introduction

The following stochastic differential equation (SDE) on $\mathbb{R}^d$ has been extensively studied for several decades:

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}L_t\ , \quad X_0 = x_0, \tag{1.1}$$

where $x_0 \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy certain regularity conditions, and the process $(L_t, t \geqslant 0)$ is a d-dimensional, rotationally invariant α-stable Lévy process with $\alpha \in (0,2]$. We refer to the books [1, 36, 38] for systematic accounts on SDEs driven by Lévy processes and to [2, 3, 5, 6, 7, 8, 16, 17, 19, 20, 30, 10, 18, 40] for more recent developments.

The Euler-Maruyama (EM) scheme of SDE (1.1) has also been studied by many authors, see [29, 30, 22, 41, 13, 42, 14, 39, 32, 4, 33]. In particular, when f is Lipschitz and g is bounded Lipschitz, by a standard method for proving the existence and

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uniqueness of SDE's strong solution, it can be shown that the EM scheme in a finite time interval $[0, T]$ strongly converges to (1.1), see for instance [26, 35, 1, 38, 36].

It is well known that, even though a (stochastic) dynamics system is well posed, its numerical schemes may blow up, see [29, 30, 24, 23, 8]. In particular, for SDE (1.1) with a Brownian motion noise, i.e., $\alpha = 2$, if the drift coefficient f is a polynomial like $-x|x|^{\theta}$ with $\theta > 0$, it has a unique strong solution with finite exponential moment. In contrast, its EM scheme has a blowing-up $L^p(\mathbb{P})$ norm with $p \geqslant 1$ as the step size $\eta \to 0$, see for instance [24, 23, 25]. When the driven noise is an α-stable process with $\alpha \in (0, 2)$, due to the heavy tailed property of stable distribution, one may expect that the corresponding EM scheme will blow up. One of the main motivation of the present paper is to verify this conjecture.

In this paper, we shall mainly consider SDE (1.1) with a special non-Lipschitz drift $-x\log(1+|x|)$, which lies at the boundary between the cases $-x|x|^{\theta}$ and $-x$. To the best of our knowledge, the behavior of the EM scheme has not been studied. Our main results, Theorems 1.1–1.3, show that the EM scheme is uniformly bounded for $\alpha = 2$, but blows up for $\alpha \in (0, 2)$. This demonstrates a phase transition as $\alpha \uparrow 2$.

1.1. **The drift $-x\log(1 + |x|)$ and $\alpha \in (0, 2]$: phase transition.** In order to demonstrate our idea, we assume for simplicity that $g(x) = I_d$ with I_d being $d \times d$ identity matrix. We expect that our results can be extended to the case in which g is bounded Lipschitz and nondegenerate.

Let us consider the following SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = -X_t \log(1 + |X_t|)\,\mathrm{d}t + \mathrm{d}L_t\ , \quad X_0 = x_0, \tag{1.2}$$

where L_t is a d-dimensional rotationally symmetric α-stable Lévy process with $\alpha \in (0, 2]$. Thanks to the dissipation of the drift term, we can show by a standard method that SDE (1.2) admits a unique strong solution, see Appendix 6 below.

The Euler-Maruyama (EM) scheme of SDE (1.2) is give by: for $k \in \mathbb{Z}_+$,

$$Y_{k+1} = Y_k - \eta Y_k \log(1 + |Y_k|) + (L_{(k+1)\eta} - L_{k\eta}), \quad Y_0 = x_0, \tag{1.3}$$

where $\eta > 0$ is the step size. We shall show that for $\alpha \in (0, 2)$, the above EM scheme $(Y_n, n \geq 1)$ blows up in $L^{\beta}(\mathbb{P})$ for any $\beta \in (0, 2)$ as $\eta \to 0$; while for $\alpha = 2$, i.e. L_t is a standard d-dimensional Brownian motion, $(Y_n, n \geq 1)$ is uniformly bounded in $L^2(\mathbb{P})$. This demonstrates a phase transition as $\alpha \uparrow 2$.

We start with the following theorem for the case of $\alpha = 2$.

Theorem 1.1. *Consider the EM scheme (1.3) with $\alpha = 2$, i.e. the driven noise is Brownian motion. Then, for any fixed initial value x_0, there exist constants $\eta_0 \leqslant \min\{(1 + |x_0|)^{-2}, \mathrm{e}^{-5}\}$ and $C > 0$ such that for all $\eta \in (0, \eta_0]$,*

$$\sup_{m \geqslant 0} \mathbb{E}\,|Y_m|^2 \leqslant C. \tag{1.4}$$

Next we consider the case $\alpha \in (0, 2)$, in which $(L_t, t \geqslant 0)$ is a rotationally symmetric α-stable process, which will be denoted by $(Z_t, t \geqslant 0)$. We refer to [37] for comprehensive account on Lévy processes, stable and more generally, infinitely divisible distributions. See also [16, 31, 28] for more recent developments.

It is known (cf. e.g., [37, Theorem 14.3] or [10, 27]) that the Lévy measure ν of the process $(Z_t, t \geqslant 0)$ is

$$\nu(\mathrm{d}z) = \frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,\mathrm{d}z,$$

where constant $C_{d,\alpha}$ is given by

$$C_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-\frac{d}{2}} \cdot \frac{\Gamma\big((d+\alpha)/2\big)}{\Gamma\big(1-\alpha/2\big)}. \tag{1.5}$$

Even though, for a general $\alpha \in (0,2)$, the transition density function $p_\alpha(t,x)$ of the α-stable processes $(Z_t, t \geqslant 0)$ does not have an explicit expression, many of its analytic or asymptotic properties have been known. In particular, it follows from [9, Theorem 2.1] that $p_\alpha(t,x)$ satisfies

$$K(d,\alpha)^{-1}\frac{t}{(t^{1/\alpha}+|x|)^{d+\alpha}} \leqslant p_\alpha(t,x) \leqslant K(d,\alpha)\frac{t}{(t^{1/\alpha}+|x|)^{d+\alpha}}, \quad x \in \mathbb{R}, \tag{1.6}$$

where $K(d,\alpha) \geqslant 1$ is a constant depending on d and α.

By the Lévy-Itô decomposition (cf. [37, Chapter 4] or [1, Chapter 2]), there exists a Poisson random measure $P(\mathrm{d}t, \mathrm{d}z)$ such that

$$\mathrm{d}Z_t = \int_{\{|z|\geqslant 1\}} zP(\mathrm{d}t, \mathrm{d}z) + \int_{\{|z|<1\}} z\widetilde{P}(\mathrm{d}t, \mathrm{d}z),$$

where $\widetilde{P}(\mathrm{d}t, \mathrm{d}z) = P(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t\,\nu(\mathrm{d}z)$ is the compensated Poisson random measure. Due to the lack of explicit representation for the probability density of the α-stable noise $Z_{(n+1)\eta} - Z_{n\eta}$, the numerical simulation becomes complicated and computationally expensive. Hence, one often does not use the scheme (1.3) directly in practice, see [34, 12] for further discussions. Since the α-stable distribution and the Pareto distribution with parameter α have the same tail behavior and the stable central limit theorem (see, e.g. [15, 21]), we can replace the stable noise $Z_{(n+1)\eta} - Z_{n\eta}$ with i.i.d. random variables with the Pareto distribution. More precisely, for any fixed time $T > 0$, the EM approximation for the SDE (1.2), denoted by mappings $\widetilde{Y}_k : \Omega \to \mathbb{R}^d$, is given by

$$\widetilde{Y}_{k+1} = \widetilde{Y}_k - \eta\widetilde{Y}_k \log\Big(1 + \big|\widetilde{Y}_k\big|\Big) + \frac{1}{\sigma}\eta^{1/\alpha}\widetilde{Z}_{k+1}, \quad \widetilde{Y}_0 := x_0, \tag{1.7}$$

for all $k \in \{0, 1, \dots, n-1\}$, $n \in \mathbb{N}$, where $\sigma^\alpha = \alpha/(s_{d-1}C_{d,\alpha})$ with $C_{d,\alpha}$ defined by (1.5), $\{\widetilde{Z}_k, k = 1, 2, \dots\}$ is a sequence of i.i.d. Pareto-distributed random variables, and the density function $p(z)$ of the Pareto distribution is given by

$$p(z) = \frac{\alpha}{s_{d-1}|z|^{\alpha+d}}\mathbb{1}_{(1,\infty)}(|z|). \tag{1.8}$$

Here, $s_{d-1} = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ represents the surface area of the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

The following theorem describes the limiting behavior of the EM schemes (1.3) and (1.7) when $\alpha \in (0,2)$. In Theorems 1.2 and 1.3, κ_α and δ_α are the constants, depending on α, given in Lemmas 2.2 and 2.3 below.

Theorem 1.2. *Let $\alpha \in (0,2)$, $\beta \in (0,\alpha)$, $T \in (0,\infty)$ be constants and let $\eta = \frac{T}{n}$ be the step size.*

(i) *For the EM scheme* (1.3)*, define* $K_1 = 2(\mathrm{e}^{\delta_\alpha/\beta} + 2)/T$*. We assume that* T *and* n *are large enough such that*

$$\frac{T}{n} \leqslant 1, \quad K_1 < \frac{\delta_\alpha - \log \delta_\alpha}{\alpha - \beta}, \quad \mathrm{e}^{nK_1} \geqslant |x_0| \big(1 + \log(1 + |x_0|)\big).$$

Then, there exist a constant $C > 0$ *and a sequence of non-empty events* $\Omega_n \subseteq \Omega$, $n \in \mathbb{N}$ *with*

$$\mathbb{P}(\Omega_n) \geqslant \frac{C\mathrm{e}^{-\alpha n K_1}}{n\delta_\alpha^n} \quad \text{and} \quad |Y_n(\omega)| \geqslant \exp\left\{nK_1 + (n-1)\frac{\delta_\alpha}{\beta}\right\}$$

for all $\omega \in \Omega_n$ *and all* $n \in \mathbb{N}$ *large enough. Consequently, we have*

$$\lim_{n\to\infty} \mathbb{E}\, |Y_n|^\beta = \infty.$$

(ii) *For the EM scheme* (1.7)*, define* $K_2 = 2(\mathrm{e}^{\kappa_\alpha/\beta} + 2/\sigma)/T$*. We assume that* T *and* n *are large enough such that*

$$\frac{T}{n} \leqslant 1, \quad K_2 < \frac{\kappa_\alpha - \log \kappa_\alpha}{\alpha - \beta}, \quad \mathrm{e}^{nK_2} \geqslant |x_0| \big(1 + \log(1 + |x_0|)\big).$$

Then, there exist a constant $C > 0$ *and a sequence of non-empty events* $\widetilde{\Omega}_n \subseteq \Omega$, $n \in \mathbb{N}$ *with*

$$\mathbb{P}(\widetilde{\Omega}_n) \geqslant \frac{C\mathrm{e}^{-\alpha n K_2}}{n\kappa_\alpha^n} \quad \text{and} \quad \big|\widetilde{Y}_n(\omega)\big| \geqslant \exp\left\{nK_2 + (n-1)\frac{\kappa_\alpha}{\beta}\right\}$$

for all $\omega \in \widetilde{\Omega}_n$ *and all* $n \in \mathbb{N}$ *large enough. Consequently, we have*

$$\lim_{n\to\infty} \mathbb{E}\big|\widetilde{Y}_n\big|^\beta = \infty.$$

1.2. **The polynomial growth drift and** $\alpha \in (0,2)$**: blow up.** By the same method for showing Theorem 1.2, we can also prove that the EM scheme of SDE (1.1) blows up if the drift has a p-order polynomial growth with $p > 1$. More precisely, we assume that $f(x)$ and $g(x)$ in SDE (1.1) satisfy the following condition: There exist constants $\gamma > \lambda > 1$ and $H \geqslant 1$ such that

$$\textbf{(A)} \qquad \max\{|f(x)|, |g(x)|\} \geqslant \frac{1}{H}|x|^\gamma, \text{ and }, \min\{|f(x)|, |g(x)|\} \leqslant H|x|^\lambda$$

for all $|x| \geqslant H$.

Assumption **(A)** is similar to the condition in [24, Theorem 1]. The EM scheme of the corresponding SDE is

$$Y_{k+1} = Y_k + \eta f\big(Y_k\big) + g\big(Y_k\big)\big(Z_{(k+1)\eta} - Z_{k\eta}\big), \quad Y_0 = x_0, \tag{1.9}$$

with step size $\eta = \frac{T}{n}$ and $k \in \{0, 1, \dots, n-1\}$. In practice, it is easier to consider the following EM scheme:

$$\widetilde{Y}_{k+1} = \widetilde{Y}_k + \eta f\big(\widetilde{Y}_k\big) + \frac{1}{\sigma}\eta^{1/\alpha} g\big(\widetilde{Y}_k\big)\widetilde{Z}_{k+1}, \quad \widetilde{Y}_0 = x_0, \tag{1.10}$$

where $\{\widetilde{Z}_k, k = 1, 2, \dots\}$ are i.i.d. Pareto distributed random variables.

The following theorem shows that for any $T > 0$, both EM schemes (1.9) and (1.10) blow up as $\eta \to 0$.

Theorem 1.3. *Consider SDE (1.1) under the assumption that L_t is a standard d-dimensional rotationally invariant α-stable process with $\alpha \in (0,2)$. We assume that* **(A)** *holds and $g(x_0) \neq 0$. Let $T > 0$ be an arbitrary number and $\eta = \frac{T}{n}$. Then, there exists a constant $c > 1$ such that*

(i) *for the standard EM scheme (1.9), there exists a sequence of non-empty events $\Omega_n \subseteq \Omega$, $n \in \mathbb{N}$ such that $\mathbb{P}(\Omega_n) \geqslant cn^{-c}\delta_\alpha^{-n}$. Furthermore, $|Y_n(\omega)| \geqslant 2^{\lambda^{n-1}}$ for all $\omega \in \Omega_n$ and all $n \in \mathbb{N}$;*

(ii) *for the EM scheme (1.10), there exists a sequence of non-empty events $\widetilde{\Omega}_n \subseteq \Omega$, $n \in \mathbb{N}$ such that $\mathbb{P}(\widetilde{\Omega}_n) \geqslant cn^{-c}\kappa_\alpha^{-n}$. Furthermore, $\big|\widetilde{Y}_n(\omega)\big| \geqslant 2^{\lambda^{n-1}}$ for all $\omega \in \widetilde{\Omega}_n$ and all $n \in \mathbb{N}$.*

Consequently, for any $\beta \in (0,\alpha)$, we have

$$\lim_{n\to\infty} \mathbb{E}\,|Y_n|^\beta = \infty, \quad \textit{and}\ , \ \lim_{n\to\infty} \mathbb{E}\big|\widetilde{Y}_n\big|^\beta = \infty.$$

The structure of this paper is outlined as follows: We introduce some useful auxiliary lemmas in the following section. In Section 3, we will prove Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are presented in Section 4. In Section 5, we provide numerical simulations that illustrate the convergence and divergence of EM schemes for $d = 1$. Finally, in Appendix 6, we establish the existence and uniqueness of the strong solution of SDE (1.2).

2. Preliminary and Auxiliary Lemmas

To prove Theorem 1.1, we will make use of the following lemmas. Since the proof of Lemma 2.1 is elementary, it is omitted.

Lemma 2.1. *If N follows a standard Gaussian distribution, then for any constants $b \geqslant a \geqslant 0$, there exists a constant C such that*

$$\mathbb{P}(a \leqslant |N| \leqslant b) \leqslant C(b-a)\mathrm{e}^{-\frac{a^2}{2}}. \tag{2.1}$$

As for the Pareto distribution with parameter $\alpha \in (0,2)$, we have the following lemma.

Lemma 2.2. *Suppose that random vector $\widetilde{Z} : \Omega \to \mathbb{R}^d$ satisfies Pareto distribution, then for all $z \in (1,\infty)$ we have*

$$\mathbb{P}(|\widetilde{Z}| \geqslant z) = \frac{1}{z^\alpha} \tag{2.2}$$

and there exists a constant $\kappa_\alpha \geqslant 1$ such that

$$\mathbb{P}(z \leqslant |\widetilde{Z}| \leqslant 2z) = \frac{1}{\kappa_\alpha z^\alpha}, \tag{2.3}$$

where constant $\kappa_\alpha = 2^\alpha/(2^\alpha - 1)$ which only depends on α.

Proof of Lemma 2.2. For all $z \in (1,\infty)$, and $\alpha \in (0,2)$, we have

$$\mathbb{P}(|\widetilde{Z}| \geqslant z) = \int_z^\infty \frac{\alpha r^{d-1}}{r^{\alpha+d}}\mathrm{d}r \int_{\mathbb{S}^{d-1}} \frac{1}{s_{d-1}}\mathrm{d}S = \frac{1}{z^\alpha}.$$

Similarly, we have

$$\mathbb{P}(z \leqslant |\widetilde{Z}| \leqslant 2z) = \left(1 - \frac{1}{2^\alpha}\right)\frac{1}{z^\alpha}.$$

Taking $\kappa_\alpha = 2^\alpha/(2^\alpha - 1)$ gives (2.3). ∎

In addition, thanks to inequality (1.6), we also have the following lemma.

Lemma 2.3. *Let $Z:\Omega\to\mathbb{R}^d$ be a standard d-dimensional rotationally invariant α-stable distribution with $\alpha\in(0,2)$. Then for all $z\in(1,\infty)$, we have*

$$\mathbb{P}(|Z|\geqslant z)\geqslant \frac{s_{d-1}}{2^{d+\alpha}K(d,\alpha)}\cdot\frac{1}{z^{\alpha}}, \tag{2.4}$$

and there exists a constant $\delta_\alpha\geqslant 1$ depending on d and α such that for all $z\in(1,\infty)$,

$$\mathbb{P}(z\leqslant |Z|\leqslant 2z)\geqslant \frac{1}{\delta_\alpha z^{\alpha}}. \tag{2.5}$$

where $\delta_\alpha=\frac{2^{d+\alpha}\alpha K(d,\alpha)}{s_{d-1}}\left(1-\frac{1}{2^\alpha}\right)^{-1}$ with $s_{d-1}=2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$, and $K(d,\alpha)$ is in (1.6).

Proof. 2.3 The proof is analogous to that of Lemma 2.2. For all $z\in(1,\infty)$, by applying inequality (1.6), we obtain

$$\begin{aligned}\mathbb{P}(|Z|\geqslant z)&\geqslant \frac{1}{K(d,\alpha)}\int_{|x|\geqslant z}\frac{\mathrm{d}x}{(1+|x|)^{d+\alpha}}\\&\geqslant \frac{s_{d-1}}{2^{d+\alpha}K(d,\alpha)}\int_z^\infty\frac{\mathrm{d}r}{r^{\alpha+1}}=\frac{s_{d-1}}{2^{d+\alpha}\alpha K(d,\alpha)}\cdot\frac{1}{z^\alpha},\end{aligned}$$

where the second inequality is due to $|x|\geqslant z>1$. Similarly,

$$\mathbb{P}(z\leqslant|Z|\leqslant 2z)\geqslant\frac{s_{d-1}}{2^{d+\alpha}\alpha K(d,\alpha)}\left(1-\frac{1}{2^\alpha}\right)\cdot\frac{1}{z^\alpha}=\frac{1}{\delta_\alpha z^\alpha}.$$

We complete the proof. □

3. EM scheme in the case of $\alpha=2$

In this section, we prove Theorem 1.1. It is easy to see that the EM scheme of (1.2) can be written as

$$Y_{k+1}=Y_k-Y_k\log(1+|Y_k|)\eta+\sqrt{\eta}N_{k+1},\quad Y_0=x_0,\quad k=0,1,2,\dots, \tag{3.1}$$

where $\{N_k,k\geqslant 1\}$ represents a sequence of i.i.d. standard Gaussian random variables and, for each $k\geq 0$, N_{k+1} is independent of $(Y_j,j\leqslant k)$.

Proof of Theorem 1.1. Let $n\in\mathbb{N}$ be arbitrary. Denote

$$A_0=\left\{\omega\in\Omega:\ \sup_{1\leqslant m\leqslant n}|N_m(\omega)|\in\left[0,|\log\eta|^2\right)\right\}.$$

We make the following claim, whose proof will be postponed.

$$\sup_{1\leqslant m\leqslant n}|Y_m(\omega)|\leqslant\eta^{-1/2}\left|\log\eta\right|^2,\quad \omega\in A_0, \tag{3.2}$$

and, for all $p\geqslant 2$, there exists a constant C_p such that

$$\mathbb{E}\left[\sup_{1\leqslant m\leqslant n}|Y_m|^p\mathbb{1}_{A_0^c}\right]\leqslant C_p. \tag{3.3}$$

Let us prove (1.4) first. It follows from (3.2) and (3.3) that

$$
\begin{aligned}
&\mathbb{E}\left[|Y_m|^2 \log^2\left(1+|Y_m|\right)\right] \\
&= \mathbb{E}\left[|Y_m|^2 \log^2\left(1+|Y_m|\right)\mathbb{1}_{A_0}\right] + \mathbb{E}\left[|Y_m|^2 \log^2\left(1+|Y_m|\right)\mathbb{1}_{A_0^c}\right] \\
&\leqslant \log^2\left(1+\eta^{-1/2}|\log\eta|^2\right)\mathbb{E}|Y_m|^2 + C \\
&\leqslant 4|\log\eta|^2\mathbb{E}|Y_m|^2 + C,
\end{aligned} \tag{3.4}
$$

where C can be taken as C_p in (3.3) with $p=3$, and the last inequality is by $1+\eta^{-1/2}|\log\eta|^2 \leqslant \eta^{-2}$ for $\eta \leqslant \frac{1}{4}$. On the other hand,

$$
\begin{aligned}
\mathbb{E}\big[|Y_m|^2\log(1+|Y_m|)\big] &\geqslant \mathbb{E}\big[|Y_m|^2\log(1+|Y_m|)\mathbb{1}_{\{|Y_m|\geqslant 2\}}\big] \\
&\geqslant \mathbb{E}\big[|Y_m|^2\mathbb{1}_{\{|Y_m|\geqslant 2\}}\big] \\
&\geqslant \mathbb{E}|Y_m|^2 - 4.
\end{aligned} \tag{3.5}
$$

By (3.1), (3.4) and (3.5), we see that for all $0 \leqslant m < n$ and $\eta \leqslant \mathrm{e}^{-5}$,

$$
\begin{aligned}
\mathbb{E}|Y_{m+1}|^2 &= \mathbb{E}|Y_m|^2 - 2\eta\mathbb{E}\left[|Y_m|^2\log(1+|Y_m|)\right] + d\eta + \eta^2\mathbb{E}\left[|Y_m|^2\log^2\left(1+|Y_m|\right)\right] \\
&\leqslant \mathbb{E}|Y_m|^2 - 2\eta\mathbb{E}\left[|Y_m|^2\right] + (d+8)\eta + \eta^2\left[4|\log\eta|^2\mathbb{E}|Y_m|^2 + C\right] \\
&\leqslant \left[1-2\eta+4\eta^2|\log\eta|^2\right]\mathbb{E}|Y_m|^2 + C\eta \\
&\leqslant \mathrm{e}^{-\eta}\mathbb{E}|Y_m|^2 + C\eta \\
&\leqslant \mathrm{e}^{-(m+1)\eta}|x_0|^2 + C\sum_{k=1}^{m+1}\eta\mathrm{e}^{(k-(m+1))\eta} \\
&\leqslant |x_0|^2 + C\mathrm{e}^{-(m+1)\eta}\int_0^{(m+1)\eta}\mathrm{e}^x\,\mathrm{d}x \leqslant |x_0|^2 + C.
\end{aligned}
$$

Since $n \in \mathbb{N}$ is arbitrary, (1.4) in Theorem 1.1 clearly holds true.

It remains to show (3.2) and (3.3). For proving (3.2), let $m_0 = \min\{1 \leqslant m \leqslant n : |Y_m| > \eta^{-1/2}|\log\eta|^2\}$ and consider

$$
Y_{m_0} = Y_{m_0-1}\big(1-\eta\log(1+|Y_{m_0-1}|)\big) + \sqrt{\eta}N_{m_0}. \tag{3.6}
$$

It is easy to check that as $\eta \leqslant \mathrm{e}^{-5}$

$$
\eta\log\left(1+\eta^{-1/2}|\log\eta|^2\right) \leqslant 1.
$$

Since $|x_0| \leqslant \eta^{-\frac{1}{2}}$, (3.2) holds for $m=0$. If $2|Y_{m_0-1}| \leqslant \eta^{-1/2}|\log\eta|^2$, then it follows from (3.6) that for every $\omega \in A_0$,

$$
|Y_{m_0}| \leqslant \frac{1}{2}\eta^{-1/2}|\log\eta|^2 + \sqrt{\eta}|\log\eta|^2 \leqslant \eta^{-1/2}|\log\eta|^2.
$$

If $2|Y_{m_0-1}| > \eta^{-1/2}\big(|\log\eta|^2\big)$, due the definition of m_0, we have

$$
\begin{aligned}
|Y_{m_0}| &\leqslant |Y_{m_0-1}|\left(1-\eta\log(1+|Y_{m_0-1}|)\right) + \sqrt{\eta}|N_{m_0}| \\
&\leqslant \eta^{-1/2}|\log\eta|^2\left[1-\eta\log\left(1+\eta^{-1/2}|\log\eta|^2/2\right)\right] + \sqrt{\eta}|\log\eta|^2 \\
&\leqslant \eta^{-1/2}|\log\eta|^2.
\end{aligned}
$$

The last inequality follows from $\eta^{-1/2}|\log\eta|^2 \geqslant 2\mathrm{e}$ for all $\eta \leqslant \mathrm{e}^{-2}$. Hence, m_0 doesn't exist and claim (3.2) holds.

In order to prove (3.3), we split A_0^c into the following disjoint events:

$$A_1 = \Big\{\omega \in \Omega : \sup_{1\leqslant m\leqslant n} |N_m(\omega)| \in \big[|\log\eta|^2, \eta^{-1}\big) \Big\};$$

$$A_k = \Big\{\omega \in \Omega : \sup_{1\leqslant m\leqslant n} |N_m(\omega)| \in \big[(k-1)\eta^{-1}, k\eta^{-1}\big) \Big\}, \quad 2 \leqslant k \leqslant \lceil \mathrm{e}^{\frac{1}{2\eta}} \rceil =: k_0;$$

$$B_k = \Big\{\omega \in \Omega : \sup_{1\leqslant m\leqslant n} |N_m(\omega)| \in \Big[k\eta^{-1}\mathrm{e}^{\frac{1}{2\eta}}, (k+1)\eta^{-1}\mathrm{e}^{\frac{1}{2\eta}}\Big) \Big\}, \quad 1 \leqslant k < \infty.$$

Firstly, under $|Y_0| = |x_0| \leqslant \eta^{-\frac{1}{2}}$, we verify that for every $1 \leqslant k \leqslant \lceil \mathrm{e}^{\frac{1}{2\eta}} \rceil$,

$$\sup_{1\leqslant m\leqslant n} |Y_m(\omega)| \leqslant \frac{k}{\eta^{3/2}}, \quad \omega \in A_k. \tag{3.7}$$

Even though this is similar to the proof of (3.2), we still give the detail here for completeness. Notice that $\eta \log\big(1 + k/(2\eta)\big) < \eta \log\big(1 + (\mathrm{e}^{\frac{1}{2\eta}} + 1)/(2\eta)\big) < 1$. Let $m_0 = \min\{1 \leqslant m \leqslant n : |Y_m| > (k+1)/\eta\}$. If $2|Y_{m_0-1}| \leqslant k\eta^{-\frac{3}{2}}$, then, due to $\eta \leqslant \mathrm{e}^{-5} \leqslant \frac{1}{2}$,

$$|Y_{m_0}| \leqslant |Y_{m_0-1}| + \sqrt{\eta}\,|N_{m_0}| \leqslant \frac{k}{2\eta^{3/2}} + \frac{k}{\sqrt{\eta}} \leqslant \frac{k}{\eta^{3/2}}.$$

On the other hand, if $2|Y_{m_0-1}| > k\eta^{-\frac{3}{2}}$, then

$$\begin{aligned} |Y_{m_0}| &\leqslant |Y_{m_0-1}|\big(1 - \eta\log(1 + k/(2\eta^{3/2}))\big) + \sqrt{\eta}\,|N_{m_0}| \\ &\leqslant \frac{k}{\eta^{3/2}}(1-\eta) + \frac{k}{\sqrt{\eta}} \leqslant \frac{k}{\eta^{3/2}}, \end{aligned}$$

where the second inequality comes from the definition of m_0 and $\eta \leqslant 2^{-2/3}$. Hence, m_0 doesn't exist and the claim (3.7) holds. Besides, Lemma 2.1 yields that

$$\mathbb{P}(A_1) \leqslant C\left(\frac{1}{\eta} - |\log\eta|^2\right)\mathrm{e}^{-|\log\eta|^2/2} \leqslant C\eta^{\frac{|\log\eta|}{2}-1},$$

$$\mathbb{P}(A_k) \leqslant Cn\cdot\frac{1}{\eta}\mathrm{e}^{-\frac{(k-1)^2}{2\eta^2}} \leqslant \frac{C}{\eta^2}\mathrm{e}^{-\frac{(k-1)^2}{2\eta^2}}, \quad \forall\ 2 \leqslant k \leqslant \lceil \mathrm{e}^{\frac{1}{2\eta}} \rceil.$$

It follows that for any $p \geqslant 1$, there is some positive constant C_p' not depending on η such that

$$\begin{aligned} \sum_{k=1}^{k_0} \mathbb{E}\left[\sup_{0\leqslant m\leqslant n} |Y_m|^p\,\mathbb{1}_{A_k}\right] &\leqslant \frac{C}{\eta^{3p/2}}\cdot\eta^{\frac{|\log\eta|}{2}-1} + \frac{C}{\eta^2}\sum_{k=2}^{\infty}\frac{k^p}{\eta^{3p/2}}\mathrm{e}^{-\frac{(k-1)^2}{2\eta^2}} \\ &\leqslant C + 2^pC\sum_{k=1}^{\infty}\left(\frac{k}{\eta}\right)^{\frac{3p}{2}+2}\mathrm{e}^{-\frac{1}{2}\left(\frac{k}{\eta}\right)^2} \\ &\leqslant C + 2^pC\int_{\lfloor\frac{1}{\eta}\rfloor-1}^{\infty} y^{\frac{3p}{2}+2}\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y \leqslant C_p', \end{aligned} \tag{3.8}$$

Here the second inequality holds since one can find $\eta \leqslant \mathrm{e}^{-(3p+2)}$ such that $\frac{|\log n|}{2} - \frac{3p+2}{2} \geqslant 0$ and the fact $(k+1)^p \leqslant (2k)^p$ for any $k \geqslant 1$ and any fixed $p \geqslant 1$. Besides, $\int_z^\infty z^q\mathrm{e}^{-z^2/2}\,\mathrm{d}z < \infty$ for any $q, z > 0$ yields the last inequality.

On the event B_k, due to EM scheme (3.1) and $|Y_0| = |x_0| < \eta^{-\frac{1}{2}}$, we have

$$|Y_1| \leqslant |Y_0| \left(1 + \eta \log(1 + |Y_0|)\right) + \sqrt{\eta}\, |N_1| \leqslant \frac{2(k+1)}{\sqrt{\eta}} \mathrm{e}^{\frac{1}{2\eta}},$$

where the last inequality comes from $\eta^{-\frac{1}{2}} + \sqrt{\eta} \log(1 + \eta^{-\frac{1}{2}}) \leqslant \eta^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2\eta}}$ as $\eta \leqslant \frac{1}{2}$. Furthermore, for $|Y_2|$ we have

$$\begin{aligned}
|Y_2| &\leqslant |Y_1| \left(1 + \eta \log(1 + |Y_1|)\right) + \sqrt{\eta}\, |N_2| \\
&\leqslant \frac{2(k+1)}{\sqrt{\eta}} (1+k) \mathrm{e}^{\frac{1}{2\eta}} + \frac{(k+1)}{\sqrt{\eta}} \mathrm{e}^{\frac{1}{2\eta}} \\
&\leqslant \left[2(k+1)^2 + (k+1)\right] \frac{1}{\sqrt{\eta}} \mathrm{e}^{\frac{1}{2\eta}}
\end{aligned}$$

Hence, by induction, we obtain that for each $k \geqslant 1$,

$$|Y_m| \leqslant \left[2(k+1)^m + (k+1)^{m-1} + \cdots + (k+1)\right] \frac{1}{\sqrt{\eta}} \mathrm{e}^{\frac{1}{2\eta}} := a_m \frac{1}{\sqrt{\eta}} \mathrm{e}^{\frac{1}{2\eta}}. \tag{3.9}$$

Hence, due to $a_m \leqslant 2m(k+1)^m$ and $\eta = O(\frac{1}{n})$, we have

$$\sup_{1 \leqslant m \leqslant n} |Y_m| \leqslant \frac{2C}{\eta^{3/2}} (k+1)^{\frac{T}{\eta}} \mathrm{e}^{\frac{1}{2\eta}}.$$

Besides, applying Lemma 2.1 again,

$$\mathbb{P}(B_k) \leqslant \frac{C}{\eta} \mathrm{e}^{-\frac{1}{2}\left(\frac{k}{\eta} \mathrm{e}^{1/(2\eta)}\right)^2}, \quad \forall\, k \geqslant 1.$$

Due to similar argument (3.8) for any $p \geqslant 1$, there exists a constant C''_p not depending on η such that

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\sup_{0 \leqslant m \leqslant n} |Y_m|^p \, \mathbb{1}_{B_k}\right] \leqslant \frac{2^p C^{p+1}}{\eta^{(3p+2)/2}} \sum_{k=1}^{\infty} (k+1)^{\frac{pC}{\eta}} \mathrm{e}^{\frac{p}{2\eta}} \cdot \mathrm{e}^{-\frac{1}{2}\left(\frac{k}{\eta} \mathrm{e}^{1/(2\eta)}\right)^2} \leqslant C''_p. \tag{3.10}$$

In sum, equations (3.8) and (3.10)yield that there exists a constant C_p not depending on η such that

$$\mathbb{E}\left[\sup_{1 \leqslant m \leqslant n} |Y_m|^p \mathbb{1}_{A_0^c}\right] \leqslant \sum_{k=1}^{k_0} \mathbb{E}\left[\sup_{0 \leqslant m \leqslant n} |Y_m|^p \, \mathbb{1}_{A_k}\right] + \sum_{k=1}^{\infty} \mathbb{E}\left[\sup_{0 \leqslant m \leqslant n} |Y_m|^p \, \mathbb{1}_{B_k}\right] \leqslant C_p\ ,$$

that is, (3.3) holds. And we complete the proof. □

4. EM scheme in the case of $\alpha \in (0, 2)$

In this section, we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We only prove Part (ii) of the theorem in detail, since Part (i) can be shown in a similar way by replacing the estimates in Lemma 2.2 with the ones in Lemma 2.3.

Recall the formulations of (1.7):

$$\widetilde{Y}_{k+1} = \widetilde{Y}_k - \eta \widetilde{Y}_k \log\left(1 + \left|\widetilde{Y}_k\right|\right) + \frac{1}{\sigma} \eta^{1/\alpha} \widetilde{Z}_{k+1}, \quad \widetilde{Y}_0 := x_0,$$

where $\eta = \frac{T}{n}$ for all $k =, 0, 1, \ldots, n-1$, and the definition of K_2 in the theorem:

$$K_2 = \frac{2}{T}\left(\mathrm{e}^{\kappa_\alpha/\beta} + \frac{2}{\sigma}\right),$$

for any fixed time $T > 0$, where κ_α is given in Lemma 2.2. Due to $\eta = \frac{T}{n} \leqslant 1$, it is easy to check that

$$\eta \cdot nK_2 - \frac{2(1+\eta)}{\sigma}\eta^{1/\alpha} - 1 = 2\mathrm{e}^{\kappa_\alpha/\beta} - 1 + \frac{4}{\sigma}(1-\eta^{1/\alpha}) \geqslant \mathrm{e}^{\kappa_\alpha/\beta}. \tag{4.1}$$

We will apply this equation repeatedly. Consider a sequence of events $\widetilde{\Omega}_n \subseteq \Omega$, $n \in \mathbb{N}$, defined by

$$\begin{aligned}\widetilde{\Omega}_n := \Bigg\{\omega \in \Omega \,\Bigg|\, & \big|\widetilde{Z}_k(\omega)\big| \in [1+\eta, 2+2\eta]\,,\ \forall k = 2, \cdots, n\ ; \\ & \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_1(\omega)\big| \geqslant |x_0|\,\big(1+\log(1+|x_0|)\big) + \exp\{nK_2\}\Bigg\}.\end{aligned} \tag{4.2}$$

For all $\omega \in \widetilde{\Omega}_n$, we verify by induction that the following holds:

$$\big|\widetilde{Y}_m(\omega)\big| \geqslant \exp\left\{\frac{\kappa_\alpha}{\beta}(m-1) + nK_2\right\}, \quad \forall\, m = 1, \ldots, n. \tag{4.3}$$

If $m = 1$, the triangle inequality yields that

$$\begin{aligned}\big|\widetilde{Y}_1(\omega)\big| &= \Big|x_0 - x_0\log(1+|x_0|) + \frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_1(\omega)\Big| \\ &\geqslant \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_1(\omega)\big| - |x_0|\,\big(1+\log(1+|x_0|)\big) \geqslant \exp\{nK_2\}.\end{aligned}$$

When $m = 2$, we have

$$\begin{aligned}\big|\widetilde{Y}_2(\omega)\big| &\geqslant \eta\big|\widetilde{Y}_1(\omega)\big|\log\Big(1+\big|\widetilde{Y}_1(\omega)\big|\Big) - \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_2(\omega)\big| - \big|\widetilde{Y}_1(\omega)\big| \\ &\geqslant \big|\widetilde{Y}_1(\omega)\big| \cdot \left[\eta\log\Big(\big|\widetilde{Y}_1(\omega)\big|\Big) - \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_2(\omega)\big| - 1\right] \\ &\geqslant \exp\{nK_2\} \cdot \left[\eta \cdot nK_2 - \frac{2(1+\eta)}{\sigma}\eta^{1/\alpha} - 1\right] \geqslant \exp\left\{\frac{\kappa_\alpha}{\beta} + nK_2\right\}.\end{aligned}$$

The second inequality is a consequence to $\big|\widetilde{Y}_1(\omega)\big| \geqslant 1$. And the third inequality arises from $\big|\widetilde{Y}_1(\omega)\big| \geqslant \mathrm{e}^{nK_2}$ and $\big|\widetilde{Z}_2(\omega)\big| \leqslant 2(1+\eta)$ for all $\omega \in \widetilde{\Omega}_n$. The last inequality follows from (4.1). Therefore, (4.3) holds for $m = 2$.

For the induction step $k \to k+1$, we assume that (4.3) holds for $\ell = 1, \cdots, k$. In particular, $\big|\widetilde{Y}_\ell(\omega)\big| \geqslant \exp\{nK_2\} \geqslant 1$ for all $\ell = 1, \cdots, k$ and all $\omega \in \widetilde{\Omega}_n$. Analogous with the case of $m = 2$, we have

$$\begin{aligned}\big|\widetilde{Y}_{k+1}(\omega)\big| &\geqslant \eta\big|\widetilde{Y}_k(\omega)\big|\log\Big(1+\big|\widetilde{Y}_k(\omega)\big|\Big) - \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_{k+1}(\omega)\big| - \big|\widetilde{Y}_k(\omega)\big| \\ &\geqslant \big|\widetilde{Y}_k(\omega)\big| \cdot \left[\eta\log\Big(\big|\widetilde{Y}_k(\omega)\big|\Big) - \frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_{k+1}(\omega)\big| - 1\right] \\ &\geqslant \exp\left\{\frac{\kappa_\alpha}{\beta}(k-1) + nK_2\right\} \cdot \left[\eta \cdot nK_2 - \frac{2\eta^{1/\alpha}}{\sigma}(1+\eta) - 1\right]\end{aligned}$$

$$\geqslant \exp\left\{\frac{\kappa_\alpha}{\beta}k + nK_2\right\}.$$

The third inequality arises from $\big|\widetilde{Y}_k(\omega)\big| \geqslant \exp\{\frac{\kappa_\alpha}{\beta}(k-1)+nK_2\}$ and $\big|\widetilde{Z}_{k+1}(\omega)\big| \leqslant 2(1+\eta)$ for all $\omega \in \widetilde{\Omega}_n$. The last inequality follows from inequality (4.1). Therefore, claim (4.3) holds. In particular, for all $n \in \mathbb{N}$,

$$\big|\widetilde{Y}_n(\omega)\big| \geqslant \exp\left\{(n-1)\frac{\kappa_\alpha}{\alpha} + nK_2\right\}, \quad \forall\, \omega \in \widetilde{\Omega}_n. \tag{4.4}$$

Next, we establish a lower bound for the probability of $\widetilde{\Omega}_n$ by using Lemma 2.2. Firstly, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{\eta^{\frac{1}{\alpha}}}{\sigma}\left|\widetilde{Z}_1(\omega)\right| \geqslant |x_0|\,\big(1+\log(1+|x_0|)\big) + \exp\{nK_2\}\right)\\
&\geqslant \frac{\eta}{\sigma^\alpha}\cdot\Big(|x_0|\,\big(1+\log(1+|x_0|)\big) + \exp\{nK_2\}\Big)^{-\alpha}\\
&\geqslant \frac{T}{2\sigma^\alpha}\cdot\left(\frac{|x_0|^\alpha\,\big(1+\log(1+|x_0|)\big)^\alpha}{\exp\{\alpha nK_2\}}+1\right)^{-1}\cdot\frac{1}{n}\exp\{-\alpha nK_2\},
\end{aligned}$$

where the last inequality comes from $(a+b)^\alpha \leqslant 2(a^\alpha+b^\alpha)$ for all $a,b>0$ and $\alpha\in(0,2)$. Besides, Lemma 2.2 also yields that for all $k=2,\dots,n$,

$$\mathbb{P}\left(\big|\widetilde{Z}_k(\omega)\big| \in [1+\eta, 2+2\eta]\right) \geqslant \frac{1}{\kappa_\alpha}(1+\eta)^{-\alpha}.$$

Thus, combining condition $\mathrm{e}^{nK_2} \geqslant |x_0|\,\big(1+\log(1+|x_0|)\big)$ for sufficiently large n,

$$\mathbb{P}\big(\widetilde{\Omega}_n\big) \geqslant \frac{T}{4\sigma^\alpha}\cdot\frac{1}{n}\exp\{-\alpha nK_2\}\cdot\frac{1}{\kappa_\alpha^n}\left(1+\frac{T}{n}\right)^{-\alpha n} \geqslant C\cdot\frac{\mathrm{e}^{-\alpha nK_2}}{n\kappa_\alpha^n} \tag{4.5}$$

holds for some constant $C>0$ independent of n. Hence, combining equation (4.4) with (4.5) leads to

$$\begin{aligned}
\mathbb{E}\left[\big|\widetilde{Y}_n\big|^\beta\right] \geqslant \mathbb{E}\left[\big|\widetilde{Y}_n\big|^\beta \mathbb{1}_{\widetilde{\Omega}_n}\right] &\geqslant \mathbb{P}\big(\widetilde{\Omega}_n\big)\cdot\exp\{\kappa_\alpha(n-1)+\beta nK_2\}\\
&\geqslant \frac{C}{n}\cdot\left(\frac{\exp\{\beta K_2+\kappa_\alpha-\alpha K_2\}}{\kappa_\alpha}\right)^n.
\end{aligned}$$

Condition $K_2 < (\kappa_\alpha - \log\kappa_\alpha)/(\alpha-\beta)$ implies $\mathrm{e}^{\beta K_2+\kappa_\alpha-\alpha K_2} > \kappa_\alpha$. Consequently, we can obtain that

$$\lim_{n\to\infty}\mathbb{E}\big|\widetilde{Y}_n\big|^\beta = \infty, \quad \forall\, \beta\in(0,\alpha). \tag{4.6}$$

The proof is completed. □

Next, we prove Theorem 1.3.

Proof of Theorem 1.3. Again, we only provide a proof for Part (ii) and omit details for proving Part (i). Recall (1.10) and that that functions f and g satisfy Assumption

(A). For all $n \in \mathbb{N}$, define r_n as

$$
\begin{aligned}
r_n := \max \Bigg\{ 2, H, \left(\frac{4H}{\eta} + \frac{4H^2}{\eta\sigma}(1+\eta)\eta^{1/\alpha} \right)^{\frac{1}{\gamma-\lambda}}, \\
\left(\sigma H \big(2+H\eta\big)\big(1+\eta\big)^{-1}\eta^{-\frac{1}{\alpha}} \right)^{\frac{1}{\gamma-\lambda}} \Bigg\} \in [2,\infty).
\end{aligned}
\tag{4.7}
$$

where $\eta = \frac{T}{n}$. The third term of the right hand side ensures that

$$
\frac{\eta}{2H} r_n^{\gamma-\lambda} \geqslant 2 + \frac{2H}{\sigma}\left(1+\eta\right)\eta^{\frac{1}{\alpha}},
\tag{4.8}
$$

and the last term ensures

$$
\frac{1}{\sigma H}(1+\eta)\eta^{\frac{1}{\alpha}} r_n^{\gamma-\lambda} \geqslant 2 + H\eta.
\tag{4.9}
$$

Both equations will be used below.

Since $g(x_0) \neq 0$, there exists a constant $M \geqslant 1$ such that $|g(x_0)| \geqslant M^{-1}$ and $|x_0| + T\,|f(x_0)| \leqslant M$. And we consider the events $\widetilde{\Omega}_n \subseteq \Omega$ for all $n \in \mathbb{N}$ defined as

$$
\begin{aligned}
\widetilde{\Omega}_n := \Bigg\{ \omega \in \Omega \;\Bigg|\; \big|\widetilde{Z}_k(\omega)\big| \in \big[1+\eta, 2+2\eta\big], \forall k = 2,\ldots,n; \\
\frac{\eta^{1/\alpha}}{\sigma}\big|\widetilde{Z}_1(\omega)\big| \geqslant M(r_n + M) \Bigg\}.
\end{aligned}
\tag{4.10}
$$

We claim that for every $\omega \in \widetilde{\Omega}_n$ and $m \in \{1,2,\cdots,n\}$,

$$
|\widetilde{Y}_m(\omega)| \geqslant r_n^{\lambda^{(m-1)}}.
\tag{4.11}
$$

By induction, in the base case $m = 1$, the triangle inequality leads to

$$
\begin{aligned}
\big|\widetilde{Y}_1(\omega)\big| =& \big|x_0 + \eta f(x_0) + \frac{\eta^{1/\alpha}}{\sigma} g(x)\widetilde{Z}_1(\omega)\big| \\
\geqslant& \frac{\eta^{1/\alpha}}{\sigma}\,|g(x_0)|\,\big|\widetilde{Z}_1(\omega)\big| - |x_0| - \eta\,|f(x_0)| \\
\geqslant& \frac{\eta^{1/\alpha}}{M\sigma}\big|\widetilde{Z}_1(\omega)\big| - M \geqslant \frac{M(r_n+M)}{M} - M \geqslant r_n,
\end{aligned}
$$

which follows from the definition (1.10) of $\widetilde{Y}_1$ and (4.10) of $\widetilde{\Omega}_n$. For the induction step $m \to m+1$, We assume that equation (4.11) holds for $k \in \{1,2,\cdots,m\}$. In particular, we can obtain $|\widetilde{Y}_k(\omega)| \geqslant r_n \geqslant H \geqslant 1$. Additionally, the EM scheme (1.10) yields that

$$
\begin{aligned}
\big|\widetilde{Y}_{m+1}(\omega)\big| =& \Big|\widetilde{Y}_m(\omega) + \eta f\big(\widetilde{Y}_m(\omega)\big) + \frac{\eta^{1/\alpha}}{\sigma} g\big(\widetilde{Y}_m(\omega)\big)\widetilde{Z}_{m+1}(\omega)\Big| \\
\geqslant& \Big|\eta f\big(\widetilde{Y}_m(\omega)\big) + \frac{\eta^{\frac{1}{\alpha}}}{\sigma} g\big(\widetilde{Y}_m(\omega)\big)\widetilde{Z}_{m+1}(\omega)\Big| - \big|\widetilde{Y}_m(\omega)\big| \\
\geqslant& \max\left\{ \eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right| \left|\frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_{m+1}(\omega)\right| \right\} \\
&- \min\left\{ \eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right| \left|\frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_{m+1}(\omega)\right| \right\} - \big|\widetilde{Y}_m(\omega)\big|,
\end{aligned}
\tag{4.12}
$$

where we have repeatedly used the triangle inequality. Since $\big|\widetilde{Z}_{m+1}(\omega)\big| \in [1+\eta, 2+2\eta]$ for all $\omega \in \widetilde{\Omega}_n$, we notice that

$$\begin{aligned} &\max\left\{\eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right|\left|\frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_{m+1}(\omega)\right|\right\} \\ \geqslant &\max\left\{\eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \frac{(1+\eta)\eta^{1/\alpha}}{\sigma}\left|g\big(\widetilde{Y}_m(\omega)\big)\right|\right\}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\min\left\{\eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right|\left|\frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_{m+1}(\omega)\right|\right\} \\ \leqslant &\min\left\{\eta\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \frac{2(1+\eta)\eta^{1/\alpha}}{\sigma}\left|g\big(\widetilde{Y}_m(\omega)\big)\right|\right\}, \end{aligned} \tag{4.14}$$

If $\alpha \in (1,2)$, we have $\frac{\eta}{2} \leqslant \frac{1}{\sigma}(1+\eta)\eta^{1/\alpha}$ for sufficiently large n. By the definition of σ, we know that $\sigma \leqslant 2$ as $\alpha = 1$. Thus $\frac{\eta}{2} \leqslant \frac{1}{\sigma}(1+\eta)\eta$ if $\alpha = 1$. Consequently, it follows from (4.12), (4.13) and (4.14) that for $\alpha \in [1,2)$,

$$\begin{aligned} \big|\widetilde{Y}_{m+1}(\omega)\big| \geqslant &\frac{\eta}{2}\max\left\{\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right|\right\} \\ &-\frac{2(1+\eta)\eta^{1/\alpha}}{\sigma}\min\left\{\left|f\big(\widetilde{Y}_m(\omega)\big)\right|, \left|g\big(\widetilde{Y}_m(\omega)\big)\right|\right\} - \big|\widetilde{Y}_m(\omega)\big| \\ \geqslant &\frac{\eta}{2H}\big|\widetilde{Y}_m(\omega)\big|^{\gamma} - \frac{2H}{\sigma}(1+\eta)\eta^{\frac{1}{\alpha}}\big|\widetilde{Y}_m(\omega)\big|^{\lambda} - \big|\widetilde{Y}_m(\omega)\big|^{\lambda} \\ = &\big|\widetilde{Y}_m(\omega)\big|^{\lambda}\left[\frac{\eta}{2H}\big|\widetilde{Y}_m(\omega)\big|^{\gamma-\lambda} - \frac{2H}{\sigma}(1+\eta)\eta^{\frac{1}{\alpha}} - 1\right] \\ \geqslant &\big|\widetilde{Y}_m(\omega)\big|^{\lambda}\left[\frac{\eta}{2H}r_n^{\gamma-\lambda} - \frac{2H}{\sigma}(1+\eta)\eta^{\frac{1}{\alpha}} - 1\right] \\ \geqslant &\big|\widetilde{Y}_m(\omega)\big|^{\lambda}, \end{aligned}$$

where the first inequality comes from Assumption **(A)** and the last inequality follows from inequality (4.8). On that other hand, in the case of $\alpha \in (0,1)$, we have $\frac{2}{\sigma}(1+\eta)\eta^{1/\alpha} \leqslant \eta$ for n large enough. Then, a similar argument leads to

$$\big|\widetilde{Y}_{m+1}(\omega)\big| \geqslant \big|\widetilde{Y}_m(\omega)\big|^{\lambda}\left[\frac{(1+\eta)\eta^{1/\alpha}}{\sigma H}r_n^{\gamma-\lambda} - H\eta - 1\right] \geqslant \big|\widetilde{Y}_m(\omega)\big|^{\lambda},$$

where the last inequality follows from (4.9). Hence, the induction hypothesis yields that for all $\alpha \in (0,2)$

$$\big|\widetilde{Y}_{m+1}(\omega)\big| \geqslant \big|\widetilde{Y}_m(\omega)\big|^{\lambda} \geqslant \left(r_n^{\lambda^{m-1}}\right)^{\lambda} = r_n^{\lambda^m}.$$

This proves the claim (4.11). In particular, since $r_n \geqslant 2$, we obtain

$$\big|\widetilde{Y}_n(\omega)\big| \geqslant r_n^{\lambda^{n-1}} \geqslant 2^{\lambda^{n-1}}, \quad \forall\, \omega \in \widetilde{\Omega}_n, \text{ and } n \in \mathbb{N} \tag{4.15}$$

Furthermore, by applying Lemma 2.2, we derive the following lower bound for the probability of $\widetilde{\Omega}_n$,

$$\begin{aligned}\mathbb{P}(\widetilde{\Omega}_n) =&\mathbb{P}\left(\left|\frac{\eta^{1/\alpha}}{\sigma}\widetilde{Z}_1\right| \geqslant M(r_n+M)\right)\prod_{k=2}^{n}\mathbb{P}\left(\left|\widetilde{Z}_k\right| \in \left[1+\eta, 2+2\eta\right]\right)\\ \geqslant&\mathbb{P}\left(\left|\widetilde{Z}_1\right| \geqslant \frac{\sigma M(r_n+M)}{\eta^{1/\alpha}}\right)\left[\mathbb{P}\left(\left|\widetilde{Z}_1\right| \in \left[1+\eta, 2+2\eta\right]\right)\right]^n\\ \geqslant&\frac{T}{n\kappa_\alpha^n}\left[\frac{1}{\sigma M(r_n+M)}\right]^\alpha\left(1+\frac{T}{n}\right)^{-\alpha n}.\end{aligned}$$

Thus, there exists a constant $c \in (1, \infty)$ such that

$$\mathbb{P}(\widetilde{\Omega}_n) \geqslant cn^{-c}\kappa_\alpha^{-n} \tag{4.16}$$

for all sufficiently large n.

Combining equations (4.15) with (4.16) gives that for any $\beta \in (0, \alpha)$

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[\left|\widetilde{Y}_n\right|^\beta\right] \geqslant \lim_{n\to\infty}\mathbb{E}\left[\left|\widetilde{Y}_n\right|^\beta\mathbb{1}_{\widetilde{\Omega}_n}\right] &\geqslant \lim_{n\to\infty}\left[\mathbb{P}[\widetilde{\Omega}_n]\, r_n^{\beta\lambda^{(n-1)}}\right]\\ &\geqslant \lim_{n\to\infty}\left(cn^{-c}\kappa_\alpha^{-n}\right)\cdot 2^{\beta\lambda^{n-1}} = \infty.\end{aligned} \tag{4.17}$$

For proving Part (i) of Theorem 1.3, we can employ a similar approach to the EM scheme . Define the parameter r_n as in (4.7) but without including the term σ. We consider a sequence of events Ω_n defined by

$$\begin{aligned}\Omega_n := \Big\{\omega \in \Omega \;\Big|\; &\left|Z_k(\omega)\right| \in \left[1+\eta, 2+2\eta\right], \forall k = 2, \ldots, n;\\ &\eta^{1/\alpha}\left|Z_1(\omega)\right| \geqslant M(r_n+M)\Big\}.\end{aligned}$$

Then, an argument similar to that for Part (ii) and Lemma 2.3 yield the desired conclusion. The proof is completed. □

5. Simulations

In this section, we present some numerical simulations that illustrate the convergence and divergence of EM scheme for $d = 1$.

Firstly, we consider SDE (1.2) with $\alpha = 2$ and the corresponding EM scheme (3.1). We set the time interval $t \in [0, T]$ with $T = 10$ and 100. For both time intervals, we take initial values Y_0 of 1, 5 and 10 respectively. And set step size η to 0.001 as $T = 10$, and to 0.01 as $T = 100$. Besides, we set $n = \frac{T}{\eta} = 10\,000$ for each case. More precisely, Figures 1 and 2 below illustrate the simulations of the EM scheme for the second absolute moment $\mathbb{E}\left|Y_k\right|^2$ over the range $0 \leqslant k \leqslant n$ with iteration steps $n = 10000$ and initial values of $Y_0 = 1$, 5, and 10. In each figure, the blue line corresponds to $Y_0 = 1$, the green to $Y_0 = 5$, and the red to $Y_0 = 10$.

Figures 1 and 2 indicate that $\{\mathbb{E}\left|Y_k\right|^2, k \leqslant n\}$ are bounded and has a clear decreasing trend with respect to $0 \leqslant k \leqslant n$ for each initial value and step size.

For SDE (1.2) with $\alpha \in (0, 2)$ and the corresponding EM scheme (1.7), due to the condition of K_2 in the proof of Theorem 1.2, we choose $T = 100$ here. And we consider three cases, that is, $\alpha = 0.5, 1.0$, and 1.5. For each case, we let β be $\frac{\alpha}{8}$, $\frac{\alpha}{4}$

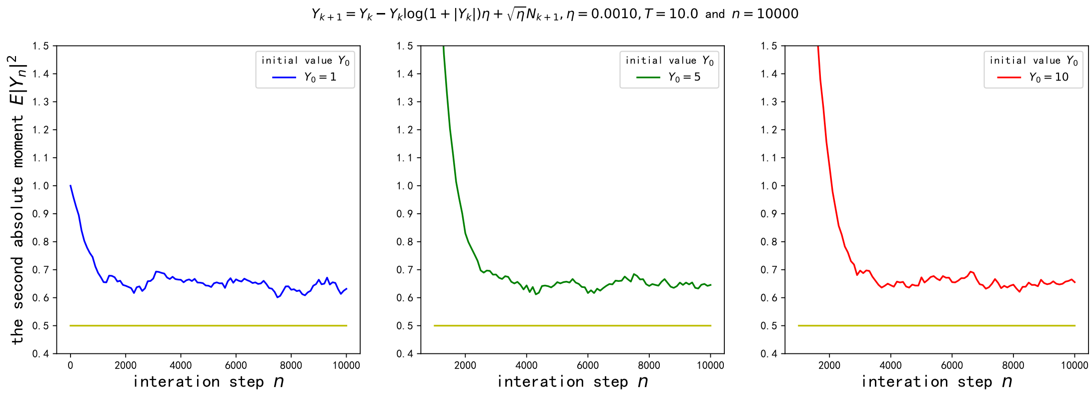

FIGURE 1. As $T = 10$, simulations values of the second absolute moment $\mathbb{E}|Y_k|^2$ for the EM scheme (3.1) with initial $Y_0 = 1, 5, 10$, $\eta = 0.001$ and iteration steps $n = 10\,000$, $0 \leqslant k \leqslant n$.

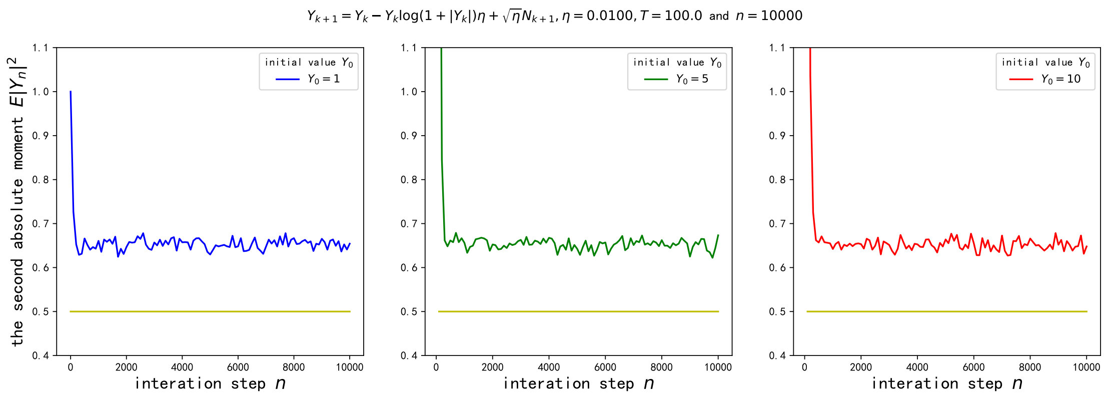

FIGURE 2. As $T = 100$, simulations values of the second absolute moment $\mathbb{E}|Y_k|^2$ for the EM scheme (3.1) with initial $Y_0 = 1, 5, 10$, $\eta = 0.01$ and iteration steps $n = 10\,000$, $0 \leqslant k \leqslant n$.

and $\frac{\alpha}{2}$. The simulated values of the β-th moment of $\widetilde{Y}_n$ in these cases are listed in Tables 1, 2, and 3.

From Tables 1, 2, and 3, we observe that the simulated values of the absolute moments of $\widetilde{Y}_n$ increase with respect to number of iteration n. In addition, they suggest that, for smaller values of $\alpha \in (0, 1)$, lager values of time T should be chosen due to constant $\kappa_\alpha = (2^\alpha/(2^\alpha - 1))$ in Lemma 2.2. Consequently, these tables indicate that the behavior of the absolute $\frac{\alpha}{2}$-th moment varies with respected to α if we choose a fixed time T.

6. APPENDIX: THE EXISTENCE AND UNIQUENESS OF STRONG SOLUTION

Since the diffusion coefficient function $f(x) = -x\log(1+|x|)$ in SDE (1.2) is only locally Lipschitz and satisfies the local growth condition in [1], Theorem 6.2.11 in [1] implies that (1.2) has a unique local solution. The following theorem strengthens this result by proving the existence and uniqueness of strong global solution of SDE (1.2).

TABLE 1. Simulated values of the absolute moment for the EM scheme (1.7) with $T = 100$, $\alpha = 0.50$ and $n = \{100, 105, 110, \ldots, 145\}$.

n	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/8}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/4}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/2}$
100	1.8×10^{13}	6.2×10^{26}	3.8×10^{55}
105	6.5×10^{13}	1.3×10^{28}	4.7×10^{57}
110	2.8×10^{14}	3.0×10^{29}	3.6×10^{60}
115	1.2×10^{15}	5.4×10^{30}	1.6×10^{63}
120	5.5×10^{15}	1.3×10^{32}	4.3×10^{65}
125	2.4×10^{16}	2.1×10^{33}	1.7×10^{68}
130	1.1×10^{17}	4.5×10^{34}	1.3×10^{71}
135	1.5×10^{18}	∞	∞
140	∞	∞	∞
145	∞	∞	∞

TABLE 2. Simulated values of the absolute moment for the EM scheme (1.7) with $T = 100$, $\alpha = 1.0$ and $n = \{100, 105, 110, \ldots, 145\}$.

n	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/8}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/4}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/2}$
100	3.8×10^{25}	7.1×10^{52}	2.2×10^{109}
105	7.7×10^{26}	1.5×10^{55}	2.3×10^{112}
110	1.3×10^{28}	1.1×10^{58}	$5,4 \times 10^{117}$
115	2.7×10^{29}	2.2×10^{60}	2.7×10^{124}
120	4.6×10^{30}	1.3×10^{63}	6.3×10^{128}
125	9.2×10^{31}	3.9×10^{65}	1.2×10^{135}
130	1.7×10^{33}	2.9×10^{68}	2.1×10^{141}
135	2.9×10^{34}	3.7×10^{71}	1.9×10^{145}
140	5.9×10^{35}	1.8×10^{73}	∞
145	∞	∞	∞

Theorem 6.1. *SDE (1.2) has an unique strong global solution. Moreover, if $\alpha = 2$, we have*

$$\mathbb{E}|X_t|^2 < \infty, \quad \forall\ t > 0;$$

if $\alpha \in (0, 2)$, then for every $\beta \in (0, \alpha)$,

$$\mathbb{E}|X_t|^\beta < \infty, \quad \forall\ t > 0.$$

By adopting the argument in [11, Section 1.6], we can show the theorem for the case of $\alpha = 2$. This argument can be extended to prove the theorem for the case of $\alpha \in (0, 2)$. However, we have not been able to find a proof in the literature. For completeness, we provide a proof of Theorem 6.1.

TABLE 3. Simulated values of the absolute moment for the EM scheme (1.7) with $T = 100$, $\alpha = 1.5$ and $n = \{100, 105, 110, \dots, 145\}$.

n	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/8}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/4}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/2}$
100	8.7×10^{37}	8.4×10^{77}	7.0×10^{158}
105	5.8×10^{39}	5.6×10^{83}	7.8×10^{168}
110	3.9×10^{41}	3.7×10^{86}	6.9×10^{174}
115	2.7×10^{43}	1.4×10^{90}	2.1×10^{182}
120	2.9×10^{45}	6.0×10^{93}	4.6×10^{192}
125	1.9×10^{47}	2.4×10^{97}	1.8×10^{200}
130	3.3×10^{49}	4.8×10^{101}	8.7×10^{207}
135	4.7×10^{51}	7.2×10^{104}	7.1×10^{215}
140	2.9×10^{53}	2.5×10^{111}	9.2×10^{219}
145	∞	∞	∞

Since the coefficient function $f(x) = -x\log(1+|x|)$ is a local Lipschitz function, For every $n \in \mathbb{N}_+$, we define the following truncated function $f_n(x)$ on $\mathbb{R}^d$ by

$$f_n(x) = \begin{cases} -x\log(1+|x|), & |x| \leqslant n; \\ -x\log(1+n), & |x| > n. \end{cases}$$

Proof. By the definition of $f_n(x)$, it can been verified that $f_n(x)$ is a global Lipschitz function with linear growth. Hence, [1, Theorem 6.2.3] implies that the SDE

$$\mathrm{d}X_{n,t} = f_n(X_{n,t})\,\mathrm{d}t + \mathrm{d}L_t \tag{6.1}$$

has a unique strong solution F_n, and

$$X_n(t, x, \omega) := F_n(x, \omega)(t).$$

Define a stopping time τ_n as

$$\tau_n = \inf\{t \geqslant 0 : |X_{n,t}| \geqslant n\}, \quad n \geqslant 2.$$

By the definition of $f_n(x)$, we have

$$\begin{aligned} \langle x, f_n(x)\rangle &= -|x|^2\log(1+|x|)\mathbb{1}_{[0,n]}(|x|) - \log(1+n)\,|x|^2\,\mathbb{1}_{(n,\infty)}(|x|) \\ &= -|x|^2\log(1+|x|)\big(\mathbb{1}_{[0,\mathrm{e}-1]}(|x|) + \mathbb{1}_{(\mathrm{e}-1],n}(|x|)\big) - |x|^2\,\mathbb{1}_{(n,\infty)}(|x|) \\ &\leqslant -|x|^2 + 1. \end{aligned}$$

In the case of $\alpha \in (0,2)$, we define function $V(x) : \mathbb{R}^d \to \mathbb{R}$ as

$$V_\beta(x) = \big(1+|x|^2\big)^{\beta/2}, \quad \beta \in (0,\alpha).$$

Then we have

$$\begin{aligned} \nabla V_\beta(x) &= \frac{\beta x}{(1+|x|^2)^{1-\beta/2}}, \\ \nabla^2 V_\beta(x) &= \frac{\beta\, I_d}{(1+|x|^2)^{1-\beta/2}} + \frac{\beta(\beta-2)xx'}{(1+|x|^2)^{2-\beta/2}}, \end{aligned}$$

where I_d is the identity matrix in $\mathbb{R}^{d\times d}$. Hence, for all $x\in\mathbb{R}^d$, we have $|x|^\beta \leqslant V_\beta(x)\leqslant 1+|x|^\beta$ and

$$|\nabla V_\beta(x)|\leqslant \beta\,|x|^{\beta-1},\quad \left\|\nabla^2 V_\beta(x)\right\|_{\mathrm{HS}}\leqslant \beta(3-\beta)\sqrt{d}.$$

Besides, the following also holds for all $x\in\mathbb{R}^d$

$$\langle\nabla V_\beta(x),f_n(x)\rangle=\frac{\beta\,\langle x,f_n(x)\rangle}{(1+|x|^2)^{1-\beta/2}}\leqslant -\beta V_\beta(x)+2\beta.$$

Itô's formula yields that

$$\begin{aligned}V_\beta(X_{n,t})&=V_\beta(x)+\int_0^t\langle\nabla V_\beta(X_{n,s}),f_n(X_{n,s})\rangle\,\mathrm{d}s\\&\quad+\int_0^t\int_{|z|<1}\left[V_\beta(X_{n,s}+z)-V_\beta(X_{n,s})\right]\widetilde{P}(\,\mathrm{d}s,\,\mathrm{d}z)\\&\quad+\int_0^t\int_{|z|\geqslant1}\left[V_\beta(X_{n,s}+z)-V_\beta(X_{n,s})\right]P(\,\mathrm{d}s,\,\mathrm{d}z)\\&\quad+\int_0^t\int_{|z|<1}\left[V_\beta(X_{n,s}+z)-V_\beta(X_{n,s})-\langle\nabla V_\beta(X_{n,s}),z\rangle\right]\frac{C_{d,\alpha}\,\mathrm{d}z\,\mathrm{d}s}{|z|^{d+\alpha}}.\end{aligned}\tag{6.2}$$

Before estimating $\mathbb{E}V_\beta(X_{n,t})$, we compute the following. If $\alpha\in(1,2)$, we let $\beta\in(1,\alpha)$, then we have that for any $x\in\mathbb{R}^d$

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus\{0\}}\left[V_\beta(x+z)-V_\beta(x)-\langle\nabla V_\beta(x),z\rangle\,\mathbb{1}_{(0,1)}(|z|)\right]\frac{\mathrm{d}z}{|z|^{d+\alpha}}\\&=\int_{|z|\geqslant1}\int_0^1\langle\nabla V_\beta(x+sz),z\rangle\frac{\mathrm{d}s\,\mathrm{d}z}{|z|^{d+\alpha}}+\int_{|z|<1}\int_0^1\int_0^s\left\langle\nabla^2V_\beta(x+uz),zz'\right\rangle_{\mathrm{HS}}\frac{\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}z}{|z|^{d+\alpha}}\\&\leqslant\beta\int_{|z|\geqslant1}(|x|^{\beta-1}|z|+|z|^\beta)\frac{\mathrm{d}z}{|z|^{d+\alpha}}+\beta(3-\beta)\sqrt{d}\int_{|z|<1}|z|^2\frac{\mathrm{d}z}{|z|^{d+\alpha}}\\&=\beta s_{d-1}\left(\frac{|x|^{\beta-1}}{\alpha-1}+\frac{1}{\alpha-\beta}\right)+\frac{\beta(3-\beta)s_{d-1}\sqrt{d}}{2(2-\alpha)}.\end{aligned}$$

On the other hand, for $\alpha\in(0,1]$, we use the inequality $(a+b)^\beta\leqslant a^\beta+b^\beta$, where $\beta\in(0,1)$ and $a,b>0$, to derive

$$\begin{aligned}\int_{|z|\geqslant1}\left[V_\beta(x+z)-V_\beta(x)\right]\frac{\mathrm{d}z}{|z|^{d+\alpha}}&\leqslant\int_{|z|\geqslant1}\left[1+|x+z|^\beta-|x|^\beta\right]\frac{\mathrm{d}z}{|z|^{d+\alpha}}\\&\leqslant\int_{|z|\geqslant1}\frac{1+|z|^\beta}{|z|^{d+\alpha}}\,\mathrm{d}z=\frac{(2\alpha-\beta)s_{d-1}}{\alpha(\alpha-\beta)}.\end{aligned}$$

Thus, an argument similar to that for the case $\alpha\in(1,2)$ implies that for $\alpha\in(0,1]$ and $\beta\in(0,\alpha)$,

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus\{0\}}\left[V_\beta(x+z)-V_\beta(x)-\langle\nabla V_\beta(x),z\rangle\,\mathbb{1}_{(0,1)}(|z|)\right]\frac{\mathrm{d}z}{|z|^{d+\alpha}}\\&\leqslant\frac{(2\alpha-\beta)s_{d-1}}{\alpha(\alpha-\beta)}+\frac{\beta(3-\beta)s_{d-1}\sqrt{d}}{2(2-\alpha)}.\end{aligned}$$

By combining the above inequalities with equation (6.2), we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{E}V_\beta(X_{n,t})}{\mathrm{d}t} &= \mathbb{E}[\langle \nabla V_\beta(X_{n,t}), f_n(X_{n,t})\rangle] \\
&\quad + C_{d,\alpha}\mathbb{E}\left[\int_{\mathbb{R}^d\setminus\{0\}} [V_\beta(X_{n,t}+z) - V_\beta(z) - \langle \nabla V_\beta(X_{n,t}), z\rangle \mathbb{1}_{(0,1)}(|z|)] \frac{\mathrm{d}z}{|z|^{d+\alpha}}\right] \\
&\leqslant -\beta\mathbb{E}V_\beta(X_{n,t}) + c_1\left[\mathbb{E}V_\beta(X_{n,t})^{\frac{\beta-1}{\beta}}\mathbb{1}_{(1,2)}(\alpha) + \mathbb{1}_{(0,1]}(\alpha)\right] + c_2 \\
&\leqslant -C_1\mathbb{E}V_\beta(X_{n,t}) + C_2,
\end{aligned} \tag{6.3}
$$

where the first inequality follows from $|x|^{\beta-1} \leqslant V_\beta(x)^{(\beta-1)/\beta}$ as $\beta > 1$, the second inequality from Hölder's inequality, and c_1, c_2, C_1, C_2 are constants independent of t and n. Hence, for any $\alpha \in (0,2)$, the differential inequality (6.3) leads to

$$
\mathbb{E}V_\beta(X_{n,t}) \leqslant V_\beta(x_0)\mathrm{e}^{-C_1 t} + C_2/C_1 \leqslant C, \quad \forall\, t \geqslant 0. \tag{6.4}
$$

Due to the definition of $V_\beta(x)$, we know that

$$
V_\beta(X_{n,\tau_n}) \geqslant (n^2+1)^{\frac{\beta}{2}},
$$

then for any $T > 0$, (6.4) and Markov's inequality lead to

$$
(n^2+1)^{\frac{\beta}{2}}\mathbb{P}(\tau_n \leqslant T) \leqslant \mathbb{E}\left[V_\beta(X_{n,\tau_n})\mathbb{1}_{\tau_n \leqslant T}\right] \leqslant C.
$$

Let $n \to \infty$, we obtain

$$
\mathbb{P}(\lim_{n\to\infty} \tau_n \leqslant T) = 0,
$$

This and the arbitrariness of T imply

$$
\lim_{n\to\infty} \tau_n = \infty. \tag{6.5}
$$

As a result, (6.5) yields that $\forall x \in \mathbb{R}^d, t \geqslant 0$,

$$
X(t,x,\omega) = \lim_{n\to\infty} X_n(t,x,\omega)
$$

exists and is continuous with respect to (t,x), which is a solution of SDE (1.2). On the other hand, let $X(t)$ and $Y(t)$ be solutions with the same initial value x. Define

$$
\gamma_n = \inf\{t : |X(t)| \geqslant n\}, \quad \theta_n = \inf\{t : |Y(t)| \geqslant n\}.
$$

Then, we have

$$
X(\gamma_n \wedge \theta_n \wedge t) - Y(\gamma_n \wedge \theta_n \wedge t) = \int_0^t f_n(X(\gamma_n \wedge \theta_n \wedge s)) - f_n(Y(\gamma_n \wedge \theta_n \wedge s))\,\mathrm{d}s,
$$

which implies that

$$
\begin{aligned}
&\mathbb{E}\left|X(\gamma_n \wedge \theta_n \wedge t) - Y(\gamma_n \wedge \theta_n \wedge t)\right| \\
\leqslant\ &\|f_n\|_{\mathrm{Lip}} \int_0^t \mathbb{E}\left|X(\gamma_n \wedge \theta_n \wedge s) - Y(\gamma_n \wedge \theta_n \wedge s)\right|\,\mathrm{d}s.
\end{aligned}
$$

Then by Grönwall's inequality, we have that

$$
\mathbb{E}\left|X(\gamma_n \wedge \theta_n \wedge t) - Y(\gamma_n \wedge \theta_n \wedge t)\right| = 0, \quad \forall\, t \geqslant 0,
$$

which implies that $X(t) = Y(t)$ on $t \leqslant \gamma_n \wedge \theta_n$. Then, let $n \to \infty$, we have $\gamma_n \wedge \theta_n \to \infty, a.s.$. Hence, $X(t) = Y(t), a.s.$ for all $t \geqslant 0$.

Finally, for the moment estimation, by the same argument for bounding $\mathbb{E}V_\beta(X_{n,t})$, we can establish that $\mathbb{E}V_\beta(X_t) \leqslant C$ for all $t > 0$, where $C > 0$ is a constant not depending on t. This implies that $\mathbb{E}|X_t|^\beta \leqslant \mathbb{E}V_\beta(X_t) \leqslant C$ for all $\beta \in (0, \alpha)$ and $t > 0$. This completes the proof. $\square$

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