# ON THE COHOMOLOGICAL DIMENSION OF KERNELS OF MAPS TO $\mathbb{Z}$

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ABSTRACT. We prove that if $G$ is a finitely generated RFRS group of cohomological dimension 2, then $G$ is virtually free-by-cyclic if and only if $b_2^{(2)}(G) = 0$. This answers a question of Wise and generalises and gives a new proof of a recent theorem of Kielak and Linton, where the same result is obtained under the additional hypotheses that $G$ is virtually compact special and hyperbolic. More generally, we show that if $G$ is a RFRS group of cohomological dimension $n$ and of type $\mathrm{FP}_{n-1}$, then $G$ admits a virtual map to $\mathbb{Z}$ with kernel of rational cohomological dimension $n-1$ if and only if $b_n^{(2)}(G) = 0$.

## 1. Introduction

There is an emerging connection between the coherence of two-dimensional groups and the vanishing of the second $\ell^2$-Betti number. Indeed, there are now many results showing that if $G$ is a group of cohomological dimension 2 and $b_2^{(2)}(G) = 0$, then $G$ is coherent [Wis20b, KKW22, KL23, JZL23]. Moreover, there are no known examples of coherent groups with non-vanishing second $\ell^2$-Betti number. Related to coherence is Wise's notion of nonpositive immersions: a 2-complex $X$ has *nonpositive immersions* if for every immersion $Y \looparrowright X$ of a compact connected complex $Y$, either $\chi(Y) \leqslant 0$ or $\pi_1(Y)$ is trivial. Wise conjectures that if $X$ is aspherical and has nonpositive immersions, then $\pi_1(X)$ is coherent [Wis20a, Conjecture 12.11], and attributes to Gromov the observation that the nonpositive immersions property should be connected to the vanishing of $b_2^{(2)}(\widetilde{X})$ (see [Wis20a, Section 16] and [Wis22b]). Wise even conjectures that having nonpositive immersions should be equivalent to the vanishing of the second $\ell^2$-Betti number [Wis22b, Conjecture 2.6]. It thus makes sense to ask the following question.

**Question 1.1.** *Let $X$ be an aspherical 2-complex. Are the following properties equivalent?*

*(1)* $b_2^{(2)}(\widetilde{X}) = 0$.
*(2)* $\pi_1(X)$ *is coherent.*
*(3)* $X$ *has nonpositive immersions.*

In this article, we focus on the class of residually finite rationally solvable (RFRS) groups, which were defined by Agol in connection with Thurston's virtual fibering conjecture [Ago08]. Notably, the class of RFRS groups contains all compact special groups, introduced by Haglund and Wise [HW08], which are fundamental groups of particularly nice nonpositively curved compact cube complexes. Compact special groups provide a rich source of examples of RFRS groups, though there are also interesting RFRS groups which are not virtually compact special ($\mathbb{Z} \wr \mathbb{Z}$ is an example–see also [AS23], which gives an example of a RFRS lattice in $\mathrm{PU}(2,1)$).

A group is *free-by-cyclic* if it is a semidirect product $F \rtimes \mathbb{Z}$, where $F$ is free (we emphasise that $F$ need not be finitely generated). Our main result characterises which RFRS groups are virtually free-by-cyclic.

**Theorem A.** *Let $G$ be a finitely generated RFRS group. Then $G$ is virtually free-by-cyclic if and only if $b_2^{(2)}(G) = 0$ and $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant 2$.*

Being virtually free-by-cyclic is a strong form of coherence. Indeed, if $G$ is free-by-cyclic, then Feighn–Handel showed that $G$ is coherent [FH99], Wise proved that $G$ is the fundamental group of a 2-complex with nonpositive immersions [Wis22a, Theorem 6.1], and Henneke–López-Álvarez showed that $k[G]$ is a pseudo-Sylvester domain for any field $k$, and is therefore coherent [HLÁ22, Theorem B, Proposition 2.9] (meaning that all of its finitely generated one-sided ideals are finitely presented). We thus obtain the following corollary of Theorem A.

**Corollary B.** *Let $G$ be a RFRS group with $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant 2$. If $b_2^{(2)}(G) = 0$, then*

*(1) $G$ is coherent;*
*(2) the group algebra $k[G]$ is a coherent ring for any field $k$;*
*(3) if $G$ is finitely generated, then $G$ contains a finite-index subgroup $H = \pi_1(X)$, where $X$ is an aspherical 2-complex with nonpositive immersions.*

Determining when two-dimensional groups are virtually free-by-cyclic is an interesting problem in its own right, and there are many results in the literature addressing this question. However, the methods used in each case are somewhat ad hoc and often implicitly rely on the vanishing of the first $\ell^2$-Betti number at some point in the arguments. We list some applications of Theorem A, which provides a uniform treatment of many of these results.

**Corollary C.** *Let $G$ be a RFRS group and additionally suppose that one of the following holds:*

*(1) $G$ admits an elementary hierarchy* [HW10a, Theorem A]*;*
*(2) $G$ is the fundamental group of a finite 2-complex $X$ such that $b_2(X) = 0$;*
*(3) $G$ is a one-relator group.*

*Then $G$ is virtually free-by-cyclic.*

Item (1) was first obtained by Hagen and Wise in [HW10a]; a group has an elementary hierarchy of length 0 if it is trivial, and an elementary hierarchy of length $n$ if it splits as a graph of groups with vertex groups admitting an elementary hierarchy of length $n-1$ and with cyclic edge groups. It is easy to show, by induction on the length of the hierarchy, that groups with an elementary hierarchy are two-dimensional and have vanishing second $\ell^2$-Betti number. Two-dimensional limit groups (equivalently, limit groups without $\mathbb{Z}^3$-subgroups) and graphs of free groups with cyclic edge groups that do not contain any Baumslag–Solitar subgroups of the form $\mathrm{BS}(m, n)$ with $|m| \neq |n|$ are examples of virtually RFRS groups with elementary hierarchies ([Wis21, Corollary 18.3] and [HW10b]). In the case of limit groups we will be able to conclude something more general: If $G$ is a limit group with $\mathrm{cd}_{\mathbb{Q}}(G) = n$, then $G$ admits a virtual map to $\mathbb{Z}$ with kernel of rational cohomological dimension $n - 1$ (this follows from [BK17, Corollary C] and Theorem D below).

In [Wis20b, Theorem 1.1, Corollary 6.2], Wise proves that if $X$ is a compact 2-complex with $\pi_1(X)$ RFRS, then $b_1^{(2)}(\widetilde{X}) \leqslant b_1(X) - 1$ and $b_2^{(2)}(\widetilde{X}) \leqslant b_2(X)$. He then uses Kielak's theorem [Kie20, Theorem 5.4] together with a result of Fel'dman

[Fel71, Theorem 2.4] to conclude that if $b_1(X) = 1$ and $b_2(X) = 0$, then $\pi_1(X)$ is virtually free-by-cyclic. In [Wis20b, Problem 6.5], it is asked whether the assumption $b_1(X) = 1$ is necessary, and item (2) of Corollary C confirms that it is not.

Since one-relator groups are of rational cohomological dimension 2 and have vanishing second $\ell^2$-Betti numbers [DL07], we see that item (3) is an immediate corollary of Corollary C. The question of which one-relator groups are virtually free-by-cyclic has received much attention. Baumslag conjectured that one-relator groups with torsion are virtually free-by-cyclic [Bau86, Problem 6], and this was later strengthened by Wise who made the same conjecture for all hyperbolic one-relator groups. In [Wis20a, Theorem 17.11], Wise observes that every special two-generator one-relator group is virtually free-by-cyclic, again as a consequence of $\ell^2$-acyclicity and the results of [Kie20] and [Fel71]. Recently, Kielak and Linton showed that if $G$ is hyperbolic and virtually compact special, then $G$ is virtually free-by-cyclic if and only if $b_2^{(2)}(G) = 0$ and $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant 2$ [KL23, Theorem 1.1]. Since one-relator groups with torsion are hyperbolic [New68] and virtually special [Wis21, Corollary 19.2], Kielak and Linton's theorem resolves Baumslag's conjecture. Note that Wise's conjecture remains open as hyperbolic one-relator groups are not known to be special.

1.1. **Summary of the proof.** Theorem A and [KL23, Theorem 1.1] are related in that they are each special cases of results that take as input a group of cohomological dimension $n$ and produce virtual maps to $\mathbb{Z}$ with kernel of cohomological dimension $n-1$ (Theorem D below and Theorem 1.11 in [KL23]). The results about free-by-cyclic groups then follow by applying these theorems at $n = 2$ and appealing to Swan's theorem [Swa69]. However, the methods in each case are quite different; we briefly review them here.

Suppose that $G$ is a compact special hyperbolic group of cohomological dimension $n$, and assume that $b_i^{(2)}(G) = 0$ for all $i > 1$. In [KL23], it is shown that $G$ embeds in an HNN extension $H = G*_F$ such that $\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{cd}_{\mathbb{Q}}(G)$ and $H$ is $\ell^2$-acyclic. Moreover, the hyperbolicity and specialness assumptions are used to show that $H$ can be arranged to be virtually compact special, and therefore admits a virtual map to $\mathbb{Z}$ with kernel $N$ of type $\mathrm{FP}(\mathbb{Q})$ by [Fis21, Theorem A]. Using Fel'dman's theorem [Fel71, Theorem 2.4], we conclude that $\mathrm{cd}_{\mathbb{Q}}(N) = n-1$. Restricting the virtual map to $G$, we conclude that $G$ admits a virtual map to $\mathbb{Z}$ with kernel of rational cohomological dimension $n-1$.

As the proof given here is entirely homological, we do not need the geometric assumptions of hyperbolicity and specialness, but only the more algebraic RFRS condition. Moreover, we will only need to assume that the top-dimensional $\ell^2$-Betti number vanishes (as opposed to all $\ell^2$-Betti numbers in dimensions greater than 1) and the proof bypasses the algebraic fibering results of Kielak and the author [Kie20, Fis21] and Fel'dman's theorem [Fel71]. Thus, we take as input a RFRS group $G$ of finite type (in fact, type $\mathrm{FP}_{n-1}(\mathbb{Q})$ suffices) and of cohomological dimension $n$ and assume that $b_n^{(2)}(G) = 0$. In [Kie20], Kielak shows that an $\ell^2$-acyclic RFRS group of finite type has a finite-index subgroup $H$ whose homology with coefficients in the *Novikov ring* $\widehat{\mathbb{Q}[H]}^{\chi}$ vanishes for many maps $\chi \colon H \to \mathbb{Z}$ (the Novikov ring is a certain completion of the group algebra $\mathbb{Q}[H]$ with respect to $\chi$; see Definition 3.1). We observe that the assumption $b_n^{(2)}(G) = 0$ also implies that the Novikov *co*homology $\mathrm{H}^n(H; \widehat{\mathbb{Q}[H]}^{\chi})$ vanishes for many maps $\chi$. Fix one such character $\chi\colon H \to \mathbb{Z}$ and denote its kernel by $N$. In Theorem 3.5, it is shown that if $H$ is a group of finite type, then $\mathrm{H}^n(H; \widehat{\mathbb{Q}[H]}^{\pm\chi}) = 0$ implies that $\mathrm{cd}_{\mathbb{Q}}(\ker\chi) < \mathrm{cd}_{\mathbb{Q}}(H)$, and from here our main results follow quickly. It is interesting to compare Theorem 3.5 with Sikorav's theorem [Sik87] (see also [Kie20, Theorem 3.11]), which states that $\mathrm{H}_1(H; \widehat{\mathbb{Q}[H]}^{\pm\chi}) = 0$ if and only if $\ker\chi$ is finitely generated.

We state the most general form of our main result, which holds over all fields $k$.

**Theorem D.** *Let $k$ be a field and let $G$ be a RFRS group of type $\mathrm{FP}_{n-1}(k)$ with $\mathrm{cd}_k(G) = n$ and let $0 \leqslant m < n$. The following are equivalent:*

*(1) $b_n^{(2)}(G; k) = 0$ and $b_i^{(2)}(G; k) = 0$ for all $i \leqslant m$;*
*(2) there is a finite-index subgroup $H \leqslant G$ and an epimorphism $\chi\colon H \to \mathbb{Z}$ such that $\mathrm{cd}_k(\ker\chi) = n - 1$ and $\ker\chi$ is of type $\mathrm{FP}_m(k)$.*

The finiteness properties $\mathrm{FP}_i(k)$ and the definition of the quantities $b_i^{(2)}(G; k)$ will be recalled in Section 2. For now, we mention that $b_i^{(2)}(G; \mathbb{Q})$ is the usual $\ell^2$-Betti number $b_i^{(2)}(G)$. Note that if the $\ell^2$-Betti numbers vanish in low dimensions as well as in the top dimension, then we can deduce finiteness properties of the kernel. This may be interesting because the $\ell^2$-Betti numbers of many groups of classical interest are concentrated in their middle dimensions. In Section 3, we will use Theorem D to deduce higher coherence properties of group algebras of RFRS groups with vanishing top-dimensional $\ell^2$-Betti numbers (see Corollary 3.7).

1.2. **Residually (locally indicable and virtually Abelian) groups.** Let $\mathcal{C}$ be the class of residually (locally indicable and virtually Abelian) groups; note that $\mathcal{C}$ contains the class of RFRS groups. Very recently, Okun and Schreve gave a simplified proof of Kielak's fibering theorem [OS24], where they show that the Linnell division ring of a group $G \in \mathcal{C}$ admits a description in terms of Novikov rings similar to that given by Kielak for RFRS groups in [Kie20]. In light of this result, we remark that our arguments apply just as well to this extended class of groups, and all the results stated above remain true when the class $\mathcal{C}$ is substituted for the class of RFRS groups.

1.3. **Acknowledgments.** The author is grateful to Dawid Kielak for many useful conversations and comments on this article and to Andrei Jaikin-Zapirain for communicating simplifications of the original argument that have led to a far more transparent and streamlined proof. The author is supported by the National Science and Engineering Research Council (NSERC) [ref. no. 567804-2022] and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 850930).

## 2. Preliminaries

Throughout the article, $k$ always denotes a field.

2.1. **Division rings.** Consider the group algebra $k[G]$ of a locally indicable group $G$. Let $\mathcal{D}$ be a division ring such that there is an embedding $\varphi\colon k[G] \hookrightarrow \mathcal{D}$. This makes $\mathcal{D}$ into a $k[G]$-bimodule. If $N$ is a subgroup of $G$, we denote the division closure of $\varphi(k[N])$ in $\mathcal{D}$ by $\mathcal{D}_N$. The embedding $\varphi$ is *Hughes-free* if, for all finitely generated $H \leqslant G$ and $N \trianglelefteqslant H$ such that $H/N \cong \mathbb{Z}$, the multiplication map

$$\mathcal{D}_N \otimes_{k[N]} k[H] \to \mathcal{D}$$

is injective. Hughes proved that if $G$ is locally indicable and $k[G]$ admits a Hughes-free embedding $\varphi \colon k[G] \hookrightarrow \mathcal{D}$, then $\mathcal{D}$ is unique up to $k[G]$-isomorphism [Hug70]. Thus, we denote the *Hughes-free division ring* of $k[G]$ by $\mathcal{D}_{k[G]}$. If $H \leqslant G$ is any subgroup, then $\mathcal{D}_H \cong \mathcal{D}_{k[H]}$.

We will consider the homology and cohomology of a Hughes-free embeddable group $G$ with coefficients in $\mathcal{D}_{k[G]}$. We define

$$\mathrm{H}_n^{(2)}(G;k) = \mathrm{Tor}_n^{k[G]}(k, \mathcal{D}_{k[G]}) \quad \text{and} \quad \mathrm{H}^n_{(2)}(G;k) = \mathrm{Ext}^n_{k[G]}(k, \mathcal{D}_{k[G]}).$$

Since modules over division rings have well-defined dimensions, we can define the Betti numbers

$$b_n^{(2)}(G;k) = \dim_{\mathcal{D}_{k[G]}} \mathrm{H}_n^{(2)}(G;k) \quad \text{and} \quad b^n_{(2)}(G;k) = \dim_{\mathcal{D}_{k[G]}} \mathrm{H}^n_{(2)}(G;k).$$

It is easy to see that if both $b_n^{(2)}(G;k)$ and $b^n_{(2)}(G;k)$ are finite, then they are equal (see [KL23, Lemma 2.2]).

2.2. **RFRS.** Residually finite rationally solvable (RFRS) groups were introduced by Agol in [Ago08], where he showed that compact irreducible 3-manifolds with $\chi(M) = 0$ virtually fiber over the circle provided that $\pi_1(M)$ is virtually RFRS. A group $G$ is *RFRS* if there is a chain $G = G_0 \geqslant G_1 \geqslant \dots$ of finite-index subgroups such that $\bigcap_{i \geqslant 0} G_i = \{1\}$ and $\ker(G_i \to G_i^{\mathsf{ab}} \otimes \mathbb{Q}) \leqslant G_{i+1}$ for all $i \geqslant 0$. The main source of examples of RFRS groups are compact special groups, defined by Haglund and Wise [HW08]. A nonpositively curved cube complex is special if it avoids certain pathological hyperplane configurations, and a group is compact special if it is the fundamental group of a compact special cube complex. We refer the reader to the original paper of Haglund and Wise for more details.

RFRS groups are locally indicable and therefore satisfy the strong Atiyah conjecture (see [JZLÁ20], though this can also be deduced from earlier work of Schick [Sch02]). The consequence of the strong Atiyah conjecture that interests us is that it implies that $\mathbb{Q}[G]$ has a Hughes-free embedding; namely the division closure of $\mathbb{Q}[G]$ in the algebra of affiliated operators $\mathcal{U}(G)$ is a Hughes-free division ring [Lin93, Proof of Lemma 3.7] (see also [Lüc02, Chapter 10]). Hence, $b_n^{(2)}(G;\mathbb{Q})$ equals the usual $\ell^2$-Betti number $b_n^{(2)}(G)$. In [JZ21, Corollary 1.3 and Proposition 4.4], it is shown that if $G$ is RFRS, then the Hughes-free division ring $\mathcal{D}_{k[G]}$ exists for all fields $k$. Thus, we will consider all the Betti numbers $b_n^{(2)}(G;k)$ below.

2.3. **Finiteness properties.** Let $R$ be a ring. An $R$-module $M$ is of *type* $\mathrm{FP}_n$ if there is a projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i$ is finitely generated for all $i \leqslant n$. If $P_i$ is finitely generated for all $i$ we say that $M$ is of *type* $\mathrm{FP}_\infty$. If, additionally, there exists $N$ such that $P_i = 0$ for all $i > N$ then $M$ is of *type* FP. A group $G$ is of *type* $\mathrm{FP}_n(R)$ (resp. $\mathrm{FP}_\infty(R)$, $\mathrm{FP}(R)$) if the trivial $R[G]$-module $R$ is of type $\mathrm{FP}_n$ (resp. $\mathrm{FP}_\infty$, FP). Note that $G$ is finitely generated if and only if $G$ is of type $\mathrm{FP}_1(R)$ for some (and hence every) ring $R$.

A group $G$ is of *cohomological dimension at most $n$ over $R$* if the trivial $R[G]$-module $R$ admits a projective resolution of length $n$. The *cohomological dimension* of $G$ over $R$ is the minimal $n$ such that $G$ is of cohomological dimension at most $n$; in this case we write $\mathrm{cd}_R(G) = n$. If no such $n$ exists, then we define $\mathrm{cd}_R(G) = \infty$.

2.4. **Higher coherence.** If $R$ is a ring, then a group is *homologically $n$-coherent over $R$* if every subgroup of type $\mathrm{FP}_n(R)$ is of type $\mathrm{FP}_{n+1}(R)$. When $n = 1$ and $R = \mathbb{Z}$, this property is called *homological coherence*. The ring $R$ is *(left) $n$-coherent* if every (left) ideal of type $\mathrm{FP}_n$ is of type $\mathrm{FP}_{n+1}$. It is not hard to see that if $k[G]$ is $n$-coherent, then $G$ is homologically $n$-coherent over $k$. It is open whether the reverse implication holds, even for $n = 1$.

A ring $R$ is of *(left) global dimension at most $n$* if every (left) $R$-module $M$ has a projective resolution of length at most $n$, and the *global dimension* of $M$ is the infimal $n$ such that $M$ is of global at most $n$. If $G$ is a group with $\mathrm{cd}_k(G) = n$, then $k[G]$ is of global dimension $n$ [JZL23, Proposition 2.2]. The *weak dimension* of a (right) $R$-module $M$ is the maximal $n$ such that there exists a left $R$-module $N$ and $\mathrm{Tor}_n^R(M, N) \neq 0$. All of these concepts can be defined with the words "left" and "right" interchanged.

The following results are proven in [JZL23, Section 3] for $n = 1$. Since the proofs are completely analogous for general $n$, we omit them.

**Proposition 2.1** ([JZL23, Corollary 3.2])**.** *Let $R$ be a ring of global dimension $n$ that embeds into a division ring $\mathcal{D}$ of weak dimension at most $n-1$ as an $R$-module. Then $R$ is $(n-1)$-coherent.*

**Proposition 2.2** ([JZL23, Theorem 3.7])**.** *Let $G$ be a RFRS group with $\mathrm{cd}_k(G) = n$. If $b_n^{(2)}(G; k) = 0$, then $G$ is homologically $(n-1)$-coherent over $k$.*

Proposition 2.2 can be generalised, though this is the statement we will need here. For instance, when $k$ is of characteristic 0, the conclusion holds for all groups satisfying the strong Atiyah conjecture.

## 3. Proof of the main result

**Definition 3.1** (The Novikov ring)**.** Let $\chi \colon G \to \mathbb{R}$ be a nontrivial homomorphism from a group $G$ to the additive group $\mathbb{R}$. The *Novikov ring* of $k[G]$ with respect to $\chi$, denoted $\widehat{k[G]}^{\chi}$, is the set of formal sums $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in k$ such that

$$\#\{g \in G : \lambda_g \neq 0 \text{ and } \chi(g) \leqslant r\} < \infty$$

for all $r \in \mathbb{R}$. The obvious addition and multiplication of elements in $\widehat{k[G]}^{\chi}$ endows $\widehat{k[G]}^{\chi}$ with the structure of a ring.

The following proposition is a cohomological analogue of [Kie20, Theorem 5.2] and is proved similarly. It relates the vanishing of $\ell^2$-cohomology with the vanishing of Novikov cohomology of a finite-index subgroup. A (not necessarily square) matrix is in *Smith normal form* if it has entries equal to 1 in some of its diagonal slots, and has entries equal to 0 in all other slots.

**Proposition 3.2.** *Let $G$ be a RFRS group and let $P_\bullet$ be a chain complex of finitely generated free $k[G]$-modules. Suppose that*

$$\mathrm{H}^n(\mathrm{Hom}_{k[G]}(P_\bullet, \mathcal{D}_{k[G]})) = 0$$

*for some $n \in \mathbb{Z}$. Then there is a finite index subgroup $H \leqslant G$ and an antipodally symmetric open subset $U \subseteq \mathrm{H}^1(H; \mathbb{R})$ such that*

*(1)* $\mathrm{H}^1(G;\mathbb{R}) \subseteq \overline{U}$ *and*
*(2)* $\mathrm{H}^n(\mathrm{Hom}_{k[H]}(P_\bullet, \widehat{k[H]}^{\chi})) = 0$ *for all* $\chi \in U$.

*Proof (sketch).* Because the modules of the chain complex $P_\bullet$ are free and finitely generated, we can identify $\mathrm{Hom}_{k[G]}(P_i, \mathcal{D}_{k[G]})$ with ${\mathcal{D}_{k[G]}}^{d_i}$ for some integer $d_i$ for each $i$. Thus, we identify the coboundary maps with matrices

$$\delta^{i+1} \colon {\mathcal{D}_{k[G]}}^{d_i} \to {\mathcal{D}_{k[G]}}^{d_{i+1}},$$

though their entries are still contained in $k[G] \subseteq \mathcal{D}_{k[G]}$. Because $\mathcal{D}_{k[G]}$ is a division ring, there are invertible matrices $M_i \in \mathrm{GL}_{d_i}(\mathcal{D}_{k[G]})$ such that $M_{i+1} \circ \delta^{i+1} \circ M_i^{-1}$ is in Smith normal form for each $i$.

For every finite index (normal) subgroup $H \trianglelefteqslant G$, the chain complex $P_\bullet$ may also be viewed as a chain complex of finitely generated free $k[H]$-modules. Moreover, $\mathcal{D}_{k[G]} \cong \mathcal{D}_{k[H]} * (G/H)$, so the matrices $M_i$ (and their inverses $M_i^{-1}$) become matrices over $\mathcal{D}_{k[H]}$. An $m$-by-$n$ matrix becomes a $[G:H]m$-by-$[G:H]n$ matrix after passing to the finite-index subgroup $H$ and each matrix $M_{i+1} \circ \delta^{i+1} \circ M_i^{-1}$ is still in Smith normal form when passing to $H$. Indeed, the entries equal to 1 get replaced by $[G:H]$-by-$[G:H]$ identity matrices, and those equal to zero get replaced by $[G:H]$-by-$[G:H]$ matrices of zeros.

By [Kie20, Theorem 4.13], there is a finite index subgroup $H \trianglelefteqslant G$ and antipodally symmetric open set $U$ such that the matrices $M_i$ and $M_i^{-1}$ can be viewed as lying over the Novikov ring $\widehat{k[H]}^{\chi}$ for every $\chi \in U$. These matrices put the coboundary maps of the cochain complex $\mathrm{Hom}_{k[H]}(P_\bullet, \widehat{k[H]}^{\chi})$ into the same Smith normal form as the corresponding maps in $\mathrm{Hom}_{k[G]}(P_\bullet, \mathcal{D}_{k[G]})$ (viewed as a complex of free $k[H]$-modules). It then follows that $\mathrm{H}^n(\mathrm{Hom}_{k[H]}(P_\bullet, \widehat{k[H]}^{\chi})) = 0$. $\square$

**Corollary 3.3.** *Let $G$ be a RFRS group of type $\mathrm{FP}_n(k)$ such that $b_i^{(2)}(G;k) = 0$ for some $i < n$. Then there is a finite-index subgroup $H \leqslant G$ and an antipodally symmetric open subset $U \subseteq \mathrm{H}^1(H;\mathbb{R})$ such that*

$$\mathrm{H}^i(H; \widehat{k[H]}^{\chi}) = 0$$

*for each $\chi \in U$ and $\mathrm{H}^1(G;\mathbb{R}) \subseteq \overline{U}$.*

*Proof.* Since $G$ is of type $\mathrm{FP}_n(k)$, there is a partial resolution

$$P_n \to \cdots \to P_0 \to k \to 0$$

of finitely generated free $k[G]$-modules. Then $\mathrm{H}^i(\mathrm{Hom}_{k[G]}(P_\bullet, \mathcal{D}_{k[G]})) = 0$, since $b_i^{(2)}(G;k) = b^i_{(2)}(G;k) = 0$. The corollary then follows from Proposition 3.2. $\square$

We follow the proof of [JZL23, Theorem 3.3] for the following lemma.

**Lemma 3.4.** *Let $G$ be RFRS and suppose that $G$ fits into a short exact sequence*

$$1 \to N \to G \to Q \to 1$$

*where $\mathrm{cd}_k(N) = n-1$ and $Q$ is torsion-free and elementary amenable. Then $\mathcal{D}_{k[G]}$ is of weak dimension at most $n-1$ as a $k[G]$-module. In particular, $b_n^{(2)}(G;k) = 0$.*

*Proof.* Let $M$ be an arbitrary $k[G]$-module. Note that $\mathcal{D}_{k[G]} \cong \mathrm{Ore}(\mathcal{D}_{k[N]} * Q)$, since twisted group algebras of torsion-free elementary amenable groups are Ore domains [KLM88]. Then

$$\mathrm{Tor}_n^{k[G]}(\mathcal{D}_{k[N]} * Q, M) \cong \mathrm{Tor}_n^{k[N]}(\mathcal{D}_{k[N]}, M) = 0$$

by Shapiro's lemma and the fact that $k[N]$ is of global dimension $n-1$. Since Ore localisation is flat, we obtain $\mathrm{Tor}_n^{k[G]}(\mathcal{D}_{k[G]}, M) = 0$ as desired. □

Fix a nontrivial character $\chi \colon G \to \mathbb{R}$. The next theorem gives a sufficient cohomological criterion for when $\mathrm{cd}_k(\ker \chi) < \mathrm{cd}_k(G)$. We thank Andrei Jaikin-Zapirain for communicating a simplification of our original proof of this theorem.

**Theorem 3.5.** *Let $G$ be a group of type $\mathrm{FP}(k)$ with $\mathrm{cd}_k(G) = n$. Suppose that $\chi \colon G \to \mathbb{R}$ is a nontrivial character such that*

$$\mathrm{H}^n(G; \widehat{k[G]}^{\pm\chi}) = 0.$$

*Then $\mathrm{cd}_k(\ker \chi) < n$.*

*Proof.* Let $N = \ker \chi$ and let $T$ be a transversal for $N$ in $G$. We will show that $\mathrm{H}^n(N; L) = 0$ for any $k[N]$-module $L$, and therefore that $\mathrm{cd}_k(N) < n$. By Shapiro's lemma, we have

$$\mathrm{H}^n(N; L) \cong \mathrm{H}^n(G; \mathrm{Hom}_{k[N]}(k[G], L)) \cong \mathrm{H}^n(G; \mathcal{L})$$

where $\mathcal{L} = \prod_T L$ is a $k[G]$-module: the elements of $N$ act factor wise via the action of $N$ on $L$ and the elements of $T$ permute the factors.

Define

$$\mathcal{L}^{\chi} = \{(x_t)_{t \in T} \in \mathcal{L} : \text{there exists } \alpha \in \mathbb{R} \text{ such that } x_t = 0 \text{ if } \chi(t) < \alpha\}$$

and

$$\mathcal{L}^{-\chi} = \{(x_t)_{t \in T} \in \mathcal{L} : \text{there exists } \alpha \in \mathbb{R} \text{ such that } x_t = 0 \text{ if } \chi(t) > \alpha\}.$$

Note that $\mathcal{L}^{\pm\chi}$ is a $\widehat{k[G]}^{\pm\chi}$-module.

We claim that the cohomology of $G$ with coefficients in $\mathcal{L}^{\pm\chi}$ vanishes. Indeed,

$$\begin{aligned} \mathrm{H}^n(G; \mathcal{L}^{\pm\chi}) &\cong \mathrm{H}^n(G; k[G]) \otimes_{k[G]} \mathcal{L}^{\pm\chi} \\ &\cong \mathrm{H}^n(G; k[G]) \otimes_{k[G]} \widehat{k[G]}^{\pm\chi} \otimes_{\widehat{k[G]}^{\pm\chi}} \mathcal{L}^{\pm\chi} \\ &\cong \mathrm{H}^n(G; \widehat{k[G]}^{\pm\chi}) \otimes_{\widehat{k[G]}^{\pm\chi}} \mathcal{L}^{\pm\chi} = 0, \end{aligned}$$

where we used the fact that $G$ is of type $\mathrm{FP}(k)$ and [Bro94, Proposition VIII.6.8].

The $k[G]$-module inclusions of $\mathcal{L}^{\pm\chi} \hookrightarrow \mathcal{L}$ induce a surjection $\mathcal{L}^{\chi} \oplus \mathcal{L}^{-\chi} \to \mathcal{L}$; denote the kernel of this map by $M$. Then we have a short exact sequence

$$0 \to M \to \mathcal{L}^{\chi} \oplus \mathcal{L}^{-\chi} \to \mathcal{L} \to 0$$

of $k[G]$-modules. The long exact sequence in $\mathrm{H}^\bullet(G; -) = \mathrm{Ext}^\bullet_{k[G]}(k, -)$ contains the portion

$$\mathrm{H}^n(G; \mathcal{L}^{\chi} \oplus \mathcal{L}^{-\chi}) \to \mathrm{H}^n(G; \mathcal{L}) \to \mathrm{H}^{n+1}(G; M).$$

But $\mathrm{H}^{n+1}(G; M) = 0$ because $\mathrm{cd}_k(G) = n$ and

$$\mathrm{H}^n(G; \mathcal{L}^{\chi} \oplus \mathcal{L}^{-\chi}) \cong \mathrm{H}^n(G; \mathcal{L}^{\chi}) \oplus \mathrm{H}^n(G; \mathcal{L}^{-\chi}) = 0$$

by [Bie81, Proposition 2.4], where again we have used that $G$ is of type $\mathrm{FP}(k)$. Hence, $\mathrm{H}^n(G; \mathcal{L}) = \mathrm{H}^n(N; L) = 0$. □

We now arrive at our main result, which follows quickly from Theorem 3.5.

**Theorem 3.6.** *Let $G$ be a RFRS group of type $\mathrm{FP}_{n-1}(k)$ with $\mathrm{cd}_k(G) = n$ and let $0 \leqslant m < n$. The following are equivalent:*

*(1) $b_n^{(2)}(G;k) = 0$ and $b_i^{(2)}(G;k) = 0$ for all $i \leqslant m$;*
*(2) there is a finite-index subgroup $H \leqslant G$ and an epimorphism $\chi\colon H \to \mathbb{Z}$ such that $\mathrm{cd}_k(\ker\chi) = n-1$ and $\ker\chi$ is of type $\mathrm{FP}_m(k)$.*

*Proof.* If (2) holds, then $b_n^{(2)}(H;k) = 0$ by Lemma 3.4 and therefore $b_n^{(2)}(G;k) = 0$ by [Fis21, Lemma 6.3]. Moreover, $b_i^{(2)}(G;k) = 0$ for $i \leqslant m$ by [Fis21, Theorem 6.4].

Now suppose that (1) holds. By Proposition 2.2, $G$ is of type $\mathrm{FP}_n(k)$, and therefore of type $\mathrm{FP}(k)$ since $\mathrm{cd}_k(G) = n$. By [Fis21, Theorem B], $G$ virtually fibers with kernel of type $\mathrm{FP}_m(k)$. Since the conclusion we want to prove is virtual, we will assume that there is an epimorphism $\varphi\colon G \to \mathbb{Z}$ with kernel of type $\mathrm{FP}_m(k)$.

By Corollary 3.3, there is a finite-index subgroup $H \leqslant G$ and an open set $U \subseteq \mathrm{H}^1(H;\mathbb{R})$ such that $\mathrm{H}^1(G;\mathbb{R}) \subseteq \overline{U}$ and

$$\mathrm{H}^n(H; \widehat{k[H]}^{\pm\chi}) = 0$$

for all characters $\chi \in U$. Fix $\chi \in U$ such that $\chi(H) \subseteq \mathbb{Z}$ (there are many such characters since $U$ is open) and let $N = \ker\chi$. Moreover, since $\pm\varphi$ is in the higher sigma invariant $\Sigma^n(H;k)$, which is open in $\mathrm{H}^1(H;\mathbb{R})$ (see [BR88]), and $\pm\varphi \in \overline{U}$, we may choose $\chi$ so that its kernel is of type $\mathrm{FP}_m(k)$. By Theorem 3.5, $\mathrm{cd}_k(N) < n$. Since $\mathrm{cd}_k(\mathbb{Z}) = 1$, we must have $\mathrm{cd}_k(N) = n-1$. □

By Proposition 2.2, if $G$ is a locally indicable group with $\mathrm{cd}_k(G) = n$ and $b_n^{(2)}(G;k) = 0$, then $G$ is $(n-1)$-homologically coherent over $k$. If we additionally assume that $G$ is RFRS, we can now conclude that $k[G]$ is $(n-1)$-coherent. Note that $(n-1)$-coherence of $k[G]$ implies $(n-1)$-homological coherence of $G$ over $k$, however, whether the converse holds in general is open, even for $n = 2$.

**Corollary 3.7.** *Let $G$ be a RFRS group of type $\mathrm{FP}_{n-1}(k)$ with $\mathrm{cd}_k(G) = n$. If $b_n^{(2)}(G;k) = 0$, then the group algebra $k[G]$ is $(n-1)$-coherent.*

*Proof.* Since group algebra coherence is a commensurability invariant, we may assume that there is a map $\chi\colon G \to \mathbb{Z}$ such that $N = \ker\chi$ satisfies $\mathrm{cd}_k(G) = n-1$. Then $\mathcal{D}_{k[G]}$ is of weak dimension at most $n-1$ by Lemma 3.4, which implies that $k[G]$ is $(n-1)$-coherent by Proposition 2.1. □

For example, if $\Gamma$ is a lattice of simplest type in $\mathrm{PO}(n,1)$, then $\Gamma$ is virtually compact special by [BHW11, Ago13] and has vanishing top-degree $\ell^2$-Betti number. Hence, $k[\Gamma]$ is $(n-1)$-coherent. Similarly, if $G$ is the fundamental group of a closed hyperbolic 3-manifold, then $k[G]$ is 2-coherent (we expect $k[G]$ should in fact be coherent).

We now specialise to dimension 2.

**Corollary 3.8.** *Let $G$ be a finitely generated RFRS group with $\mathrm{cd}_k(G) \leqslant 2$. Then $G$ is virtually free-by-cyclic if and only if $b_2^{(2)}(G;k) = 0$.*

*Proof.* This follows immediately from Theorem 3.6 applied at $n = 2$ and Swan's theorem [Swa69], which states that torsion-free groups of $k$-cohomological dimension 1 are free. □

**Corollary 3.9.** *If $G$ is RFRS, $\mathrm{cd}_k(G) \leqslant 2$, and $b_2^{(2)}(G;k) = 0$, then $G$ and $k[G]$ are coherent. If, additionally, $G$ is finitely generated, then $G$ has a finite-index subgroup $H = \pi_1(X)$ where $X$ is a 2-complex with nonpositive immersions.*

*Proof.* Let $H \leqslant G$ be a finitely generated subgroup. Then $b_2^{(2)}(H;k) = 0$ by [FM23, Lemma 3.21]. Then $H$ is virtually free-by-cyclic by Corollary 3.8 and hence finitely presented by [FH99].

Let $I \trianglelefteq k[G]$ be a finitely generated (left) ideal with finite generating set $S \subseteq k[G]$. Note that there is a finitely generated subgroup $H \leqslant G$ such that $S \subseteq k[H]$. Then $H$ is virtually free-by-cyclic by the argument above. Let $I_H$ be the left ideal of $k[H]$ generated by $S$. It follows from [HLÁ22, Theorem B, Proposition 2.9] that group algebras of free-by-cyclic groups are coherent, so there is an exact sequence

$$k[H]^m \to k[H]^n \to I_H \to 0$$

for some integers $m$ and $n$. Since $k[G]$ is free as a $k[H]$-module, it is also flat and thus tensoring with $k[G]$ gives the exact sequence

$$k[G]^m \to k[G]^n \to k[G] \otimes_{k[H]} I_H \to 0.$$

We verify that the multiplication map $m \colon k[G] \otimes_{k[H]} I_H \to I$ is an isomorphism, which will conclude the proof that $k[G]$ is coherent. The generating set $S$ is clearly in the image of $m$, so $m$ is surjective. Let $T$ be a left transversal for $H$ in $G$. Then

$$k[G] \otimes_{k[H]} I_H \cong \bigoplus_{t \in T} t \cdot k[H] \otimes_{k[H]} I_H \cong \bigoplus_{t \in T} t \cdot I_H \subseteq \bigoplus_{t \in T} t \cdot k[H] \cong k[G],$$

and therefore $m$ is injective.

The final claim follows from Corollary 3.8 and [Wis22a, Theorem 6.1], which states that ascending HNN extensions of free groups are fundamental groups of 2-complexes with nonpositive immersions. $\square$

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