A DIMCA-GREUEL TYPE INEQUALITY FOR FOLIATIONS

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Abstract. Let $\mathcal{F}$ be a holomorphic foliation at $p \in \mathbb{C}^2$, and B be a separatrix of $\mathcal{F}$. We prove the following Dimca-Greuel type inequality $\frac{\mu_p(\mathcal{F},B)}{\tau_p(\mathcal{F},B)} < 4/3$, where $\mu_p(\mathcal{F},B)$ is the multiplicity of $\mathcal{F}$ along B and $\tau_p(\mathcal{F},B)$ is the dimension of the quotient of $\mathbb{C}[[x,y]]$ by the ideal generated by the components of any 1-form defining $\mathcal{F}$ and any equation of B. As a consequence, we provide a new proof of the $\frac{4}{3}$-Dimca-Greuel's conjecture for singularities of irreducible plane curve germs, with foliations ingredients, that differs from those given by Alberich-Carramiñana, Almirón, Blanco, Melle-Hernández and Genzmer-Hernandes but it is in line with the idea developed by Wang.

1. Introduction

Let $\mathbb{C}\{x,y\}$ be the ring of complex convergent power series in two variables. Consider the germ of isolated plane curve singularity C defined by the non-unit $f(x,y) \in \mathbb{C}\{x,y\}$. The *Milnor number* of C is

$$\mu(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(\partial_x f, \partial_y f)$$

and its *Tjurina number* is

$$\tau(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(f, \partial_x f, \partial_y f).$$

Is clear from their definitions that $\tau(C) \leq \mu(C)$.

In [6, Question 4.2], Dimca and Greuel present a family of curves $\{C_m\}_{m\geq 1}$ such that the quotient $\frac{\mu(C_m)}{\tau(C_m)}$ is strictly increasing with limit $4/3$ and they ask if this property is true for any plane curve singularity. The question was affirmatively answered in [1], [10], and [15] for the irreducible case and in [2] for the reduced case.

It is natural to wonder whether this bound also holds if we consider *germs of singular foliations* instead of plane curve germs. More precisely, if $\mathcal{F}$ is a germ of

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singular foliation at $(\mathbb{C}^2, p)$ given by the 1-form $\omega := P(x,y)dx + Q(x,y)dy$, with $P(x,y), Q(x,y) \in \mathbb{C}[[x,y]]$, then the *Milnor number* of $\mathcal{F}$ is by definition the dimension of $\mathbb{C}[[x,y]]/(P,Q)$ and we denote it by $\mu_p(\mathcal{F})$. Moreover if $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ is a balanced divisor of separatrices of $\mathcal{F}$ (see [9], [11]), the *Tjurina number* of $\mathcal{F}$ with respect to $\mathcal{B}_0 : f(x,y) = 0$ is the dimension $\tau_p(\mathcal{F}, \mathcal{B}_0) := \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(P,Q,f)$. If $\mathcal{F}$ is the *hamiltonian foliation* given by the 1-form df for some $f \in \mathbb{C}\{x,y\}$, then the ratio

$$\frac{\mu_p(\mathcal{F})}{\tau_p(\mathcal{F}, \mathcal{B}_0)} \tag{1}$$

equals $\frac{\mu_p(C)}{\tau_p(C)}$, where C is the germ of the curve of equation $f(x,y) = 0$.

If the foliation $\mathcal{F}$ is of the second type and $\mathcal{B}$ is a balanced divisor of separatrices for $\mathcal{F}$ then, in [8, Proposition 4.1], we have proved the inequality $\mu_p(\mathcal{F}) \leq 2\tau_p(\mathcal{F}, \mathcal{B}_0)$ generalizing to foliations of second type this well-known inequality in the context of germs of reduced plane curves (see for instance [14] where the general case of hypersurfaces is studied). It is natural to wonder if we can improve the bound 2 for the ratio (1).

In general, the Dimca-Greuel bound does not hold for foliations, as the following example illustrates: if we borrow from [7, Example 6.5] the family of dicritical foliations $\mathcal{F}_k$, $k \geq 3$, given by the 1-form

$$\omega_k = y\left(2x^{2k-2} + 2(\lambda+1)x^2y^{k-2} - y^{k-1}\right)dx + x\left(y^{k-1} - (\lambda+1)x^2y^{k-2} - x^{2k-2}\right)dy,$$

then for the effective balanced divisor of separatrices $\mathcal{B} = \mathcal{B}_0 : xy = 0$ of $\mathcal{F}_k$ we have $\tau_0(\mathcal{F}_k, \mathcal{B}_0) = 3k - 2$ and $\mu_0(\mathcal{F}_k) = k(2k-1)$; hence for the family $\mathcal{F}_k$ of foliations the ratio (1) goes to infinity when k grows.

However, if in (1), we change $\mu_p(\mathcal{F})$ to the *multiplicity of the singular foliation $\mathcal{F}$ along a separatrix B* of $\mathcal{F}$, $\mu_p(\mathcal{F}, B)$ (see Section 2 for definition), and we put in the denominator $\tau_p(\mathcal{F}, B)$ then we get a bound of the Dimca-Greuel type for foliations.

Our main theorem is

Theorem A. *Let $\mathcal{F}$ be a germ of a singular foliation on $(\mathbb{C}^2, p)$, and B be a separatrix of $\mathcal{F}$ at p. Then*

$$\frac{\mu_p(\mathcal{F}, B)}{\tau_p(\mathcal{F}, B)} < \frac{4}{3}.$$

We will prove Theorem A in Section 3.

As a consequence of Theorem A we deduce (see Corollary 3.5) the Dimca-Greuel bound for the case of irreducible plane curve germs, proved simultaneously in [1], [10] and [15].

2. Multiplicity of a foliation along a divisor of separatrices

Throughout this text $\mathcal{F}$ denotes a germ of a singular (holomorphic or formal) foliation at $(\mathbb{C}^2, p)$, given in local coordinates (x, y) centered at p by a (holomorphic or formal) 1-form $\omega := P(x, y)dx + Q(x, y)dy$, with $P(x, y), Q(x, y) \in \mathbb{C}[[x, y]]$ are coprime; or by its dual vector field

$$v := -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}.$$

The *algebraic multiplicity* $\nu_p(\mathcal{F})$ of $\mathcal{F}$ is the minimum of the orders $\mathrm{ord}_p(P)$ and $\mathrm{ord}_p(Q)$.

Remember that a plane curve germ $f(x, y) = 0$ is an $\mathcal{F}-$*invariant curve* if $\omega \wedge df = (f.h)dx \wedge dy$, for some $h(x, y) \in \mathbb{C}[[x, y]]$ and a *separatrix* of $\mathcal{F}$ is an irreducible $\mathcal{F}$-invariant curve.

Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$ induced by the vector field v and B be a separatrix of $\mathcal{F}$. Let $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$ be a primitive parametrization of B, we can consider the *multiplicity of* $\mathcal{F}$ *along* B *at* p defined by

$$\mu_p(\mathcal{F}, B) = \mathrm{ord}_t\theta(t), \tag{2}$$

where $\theta(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_*\theta(t) = v \circ \gamma(t)$, see for instance [4, page 159]. If $\omega = P(x, y)dx+Q(x, y)dy$ is a 1-form inducing $\mathcal{F}$ and $\gamma(t) = (x(t), y(t))$, then

$$\theta(t) = \begin{cases} -\frac{Q(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0 \\ \frac{P(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases} \tag{3}$$

We extend the notion of multiplicity of $\mathcal{F}$ along any nonempty divisor $\mathcal{B} = \sum_B a_B B$ of separatrices of $\mathcal{F}$ at p in the following way:

$$\mu_p(\mathcal{F}, \mathcal{B}) = \left(\sum_B a_B \mu_p(\mathcal{F}, B)\right) - \deg(\mathcal{B}) + 1. \tag{4}$$

By convention we put $\mu_p(\mathcal{F}, \mathcal{B}) = 1$ for any empty divisor $\mathcal{B}$. In particular, when $\mathcal{B} = B_1 + \cdots + B_r$ is an effective divisor of separatrices of the singular foliation $\mathcal{F}$ at p, we rediscover [13, Equation (2.2), p. 329] (for the reduced plane curve $C := \cup_{i=1}^r B_i$). Hence, if we write $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ where $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are effective divisors we get

$$\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}, \mathcal{B}_0) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + 1. \tag{5}$$

Since $\mu(\mathcal{F}, \mathcal{B}_\infty) \geq 1$ we get

$$\mu_p(\mathcal{F}, \mathcal{B}_0) \geq \mu_p(\mathcal{F}, \mathcal{B}). \tag{6}$$

Denote by $GSV_p(\mathcal{F},\mathcal{B})$ the *Gómez-Mont-Seade-Verjovsky index* of the foliation $\mathcal{F}$ at $(\mathbb{C}^2,p)$ (GSV-index) with respect to the effective divisor $\mathcal{B}$ of separatrices of $\mathcal{F}$. See [12] for details.

Remark 2.1. *Let $B : f(x,y)=0$ be a separatrix of $\mathcal{F}$ and let $\mathcal{G}_f$ be the hamiltonian foliation defined by $df=0$. From* (2) *and* (3), *we have*

$$\mu_p(\mathcal{G}_f,B)=\begin{cases} ord_t\partial_y f(\gamma(t))-ord_t x(t)+1 & \text{if } x(t)\neq 0\\ ord_t\partial_x f(\gamma(t))-ord_t y(t)+1 & \text{if } y(t)\neq 0,\end{cases} \tag{7}$$

where $\gamma(t)=(x(t),y(t))$ is a Puiseux parametrization of B. By [7, Proposition 5.7], *we get $\mu_p(\mathcal{F},B)=GSV_p(\mathcal{F},B)+\mu_p(\mathcal{G}_f,B)$. But, since B is irreducible, after* [7, Proposition 4.7], *$\mu_p(\mathcal{G}_f,B)=\mu_p(B)$ and so that $\mu_p(\mathcal{F},B)=GSV_p(\mathcal{F},B)+\mu_p(B)$.*

3. Proof of Theorem A

Let C be an irreducible plane curve of multiplicity $\nu_p(C)=n$. Denote by $\tilde{C}$ the strict transform of C by a blow-up and $\bar{C}$ the normalization of C. We have natural morphisms

$$\pi:\bar{C}\xrightarrow{\rho}\tilde{C}\xrightarrow{\sigma}C. \tag{8}$$

Consider $(C,p)\xrightarrow{i}(\mathcal{C},x_0)\xrightarrow{\phi}(T,t_0)$ the miniversal deformation of C. We extend to $\mathcal{C}$ the morphisms in (8):

$$\pi:\bar{\mathcal{C}}\xrightarrow{\rho}\tilde{\mathcal{C}}\xrightarrow{\sigma}\mathcal{C}. \tag{9}$$

We have two other natural morphisms $\tilde{C}\xrightarrow{\tilde{\phi}}T$ and $\bar{C}\xrightarrow{\bar{\phi}}T$ and its respectively extensions to the deformation, that is $\tilde{\mathcal{C}}\xrightarrow{\tilde{\phi}}T$ and $\bar{\mathcal{C}}\xrightarrow{\bar{\phi}}T$.

Let $\Omega_{\mathcal{C}/T}$ (respectively, $\Omega_{\tilde{\mathcal{C}}/T}$, respectively, $\Omega_{\bar{\mathcal{C}}/T}$) denote the $\mathcal{O}_{\mathcal{C}}$-module of relative Kähler differentials of $\mathcal{C}$ over T (respectively, the $\mathcal{O}_{\tilde{\mathcal{C}}}$-module of relative Kähler differentials of $\tilde{\mathcal{C}}$ over T, respectively the $\mathcal{O}_{\bar{\mathcal{C}}}$-module of relative Kähler differentials of $\bar{\mathcal{C}}$ over T). Denote by $G:=\rho^*\Omega_{\tilde{\mathcal{C}}/T}/\pi^*\Omega_{\mathcal{C}/T}$. If C is singular then G is not zero and $\bar{\phi}_*G$ is a finite $\mathcal{O}_T$-module and after Wang ([15]) we put $\mathcal{D}(t)=\text{length}_{\mathcal{O}_{T,t}}((\bar{\phi}_*G)_t)$ and $\alpha:=\min_{t\in T}\mathcal{D}(t)$. By [15, Claim 4.4] we have

$$\alpha>\frac{\nu_p(C)(\nu_p(C)-1)}{4}, \tag{10}$$

for $\nu_p(C)\geq 2$.

We will need the following lemma.

Lemma 3.1. *Let $\mathcal{F}$ be a germ of a singular foliation on $(\mathbb{C}^2, p)$, and B be a separatrix of $\mathcal{F}$ at p. If $\tilde{\mathcal{F}}$ (respectively $\tilde{B}$) is the strict transform of $\mathcal{F}$ (respectively of B) at the point q, then*

$$3\mu_q(\tilde{\mathcal{F}}, \tilde{B}) - 4\tau_q(\tilde{\mathcal{F}}, \tilde{B}) + GSV_q(\tilde{\mathcal{F}}, \tilde{B}) \geq 3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B). \tag{11}$$

Moreover, the equality holds if and only if B is smooth.

Proof. Suppose first $\nu_p(B) \geq 2$. By [5, Equality (1.2), p. 291] we get

$$\mu_q(\tilde{\mathcal{F}}, \tilde{B}) = \mu_p(\mathcal{F}, B) - \nu_p(B)(m_p(\mathcal{F}) - 1), \tag{12}$$

where

$$m_p(\mathcal{F}) = \begin{cases} \nu_p(\mathcal{F}) + 1 & \text{if } \sigma \text{ is dicritical} \\ \nu_p(\mathcal{F}) & \text{if } \sigma \text{ is nondicritical.} \end{cases} \tag{13}$$

On the other hand by the behaviour under blow-up of the GSV index [3, p.30] we get

$$GSV_p(\tilde{\mathcal{F}}, \tilde{B}) = GSV_p(\mathcal{F}, B) + \nu_p(B)(\nu_p(B) - m_p(\mathcal{F})), \tag{14}$$

and by [7, Proposition 6.2] we have

$$GSV(\mathcal{F}, B) = \tau_p(\mathcal{F}, B) - \tau_p(B). \tag{15}$$

After [15, Equation (4)] we get:

$$\tau_p(B) - \tau_q(\tilde{B}) = \frac{\nu_p(B)(\nu_p(B) - 1)}{2} + \alpha, \tag{16}$$

where α was defined in (10). From (15), (14), and (16) we deduce

$$\tau_q(\tilde{\mathcal{F}}, \tilde{B}) = \tau_p(\mathcal{F}, B) + \frac{\nu_p(B)[\nu_p(B) + 1 - 2m_p(\mathcal{F})]}{2} + \alpha. \tag{17}$$

By (12), (14) and (17) it holds

$$\begin{aligned} 3\mu_q(\tilde{\mathcal{F}}, \tilde{B}) - 4\tau_q(\tilde{\mathcal{F}}, \tilde{B}) + GSV_q(\tilde{\mathcal{F}}, \tilde{B}) &= 3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B) \\ &- \nu_p(B)(\nu_p(B) - 1) + 4\alpha. \end{aligned} \tag{18}$$

Hence, if B is a singular curve then the lemma follows by (10).

Suppose now that B is a non-singular curve. By a change of coordinates, if necessary, we can suppose that B is given by $x = 0$. Since B is a separatrix of $\mathcal{F} : \omega = Pdx + Qdy$ then $\omega \wedge dx = Qdx \wedge dy$. Hence $Q = xh$ for some formal power series $h(x, y) \in \mathbb{C}[[x, y]]$ such that $h(0, y) \neq 0$ and $\mathcal{F} : \omega = P(x, y)dx + xh(x, y)dy$. Since $\gamma(t) = (0, t)$ is a parametrization of B, then $\mu_p(\mathcal{F}, B) = \mathrm{ord}_t P(0, t)$ and

$$\tau_p(\mathcal{F}, B) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(P, xh, x) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(P, x) = \mu_p(\mathcal{F}, B). \tag{19}$$

Moreover by (15) we have the equality $GSV_p(\mathcal{F},B) = \tau_p(\mathcal{F},B)$. Thus if $\nu_p(B) = 1$ then $\nu_p(\tilde{B}) = 1$, $\mu_p(\mathcal{F},B) = \tau_p(\mathcal{F},B) = GSV_p(\mathcal{F},B)$ and $\mu_p(\tilde{\mathcal{F}},\tilde{B}) = \tau_p(\tilde{\mathcal{F}},\tilde{B}) = GSV_p(\tilde{\mathcal{F}},\tilde{B})$, so (11) becomes an equality.

Moreover, if (11) is an equality, then B is necessarily smooth. Indeed if B were singular, by (10) the number $-\nu_p(B)(\nu_p(B)-1)+4\alpha$ is positive and from the equality (18) we have that (11) is a strict inequality that is a contradiction. Hence the lemma follows. □

Corollary 3.2. *Let $\mathcal{F}$ be a germ of a singular foliation on $(\mathbb{C}^2,p)$, and B be a separatrix of $\mathcal{F}$ at p. If $\tilde{\mathcal{F}}$ (respectively $\tilde{B}$) is the strict transform of $\mathcal{F}$ (respectively of B) at the point q after a composition of ℓ blows-up then*

$$3\mu_q(\tilde{\mathcal{F}},\tilde{B}) - 4\tau_q(\tilde{\mathcal{F}},\tilde{B}) + GSV_q(\tilde{\mathcal{F}},\tilde{B}) \geq 3\mu_p(\mathcal{F},B) - 4\tau_p(\mathcal{F},B) + GSV_p(\mathcal{F},B). \tag{20}$$

Proof. It follows applying Lemma 3.1 ℓ times. □

Remark 3.3. *By definition, for any branch B we have $\tau_p(B) \leq \mu_p(B)$, so after* [7, Proposition 6.2] *and* [7, Propositon 5.7] *we get $\tau_p(\mathcal{F},B) - GSV_p(\mathcal{F},B) \leq \mu_p(\mathcal{F},B) - GSV_p(\mathcal{F},B)$ and*

$$\tau_p(\mathcal{F},B) \leq \mu(\mathcal{F},B). \tag{21}$$

In addition, by (19) *if B is smooth then $\tau_p(\mathcal{F},B) = \mu(\mathcal{F},B)$.*

Remark 3.4. *If $\mathcal{F}$ is a non-dicritical second type foliation at $p \in \mathbb{C}^2$, and B is the only separatrix of $\mathcal{F}$, then $\mathcal{F}$ is a generalized curve foliation. In fact, assume by contradiction that $\mathcal{F}$ admits a saddle-node singularity in its reduction process with strong separatrix on the exceptional divisor. Without lost of generality, we can suppose that $\mathcal{F}$ is a reduced foliation after a blow-up σ at p. Since $\mathcal{F}$ is of second type, after a blow-up, the strong separatrix of the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$ through $q \in \sigma^{-1}(p)$ is contained in $E = \sigma^{-1}(p)$, thus the Camacho-Sad index is $CS_q(\tilde{\mathcal{F}},E) = 0$, but it is a contradiction with Camacho-Sad's formula*

$$CS_q(\tilde{\mathcal{F}},E) = E \cdot E = -1.$$

Proof of Theorem A: If B is a non-singular curve then, by Remark 3.3, $\mu_p(\mathcal{F},B) = \tau_p(\mathcal{F},B)$ and the theorem follows.

Let us suppose now that B is a singular separatrix, that is $\nu_p(B) \geq 2$. After Corollary 3.2 we can suppose, without lost of generality that the number of blows-up n needed in the process of the reduction of the singularity of B is one. In particular, after a blow-up σ the strict transform $\tilde{B}$ of B is smooth, that is the algebraic multiplicity of $\tilde{B}$ is one.

Applying Lemma 3.1 and after (14) we get

$$[3\mu_q(\tilde{\mathcal{F}},\tilde{B}) - 4\tau_q(\tilde{\mathcal{F}},\tilde{B})] - [3\mu_p(\mathcal{F},B) - 4\tau_p(\mathcal{F},B)] > \nu_p(B)(m_p(\mathcal{F}) - \nu_p(B)). \tag{22}$$

Since $\tilde{B}$ is smooth, from (22) and the equality $\tau_q(\tilde{\mathcal{F}},\tilde{B}) = \mu_q(\tilde{\mathcal{F}},\tilde{B})$ (see Remark 3.3) we obtain

$$-\tau_q(\tilde{\mathcal{F}},\tilde{B}) - [3\mu_p(\mathcal{F},B) - 4\tau_p(\mathcal{F},B)] > \nu_p(B)(m_p(\mathcal{F}) - \nu_p(B)). \tag{23}$$

Moreover, by the definition of $m_p(\mathcal{F})$ and after [11, Proposition 3.3], if $\mathcal{B} = B + \mathcal{B}'$ is a balanced divisor of separatrices of $\mathcal{F}$ (for some divisor $\mathcal{B}'$, if B is the only separatrix of $\mathcal{F}$ we consider for $\mathcal{B}'$ the empty divisor) then we have

$$m_p(\mathcal{F}) - \nu_p(B) = \begin{cases} \nu_p(\mathcal{B}') - 1 + \xi_p(\mathcal{F}) & \text{if } \sigma \text{ is non-dicritical} \\ \nu_p(\mathcal{B}') + \xi_p(\mathcal{F}) & \text{if } \sigma \text{ is dicritical,} \end{cases}$$

where $\xi_p(\mathcal{F})$ is the tangency excess of $\mathcal{F}$ (see for example [7, Definition 2.3]). So $m_p(\mathcal{F}) - \nu_p(B) \geq 0$ except in the case that $\mathcal{F}$ is a foliation of the second type ($\xi_p(\mathcal{F}) = 0$) and B is its only separatrix ($\nu_p(\mathcal{B}') = 0$) in which case we get $m_p(\mathcal{F}) - \nu_p(B) = -1$. In the case $m_p(\mathcal{F}) - \nu_p(B) \geq 0$, from (23) we obtain $3\mu_p(\mathcal{F},B) - 4\tau_p(\mathcal{F},B) < 0$ and the theorem follows. Otherwise, assume that $\mathcal{F}$ is of the second type and that B is its only separatrix. By Remark 3.4, $\mathcal{F}$ is a generalized curve foliation, so $GSV_p(\mathcal{F},B) = 0$. Moreover, since $\tilde{B}$ is smooth then the left-term of (11) is zero, so the theorem follows by Lemma (3.1) (since B is singular the inequality (11) is strict) and Remark 3.3 ($\tilde{B}$ is smooth). □

From Theorem A we deduce the Dimca-Greuel bound for the case of irreducible plane curve germs, proved simultaneously in [1], [10] and [15]:

Corollary 3.5. *Let $f(x,y) \in \mathbb{C}[[x,y]]$ an irreducible power series. Put $C : f(x,y) = 0$ the branch defined by f. Then*

$$\frac{\mu(C)}{\tau(C)} < \frac{4}{3}.$$

Proof. It is a consequence of Remark 2.1 and of applying Theorem A to foliation $\mathcal{F}$: $df = 0$. □

Remark 3.6. *In* [15] *Wang proved that the number $3\mu_p(B) - 4\tau_p(B)$ is a non-negative number, non-decreasing under blowups and strictly increasing unless the curve B is non-singular. The result of Wang is an special case of Lemma 3.1, taking $\mathcal{F} = df$ where B is given by $f(x,y) = 0$. Lemma 3.1 and Theorem A generalizes this for the number $3\mu_p(\mathcal{F},B) - 4\tau_p(\mathcal{F},B)$.*

Now, if in (1), we change $\mu_p(\mathcal{F})$ to the multiplicity of the singular foliation $\mathcal{F}$ along a reduced curve C, $\mu_p(\mathcal{F},C)$ (see Section 2), and we put in the denominator the sum $T_p(\mathcal{F},C) := \sum_B \tau_p(\mathcal{F},B)$ of Tjurina numbers of $\mathcal{F}$ along the irreducible components of C, then we get the following inequality of the Dimca-Greuel type for foliations:

Corollary 3.7. *Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$ at p. Then*

$$\frac{\mu_p(\mathcal{F}, \mathcal{B})}{T_p(\mathcal{F}, \mathcal{B}_0)} < \frac{4}{3}. \tag{24}$$

Proof. By (6) we have $\frac{\mu_p(\mathcal{F},\mathcal{B})}{T_p(\mathcal{F},\mathcal{B}_0)} \leq \frac{\mu_p(\mathcal{F},\mathcal{B}_0)}{T_p(\mathcal{F},\mathcal{B}_0)}$. Hence we will prove that

$$\frac{\mu_p(\mathcal{F}, \mathcal{B}_0)}{T_p(\mathcal{F}, \mathcal{B}_0)} < \frac{4}{3}. \tag{25}$$

Suppose that $\mathcal{B}_0 = \sum_{j=1}^{\ell} B_j$, where each B_j is a separatrix for $\mathcal{F}$. It follows from Theorem A that

$$\sum_{j=1}^{\ell} (3\mu_p(\mathcal{F}, B_j) - 4\tau_p(\mathcal{F}, B_j)) \leq -\ell < -\ell + 1.$$

Thus

$$3\sum_{j=1}^{\ell} \mu_p(\mathcal{F}, B_j) - 4\sum_{j=1}^{\ell} \tau_p(\mathcal{F}, B_j) < -\ell + 1 = 3\ell - 4\ell + (4 - 3)$$

and

$$3\left(\sum_{j=1}^{\ell} \mu_p(\mathcal{F}, B_j) - \ell + 1\right) < 4\left(\sum_{j=1}^{\ell} \tau_p(\mathcal{F}, B_j) - \ell + 1\right),$$

so

$$\frac{\mu_p(\mathcal{F}, \mathcal{B}_0)}{T_p(\mathcal{F}, \mathcal{B}_0) - \ell + 1} < \frac{4}{3}.$$

Since $\ell \geq 1$, we obtain $\frac{\mu_p(\mathcal{F},\mathcal{B}_0)}{T_p(\mathcal{F},\mathcal{B}_0)} \leq \frac{\mu_p(\mathcal{F},\mathcal{B}_0)}{T_p(\mathcal{F},\mathcal{B}_0)-\ell+1} < \frac{4}{3}$. □

The following example illustrates Corollary 3.7.

Example 3.8. *Let $\mathcal{F} : \omega = (2x^7+5y^5)dx - xy^2(5y^2+3x^5)dy$, whose leaves are contained in the connected components of the curves $C_{\lambda,\zeta} : \zeta(y^5 - x^7 + x^5y^3) - \lambda x^5 = 0$, where $(\zeta, \lambda) \neq (0, 0)$.*

Let $C = \{x = 0\}$, $\mathcal{B}_0 = C + C_{0,1} + C_{1,1}$, $\mathcal{B}_\infty = C_{2,1}$, then $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ is a balanced divisor of separatrices for $\mathcal{F}$. We have

$$\mu_0(\mathcal{F}, \mathcal{B}_0) = \mu_0(\mathcal{F}, C) + \mu_0(\mathcal{F}, C_{0,1}) + \mu_0(\mathcal{F}, C_{1,1}) - deg(\mathcal{B}) + 1 = 5 + 29 + 21 - 2 + 1 = 54,$$
$$\mu_0(\mathcal{F}, \mathcal{B}_\infty) = \mu_0(\mathcal{F}, C_{2,1}) = 21,$$

and

$$T_0(\mathcal{F}, \mathcal{B}_0) = \tau_0(\mathcal{F}, C) + \tau_0(\mathcal{F}, C_{0,1}) + \tau_0(\mathcal{F}, C_{1,1}) = 5 + 28 + 21 = 54.$$

Hence

$$\mu_0(\mathcal{F},\mathcal{B}) = \mu_0(\mathcal{F},\mathcal{B}_0) - \mu_0(\mathcal{F},\mathcal{B}_\infty) + 1 = 54 - 21 + 1 = 34,$$

and

$$\frac{\mu_0(\mathcal{F},\mathcal{B})}{T_0(\mathcal{F},\mathcal{B}_0)} = \frac{34}{54} < \frac{4}{3}.$$

We conclude the paper with the following question:
Question: Is the inequality (24) true if instead of $T_p(\mathcal{F},\mathcal{B}_0)$ we write $\tau_p(\mathcal{F},\mathcal{B}_0)$?

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