

MODULAR REPRESENTATIONS OF THE YANGIAN Y_2

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ABSTRACT. Let Y_2 be the Yangian associated to the general linear Lie algebra \mathfrak{gl}_2 , defined over an algebraically closed field \mathbb{k} of characteristic $p > 0$. In this paper, we study the representation theory of the restricted Yangian $Y_2^{[p]}$. This gives a description of the representations of \mathfrak{gl}_{2n} , whose p -character is a nilpotent whose Jordan type is the two-row partition (n, n) .

INTRODUCTION

For each simple finite-dimensional Lie algebra \mathfrak{a} over the field of complex numbers, the corresponding Yangian $Y(\mathfrak{a})$ was defined by Drinfeld in [D1] as a canonical deformation of the universal enveloping algebra $U(\mathfrak{a}[x])$ for the current Lie algebra $\mathfrak{a}[x]$. The Yangian $Y_N := Y(\mathfrak{gl}_N)$ associated to the Lie algebra \mathfrak{gl}_N was earlier considered in the works of mathematical physicists from St.-Petersburg around 1980; see for instance, [TF]. Finite dimensional irreducible representations of the Yangians were classified by Drinfeld [D2]. In the case $N = 2$, classification of the finite dimensional irreducible representations of Y_2 is due to Tarasov [Ta1, Ta2], see also [Mol1]. In this paper, we initiate a study of the representation theory of the Yangian Y_2 , over an arbitrary field \mathbb{k} of positive characteristic $p > 0$.

Let us explain our motivation. Over the field of complex numbers, Brundan–Kleshchev [BK1] showed the shifted Yangians have some truncations which are isomorphic to the *finite W -algebras* associated to nilpotent orbits in \mathfrak{gl}_N , as defined by Premet [Pre1]. Premet’s motivation came from the representation theory of Lie algebras in positive characteristic, he also discovered some remarkable finite dimensional *restricted finite W -algebras*, which are Morita equivalent to the reduced enveloping algebras of modular reductive Lie algebras. In [BT], Brundan and Topley developed the theory of the shifted Yangian $Y_N(\sigma)$ over \mathbb{k} . In particular, they gave a description of the centre $Z(Y_N(\sigma))$ of $Y_N(\sigma)$. One of the key features which differs from characteristic zero is the existence of a large central subalgebra $Z_p(Y_N(\sigma))$, called the p -center. Moreover, in their paper [GT2], the authors showed that Brundan–Kleshchev’s isomorphism descends to positive characteristic, i.e., the modular finite W -algebra is a truncation of the modular shifted Yangian. Both algebras admit the p -center, the restricted version of Brundan–Kleshchev’s isomorphism was established by Goodwin and Topley (see [GT3, Theorem 1.1]). Premet’s restricted finite W -algebras can be recovered from the modular shifted Yangians. Therefore, we may understand the representations of certain reduced enveloping algebras from the representation theory of Yangians.

As a first step towards developing the representation theory of modular Yangian, we investigate the finite dimensional irreducible representations of the restricted Yangian

$Y_2^{[p]}$ in detail. By definition, the *restricted Yangian* $Y_2^{[p]} := Y_2/Y_2Z_p(Y_2)_+$ is the quotient of Y_2 by the ideal generated by the generators of the p -center $Z_p(Y_2)$ (see Subsection 1.2). Fix $n \in \mathbb{Z}_{\geq 1}$. Let \mathfrak{gl}_{2n} be the general linear Lie algebra consisting of $2n \times 2n$ -matrices and $e \in \mathfrak{gl}_{2n}$ the $2 \times n$ -rectangular nilpotent element (see Subsection 3.1). Our applications will rely on the following isomorphism ([GT3, Theorem 1.1]):

$$Y_{2,n}^{[p]} \cong U^{[p]}(\mathfrak{gl}_{2n}, e), \quad (0.1)$$

where $Y_{2,n}^{[p]}$ is the restricted truncated Yangian of level n and $U^{[p]}(\mathfrak{gl}_{2n}, e)$ is the restricted finite W -algebra associated to e (see [GT3, §4]). Recently, the modular representations of type A Lie algebras with a particular two-row nilpotent central character are studied, the characteristic of the ground field are assumed to be large enough (see [DNY]).

We organize this article in the following manner. In Section 1, we recall some preliminaries about the modular Yangian Y_2 and its p -center. In Section 2, we define the baby Verma modules for $Y_2^{[p]}$ and prove that every finite dimensional irreducible representation is isomorphic to the simple head of some baby Verma module. We give the necessary and sufficient condition for an irreducible representation to be finite dimensional. In Section 3, we give applications. Using (0.1), the irreducible modules of $Y_{2,n}^{[p]}$ can be constructed from a certain Levi subalgebra. we also determine the irreducible modules for reduced enveloping algebras $U_\chi(\mathfrak{gl}_{2n})$ and their dimensions, where χ is the p -character corresponding to e .

Throughout this paper, \mathbb{k} denotes an algebraically closed field of characteristic $\text{char}(\mathbb{k}) =: p > 0$.

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1. RESTRICTED YANGIAN

In [BT], Brundan and Topley developed the theory of the Yangian Y_N over a field of positive characteristic. We only need here the special case of $N = 2$.

1.1. Modular Yangian Y_2 . The Yangian associated to the general linear Lie algebra \mathfrak{gl}_2 , denoted by Y_2 , is the associated algebra over \mathbb{k} with the *RTT generators* $\{t_{i,j}^{(r)}; 1 \leq i, j \leq 2, r > 0\}$ subject the following relations:

$$\left[t_{i,j}^{(r)}, t_{k,l}^{(s)} \right] = \sum_{t=0}^{\min(r,s)-1} \left(t_{k,j}^{(t)} t_{i,l}^{(r+s-1-t)} - t_{k,j}^{(r+s-1-t)} t_{i,l}^{(t)} \right) \quad (1.1)$$

for $1 \leq i, j, k, l \leq 2$ and $r, s > 0$. By convention, we set $t_{i,j}^{(0)} := \delta_{i,j}$. We often put the generators $t_{i,j}^{(r)}$ for all $r \geq 0$ together to form the generating function

$$t_{i,j}(u) := \sum_{r \geq 0} t_{i,j}^{(r)} u^{-r} \in Y_2[[u^{-1}]].$$

Then Y_2 is a Hopf algebra with comultiplication Δ given in terms of generating functions by

$$\Delta(t_{i,j}(u)) = \sum_{k=1}^2 t_{i,k}(u) \otimes t_{k,j}(u). \quad (1.2)$$

It is easily seen that, in terms of the generating series, the initial defining relation (1.1) may be rewritten as follows:

$$(u - v)[t_{i,j}(u), t_{k,l}(v)] = t_{k,j}(u)t_{i,l}(v) - t_{k,j}(v)t_{i,l}(u). \quad (1.3)$$

We need another set of generators for Y_2 called *Drinfeld generators*. To define these, we consider the Gauss factorization $T(u) = F(u)D(u)E(u)$ of the matrix

$$T(u) := \begin{pmatrix} t_{1,1}(u) & t_{1,2}(u) \\ t_{2,1}(u) & t_{2,2}(u) \end{pmatrix}.$$

This defines power series $d_i(u), e(u), f(u) \in Y_2[[u^{-1}]]$ such that

$$D(u) = \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix}, E(u) = \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}, F(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix},$$

Then we have that

$$t_{1,1}(u) = d_1(u), t_{2,2}(u) = f(u)d_1(u)e(u) + d_2(u), \quad (1.4)$$

$$t_{1,2}(u) = d_1(u)e(u), t_{2,1}(u) = f(u)d_1(u). \quad (1.5)$$

The Drinfeld generators are the elements $d_i^{(r)}, e^{(r)}$ and $f^{(r)}$ of Y_2 defined from the expansions $d_i(u) = \sum_{r \geq 0} d_i^{(r)} u^{-r}$, $e(u) = \sum_{r > 0} e^{(r)} u^{-r}$ and $f(u) = \sum_{r > 0} f^{(r)} u^{-r}$. Also define $d_i^{\prime(r)}$ from the identity $d_i^{\prime(r)}(u) = \sum_{r \geq 0} d_i^{\prime(r)} u^{-r} =: d_i(u)^{-1}$.

Theorem 1.1. [BT, Theorem 4.3] *The algebra Y_2 is generated by the elements $\{d_i^{(r)}, d_i^{\prime(r)}; 1 \leq i \leq 2, r > 0\}$ and $\{e^{(r)}, f^{(r)}; r > 0\}$ subject to the following relations:*

$$d_i^{(0)} = 1, \sum_{t=0}^r d_i^{(t)} d_i^{\prime(r-t)} = \delta_{r0}; \quad (1.6)$$

$$[d_i^{(r)}, d_j^{(s)}] = 0; \quad (1.7)$$

$$[d_i^{(r)}, e^{(s)}] = (\delta_{i1} - \delta_{i2}) \sum_{t=0}^{r-1} d_i^{(t)} e^{(r+s-1-t)}; \quad (1.8)$$

$$[d_i^{(r)}, f^{(s)}] = (\delta_{i2} - \delta_{i1}) \sum_{t=0}^{r-1} f^{(r+s-1-t)} d_i^{(t)}; \quad (1.9)$$

$$[e^{(r)}, f^{(s)}] = - \sum_{t=0}^{r+s-1} d_1^{(t)} d_2^{(r+s-1-t)}; \quad (1.10)$$

$$[e^{(r)}, e^{(s)}] = \sum_{t=1}^{s-1} e^{(t)} e^{(r+s-1-t)} - \sum_{t=1}^{r-1} e^{(t)} e^{(r+s-1-t)}; \quad (1.11)$$

$$[f^{(r)}, f^{(s)}] = \sum_{t=1}^{r-1} f^{(t)} f^{(r+s-1-t)} - \sum_{t=1}^{s-1} f^{(t)} f^{(r+s-1-t)}. \quad (1.12)$$

For any power series $f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$, it follows from the defining relation (1.3) that there is an automorphism defined via

$$\mu_f : Y_2 \rightarrow Y_2; \quad t_{i,j}(u) \mapsto f(u)t_{i,j}(u). \quad (1.13)$$

On Drinfeld generators, we have that

$$\mu_f(d_i(u)) = f(u)d_i(u), \mu_f(e_i(u)) = e_i(u) \text{ and } \mu_f(f_i(u)) = f_i(u). \quad (1.14)$$

Here is the *PBW theorem* for Y_2 .

Theorem 1.2. [BT, Theorem 4.14] *Ordered monomias in the elements*

$$\{d_i^{(r)}; 1 \leq i \leq 2, r > 0\} \cup \{e^{(r)}, f^{(r)}; r > 0\} \quad (1.15)$$

taken in any fixed ordering form a basis for Y_2 .

1.2. Restricted Yangian $Y_2^{[p]}$. We proceed to recall the description of the p -central elements of Y_2 given in [BT]. For $i = 1, 2$, we define

$$b_i(u) = \sum_{r \geq 0} b_i^{(r)} u^{-r} := d_i(u) d_i(u-1) \cdots d_i(u-p+1). \quad (1.16)$$

By [BT, Theorem 5.11(2)] the elements in

$$\{b_i^{(rp)}; 1 \leq i \leq 2, r > 0\} \cup \{(e^{(r)})^p, (f^{(r)})^p; 1 \leq i < j \leq 2, r > 0\} \quad (1.17)$$

are algebraically independent, and lie in the center $Z(Y_2)$ of Y_2 . The subalgebra they generated is called p -center of Y_2 and is denoted by $Z_p(Y_2)$. According to [BT, Corollary 5.13], the Yangian Y_2 is free as a module over $Z_p(Y_2)$ with basis given by the ordered monomials in the generators in (1.15) in which no exponent is p or more, we refer to such monomials as p -restricted monomials. We let $Z_p(Y_2)_+$ be the maximal ideal of $Z_p(Y_2)$ generated by the elements given in (1.17). The *restricted Yangian* is defined by $Y_2^{[p]} := Y_2 / Y_2 Z_p(Y_2)_+$ (see [GT3, §4.3]). Clearly, the images in $Y_2^{[p]}$ of the p -restricted monomials in the Drinfeld generators of Y_2 form a basis of $Y_2^{[p]}$. When working with $Y_2^{[p]}$ we often abuse notation by using the same symbols $d_i^{(r)}, e^{(r)}, f^{(r)}, t_{i,j}^{(r)}$ to refer to the elements of Y_2 and their images in $Y_2^{[p]}$.

Remark 1. In fact, the elements $\{b_i^{(r)}; i = 1, 2, r > 0\}$ also belong to the p -center $Z_p(Y_2)$ (see [BT, Lemma 5.7, Theorem 5.8]). Hence $1 = b_i(u) \in Y_2^{[p]}[[u^{-1}]]$.

1.3. Reduced enveloping algebras. A *restricted Lie algebra* is a Lie algebra \mathfrak{l} with a map $\mathfrak{l} \rightarrow \mathfrak{l}$ sending $x \mapsto x^{[p]}$ such that $x^p - x^{[p]} \in Z(\mathfrak{l})$ for all $x \in \mathfrak{l}$, where $Z(\mathfrak{l})$ is the center of the enveloping algebra $U(\mathfrak{l})$. The p -center of $U(\mathfrak{l})$ is the subalgebra $Z_p(\mathfrak{l})$ of $Z(\mathfrak{l})$ generated by $\{x^p - x^{[p]}; x \in \mathfrak{l}\}$. Given $\chi \in \mathfrak{l}^*$, we define J_χ to be the ideal of $U(\mathfrak{l})$ generated by $\{x^p - x^{[p]} - \chi(x)^p; x \in \mathfrak{l}\}$, and the *reduced enveloping algebra corresponding to χ* to be $U_\chi(\mathfrak{l}) := U(\mathfrak{l})/J_\chi$. For $\chi = 0$ one usually calls $U_0(\mathfrak{l})$ the *restricted enveloping algebra* of \mathfrak{l} (see [Jan, §2.7]).

Let $N \in \mathbb{Z}_{\geq 1}$ and \mathfrak{gl}_N be the general linear Lie algebra, which is spanned by the matrix units $\{e_{i,j}; 1 \leq i, j \leq N\}$. Then \mathfrak{gl}_N is a restricted Lie algebra with the p -map $\mathfrak{gl}_N \rightarrow \mathfrak{gl}_N; x \mapsto x^{[p]}$ where $x^{[p]}$ denotes the p th matrix power of $x \in \mathfrak{gl}_N$. In particular, we note that $e_{i,j}^{[p]} = \delta_{ij}e_{i,j}$ for $1 \leq i, j \leq N$. For the case $N = 2$, the *evaluation homomorphism*

$$\text{ev} : t_{i,j}(u) \mapsto \delta_{i,j} + e_{i,j}u^{-1} \quad (1.18)$$

defines a surjective homomorphism $Y_2 \rightarrow U(\mathfrak{gl}_2)$. By its virtue, any representation of Lie algebra \mathfrak{gl}_2 can be regarded as a representation of Y_2 , and any irreducible representation of \mathfrak{gl}_2 remains irreducible over Y_2 . Moreover, by [GT3, Theorem 1.1] the homomorphism π induces an surjective algebra homomorphism

$$\text{ev}^{[p]} : Y_2^{[p]} \rightarrow U_0(\mathfrak{gl}_2). \quad (1.19)$$

By the same token, the homomorphism (1.19) allows us to equip any $U_0(\mathfrak{gl}_2)$ -module with a structure of $Y_2^{[p]}$ -module.

2. REPRESENTATIONS OF $Y_2^{[p]}$

In this section, we study the representation of the restricted Yangian $Y_2^{[p]}$. To describe the finite dimensional irreducible modules of $Y_2^{[p]}$, we need to construct analogues of highest weight representations in characteristic 0.

2.1. Baby Verma modules. We first give some notation for the PBW basis of $Y_2^{[p]}$. Let

$$\mathbf{I}_{\mathbb{N}} := \{(i_1, i_2, \dots); i_k \in \mathbb{Z}_{\geq 0} \text{ and only finitely many are nonzero}\}$$

and $|I| := \sum_{k \geq 1} i_k$ for $I \in \mathbf{I}_{\mathbb{N}}$. Given $I = (i_1, i_2, \dots) \in \mathbf{I}_{\mathbb{N}}$, set

$$f^I := \prod_{r > 0} (f^{(r)})^{i_r} \in Y_2^{[p]}$$

and we may similarly define the elements d_1^I, d_2^I and e^I . For $I = (i_1, i_2, \dots) \in \mathbf{I}_{\mathbb{N}}$, note that e^I and f^I are zero if some $i_k \geq p$. Consider the subset of $\mathbf{I}_{\mathbb{N}}$

$$\mathbf{I}_p := \{(i_1, i_2, \dots) \in \mathbf{I}_{\mathbb{N}}; 0 \leq i_k < p\}.$$

So that

$$\{f^{I_1} d_1^{I_2} d_2^{I_3} e^{I_4}; I_1, I_2, I_3, I_4 \in \mathbf{I}_p\} \quad (2.1)$$

is the PBW basis of $Y_2^{[p]}$ (see Section 1.2).

Let $Y_2^{[p],-}$ denote the subalgebra of $Y_2^{[p]}$ generated by all the f 's and $(Y_2^{[p],-})_+$ be its maximal ideal generated by the elements $\{f^{(r)}; r > 0\}$.

Lemma 2.1. *For any $x \in (Y_2^{[p],-})_+$, there exists a positive integer n such that $x^n = 0$.*

Proof. We fix an order on the set $\{f^{(r)}; r > 0\}$ where the ordering is specified by $f^{(1)} < f^{(2)} < \dots$. Given $f^{(r_1)}, f^{(r_2)}, \dots, f^{(r_t)} \in \{f^{(r)}; r > 0\}$, we consider the element

$$f^{(r_1)} f^{(r_2)} \dots f^{(r_t)} \in Y_2^{[p],-}.$$

Using the relation (1.12), we may write

$$f^{(r_1)} f^{(r_2)} \dots f^{(r_t)} = \sum_{I \in \mathcal{I}} c_I f^I, \quad (2.2)$$

where $c_I \in \mathbb{k}$ and \mathcal{I} is a finite subset of $\mathbf{I}_{\mathbb{N}}$. Since both sides of (1.12) are homogeneous, it follows that the f^I in (2.2) is homogeneous with $|I| = t$ for all $I \in \mathcal{I}$. We put $k := \min\{r_1, \dots, r_t\}$ as well as $l := \max\{r_1, \dots, r_t\}$. In view of (1.12), the Drinfeld generators appearing in (2.2) are contained in the set $\{f^{(r)}; k \leq r \leq l\}$.

Let $x = \sum_{I \in \mathcal{J}} a_I f^I \in (Y_2^{[p],-})_+$ be a non-zero element, where $\mathcal{J} \subseteq \mathbf{I}_{\mathcal{P}}$ is a finite subset and $|I| \neq 0$ for all $I \in \mathcal{J}$. Assume that the Drinfeld generators in x are $f^{(s_1)}, f^{(s_2)}, \dots, f^{(s_m)}$ with $1 \leq s_1 < s_2 < \dots < s_m$. Put $q := \min\{|I|; I \in \mathcal{J}\}$. Apply again the relation (1.12) one obtains that x^n has the form

$$(*) \quad \sum_{J \in \mathcal{J}} b_J f^J,$$

where \mathcal{J} is a finite subset of $\mathbf{I}_{\mathbb{N}}$ and $|J| \geq nq$ for all $J \in \mathcal{J}$. On the other hand, the foregoing observation implies that the Drinfeld generators in $(*)$ must be contained in the set $\{f^{(r)}; s_1 \leq r \leq s_m\}$. Given $J = (j_1, j_2, \dots) \in \mathcal{J}$, we have the summand

$$b_J f^J = b_J (f^{(s_1)})^{j_{s_1}} (f^{(s_1+1)})^{j_{s_1+1}} \dots (f^{(s_m)})^{j_{s_m}}.$$

Consequently, for large enough n , it must be zero, as desired. \square

Definition. A formal power series $f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$ is called *restricted*, provided

$$f(u)f(u-1) \dots f(u-p+1) = 1.$$

Given two formal series

$$\lambda_1(u) = 1 + \lambda_1^{(1)}u^{-1} + \lambda_1^{(2)}u^{-1} + \dots$$

and

$$\lambda_2(u) = 1 + \lambda_2^{(1)}u^{-1} + \lambda_2^{(2)}u^{-1} + \dots,$$

we say that the tuple $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ is restricted if both $\lambda_1(u)$ and $\lambda_2(u)$ are restricted. The *baby Verma module* $Z^{[p]}(\lambda(u))$ corresponding to $\lambda(u)$ is the quotient of $Y_2^{[p]}$ by the left ideal generated by the elements $e^{(r)}$ with $r > 0$, and by $d_i^{(r)} - \lambda_i^{(r)}$ with $i = 1, 2$ and $r > 0$.

Given a baby Verma module $Z^{[p]}(\lambda(u))$, we denote by $1_{\lambda(u)}$ the image of the element $1 \in Y_2^{[p]}$ in the quotient. Clearly, $Z^{[p]}(\lambda(u)) = Y_2^{[p]}.1_{\lambda(u)}$. Owing to (2.1), there is an isomorphism

$$Z^{[p]}(\lambda(u)) \cong Y_2^{[p],-} \otimes_{\mathbb{k}} \mathbb{k}1_{\lambda(u)}$$

of vector spaces.

Alternatively, the baby Verma modules can be described in terms of the RTT presentation of Y_2 (cf. [Mol2, Proposition 3.2.2]).

Proposition 2.2. *Let $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ be restricted. The baby Verma module $Z^{[p]}(\lambda(u))$ equals to the quotient of $Y_2^{[p]}$ by the left ideal J which is generated by the elements $t_{1,2}^{(r)}$ with $r > 0$ and by $t_{i,i}^{(r)} - \lambda_i^{(r)}$ with $i = 1, 2$ and $r > 0$.*

Proof. Let I be the left ideal generated by the elements $e^{(r)}$ with $r > 0$, and by $d_i^{(r)} - \lambda_i^{(r)}$ with $i = 1, 2$ and $r > 0$. It suffices to show that $I = J$.

Since $e(u) \equiv 0 \pmod{I}$, it follows from (1.5) that

$$t_{12}(u) = d_1(u)e(u) \equiv 0 \pmod{I},$$

so that $t_{1,2}^{(r)} \in I$. Note that $t_{1,1}(u) = d_1(u)$, it is obvious that $t_{1,1}^{(r)} - \lambda_1^{(r)} \in I$. Using (1.4) one obtains

$$t_{2,2}(u) - \lambda_2(u) = f(u)d_1(u)e(u) + d_2(u) - \lambda_2(u) \equiv d_2(u) - \lambda_2(u) \equiv 0 \pmod{I}.$$

Consequently, $J \subseteq I$.

One argues similarly for $I \subseteq J$ applying again (1.4) and (1.5). \square

Proposition 2.3. *$Z^{[p]}(\lambda(u))$ has a unique maximal submodule.*

Proof. We shall prove any proper submodule M of $Z^{[p]}(\lambda(u))$ is contained in $(Y_2^{[p],-})_+ \cdot 1_{\lambda(u)}$. Suppose that $M \not\subseteq (Y_2^{[p],-})_+ \cdot 1_{\lambda(u)}$, then there is a nonzero element $y = (1 - x) \cdot 1_{\lambda(u)} \in M$ and $x \in (Y_2^{[p],-})_+$. Lemma 2.1 provides an integer n such that $x^n = 0$. We have for $x' := 1 + x + \dots + x^{n-1}$, $x'y = 1_{\lambda(u)} \in M$, so that $M = Z^{[p]}(\lambda(u))$, a contradiction. Obviously, $(Y_2^{[p],-})_+ \cdot 1_{\lambda(u)}$ is a proper subspace of $Z^{[p]}(\lambda(u))$. Hence, the sum of all proper submodules of $Z^{[p]}(\lambda(u))$ is the unique maximal submodule. \square

Definition. Given a restricted tuple $\lambda(u) = (\lambda_1(u), \lambda_2(u))$, the irreducible representation $L^{[p]}(\lambda(u))$ of $Y_2^{[p]}$ is defined as the quotient of the baby Verma module $Z^{[p]}(\lambda(u))$ by the unique maximal proper submodule.

Let $Y_2^{[p],0}$ denote the subalgebra of $Y_2^{[p]}$ generated by all d 's. The one dimensional quotient $L^{[p]}(\lambda(u)) / (Y_2^{[p],-})_+ L^{[p]}(\lambda(u))$ can be viewed as $Y_2^{[p],0}$ -module and $d_i(u)$ acts on this space via $\lambda_i(u)$. So we get $L^{[p]}(\lambda(u)) \cong L^{[p]}(\nu(u))$ if and only if $\lambda(u) = \nu(u)$.

Let L be a finite dimensional representation of $Y_2^{[p]}$. We define the subspace L^0 of L via

$$L^0 := \{v \in L; e(u)v = 0\}.$$

Lemma 2.4. *If L is a finite dimensional representation of $Y_2^{[p]}$, then the space L^0 is nonzero.*

Proof. Suppose that $L^0 = (0)$. Since $e^{(1)}$ is nilpotent, there exists a nonzero vector v_1 such that $e^{(1)}v_1 = 0$. By assumption, we can find an positive integer n_1 such that $e^{(r)}v_1 = 0$ for all $1 \leq r \leq n_1$ and $e^{(n_1+1)}v_1 \neq 0$.

Again, the element $e^{(n_1+1)}$ is nilpotent. There exists some integer t with $1 \leq t \leq p-1$ such that $(e^{(n_1+1)})^l v_1 \neq 0$ for all $1 \leq l \leq t$ and $(e^{(n_1+1)})^{t+1} v_1 = 0$. Setting $v_2 := (e^{(n_1+1)})^t v_1 \neq$

0, we obtain $e^{(n_1+1)}v_2 = 0$. For $1 \leq i \leq n_1$, the relation (1.11) yields

$$e^{(i)}v_2 = e^{(i)}(e^{(n_1+1)})^t v_1 = \sum c_{a_1, \dots, a_k} (e^{(n_1+1)})^{a_1} (e^{(n_1)})^{a_2} \dots (e^{(i)})^{a_k} v_1,$$

where $k = n_1 + 2 - i$ and $a_1 + a_2 + \dots + a_k = t + 1$. The choice of v_2 implies that the above sum is zero. This show that $e^{(r)}v_2 = 0$ for all $1 \leq r \leq n_1 + 1$. So that we can find an positive integer $n_2 > n_1$ such that $e^{(r)}v_2 = 0$ for all $1 \leq r \leq n_2$ and $e^{(n_2+1)}v_2 \neq 0$.

Repeat the above argument, we obtain two sequences v_1, v_2, \dots and $1 \leq n_1 < n_2 < \dots$ which satisfy (a) $v_i \neq 0$; (b) $e^{(1)}v_i = e^{(2)}v_i = \dots = e^{(n_i)}v_i = 0$; (c) $e^{(n_i+1)}v_i \neq 0$ for every positive integer i . Let m be an arbitrary positive integer. Assume that $c_1v_1 + c_2v_2 + \dots + c_mv_m = 0$. It follows from (b) that $c_1e^{(n_1+1)}v_1 = 0$, and implication (c) gives $c_1 = 0$. By the same token, we obtain all c_i are zero. Therefore v_1, v_2, \dots, v_m are linearly independent. As L is finite dimensional, we arrive at a contradiction. \square

Theorem 2.5. *Every finite dimensional irreducible representation L of $Y_2^{[p]}$ is isomorphic to some $L^{[p]}(\lambda(u))$.*

Proof. This is very similar to the proof in the characteristic zero explained in [Mol2, Theorem 3.2.7]. We just give a brief account by using the Drinfeld generators.

By Lemma 2.4, we know that $L^0 \neq (0)$. We show that the subspace L^0 is invariant with respect the action of all elements $d_i^{(r)}$. If $v \in L^0$, then (1.8) implies that $[d_i^{(r)}, e^{(s)}]v = 0$ for all $r, s > 0$. It follows that $e^{(s)}d_i^{(r)}v = 0$, so that $d_i^{(r)}v \in L^0$.

Furthermore, (1.7) implies that the elements $d_i^{(r)}$ with $i = 1, 2$ and $r > 0$ act on L^0 as pairwise commuting operators. Hence, there exists a nonzero vector $\zeta \in L^0$ such that $d_i^{(r)}\zeta = \lambda_i^{(r)}\zeta$, where $\lambda_i^{(r)} \in \mathbb{k}$. Letting $\lambda_i(u) := 1 + \lambda_i^{(1)}u^{-1} + \lambda_i^{(2)}u^{-2} + \dots \in \mathbb{k}[[u^{-1}]]$, we put $\lambda(u) = (\lambda_1(u), \lambda_2(u))$. Then $d_i(u)\zeta = \lambda_i(u)\zeta$. Since $1 = b_i(u) = d_i(u)d_i(u-1) \dots d_i(u-p+1)$ (Remark 1), it follows that $\lambda(u)$ is restricted. Note that L is irreducible and $L = Y_2^{[p]}\zeta$. Clearly, there is a surjective homomorphism $Z^{[p]}(\lambda(u)) \twoheadrightarrow L$; $1_{\lambda(u)} \mapsto \zeta$. Proposition 2.3 now yields isomorphism $L \cong L^{[p]}(\lambda(u))$. \square

2.2. Evaluation modules. For any $\alpha, \beta \in \mathbb{F}_p$, we consider the irreducible $U_0(\mathfrak{gl}_2)$ -module $L(\alpha, \beta)$. The module $L(\alpha, \beta)$ is generated by a vector ξ and

$$e_{1,1}\xi = \alpha\xi, e_{2,2}\xi = \beta\xi \text{ and } e_{1,2}\xi = 0. \quad (2.3)$$

For $n \in \mathbb{F}_p$, we denote by $[n] \in \mathbb{N}$ the minimal element such that $[n] \equiv n \pmod{p}$. The module $L(\alpha, \beta)$ has a basis $e_{2,1}^k \xi$ for k from 0 to $[\alpha - \beta]$ (cf. [Jan, §5.2]). The homomorphism (1.19) allows us to view $L(\alpha, \beta)$ as a $Y_2^{[p]}$ -module. Namely, for any indices $i, j \in \{1, 2\}$ the generator $t_{i,j}^{(1)}$ acts on $L(\alpha, \beta)$ as $e_{i,j}$, while any generator $t_{i,j}^{(r)}$ with $r \geq 2$ acts as the zero operator. We will keep the same notation $L(\alpha, \beta)$ for this $Y_2^{[p]}$ -module and call it the *evaluation module*. By (2.2), it is clear that $L(\alpha, \beta)$ is isomorphic to the module $L^{[p]}(\lambda_1(u), \lambda_2(u))$, where $\lambda_1(u) = 1 + \alpha u^{-1}$ and $\lambda_2(u) = 1 + \beta u^{-1}$.

Let L and M be two Y_2 -modules. Since the algebra Y_2 is a Hopf algebra, the tensor product space $L \otimes M$ can be equipped with a Y_2 -action with the use of the comultiplication Δ on Y_2 (see (1.2)) by the rule

$$x.(a \otimes b) := \Delta(x)(a \otimes b), \quad x \in Y_2, \quad a \in L, \quad b \in M.$$

Proposition 2.6. *Given two sequences α_i, β_i of elements of \mathbb{F}_p for $i = 1, \dots, k$, renumerate them in such a way that $[\alpha_i - \beta_i]$ is minimal among all $[\alpha_j - \beta_l]$ for $i \leq j, l \leq k$. Then the tensor product*

$$L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k) \quad (2.4)$$

is an irreducible Y_2 -module.

Proof. We can use the same proof [Mol2, Proposition 3.3.2] (see also [Kal, Proposition 3.2.3]).

Denote the module (2.4) by L . Let ξ_i be the generating vector of $L(\alpha_i, \beta_i)$ (2.3) for $i = 1, \dots, k$. Using the definition (1.2) of Δ , we obtain $t_{1,2}(u)\xi = 0$, where $\xi = \xi_1 \otimes \cdots \otimes \xi_k$. It suffices to show that any vector $\zeta \in L$ satisfying $t_{1,2}(u)\zeta = 0$ is proportional to ξ . Now proceed by induction on k . Write any such vector $\zeta = \sum_{r=0}^s e_{1,2}^r \xi_1 \otimes \zeta_r$ and ζ_r some elements of $L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k)$. We may assume that $s \leq [\alpha_1 - \beta_1]$. Then repeating all the steps of the proof of [Mol2, Proposition 3.3.2], we obtain the following relation:

$$s(\alpha_1 - \beta_1 - s + 1)(\alpha_1 - \beta_2 - s + 1) \cdots (\alpha_1 - \beta_k - s + 1) = 0.$$

Since $0 \leq [\alpha_1 - \beta_1] \leq p - 1$ and $0 \leq s \leq [\alpha_1 - \beta_1]$, it follows that $(\alpha_1 - \beta_1 - s + 1) = 0$ only if $s = 0$ and $[\alpha_1 - \beta_1] = p - 1$. Our assumption on the parameter α_i and β_i now implies that $[\alpha_1 - \beta_j] \geq [\alpha_1 - \beta_1]$ for all other j . So that $(\alpha_1 - \beta_j - s + 1) = 0$ can be zero again only for $s = 0$ and the claim follows. \square

We still denote by L the tensor product (2.4). Let $\xi = \xi_1 \otimes \cdots \otimes \xi_k$, where ξ_i be the generating vector of $L(\alpha_i, \beta_i)$ (2.3). As L is irreducible, we have $L = Y_2 \xi$. We want to show that L is a $Y_2^{[p]}$ -module. To do this, we need another description of the p -center of Y_2 in terms of the RTT generators. Let

$$s_{i,j}(u) = \sum_{r \geq 0} s_{i,j}^{(r)} u^{-r} := t_{i,j}(u) t_{i,j}(u-1) \cdots t_{i,j}(u-p+1) \in Y_2[[u^{-1}]].$$

According to [BT, Lemma 6.8, Theorem 6.9], the p -center $Z_p(Y_2)$ is generated by $\{s_{i,j}^{(r)}; 1 \leq i, j \leq 2, r > 0\}$. Write down

$$\lambda_1(u) := (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_k u^{-1})$$

and

$$\lambda_2(u) := (1 + \beta_1 u^{-1}) \cdots (1 + \beta_k u^{-1}).$$

Clearly both $\lambda_1(u)$ and $\lambda_2(u)$ are restricted, because all the elements α_i and β_i belong to \mathbb{F}_p .

Proposition 2.7. *Suppose L is the tensor product module (2.4) satisfying the conditions of Proposition 2.6. Then L is a $Y_2^{[p]}$ -module and is isomorphic to $L^{[p]}(\lambda_1(u), \lambda_2(u))$.*

Proof. By the above discussion, we need to prove that

$$(i) \quad s_{1,2}(u)\xi = 0;$$

$$(ii) \quad t_{i,i}(u)\xi = \lambda_i(u)\xi, \quad i = 1, 2;$$

$$(iii) \quad s_{i,i}(u)\xi = \xi, \quad i = 1, 2;$$

(iv) $s_{2,1}(u)\xi = 0$.

Using the definition of Δ , we have

$$t_{i,j}(u)\xi = \sum_{a_1, \dots, a_{k-1}} t_{i,a_1}(u)\xi_1 \otimes t_{a_1,a_2}(u)\xi_2 \otimes \cdots \otimes t_{a_{k-1},j}(u)\xi_k, \quad (2.5)$$

summed over $a_1, \dots, a_{k-1} \in \{1, 2\}$. If $(i, j) = (1, 2)$, then each summand in the sum (2.5) is zero because it contains a factor of the form $t_{1,2}(u)\xi_m$, which is zero. We obtain (i). If $i = j$, then the only nonzero summand corresponds to the case where each index a_m equals i . This yields (ii) and (iii). As $s_{i,j}^{(r)} \in Z_p(Y_2)$, direct computation shows that $t_{1,2}(u)s_{i,j}^{(r)}\xi = 0$. The proof of Proposition 2.6 implies that $s_{i,j}^{(r)}\xi$ is proportional to ξ . Hence we have $s_{2,1}(u)\xi = f(u)\xi$ for some $f(u) \in \mathbb{k}[[u^{-1}]]$. Note that $t_{2,1}(u)$ acts nilpotently on the module L , so is $s_{2,1}(u)$. Thus, $f(u) = 0$. This proves (iv). \square

From Theorem 2.5, we know that every finite dimensional irreducible module of $Y_2^{[p]}$ is isomorphic to a unique simple quotient module $L^{[p]}(\lambda_1(u), \lambda_2(u))$ of the baby Verma module $Z^{[p]}(\lambda_1(u), \lambda_2(u))$. We first consider the case that both $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1} .

Lemma 2.8. *Let $\lambda(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$ be a polynomial with the decomposition*

$$\lambda(u) = (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_k u^{-1}). \quad (2.6)$$

If $\lambda(u)$ is restricted, then all the elements α_i belong to \mathbb{F}_p .

Proof. Multiplying u^k on both sides of (2.6), we have

$$u^k \lambda(u) = (u + \alpha_1) \cdots (u + \alpha_k).$$

It follows that

$$\prod_{i=0}^{p-1} (u - i)^k \prod_{i=0}^{p-1} \lambda(u - i) = \prod_{j=0}^k \prod_{i=0}^{p-1} (u + \alpha_j - i).$$

Note that $\prod_{i=0}^{p-1} \lambda(u - i) = 1$ because $\lambda(u)$ is restricted. Then the assertion follows by comparing the roots of the above equation. \square

Theorem 2.9. *Suppose that $\lambda_1(u)$ and $\lambda_2(u)$ are restricted polynomials. Then $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is isomorphic to some tensor product of evaluation modules. In particular, $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is finite dimensional.*

Proof. Write the decomposition $\lambda_1(u) = (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_k u^{-1})$ and $\lambda_2(u) = (1 + \beta_1 u^{-1}) \cdots (1 + \beta_k u^{-1})$ for some $k \geq 0$ and some parameters α_i, β_i . By Lemma 2.8, all of the elements α_i, β_i belong to the field \mathbb{F}_p . Renumerate them in a way consistent with Proposition 2.6. Now Proposition 2.7 yields the isomorphism

$$L^{[p]}(\lambda_1(u), \lambda_2(u)) \cong L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k).$$

\square

Lemma 2.10. *If $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is finite dimensional then there is a formal series $f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$ such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in u^{-1} .*

Proof. we can repeat the proof of [Mol2, Proposition 3.3.1] for $L^{[p]}(\lambda_1(u), \lambda_2(u))$ without any change. \square

Remark 2. It should be notice that the $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ in the above Lemma might no longer be restricted. Given a formal power series $f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$ and we recall that the automorphism μ_f (1.13) of Y_2 . If μ_f factors to an automorphism of $Y_2^{[p]}$, then clearly f must be restricted. For general $\lambda(u) = (\lambda_1(u), \lambda_2(u))$, there may not exist a restricted power series $f(u)$ such $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials even $L^{[p]}(\lambda(u))$ is finite dimensional (see Remark 4).

Theorem 2.11. *The irreducible representation $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is finite dimensional if and only if there exists a monic polynomial $P(u)$ in u such that*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}. \quad (2.7)$$

Proof. Suppose that the representation $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is finite dimensional. Then by Lemma 2.10 we can find a formal series $f(u)$ such that $f(u)\lambda_1(u) = (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_k u^{-1})$ and $f(u)\lambda_2(u) = (1 + \beta_1 u^{-1}) \cdots (1 + \beta_k u^{-1})$ for some $k \geq 0$ and some parameters α_i, β_i . Note that $\lambda_1(u)/\lambda_2(u) = f(u)\lambda_1(u)/f(u)\lambda_2(u)$ is restricted, so that for each α_i , there exists β_j such that $\alpha_i - \beta_j \in \mathbb{F}_p$. Renumerating them if necessary, we may thus assume that $\alpha_i - \beta_i \in \mathbb{F}_p$. The the polynomial

$$P(u) = \prod_{i=1}^k (u + \beta_i)(u + \beta_i + 1) \cdots (u + \alpha_i - 1)$$

obviously satisfies (2.7).

Conversely, suppose (2.7) holds for a polynomial $P(u) = (u + \gamma_1) \cdots (u + \gamma_s)$. Set

$$\begin{aligned} \mu_1(u) &= (1 + (\gamma_1 + 1)u^{-1}) \cdots (1 + (\gamma_s + 1)u^{-1}), \\ \mu_2(u) &= (1 + \gamma_1 u^{-1}) \cdots (1 + \gamma_s u^{-1}). \end{aligned}$$

For each γ_i , we consider the \mathfrak{gl}_2 -module $L(\gamma_i + 1, \gamma_i)$ which is generated by ξ_i and the module structure is given by

$$e_{1,1}\xi_i = (\gamma_i + 1)\xi_i, e_{2,2}\xi_i = \gamma_i\xi_i, e_{1,2}\xi_i = 0, e_{2,1}^2\xi_i = 0. \quad (2.8)$$

In particular, the module $L(\gamma_i + 1, \gamma_i)$ is two dimensional with basis $\{\xi_i, e_{2,1}\xi_i\}$. We can regard them as Y_2 -modules via (1.18) and consider the tensor product module

$$L := L(\gamma_1 + 1, \gamma_1) \otimes \cdots \otimes L(\gamma_s + 1, \gamma_s).$$

Clearly, $\dim_{\mathbb{k}} L = 2^s$. We put $\xi := \xi_1 \otimes \cdots \otimes \xi_s$. Using the comultiplication (1.2) in conjunction with (2.8) one can show by direct computation that

$$t_{1,2}(u)\xi = 0, t_{1,1}(u)\xi = \mu_1(u)\xi, t_{2,2}(u)\xi = \mu_2(u)\xi$$

and $t_{2,1}(u)$ acts nilpotently on the module L . Let $M := Y_2\xi \subseteq L$ be the cyclic submodule of L . By twisting the action of Y_2 on M by the automorphism (1.13) with $f(u) = \mu_2(u)^{-1}$, we obtain a module over Y_2 which is still denoted by \widetilde{M} . It is clear that \widetilde{M} is also generated by ξ and

$$(*) \quad t_{1,2}(u)\xi = 0, t_{1,1}(u)\xi = \frac{\mu_1(u)}{\mu_2(u)}\xi = \frac{\lambda_1(u)}{\lambda_2(u)}\xi, t_{2,2}(u)\xi = \xi.$$

We define a subspace V of \widetilde{M}

$$V := \text{Span}\{s_{2,1}^{(r_1)} s_{2,1}^{(r_2)} \cdots s_{2,1}^{(r_m)} \xi; m \geq 0, r_1, r_2, \dots, r_m \geq 1\}.$$

Since the set $\{s_{2,1}^{(r)}; r \geq 1\}$ is a Lie subset that acts on V by nilpotent transformations. Thus, the Engel-Jacobson theorem provides a vector $\eta \in V$ such that $s_{2,1}(u)\eta = 0$. Then we consider the submodule $Y_2\eta$ of the module \widetilde{M} . By $(*)$, we obtain

$$t_{1,2}(u)\eta = 0, t_{1,1}(u)\eta = \frac{\mu_1(u)}{\mu_2(u)}\eta = \frac{\lambda_1(u)}{\lambda_2(u)}\eta, t_{2,2}(u)\eta = \eta.$$

There results a surjective homomorphism

$$Z^{[p]}(\frac{\lambda_1(u)}{\lambda_2(u)}, 1) \twoheadrightarrow Y_2\eta; 1_{\lambda_1(u)/\lambda_2(u)} \mapsto \eta.$$

As a result, $L^{[p]}(\frac{\lambda_1(u)}{\lambda_2(u)}, 1)$ is finite dimensional. Since $\lambda_2(u)$ is restricted, we apply again the twisted action of $Y_2^{[p]}$ on $L^{[p]}(\frac{\lambda_1(u)}{\lambda_2(u)}, 1)$ with $f(u) = \lambda_2(u)$ (see Remark 2) to obtain the module $L^{[p]}(\lambda_1(u), \lambda_2(u))$. Thus, the module $L^{[p]}(\lambda_1(u), \lambda_2(u))$ is also finite dimensional. \square

Remark 3. As observed by Kalinov ([Kal, Remark, p. 6985]), the polynomial $P(u)$ in Theorem 2.11 is not unique. Suppose that $Q(u)$ is another monic polynomial in u and $\frac{P(u+1)}{P(u)} = \frac{Q(u+1)}{Q(u)}$. It follows that $\frac{P(u)}{Q(u)} =: F(u)$ satisfies $F(u+1) = F(u)$. Thus $F(u)$ is a ratio of products of expressions of the form $(u + \alpha)^p - (u + \alpha)$ for some $\alpha \in \mathbb{k}$.

Remark 4. Let $\alpha \notin \mathbb{F}_p$. We consider the 2-dimensional \mathfrak{gl}_2 -module $L(\alpha + 1, \alpha)$ as defined in (2.8). By twisting the action of Y_2 on $L(\alpha + 1, \alpha)$ by the automorphism (1.13) with $f(u) = (1 + \alpha u^{-1})^{-1}$, we obtain a module over $Y_2^{[p]}$ which is isomorphic to the irreducible module $L^{[p]}(\frac{1+(\alpha+1)u^{-1}}{1+\alpha u^{-1}}, 1)$. However, there does not exist a restricted polynomial $g(u)$ in u^{-1} such that $g(u)\frac{1+(\alpha+1)u^{-1}}{1+\alpha u^{-1}}$ is a restricted polynomial.

3. FINITE W -ALGEBRAS

We turn to the W -algebra side. We fix $n \in \mathbb{Z}_{\geq 1}$. Let $G := GL_{2n}(\mathbb{k})$ be the general linear group of degree $2n$ with Lie algebra $\mathfrak{g} := \text{Lie}(G) = \mathfrak{gl}_{2n}$. We write $\{e_{i,j}; 1 \leq i, j \leq 2n\}$ for the standard basis of \mathfrak{g} consisting of matrix units. In this section, we only consider the W -algebras associated to $2 \times n$ -rectangular nilpotent elements in \mathfrak{g} . The reader is referred to [GT1] for the theory of modular finite W -algebras.

3.1. Restricted finite W -algebras. We consider the partition $(n, n) \vdash 2n$ of $2n$. Let π be the corresponding pyramid. The boxes in the pyramid are numbered along rows from left to right and from top to bottom. For example, if $n = 5$, then the pyramid associated to $(5, 5)$ is

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline \end{array}.$$

For $1 \leq i \leq n$, we use the notation $i' := i + n$. The box in π containing i is referred to as the i th box, and let $\text{row}(i)$ and $\text{col}(i)$ denote the *row* and *column* numbers of the brick in which

i appears, respectively. We therefore have $\text{row}(i) = 1, \text{row}(i') = 2$ and $\text{col}(i) = \text{col}(i') = i$ for every $1 \leq i \leq n$.

The pyramid π is used to determine the nilpotent element

$$e := \sum_{1 \leq i \leq n-1} e_{i,i+1} + \sum_{1 \leq i \leq n-1} e_{i',(i+1)'} \in \mathfrak{g}, \quad (3.1)$$

which has Jordan type (n, n) . We call it $2 \times n$ -rectangular nilpotent element.

Consider the cocharacter $\mu : \mathbb{k}^\times \rightarrow T \subseteq G$ defined by

$$\mu(t) = \text{diag}(t^{-1}, t^{-2}, \dots, t^{-n}, t^{-1}, t^{-2}, \dots, t^{-n}),$$

where T is the maximal torus of G of diagonal matrices. Using μ we define the \mathbb{Z} -grading

$$\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}(r) \quad \text{where} \quad \mathfrak{g}(r) := \{x \in \mathfrak{g}; \mu(t)x = t^r x \text{ for all } t \in \mathbb{k}^\times\}. \quad (3.2)$$

Since the adjoint action of $\mu(t)$ on a matrix unit is given by $\mu(t)e_{i,j} = t^{\text{col}(j) - \text{col}(i)}e_{i,j}$, we have $\mathfrak{g}(r) = \text{span}\{e_{i,j}; \text{col}(j) - \text{col}(i) = r\}$.

We define the following subalgebras of \mathfrak{g} :

$$\mathfrak{p} := \bigoplus_{r \geq 0} \mathfrak{g}(r), \quad \mathfrak{h} := \mathfrak{g}(0), \quad \text{and} \quad \mathfrak{m} := \bigoplus_{r < 0} \mathfrak{g}(r). \quad (3.3)$$

Then \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} with Levi factor \mathfrak{h} and \mathfrak{m} is the nilradical of the opposite parabolic to \mathfrak{p} . We see that \mathfrak{h} is isomorphic to the direct sum of n copies of \mathfrak{gl}_2 .

The finite W -algebra $U(\mathfrak{g}, e)$ associated to e is a subalgebra of $U(\mathfrak{p})$ generated by

$$\{d_i^{(r)}; 1 \leq i \leq 2, r > 0\} \cup \{e^{(r)}, f^{(r)}; r > 0\}. \quad (3.4)$$

These elements were defined by remarkable formulas, given in [BK2, §9]; see also [GT2, §4]. We note there is an abuse of notation as these generators of $U(\mathfrak{g}, e)$ have the same names as the generators for Y_2 given in (1.15), this overloading of notation will be justified in the next subsection.

Now let \mathfrak{t} be the Lie algebra of T , and write $\{\epsilon_1, \dots, \epsilon_{2n}\}$ for the standard basis of \mathfrak{t}^* . We define the weight $\eta \in \mathfrak{t}^*$ by

$$\eta := \sum_{i=1}^n 2(i-n)(\epsilon_i + \epsilon_{i'}), \quad (3.5)$$

and we note that η extends to a character of \mathfrak{p} . For $e_{i,j} \in \mathfrak{p}$ define

$$\tilde{e}_{i,j} := e_{i,j} + \eta(e_{i,j}).$$

Then by definition

$$d_i^{(r)} := \sum_{s=1}^r (-1)^{r-s} \sum_{\substack{i_1, \dots, i_s \\ j_1, \dots, j_s}} (-1)^{|\{t=1, \dots, s-1 | \text{row}(j_t) \leq i-1\}|} \tilde{e}_{i_1, j_1} \cdots \tilde{e}_{i_s, j_s} \in U(\mathfrak{p}) \quad (3.6)$$

where the sum is taken over all $1 \leq i_1, \dots, i_s, j_1, \dots, j_s \leq 2n$ such that

- (a) $\text{col}(j_1) - \text{col}(i_1) + \dots + \text{col}(j_s) - \text{col}(i_s) + s = r$;
- (b) $\text{col}(i_t) \leq \text{col}(j_t)$ for each $t = 1, \dots, s$;
- (c) if $\text{row}(j_t) \geq i$, then $\text{col}(j_t) < \text{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;

- (d) if $\text{row}(j_t) < i$ then $\text{col}(j_t) \geq \text{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;
- (e) $\text{row}(i_1) = i, \text{row}(j_s) = i$;
- (f) $\text{row}(j_t) = \text{row}(i_{t+1})$ for each $t = 1, \dots, s-1$.

The expressions for the elements $e^{(r)} \in U(\mathfrak{p})$ and $f^{(r)} \in U(\mathfrak{p})$ are given by similar formulas, see [BK2, §9] or [GT2, §4] for more details.

Since \mathfrak{p} is a restricted subalgebra of \mathfrak{g} . We write $Z_p(\mathfrak{p})_+$ for the ideal of $Z_p(\mathfrak{p})$ generated by $\{x^p - x^{[p]}; x \in \mathfrak{p}\}$, so the restricted enveloping algebra of \mathfrak{p} is $U_0(\mathfrak{p}) = U(\mathfrak{p})/U(\mathfrak{p})Z_p(\mathfrak{p})_+$ (see Section 1.3). Then the *restricted W -algebra* is defined as

$$U^{[p]}(\mathfrak{g}, e) := U(\mathfrak{g}, e)/(U(\mathfrak{g}, e) \cap U(\mathfrak{p})Z_p(\mathfrak{p})_+).$$

Since the kernel of the restriction of the projection $U(\mathfrak{p}) \rightarrow U_0(\mathfrak{p})$ to $U(\mathfrak{g}, e)$ is $U(\mathfrak{g}, e) \cap U(\mathfrak{p})Z_p(\mathfrak{p})_+$, we can identify $U^{[p]}(\mathfrak{g}, e)$ with the image of $U(\mathfrak{g}, e)$ in $U_0(\mathfrak{p})$.

3.2. $U^{[p]}(\mathfrak{g}, e)$ as restricted truncated Yangian. Let $I_{2,n}^{[p]}$ be the ideal of $Y_2^{[p]}$ generated by $\{d_1^{(r)} + Z_p(Y_2)_+; r > n\}$. The *restricted truncated Yangian* $Y_{2,n}^{[p]}$ is defined to be the quotient of $Y_2^{[p]}$ by the ideal $I_{2,n}^{[p]}$ ([GT3, (4.13)]). As before, we will use the same symbols $d_i^{(r)}, e^{(r)}, f^{(r)}$ for their canonical images in the quotient $Y_{2,n}^{[p]}$ and $U^{[p]}(\mathfrak{g}, e)$, respectively. According to [GT3, Theorem 1.1], the map from $Y_{2,n}^{[p]}$ to $U^{[p]}(\mathfrak{g}, e)$, determined by sending the generators $e^{(r)}, d_i^{(r)}, f^{(r)}$ of $Y_{2,n}^{[p]}$ to the generators (3.4) of $U^{[p]}(\mathfrak{g}, e)$ with the same names, defined an isomorphism

$$\phi^{[p]} : Y_{2,n}^{[p]} \rightarrow U^{[p]}(\mathfrak{g}, e). \quad (3.7)$$

3.3. Irreducible representations for $Y_{2,n}^{[p]}$. Recall from Theorem 2.5 each finite dimensional simple $Y_{2,n}^{[p]}$ -module has the form $L^{[p]}(\lambda_1(u), \lambda_2(u))$. Our next proposition determines for which of these simple modules the action of $Y_{2,n}^{[p]}$ factors through the quotient $Y_2^{[p]} \twoheadrightarrow Y_{2,n}^{[p]}$.

Proposition 3.1. *Let $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ be restricted. Then $L^{[p]}(\lambda(u))$ factors to a module for $Y_{2,n}^{[p]}$ if and only if $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1} and $\deg \lambda_i(u) \leq n$.*

Proof. Suppose that both $\lambda_1(u)$ and $\lambda_2(u)$ are restricted polynomials. Thanks to Theorem 2.9, $L^{[p]}(\lambda(u))$ is isomorphic to some tensor product of evaluation modules which has at most n tensor factors. Using the comultiplication (1.2), we see that every generator $t_{i,j}^{(r)}$ with $r > n$ acts on the module as the zero operator. Note that $t_{1,1}^{(r)} = d_1^{(r)}$ (1.4). We thus obtain $I_{2,n}^{[p]} \cdot L^{[p]}(\lambda(u)) = (0)$.

On the other hand, Proposition 2.2 implies that $L^{[p]}(\lambda(u))$ is generated by ζ and $t_{i,i}(u)\zeta = \lambda_i(u)\zeta$. Now our assertion follows from the fact that $t_{i,i}^{(r)} = 0$ in $Y_{2,n}^{[p]}$ for $r > n$ ([GT3, Corollary 3.6]). \square

In conjunction with the isomorphism $\phi^{[p]}$, this also shows the following:

Corollary 3.2. *The isomorphism classes of finite dimensional irreducible representations of the restricted W -algebra $U^{[p]}(\mathfrak{g}, e)$ are parameterized by the set*

$$\{(\lambda_1(u), \lambda_2(u)); \prod_{j=1}^p \lambda_i(u - j + 1) = 1, \deg \lambda_i(u) \leq n, i = 1, 2\}.$$

4. MODULES FOR REDUCED ENVELOPING ALGEBRAS

In this section, we continue to use the notation from Section 3. We will determine the irreducible modules for the reduced enveloping algebra of \mathfrak{g} associated with the $2 \times n$ -rectangular nilpotent element.

4.1. Baby Verma modules for $U^{[p]}(\mathfrak{g}, e)$. We recall that the grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ from (3.2) and the notation $\mathfrak{h} = \mathfrak{g}(0)$ and $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ from (3.3). Note that \mathfrak{h} is reductive and is isomorphic to the n copies of \mathfrak{gl}_2 . We let $\mathfrak{b}_{\mathfrak{h}}$ be the Borel subalgebra of \mathfrak{h} with basis

$$\{e_{i,i}; 1 \leq i \leq n\} \cup \{e_{i',i'}; 1 \leq i \leq n\} \cup \{e_{i,i'}; 1 \leq i \leq n\}, \quad (4.1)$$

which is the direct sum of the Borel subalgebras of upper triangular matrices in each of the \mathfrak{gl}_2 .

Given two n -tuples $\alpha := (\alpha_1, \dots, \alpha_n)$ and $\beta := (\beta_1, \dots, \beta_n)$ of elements of \mathbb{F}_p , we define the weight $\lambda_{\alpha,\beta} \in \mathfrak{t}^*$ by

$$\lambda_{\alpha,\beta} := \sum_{i=1}^n \alpha_i \epsilon_i + \sum_{i=1}^n \beta_i \epsilon_{i'}.$$

We define $\mathbb{k}_{\alpha,\beta} = \mathbb{k} \cdot 1_{\alpha,\beta}$ to be the 1-dimensional \mathfrak{t} -module on which \mathfrak{t} acts via $\lambda_{\alpha,\beta} - \eta$, where we recall that η is defined in (3.5). It is obvious that $\mathbb{k}_{\alpha,\beta}$ is a $U_0(\mathfrak{t})$ -module. Furthermore, we view it as module for $U_0(\mathfrak{b}_{\mathfrak{h}})$ on which the nilradical acts trivially. Then we define the *baby Verma module*

$$Z_{\mathfrak{h}}(\alpha, \beta) := U_0(\mathfrak{h}) \otimes_{U_0(\mathfrak{b}_{\mathfrak{h}})} \mathbb{k}_{\alpha,\beta}.$$

We put $z_{\alpha,\beta} := 1 \otimes 1_{\alpha,\beta}$. We may view $Z_{\mathfrak{h}}(\alpha, \beta)$ as a $U_0(\mathfrak{p})$ -module on which the nilradical $\bigoplus_{i > 0} \mathfrak{g}(i)$ of \mathfrak{p} acts trivially. As the restricted W -algebra $U^{[p]}(\mathfrak{g}, e)$ is a subalgebra of $U_0(\mathfrak{p})$, we restrict $Z_{\mathfrak{h}}(\alpha, \beta)$ to $U^{[p]}(\mathfrak{g}, e)$ and write $\overline{Z}_{\mathfrak{h}}(\alpha, \beta)$ for the restriction and $\overline{z}_{\alpha,\beta}$ for $z_{\alpha,\beta}$ viewed as an element of $\overline{Z}_{\mathfrak{h}}(\alpha, \beta)$.

Remark 5. In [GT3], the authors defined the weight $\rho_{\mathfrak{h}}$. In our situation, the weight $\rho_{\mathfrak{h}}$ is just equal to $-\sum_{i=1}^n \epsilon_{i'}$. They defined 1-dimensional \mathfrak{t} -module on which \mathfrak{t} acts via $\lambda_{\alpha,\beta} - \eta - \rho_{\mathfrak{h}}$. Equivalently, the n -tuple β is replaced by $(\beta_1 - 1, \dots, \beta_n - 1)$.

We let e_r denote the r th elementary symmetric polynomial. The proof of the following result is based on [GT3, Lemma 5.6].

Lemma 4.1. *Let $\alpha := (\alpha_1, \dots, \alpha_n)$ and $\beta := (\beta_1, \dots, \beta_n)$ of elements of \mathbb{F}_p and let $\overline{z}_{\alpha,\beta}$ be as defined above. Then*

- (a) $e^{(r)} \overline{z}_{\alpha,\beta} = 0$ for all $r > 0$;
- (b) $d_1^{(r)} \overline{z}_{\alpha,\beta} = e_r(\alpha_1, \dots, \alpha_n) \overline{z}_{\alpha,\beta}$ for all $0 < r \leq n$; and
- (c) $d_2^{(r)} \overline{z}_{\alpha,\beta} = e_r(\beta_1, \dots, \beta_n) \overline{z}_{\alpha,\beta}$ for all $0 < r \leq n$.

Proof. Instead of the construction of $Z_{\mathfrak{h}}(\alpha, \beta)$, we define the Verma module $M_{\mathfrak{h}}(\alpha, \beta) := U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\mathfrak{h}})} \mathbb{k}_{\alpha, \beta}$ and write $m_{\alpha, \beta} := 1 \otimes 1_{\alpha, \beta}$. We can inflate it to a $U(\mathfrak{p})$ -module and then restrict it to $U(\mathfrak{g}, e) \subseteq U(\mathfrak{p})$. We write $\overline{M}_{\mathfrak{h}}(\alpha, \beta)$ for the restriction and $\overline{m}_{\alpha, \beta}$ for $m_{\alpha, \beta}$ viewed as an element of $\overline{M}_{\mathfrak{h}}(\alpha, \beta)$. There is a surjective homomorphism $M_{\mathfrak{h}}(\alpha, \beta) \twoheadrightarrow Z_{\mathfrak{h}}(\alpha, \beta)$; $m_{\alpha, \beta} \mapsto z_{\alpha, \beta}$ of $U(\mathfrak{p})$ -modules. As $U(\mathfrak{g}, e) \subseteq U(\mathfrak{p})$, this gives a surjective homomorphism

$$\overline{M}_{\mathfrak{h}}(\alpha, \beta) \twoheadrightarrow \overline{Z}_{\mathfrak{h}}(\alpha, \beta); \overline{m}_{\alpha, \beta} \mapsto \overline{z}_{\alpha, \beta}$$

of $U(\mathfrak{g}, e)$ -modules. Now [GT3, Lemma 5.6(a)] and its proof imply (a). For (b) and (c), this follows from [GT3, Lemma 5.6(b)] in conjunction with the foregoing observation (Remark 5). \square

We denote by $L_{\mathfrak{h}}(\alpha, \beta)$ the unique simple quotient of the baby Verma module $Z_{\mathfrak{h}}(\alpha, \beta)$ (see [Jan, 10.2]). Recall that $\mathfrak{h} \cong \mathfrak{gl}_2^{\oplus n}$. For $1 \leq i \leq n$, we write \mathfrak{g}_i for the i th \mathfrak{gl}_2 corresponding to the i th column. It follows that \mathfrak{g}_i has basis $\{e_{i,i}, e_{i',i'}, e_{i,i'}, e_{i',i}\}$. For each i , we have $(\lambda_{\alpha, \beta} - \eta)(e_{i,i}) = \alpha_i + 2(n-i)$ and $(\lambda_{\alpha, \beta} - \eta)(e_{i',i'}) = \beta_i + 2(n-i)$. Consequently, we obtain

$$L_{\mathfrak{h}}(\alpha, \beta) \cong L(\alpha_1 + 2(n-1), \beta_1 + 2(n-1)) \otimes \cdots \otimes L(\alpha_n, \beta_n), \quad (4.2)$$

where $L(\alpha_i + 2(n-i), \beta_i + 2(n-i))$ is the irreducible $U_0(\mathfrak{g}_i)$ -module and $\dim_{\mathbb{k}} = [\alpha_i - \beta_i] + 1$ (see Section 2.2). As before, we restrict $L_{\mathfrak{h}}(\alpha, \beta)$ to $U^{[p]}(\mathfrak{g}, e)$ and write $\overline{L}_{\mathfrak{h}}(\alpha, \beta)$ for the restriction. We denote by $\overline{l}_{\alpha, \beta}$ the image of $\overline{z}_{\alpha, \beta}$ in $\overline{L}_{\mathfrak{h}}(\alpha, \beta)$. Also, we can view $\overline{L}_{\mathfrak{h}}(\alpha, \beta)$ as a $Y_{2,n}^{[p]}$ -module via the isomorphism (3.7).

Given two n -tuples $\alpha := (\alpha_1, \dots, \alpha_n)$ and $\beta := (\beta_1, \dots, \beta_n)$ of elements of \mathbb{F}_p , we define

$$\begin{aligned} \lambda_{\alpha}(u) &= (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_n u^{-1}), \\ \lambda_{\beta}(u) &= (1 + \beta_1 u^{-1}) \cdots (1 + \beta_n u^{-1}). \end{aligned}$$

Theorem 4.2. *Given two n -tuples $\alpha := (\alpha_1, \dots, \alpha_n)$ and $\beta := (\beta_1, \dots, \beta_n)$ of elements of \mathbb{F}_p , and let $\lambda_{\alpha}(u)$ and $\lambda_{\beta}(u)$ be as defined above. Suppose that for every $i = 1, \dots, n$, the following condition hold: $[\alpha_i - \beta_i]$ is minimal among all $[\alpha_j - \beta_l]$ for $i \leq j, l \leq n$. Then the irreducible module $L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u))$ is isomorphic to $\overline{L}_{\mathfrak{h}}(\alpha, \beta)$.*

Proof. Thanks to Lemma 4.1, there is a well-defined homomorphism

$$Z^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)) \rightarrow \overline{L}_{\mathfrak{h}}(\alpha, \beta); 1_{(\lambda_{\alpha}(u), \lambda_{\beta}(u))} \mapsto \overline{l}_{\alpha, \beta}$$

of $L_2^{[p]}$ -modules. We let $L_2^{[p]} \cdot \overline{l}_{\alpha, \beta}$ be the cyclic submodule of $\overline{L}_{\mathfrak{h}}(\alpha, \beta)$ generated by $\overline{l}_{\alpha, \beta}$. As $L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u))$ is the simple quotient of $Z^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u))$. There results a surjective homomorphism

$$\overline{L}_{\mathfrak{h}}(\alpha, \beta) \supseteq L_2^{[p]} \cdot \overline{l}_{\alpha, \beta} \twoheadrightarrow L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)). \quad (4.3)$$

A consecutive application of Theorem 2.9 and Proposition 2.6 implies

$$L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)) \cong L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_n, \beta_n).$$

We note that the evaluation module $L(\alpha_i, \beta_i)$ has dimension $[\alpha_i - \beta_i] + 1$ (see Subsection 2.2). As a result, $\dim_{\mathbb{k}} L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)) = ([\alpha_1 - \beta_1] + 1) \cdots ([\alpha_n - \beta_n] + 1)$. Moreover, observing (4.2) one gets $\dim_{\mathbb{k}} L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)) = \dim \overline{L}_{\mathfrak{h}}(\alpha, \beta)$ and (4.3) ensures that $L^{[p]}(\lambda_{\alpha}(u), \lambda_{\beta}(u)) \cong \overline{L}_{\mathfrak{h}}(\alpha, \beta)$. \square

4.2. Simple $U_\chi(\mathfrak{g})$ -modules. We recall $e \in \mathfrak{g}$ the $2 \times n$ -rectangular nilpotent element from (3.1). Let $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{k}$ denote the trace form associated to the natural representation of \mathfrak{g} . We define $\chi \in \mathfrak{g}^*$ to be the element dual to e via the trace form (\cdot, \cdot) , i.e. $\chi = (e, \cdot)$. Recall that

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g})/J_\chi = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p; x \in \mathfrak{g}).$$

To describe the simple $U_\chi(\mathfrak{g})$ -modules, we require some notations. Since $e \in \mathfrak{g}(1)$, we have that χ vanishes on $\mathfrak{g}(k)$ for $k \neq -1$. Therefore χ restricts to a character of \mathfrak{m} , so that χ defines a one dimensional representation $\mathbb{k}_\chi = \mathbb{k} \cdot 1_\chi$ of $U_\chi(\mathfrak{m})$. We define the *restricted Gelfand–Graev module* to be

$$Q^\chi := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{m})} \mathbb{k}_\chi \cong U_\chi(\mathfrak{g})/U_\chi(\mathfrak{g})\mathfrak{m}_\chi, \quad (4.4)$$

where we recall that $\mathfrak{m}_\chi = \{x - \chi(x); x \in \mathfrak{m}\}$. Note that Q^χ is a left $U_\chi(\mathfrak{g})$ -module and a right $U^{[p]}(\mathfrak{g}, e)$ -module.

We recall that the following Premet equivalence. This theorem is based on [Pre1, Theorem 2.4], see also [Pre2, Proposition 4.1] and [GT3, Theorem 2.4].

Theorem 4.3. *The functor from $U^{[p]}(\mathfrak{g}, e)$ -mod to $U_\chi(\mathfrak{g})$ -mod given by*

$$M \mapsto Q^\chi \otimes_{U^{[p]}(\mathfrak{g}, e)} M$$

is an equivalence of categories of quasi-inverse given by

$$V \mapsto V^{\mathfrak{m}_\chi} := \{v \in V; \mathfrak{m}_\chi v = (0)\}.$$

Taking into account Theorem 4.2, we obtain the following corollary:

Corollary 4.4. *Let $\alpha := (\alpha_1, \dots, \alpha_n)$ and $\beta := (\beta_1, \dots, \beta_n)$ be two n -tuples of elements of \mathbb{F}_p . Suppose that for every $i = 1, \dots, n$, the following condition hold: $[\alpha_i - \beta_i]$ is minimal among all $[\alpha_j - \beta_l]$ for $i \leq j, l \leq n$. Then $Q^\chi \otimes_{U^{[p]}(\mathfrak{g}, e)} \overline{L}_\mathfrak{h}(\alpha, \beta)$ is a simple $U_\chi(\mathfrak{g})$ -module and $\dim_{\mathbb{k}} Q^\chi \otimes_{U^{[p]}(\mathfrak{g}, e)} \overline{L}_\mathfrak{h}(\alpha, \beta) = p^{2n^2-2n}([\alpha_1 - \beta_1] + 1) \cdots ([\alpha_n - \beta_n] + 1)$. In particular, the isomorphism classes of simple $U_\chi(\mathfrak{g})$ -modules are parameterized by the set*

$$\{(\lambda_1(u), \lambda_2(u)); \prod_{j=1}^p \lambda_i(u - j + 1) = 1, \deg \lambda_i(u) \leq n, i = 1, 2\}.$$

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