Lattice study of disordering of inhomogeneous condensates and the Quantum Pion Liquid in effective $O(N)$ model*

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In this talk, we study a scalar $O(N)$ model with a so-called moat regime – a regime with negative bosonic wave function renormalization – using lattice field theory. For negative bare wave function renormalization, inhomogeneous condensates are solutions of the classical equations of motions. Using hybrid Monte Carlo simulations we demonstrate how bosonic quantum fluctuations disorder the inhomogeneous condensate. Instead, one finds a so-called Quantum Pion Liquid, where bosonic correlation functions are spatially oscillating, but also exponentially decaying.

1. Introduction

Scalar $O(N)$ field theories are a useful tool for the study of phase transitions. This is especially true in the case of Quantum Chromodynamics (QCD) at finite density where 1) Lorentz symmetry is broken explicitly and therefore new phases can arise and 2) Monte-Carlo simulations are not available due to the sign problem. In particular, QCD might exhibit the so-called inhomogeneous phase (IP) – where in addition to chiral symmetry also translational invariance is spontaneously broken [1]. In order to study this phenomenon one can consider effective models such as Nambu-Jona-Lasinio (NJL) [2] and scalar $O(N)$ models [3]. Instead (or in addition to) of critical end-point of $O(4)$ universality class one might expect to find so-called Lisfhitz point where three phases meet: disordered, ordered and the IP.

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In previous studies of an NJL-type model in 1+1 dimensions it was found that the IP coincides with so-called moat regime [2], where the dispersion relation of quasi-particles exhibits minimum at non-zero momentum. This unusual feature manifests itself as periodic spatial oscillations of two-point correlation functions. However, the connection of the moat regime to IP is not straightforward. Quantum fluctuations are expected to weaken ordered phases such as the IP [4, 5], and in fact they can even eliminate them. In particular, it was argued that in $3+1$ dimensional scalar $O(N)$ model one can find so-called Quantum Pion Liquid (QπL) analogous to Quantum Spin Liquid (QSL) instead of IP [3] due to transverse fluctuations. In this case, QπL is characterized by small yet always finite mass and is accompanied by the moat regime. The QπL regime can also be generated through mixing effects of scalar and vector modes [6, 7].

In this work we set ourselves to go beyond limitations of previous works and address the problem of inhomogeneous phase using ab-initio lattice calculations of scalar $O(N)$ model at finite N.

2. Model and algorithm

We consider an $O(N)$ model in $3+1$ dimensions with spatial higher-derivative terms as defined in [3] and take into account only the static Matsubara mode $\beta\omega_0 = 0$ where $\beta = 1/T$. In this case, an effective 3-dimensional Lagrangian of the static mode reads as

$$L_{\text{eff}} = \frac{Z}{2}\left(\partial_j\vec{\phi}\right)^2 + \frac{1}{2M^2}\left(\sum_j \partial_j^2\vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2. \quad (1)$$

This Lagrangian can be analyzed in mean-field approximation and in the large-N limit. In the mean field approach, it features a Lifshitz point at $Z = 0$ and $m^2 = 0$ where three phases meet: disordered, ordered and the IP. The ground state of the IP is a chiral spiral

$$\vec{\phi} = \phi_0\left(\cos(k_0 z), \sin(k_0 z), \phi_\perp = \vec{0}\right)^T \quad (2)$$

where k_0 and ϕ_0 are such that they minimize the free energy. In contrast, in the large-N limit the Lifshitz point is gone. The IP is replaced by a QπL characterized by dynamically generated mass gap and oscillating two-point function for certain regions in the (m^2, Z) plane. The QπL can exist for both the bare parameter $Z < 0$ but also for positive Z. As was argued in Ref. [3], even at finite values of N transverse fluctuations should be strong enough to disorder the condensate and transform the IP into a QπL.

In order to write down a Lagrangian suitable for numerical calculations we discretize derivatives:

$$[\Delta_i]_{x,y} = \frac{1}{12a^2} \left(-\delta_{y,x+2e_i} + 16\delta_{y,x+e_i} - 30\delta_{y,x} + 16\delta_{y,x-e_i} - \delta_{y,x-2e_i}\right) \quad (3)$$

where e_i is a unit lattice step in the i-th direction, and perform standard Hybrid Monte-Carlo (HMC) simulations. We run the calculations for $N = 1, 2, 4$ and lattice sizes $V = 12^3, 16^3, 20^3$ and various values of coupling constants. We generated around $3000 - 7000$ independent configurations for each point of the phase diagram and used Jackknife algorithm for error estimation.

3. Results

As discussed above, one expects different alternative scenarios to an IP from analytical approximations to the partition function of model (1). Both the IP as well as the Quantum Pion Liquid regime are characterized by a particular behavior of bosonic correlations functions. Also, direct access of $\langle\vec{\phi}(x)\rangle$ or $\langle|\vec{\phi}|(x)\rangle$ is not suitable for detecting IPs due to destructive interference [4]. Thus, a straightforward choice for an observable characterizing these different regimes are the spatial correlation functions between the bosonic fields $\vec{\phi}$

$$C^{ij}(\vec{x}) = \langle c^{ij}(\vec{x})\rangle = \frac{1}{V} \sum_{\vec{y}} \langle\phi^i(\vec{y}+\vec{x})\phi^j(\vec{y})\rangle, \quad (4)$$

where the sum over lattice sites y is used to get more statistics. In order to characterize the different regimes, we use fits of C^{jj} for the respective regimes:

- Decaying oscillations $C^{ij}(x) \sim \delta_{ij} e^{-m_r x} \cos(kx)$ for the QπL, ordinary symmetry-restored phase (OSP) (using $k = 0$) and the IP (using $m_r = 0$).

- Algebraically decaying oscillations $C^{ij}(x) \sim \delta_{ij}\frac{\cos(kx)}{x^\alpha}$ for possible quasi-long range order (a.k.a liquid crystal)

As a criterion to compare the fit qualities we use the coefficient of determination. In the following, we will present results for $M^2 = 1.0$, $\lambda N = 1.0$ and $V = 20^3$. However, we note that our findings are stable among different volumes $V = 12^3, 16^3, 20^3$ except for very small, negative Z, where the decay rates are getting small and larger volumes are needed.

In fig. 1, we plot C^{11}, see eq. (4), as well as the preferred fit scenario for $Z = -1.0, m^2 = 0.0$ and $N = 1, 2, 4$. As one can see, the Quantum
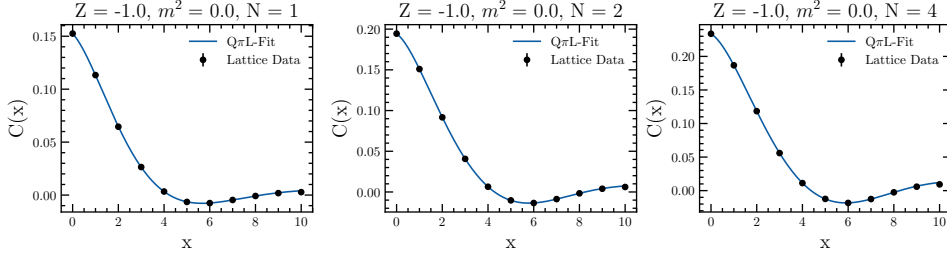
Fig. 1. Plot of $C(x) = C^{11}((x, 0, 0))$ for $Z = -1.0, m^2 = 0.0$. The preferred fit scenario is determined using the coefficient of determination. **(left)** $N = 1$. **(middle)** $N = 2$. **(right) N = 4**.

Pion Liquid fit scenario is preferred independently of N. From Ref. [3], one would have expected to obtain an IP at least for $N = 1$ since there is no disordering through Goldstone modes of $O(N)$ symmery breaking. When further decreasing Z the obtained exponential decay rate gets smaller in consistency with the predictions from Ref. [3] from the large-N limit.

In general, the observation of the different regimes in the (m^2, Z) plane is similar to the large-N findings [3] for all studied N. For fig. 2, we depict the found regimes for $N = 1$ in comparison to the large-N boundary lines. Our findings are in agreement with the large-N prediction, although one has to note that the comparison of different fit scenarios does not yield conclusive results near any of the phase boundaries. The phase diagram in the (m^2, Z) plane seems to be identical for all studied values of Z, m^2 and N up to the current status of investigation.

3.1. Infinite volume limit with the external field

The above findings can be seen as an indication that the IP might not exist in the (m^2, Z) phase diagram of eq. (1). However, one still has to perform the thermodynamic limit, i.e. $V \to \infty$, in order to identify phase transitions. On the lattice we only simulate finite volumes. Thus, we are in the need of an extrapolation method of our findings to the infinite volume. While a finite volume scaling of the exponential decay rate m_r in the Quantum Pion Liquid regime is plagued by fitting and statistical errors, one can go the traditional route of studying phase transition using an external symmetry breaking parameter. This is introduced for $N = 1$ by modifying eq. (1) using an oscillating external field

$$L_{h_0} = L_{\text{eff}} - h(x)\phi_1(x), \quad h(x) = \frac{h_0}{\sqrt{2\pi}L\,\sigma_0}\sum_{n=0}^{L-1}e^{-\frac{1}{2\sigma}(p_n - k_0)^2}\cos(p_n x_2) \quad (5)$$
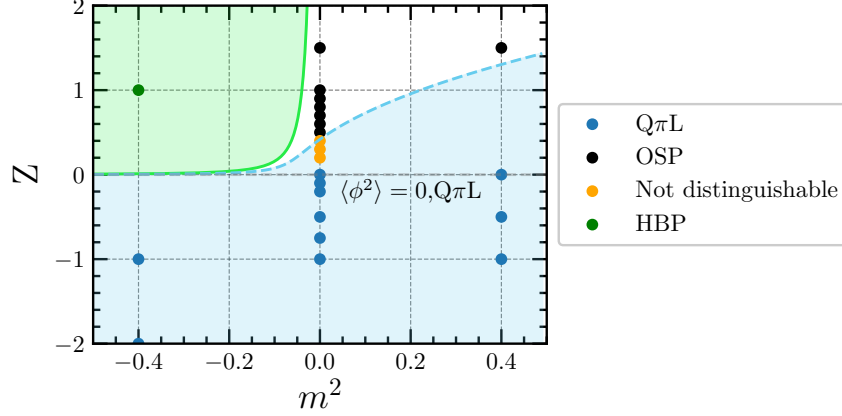
Fig. 2. Phase diagram from fits for $N = 1$. The green color denotes Homogeneous Broken Phase (HBP), white — Ordinary Symmetric Phase (OSP) and blue — Quantum Pion Liquid (QπL) regions according to the large-N results from Ref. [3].

with Gaussian distributed momenta, where $p_n = 2\pi n/V^{1/3}$ and $\sigma_0 = 0.1$. In order to determine the peak of the Gaussian in momentum space, we extract k_0 from the simulations with $h_0 = 0$. For $Z = -1.0, m^2 = 0.0$ and $V = 20^3$ as used in fig. 3 we determine $k_0 = 0.942 \approx 3 \times 2\pi/20$. Since the introduction of the external symmetry breaking term does not only break translational but also rotational invariance for $h_0 \neq 0$, we expect that the correlation function still depends on the relative differences in the x_0 and x_1 directions. Translational symmetry breaking should then have an impact on its dependence on the x_2 directions, i.e., one expects

$$C(x_0 - y_0, x_1 - y_1, x_2, y_2) = C^{11}(x, y). \tag{6}$$

In fig. 3, we plot $C(0, 0, x, y)$ as a color map. For the left plot with $h_0 = 0.04$, one can directly see the signs of translational symmetry breaking due to external field. On the other hand, translational invariance is restored for $h_0 = 0.01$ on the right plot of fig. 3 since there clearly is only a dependence of C on $x - y$. It is important to determine the transition point $h_{0,c} \in (0.01, 0.04)$ between these behaviors as a function of the volume V in order to clarify the fate of translation symmetry. If an extrapolation of the computed values of $h_{0,c}$ reveals that $h_{0,c}(V) = 0.0$ for $V \to \infty$, the phase diagram of model (1) would in turn consist of an IP for $N = 1$ in the thermodynamic limit. Alternatively, one could rigorously establish that inhomogeneous condensates can simply be disordered by the inclusion of bosonic quantum fluctuations. It would also be interesting to study the
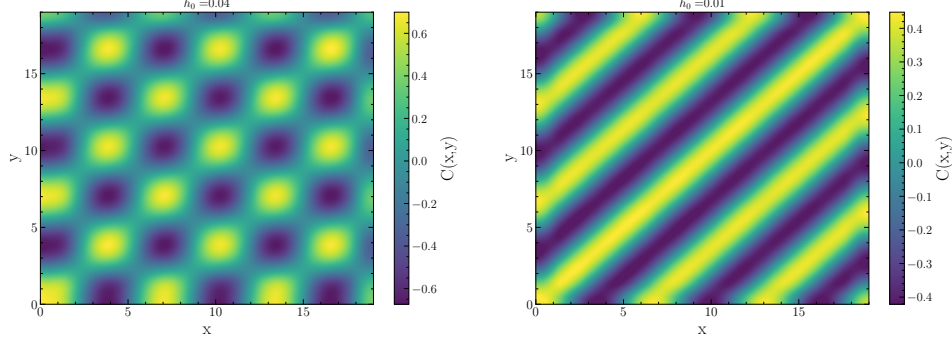
Fig. 3. Color code of $C(0, 0, x, y)$ for $Z = -2.0, m^2 = 0.0$. Note that the scaling of the color bar differs in both plots. **(left)** $h_0 = 0.04$. **(right)** $h_0 = 0.01$.

dependence of $C^{ij}(x, y)$ on directions which are transverse to $h(x)$ in order to study spontaneous breaking of the rotational symmetry. This work is planned for a future publication.

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