CHARACTERIZATION OF GENUINE RAMIFICATION USING FORMAL ORBIFOLDS

INDRANIL BISWAS, MANISH KUMAR, AND A. J. PARAMESWARAN

Abstract. We give a characterization of genuinely ramified maps of formal orbifolds in the Tannakian framework. In particular we show that a morphism is genuinely ramified if and only if the pullback of every stable bundle remains stable in the orbifold category. We also give some other characterizations of genuine ramification. This generalizes the results of [BKP1] and [BP1]. In fact, it is a positive characteristic analogue of results in [BKP2].

1. Introduction

Let $f : Y \longrightarrow X$ be a finite generically smooth morphism of smooth projective connected curves over an algebraically closed field k. Then f is called genuinely ramified if there is no intermediate nontrivial étale cover of X. This notion of genuine ramification admits several equivalent formulations. For instance f is genuinely ramified if and only if the maximal semistable subsheaf of $f_*\mathcal{O}_Y$ is $\mathcal{O}_X$, or if and only if the induced map of the étale fundamental groups is surjective, or if and only if f^*E is stable for every stable vector bundle E on X, etcetera (see [BP1]).

This notion of genuine ramification extends to the more general context of orbifolds. A formal orbifold curve is a pair $(X,\, P)$, where X is a smooth projective curve and P is a "branch data" on X (the definition is recalled in Section 2; see [KP] for more details). Since we will only be dealing with curves, for us a formal orbifold will mean a formal orbifold curve. A morphism of formal orbifolds $(Y,\, Q) \longrightarrow (X,\, P)$ is a finite generically smooth morphism of curves $f : Y \longrightarrow X$ such that $Q(y) \supset P(f(y))$ for all closed points $y \in Y$. This morphism of formal orbifolds is étale if the equality $Q(y) = P(f(y))$ holds for all $y \in Y$. A morphism of formal orbifolds is called genuinely ramified if there is no intermediate nontrivial étale cover (see Definition 3.1).

A "geometric" formal orbifold $(X,\, P)$ is one for which there exists a Galois étale cover of formal orbifolds $g : (W,\, O) \longrightarrow (X,\, P)$, where O denotes the trivial branch data. In [KP] a vector bundle on such a pair $(X,\, P)$ was defined to be a G-equivariant vector bundle on W, where G is the Galois group for the map g. The notions of degree, slope, semistability etcetera were also defined in [KP]. When the characteristic of the base field k is zero, a branch data on X is the same as assigning integers greater than 1 at finitely many closed points of X. This in turn is equivalent to giving X an orbifold structure. When the characteristic of the base field k is zero, the equivariant bundles are also called orbifold bundles and they are equivalent to (see [Bi]) the parabolic bundles introduced by Mehta and Seshadri ([MS], see also [MY]). In positive characteristic this result was

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proved in [KM] where the notion of parabolic bundles is different and it is a slight variant of Nori's definition (given in [No]).

The following results were proved in [BKP2]. However their formulation in [BKP2] involved the terminology of orbifolds.

Proposition 1.1. *Let $f : Y \longrightarrow X$ be a morphism of irreducible smooth curves over an algebraically closed field of characteristic zero. Let X be equipped with a branch data P. Then $f_*\mathcal{O}_Y$ is a semistable "parabolic bundle" of degree zero belonging to* $\mathrm{Vect}(X,\, B_f)$ *and so it is semistable in* $\mathrm{Vect}(X,\, PB_f)$. *Then the maximal subbundle of $f_*\mathcal{O}_Y$ belonging to* $\mathrm{Vect}(X,\, P)$ *is a sheaf of subalgebras in $f_*\mathcal{O}_Y$ under parabolic tensor product. Moreover the corresponding spectrum defines the maximal cover $g : Z \longrightarrow X$ such that $(Z,\, g^*P) \longrightarrow (X,\, P)$ is étale and f dominates g.*

Theorem 1.2. *Let $f : Y \longrightarrow X$ be a morphism of irreducible smooth curves over an algebraically closed field of characteristic zero, and let P be a branch data on X. Then the following statements are equivalent:*

1. *The induced map $\pi(f) : \pi_1(Y,\, f^*P) \longrightarrow \pi_1(X,\, P)$ is surjective.*
2. *Every stable parabolic bundle in* $\mathrm{Vect}(X,\, P)$ *pulls back to a stable parabolic vector bundle in* $\mathrm{Vect}(Y,\, f^*P)$.
3. *The maximal subbundle of $f_*\mathcal{O}_Y$ of degree zero belonging to* $\mathrm{Vect}(X,\, P)$ *is $\mathcal{O}_X$.*

We extend Proposition 1.1 and Theorem 1.2 to the fields of positive characteristic. More precisely, in Theorem 3.3 we show that the above (1) and (2) are equivalent. This improves the main result of [BKP1]. In Theorem 5.2, (1) and (3) are shown to be equivalent under the additional hypothesis that f is Galois. In Proposition 4.4, a characterization of genuinely ramified morphisms is given in the Tannakian framework.

2. Vector bundles over orbifolds

Let X be a smooth projective connected curve defined over an algebraically closed field k. We briefly recall some basic definitions from [KP, Section 2]. For a closed point $x \in X$, let $\mathcal{K}_{X,x}$ denote the fraction field of $\widehat{\mathcal{O}}_{X,x}$. A *branch data* P on X assigns to each closed point $x \in X$ a finite Galois extension $P(x)$ of $\mathcal{K}_{X,x}$, satisfying the condition that the extension is trivial for all but finitely many points. The support $\mathrm{Supp}(P)$ of the branch data P is the finite subset of closed points where the field extension is actually nontrivial.

Given any two branch data P_1 and P_2 on X, we say that $P_1 \leq P_2$ if $P_1(x) \subset P_2(x)$ for all closed points $x \in X$. We can define their intersection $P_1 \bigcap P_2$ by $(P_1 \bigcap P_2)(x) := P_1(x) \bigcap P_2(x)$ for all closed points $x \in X$. Here the intersection is taken in a fixed algebraic closure of $\mathcal{K}_{X,x}$. Note that

$$\mathrm{Supp}(P_1 \cap P_2) \ \subseteq\ \mathrm{Supp}(P_1) \cap \mathrm{Supp}(P_2).$$

Similarly we also define their compositum P_1P_2 by

$$(P_1P_2)(x) \,:=\, P_1(x) \cdot P_2(x)$$

for all closed points $x \in X$. Note that $\mathrm{Supp}(P_1P_2) \,=\, \mathrm{Supp}(P_1) \cup \mathrm{Supp}(P_2)$. Also we have

$$P_1 \cap P_2 \ \leq\ P_i \ \leq\ P_1P_2$$

for $i = 1, 2$.

The trivial branch data is the one where all the field extensions are trivial; the trivial branch data is denoted by O. Let $f : Y \longrightarrow X$ be a finite generically smooth map and P a branch data on X. Then there is a natural branch data f^*P on Y constructed as follows: For any closed point $y \in Y$, the field $f^*P(y)$ is the compositum $P(f(y))\mathcal{K}_{Y,y}$.

Let $f : Y \longrightarrow X$ be a finite generically smooth map. For a closed point $x \in X$, let $B_f(x)$ be the compositum of the Galois closures of the field extensions $\{\mathcal{K}_{Y,y}\}_{y \in f^{-1}(x)}$ of $\mathcal{K}_{X,x}$. So we have a branch data B_f that assigns $B_f(x)$ to any $x \in X$. The support of B_f is evidently the subset over which f is ramified. Note that if f is also Galois, then $B_f(x) = \mathcal{K}_{Y,y}$ for any $y \in Y$ with $f(y) = x$.

A formal orbifold curve is a pair (X, P), where X is a smooth projective curve with P being a branch data on X. A *morphism* of orbifold curves

$$f : (Y, Q) \longrightarrow (X, P)$$

is a finite generically smooth morphism $f : Y \longrightarrow X$ such that $Q(y) \supset P(f(y))$ for every closed point $y \in Y$. The map f is said to be *étale* if $Q(y) = P(f(y))$ for all closed points $y \in Y$.

For any finite generically smooth morphism $f : Y \longrightarrow X$,

$$f : (Y, f^*B_f) \longrightarrow (X, B_f)$$

is an étale cover ([KP, Lemma 2.12]). We note that a Galois morphism $f : (Y, O) \longrightarrow (X, P)$ is étale if and only if $P = B_f$. Also, we have $f^*B_f = O$ for any Galois étale morphism f.

A branch data P on X is said to be *geometric* if there exists an étale Galois covering map $f : (Y, O) \longrightarrow (X, P)$ of formal orbifolds. If P is a geometric branch data on X, then (X, P) is called a *geometric formal orbifold*.

Take a geometric formal orbifold (X, P). Fix a Galois étale covering

$$f : (W, O) \longrightarrow (X, P);$$

the Galois group of f will be denoted by G. An object $\mathcal{V}$ in the category $\mathrm{Vect}(X, P)$ is a G-equivariant vector bundle V on W, while morphisms in $\mathrm{Vect}(X, P)$ are the equivariant morphisms of G-equivariant vector bundles on W. It should be clarified that the category $\mathrm{Vect}(X, P)$ is actually independent of the choice of f (see [KP, Proposition 3.6]).

Let

$$\deg(\mathcal{V}) := \frac{\deg(V)}{\deg(f)}$$

be the degree of the object $\mathcal{V}$ of the category $\mathrm{Vect}(X, P)$. Also define the slope

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\mathrm{rank}(V)}.$$

The object $\mathcal{V}$ is called *stable* (respectively, *semistable*) if $\mu(\mathcal{F}) < \mu(\mathcal{V})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{V})$) for all nonzero subobjects $\mathcal{F} \subset \mathcal{V}$ of smaller rank. A semistable vector bundle $\mathcal{V}$ is called *strongly semistable* if $\mathcal{V}^{\otimes j}$ is semistable for all $j \geq 1$.

Recall from [No] that a vector bundle V is called finite if $p(V) = q(V)$ for two distinct polynomials p and q with nonnegative integer coefficients. For a and n nonnegative integers, aV^n means $\bigoplus_{i=1}^{a} V^{\otimes n}$. Any vector bundle of degree zero which is isomorphic to a subbundle of a quotient bundle of degree zero of a finite bundle is called an essentially finite bundle. Let $\text{Vect}^f(X,\, P)$ (respectively, $\text{Vect}^{ss}(X,\, P)$) denote the full subcategory of $\text{Vect}(X,\, P)$ consisting of essentially finite (respectively, strongly semistable of degree zero) vector bundles. Also, $\text{Vect}^{et}(X,\, P)$ denotes the full subcategory of $\text{Vect}(X,\, P)$ consisting of étale trivial vector bundles.

Let $x \in X$ be a closed point outside the support of P. Note that each of $\text{Vect}^{ss}(X,\, P)$, $\text{Vect}^f(X,\, P)$ and $\text{Vect}^{et}(X,\, P)$, equipped with the natural fiber functor associated to x, is a neutral Tannakian category; their Tannaka duals are denoted by $\pi^S(X,\, P)$, $\pi^N(X,\, P)$ and $\pi_1^{et}(X,\, P)$ respectively.

In [KP, Section 3.2], for an étale morphism $f\,:\,(X_1,\, P_1)\,\longrightarrow\,(X_2,\, P_2)$ a pushforward functor $\text{Vect}(X_1,\, P_1)\,\longrightarrow\,\text{Vect}(X_2,\, P_2)$ was defined. To explain this construction, let $(Y_2,\, O)\,\longrightarrow\,(X_2,\, P_2)$ be a Galois étale covering map; denote by Γ the Galois group of this map. Let Y_1 be the normalization of the fiber product $\widetilde{X_1 \times_{X_2} Y_2}$ over X_2. Then the natural projection $\mathfrak{f}\,:\,Y_1\,\longrightarrow\,Y_2$ is an étale cover. Also, $\mathfrak{f}$ is a Γ-equivariant morphism. Moreover, the natural map $(Y_1,\, O)\,\longrightarrow\,(X_1,\, P_1)$ is an étale Galois covering with Galois group Γ. Hence an object $\mathcal{V}$ of $\text{Vect}(X_1,\, P_1)$ is a Γ-equivariant vector bundle V on Y_1. The direct image $\mathfrak{f}_*V$ is a Γ-equivariant vector bundle on Y_2, and hence it is an object of $\text{Vect}(X_2,\, P_2)$. We define this object of $\text{Vect}(X_2,\, P_2)$ as the *direct image* of $\mathcal{V}$; this direct image will be denoted by

$$\widehat{f_*}\mathcal{V}. \tag{2.1}$$

It should be mentioned that the object $\widehat{f_*}\mathcal{V}$ of $\text{Vect}(X_2,\, P_2)$ is actually independent of the choice of the Galois étale covering map $(Y_2,\, O)\,\longrightarrow\,(X_2,\, P_2)$.

Lemma 2.1. *Let $P_1\,\leq\,P_2$ be two branch data. Then there is a natural fully faithful functor* $\text{Vect}(X,\, P_1)\,\longrightarrow\,\text{Vect}(X,\, P_2)$. *Moreover, a vector bundle $\mathcal{V}$ in* $\text{Vect}(X,\, P_1)$ *is semistable if and only if it is semistable as a vector bundle in* $\text{Vect}(X,\, P_2)$.

Proof. The first statement is proved in [KP, Theorem 3.7] (also see [BKP3, Theorem 2.5]). The second statement follows from [KP, Lemma 3.10]. □

Definition 2.2. Let P_1 and P_2 be two geometric branch data on X. An object

$$\mathcal{E}\,\in\,\text{Vect}(X,\, P_1)$$

is said to be from $\text{Vect}(X,\, P_2)$ if there exists an object $\mathcal{E}'\,\in\,\text{Vect}(X,\, P_2)$ such that the images of $\mathcal{E}$ and $\mathcal{E}'$ in $\text{Vect}(X,\, P_1P_2)$ — under the functors $\text{Vect}(X,\, P_1)\,\longrightarrow\,\text{Vect}(X,\, P_1P_2)$ and $\text{Vect}(X,\, P_2)\,\longrightarrow\,\text{Vect}(X,\, P_1P_2)$ in Lemma 2.1 — are isomorphic.

Lemma 2.3. *Let $f\,:\,Y\,\longrightarrow\,X$ be a finite generically smooth morphism of smooth projective curves. Consider $\mathcal{O}_Y$ as an object of* $\text{Vect}(Y,\, f^*B_f)$. *Then $\widehat{f_*}\mathcal{O}_Y$ (see* (2.1)*) is a semistable vector bundle in* $\text{Vect}(X,\, B_f)$ *of degree zero.*

Proof. When the characteristic of k is 0, this is proved in [Pa].

The morphism f is actually an étale morphism of orbifolds $f\,:\,(Y,\, f^*B_f)\,\longrightarrow\,(X,\, B_f)$. Let

$$X_e\,\longrightarrow\,Y\,\longrightarrow\,X \tag{2.2}$$

be the morphism of smooth projective curves constructed by setting $k(X_e)$ to be the Galois closure of $k(Y)/k(X)$. Both the maps in

$$(X_e,\, O) \longrightarrow (Y,\, f^*B_f) \longrightarrow (X,\, B_f)$$

are étale. Denote $G = \mathrm{Gal}(k(X_e)/k(X))$. Note that $\mathcal{O}_Y$ is in $\mathrm{Vect}(Y,\, O)$ and hence it is also in $\mathrm{Vect}(Y,\, f^*B_f)$.

Let $Y_e = \widetilde{Y \times_X X_e}$ be the normalization of the fiber product $Y \times_X X_e$ over X. The action of G on X_e and the trivial action of G on Y together produce an action of G on Y_e. The projection

$$h \,:\, (Y_e,\, O) \longrightarrow (Y,\, f^*B_f)$$

is an étale Galois covering with Galois group G. Clearly, $\mathcal{O}_{Y_e} = h^*\mathcal{O}_Y$ is a G-equivariant line bundle. Hence $\mathcal{O}_Y$ — as an object of $\mathrm{Vect}(Y,\, f^*B_f)$ — is the G-equivariant bundle $\mathcal{O}_{Y_e}$. Let $\mathfrak{f} \,:\, Y_e = \widetilde{Y \times_X X_e} \longrightarrow X_e$ be the natural projection. By [KP, Section 3.2], the direct image $\widehat{f}_*\mathcal{O}_Y$ (see (2.1)) is the G-equivariant vector bundle $\mathfrak{f}_*\mathcal{O}_{Y_e}$ on X_e. Since $\mathfrak{f}$ is étale, the direct image $\mathfrak{f}_*\mathcal{O}_{Y_e}$ is semistable of degree zero [BP1, p. 12825, Lemma 2.3]. This proves the lemma. $\square$

Lemma 2.4. *Let $P_1 \le P_2$ be two branch data on X. Let $\mathcal{V} \in \mathrm{Vect}(X,\, P_2)$ be a semistable vector bundle admitting a subbundle $\mathcal{V} \supset \mathcal{V}' \in \mathrm{Vect}(X,\, P_1)$ (see Definition 2.2) such that $\mu(\mathcal{V}) = \mu(\mathcal{V}')$. Then there is a unique maximal semistable subbundle $\mathcal{V}_1 \subset \mathcal{V}$ such that*

(1) $\mu(\mathcal{V}_1) = \mu(\mathcal{V})$, *and*
(2) $\mathcal{V}_1 \in \mathrm{Vect}(X,\, P_1)$.

Proof. Since $\mathcal{V}$ is semistable, and the slope of the above vector bundle $\mathcal{V}'$ coincides with that of $\mathcal{V}$, it follows that $\mathcal{V}'$ is also semistable. If $\mathcal{V}'$ and $\mathcal{V}''$ are two subbundles of $\mathcal{V}$ lying in $\mathrm{Vect}(X,\, P_1)$ such that $\mu(\mathcal{V}'') = \mu(\mathcal{V}) = \mu(\mathcal{V}')$, then their sum $\mathcal{V}' + \mathcal{V}'' \subset \mathcal{V}$ is again a subbundle with $\mu(\mathcal{V}' + \mathcal{V}'') = \mu(\mathcal{V})$. Indeed, $\mathcal{V}' + \mathcal{V}''$ is a quotient of $\mathcal{V}' \oplus \mathcal{V}''$, so $\mu(\mathcal{V}' + \mathcal{V}'') \ge \mu(\mathcal{V}') = \mu(\mathcal{V}'')$, on the other hand, $\mathcal{V}' + \mathcal{V}''$ is a subsheaf of $\mathcal{V}$, so $\mu(\mathcal{V}' + \mathcal{V}'') \le \mu(\mathcal{V})$. We also have $\mathcal{V}' + \mathcal{V}'' \in \mathrm{Vect}(X,\, P_1)$. This proves the existence and uniqueness of a maximal subbundle $\mathcal{V}_1 \subset \mathcal{V}$ as in the lemma. $\square$

3. Pullback of stable bundles

Let $f \,:\, Y \longrightarrow X$ be a finite generically smooth morphism between smooth connected projective curves. Let P be a geometric branch data on X.

Definition 3.1. We say $f \,:\, (Y,\, f^*P) \longrightarrow (X,\, P)$ to be a genuinely ramified map of formal orbifolds if there is no intermediate cover

$$(Y,\, f^*P) \longrightarrow (Z,\, Q) \longrightarrow (X,\, P)$$

where $(Z,\, Q) \longrightarrow (X,\, P)$ is a nontrivial étale cover of formal orbifold curves.

Lemma 3.2. *Let $f \,:\, (Y,\, f^*P) \longrightarrow (X,\, P)$ be genuinely ramified. Let $(W,\, O) \longrightarrow (X,\, P)$ be an étale Galois cover with Galois group Γ. Let*

$$g \,:\, Z := \widetilde{W \times_X Y} \longrightarrow W$$

be the normalization of the fiber product $W \times_X Y$. Then g is a genuinely ramified morphism. Also the morphism $(Z, O) \longrightarrow (Y, f^*P)$ is a Galois étale cover with Galois group Γ.

Proof. If $L = k(W) \cap k(Y) \supsetneq k(X)$, then the normalization

$$f' : Y' \longrightarrow X$$

of X in L is of degree at least two. Note that f' is an essentially étale cover of (X, P) (see [KP, Definition 2.6(3)]) because f' is dominated by $W \longrightarrow X$. Hence by [KP, Lemma 2.12]

$$f' : (Y', f'^*P) \longrightarrow (X, P)$$

is an étale cover, which contradicts the given condition that $f : (Y, f^*P) \longrightarrow (X, P)$ is a genuinely ramified morphism. So $k(W) \cap k(Y) = k(X)$.

Moreover, since $k(W)/k(X)$ is Galois, it follows that $k(W)$ and $k(Y)$ are linearly disjoint over $k(X)$. Hence the morphism $g : Z \longrightarrow W$ in the statement of the lemma is a finite generically smooth morphism of connected nonsingular curves, and $Z \longrightarrow Y$ is a Galois cover with Galois group Γ. Furthermore, since $(W, O) \longrightarrow (X, P)$ is étale, the pullback $(Z, O) \longrightarrow (Y, f^*P)$ is also étale by [KP, Proposition 2.14 and Proposition 2.16].

Suppose that g is not genuinely ramified. Let $h : W' \longrightarrow W$ be the maximal étale cover of W dominated by g. For $\gamma \in \Gamma$, the automorphism of W given by γ will also be denoted by γ. The pullback of $h : W' \longrightarrow W$ by γ is again étale. Since W' is maximal étale, the pullback of $h : W' \longrightarrow W$ is again h. Hence Γ acts on W'. Let $f'' : Y'' \longrightarrow X$ be the normalization of X in $k(W')^\Gamma$. Then $Y'' \longrightarrow X$ is dominated by $f : Y \longrightarrow X$. The following diagram summarizes the situation:

$$\begin{tikzcd}
Z \arrow[r, "\Gamma"] \arrow[d] \arrow[dd, bend right, "g"'] & Y \arrow[d] \arrow[dd, bend left, "f"] \\
W' \arrow[r] \arrow[d] & Y'' \arrow[d] \\
W \arrow[r, "\Gamma"] & X
\end{tikzcd} \tag{3.1}$$

Also $(W', O) \longrightarrow (X, P)$ being the composition of two étale maps is also étale. Hence $Y'' \longrightarrow X$ is essentially étale for (X, P). So again by [KP, Lemma 2.12]

$$f'' : (Y'', f''^*P) \longrightarrow (X, P)$$

is étale, which gives a contradiction. Hence g is genuinely ramified. This completes the proof. $\square$

Theorem 3.3. *Let $f : (Y, f^*P) \longrightarrow (X, P)$ be a morphism of formal orbifolds. Then f is genuinely ramified if and only if the pullback of every stable object in* $\mathrm{Vect}(X, P)$ *is stable in* $\mathrm{Vect}(Y, f^*P)$.

Proof. Let $a : (W, O) \longrightarrow (X, P)$ be a Galois étale covering with Galois group Γ. A stable object of $\mathrm{Vect}(X, P)$ is a Γ-stable vector bundle E on W. Let

$$g : Z := \widetilde{W \times_X Y} \longrightarrow W$$

be the normalized pullback of f (see Lemma 3.2). By Lemma 3.2, the morphism $g : Z \longrightarrow Y$ is genuinely ramified. Also note that g is a Γ-equivariant morphism. By [BKP1, Proposition 4.2], the pullback g^*E is a Γ-stable vector bundle on Z. The morphism $g : (Z, O) \longrightarrow (Y, f^*P)$ is Galois étale with Galois group Γ. Hence g^*E is a stable object in $\mathrm{Vect}(Y, f^*P)$.

If f is not genuinely ramified, there is a nontrivial étale cover of formal orbifold curves

$$h : (Z, Q) \longrightarrow (X, P)$$

such that f factors as

$$(Y, f^*P) \xrightarrow{\phi} (Z, Q) \xrightarrow{h} (X, P).$$

Take any line bundle L on (Z, Q) of degree one. Then $\widehat{h}_*L$ is a stable vector bundle on (X, P). But $h^*\widehat{h}_*L$ is not stable because L is a quotient of it. Since $h^*\widehat{h}_*L$ is not stable, we conclude that $\phi^*h^*\widehat{h}_*L = f^*\widehat{h}_*L$ is not stable. $\square$

4. Tannakian characterization

Let $f : Y \longrightarrow X$ be a finite generically smooth morphism between smooth connected projective curves. As before, the branch data on X given by f will be denoted by B_f. As in (2.2), let $b : X_e \longrightarrow X$ be the Galois closure of f with Galois group G. Consider the normalization

$$\mathfrak{f} : Y_e := \widetilde{Y \times_X X_e} \longrightarrow X_e \tag{4.1}$$

of the fiber product $Y \times_X X_e$. Note that Y_e is $\deg(f)$-copies of X_e. As noted in the proof of Lemma 2.3, the group G acts on Y_e. The direct image

$$E := \mathfrak{f}_*\mathcal{O}_{Y_e}, \tag{4.2}$$

where $\mathfrak{f}$ is the map in (4.1), is a G-equivariant vector bundle on X_e, because $\mathcal{O}_{Y_e}$ is a G-equivariant vector bundle and the projection $\mathfrak{f}$ is a G-equivariant morphism. We have the diagram:

$$\begin{tikzcd}
 & Y_e \arrow[dl, "\mathfrak{f}"'] \arrow[ddr, "a"', "G"] & \\
X_e \arrow[drr, dashed] \arrow[ddr, "b"', "G"] & & \\
 & & Y \arrow[dl, "f"] \\
 & X &
\end{tikzcd} \tag{4.3}$$

We saw in the proof of Lemma 2.3 that E is $\widehat{f}_*\mathcal{O}_Y \in \mathrm{Vect}(X, B_f)$.

Remark 4.1. If the map f is Galois, then $Y = X_e$. So $Y_e = G \times X_e$, where $G = \mathrm{Gal}(f)$. Hence in that case we have

$$E = \mathfrak{f}_*\mathcal{O}_{Y_e} = k[G] \otimes_k \mathcal{O}_{X_e}.$$

Let $\mathcal{C}(f)$ be the neutral Tannakian subcategory of $\text{Vect}^{ss}(X,\, B_f)$ defined by the full subcategory generated by E in (4.2).

Note that $\mathcal{O}_{X_e} \in \text{Vect}^{ss}(X,\, B_f)$. Let $\mathcal{C}(b)$ denote the full neutral Tannakian subcategory of $\text{Vect}^{ss}(X,\, B_f)$ generated by $k[G]\otimes_k \mathcal{O}_{X_e}$.

Proposition 4.2. *The equality $\mathcal{C}(f) \,=\, \mathcal{C}(b)$ holds. Hence the Tannaka dual of $\mathcal{C}(f)$ is the Galois group G.*

Proof. Let $\widetilde{f} \,:\, X_e \,\longrightarrow\, Y$ be the map in (4.3). The normalization of the fiber product $X_e\times_X X_e$ will be denoted by M. Let $\varphi \,:\, M \,\longrightarrow\, X_e$ be the projection to the second factor. The map

$$\widetilde{f}\times \text{Id} \;:\; X_e\times_X X_e \,\longrightarrow\, Y\times_X X_e, \quad (x_1,\, x_2)\,\longmapsto\, (\widetilde{f}(x_1),\, x_2)$$

produces a map $g\,:\, M\,\longrightarrow\, Y_e$. This map g satisfies the equation

$$\mathfrak{f}\circ g \,=\, \varphi\, ,$$

where $\mathfrak{f}$ is the map in (4.1). This implies that

$$E\,\subset\, \varphi_*\mathcal{O}_M \tag{4.4}$$

(see (4.2)). But $\varphi_*\mathcal{O}_M \,=\, k[G]\otimes_k \mathcal{O}_{X_e}$ because the map b in (4.3) is Galois with Galois group G. So from (4.4) it follows that

$$\mathfrak{f}_*\mathcal{O}_{Y_e}\,\subset\, k[G]\otimes_k \mathcal{O}_{X_e}\, . \tag{4.5}$$

From (4.5) it follows immediately that $\mathcal{C}(f)$ is a full subcategory of $\mathcal{C}(b)$.

Consider the neutral Tannakian category $\text{Rep}(G)$ defined by all algebraic representations of G in finite dimensional k–vector spaces. Consider the subgroup

$$H\,:=\, \text{Gal}(a)\,\subset\, G$$

which is the Galois group of the morphism $a\,:\, Y_e\,\longrightarrow\, Y$ in (4.3). The left-translation action of G on G/H makes $k[G/H]\,\in\, \text{Rep}(G)$. Let $\mathcal{C}(G/H)$ be the full neutral Tannakian subcategory of $\text{Rep}(G)$ generated by $k[G/H]$. Since $b\,:\, X_e\,\longrightarrow\, X$ is the Galois closure of f, it follows that

$$\mathcal{C}(G/H)\,=\, \text{Rep}(G)\, .$$

From this it follows that the subcategory $\mathcal{C}(f)$ of $\mathcal{C}(b)$ actually coincides with $\mathcal{C}(b)$. $\square$

In the set-up of Proposition 4.2, let P be a geometric branch data on X. Let $\mathcal{C}_P(f)$ be the full neutral Tannakian subcategory of $\mathcal{C}(f)$ consisting of objects which are from $\text{Vect}(X,\, P)$ (in the sense of (2.2)). Let A be the Tannaka dual of $\mathcal{C}_P(f)$. Proposition 4.2 says that the Tannaka dual of $\mathcal{C}(f)$ is G. So we have a natural epimorphism

$$\alpha\,:\, G\,\longrightarrow\, A. \tag{4.6}$$

Lemma 4.3. *Let $H'\,\subset\, G$ be the kernel of the homomorphism α in (4.6). Let $\phi\,:\, Y'\,\longrightarrow\, X$ be the normalization of X in $k(X_e)^{H'}$, so $Y'\,=\, X_e/H'$. Then $\phi\,:\, (Y',\, \phi^*P)\,\longrightarrow\, (X,\, P)$ is the unique maximal étale cover of $(X,\, P)$ dominated by X_e.*

Proof. Let Q denote the branch data on Y' given by the quotient map

$$X_e\,\longrightarrow\, X_e/H'\,=\, Y'.$$

The category $\mathcal{C}(b)$ in Proposition 4.2 contains only étale trivial bundles, and hence from Proposition 4.2 it follows that $\mathcal{C}(f)$ contains only étale trivial bundles. Therefore, $\mathcal{C}_P(f)$ contains only étale trivial bundles. In other words, $\mathcal{C}_P(f)$ is a full subcategory of the neutral Tannakian category $\mathrm{Vect}^{et}(X, P)$. Consequently, there is a natural surjection between their Tannaka duals

$$\pi_1^{et}(X, P) \longrightarrow A \longrightarrow e,$$

where A is as in (4.6). Hence the induced A-cover $(Y', Q) \longrightarrow (X, P)$ is étale, where Q is defined above.

Let $\psi : (Z, P') \longrightarrow (X, P)$ be any étale cover which is dominated by X_e. Let $\mathcal{C}(\psi)$ denote the neutral Tannakian subcategory of $\mathrm{Vect}^{ss}(X, B_f)$ defined by the full subcategory generated by $\psi_* \mathcal{O}_Z$. Then $\mathcal{C}(\psi)$ is a subcategory of $\mathcal{C}_P(f)$. Hence Z is dominated by Y' establishing that $\phi : (Y', \phi^* P) \longrightarrow (X, P)$ is the maximal étale cover of (X, P) dominated by X_e. □

Proposition 4.4. *Let $f : Y \longrightarrow X$ be a finite generically smooth morphism, and let P be a geometric branch data on X. Let $\mathcal{C}_P(f)$ be the full Tannakian subcategory of $\mathcal{C}(f)$ consisting of objects which are from* $\mathrm{Vect}(X, P)$ *(in the sense of* (2.2)*). Let A be the Tannaka dual of $\mathcal{C}_P(f)$, and let $\alpha : G \longrightarrow A$ be the natural epimorphism. Then the following five statements are equivalent:*

(1) *The morphism $(Y, f^* P) \longrightarrow (X, P)$ is genuinely ramified.*
(2) $\alpha(\mathrm{Gal}(k(X_e)/Y)) = A$.
(3) $\pi_1(f) : \pi_1^S(Y, f^* P) \longrightarrow \pi_1^S(X, P)$ *is surjective.*
(4) $\pi_1(f) : \pi_1^N(Y, f^* P) \longrightarrow \pi_1^N(X, P)$ *is surjective.*
(5) $\pi_1(f) : \pi_1^{et}(Y, f^* P) \longrightarrow \pi_1^{et}(X, P)$ *is surjective.*

Proof. The equivalence of (1) and (5) is trivial. Also note that (3) implies (4) and (4) implies (5).

Let $H' = \mathrm{kernel}(\alpha)$, and let $\phi : Y' \longrightarrow X$ be the normalization of X in $k(X_e)^{H'}$. By Lemma 4.3,

$$\phi : (Y', \phi^* P) \longrightarrow (X, P)$$

is the maximal étale cover of (X, P) dominated by X_e. Hence (1) is equivalent to the statement that $k(Y') \cap k(Y) = k(X)$. But this is equivalent to the subgroup $H' \subset \mathrm{Gal}(k(X_e)/k(Y))$ being the whole group G, which in turn is equivalent to (2).

To prove that (5) implies (3) we need to show that

(i) the functor $f^* : \mathrm{Vect}^{ss}(X, P) \longrightarrow \mathrm{Vect}^{ss}(Y, f^* P)$ is fully faithful, and
(ii) subobjects of $f^* \mathcal{E}$ are pullback bundles.

(See [DM, p. 139, Proposition 2.21(a)].)

Since $f : (Y, f^* P) \longrightarrow (X, P)$ is genuinely ramified, the map $g : Z \longrightarrow W$ in (3.1) is genuinely ramified by Lemma 3.2. Hence the natural homomorphism

$$H^0(W,\ Hom(V_1, V_2)) \longrightarrow H^0(Z,\ Hom(g^* V_1, g^* V_2))$$

is an isomorphism by [BP1, Lemma 4.3]. When V_1 and V_2 are Γ-equivariant, the above natural map is Γ-equivariant; consequently, we have

$$H^0(W, Hom(V_1, V_2))^\Gamma \cong H^0(Z, Hom(g^* V_1, g^* V_2))^\Gamma.$$

We will now show that all subobjects of $f^*\mathcal{E}$ are of the form $f^*\mathcal{V}$, where $\mathcal{V} \subset \mathcal{E}$ is a subobject. Let E be the Γ–bundle on W representing $\mathcal{E}$. First assume that $\mathcal{E}$ is stable. Then $f^*\mathcal{E}$ is stable by Theorem 3.3. Therefore, any subobject of $f^*\mathcal{E}$ is either $f^*\mathcal{E}$ or 0. Hence all subobjects of $f^*\mathcal{E}$ are of the form $f^*\mathcal{V}$, where $\mathcal{V} \subset \mathcal{E}$. Next assume that $\mathcal{E}$ is polystable. So

$$E = \bigoplus_{i=1}^{n} E_i \otimes T_i ,$$

where $E_1, \cdots, E_n$ are stable Γ–bundles such that $E_i \neq E_j$ if $i \neq j$, and $T_1, \cdots, T_n$ are trivial Γ–bundles on W. In fact T_i is the trivial Γ–bundle on W with fiber

$$H^0(W, \mathrm{Hom}(E_i, E))^{\Gamma}.$$

Let r_i be the rank of T_i. Consider the pullback

$$g^*E = \bigoplus_{i=1}^{n} g^*E_i \otimes \mathcal{O}_Z^{r_i}.$$

From Theorem 3.3 it follows that each g^*E_i is a stable Γ–bundle. For $1 \leq i, j \leq n$, we again have (by [BP1, Lemma 4.3]),

$$H^0(W, \mathrm{Hom}(E_i, E_j)) \cong H^0(Z, \mathrm{Hom}(g^*E_i, g^*E_j)),$$

and hence

$$H^0(Z, \mathrm{Hom}(g^*E_i, g^*E_j))^{\Gamma} = H^0(W, \mathrm{Hom}(E_i, E_j))^{\Gamma} = 0,$$

because $E_1, \cdots, E_n$ are pairwise non-isomorphic stable Γ–bundles. Hence any subobject of g^*E is of the form $\bigoplus_{i=1}^{n} g^*E_i \otimes T'_i$, where $T'_i \subset \mathcal{O}_Z^{r_i}$ is a trivial Γ-subbundle. Consequently, all subobjects of $f^*\mathcal{E}$ are of the form $f^*\mathcal{V}$, where $\mathcal{V} \subset \mathcal{E}$.

Finally, for a general subobject of $\mathcal{E}$, let

$$0 \subset F_1 \subset \cdots \subset F_{\ell-1} \subset F_\ell = E \tag{4.7}$$

be the Jordan–Hölder filtration of the Γ–bundle E; so for any $1 \leq i \leq \ell$, the quotient bundle F_i/F_{i-1} is the unique maximal polystable subbundle of E/F_{i-1} [HL, p. 24, Lemma 1.5.5]. This uniqueness ensures that each F_i is preserved by the action of Γ on E. Let

$$V \subset g^*E$$

be a semistable Γ-subbundle over Z of degree zero. Let

$$V' \subset g_*V$$

be the maximal semistable subsheaf (the first nonzero term in the Harder–Narasimhan filtration). Then using the above observations if follows that $\mathrm{degree}(V') = 0$ and $\mathrm{rank}(V') = \mathrm{rank}(V)$ (see [BP2]). Moreover, the natural homomorphism $g^*g_*V \longrightarrow V$ has the property that its restriction to g^*V' is an isomorphism. This completes the proof. $\square$

Note that the category $\mathrm{Vect}(X, O)$ is the same as $\mathrm{Vect}(X)$, the category of vector bundles on X.

Proposition 4.4 has the following immediate consequence:

Corollary 4.5. *Let $\mathcal{C}_0(f)$ be the full Tannakian subcategory of $\mathcal{C}(f)$ consisting of objects from* $\mathrm{Vect}(X)$*. Then $\mathcal{C}_0(f)$ is a Tannakian category. Let A be its Tannaka dual and $a : G \longrightarrow A$ the natural epimorphism. Let $H = \mathrm{Gal}(k(X_e)/k(Y))$ and let B be the image of H in A. Then the following are equivalent:*

(1) *$f : Y \longrightarrow X$ is genuinely ramified.*
(2) *$B = A$.*

5. Pushforward of the structure sheaf

Consider the diagram in (4.3). Since $\mathcal{O}_{X_e} = b^*\mathcal{O}_X$, it follows that $\mathcal{O}_{X_e}$ has a natural G-equivariant structure. Note that $\mathcal{O}_{X_e}$ is a subbundle of E (defined in (4.2)) preserved by the action of G. Since $\mathcal{O}_{X_e}$ is in $\mathrm{Vect}(X, O)$, by Lemma 2.1 it is also in $\mathrm{Vect}(X, P \cap B_f)$ and its degree is 0. So by Lemma 2.4 there exists a unique maximal semistable G-equivariant subbundle

$$F \subset E \tag{5.1}$$

of degree 0 such that $F \in \mathrm{Vect}(X, P \cap B_f)$.

Let U be the normalized fiber product of $b : X_e \longrightarrow X$ and $\alpha : W \longrightarrow X$. Let $a : U \longrightarrow W$ and $\beta : U \longrightarrow X_e$ be the natural projections. So we have the following diagram:

$$\begin{tikzcd}
 & & Y_e \arrow[dl, "\mathfrak{f}"'] \arrow[dd, "G"] \\
U \arrow[r, "\beta"] \arrow[dd, "a"'] & X_e \arrow[dr, dashed] \arrow[dd, "b"', "G"] & \\
 & & Y \arrow[dl, "f"] \\
W \arrow[r, "\alpha"] & X &
\end{tikzcd} \tag{5.2}$$

Lemma 5.1. *Let F_W be the maximal degree zero Γ-subbundle of $(a_*\beta^*E)^G$. The $\Gamma \times G$-equivariant bundles β^*F (see (5.1)) and a^*F_W are isomorphic.*

Proof. Note that F_W is a slope zero subbundle of $a_*\beta^*E$. Since F is the maximal subbundle of E of slope zero, and E/F is a semistable bundle of negative slope, the image of F_W in $a_*\beta^*E$ lies in $a_*\beta^*F$. By adjointness we get a natural map of bundles

$$a^*F_W \longrightarrow \beta^*F \tag{5.3}$$

over U.

It can be shown that away from the preimage of $B_f \cup P$ the homomorphism in (5.3) is an isomorphism. Indeed, this follows immediately from the following two facts:

(1) For any generically smooth surjective map $\varphi : Z_1 \longrightarrow Z_2$ of smooth projective curves, and any vector bundle E on Z_2, the Harder–Narasimhan filtration of φ^*E is the pullback of the Harder–Narasimhan filtration of E (see [BP1, p. 12823–12824, Remark 2.1]).

(2) If $\varphi : Z_1 \longrightarrow Z_2$ is an étale Galois covering map of smooth curves with Galois group G, and E is a G-equivariant vector bundle on Z_1, then the natural map
$$\varphi^*((\varphi_* E)^G) \longrightarrow E$$
is an isomorphism.

Since both $a^* F_W$ and $\beta^* F$ are degree zero bundles, the generically isomorphic homomorphism in (5.3) is actually an isomorphism. Indeed, the cokernel of this map is a torsion sheaf and its degree is $\text{degree}(\beta^* F) - \text{degree}(a^* F_W) = 0$, and hence the cokernel is the zero sheaf. Consequently, the homomorphism in (5.3) is an isomorphism. □

The goal is to prove the following:

Theorem 5.2. *Let $f : (Y, f^*P) \longrightarrow (X, P)$ be a morphism of formal orbifolds. Assume f is also a Galois cover. Then the following are equivalent:*

(1) *$f : (Y, f^*P) \longrightarrow (X, P)$ is genuinely ramified.*
(2) *$F = \mathcal{O}_{X_e}$.*
(3) *$F_W = \mathcal{O}_W$.*

Proof. The equivalence of (2) and (3) is a consequence of Lemma 5.1.

Since f is Galois $Y = X_e$ and $E = \mathcal{O}_{X_e} \times k[G]$. Note that
$$a_* \beta^* E = a_* \beta^* \mathcal{O}_{X_e} \otimes_k k[G] = a_* \mathcal{O}_U \otimes_k k[G]$$
(see (5.2)). Hence $[a_* \beta^* E]^G = a_* \mathcal{O}_U$.

Since f is genuinely ramified by Lemma 3.2, the map $a : U \longrightarrow W$ is genuinely ramified. Therefore the degree zero part of the Harder–Narasimhan filtration of $a_* \mathcal{O}_U$ is actually $\mathcal{O}_W$ (by [BP1]). So F_W, which is the degree zero part of the Harder–Narasimhan filtration of $[a_* \beta^* E]^G$, is also $\mathcal{O}_W$. This proves that (1) implies (3).

Suppose that $f : (Y, f^*P) \longrightarrow (X, P)$ is not genuinely ramified. Let
$$g : Y' \longrightarrow X$$
be the maximal intermediate cover such that $g : (Y', g^*P) \longrightarrow (X, P)$ is étale. Let Z' be the normalized fiber product of Y' and W over X, and let $h : Z' \longrightarrow W$ be the natural projection. Then h is étale and it is dominated by a. Hence $h_* \mathcal{O}_{Z'}$ is a subbundle of $a_* \mathcal{O}_U$ of degree zero, and its rank is the same as the degree of h. But F_W is the maximal degree zero subsheaf of $[a_* \beta^* E]^G = a_* \mathcal{O}_U$. This contradicts (3). Hence (3) implies (1). □

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Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India

Email address: indranil.biswas@snu.edu.in, indranil29@gmail.com

Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India

Email address: manish@isibang.ac.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Email address: param@math.tifr.res.in