

# DUALITY FOR HODGE-WITT COHOMOLOGY WITH MODULUS

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ABSTRACT. Given an effective Cartier divisor  $D$  with simple normal crossing support on a smooth and proper scheme  $X$  over a perfect field of positive characteristic  $p$ , there is a natural notion of de Rham-Witt sheaves on  $X$  with zeros along  $D$ . We show that these sheaves correspond under Grothendieck duality for coherent sheaves to de Rham-Witt sheaves on  $X$  with modulus  $(X, D)$ , as defined in the theory of cube invariant modulus sheaves with transfers developed by Kahn-Miyazaki-Saito-Yamazaki. From this we deduce refined versions of Ekedahl - and Poincaré duality for crystalline cohomology generalizing results of Mokrane and Nakkajima for reduced  $D$ , and a modulus version of Milne-Kato duality for étale motivic cohomology with  $p$ -primary torsion coefficients, which refines a result of Jannsen-Saito-Zhao. We furthermore get new integral models for rigid cohomology with compact supports on the complement of  $D$  and a modulus version of Milne's perfect Brauer group pairing for smooth projective surfaces over finite fields.

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## 1. INTRODUCTION

Let  $k$  be a perfect field of positive characteristic  $p > 0$ . There is a wealth of duality results for smooth and proper  $k$ -schemes with  $p$ -torsion - or  $p$ -adic coefficients, such as Serre duality for differential forms, Poincaré duality for crystalline cohomology [Ber74], or Milne duality [Mil76], see also [Kat86], [Kat87], which by [GL00] can be interpreted as a duality for étale motivic cohomology with mod  $p^n$  coefficients. For surfaces over a finite field the latter induces a pairing for the Brauer group [Mil75]. These pairings can all be understood via Grothendieck-Ekedahl duality [Eke84], [Gro85] which is a duality statement for the de Rham-Witt complex of Bloch-Deligne-Illusie [Blo77], [Ill79].

A generalization of these duality statements to open subschemes (say of a smooth and proper  $k$ -scheme) requires the use of cohomology with compact support and relies on considering pairings between certain pro-groups (on the compact support side) and certain ind-groups. For example there is the Deligne-Hartshorne duality for compactly supported cohomology of coherent sheaves [Har72], Berthelot's duality for rigid cohomology [Ber97], and the generalization of Milne duality by Jannsen-Saito-Zhao [JSZ18] and Gupta-Krishna [GK], which generalizes the  $p$ -primary torsion part of geometric global class field theory.

As the (compact supported) cohomology groups of an open in the above situations are very often not finitely generated over the base ring at hand, e.g.,  $k$ ,  $W_n(k)$ ,  $W(k)$ , or  $\mathbb{Z}/p^n$ , it is desirable to have more precise pairings which hold before taking the colimit or the limit to go all the way to compactly supported cohomology. This requires to have a sturdy notion of poles and pole orders along the complement of an open immersion  $U \hookrightarrow X$  between smooth  $k$ -schemes and also a dual notion of zeros and vanishing order along the same closed subset. Candidates appear at various places in the literature: in [Bry83], [Kat89], [Mat97], Brylinski, Kato, and Matsuda introduce pole order filtrations on Witt vectors which under the Artin-Schreier-Witt sequence measure the ramification of  $\mathbb{Z}/p^n$ -Galois coverings of  $U$  along  $X \setminus U$ . In [KS14] these filtrations are used to prove a Lefschetz-type result for the abelian fundamental group with modulus. In [Tan22] Tanaka introduces Witt divisorial sheaves to obtain Witt versions of classical vanishing theorems, similar sheaves appear also in [JSZ18], [GK], [KSY16]. Hodge-Witt sheaves with vanishing order along a (non-reduced) closed subscheme also play an important role in the work [Mor19].

In all these cases we would like to have a precise duality pairing which matches Witt differential forms with certain poles along  $X \setminus U$  with corresponding zeros along the same closed subset. In this paper we are going to construct such pairings in the case  $X \setminus U$  is the support of a simple normal crossing divisor.

Let  $X$  be a smooth  $k$ -scheme (not necessarily proper) of pure dimension  $N$  and let  $D$  be an effective Cartier divisor such that the underlying reduced divisor  $D_{\text{red}}$  has simple normal crossings and denote the complement by  $U = X \setminus D$ . The question is now how to define appropriate notions of poles bounded by  $D$  and vanishing along  $D$  for the Hodge-Witt sheaves  $W_n \Omega_X^q$ . One natural way is to consider the multiplicative lift  $\text{Pic}(X) \rightarrow \text{Pic}(W_n X)$ . Then  $D$  gives rise to an invertible  $W_n \mathcal{O}_X$ -module  $W_n \mathcal{O}(D)$  and we can consider  $W_n \Omega_X^q \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}(\pm D)$  for poles (+) and zeros (-). This approach is taken for example in [Tan22] and at least for the pole side also in [JSZ18], [GK], [KSY16, Appendix]. The advantage of this choice is that by Ekedahl duality multiplication induces immediately an isomorphism

$$W_n \Omega_X^q \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}(D) \cong R\mathcal{H}om_{W_n \mathcal{O}_X}(W_n \Omega_X^{N-q} \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}(-D), W_n \Omega_X^N),$$

and this clearly works for any Cartier divisor  $D$ . The drawback of this choice is that the differential  $d$ , the Verschiebung  $V$ , and the Frobenius  $F$  which are defined on the de Rham-Witt complex do not extend to endomorphisms of the graded pro-object  $W_\bullet \Omega_X^* \otimes_{W_\bullet \mathcal{O}_X} W_\bullet \mathcal{O}(D)$ . Hence we cannot extend this to a pairing for crystalline cohomology or to a Milne type pairing. Furthermore, this makes it hard to understand the duality in the limit over  $n$ , see, e.g., [Lem22], where the product  $\prod_r W_\bullet(\mathcal{O}(p^r D))$  is considered to accomplish this for  $q = 0$ . Also considering  $D$  as a (non-reduced) scheme the complex  $W_n \Omega_D^*$  is defined and studying it requires to consider  $\text{Ker}(W_n \Omega_X^* \rightarrow W_n \Omega_D^*)$ , which is not the same as  $W_\bullet \Omega_X^* \otimes_{W_\bullet \mathcal{O}_X} W_\bullet \mathcal{O}(-D)$ .<sup>1</sup> In case  $D$  is *reduced* the log de Rham-Witt complex [HK94], [Mat17] has all the desired properties.

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<sup>1</sup>Though the pro-systems over  $n$  are isomorphic.

Furthermore, [Mok93] constructed a de Rham-Witt complex with zeros along the reduced divisor  $D$ , which by [Hyo88] and [Nak05] is dual to the log de Rham-Witt. But at least for the pole side it was not clear how to define a notion of poles bounded by a non-reduced divisor.

The notion of pole orders we choose in this paper relies on the theory of motives with modulus developed by Kahn-Miyazaki-Saito-Yamazaki in [KMSY21a], [KMSY21b], [KMSY22]. We will explain the relation more precisely a little later. For now we make the following a bit more direct definition which comes from the work [RS] and is inspired by [AS11]: Let  $F$  be a Nisnevich sheaf of abelian groups on all finite type and separated  $k$ -schemes. Let  $X$  and  $D$  be as above. We denote by  $F_X$  the Nisnevich sheaf on the small site  $X_{\text{Nis}}$ .

**Poles along  $D$ .** Denote by  $\text{Bl}_D(X \times X)$  the blow-up in  $D$  diagonally embedded into  $X \times X$  and denote by  $P_X^{(D)}$  the complement of the union of the strict transforms of  $X \times D$  and  $D \times X$  in  $\text{Bl}_D(X \times X)$ . The open embedding  $U \times U \hookrightarrow X \times X$  extends to an embedding  $U \times U \hookrightarrow P_X^{(D)}$ . We set

$$F(X, D) := \text{Equalizer} \left( F(U) \begin{array}{c} \xrightarrow{p_1^*} \\ \xrightarrow{p_2^*} \end{array} F(U \times U) / F(P_X^{(D)}) \right)$$

and denote by  $F_{(X, D)}$ <sup>2</sup> the Nisnevich sheaf  $(V \rightarrow X) \mapsto F(V, D|_V)$ , see 5.3 for details.

**Zeros along  $D$ .** Let  $D = \sum_i D_i$  with  $D_{i, \text{red}}$  smooth. We set

$$F_{(X, -D)} := \text{Ker} \left( F_X \rightarrow \bigoplus_i F_{D_i} \right).$$

Note that both definitions are functorial in  $F$ . One of the main results of this paper is now:

**Theorem 1** (Theorem 9.3). *Multiplication induces isomorphisms, for all  $q \geq 1$  and  $n$ ,*

$$W_n \Omega_{(X, D)}^q \xrightarrow{\cong} R \mathcal{H}om_{W_n \mathcal{O}_X}(W_n \Omega_{(X, -D)}^{N-q}, W_n \Omega_X^N)$$

and

$$W_n \Omega_{(X, -D)}^{N-q} \xrightarrow{\cong} R \mathcal{H}om_{W_n \mathcal{O}_X}(W_n \Omega_{(X, D)}^q, W_n \Omega_X^N).$$

From Ekedahl's trace isomorphism

$$(1.0.1) \quad \pi_n^! W_n(k) \cong W_n \Omega^N[N],$$

where  $\pi_n : W_n(X) \rightarrow \text{Spec } W_n(k)$  is the structure map, together with Grothendieck duality we immediately get:

**Corollary 1.** *Assume additionally that  $X$  is proper over  $k$ . Then there is a canonical isomorphism of finite  $W_n(k)$ -modules*

$$H^{N-i}(X, W_n \Omega_{(X, -D)}^{N-q}) \cong H^i(X, W_n \Omega_{(X, D)}^q)^\vee,$$

where  $(-)^{\vee}$  denotes the dual  $W_n(k)$ -module.

We make some comments on the above result:

- It is not obvious from the definition given above that  $W_n \Omega_{(X, D)}^q$  is a  $W_n \mathcal{O}_X$ -module. And in fact this is not true for  $q = 0$ , which is shown in [RS]. Using a certain conductor as defined in [RS21], we can also define the correct  $W_n \Omega_{(X, D)}^0$ , so that the isomorphisms in Theorem 1 hold for  $q = 0$  as well.

<sup>2</sup>In the body of the text these sheaves will be denoted by  $F_{(X, D)}^{\text{AS}}$ , but we prefer to have this simpler notation in the introduction.

- We have  $Rj_* W_n \Omega_U^q = \operatorname{colim}_r W_n \Omega_{(X, rD)}^q$ , where  $j : U \hookrightarrow X$  is the open immersion, hence taking the limit over  $\{rD\}_r$  in Corollary 1 yields

$$H_c^{N-i}(U, W_n \Omega_U^{N-q}) \cong H^i(U, W_n \Omega_U^q)^\vee,$$

where the left hand side is the compactly supported cohomology from [Har72].

- By Ekedahl's isomorphism (1.0.1) the  $W_n \mathcal{O}_X$ -module  $W_n \Omega_X^N$  is a dualizing complex in the sense of Grothendieck. Hence the two isomorphisms in Theorem 1 imply each other.
- For  $D = \emptyset$ , Theorem 1 is due to Ekedahl. We have  $W_n \Omega_{(X, D_{\text{red}})}^q = W_n \Omega_X^q(\log D_{\text{red}})$  and hence for  $D$  reduced Theorem 1 recovers the duality from [Nak05, Theorem 5.3(1)], [Hyo88, (3.3.1)].
- Part of the statement of Theorem 1 is also the vanishing

$$\mathcal{E}xt^i(W_n \Omega_{(X, \pm D)}^q, W_n \Omega_X^N) = 0 \quad \text{for } i \geq 1.$$

Note that this is not automatic as  $W_n \Omega_{(X, \pm D)}^q$  is not a locally free  $W_n \mathcal{O}_X$ -module. The idea to prove this vanishing is similar as in [Eke84] but the actual computations are more involved, see Theorem 7 below.

- It is not hard to guess from the existing literature, e.g., [JSZ18], [Mor19], [GK], [RS], that there is an isomorphism of pro-systems over  $\{rD\}_r$  as in the second isomorphism of Theorem 1, but that these two abstract notions of poles and zeros introduced above for general sheaves correspond precisely under multiplication came as a surprise to the authors and indicates that there might be some more abstract yet to discover duality in the background. We want to stress however, that our proof essentially proceeds by computing the pole and the zero side completely and then show by hand that they match.

Before we give more applications we consider the pole part in greater detail. Recall the notion of cube invariant sheaves with transfers  $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$  from [KSY22], [Sai20], see also 5.1 and 7.1 for some more details and references. In particular we have a functor

$$\underline{\omega}^{\mathbf{CI}} : \mathbf{NST} \rightarrow \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$$

from the category of Nisnevich sheaves with transfers to (certain) cube invariant sheaves with transfers which in case  $X$  is proper is given by

$$\underline{\omega}^{\mathbf{CI}}(F)(X, D) = \operatorname{Hom}_{\mathbf{NST}}(h_0(X, D), F).$$

Here  $h_0(X, D)$  is a certain Nisnevich sheaf with transfers which on a function field  $K/k$  is equal to  $\operatorname{CH}_0(X_K, D_K)$  the Chow group with modulus as introduced in [KS16], where the index  $K$  indicates the base change over  $K$ . Moreover, if for each henselian discrete valuation field  $L$  of geometric type over  $k$ , we are given an increasing filtration  $\{\operatorname{Fil}_r F(L)\}_{r \geq 0}$  which defines a conductor  $c = \{c_L : F(L) \rightarrow \mathbb{N}\}_L$  in the sense of [RS21], then there is a cube invariant sheaf with transfers  $F_c$  which for  $X$  proper is given by

$$F_c(X, D) = \{a \in F(U) \mid \rho^*(a) \in \operatorname{Fil}_{v_L(\rho^* D)} F(L) \quad \text{for all } \rho : \operatorname{Spec} \mathcal{O}_L \rightarrow X\},$$

see 3.2 for details and references. By [RS21, Theorem 4.15] and [RS, Theorem 2.6] we have

$$(1.0.2) \quad F_{c, (X, D)} \subset \underline{\omega}^{\mathbf{CI}} F_{(X, D)} \subset F_{(X, D)},$$

where the right hand side is *poles along  $D$*  as defined above. Furthermore by [RS, Theorem 2.10] the second inclusion is an equality in case  $X$  is projective and smooth.

For  $L$  as above we define for  $r, q \geq 0$

$$\mathrm{fil}_r W_n \Omega_L^q := \mathrm{fil}_{r-1}^{\log} W_n(L) \cdot \mathrm{dlog}(K_q^M(L)) + V^{n-m} \left( \mathrm{fil}_r^{\log} W_n(L) \cdot \mathrm{dlog} K_q^M(\mathcal{O}_L) \right),$$

where  $\mathrm{fil}_r^{\log} W_n(L)$  is the Brylinski-Kato filtration,  $m = \min\{n, v_p(r)\}$ , and  $K_q^M$  denotes the  $q$ th Milnor  $K$ -theory, and

$$\mathrm{Fil}_r^p W_n \Omega_L^q = \sum_{s \geq 0} p^s (\mathrm{fil}_{p^s r} W_n \Omega_L^q + d(\mathrm{fil}_{p^s r} W_n \Omega_L^q)),$$

see section 2 for details. The second main result of this paper, which is also essential for the proof of Theorem 1, is:

**Theorem 2** (Theorem 3.3 and Theorem 5.4). *The filtration  $\{\mathrm{Fil}_r^p W_n \Omega_L^q\}_{r,L}$  defines a conductor  $c$ . Moreover, for  $q \geq 1$  and  $X$  smooth (not necessarily proper)*

$$W_n \Omega_{c,(X,D)}^q = \underline{\omega}^{\mathrm{CI}} W_n \Omega_{(X,D)}^q = W_n \Omega_{(X,D)}^q.$$

For  $q = 0$ , the correct definition for the pole side from the point of view of Theorem 1 is given by  $W_n \Omega_{c,(X,D)}^0$ . In view of the first part of this theorem the proof of the second part is essentially by (1.0.2) reduced to show that the Nisnevich stalk  $(W_n \Omega_{(X,D)}^q)_\eta^h$  in a generic point  $\eta$  of  $D$  is contained in  $\mathrm{Fil}_r^p W_n \Omega_L^q$ , where  $L = \mathrm{Frac}(\mathcal{O}_{X,\eta}^h)$ . The proof of this result takes the sections 2 - 5. It relies crucially on the construction of some ad hoc characteristic form (see Lemma 4.1). We remark that for  $n = 1$  the above result is proven in [RS, Theorem 6.6] and our proof makes use of this result. Note that the  $p$ -saturation is not visible in the case  $n = 1$ .

Though the definition of the filtration  $\mathrm{Fil}_r^p W_n \Omega_L^q$  looks a bit unmotivated it has several universal interpretations, e.g., Theorem 2 states ( $q \geq 1$ )

$$\mathrm{Fil}_r^p W_n \Omega_L^q = \underline{\omega}^{\mathrm{CI}} W_n \Omega^q(\mathcal{O}_L, \mathfrak{m}_L^{-r}),$$

which also implies that it defines the motivic conductor in the sense of [RS21]. Furthermore Theorem 1 gives the isomorphism (which also holds for  $q = 0$ )

$$(1.0.3) \quad \mathrm{Fil}_r^p W_n \Omega_L^q \xrightarrow{\sim} \mathrm{Hom}_{W_n \mathcal{O}_L} (W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^{N-q}, W_n \Omega_L^N),$$

where  $W_n \Omega^{N-q}(\mathcal{O}_L, \mathfrak{m}_L^r) = \mathrm{Ker}(W_n \Omega_{\mathcal{O}_L}^{N-q} \rightarrow W_n \Omega_{\mathcal{O}_L/\mathfrak{m}_L^r}^{N-q})$  and  $N = \mathrm{trdeg}(L/k)$ . If  $f_n : \mathrm{Spec} W_n(\mathcal{O}_L/\mathfrak{m}_L^r) \rightarrow \mathrm{Spec} W_n(k)$  is the structure map, then we get an isomorphism

$$f_n^! W_n(k) \cong \frac{\mathrm{Fil}_r^p W_n \Omega_L^N}{W_n \Omega_{\mathcal{O}_L}^N} [N-1],$$

see Corollary 9.8. Also note that this filtration is related to Kato's ramification filtration for étale motivic cohomology with  $p$ -torsion coefficients from [Kat89], see Lemma 11.8 for a precise statement. However we remark, that there are several ways to lift  $W_n \Omega^q$  to a cube invariant sheaf with transfers (corresponding to different conductors), and that this particular lift we are using here is the maximal one and turns out to be suitable for duality, but has the disadvantage, as already remarked in [RS, 6.9] (there for  $n = 1$ ), that the functor

$$(X, D) \mapsto R\Gamma(X, \underline{\omega}^{\mathrm{CI}} W_n \Omega_{(X,D)}^q)$$

does not factor through the triangulated category of motives with modulus  $\underline{\mathbf{MDM}}^{\mathrm{eff}}$  defined in [KMSY22]. A sufficient criterion for a cube invariant sheaf to have this property is given by Koizumi in [Koi]. There it is also shown that the filtration  $\{\mathrm{fil}_r^{\log} W_n(L)\}_r$  gives rise to a

structure of cube invariant Nisnevich sheaf with transfers on  $W_n\mathcal{O}$  which defines a realization of Witt vector cohomology from  $\underline{\text{MDM}}^{\text{eff}}$ .

In the following, we write  $W_n\Omega_{(X,D)}^q$  for  $W_n\Omega_{c,(X,D)}^q$  and allow also  $q = 0$ . In particular we get that  $W_n\Omega_{(X,D)}^q$  is a  $W_n\mathcal{O}_X$ -module, that  $\text{colim}_r W_n\Omega_{(X,rD)}^q = j_* W_n\Omega_U^q$ , and that the equality

$$W_n\Omega_{(X,D_{\text{red}})}^q = W_n\Omega_X^q(\log D_{\text{red}})$$

holds. By Theorem 2 this sheaf has transfers, which gives a new proof of a result by Merici [Mer], see Remark 5.5. We also recall the formula for  $n = 1$ , which was proven in [RS],

$$W_1\Omega_{(X,D)}^q = \Omega_{(X,D)}^q = \Omega_X^q(\log D_0) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_0 - D_{0,\text{red}} + pD_1),$$

where we write  $D = D_0 + pD_1$  such that  $D_0$  and  $D_1$  have no irreducible components in common and  $D_0$  has none of its multiplicities divisible by  $p$ .

Using the explicit description of the filtration we can check that multiplication induces the map in (1.0.3) which also induces the natural maps in Theorem 1. To prove that the pairing is perfect and that the higher ext-groups vanish requires some further analysis of the structure of  $W_n\Omega_{(X,\pm D)}^q$ , see Theorem 7 below. We first give some applications.

In Theorem 7.3 we use Theorem 7 plus a result from [BRS22] to show

$$\underline{\text{Hom}}_{\underline{\text{MNST}}}(K_r^M, \underline{\omega}^{\text{CI}} W_n\Omega^q)_{(X,D)} \cong W_n\Omega_{(X,D)}^{q-r}.$$

For  $D = \emptyset$ , this was proven in [BRS22]. Hence we can apply the results of [BRS22] to obtain a projective bundle formula and a blow-up formula for  $W_n\Omega_{(X,D)}^q$ , see 7.4, and we can apply Theorem 1 to get these formulas as well for  $D$  replaced by  $-D$ , see Corollary 9.4. These generalize results by Gros [Gro85] for the case  $D = \emptyset$ . We furthermore get a Gysin triangle (for poles): Let  $Z \hookrightarrow X$  be a closed immersion of pure codimension  $r$  between smooth  $k$ -schemes, which intersects  $D$  transversally ( $D$  may be non-reduced). Denote by  $\rho: \tilde{X} \rightarrow X$  the blow-up of  $X$  in  $Z$  and by  $E$  the exceptional divisor. Then there is a canonical distinguished triangle in  $D(W\mathcal{O}_X)$

$$i_* W_n\Omega_{(Z,D|_Z)}^{q-r}[-r] \xrightarrow{g} W_n\Omega_{(X,D)}^q \xrightarrow{\rho^*} R\rho_* W_n\Omega_{(\tilde{X}, \rho^*D+E)}^q \xrightarrow{\partial} i_* W_n\Omega_{(Z,D|_Z)}^{q-r}[-r+1].$$

As an application we obtain the following Lefschetz-type statement for the cohomology of the top Hodge-Witt forms:

**Theorem 3** (Theorem 7.5). *Assume that  $X$  is additionally projective and let  $H \subset X$  be a smooth hypersurface section which intersects  $D$  transversally and satisfies*

$$H^j(X, \Omega_{(X,D)}^N \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)) = 0, \quad \text{for all } j \geq 1.$$

*Then the Gysin map*

$$g: H^{j-1}(H, W_n\Omega_{(H,D|_H)}^{N-1}) \longrightarrow H^j(X, W_n\Omega_{(X,D)}^N)$$

*is an isomorphism for  $j \geq 2$  and is surjective for  $j = 1$ .*

We will not spell it out in the following, but the reader should keep in mind that the projective bundle formula, the blow-up formula and the Gysin triangle are compatible with Frobenius, Verschiebung, differential, etc. and that these formulas continue to hold in a suitably modified sense in the crystalline cohomology and étale motivic cohomology with poles and zeros which we will discuss next.

The two isomorphisms in Theorem 1 behave quite differently when taking the limit over  $n$ . Indeed, by Proposition 10.6,

$$R \lim_n W_n \Omega_{(X,D)}^q = W \Omega_X^q(\log D_{\text{red}}).$$

Hence taking limits over  $n$ , the first isomorphism in Theorem 1 together with Ekedahl's duality formalism yields in case  $X$  is proper the Poincaré duality, see 10.12(2),

$$(1.0.4) \quad R\Gamma(X, W\Omega_{(X, -D_{\text{red}})}^\bullet) \cong R\text{Hom}_W(R\Gamma_{\log\text{-crys}}((X, D_{\text{red}})/W), W)[-2N],$$

where we use that the de Rham-Witt complex with log poles along  $D_{\text{red}}$  computes log-crystalline cohomology of the smooth log-scheme  $(X, D_{\text{red}})$ . On finite level this statement is also proven in [Hyo88, Proposition (3.3)] and [Nak05, Theorem 5.3]. From the above isomorphism (or rather a version before taking the total complex) and weight filtration arguments [Mok93], [Nak05] we deduce that  $R\Gamma(X, W\Omega_{(X, \pm D_{\text{red}})}^*)$ <sup>3</sup> are coherent complexes of modules over the Cartier-Dieudonné-Raynaud ring  $\mathcal{R}$ , in the sense of Illusie-Raynaud, Ekedahl. We do not know if  $R\Gamma(X, W\Omega_{(X, -D)}^*)$  is a coherent  $\mathcal{R}$ -complex for  $D$  non-reduced, in fact we do not know if it is a complete  $\mathcal{R}$ -complex, see Remark 10.13. As the natural map

$$R\Gamma(X, W\Omega_{(X, -D)}^*) \rightarrow R\Gamma(X, W\Omega_{(X, -D_{\text{red}})}^*)$$

is an isomorphism up to bounded  $p$ -primary torsion (see 10.12(3)) it follows from Shiho's comparison of log-crystalline cohomology with rigid cohomology, Berthelot's Poincaré duality for rigid cohomology, and the isomorphism (1.0.4) that  $R\Gamma(X, W\Omega_{(X, -D)}^\bullet)$  is an integral model for compactly supported rigid cohomology of  $U$ , for any  $D$  with  $U = X \setminus D_{\text{red}}$ . There are now several corollaries one can draw from this concerning the degeneration of the slope spectral sequence and vanishing results and such, but which all work up to bounded torsion and hence only require to work with  $D_{\text{red}}$  in which case they can be found at least implicitly in the literature, see 10.12 for more details.

However the multiplicities of  $D$  remain visible when we take the limit in the second isomorphism of Theorem 1. To state the result set

$$W_\infty \Omega_{(X,D)}^* := \text{colim}_{\underline{p}} W_n \Omega_{(X,D)}^q \quad \text{and} \quad W\Omega_{(X,-D)}^* := \text{Ker} \left( W\Omega_X^* \rightarrow \bigoplus_i W\Omega_{D_i}^* \right),$$

where the colimit is over the map  $\underline{p}$  which is "lift and multiply by  $p$ " and  $D = \sum_i D_i$  with  $D_{i,\text{red}}$  smooth. We define *co-crystalline cohomology with modulus*  $(X, D)$  by

$$R\Gamma_{\text{crys}}((X, D)/W_\infty) := R\Gamma(X, W_\infty \Omega_{(X,D)}^\bullet),$$

which has an operator  $\mathcal{V}$  on it induced by  $p^{N-q}V$  in degree  $q$ . We furthermore define *crystalline cohomology with zeros along  $D$*  by

$$R\Gamma_{\text{crys}}((X, -D)/W) := R\Gamma(X, W\Omega_{(X,-D)}^\bullet),$$

which has a Frobenius  $F$  on it induced by the absolute Frobenius on  $X$ . Theorem 1 together with Ekedahl's duality formalism yields the following version of Poincaré duality.

**Theorem 4** (Corollary 10.9). *Assume  $X$  is proper. There is a canonical isomorphism in  $D^b(W[F])$*

$$R\Gamma_{\text{crys}}((X, -D)/W) \cong R\text{Hom}_W(R\Gamma_{\text{crys}}((X, D)/W_\infty), K/W)[-2N],$$

where the action of  $F$  on the right hand side is induced by  $\mathcal{V}^\vee$ .

<sup>3</sup>If  $C^*$  is a dga of sheaves, we use  $R\Gamma(X, C^*)$  to denote the dga in complexes, and we use  $R\Gamma(X, C^\bullet)$  to denote the total complex of the double complex  $R\Gamma(X, C^*)$ .

We have  $R\Gamma_{\text{crys}}((X, D_{\text{red}})/W_n) = R\Gamma(X, W\Omega_{(X, D_{\text{red}})}^\bullet) \otimes_W^L W_n$  (cf. Corollary 10.11) and from this one can deduce that the isomorphism in Theorem 4 for  $D_{\text{red}}$  is induced by (1.0.4). This is not the case if  $D$  is not reduced, as the left hand side of the isomorphism in Theorem 4 depends on the multiplicities of  $D$ . See 10.12 for more comments and consequences such as vanishing statements. Finally we remark that it is an intriguing problem to investigate the relation of  $R\Gamma_{\text{crys}}((X, -D)/W)$  or  $R\Gamma_{\text{crys}}((X, D)/W_\infty)$  with the edged crystalline cohomology from [D'A].

Next we explain a modulus version of Milne duality which we get from Theorem 1. Consider the following two complexes of abelian sheaves on  $X_{\text{ét}}$

$$\mathbb{Z}/p^n(q)_{(X, D)} := \left( \underline{\omega}^{\text{CI}}(FW_{n+1}\Omega^q) \xrightarrow{1-C} \underline{\omega}^{\text{CI}}W_n\Omega^q \right)_{(X, D)} [-q]$$

and

$$\mathbb{Z}/p^n(q)_{(X, -D)} := \left( W_n\Omega_{(X, -D)}^q \xrightarrow{C^{-1}-1} \frac{W_n\Omega_{(X, -D)}^q}{dV^{n-1}\Omega_X^{q-1} \cap W_n\Omega_{(X, -D)}^q} \right) [-q].$$

Both complexes sit in degree  $[q, q+1]$  and are for  $D = \emptyset$  quasi-isomorphic to the étale motivic complex with  $\mathbb{Z}/p^n$ -coefficients by [GL00]. Complexes similar to  $\mathbb{Z}/p^n(q)_{(X, -D)}$  are for example considered in [JSZ18], [Mor19], and [GK]. A complex similar to  $\mathbb{Z}/p^n(0)_{(X, D)}$  is considered in [KS14]. We remark that  $\mathbb{Z}/p^n(q)_{(X, -D)}$  is quasi-isomorphic to a sheaf concentrated in degree  $q$ , which is denoted by  $W_n\Omega_{(X, -D), \log}^q$ . In the *pole-case* this is also true if  $D$  is reduced but not in general. The local sections of the étale sheaf  $\mathcal{H}^{q+1}(\mathbb{Z}/p^n(q)_{(X, D)})$  only vanish after a finite cover which ramifies along  $D$ , see Lemma 11.6 and Remark 11.7.

In order to obtain a version of Kato's generalization of Milne's duality we denote by  $\mathbb{Z}/p^n(q)_{(X, \pm D)}^{\text{FRP}}$  the extension to the flat relatively perfect site  $X_{\text{FRP}}$ .

**Theorem 5** (Theorem 11.15). *There are isomorphisms*

$$\begin{aligned} \mathbb{Z}/p^n(q)_{(X, D)}^{\text{FRP}}[q] &\xrightarrow{\sim} \mathbb{D}_{n, X} \left( \mathbb{Z}/p^n(N-q)_{(X, -D)}^{\text{FRP}}[N-q] \right), \\ \mathbb{Z}/p^n(N-q)_{(X, -D)}^{\text{FRP}}[N-q] &\xrightarrow{\sim} \mathbb{D}_{n, X} \left( \mathbb{Z}/p^n(q)_{(X, D)}^{\text{FRP}}[q] \right), \end{aligned}$$

where  $\mathbb{D}_{X, n}(-)$  is Kato's dualizing functor, see 11.10.

**Corollary 2** (Corollary 11.16). *Assume additionally that  $k$  is a finite field and  $X$  is proper over  $k$ . Then there is an isomorphism in  $D(\mathbb{Z}/p^n)$*

$$R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(q)_{(X, D)}) \cong R\text{Hom}_{\mathbb{Z}/p^n} \left( R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(N-q)_{(X, -D)}), \mathbb{Z}/p^n \right) [-2N-1].$$

In particular we obtain isomorphisms of finite groups for all  $i$

$$H^{i+q}(X_{\text{ét}}, \mathbb{Z}/p^n(q)_{(X, D)}) \cong \text{Hom}_{\mathbb{Z}/p^n} (H^{2N-i-q+1}(X_{\text{ét}}, \mathbb{Z}/p^n(N-q)_{(X, -D)}), \mathbb{Z}/p^n).$$

Taking the limit over  $\{rD\}_r$  in the second isomorphism above yields [JSZ18, Theorem 2], see Remark 11.17. Denote by  $\pi_1(X, D)^p$  the étale fundamental group which classifies abelian  $p$ -covers of  $U$  with ramification bounded by  $D$  from [KS16]. Then the case  $q = 0$  and  $i = 1$  in the above corollary yields (assuming  $k$  finite and  $X$  proper)

$$\pi_1^{\text{ab}}(X, D)^p = H^N(X_{\text{ét}}, W\Omega_{(X, -D), \log}^N).$$

As last application we mention a refinement of Milne's pairing for the Brauer group of a smooth projective surface over a finite field to the ramified situation. The Brauer group of  $X$  with ramification bounded by  $D$ , is defined by

$$\text{Br}(X, D) := H^0(U, R^2\epsilon_*\mathbb{Q}/\mathbb{Z}(1)'_U) \oplus H^0(X, R^2\epsilon_*\mathbb{Q}_p/\mathbb{Z}_p(1)_{(X, D)}),$$

where  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the change of sites map, and

$$\mathbb{Q}/\mathbb{Z}(1)'_U = \text{colim}_{n'} \mu_{n',U}, \quad \mathbb{Q}_p/\mathbb{Z}_p(1)_{(X,D)} = \text{colim}_{\underline{p}} \mathbb{Z}/p^n(1)_{(X,D)}$$

with the colimit on the left over all  $n'$  which are prime to  $p$ . The Brauer group with zeros along  $D$  is defined by

$$\text{Br}(X, -D) := H^2(X_{\text{ét}}, \mathcal{O}_{(X,-D)}^\times),$$

where  $\mathcal{O}_{(X,-D)}^\times = \text{Ker}(\mathcal{O}_X^\times \rightarrow \mathcal{O}_D^\times)$ . There are natural inclusions  $\text{Br}(X) \subset \text{Br}(X, D) \subset \text{Br}(U)$  and  $\text{colim}_r \text{Br}(X, rD) = \text{Br}(U)$ . Furthermore there is an exact sequence

$$\text{Pic}(X) \rightarrow \text{Pic}(D) \rightarrow \text{Br}(X, -D) \rightarrow \text{Br}(X) \rightarrow \text{Br}(D),$$

where  $\text{Br}(Z) = H^2(Z_{\text{ét}}, \mathbb{G}_m)$ . For  $D = \emptyset$  the following duality statement is the  $p$ -part of Milne's duality [Mil75, Theorem 2.4].

**Theorem 6** (Theorem 12.13). *Assume  $k$  is a finite field and  $X$  is a smooth proper surface. Then there is a canonical isomorphism of profinite groups*

$$\frac{\text{Br}(X, -D)[p^\infty]}{(\text{Br}(X, -D)[p^\infty])_{\text{div}}} \xrightarrow{\sim} \text{Hom} \left( \frac{\text{Br}(X, D)[p^\infty]}{(\text{Br}(X, D)[p^\infty])_{\text{div}}}, \mathbb{Q}/\mathbb{Z} \right),$$

where the index “div” refers to the divisible part and  $M[p^\infty]$  is the  $p$ -primary torsion in  $M$ .

For example, if  $A$  is an abelian surface over a finite field, it follows from the finiteness of  $\text{Br}(A)$ , proved by Tate and Milne, that  $\text{Br}(A, -D)[p^\infty]$  is finite as well and hence the above duality yields that  $\text{Br}(A, D)[p^\infty]$  is the direct sum of a divisible group with a finite group, for all effective Cartier divisors  $D$  on  $A$  with  $D_{\text{red}}$  a simple normal crossing divisor, see Remark 12.14.

Finally we state the key results which are needed to complete the proof of Theorem 1.

**Theorem 7** (Theorem 6.4, Theorem 8.7). *There are short exact sequences of  $W_{n+1}\mathcal{O}_X$ -modules*

$$0 \longrightarrow B_n \Omega_{(X,D)}^{q+1} \longrightarrow \frac{W_{n+1} \Omega_{(X,D)}^q}{\underline{p} W_n \Omega_{(X,D)}^q} \xrightarrow{F^n} Z_n \Omega_{(X,D)}^q \longrightarrow 0,$$

and

$$0 \longrightarrow (\Omega/B)_{n,(X,-D)}^q \xrightarrow{V^n} \text{Ker} \left( W_{n+1} \Omega_{(X,-D)}^q \xrightarrow{R} W_n \Omega_{(X,-D)}^q \right) \longrightarrow (\Omega/Z)_{n,(X,-D)}^{q-1} \longrightarrow 0,$$

where in both cases the outer terms are locally free  $\mathcal{O}_X$ -modules. Moreover, multiplication induces  $\mathcal{O}_X$ -linear isomorphisms

$$(\Omega/B)_{n,(X,-D)}^q \xrightarrow{\sim} \mathcal{H}om(Z_n \Omega_{(X,D)}^{N-q}, \Omega_X^N), \quad (\Omega/Z)_{n,(X,-D)}^{q-1} \xrightarrow{\sim} \mathcal{H}om(B_n \Omega_{(X,D)}^{N-q+1}, \Omega_X^N).$$

Recall that using Theorem 2 we can show that there are well-defined maps as in Theorem 1. With Theorem 7 at hand it is direct to check that the same arguments from Ekedahl's proof of Theorem 1 for  $D = \emptyset$  work in the general case as well.

The outer terms of the two exact sequences in Theorem 7 are defined in (6.4.1) and (8.6.3). For  $D = \emptyset$  the exact sequences were proven by Illusie and Ekedahl and the fact that the outer terms are locally free  $\mathcal{O}_X$ -modules is proven by Cartier operator calculus. The strategy

for  $D \neq \emptyset$  is the same, but it turns out to be more involved. For example, by definition of  $(\Omega/B)_{n,(X,-D)}^q$  there is a cartesian diagram of  $W_{n+1}\mathcal{O}_X$ -modules

$$\begin{array}{ccc} (\Omega/B)_{n,(X,-D)}^q & \longrightarrow & W_{n+1}\Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ \frac{F_{X*}^n \Omega_X^q}{B_n \Omega_X^q} & \xrightarrow{V^n} & W_{n+1}\Omega_X^q. \end{array}$$

But in general  $(\Omega/B)_{n,(X,-D)}^q$  is not a quotient of

$$\Omega_{(X,-D)}^q = \Omega_X^q(\log D_{0,\text{red}})(-D),$$

where  $D = D_0 + pD_1$  and  $D_0$  has none of its multiplicities divisible by  $p$ . It rather depends on the whole  $p$ -divisibility decomposition of  $D$  up to length  $n+1$ . More precisely, write

$$D = D_0 + pD_1 + \dots + p^{n+1}D_{n+1},$$

where  $D_i$  and  $D_j$  have no irreducible components in common for  $i \neq j$ , and  $p$  does not divide any multiplicity of  $D_i$ , for  $i = 0, \dots, n$ . Then there is a surjection

$$\bigoplus_{j=0}^{n-j} F_{X*}^{n-j} (\Omega_X^q(\log \underline{D}_{j,\text{red}}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-[\underline{D}_j/p^j] - \underline{D}^{n+1-j})) \xrightarrow{\oplus_j C^{-j}} (\Omega/B)_{n,(X,-D)}^q,$$

where  $\underline{D}_j = D_0 + \dots + p^j D_j$  and  $\underline{D}^{n+1-j} = pD_{j+1} + \dots + p^{n+1-j}D_{n+1}$ , see Remark 8.22. The proof of Theorem 7 takes almost all of the sections 6 and 8.

**General conventions.** Throughout the whole paper,  $k$  is a perfect field of characteristic  $p > 0$ . We denote by  $\mathbf{Sm}$  the category of separated schemes which are smooth and of finite type over  $k$ . If  $F$  is a Nisnevich sheaf on  $\mathbf{Sm}$  and  $X \in \mathbf{Sm}$ , then we denote by  $F_X$  the restriction of  $F$  to the small Nisnevich site  $X_{\text{Nis}}$ . If  $x \in X$  is a point, then we denote by  $F_{X,x}^h$  the Nisnevich stalk at  $x$ .

## 2. THE $p$ -SATURATED FILTRATION

We start by fixing some standard notation and recalling some results on the de Rham-Witt complex. In 2.5 and Definition 2.6 we introduce a filtration which will play an essential role throughout the rest of the paper.

**2.1.** Let  $X \in \mathbf{Sm}$ . We denote by  $W_n\Omega_X^\bullet$  the de Rham-Witt complex of Bloch-Deligne-Illusie of length  $n$  on  $X$  (see [Ill79]). We denote by  $R : W_{n+1}\Omega_X^\bullet \rightarrow W_n\Omega_X^\bullet$ ,  $V : W_n\Omega_X^\bullet \rightarrow W_{n+1}\Omega_X^\bullet$ ,  $F : W_{n+1}\Omega_X^\bullet \rightarrow W_n\Omega_X^\bullet$ , the restriction, the Verschiebung, and the Frobenius morphism, respectively, which are part of the structure of the de Rham-Witt complex. Furthermore, we have the map

$$(2.1.1) \quad \underline{p} : W_n\Omega_X^\bullet \rightarrow W_{n+1}\Omega_X^\bullet$$

which is given by “lifting to level  $n+1$  and multiply by  $p$ ”, it is well-defined and injective by [Ill79, I, Proposition 3.4]. Recall that  $W_n\Omega_X^\bullet$  is a differential graded  $W_n(k)$ -algebra; we denote by  $d : W_n\Omega_X^\bullet \rightarrow W_n\Omega_X^{\bullet+1}$  the differential and by  $W_n\Omega_X^q$  its degree  $q$  part. Also recall that  $W_n\Omega_X^0 = W_n\mathcal{O}_X$  is the sheaf of Witt vectors of length  $n$  on  $X$  and that we have the multiplicative Teichmüller lift  $[-] = [-]_n : \mathcal{O}_X \rightarrow W_n\mathcal{O}_X$  at our disposal. By [KSY22, Corollary 3.2.5(3)] the functor

$$X \mapsto H^0(X, W_n\Omega_X^q)$$

defines a Nisnevich sheaf with transfers for all  $q \geq 0$ , which has SC-reciprocity; we also denote it by  $W_n \Omega^q \in \mathbf{RSC}_{\text{Nis}}$ . See also [CR12] for details on how to define the transfers structure and see 5.1 for the definition of  $\mathbf{RSC}_{\text{Nis}}$ . If  $f : X \rightarrow Y$  is a morphism in  $\mathbf{Sm}$ , then the morphism

$$\Gamma_f^* = f^* : W_n \Omega^q(Y) \rightarrow W_n \Omega^q(X)$$

induced by its graph  $\Gamma_f \in \mathbf{Cor}(X, Y)$ , is the natural pullback morphism induced by the functoriality of the de Rham-Witt complex. If  $f$  is finite and surjective, then the transpose of the graph defines an element  $\Gamma_f^t \in \mathbf{Cor}(Y, X)$  and  $\Gamma_f^{t*} = f_*$ , where  $f_*$  is the pushforward defined using duality theory. We list some properties of the transfers structure which will be used later, we refer to [RS21, Lemmas 7.7] for proofs of the first two and further references to the literature:

- (a) The restriction, Verschiebung, Frobenius,  $\underline{p}$ , and the differential define morphisms in  $\mathbf{RSC}_{\text{Nis}}$

$$R : W_{n+1} \Omega^q \rightarrow W_n \Omega^q, \quad V : W_n \Omega^q \rightarrow W_{n+1} \Omega^q, \quad F : W_{n+1} \Omega^q \rightarrow W_n \Omega^q,$$

$$\underline{p} : W_n \Omega^q \rightarrow W_{n+1} \Omega^{q+1}, \quad d : W_n \Omega^q \rightarrow W_n \Omega^{q+1}.$$

- (b) The Nisnevich sheaf with transfers  $W_n \Omega^0 = W_n \mathcal{O}$  coincides with the Nisnevich sheaf with transfers defined by the algebraic group  $W_n$  in [SS03, Proof of Lemma 3.2].  
(c) Denote by  $K_q^M$ ,  $q \geq 0$ , the restriction of the improved Milnor  $K$ -theory from [Ker10] to  $\mathbf{Sm}$ . It is an  $\mathbb{A}^1$ -invariant Nisnevich sheaf with transfers, in particular  $K_q^M \in \mathbf{RSC}_{\text{Nis}}$  and the map

$$\text{dlog} : K_q^M \rightarrow W_n \Omega^q, \quad u = \{u_1, \dots, u_q\} \mapsto \text{dlog } u := \text{dlog}_n u := \frac{d[u_1]_n}{[u_1]} \dots \frac{d[u_q]_n}{[u_q]}, \quad u_i \in \mathcal{O}^\times,$$

defines a morphism in  $\mathbf{RSC}_{\text{Nis}}$ , e.g. [BRS22, 11.1(4)]. In particular if  $f : Y \rightarrow X$  is a finite and surjective morphism in  $\mathbf{Sm}$ , then

$$f_* \text{dlog } u = \text{dlog}(\text{Nm}(u)) \quad \text{in } W_n \Omega^1(X), \quad u \in \mathcal{O}_Y^\times(Y),$$

where  $\text{Nm} : f_* \mathcal{O}_Y^\times \rightarrow \mathcal{O}_X^\times$  denotes the usual norm.

Another property of the de Rham-Witt forms which will be important in the following is that the natural map of Nisnevich sheaves (without transfers)

$$(2.1.2) \quad (W_n \mathcal{O} \otimes_{\mathbb{Z}} K_q^M) \oplus (W_n \mathcal{O} \otimes_{\mathbb{Z}} K_{q-1}^M) \rightarrow W_n \Omega^q, \quad (a \otimes u, b \otimes v) \mapsto a \text{dlog } u + db \text{dlog } v,$$

is surjective. This follows easily by induction over  $n$  from the corresponding fact for  $\Omega^q$ , the exact sequence of Nisnevich sheaves on  $\mathbf{Sm}$  (see [Ill79, I, Proposition 3.2])

$$\Omega^q \oplus \Omega^{q-1} \xrightarrow{V^{n-1} + dV^{n-1}} W_n \Omega^q \xrightarrow{R} W_{n-1} \Omega^q \rightarrow 0,$$

and the formulas  $V(a \text{dlog } u) = V(a) \text{dlog } u$  and  $Vd = pdV$ , cf. also [HK94, Proposition (4.6)].

The following notation will be used throughout.

**Notation 2.2.** A *henselian discrete valuation field of geometric type over  $k$*  is a field

$$L = \text{Frac}(\mathcal{O}_{U,z})^h,$$

where  $U \in \mathbf{Sm}$ ,  $z \in U^{(1)}$  is a point of codimension 1, and  $(\mathcal{O}_{U,z})^h$  denotes the henselization of the local ring  $\mathcal{O}_{U,z}$ . We denote the set of all such  $L$  by  $\Phi$ . For  $L \in \Phi$  we denote by  $\mathcal{O}_L$ ,  $v_L$ ,  $\mathfrak{m}_L$ ,  $\kappa_L$  the ring of integers, the normalized discrete valuation, the maximal ideal, and the residue field, respectively. In case there is no ambiguity we also write  $\mathfrak{m}$  instead of  $\mathfrak{m}_L$  and  $\kappa$  instead of  $\kappa_L$ . For each  $L \in \Phi$ , we pick a local parameter  $z = z_L \in \mathcal{O}_L$ .

**2.3.** Let  $L \in \Phi$  (see 2.2) and pick a local parameter  $z \in \mathfrak{m}_L$ . Recall that the *Brylinski-Kato filtration* of  $W_n(L)$  is defined as follows (see [Bry83]):

$$\begin{aligned} \mathrm{fil}_r^{\log} W_n(L) &:= \{(a_0, \dots, a_{n-1}) \mid p^{n-1-i} v_L(a_i) \geq -r\} \\ &= \{a \in W_n(L) \mid [z]^r \bar{F}^{n-1}(a) \in W_n \mathcal{O}_L\}, \quad r \geq 0. \end{aligned}$$

We have

$$\mathrm{fil}_0^{\log} W_n(L) = W_n \mathcal{O}_L, \quad \mathrm{fil}_r^{\log} L = \mathrm{fil}_r^{\log} W_1(L) = \mathfrak{m}_L^{-r}.$$

Furthermore, the maps  $V$ ,  $R$ ,  $F$  restrict to

$$(2.3.1) \quad \begin{array}{ccc} & \xleftarrow{R} & \\ \mathrm{fil}_r^{\log} W_n(L) & \xrightarrow{V} & \mathrm{fil}_r^{\log} W_{n+1}(L) \\ & \xrightarrow{F} & \end{array}$$

and more precisely,

$$(2.3.2) \quad R(\mathrm{fil}_r^{\log} W_{n+1}(L)) \subset \mathrm{fil}_{\lfloor r/p \rfloor}^{\log} W_n(L) \subset \mathrm{fil}_r^{\log} W_n(L).$$

It follows from the formula  $V^i([a]) \cdot V^j([b]) = V^{i+j}([a^p]^i [b^p]^j)$  that we have

$$(2.3.3) \quad \mathrm{fil}_r^{\log} W_n(L) \cdot \mathrm{fil}_s^{\log} W_n(L) \subset \mathrm{fil}_{r+s}^{\log} W_n(L), \quad r, s \geq 0.$$

In particular,  $\mathrm{fil}_r^{\log} W_n(L)$  is a  $W_n \mathcal{O}_L$ -submodule of  $W_n(L)$ . Moreover the injective map  $\underline{p} : W_n(L) \rightarrow W_{n+1}(L)$  induces an isomorphism

$$(2.3.4) \quad \underline{p} : \mathrm{fil}_r^{\log} W_n(L) \xrightarrow{\sim} p \cdot \mathrm{fil}_{rp}^{\log} W_{n+1}(L).$$

**Definition 2.4.** Let  $L \in \Phi$ . For  $q, r \geq 0$  we define

$$\begin{aligned} \mathrm{fil}_r^{\log} W_n \Omega_L^q &:= \mathrm{Im} \left( \mathrm{fil}_r^{\log} W_n(L) \otimes_{\mathbb{Z}} K_q^M(L) \xrightarrow{\mathrm{id} \otimes \mathrm{dlog}} W_n \Omega_L^q \right), \\ \mathrm{fil}_r^{\log'} W_n \Omega_L^q &:= \mathrm{Im} \left( \mathrm{fil}_r^{\log} W_n(L) \otimes_{\mathbb{Z}} K_q^M(\mathcal{O}_L) \xrightarrow{\mathrm{id} \otimes \mathrm{dlog}} W_n \Omega_L^q \right), \end{aligned}$$

and

$$\mathrm{fil}_r W_n \Omega_L^q := \mathrm{fil}_{r-1}^{\log} (W_n \Omega_L^q) + V^{n-m} (\mathrm{fil}_r^{\log'} W_m \Omega_L^q),$$

where  $m = \min\{v_p(r), n\}$  with  $v_p$  the normalized  $p$ -adic valuation. We have  $\mathrm{fil}_0 W_n \Omega_L^q = \mathrm{fil}_0^{\log'} W_n \Omega_L^q$  (by convention). Finally

$$\mathrm{Fil}_r W_n \Omega_L^q := \mathrm{fil}_r W_n \Omega_L^q + d(\mathrm{fil}_r W_n \Omega_L^{q-1}).$$

**2.5.** We make some comments and list some easy properties of the above defined filtrations:

- (1) The filtration  $\{\mathrm{Fil}_r W_n \Omega_L^0\}_{r \geq 0} = \{\mathrm{fil}_r W_n(L)\}_{r \geq 0}$  coincides with the Kato-Matsuda filtration [Mat97] (with the notational conventions from [KS16, 2.1]).
- (2) The family  $\{\mathrm{Fil}_r W_n \Omega_L^q\}$  defines an increasing and exhaustive filtration of  $W_n \Omega_L^q$

$$\mathrm{Fil}_0 W_n \Omega_L^q \subset \mathrm{Fil}_1 W_n \Omega_L^q \subset \dots \subset \mathrm{Fil}_r W_n \Omega_L^q \subset \dots \subset W_n \Omega_L^q.$$

(The filtration is exhaustive by the surjectivity of (2.1.2).)

- (3) The surjectivity of (2.1.2) yields

$$\mathrm{Fil}_0 W_n \Omega_L^q = W_n \Omega_{\mathcal{O}_L}^q, \quad \mathrm{Fil}_1 W_n \Omega_L^q = W_n \Omega_{\mathcal{O}_L}^q(\log),$$

where

$$W_n \Omega_{\mathcal{O}_L}^q(\log) = W_n \Omega_{\mathcal{O}_L}^q + W_n \Omega_{\mathcal{O}_L}^{q-1} \mathrm{dlog} z \subset W_n \Omega_L^q,$$

which is also equal to the  $q$ th de Rham-Witt forms of the log ring  $(\mathcal{O}_L, \lambda)$  over  $k$  with trivial log structure, where  $\lambda$  is the logarithmic structure associated to  $\mathbb{N} \rightarrow \mathcal{O}_L$ ,  $1 \mapsto z$ , see e.g. [Mat17].

(4) We have

$$\mathrm{fil}_r W_n \Omega_L^q = \begin{cases} \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^q & \text{if } v_p(r) = 0 \\ \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^q + \mathrm{fil}_r^{\log'} W_n \Omega_L^q & \text{if } v_p(r) \geq n. \end{cases}$$

(5) For  $j \geq 0$  we have a natural map

$$\mathrm{Fil}_r W_n \Omega_L^j \otimes_{\mathbb{Z}} K_{q-j}^M(\mathcal{O}_L) \rightarrow \mathrm{Fil}_r W_n \Omega_L^q, \quad a \otimes u \mapsto a \, d \log u,$$

which is surjective for  $j \geq 1$ . (Use the formulas  $V(a \, d \log u) = V(a) \, d \log u$  and  $d \, d \log u = 0$  and the surjectivity of  $L^\times \otimes_{\mathbb{Z}} K_{q-1}^M(\mathcal{O}_L) \rightarrow K_q^M(L)$ ,  $x \otimes u \mapsto \{x, u\}$ .)

**Definition 2.6.** Let  $\{G_r\}_{r \geq 0}$  be a family of subgroups of  $W_n \Omega_L^q$  with  $G_r \subset G_{r+1}$ . The  $p$ -saturation of  $G_r$  is then defined by

$$G_r^p := \sum_{s=0}^{n-1} p^s G_{rp^s} \subset W_n \Omega_L^q.$$

In particular we have

$$\mathrm{Fil}_r^p W_n \Omega_L^q = \sum_{s \geq 0} p^s \mathrm{Fil}_{rp^s} W_n \Omega_L^q = \mathrm{fil}_r^p W_n \Omega_L^q + d(\mathrm{fil}_r^p W_n \Omega_L^{q-1}).$$

We immediately get from 2.5 that  $\{\mathrm{Fil}_r^p W_n \Omega_L^q\}_{r \geq 0}$  is an increasing and exhaustive filtration of  $W_n \Omega_L^q$ , which comes with a natural map

$$(2.6.1) \quad \mathrm{Fil}_r^p W_n \Omega_L^j \otimes_{\mathbb{Z}} K_{q-j}^M(\mathcal{O}_L) \rightarrow \mathrm{Fil}_r^p W_n \Omega_L^q,$$

which is surjective for  $j \geq 1$ .

**Lemma 2.7.** For all  $0 \leq s \leq n-1$  and all  $r \geq 1$  we have

$$p^s \mathrm{fil}_{(r-1)p^s}^{\log} W_n \Omega_L^q = p^s \mathrm{fil}_{rp^{s-1}}^{\log} W_n \Omega_L^q, \quad \text{and} \quad p^s \mathrm{fil}_{(r-1)p^s}^{\log'} W_n \Omega_L^q = p^s \mathrm{fil}_{rp^{s-1}}^{\log'} W_n \Omega_L^q.$$

*Proof.* Clearly we have this  $\subset$  inclusion in both cases. For the other direction it suffices to consider  $q = 0$ . Let  $y \in L$  with  $p^{n-j-1} v_L(y) \geq -rp^s + 1$ . If  $s \geq n-j$ , then  $p^s V^j([y]) = 0$ ; if  $s \leq n-j-1$ , then  $p^{n-j-s-1} v_L(y)$  is an integer  $\geq -r + 1/p^s$  and hence it is  $\geq -r + 1$ . This yields this  $\supset$  inclusion in both cases.  $\square$

**Lemma 2.8.** We have

$$\mathrm{Fil}_0^p W_n \Omega_L^q = W_n \Omega_{\mathcal{O}_L}^q, \quad \mathrm{Fil}_1^p W_n \Omega_L^q = W_n \Omega_{\mathcal{O}_L}^q(\log).$$

*Proof.* By 2.5(3) it remains to show the inclusion  $p^s \mathrm{fil}_{p^s} W_n \Omega_L^q \subset W_n \Omega_{\mathcal{O}_L}^q(\log)$ , for  $s \in [1, n-1]$ . Since  $p^s V^{n-s}(W_s \Omega_L^q) = 0$ , it suffices to show

$$p^s \mathrm{fil}_{p^{s-1}}^{\log} W_n \Omega_L^q \subset W_n \Omega_{\mathcal{O}_L}^q(\log),$$

which follows from Lemma 2.7.  $\square$

**Lemma 2.9.** Set  $w_n := W_n \Omega_L^q$ . We have

$$(2.9.1) \quad V(\mathrm{fil}_r^{\log} w_n) \subset \mathrm{fil}_r^{\log} w_{n+1}, \quad V(\mathrm{fil}_r^{\log'} w_n) \subset \mathrm{fil}_r^{\log'} w_{n+1}, \quad V(\mathrm{fil}_r w_n) \subset \mathrm{fil}_r w_{n+1},$$

$$(2.9.2) \quad \underline{p}(\mathrm{fil}_r^{\log} w_n) = p \cdot \mathrm{fil}_{rp}^{\log} w_{n+1}, \quad \underline{p}(\mathrm{fil}_r^{\log'} w_n) = p \cdot \mathrm{fil}_{rp}^{\log'} w_{n+1}, \quad \underline{p}(\mathrm{fil}_r w_n) = p \cdot \mathrm{fil}_{rp} w_{n+1},$$

and for  $\varphi \in \{R, F\}$

$$(2.9.3) \quad \varphi(\text{fil}_r^{\log} w_{n+1}) \subset \text{fil}_r^{\log} w_n, \quad \varphi(\text{fil}_r^{\log'} w_{n+1}) \subset \text{fil}_r^{\log'} w_n, \quad \varphi(\text{fil}_r w_{n+1}) \subset \text{fil}_r w_n.$$

Furthermore,

$$(2.9.4) \quad Fd(\text{fil}_r^{\log} W_{n+1} \Omega_L^{q-1}) \subset \text{fil}_r^{\log} W_n \Omega_L^q + d(\text{fil}_r^{\log} W_n \Omega_L^{q-1})$$

and if  $v_p(r) \geq n+1$ , then

$$(2.9.5) \quad Fd(\text{fil}_r^{\log'} W_{n+1} \Omega_L^{q-1}) \subset p \cdot \text{fil}_r^{\log} W_n \Omega_L^q + \text{Fil}_r W_n \Omega_L^q.$$

*Proof.* The inclusions (2.9.1) and (2.9.3) follow immediately from (2.3.1) and the formula  $\varphi(a \text{dlog } x) = \varphi(a) \text{dlog } x$ , for  $\varphi \in \{F, R, V\}$ , the first two equalities in (2.9.2) follow from (2.3.4), the last equality from this and Lemma 2.7. Note that any element in  $\text{fil}_r^{\log} W_{n+1} \Omega_L^{q-1}$  is a sum of elements  $\omega = a \text{dlog } x$ , with  $a \in \text{fil}_r^{\log} W_{n+1}(L)$  and  $x \in K_{q-1}^M(L)$ . Write  $a = [a_0] + V(b)$ , so that  $[a_0] \in \text{fil}_r^{\log} W_{n+1}(L)$  and  $b \in \text{fil}_r^{\log} W_n(L)$ . Thus (2.9.4) follows from

$$Fd(\omega) = F([a_0]) \text{dlog}\{a_0, x\} + d(b \text{dlog } x) \in \text{fil}_r^{\log} W_n \Omega_L^q + d(\text{fil}_r^{\log} W_n \Omega_L^{q-1}).$$

Finally, (2.9.5). We assume  $v_p(r) \geq n+1$ . Let  $\omega$  be as above but this time  $x \in K_{q-1}^M(\mathcal{O}_L)$ , and write  $a_0 = z^e u$ , with  $u \in \mathcal{O}_L^\times$  and  $ep^n \geq -r$ . We get

$$Fd(\omega) = \underbrace{eF([a_0]) \text{dlog}\{z, x\}}_{=\omega_1} + \underbrace{F([a_0]) \text{dlog}\{u, x\} + d(b \text{dlog } x)}_{\in \text{Fil}_r W_n \Omega_L^q}.$$

If  $ep^n > -r$ , then  $[a_0] \in \text{fil}_{r-1}^{\log} W_{n+1}(L)$  and hence  $\omega_1 \in \text{fil}_{r-1}^{\log} W_n \Omega_L^q$ . If  $ep^n = -r$ , then  $p|e$  and hence  $\omega_1 \in p \cdot \text{fil}_r^{\log} W_n \Omega_L^q$ . This yields the statement.  $\square$

**Corollary 2.10.** *The maps  $F, R, V, \underline{p}, d$  on  $W_\bullet \Omega_L^*$  induce well-defined maps*

$$\text{Fil}_r^p W_n \Omega_L^q \xrightleftharpoons[V, R]{V, \underline{p}} \text{Fil}_r^p W_{n+1} \Omega_L^q, \quad d : \text{Fil}_r^p W_n \Omega_L^q \rightarrow \text{Fil}_r^p W_n \Omega_L^{q+1}.$$

Furthermore,

$$\underline{p}(\text{Fil}_r^p W_{n-1} \Omega_L^q) = p \text{Fil}_{pr}^p W_n \Omega_L^q.$$

*Proof.* This holds for  $d$  by definition. The well-definedness of  $\underline{p}$  and the final statement follow directly from (2.9.2). For  $V$  and  $R$  the statement follows from (2.9.1) and (2.9.3) and the formulas  $Vd = p d V$  and  $Rd = d R$ . For  $F$  it follows from (2.9.3) - (2.9.5) and the observation

$$\text{fil}_{rp^s}^{\log} W_n \Omega_L^q \subset \text{fil}_{rp^{s+1}-1}^{\log} W_n \Omega_L^q \subset \text{fil}_{rp^{s+1}} W_n \Omega_L^q.$$

$\square$

It will be useful to have a different presentation of  $\text{Fil}_r^p W_n \Omega_L^q$  which is provided in the following lemma.

**Lemma 2.11.** *Let  $r \geq 1$  and let  $z \in \mathcal{O}_L$  be a local parameter.*

(1) *For  $0 \leq j \leq n-1$ ,  $i \in \mathbb{Z}$ ,  $\alpha \in W_{n-j} \Omega_{\mathcal{O}_L}^q$ , and  $\beta \in W_{n-j} \Omega_{\mathcal{O}_L}^{q-1}$  we have*

$$(2.11.1) \quad V^j([z]^i \alpha) \in \text{fil}_{r-1}^{\log} W_n \Omega_L^q + d(\text{fil}_{r-1}^{\log'} W_n \Omega_L^{q-1}), \quad \text{if } ip^{n-1-j} \geq -r+1,$$

$$(2.11.2) \quad V^j([z]^i \text{dlog } z \cdot \beta) \in \text{fil}_{r-1}^{\log} W_n \Omega_L^q + d(\text{fil}_{r-1}^{\log} W_n \Omega_L^{q-1}), \quad \text{if } ip^{n-1-j} \geq -r+1,$$

*Furthermore, if  $n-m \leq j \leq n-1$ , where  $m = \min\{v_p(r), n\} \geq 1$ , then*

$$(2.11.3) \quad V^j([z]^i \cdot \alpha) \in p \cdot V^{n-m}(\text{fil}_r^{\log} W_m \Omega_L^q) + dV^{n-m}(\text{fil}_r^{\log'} W_m \Omega_L^{q-1}), \quad \text{if } ip^{n-j-1} = -r.$$

(2) Let  $H_r^q \subset W_n \Omega_L^q$  be the subgroup generated by the elements (2.11.1) - (2.11.3). Then

$$\mathrm{fil}_r W_n \Omega_L^q \subset H_r^q \subset \mathrm{Fil}_r W_n \Omega_L^q + p \cdot \mathrm{Fil}_{rp} W_n \Omega_L^q.$$

In particular

$$\mathrm{Fil}_r^p W_n \Omega_L^q = \sum_{s=0}^{n-1} p^s (H_{rp^s}^q + d(H_{rp^s}^{q-1})).$$

*Proof.* (1). If  $\alpha$  (resp.  $\beta$ ) is equal to  $a \mathrm{dlog} u$  with  $a \in W_{n-j}(\mathcal{O}_L)$  and  $u \in K_q^M(\mathcal{O}_L)$  (resp.  $u \in K_{q-1}^M(\mathcal{O}_L)$ ), then (2.11.1) - (2.11.3) follow directly from 2.3 and Definition 2.4. By the surjectivity of (2.1.2) and 2.5(5) it remains to consider the case where  $\alpha$  (resp.  $\beta$ ) is equal to  $db$  with  $b \in W_{n-j}(\mathcal{O}_L)$  and  $q = 1$  (resp.  $q = 2$ ). We compute

$$V^j([z]^i db) = p^j dV^j(b[z]^i) - V^j(ib[z]^i) \mathrm{dlog} z,$$

where the equality follows from the Leibniz rule and the formula  $Vd = pdV$ . This yields (2.11.1). Observe that in the situation of (2.11.3) the integer  $i$  is divisible by  $p$ , which yields (2.11.3). Similarly we get

$$V^j([z]^i \mathrm{dlog}(z)db) = -p^j dV^j(b[z]^i) \mathrm{dlog} z,$$

whence (2.11.2). Concerning the chain of inclusions in (2) observe that the left inclusion holds by definition and that the right inclusion follows from (1), where for the elements of type (2.11.3) we observe that

$$p \cdot \mathrm{fil}_r^{\mathrm{log}} W_n \Omega_L^q \subset p \cdot \mathrm{fil}_{r+1} W_n \Omega_L^q \subset p \cdot \mathrm{fil}_{rp} W_n \Omega_L^q.$$

This completes the proof.  $\square$

**Corollary 2.12.** *The group  $\mathrm{Fil}_r^p W_n \Omega_L^q$  is a finitely generated  $W_n \mathcal{O}_L$ -submodule of  $W_n \Omega_L^q$ , for all  $r, q \geq 0$  and  $n \geq 0$ . Moreover we have inclusions of  $W_n \mathcal{O}_L$ -modules*

$$(2.12.1) \quad W_n \Omega_{\mathcal{O}_L}^q \subset \mathrm{Fil}_r^p W_n \Omega_L^q \subset W_n \Omega_{\mathcal{O}_L}^q \cdot \frac{1}{[z]^{rp^{n-1}}},$$

where  $z \in \mathcal{O}_L$  denotes a local parameter.

*Proof.* Let  $H_r^q$  be as in Lemma 2.11. It is a  $W_n \mathcal{O}_L$ -submodule of  $W_n \Omega_L^q$ , by the formula  $aV^j(x) = V^j(F^j(a)x)$ . Using additionally the Leibniz rule we find for  $\gamma \in W_{n-j} \Omega_L^{q-1}$  and  $a \in W_n \mathcal{O}_L$

$$adV^j([z]^i \gamma) = dV^j([z]^i F^j(a)\gamma) - V^j([z]^i F^j(da)\gamma).$$

Hence  $H_r^q + d(H_r^q)$  is a  $W_n \mathcal{O}_L$ -submodule of  $W_n \Omega_L^q$ , for all  $r \geq 0$ , and hence so is  $\mathrm{Fil}_r^p W_n \Omega_L^q$ , by Lemma 2.11. The first inclusion in (2.12.1) holds by Lemma 2.8, the second inclusion follows from  $[z]^t \cdot \mathrm{fil}_t^{\mathrm{log}} W_n(L) \subset W_n \mathcal{O}_L$ , for all  $t \geq 0$ . Since  $W_n \mathcal{O}_L$  is noetherian (e.g. [LZ04, Prop A.4]) and  $W_n \Omega_{\mathcal{O}_L}^q$  is a finite  $W_n \mathcal{O}_L$ -module (where we use that  $L$  is of geometric type), we obtain that  $\mathrm{Fil}_r^p W_n \Omega_L^q$  is a finite  $W_n \mathcal{O}_L$ -module as well.  $\square$

### 3. THE $p$ -SATURATED FILTRATION DEFINES A CONDUCTOR

In this section we show that the  $p$ -saturated filtration from section 2 defines a conductor in the sense of [RS21].

**3.1.** Following [KMSY21a] we call a pair  $(X, D)$  a *modulus pair*, if  $X$  is separated and of finite type over  $k$  and  $D$  is an effective Cartier divisor on  $X$ , such that  $X \setminus D$  is smooth. The modulus pair  $(X, D)$  is called *proper* if  $X$  is a proper  $k$ -scheme. A *compactification* of a modulus pair  $(X, D)$  is a proper modulus pair  $(\overline{X}, D' + X_\infty)$  such that  $\overline{X} \setminus X_\infty = X$  and

$D'_|X = D$ . All compactifications of a fixed modulus pair form a cofiltered set, see [KMSY21a, Lemma 1.8.2].

**3.2.** Recall from [RS21, Definition 4.3] that a *conductor* on a presheaf with transfers  $F$  on  $\mathbf{Sm}$  is a collection of set maps

$$c = \{c_L : F(L) \rightarrow \mathbb{N}_0\}_{L \in \Phi},$$

where  $\Phi$  denotes the set of henselian discrete valuation fields of geometric type over  $k$ , see Notation 2.2, satisfying the following properties for all  $L \in \Phi$  and all  $X \in \mathbf{Sm}$ :

- (c1)  $c_L(a) = 0 \Rightarrow a \in \text{Im}(F(\mathcal{O}_L) \rightarrow F(L))$ .
- (c2)  $c_L(a + b) \leq \max\{c_L(a), c_L(b)\}$ .
- (c3) For any finite extension  $L'/L$  of ramification degree  $e$  and any  $a \in F(L')$  we have

$$c_L(\text{Tr}_{L'/L} a) \leq \left\lceil \frac{c_{L'}(a)}{e} \right\rceil,$$

where  $\text{Tr}_{L'/L}$  is the trace which is given by the pullback along the transpose of the graph of  $\text{Spec } L' \rightarrow \text{Spec } L$  (viewed as a finite correspondence) and  $\lceil - \rceil$  is the round up.

- (c4) Let  $a \in F(\mathbb{A}_X^1)$ . Then

$$c_{k(x)(t)_\infty}(\rho_x^* a) \leq 1, \quad \forall x \in X \implies a \in F(X),$$

where  $k(x)(t)_\infty := \text{Frac}(\mathcal{O}_{\mathbb{P}_x^1, \infty}^h)$ ,  $\rho_x : \text{Spec } k(x)(t)_\infty \rightarrow \mathbb{A}_X^1$  is the natural map, and we identify  $F(X)$  with its image in  $F(\mathbb{A}_X^1)$  under the pullback along  $\mathbb{A}_X^1 \rightarrow X$ .

- (c5) For any  $a \in F(X)$  there exists a proper modulus pair  $(\overline{X}, D)$  with  $X = \overline{X} \setminus D$ , such that for all  $k$ -morphisms  $\rho : \text{Spec } L \rightarrow X$  we have

$$c_L(\rho^* a) \leq v_L(D_L),$$

where  $D_L$  denotes the pullback of  $D$  under the unique extension  $\text{Spec } \mathcal{O}_L \rightarrow \overline{X}$  of  $\rho$  and  $v_L(D_L)$  denotes its multiplicity.

We say a conductor has *level*  $n$  if in (c4) it suffices to consider points

$$x \in X_{(\leq n-1)} = \{\text{points of dimension } \leq n-1\}.$$
<sup>4</sup>

We say  $c$  is *semi-continuous* if it satisfies the following condition:

- (c6) Let  $X \in \mathbf{Sm}$  and let  $Z \subset X$  be a smooth prime divisor with generic point  $z$  and  $K = \text{Frac}(\mathcal{O}_{X,z}^h)$ . Then for any  $a \in F(X \setminus Z)$  with  $c_K(a_K) \leq r$  there exists a Nisnevich neighborhood  $u : U \rightarrow X$  of  $z$  and a compactification  $(Y, E)$  of the modulus pair  $(U, r \cdot u^* Z)$  such that

$$c_L(\rho^* a_U) \leq v_L(E_L), \quad \text{for all } L \in \Phi \text{ and all } \rho : \text{Spec } L \rightarrow U,$$

where  $a_U$  (resp.  $a_K$ ) denotes the restriction of  $a$  to  $U$  (resp.  $K$ ).

**Theorem 3.3.** *The collection  $c = \{c_L : W_n \Omega_L^q \rightarrow \mathbb{N}_0\}_{L \in \Phi}$  with*

$$(3.3.1) \quad c_L(a) = \min \{r \geq 0 \mid a \in \text{Fil}_r^p W_n \Omega_L^q\}$$

*defines a semi-continuous conductor of level  $q + 1$ . (See Definition 2.6 for  $\text{Fil}_r^p W_n \Omega_L^q$ .)*

<sup>4</sup>In fact in [RS21] a conductor of level  $n$  is only required to be defined for  $L \in \Phi$  of transcendence degree over  $k$  at most  $n$ . The restriction of conductor of level  $n$  in the above sense to those  $L$  is a conductor of level  $n$  in the sense of [RS21].

*Proof.* (c1) holds by Lemma 2.8 and (c2) is clear. The proof of (c3) requires some more lemmas and will be given at the end of the next subsection. (c4) Let  $X \in \mathbf{Sm}$  and  $a \in W_n \Omega^q(\mathbb{A}_X^1)$ . For  $x \in X$  let  $\rho_x : \operatorname{Spec} k(x)(t)_\infty \rightarrow \mathbb{A}_X^1$  be as in (c4). Assume  $c_{k(x)(t)_\infty}(\rho_x^* a) \leq 1$ , for all points  $x \in X_{(\leq q)}$ . Since  $W_n \Omega_{\mathbb{P}_x^1}^q(\log \infty)$  is a Nisnevich sheaf we find by Lemma 2.8

$$a|_{\mathbb{A}_x^1} \in \operatorname{Ker} \left( W_n \Omega^q(\mathbb{A}_x^1) \rightarrow \frac{W_n \Omega_{k(x)(t)_\infty}^q(\log \infty)}{(W_n \Omega_{\mathbb{P}_x^1, \infty}^q(\log \infty))^h} \right) = W_n \Omega^q(\log \infty)(\mathbb{P}_x^1).$$

We have an exact sequence

$$0 \rightarrow W_n \Omega^q(\mathbb{P}_x^1) \rightarrow W_n \Omega^q(\log \infty)(\mathbb{P}_x^1) \xrightarrow{\operatorname{Res}} W_n \Omega^{q-1}(x) \xrightarrow{\delta} H^1(\mathbb{P}_x^1, W_n \Omega_{\mathbb{P}_x^1}^q),$$

see [Mat17, p. 68]. By the projective bundle formula (see [Gro85, I, Corollaire 4.2.13]) the map  $\delta$  is a split injection and we find

$$a|_{\mathbb{A}_x^1} \in W_n \Omega^q(\log \infty)(\mathbb{P}_x^1) = W_n \Omega^q(\mathbb{P}_x^1) = W_n \Omega^q(x),$$

i.e.,  $a|_{\mathbb{A}_x^1}$  is pulled-back from  $W_n \Omega^q(x)$ . Let  $s : X \hookrightarrow \mathbb{A}_X^1$  be a section of the projection  $\pi : \mathbb{A}_X^1 \rightarrow X$  and consider  $b := a - \pi^* s^* a \in W_n \Omega^q(\mathbb{A}_X^1)$ . By the above,

$$(3.3.2) \quad b|_{\mathbb{A}_x^1} = 0, \quad \text{for all } x \in X_{(\leq q)}.$$

We claim that (3.3.2) implies  $b = 0$ . To prove this latter claim we may assume  $X = \operatorname{Spec} A$ , with  $A$  a smooth integral  $k$ -algebra and  $b \in W_n \Omega_{A[t]}^q$ . By [HM04, Theorem B] (for  $p$  odd) and [Cos08, Theorem 4.3] (for  $p = 2$ ) the abelian group  $W_n \Omega_{A[t]}^q$  is isomorphic to a direct sum of the groups  $W_m \Omega_A^q$  and  $W_m \Omega_A^{q-1}$ , for various  $m \in [0, n]$ . Furthermore this isomorphism is natural in  $A$ . The claim thus follows from Lemma 3.4 below.

(c5). Let  $X \in \mathbf{Sm}$  and  $a \in W_n \Omega^q(X)$ . Let  $(\overline{X}, D)$  be a proper modulus pair with  $X = \overline{X} \setminus D$ . By the surjectivity of (2.1.2) we find an open cover  $\overline{X} = \cup_i V_i$  inducing an open cover  $X = \cup_i U_i$ , with  $U_i = V_i \setminus D$ , and finitely many elements  $a_{i,j} \in W_n \mathcal{O}(U_i)$ ,  $b_{i,j} \in W_n \mathcal{O}(U_i)$ ,  $u_{i,j} \in K_q^M(U_i)$ , and  $v_{i,j} \in K_{q-1}^M(U_i)$  such that

$$a_{U_i} = \sum_j a_{i,j} \operatorname{dlog} u_{i,j} + \sum_j db_{i,j} \operatorname{dlog} v_{i,j},$$

where  $a_{U_i}$  denotes the restriction of  $a$  to  $U_i$ . For  $N \geq 1$  big enough we have

$$a_{i,j}, b_{i,j} \in H^0(V_i, W_n \mathcal{O}_{V_i}(N \cdot D|_{V_i})), \quad \text{for all } i, j.$$

Here for a divisor  $E$  on a finite type  $k$ -scheme  $Y$ , we denote by  $W_n \mathcal{O}_Y(E)$  the invertible  $W_n \mathcal{O}_Y$ -module, which on an open  $V \subset Y$  with  $E|_V = \operatorname{Div}(e)$  is equal to  $W_n \mathcal{O}_V \cdot \frac{1}{[e]}$ . By the proof of [RS21, Claim 7.5.1] we obtain for  $r > p^{n-1}N$

$$\rho^*(a_{i,j}), \rho^*(b_{i,j}) \in \operatorname{fil}_{rm_{i,L}-1}^{\log} W_n(L), \quad \text{for all } \rho : \operatorname{Spec} \mathcal{O}_L \rightarrow V_i, \text{ with } L \in \Phi, \text{ all } i, j,$$

where  $m_{i,L} = v_L(\rho^* D|_{V_i})$ . By definition of  $\operatorname{Fil}_r^p$  we find for all  $\rho : \operatorname{Spec} \mathcal{O}_L \rightarrow V_i$

$$(3.3.3) \quad c_L(\rho^* a_{U_i}) \leq v_L(r D|_{V_i}).$$

This yields (c5) as any  $\operatorname{Spec} L \rightarrow X$  extends uniquely to  $\operatorname{Spec} \mathcal{O}_L \rightarrow \overline{X}$  and factors via some  $V_i \hookrightarrow \overline{X}$ .

Finally (c6). Let  $Z \subset X$ ,  $z \in Z$  and  $K = \operatorname{Frac}(\mathcal{O}_{X,z}^h)$  be as in (c6). Let  $a \in W_n \Omega^q(X \setminus Z)$  with  $c_K(a_K) \leq r$ . By Definition 2.6 and (2.9.2)

$$a_K = \sum_{s=0}^{n-1} p_s^s (\alpha_s + d\beta_s)$$

with  $\alpha_s \in \text{fil}_r W_{n-s} \Omega_K^q$  and  $\beta_s \in \text{fil}_r W_{n-s} \Omega_K^{q-1}$ . Replacing  $X$  by a Nisnevich neighborhood of  $z$ , we can assume that  $\alpha_s$  and  $\beta_s$  are the restriction to  $K$  of de Rham Witt forms on  $X \setminus Z$ . As the set of compactifications of  $(U, rZ)$  is cofiltered, for  $U$  a Nisnevich neighborhood of  $z$ , see 3.1, it suffices to prove the following, for all  $q \geq 0$  and  $n \geq 1$ : Assume  $a$  as above satisfies  $a_K \in \text{fil}_r W_n \Omega_K^q$ . Then there exists a Nisnevich neighborhood  $U \rightarrow X$  and a compactification  $(Y, E)$  of  $(U, rZ_U)$  such that

$$(3.3.4) \quad \rho^* a \in \text{fil}_{v_L(E_L)} W_n \Omega_L^q, \quad \text{for all } \rho : \text{Spec } L \rightarrow U.$$

Let  $m = \min\{v_p(r), n\}$ . We find an open neighborhood  $U \subset X$  of  $z$  such that

$$a_U = \sum_j a_j \text{dlog}(\lambda_j) + V^{n-m}(b_j \text{dlog}(u_j)),$$

with  $a_j \in W_n(U \setminus Z_U)$ ,  $b_j \in W_m(U \setminus Z_U)$ ,  $\lambda_j \in K_q^M(U \setminus Z_U)$ , and  $u_j \in K_q^M(U)$ , and

$$a_{j,K} \in \text{fil}_{r-1}^{\log} W_n(K), \quad \text{and} \quad b_{j,K} \in \text{fil}_r^{\log} W_m(K), \quad \text{all } j.$$

If  $r = 0$ , then we can take  $a_j = 0 = \lambda_j$ . Let  $(Y, Z' + Y_\infty)$  be a compactification of  $(U, Z_U)$  with  $Y$  normal. By the proof of (c6) in [RS21, Proof of Proposition 7.5] we have for any  $N \gg 0$  with  $p^n | N$ , and for any map  $\rho : \text{Spec } L \rightarrow U$ , with  $L \in \Phi$ ,

$$\rho^*(a_j) \in \text{fil}_{v_L((r-1)Z' + (N-1)Y_\infty)}^{\log} W_n(L) \subset \text{fil}_{v_L(rZ' + NY_\infty)-1}^{\log} W_n(L)$$

and

$$V^{n-m}(\rho^* b_j) \in V^{n-m_0}(\text{fil}_{v_L(rZ' + NY_\infty)}^{\log} W_{m_0}(L)),$$

where  $m_0 = \min\{v_p(v_L(rZ' + NY_\infty)), n\}$ . Hence condition (3.3.4) is satisfied for  $(Y, E)$  with  $E = rZ' + NY_\infty$ ,  $N$  as above, which is a modulus compactification of  $(U, rZ_U)$ .

It remains to prove (c3), which is done at the end of the next subsection.  $\square$

**Lemma 3.4.** *Let  $X$  be a smooth  $k$ -scheme and  $a \in W_n \Omega^q(X)$ , for some  $n \geq 1$  and  $q \geq 0$ . If  $a$  is nonzero, then there exists a point  $x \in X_{(\leq q)}$ , such that the image of  $a$  in  $W_n \Omega^q(x)$  is nonzero.*

*Proof.* First note that the restriction  $W_n \Omega^q(X) \rightarrow W_n \Omega^q(U)$  along a dense open subset  $U \subset X$  is injective, which follows from the fact that  $W_n \Omega_X^q$  is a successive extension of locally free  $\mathcal{O}_X$ -modules, see [Ill79, I, Corollaire 3.9]. Since  $W_n \Omega_X^q$  is moreover an étale sheaf we may assume  $k$  is algebraically closed and  $X = \text{Spec } A$  with  $A$  smooth and étale over  $k[t_1, \dots, t_e]$  with  $e = \dim X > q$ .

For  $n = 1$ , the differential forms  $dt_{i_1} \cdots dt_{i_q}$ ,  $1 \leq i_1 < \dots < i_q \leq e$ , form a basis of the  $A$ -module  $\Omega_A^q$ . Assume for  $a \in \Omega_A^q$  we have  $b := (dt_{i_1} \cdots dt_{i_q})^\vee(a) \neq 0$  for some sequence  $(i_1, \dots, i_q)$  as above. As  $k$  is algebraically closed we find elements  $\lambda_j \in k$  such that  $b \neq 0$  in  $B := A/(t_j - \lambda_j | j \in \{1, \dots, e\} \setminus \{i_1, \dots, i_q\})$ . Then  $a(x) \neq 0$  in  $\Omega_{k(x)}^q$ , for some generic point  $x$  of  $\text{Spec } B$ , which is a point of dimension  $q$  in  $X$ .

By induction over  $n$  we assume the statement is proven for  $W_m \Omega_Y^q$ , for  $m \leq n - 1$  and all  $Y \in \mathbf{Sm}$ . For an étale sheaf of abelian groups  $F$  on  $\mathbf{Sm}$  consider the property:

- (\*) For each non-zero element  $a \in F(X)$ , there exists a morphism  $f : Z \rightarrow X$  in  $\mathbf{Sm}$  such that  $\dim Z \leq e - 1$  and  $f^* a \neq 0$  in  $F(Z)$ .

By the above, property (\*) holds for  $W_m \Omega^q$ , for  $m \leq n - 1$ , and it suffices to show that  $F = W_n \Omega^q$  has property (\*). To this end observe that if we have an exact sequence of sheaves

$$0 \rightarrow F' \rightarrow F \rightarrow F'',$$

and  $F$  satisfies (\*), then so does  $F'$ ; moreover if  $F''$  and  $F'$  satisfy (\*), then so does  $F$ . By [Ill79, I, Corollaire 3.9] we are reduced to show that  $\Omega^{q-1}/Z_n \Omega^{q-1}$  and  $\Omega^q/B_n \Omega^q$  satisfy (\*),

for all  $n$  (with the notation from *loc. cit.*). Since we have an injection  $\Omega^{q-1}/Z_1\Omega^{q-1} \hookrightarrow \Omega^q$  (via  $d$ ) and isomorphisms induced by the Cartier operator

$$(3.4.1) \quad Z_{n-1}\Omega^{q-1}/Z_n\Omega^{q-1} \cong Z_{n-2}\Omega^{q-1}/Z_{n-1}\Omega^{q-1} \cong \dots \cong \Omega^{q-1}/Z_1\Omega^{q-1},$$

the quotient  $\Omega^{q-1}/Z_n\Omega^{q-1}$  satisfies  $(*)$ . Next we consider  $\Omega^q/B_n\Omega^q$ . Let  $t := t_1$  and set  $A_0 := A/tA$ . Up to replacing  $X$  by a Nisnevich neighborhood of  $\text{Spec } A_0 \subset X$ , we can assume that we have a  $k$ -algebra morphism  $\sigma : A_0 \rightarrow A$  which composed with the quotient map  $A \rightarrow A_0$  is the identity on  $A_0$ , e.g. [BRS22, Lemma 7.14]. Note that  $A_0[t] \rightarrow A$  is étale and induces an identification  $\hat{A} := \varprojlim_n A/t^n A \cong A_0[[t]]$ . As  $B_n\Omega_X^q$  has the structure of a coherent locally free  $\mathcal{O}_X$ -submodule of  $F_{X*}^n\Omega_X^q$  we have

$$(\Omega^q/B_n\Omega^q)(X) = \Omega_A^q/B_n\Omega_A^q =: M_A.$$

Let  $a \in M_A$  be a nonzero element. We have to show there exists a map  $\varphi : A \rightarrow C$  with  $C$  a smooth  $k$ -algebra of dimension  $\leq e-1$  such that  $\varphi(a) \neq 0$  in  $M_C$ . Since  $M_A$  is a projective  $A$ -module of finite rank (with module structure induced by the  $p^n$ -power map) we find that the natural map

$$M_A \rightarrow \varinjlim_n M/t^n M \cong M_{\hat{A}}$$

is injective, e.g., [Mat89, Theroem 8.9]. Denote by  $\hat{a}$  the image of  $a$  in  $M_{\hat{A}}$ , which is nonzero. It suffices to show:

**Claim 3.4.1.** *Either the image of  $\hat{a}$  in  $M_{\hat{A}/(t-\lambda)}$  is nonzero, for some  $\lambda \in k$ , or there exists a map  $\varphi_0 : A_0 \rightarrow C_0$ , with  $C_0$  smooth of dimension  $\dim C_0 \leq e-2$ , such that the image of  $\hat{a}$  in  $M_{C_0[[t]]}$  is nonzero.*

Indeed, in this case  $C := A/(t-\lambda)$  (resp.  $C := A \otimes_{A_0} C_0$ ) is smooth of dimension  $\leq e-1$  and the  $t$ -adic completion of the natural map  $\varphi : A \rightarrow C$  is the morphism  $\hat{A} \rightarrow \hat{A}/(t-\lambda)$  (resp.  $A_0[[t]] \rightarrow C_0[[t]]$ ), hence  $\varphi(a) \neq 0$  in  $M_C$ .

We prove the claim. Note that we can write any element  $\Omega_{A_0[[t]]}^q$  uniquely as a sum

$$(3.4.2) \quad \sum_{l=0}^{\infty} b_l t^l + \sum_{l=1}^{\infty} c_l t^l d \log t, \quad b_l \in \Omega_{A_0}^q, c_l \in \Omega_{A_0}^{q-1}.$$

We characterize those elements which are in  $B_n\Omega_{A_0[[t]]}^q$ . To this end, note that  $B_n\Omega_{A_0[[t]]}^q = F^{n-1}dW_n\Omega_{A_0[[t]]}^{q-1}$ , by [Ill79, I, Proposition 3.11]. By [GH06, Theorem B] each element in  $W_n\Omega_{A_0[[t]]}^{q-1}$  can uniquely be written as an infinite sum of elements of the form

$$b_0[t]^i, \quad c_0[t]^i d \log[t], \quad (i \geq 1), \quad V^s(b_s[t]^j), \quad dV^s(c_s[t]^j),$$

where  $i \geq 0$ ,  $j \geq 1$  with  $(j, p) = 1$ ,  $s \in \{1, \dots, n-1\}$ , and  $b_r \in W_{n-r}\Omega_{A_0}^q$ ,  $c_r \in W_{n-r}\Omega_{A_0}^{q-1}$ . It follows that an element (3.4.2) lies in  $B_n\Omega_{A_0[[t]]}^q$  if and only if the following conditions are satisfied:

1st case:  $l = p^r l_0$  with  $r \in [0, n-2]$  and  $(l_0, p) = 1$ . Then

$$\exists \gamma \in W_{r+1}\Omega_{A_0}^{q-1} \text{ such that } b_l = F^r d(\gamma) \text{ and } c_l = l_0 F^r(\gamma).$$

2nd case:  $l = p^{n-1} l_0$  with  $(l_0, p) = 1$ . Then

$$\exists \gamma \in W_n\Omega_A^q \text{ and } \delta \in B_{n-1}\Omega_{A_0}^{q-1} \text{ such that } b_l = F^{n-1}d(\gamma) \text{ and } c_l = l_0 F^{n-1}(\gamma) + \delta.$$

3rd case:  $p^n | l$  (including the case  $l = 0$ ). Then

$$b_l \in B_n\Omega_{A_0}^q \text{ and } c_l \in B_n\Omega_{A_0}^{q-1} \quad (\text{where } c_l = 0, \text{ if } l = 0).$$

Let  $\tilde{a} \in \Omega_{A_0[[t]]}^q$  be a representative of  $\hat{a} \in M_{A_0[[t]]}$  and write  $\tilde{a}$  in the form (3.4.2). As  $\hat{a} \neq 0$ , we find  $l_1 \geq 0$  minimal with the property

$$b_{l_1} t^{l_1} + c_{l_1} t^{l_1} d\log(t) \notin B_n \Omega_{A_0[[t]]}^q.$$

First assume  $p^n | l_1$ . Then either  $b_{l_1} \notin B_n \Omega_{A_0}^q$  or  $c_{l_1} \notin B_n \Omega_{A_0}^{q-1}$ . If  $b_{l_1} \notin B_n \Omega_{A_0}^q$ , then taking  $\lambda = 1$  in Claim 3.4.1 will work; if  $c_{l_1} \notin B_n \Omega_{A_0}^{q-1}$  we find by induction over  $q$  a map  $\varphi_0 : A_0 \rightarrow C_0$  as in the claim such that  $\varphi(c_{l_1}) \notin B_n \Omega_{C_0}^{q-1}$ , whence  $\hat{a}$  does not map to 0 in  $M_{C_0[[t]]}$ .

Now assume  $l_1 = p^r l_0$  with  $r \in [0, n-2]$  and  $(l_0, p) = 1$ . If  $b_{l_1} \notin B_{r+1} \Omega_{A_0}^q$ , again taking  $\lambda = 1$  in the claim will work. Else there exists a  $\gamma \in W_{r+1} \Omega_{A_0}^{q-1}$ , such that  $b_{l_1} = F^r d(\gamma)$ . If  $\gamma' \in W_{r+1} \Omega_{A_0}^{q-1}$  has the same property, then  $\gamma - \gamma' \in F W_{r+2} \Omega_{A_0}^{q-1}$ , by [III79, I, (3.21.1.1)]. Since  $Z_{r+1} \Omega^{q-1} = F^{r+1} W_{r+2} \Omega_{A_0}^{q-1}$ , see [III79, I, (3.11.3)], we deduce from the 1st case above that we have

$$c_{l_1} - l_0 F^r(\gamma) \neq 0 \quad \text{in } \Omega_{A_0}^{q-1} / Z_{r+1} \Omega_{A_0}^{q-1}.$$

By (\*) for  $\Omega^{q-1} / Z_{r+1} \Omega^{q-1}$ , proven above, we find a map  $\varphi_0 : A_0 \rightarrow C_0$  as in the claim such that,  $\varphi_0(c_{l_1} - l_0 F^r(\gamma)) \neq 0$  in  $\Omega_{C_0}^{q-1} / Z_{r+1} \Omega_{C_0}^{q-1}$ . Thus the image of  $\hat{a}$  in  $M_{C_0[[t]]}$  is nonzero.

Finally assume  $l_1 = p^{n-1} l_0$  with  $(l_0, p) = 1$ . In the case  $b_{l_1} \notin B_n \Omega_{A_0}^q$  we take  $\lambda = 1$  as above. Otherwise we can argue as in the 2nd case, as  $B_{n-1} \Omega_{A_0}^{q-1} \subset Z_n \Omega_{A_0}^{q-1}$ . This completes the proof of Claim 3.4.1 and hence of the lemma.  $\square$

**Remark 3.5.** As the above proof shows, the statement of Lemma 3.4 also holds for  $\Omega^q / B_n \Omega^q$  instead of  $W_n \Omega^q$ , for  $n \geq 1$ . But note that the same statement does not hold for  $\Omega^q / Z_n \Omega^q$ , as the latter sheaf vanishes on all smooth  $k$ -schemes of dimension  $\leq q$ .

**3.1. The proof of (c3).** In the following, we let  $L \in \Phi$  and denote by  $\mathcal{O}_L$  its ring of integers with maximal ideal  $\mathfrak{m}_L$  and we let  $z \in \mathfrak{m}_L$  be a fixed local parameter.

**Lemma 3.6.** *Let  $K \rightarrow \mathcal{O}_L$  be a coefficient field. Then modulo  $W_n \mathcal{O}_L$  any element of  $\text{fil}_r^{\log} W_n L$ ,  $r \geq 1$ , is a sum of elements*

$$V^j([\mu z^i]) \quad \text{with } \mu \in K, 0 \leq j \leq n-1, -r \leq ip^{n-j-1} < 0.$$

*Proof.* This is immediate for  $n = 1$ . For  $n \geq 2$  we can write  $a = [a_0] + V(b)$ , where  $a_0 \in L$  with  $p^{n-1} v_p(a_0) \geq -r$  and  $b \in \text{fil}_r^{\log} W_{n-1} L$ . We find  $\mu_i \in K$  and  $c \in \mathcal{O}_L$  such that  $a_0 = \sum_{0 > ip^{n-1} \geq -r} \mu_i z^i + c$ . Note that

$$[a_0] - \sum_{0 > ip^{n-1} \geq -r} [\mu_i z^i] - [c] \in (\text{fil}_r^{\log} W_n \mathcal{O}_L \cap V W_{n-1} L) = V \text{fil}_r^{\log} W_{n-1} L.$$

Hence

$$a = \sum_{0 > ip^{n-1} \geq -r} [\mu_i z^i] \quad \text{mod } V \text{fil}_r^{\log} W_{n-1}(L) + W_n \mathcal{O}_L.$$

The statement follows by induction over  $n$ .  $\square$

**Lemma 3.7.** *For  $q \geq 1$  denote by  $\mathcal{V}^1 K_q^M(\mathcal{O}_L)$  the image under the map*

$$(1 + \mathfrak{m}_L) \otimes_{\mathbb{Z}} K_{q-1}^M(\mathcal{O}_L) \rightarrow K_q^M(\mathcal{O}_L), \quad (1 + b) \otimes u \mapsto \{1 + b, u\}.$$

*The multiplication 2.5(5) induces maps*

$$(3.7.1) \quad \text{fil}_r^{\log} W_n(L) \otimes_{\mathbb{Z}} \mathcal{V}^1 K_q^M(\mathcal{O}_L) \longrightarrow \text{fil}_{r-1}^{\log} W_n \Omega_L^q + d(\text{fil}_{r-1}^{\log} W_n \Omega_L^{q-1})$$

*and*

$$(3.7.2) \quad \mathrm{fil}_r^p W_n(L) \otimes_{\mathbb{Z}} \mathcal{V}^1 K_q^M(\mathcal{O}_L) \\ \longrightarrow \mathrm{Fil}_{r-1}^p W_n \Omega_L^q + \sum_{s=0}^{n-1} p^s \left( V^{n-s-1}(\mathrm{fil}_{(r-1)p^s}^{\log'} W_{s+1} \Omega_L^q) + dV^{n-s-1}(\mathrm{fil}_{(r-1)p^s}^{\log'} W_{s+1} \Omega_L^{q-1}) \right).$$

*Proof.* It suffices to consider the case  $q = 1$ , cf. 2.5(5). For (3.7.1) we have to show

$$V^j([x]) \mathrm{dlog}(1-b) \in \left( \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^1 + d(\mathrm{fil}_{r-1}^{\log} W_n(L)) \right),$$

for  $x \in L$  with  $p^{n-1-j}v_L(x) \geq -r$ , and  $b \in \mathfrak{m}_L$ . In view of  $V^j(y) \mathrm{dlog} z = V^j(y \mathrm{dlog} z)$  and (2.9.1) it suffices to consider the case  $j = 0$ . By [RS21, Lemma 7.13] we have in  $W_n \Omega_{\widehat{\mathcal{O}}_L}^q$ , where  $\widehat{\mathcal{O}}_L = \varprojlim_s \mathcal{O}_L/\mathfrak{m}_L^s$ ,

$$\mathrm{dlog}(1-b) = - \sum_{i \geq 0} [b]^i d[b] - \sum_{j=1}^{n-1} \sum_{(i,p)=1} \frac{1}{i} dV^j([b]^i).$$

Thus it remains to show for  $x \in L$  with  $p^{n-1}v_L(x) \geq -r$  and  $0 \neq b \in \mathfrak{m}_L$

- (a)  $[x][b]^i d[b] \in \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^1$ , for all  $i \geq 0$ ;
- (b)  $[x]dV^j([b]^i) \in \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^1 + d(\mathrm{fil}_{r-1}^{\log} W_n(L))$ , for all  $0 \leq j \leq n-1$ , and all  $i \geq 1$  with  $(i, p) = 1$ .

For (a) it suffices to consider  $i = 0$ , in which case the claim is immediate, as  $[x]d[b] = [xb] \mathrm{dlog} b$  and  $p^{n-1}v_L(xb) \geq -r + p^{n-1} \geq -r + 1$ . For (b) we compute (using the Leibniz rule)

$$[x]dV^j([b]^i) = dV^j([x]^{p^j}[b]^i) - V^j([b]^i[x]^{p^j}) \mathrm{dlog} x.$$

Since  $p^{n-j-1}(p^j v_L(x) + i) \geq -r + 1$  it lies in the right hand side of (3.7.1).

We show (3.7.2). By Definition 2.6 and Lemma 2.7

$$\mathrm{fil}_r^p W_n(L) = \sum_{s=0}^{n-1} p^s \left( \mathrm{fil}_{(r-1)p^s}^{\log} W_n(L) + V^{n-m_s}(\mathrm{fil}_{rp^s}^{\log} W_{m_s}(L)) \right),$$

where  $m_s = \min\{v_p(rp^s), n\} = \min\{v_p(r), n-s\} + s$ . By (3.7.1) it suffices to consider  $p^s V^{n-m_s}(\mathrm{fil}_{rp^s}^{\log} W_{m_s}(L))$  and by (2.9.3) we may assume  $s = 0$ . We can furthermore assume  $m := m_0 \geq 1$ . It remains to show that  $V^j([y]) \mathrm{dlog}(1-b)$  lies in the right hand side of (3.7.2) for  $n-m \leq j \leq n-1$ ,  $y \in L$  with  $p^{n-j-1}v_L(y) = -r$ , and  $b \in \mathfrak{m}_L$ . By Corollary 2.10 we may as before assume  $j = 0$  (and hence  $m = n$ ) and consider the elements in the cases (a) and (b) from above with  $x$  replaced by  $y$  with  $p^{n-1}v_L(y) = -r$ . Note  $v_p(r) \geq n$  hence  $p|v_L(y) =: e$ . In case (a) (with  $i = 0$ ) write  $b = z^c u$  and  $y = z^e v$ , with  $u, v \in \mathcal{O}_L^\times$  and  $c \geq 1$ . We obtain

$$[y]d[b] = c[uv][z]^{e+c} \mathrm{dlog} z + [uv][z]^{e+c} \mathrm{dlog} u.$$

It follows that, if  $n \geq 2$  or  $c \geq 2$ , then

$$[y]d[b] \in \mathrm{fil}_{r-2}^{\log} W_n \Omega_L^1 \subset \mathrm{Fil}_{r-1}^p W_n \Omega_L^1.$$

On the other hand if  $n = 1 = c$ , then

$$[y]d[b] = ydb = \frac{1}{e+1} d(uvz^{e+1}) - uvz^{e+1} \left( \frac{1}{e+1} \mathrm{dlog}(uv) - \mathrm{dlog} u \right),$$

which lies in  $\mathrm{fil}_{r-1}^{\log'} W_1 \Omega_L^1 + d(\mathrm{fil}_{r-1}^{\log'} W_1(L))$ , which lies in the right hand side of (3.7.2).

Finally we consider (b) for  $y \in L$  as above. We have

$$[y]dV^j([b]^i) = dV^j([y]^{p^j}[b]^i) - V^j([b]^i[y]^{p^j}) \mathrm{dlog} y.$$

If  $j < n - 1$  or  $i \geq 2$ , then it follows

$$[y]dV^j([b]^i) \in \text{fil}_{r-2}^{\log} W_n \Omega_L^1 + d(\text{fil}_{r-2}^{\log} W_n(L)) \subset \text{Fil}_{r-1}^p W_n \Omega_L^1.$$

If  $j = n - 1$  and  $i = 1$ , then we write  $y = z^e v$  with  $v \in \mathcal{O}_L^\times$  and obtain

$$[y]dV^{n-1}([b]^i) = dV^{n-1}([y]^{p^{n-1}}[b]) - V^{n-1}([b][y]^{p^{n-1}}) \text{dlog } v,$$

where we use that  $p$  divides  $e$  and hence  $e \cdot V^{n-1}(W_1 \mathcal{O}_L) = 0$ . Thus  $[y]dV^{n-1}([b])$  lies in the right hand side of (3.7.2), in this case as well.  $\square$

**Lemma 3.8.** *Let  $K \hookrightarrow \mathcal{O}_L$  be a coefficient field. For  $r \geq 1$  set*

$$E := \text{fil}_r^{\log'} W_n \Omega_L^1 + W_n \Omega_{\mathcal{O}_L}^1(\log).$$

*The following two subgroups of  $W_n \Omega_L^1$  are equal:*

- (1)  $\text{fil}_r^{\log} W_n \Omega_L^1 + d(\text{fil}_r^{\log} W_n(L))$ .
- (2) *The subgroup  $G$  generated by  $E$  and the elements*

$$\mu F^e(da), \quad a \in \text{fil}_r^{\log} W_{n+e}(L), \mu \in W_n(K), e \geq 0.$$

*Proof.* Clearly,  $d(\text{fil}_r^{\log} W_n(L)) \subset G$ . We show  $\text{fil}_r^{\log} W_n \Omega_L^1 \subset G$ . Since  $W_n \Omega_{\mathcal{O}_L}^1(\log)$  is contained in both groups it suffices by Lemma 3.6 to show

$$V^j([\nu z^i]) \text{dlog } b \in G, \quad \text{for } 0 \leq j \leq n-1, b \in L^\times, \nu \in K^\times, 0 > ip^{n-j-1} \geq -r.$$

Write  $b = z^c u$  with  $u \in \mathcal{O}_L^\times$  and  $c \geq 0$  and  $i = p^e i_0$  with  $(i_0, p) = 1$  and  $e \geq 0$ . We obtain

$$V^j([\nu z^i]) \text{dlog } b = \frac{c}{i_0} V^j([\nu]) F^e d(V^j([z]^{i_0})) + V^j([\nu z^i]) \text{dlog } u \in G,$$

where we use

$$[z]^i \text{dlog } z^c = F^e([z]^{i_0} \text{dlog } z^c) = \frac{c}{i_0} F^e d([z]^{i_0}) = \frac{c}{i_0} F^{e+j} dV^j([z]^{i_0}).$$

We show the other inclusion  $G \subset \text{fil}_r^{\log} W_n \Omega_L^1 + d(\text{fil}_r^{\log} W_n(L))$ . By 2.5(5) and Lemma 3.6 it suffices to show

$$(3.8.1) \quad \mu F^e dV^j([\nu z^i]) \in \text{fil}_r^{\log} W_n \Omega_L^1 + d(\text{fil}_r^{\log} W_n(L)),$$

for  $\mu \in W_n(K)$ ,  $e \geq 0$ ,  $0 \leq j \leq n+e-1$ ,  $\nu \in K^\times$ , and  $ip^{n+e-j-1} \geq -r$ . There are two cases. First assume  $e < j$ . Then

$$\mu F^e dV^j([\nu z^i]) = \mu dV^{j-e}([\nu z^i]) = dV^{j-e}(F^{j-e}(\mu)[\nu z^i]) - V^{j-e}([\nu z^i] F^{j-e} d\mu).$$

Hence (3.8.1) holds in this case by (2.11.1). Now assume  $e \geq j$ . We obtain

$$\mu F^e dV^j([\nu z^i]) = \mu[\nu z^i]^{p^{e-j}} \text{dlog } \nu + i\mu[\nu z^i]^{p^{e-j}} \text{dlog } z$$

which satisfies (3.8.1) by Lemma 2.11.  $\square$

**Lemma 3.9.** *Let  $K \hookrightarrow \mathcal{O}_L$  be a coefficient field and  $r \geq 2$ . The following elements are contained in  $\text{Fil}_r^p W_n \Omega_L^q$ :*

$$(3.9.1) \quad a \text{dlog } u, \quad a \in \text{fil}_r^p W_n(L), u \in K_q^M(K),$$

$$(3.9.2) \quad da \text{dlog } v, \quad a \in \text{fil}_r^p W_n(L), v \in K_{q-1}^M(K),$$

$$(3.9.3) \quad (F^e db) \cdot \gamma, \quad b \in \text{fil}_r^p W_{n+e}(L), \gamma \in W_n \Omega_K^{q-1}, e \geq 0.$$

Furthermore the abelian group

$$\frac{\text{Fil}_r^p W_n \Omega_L^q}{\text{Fil}_{r-1}^p W_n \Omega_L^q}$$

is generated by the elements (3.9.1) and (3.9.2) and those elements (3.9.3), which have  $b \in \mathrm{fil}_{r-1}^{\log, p} W_{n+e}(L)$ .

*Proof.* As we have by definition  $p^s \mathrm{Fil}_{p^s r}^p W_n \Omega_L^q \subset \mathrm{Fil}_r^p W_n \Omega_L^q$ , it suffices to prove the first statement with  $\mathrm{fil}_r^p$  in (3.9.1) - (3.9.3) replaced by  $\mathrm{fil}_r$ . In this case we find (3.9.1), (3.9.2)  $\in \mathrm{Fil}_r W_n \Omega_L^q$ . It remains to consider (3.9.3) for which it suffices by the surjectivity of (2.1.2) and by 2.5(5) to show

- (a)  $F^e(db)d\mu \in \mathrm{Fil}_r^p W_n \Omega_L^2$ ,
- (b)  $\mu F^e(db) \in \mathrm{Fil}_r^p W_n \Omega_L^1$ ,

where  $b \in \mathrm{fil}_r W_{n+e}(L)$ ,  $\mu \in W_n(K)$ , and  $e \geq 0$ . In case (a) we have

$$F^e(db)d\mu = F^e d(bdV^e(\mu)).$$

As  $bdV^e(\mu) \in \mathrm{Fil}_r^p W_{n+e} \Omega_L^1$ , by Lemma 2.11, the compatibility of  $\mathrm{Fil}_r^p$  with  $F$  and  $d$ , see Corollary 2.10, yields the claim in this case. Similarly we have  $F^e(db) \in \mathrm{Fil}_r^p W_n \Omega_L^1$ , since  $\mathrm{Fil}_r^p W_n \Omega_L^1$  is an  $W_n \mathcal{O}_L$ -module by Corollary 2.12, we get (b).

It remains to show the second statement on the generators of the quotient. We assume  $q \geq 1$  as there is nothing to show for  $q = 0$ . We define the following subgroups of  $W_n \Omega_L^q$

$$\mathcal{V}^1 \mathrm{fil}_r^p W_n \Omega_L^q := \mathrm{Im}(\mathrm{fil}_r^p W_n(L) \otimes \mathcal{V}^1 K_q^M(\mathcal{O}_L) \rightarrow W_n \Omega_L^q),$$

see Lemma 3.7 for the definition of  $\mathcal{V}^1 K_q^M(\mathcal{O}_L)$ , and

$$\mathcal{V}^1 \mathrm{Fil}_r^p W_n \Omega_L^q := \mathcal{V}^1 \mathrm{fil}_r^p W_n \Omega_L^q + d(\mathcal{V}^1 \mathrm{fil}_r^p W_n \Omega_L^{q-1}).$$

Furthermore denote by  $G_0^q \subset W_n \Omega_L^q$  the subgroup generated by all the elements (3.9.1) and (3.9.2). We claim

$$(3.9.4) \quad \mathcal{V}^1 \mathrm{Fil}_r^p W_n \Omega_L^q \subset G_0^q + \mathrm{Fil}_{r-1}^p W_n \Omega_L^q.$$

By Lemma 3.7 and the fact that  $d(G_0^{q-1} + \mathrm{Fil}_{r-1}^p W_n \Omega_L^{q-1}) \subset G_0^q + \mathrm{Fil}_{r-1}^p W_n \Omega_L^q$  it suffices to show

$$p^s V^{n-s-1}(\mathrm{fil}_{(r-1)p^s}^{\log'} W_{s+1} \Omega_L^q) \subset G_0^q + \mathrm{Fil}_{r-1}^p W_n \Omega_L^q.$$

By (2.9.2) we can assume  $s = 0$ . By the decomposition

$$(3.9.5) \quad \mathcal{O}_L^\times = K^\times \cdot (1 + \mathfrak{m}_L)$$

every element in the left hand side (for  $s = 0$ ) can be written as a sum of elements

- (a)  $V^{n-1}(a \, d \log u) = V^{n-1}(a) \, d \log u$ ,
- (b)  $V^{n-1}(a \, d \log \beta) = V^{n-1}(a) \, d \log \beta$ ,

where  $a \in \mathrm{fil}_{r-1}^{\log} W_n(L)$ ,  $u \in K_q^M(K)$ , and  $\beta \in \mathcal{V}^1 K_q^M(\mathcal{O}_L)$ . The elements (a) are clearly in  $G_0^q$ , and the elements (b) are in  $\mathrm{fil}_{r-2}^{\log} W_n \Omega_L^q + d(\mathrm{fil}_{r-2}^{\log} W_n \Omega_L^q) \subset \mathrm{Fil}_{r-1}^p W_n \Omega_L^q$ , by (3.7.1). This shows Claim (3.9.4).

As any element in  $K_q^M(L)/\mathcal{V}^1 K_q^M(\mathcal{O}_L)$  can be represented by a sum of elements  $u \in K_q^M(K)$  and  $\{z, v\}$  with  $v \in K_{q-1}^M(K)$  we find by (3.9.4) and the definition of  $G_0^q$  a surjection

$$H_0^q + d(H_0^{q-1}) \twoheadrightarrow \frac{\mathrm{Fil}_r^p W_n \Omega_L^q}{G_0^q + \mathrm{Fil}_{r-1}^p W_n \Omega_L^q},$$

where  $H_0^q \subset W_n \Omega_L^q$  is the subgroup generated by elements

$$p^s a \, d \log \{z, u\} \quad \text{for } 0 \leq s \leq n-1, a \in \mathrm{fil}_{(r-1)p^s}^{\log} W_n(L), u \in K_{q-1}^M(K).$$

Denote by  $H_1^q$  the subgroup of  $W_n\Omega_L^q$  generated by the elements (3.9.3), which have  $b \in \text{fil}_{r-1}^{\log, p} W_{n+e}(L)$ . It remains to show

$$H_0^q + dH_0^{q-1} \subset R^q =: H_1^q + G_0^q + \text{Fil}_{r-1}^p W_n\Omega_L^q.$$

As  $d(R^{q-1}) \subset R^q$  (for  $q \geq 2$ ) it suffices to show  $H_0^q \subset R^q$ , for all  $q \geq 1$ . Since the multiplication  $W_n\Omega_L^1 \otimes K_{q-1}^M(K) \rightarrow W_n\Omega_L^q$  induces a surjection  $H_0^1 \otimes K_{q-1}^M(K) \twoheadrightarrow H_0^q$  and a well-defined map  $R^1 \otimes K_{q-1}^M(K) \rightarrow R^q$  it suffices to consider the case  $q = 1$ . By (3.7.1), the decomposition (3.9.5), (2.9.2), and 2.8 we have

$$p^s \text{fil}_{(r-1)p^s}^{\log'} W_n\Omega_L^1 + W_n\Omega_{\mathcal{O}_L}^1(\log) \subset R^1, \quad \text{for all } s.$$

Hence the inclusion  $H_0^1 \subset R^1$  follows from Lemma 3.8.  $\square$

**Lemma 3.10.** *Let  $L'/L$  be a finite extension with ramification degree  $e$ . Denote by  $\text{Tr} : W_n(L') \rightarrow W_n(L)$  the trace. Then*

$$\text{Tr}(\text{fil}_r W_n(L')) \subset \text{fil}_{\lceil \frac{r}{e} \rceil} W_n(L) \quad \text{and} \quad \text{Tr}(\text{fil}_r^p W_n(L')) \subset \text{fil}_{\lceil \frac{r}{e} \rceil}^p W_n(L).$$

*Proof.* By (2.9.2) we have  $\text{fil}_r^p W_n(L') = \sum_{s=0}^{n-1} p^s \text{fil}_r W_{n-s}(L')$ . Hence it suffices to prove the first statement. This is proven in [RS21, (7.5.1)] for the  $F$ -saturated filtration  $\text{fil}_r^F W_n(L)$ . We check that it also works without  $F$ -saturation. Set  $s = \lceil \frac{r}{e} \rceil$ . By the same argument as in *loc. cit.* (below (7.5.1)) we find  $\text{Tr}(a) \in \text{fil}_s^{\log} W_n(L)$ . Set  $m := \min\{v_p(s), n\}$ . If  $m = n$ , then  $\text{fil}_s^{\log} W_n(L) = \text{fil}_s W_n(L)$  and we are done. If  $m \leq n-1$ , then it follows directly from the definition that we have an isomorphism

$$(3.10.1) \quad K \xrightarrow{\sim} \frac{\text{fil}_s^{\log} W_n(L)}{\text{fil}_s W_n(L)}, \quad x \mapsto V^{n-m-1}([\tilde{x}z^{-s_0}]),$$

where  $s_0 = s/p^m$  (which is prime to  $p$ ) and  $\tilde{x} \in \mathcal{O}_L^\times$  is any lift of  $x$ . The map

$$F^{n-1}d : W_n(L) \rightarrow \Omega_L^1, \quad (a_0, \dots, a_{n-1}) \mapsto \sum_{j=0}^{n-1} a_j^{p^{n-j-1}} d \log a_j,$$

clearly induces maps

$$\text{fil}_s^{\log} W_n(L) \rightarrow \mathfrak{m}_L^{-s} \Omega_{\mathcal{O}_L}^1(\log) \quad \text{and} \quad \text{fil}_s W_n(L) \rightarrow \mathfrak{m}_L^{-s} \Omega_{\mathcal{O}_L}^1.$$

By (3.10.1) the induced map on the quotient is injective,

$$\frac{\text{fil}_s^{\log} W_n(L)}{\text{fil}_s W_n(L)} \longrightarrow \frac{\mathfrak{m}_L^{-s} \Omega_{\mathcal{O}_L}^1(\log)}{\mathfrak{m}_L^{-s} \Omega_{\mathcal{O}_L}^1}, \quad V^{n-m-1}(\tilde{x}z^{-s_0}) \mapsto -s_0 \tilde{x}^{p^m} z^{-s} d \log z.$$

Thus the statement follows from  $\mathfrak{m}_L^s \cdot F^{n-1}d(\text{Tr}(a)) \subset \Omega_{\mathcal{O}_L}^1$ , see [RS21, (7.5.2)].  $\square$

Now we can complete the proof of Theorem 3.3.

*Proof of (c3).* Let  $L'/L$  be a finite extension with ramification index  $e$  and denote by  $\text{Tr} : W_n\Omega_{L'}^q \rightarrow W_n\Omega_L^q$  the trace. We have to show

$$(3.10.2) \quad \text{Tr}(\text{Fil}_r^p W_n\Omega_{L'}^q) \subset \text{Fil}_{\lceil \frac{r}{e} \rceil}^p W_n\Omega_L^q.$$

As  $\text{Tr}$  restricts to  $\text{Tr} : W_n\Omega_{\mathcal{O}_{L'}}^q \rightarrow W_n\Omega_{\mathcal{O}_L}^q$  we can assume  $r \geq 1$ . We consider several cases. In the following we will use that  $\mathcal{O}_L$  is excellent and that henceforth for every finite extension  $L'/L$  we have  $[L' : L] = e(L'/L)f(L'/L)$ .

*0th case:* We have finite field extensions  $L' \supset E \supset L$ , and (3.10.2) holds for  $L'/E$  and  $E/L$ . Then it holds for  $L'/L$ . This follows from the transitivity of the trace and the formula  $\lceil \lceil r/e_1 \rceil / e_2 \rceil = \lceil r/e_1 e_2 \rceil$ .

*1st case:*  $e = 1$ . In this case the local parameter  $z \in \mathcal{O}_L$  is local parameter of  $\mathcal{O}_{L'}$  as well and hence (3.10.2) follows from Lemma 2.11(2) and the fact that  $\text{Tr}$  commutes with  $V$  and  $d$  (see 2.1) and satisfies a projection formula.

*2nd case:*  $e = [L' : L]$ . Let  $K \hookrightarrow \mathcal{O}_L$  be a coefficient field, by assumption the composition  $K \hookrightarrow \mathcal{O}_L \hookrightarrow \mathcal{O}_{L'}$  is a coefficient field of  $\mathcal{O}_{L'}$ . Let  $t \in \mathcal{O}_{L'}$  be a local parameter. Any element in  $W_n \Omega_{\mathcal{O}_{L'}}^q(\log) / W_n \Omega_{\mathcal{O}_{L'}}^q$  admits a representative which is a sum of elements  $\alpha \text{dlog } t$  with  $\alpha \in W_n \Omega_K^{q-1}$ . We have  $\text{Tr}(\alpha \text{dlog } t) = \alpha \text{dlog}(\text{Nm}(t))$ , see 2.1(c). Thus in this case (3.10.2) holds for  $r = 0, 1$ , by Lemma 2.8. For  $r \geq 2$  condition (3.10.2) follows by induction over  $r$ , from the Lemmas 3.9 and 3.10, and the fact that  $\text{Tr}$  commutes with  $F$ ,  $d$ , and satisfies a projection formula.

*3rd case:*  $L'/L$  purely inseparable. In this case we can refine  $L'/L$  into a tower of subextensions of degree  $p$ . Thus the statement follows from the above cases.

*4th case:*  $L'/L$  separable. In this case we use the  $p$ -extension trick from [BT73, 5.9]. Let  $H$  a  $p$ -Sylow subgroup in the Galois group of some finite Galois extension  $M/L$  containing  $L'$ . Set  $E := M^H$ . Then every finite extension of  $E$  inside  $M$  has  $p$ -power degree and  $([E : L], p) = 1$ . Since  $L'/L$  is separable we obtain decomposition  $L' \otimes_L E = \bigoplus_i E_i$  with  $E_i/E$  a finite field extension of  $p$ -power degree. We obtain a commutative diagram

$$(3.10.3) \quad \begin{array}{ccc} W_n \Omega_{L'}^q & \xrightarrow{\oplus_i \varphi_i} & \bigoplus_i W_n \Omega_{E_i}^q \\ \text{Tr}_{L'/L} \downarrow & & \downarrow \sum_i \text{Tr}_{E_i/E} \\ W_n \Omega_L^q & \xrightarrow{\varphi} & W_n \Omega_E^q. \end{array}$$

By the cases 0 - 2, (3.10.2) holds for  $E/L$  since it is tamely ramified; similarly (3.10.2) holds for  $E_i/E$  for all  $i$ , since a  $p$ -power extension can be refined to a tower of degree  $p$ -extensions. Now let  $a \in \text{Fil}_r^p W_n \Omega_{L'}^q$ , by definition, or (c6), we have

$$c_{E_i}(\varphi_i(a)) \leq e(E_i/L') \cdot r,$$

by (c3) for  $E_i/E$  and  $e(L'/L)e(E_i/L') = e(E/L)e(E_i/E)$  we have

$$c_E(\text{Tr}_{E_i/E}(\varphi_i(a))) \leq \left\lceil \frac{e(E/L) \cdot r}{e(L'/L)} \right\rceil,$$

by (c3) for  $E/L$  we have

$$(3.10.4) \quad c_L(\text{Tr}_{E/L} \text{Tr}_{E_i/E}(\varphi_i(a))) \leq \left\lceil \frac{e(E/L) \cdot r}{e(L'/L)e(E/L)} \right\rceil = \left\lceil \frac{r}{e(L'/L)} \right\rceil.$$

Thus altogether

$$\begin{aligned} c_L(\text{Tr}_{L'/L}(a)) &= c_L([E : L] \cdot \text{Tr}_{L'/L}(a)), & \text{since } [E : L] \in \mathbb{Z}_p^\times, \\ &= c_L(\text{Tr}_{E/L} \varphi(\text{Tr}_{L'/L}(a))), & \text{by } \text{Tr}_{E/L} \circ \varphi = [E : L], \\ &= c_L\left(\sum_i \text{Tr}_{E/L} \text{Tr}_{E_i/E} \varphi_i(a)\right), & \text{by (3.10.3)} \\ &\leq \max\{c_L(\text{Tr}_{E/L} \text{Tr}_{E_i/E} \varphi_i(a))\}, & \text{by (c2)} \\ &\leq \left\lceil \frac{r}{e(L'/L)} \right\rceil, & \text{by (3.10.4).} \end{aligned}$$

5th case: For an arbitrary finite field extension  $L'/L$  property (c3) follows from the cases 0, 3, and 4 above. This completes the proof of Theorem 3.3.  $\square$

#### 4. SOME MORE PROPERTIES OF THE $p$ -SATURATED FILTRATION

For later use we analyze in this section the  $p$ -saturated filtration further.

**Lemma 4.1.** *Let  $L \in \Phi$  and  $r \geq 2$ . The map*

$$(4.1.1) \quad F^{n-1} \oplus F^{n-1}d : \frac{\text{Fil}_r^p W_n \Omega_L^q}{\text{Fil}_{r-1}^p W_n \Omega_L^q + p(\text{Fil}_r^p W_{n-1} \Omega_L^q)} \longrightarrow \frac{\Omega_L^q}{\text{Fil}_{r-1}^p \Omega_L^q} \oplus \frac{\Omega_L^{q+1}}{\text{Fil}_{r-1}^p \Omega_L^{q+1}}$$

is injective.

*Proof.* First note that the map in the statement is well-defined by Corollary 2.10. Let  $K \hookrightarrow \mathcal{O}_L$  be a coefficient field. Let  $\phi$  be an element in the source of (4.1.1) and assume

$$(4.1.2) \quad (F^{n-1}(\phi), F^{n-1}d\phi) = 0 \quad \text{in} \quad \frac{\Omega_L^q}{\text{Fil}_{r-1}^p \Omega_L^q} \oplus \frac{\Omega_L^{q+1}}{\text{Fil}_{r-1}^p \Omega_L^{q+1}}.$$

By Lemma 3.9 we can assume that  $\phi$  is a sum

$$(4.1.3) \quad \phi = \sum_i a_i \text{dlog } u_i + \sum_j da'_j \text{dlog } v_j + \sum_l (F^{c_l} db_l) \cdot \delta_l,$$

with

$$a_i, a'_j \in \text{gr}_r W_n L := \frac{\text{fil}_r W_n L}{\text{fil}_{r-1} W_n L}, \quad b_l \in \frac{\text{fil}_{r-1}^{\log} W_{n+c_l} L}{\text{fil}_{r-1} W_{n+c_l} L \cap \text{fil}_{r-1}^{\log} W_{n+c_l} L},$$

$$u_i \in K_q^M(K), \quad v_j \in K_{q-1}^M(K), \quad \delta_l \in W_n \Omega_K^{q-1}, \quad c_l \geq 0.$$

Write

$$e = v_p(r), \quad e_1 = v_p(r-1), \quad r = r_0 p^e, \quad r-1 = r_1 p^{e_1}.$$

We consider four cases.

1st case:  $e = 0$ ,  $e_1 \in [0, n-1]$ . By [RS21, 7.18(1)] we can assume

$$a_i = V^{n-1-e_1}([\alpha_i z^{-r_1}]), \quad a'_j = V^{n-1-e_1}([\alpha'_j z^{-r_1}]), \quad b_l = V^{n+c_l-1-e_1}([\beta_l z^{-r_1}]),$$

for some  $\alpha_i, \alpha'_j, \beta_l \in K$ . Hence

$$\phi = V^{n-1-e_1}([z]^{-r_1} A) + dV^{n-1-e_1}([z]^{-r_1} B),$$

where

$$A := \sum_i [\alpha_i] \text{dlog } u_i - \sum_l [\beta_l] F^{n-1-e_1} d\delta_l \in W_{e_1+1} \Omega_K^q,$$

$$B := \sum_j [\alpha'_j] \text{dlog } v_j + \sum_l [\beta_l] F^{n-1-e_1}(\delta_l) \in W_{e_1+1} \Omega_K^{q-1}.$$

Therefore

$$F^{n-1}d\phi = z^{-(r-1)} F^{e_1} dA + (-r_1) z^{-(r-1)} \text{dlog } z \cdot F^{e_1}(A),$$

$$F^{n-1}(\phi) = z^{-(r-1)} (p^{n-1-e_1} F^{e_1}(A) + F^{e_1} dB) + (-r_1) z^{-(r-1)} \text{dlog } z \cdot F^{e_1}(B).$$

If  $e_1 = 0$ , then

$$\text{Fil}_{r-1}^p \Omega_L^q = \frac{1}{z^{r-2}} \Omega_{\mathcal{O}_L}^q(\log z)$$

and the vanishing (4.1.2) yields  $A = 0$  and  $B = 0$ , whence  $\phi = 0$ . If  $e_1 \in [1, n-1]$ , then

$$\mathrm{Fil}_{r-1}^p \Omega_L^q = \frac{1}{z^{r-1}} \Omega_{\mathcal{O}_L}^q$$

and the vanishing (4.1.2) yields  $F^{e_1}(A) = 0$  and  $F^{e_1}(B) = 0$ . By [III79, I, (3.11.3)] we find  $A' \in W_{e_1} \Omega_K^q$  and  $B' \in W_{e_1} \Omega_K^{q-1}$  with  $V(A') = A$  and  $V(B') = B$ . Therefore

$$\phi = V^{n-e_1}([z^{-r_1 p}]A') + dV^{n-e_1}([z^{-r_1 p}]B').$$

Thus  $\phi \in \mathrm{Fil}_{r-1}^p W_n \Omega_L^q$ , by Lemma 2.11.

*2nd case:*  $e = 0$ ,  $e_1 \geq n$ . In this case  $\mathrm{gr}_r W_n L = 0 = \mathrm{gr}_r W_{n+c_l} L$ , for all  $c_l$  with  $e_1 \geq n + c_l$ , by [RS21, Cor. 7.18(1)]. Hence we can assume  $\phi = \sum_l (F^{c_l} db_l) \cdot \delta_l$ , with  $e_1 < n + c_l$  for all  $l$ , and  $b_l = V^{n+c_l-1-e_1}([\beta_l z^{-r_1}])$ , with  $\beta_l \in K$ . Hence

$$\phi = [z]^{-\frac{r-1}{p^{n-1}}} A + [z]^{-\frac{r-1}{p^{n-1}}} d \log z \cdot B,$$

where

$$A = \sum_l F^{e_1-(n-1)} d[\beta_l] \cdot \delta_l \in W_n \Omega_K^q \quad \text{and} \quad B = \sum_l F^{e_1-(n-1)}([\beta_l]) \delta_l \in W_n \Omega_K^{q-1}.$$

By Lemma 2.11 we have

$$[z]^{-\frac{r-1}{p^{n-1}}} A \in p \cdot \mathrm{fil}_{r-1}^{\log} W_n \Omega_L^q + d(\mathrm{fil}_r^{\log'} W_n \Omega_L^{q-1}) \subset \mathrm{Fil}_{r-1}^p W_n \Omega_L^q.$$

Thus we can assume  $A = 0$  and hence

$$F^{n-1} d\phi = -z^{-(r-1)} d \log z \cdot F^{n-1} dB \quad \text{and} \quad F^{n-1} \phi = z^{-(r-1)} d \log z \cdot F^{n-1}(B).$$

As in the case under consideration

$$\mathrm{Fil}_{r-1}^p \Omega_L^q = \frac{1}{z^{r-1}} \Omega_{\mathcal{O}_L}^q$$

the vanishing (4.1.2) yields

$$(4.1.4) \quad B \in \mathrm{Ker}(F^{n-1}) \cap \mathrm{Ker}(F^{n-1}d) = pW_n \Omega_K^{q-1},$$

see e.g. [Eke84, (0.6.3)] for the equality<sup>5</sup>. By Lemma 2.11 and Corollary 2.10 we find

$$\phi \in p \cdot \mathrm{Fil}_r^p W_n \Omega_L^q \subset p \mathrm{Fil}_{pr}^p W_n \Omega_L^q = \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q).$$

*3rd case:*  $e \in [1, n-1]$ ,  $e_1 = 0$ . By [RS21, Cor. 7.18(2) and Cor 7.17(1)], we can assume

$$\alpha_i = V^{n-1}(\alpha_i z^{-(r-1)}) + V^{n-e}(\beta_i [z]^{-r_0 p}), \quad \alpha'_j = V^{n-1}(\alpha'_j z^{-(r-1)}) + V^{n-e}(\beta'_j [z]^{-r_0 p}),$$

and

$$b_l = V^{n+c_l-1}(\gamma_l z^{-(r-1)}),$$

for some  $\alpha_i, \alpha'_j, \gamma_l \in K$ ,  $\beta_i, \beta'_j \in W_e K$ . Hence

$$\phi = V^{n-1}(z^{-(r-1)} A) + dV^{n-1}(z^{-(r-1)} B) + V^{n-e}([z]^{-r_0 p} C) + dV^{n-e}([z]^{-r_0 p} D),$$

where

$$\begin{aligned} A &:= \sum_i \alpha_i d \log u_i - \sum_l \gamma_l F^{n-1} d \delta_l \in \Omega_K^q, \\ B &:= \sum_j \alpha'_j d \log v_j + \sum_l \gamma_l F^{n-1}(\delta_l) \in \Omega_K^{q-1}, \\ C &:= \sum_i \beta_i d \log u_i \in W_e \Omega_K^q, \quad D := \sum_j \beta'_j d \log v_j \in W_e \Omega_K^{q-1}. \end{aligned}$$

<sup>5</sup>It is easily deduced from [III79, I, Proposition 3.11]

Therefore

$$\begin{aligned} F^{n-1}d\phi &= z^{-(r-1)}dA - (r-1)z^{-(r-1)}\mathrm{dlog} z \cdot A + z^{-r}F^{e-1}dC, \\ F^{n-1}\phi &= z^{-(r-1)}dB - (r-1)z^{-(r-1)}\mathrm{dlog} z \cdot B + z^{-r}F^{e-1}dD. \end{aligned}$$

As in the case under consideration

$$\mathrm{Fil}_{r-1}^p \Omega_L^q = \frac{1}{z^{r-2}} \Omega_{\mathcal{O}_L}^q(\mathrm{log} z)$$

the vanishing (4.1.2) yields

$$A = 0, \quad B = 0, \quad F^{e-1}dC = 0, \quad F^{e-1}dD = 0.$$

By [Ill79, (3.11.4)] we find  $C' \in W_{e+1}\Omega_K^q$  and  $D' \in W_{e+1}\Omega_K^{q-1}$  such that

$$C = F(C') \quad \text{and} \quad D = F(D').$$

Altogether

$$\phi = pV^{n-e-1}([z]^{-r_0}C') + p dV^{n-e-1}([z]^{-r_0}D').$$

Hence  $\phi \in p\mathrm{Fil}_{r+1}^p W_n \Omega_L^q$ , by Lemma 2.11. As  $r+1 \leq pr$ , Corollary 2.10 yields

$$\phi \in p\mathrm{Fil}_{pr}^p W_n \Omega_L^q = \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q).$$

4th case:  $e \geq n, e_1 = 0$ . By [RS21, Cor. 7.18(3) and Cor. 7.17(1)], we can assume

$$a_i = V^{n-1}(\alpha_i z^{-(r-1)}) + \beta_i [z]^{-\frac{r}{p^{n-1}}}, \quad a'_j = V^{n-1}(\alpha'_j z^{-(r-1)}) + \beta'_j [z]^{-\frac{r}{p^{n-1}}},$$

and

$$\beta_l = V^{n+c_l-1}(\gamma_l z^{-(r-1)}),$$

for some  $\alpha_i, \alpha'_j, \gamma_l \in K, \beta_i, \beta_j \in W_n K$ . Similarly as in the 3rd case we find

$$\phi = V^{n-1}(z^{-(r-1)}A) + dV^{n-1}(z^{-(r-1)}B) + [z]^{-\frac{r}{p^{n-1}}}C + p[z]^{-\frac{r}{p^{n-1}}}\mathrm{dlog} z \cdot D,$$

where  $A \in \Omega_K^q, B \in \Omega_K^{q-1}, C \in W_n \Omega_K^q$ , and  $D \in W_n \Omega_K^{q-1}$  and where we use that  $r/p^{n-1}$  is divisible by  $p$  to get the  $p$  in front of the last summand. Hence

$$\begin{aligned} F^{n-1}d\phi &= -(r-1)z^{-(r-1)}\mathrm{dlog} z \cdot A + z^{-(r-1)}dA + z^{-r}F^{n-1}(dC), \\ F^{n-1}\phi &= -(r-1)z^{-(r-1)}\mathrm{dlog} z \cdot B + z^{-(r-1)}dB + z^{-r}F^{n-1}(C). \end{aligned}$$

As in the case under consideration

$$\mathrm{Fil}_{r-1}^p \Omega_L^q = \frac{1}{z^{r-2}} \Omega_{\mathcal{O}_L}^q(\mathrm{log} z)$$

the vanishing (4.1.2) yields

$$A = 0, \quad B = 0, \quad C \in \mathrm{Ker}(F^{n-1}) \cap \mathrm{Ker}(F^{n-1}d).$$

Thus

$$\phi \in p\mathrm{Fil}_{r+1}^p W_n \Omega_L^q \subset \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q),$$

by (4.1.4), Lemma 2.11, and Corollary 2.10. This completes the proof.  $\square$

**Lemma 4.2.** *Let  $L \in \Phi$  and  $r \geq 2$ . The sequences*

$$(4.2.1) \quad F(W_{n+1}\Omega_L^q) \cap \mathrm{Fil}_r^p W_n \Omega_L^q \longrightarrow \frac{\mathrm{Fil}_r^p W_n \Omega_L^q}{\mathrm{Fil}_{r-1}^p W_n \Omega_L^q + \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q)} \xrightarrow{F^{n-1}d} \frac{\Omega_L^{q+1}}{\mathrm{Fil}_{r-1}^p \Omega_L^{q+1}}$$

and

$$(4.2.2) \quad V(W_{n-1}\Omega_L^q) \cap \mathrm{Fil}_r^p W_n \Omega_L^q \longrightarrow \frac{\mathrm{Fil}_r^p W_n \Omega_L^q}{\mathrm{Fil}_{r-1}^p W_n \Omega_L^q + \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q)} \xrightarrow{F^{n-1}} \frac{\Omega_L^q}{\mathrm{Fil}_{r-1}^p \Omega_L^q}$$

are exact, the maps on the left being induced by the quotient maps.

*Proof.* We use the notation from the proof of Lemma 4.1. Clearly the sequences from the statement form a complex. Write

$$e = v_p(r), \quad e_1 = v_p(r-1), \quad r = r_0 p^e, \quad r-1 = r_1 p^{e_1}.$$

We show the exactness of (4.2.1). Let  $\phi \in \text{Fil}_r^p W_n \Omega_L^q$  be as in (4.1.3) and assume

$$F^{n-1}d(\phi) = 0 \quad \text{in} \quad \frac{\Omega_L^{q+1}}{\text{Fil}_{r-1}^p \Omega_L^{q+1}}.$$

We consider the same four cases as in Lemma 4.1 to show that the image of  $\phi$  in the quotient in the middle of (4.2.1) lies in the image of  $F(W_{n+1} \Omega_L^q)$ .

*1st case:*  $e = 0, e_1 \in [0, n-1]$ . With the notation from the corresponding case in Lemma 4.1 we find: if  $e_1 = 0$ , then  $A = 0$  and hence

$$\phi = dV^{n-1}([z]^{-r_1} B) = FdV^n([z]^{-r_1} B);$$

if  $e_1 \in [1, n-1]$ , then

$$A \in \text{Ker}(F^{e_1}) \cap \text{Ker}(dF^{e_1}) \subset pW_n \Omega_K^q,$$

see (4.1.4) for the inclusion, hence writing  $A = pA' = FV(A')$  yields

$$\phi = F\left(V^{n-e_1}([z]^{-r_1} A') + dV^{n-e_1}([z]^{-r_1} B)\right).$$

*2nd case:*  $e = 0, e_1 \geq n$ . The argument from the second case of the proof of Lemma 4.1 yields

$$\phi = [z]^{-\frac{r-1}{p^{n-1}}} \text{dlog } z \cdot B, \quad \text{with } F^{n-1}d(B) = 0.$$

By [Ill79, I, (3.11.4)] we have  $B = F(B')$  and hence

$$\phi = F([z]^{-\frac{r-1}{p^n}} \text{dlog } z \cdot B').$$

*3rd case:*  $e \in [1, n-1], e_1 = 0$ . As in the third case of the proof of Lemma 4.1 we find  $A = 0$  and  $C = F(C')$ . Thus

$$\phi = F\left(dV^n(z^{-(r-1)} B) + V^{n-e}([z]^{-r_0} C') + dV^{n-e}([z]^{-r_0} V(D))\right).$$

*4th case:*  $e \geq n, e_1 = 0$ . Similarly as in the fourth case of Lemma 4.1, we get  $A = 0$  and  $C = F(C')$ . Thus

$$\phi = F\left(dV^n(z^{-(r-1)} B) + [z]^{-\frac{r}{p^n}} C' + [z]^{-\frac{r}{p^n}} \text{dlog } z \cdot V(D)\right).$$

This completes the proof of the exactness of (4.2.1).

We show the exactness of (4.2.2). We assume  $n \geq 2$  as the statement is trivial for  $n = 1$ . Let  $\phi \in \text{Fil}_r^p W_n \Omega_L^q$  be as in (4.1.3) and assume

$$F^{n-1}(\phi) = 0 \quad \text{in} \quad \frac{\Omega_L^q}{\text{Fil}_{r-1}^p \Omega_L^q}.$$

We consider the same four cases as above to show that the image of  $\phi$  in the quotient lies in the image of  $V(W_{n-1} \Omega_L^q)$  under the quotient map.

*1st case:*  $e = 0, e_1 \in [0, n-1]$ . With the notation from the corresponding case in Lemma 4.1 we find: if  $e_1 = 0$ , then  $B = 0$  and hence

$$\phi = V^{n-1}([z]^{-r_1} A);$$

if  $e_1 \in [1, n-1]$ , then  $F^{e_1}(B) = 0$ , hence  $B = V(B')$ , by [Ill79, I, (3.11.3)]. As

$$dV^{n-e_1-1}([z]^{-r_1} B) = dV^{n-e_1}([z]^{-r_1 p} B') \in \text{Fil}_{r-1}^p W_n \Omega_L^q,$$

by Lemma 2.11, we obtain that the image of  $\phi$  in the quotient is equal to

$$\phi = V^{n-1-e_1}([z]^{-r_1}A).$$

*2nd case:*  $e = 0, e_1 \geq n$ . The argument from the second case of the proof of Lemma 4.1 yields

$$\phi = [z]^{-\frac{r-1}{p^{n-1}}} \mathrm{dlog} z \cdot B, \quad \text{with } F^{n-1}(B) = 0.$$

By [Ill79, I, (3.11.3)] we have  $B = V(B')$  and hence

$$\phi = V([z]^{-\frac{(r-1)p}{p^n}} \mathrm{dlog} z \cdot B').$$

*3rd case:*  $e \in [1, n-1], e_1 = 0$ . As in the third case of the proof of Lemma 4.1 we find  $B = 0$  and  $F^{e-1}dD = 0$ . Thus  $D = F(D')$  and

$$\phi = V^{n-1}(z^{-(r-1)}A) + V^{n-e}([z]^{-r_0p}C) + V(FdV^{n-e-1}([z]^{-r_0}D')).$$

*4th case:*  $e \geq n, e_1 = 0$ . Similarly as in the fourth case of Lemma 4.1, we get  $B = 0$  and  $C = V(C')$ . Thus

$$\phi = V^{n-1}(z^{-(r-1)}A) + V([z]^{-\frac{pr}{p^{n-1}}}C') + VF([z]^{-\frac{r}{p^{n-1}}} \mathrm{dlog} z \cdot D).$$

This completes the proof of the exactness of (4.2.2).  $\square$

## 5. HODGE-WITT SHEAVES WITH MODULUS

We begin this section by recalling the notion of reciprocity sheaves and several a priori different ways to assign a modulus to a section of certain presheaves, which appear in [KSY22], [RS21], and [RS]. The main result of this section is Theorem 5.4, which says that these different notions of modulus agree for  $W_n\Omega^q$ ,  $q \geq 1$ . Theorem 3.3 and Lemma 4.1 are crucial ingredients in the proof of that theorem.

**5.1.** We recall the definition of a *reciprocity sheaf* from [KSY22, Definition 2.2.4 and Section 2.4], where these are called sheaves with SC-reciprocity. Denote by **PST** the category of presheaves with transfers on **Sm** in the sense of Voevodsky and by **NST** its full subcategory of Nisnevich sheaves. For  $U \in \mathbf{Sm}$  we denote by  $\mathbb{Z}_{\mathrm{tr}}(U)$  the presheaf with transfers represented by  $U$ . Let  $F \in \mathbf{PST}$  and let  $(X, D)$  be a proper modulus pair (see 3.1) with  $X \setminus D = U$ . A section  $a \in F(U)$  has modulus  $(X, D)$  if the map  $a : \mathbb{Z}_{\mathrm{tr}}(U) \rightarrow F$ , defined by  $a$  via the Yoneda embedding, factors in **PST** as

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{tr}}(U) & \longrightarrow & F \\ \downarrow & \nearrow & \\ h_0(X, D) & & \end{array}$$

Here  $h_0(X, D)$  is the presheaf with transfers given on  $S \in \mathbf{Sm}$  by

$$h_0(X, D)(S) = \mathrm{Coker}(\mathbf{Cor}((\mathbb{P}_S^1, \infty_S), (X, D)) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\mathrm{tr}}(U)(S)),$$

where  $i_\epsilon : \{\epsilon\}_S \hookrightarrow \mathbb{P}_S^1$  denotes the closed immersion, for  $\epsilon \in \{0, 1\}$ , and  $\mathbf{Cor}((\mathbb{P}_S^1, \infty_S), (X, D))$  denotes the free abelian group generated on integral closed subschemes  $Z \subset \mathbb{A}_S^1 \times_k U$  which are surjective and finite over a connected component of  $U$  such that

$$\nu^* \infty_S \geq \nu^* D,$$

where  $\nu : \tilde{Z} \rightarrow \mathbb{P}_S^1 \times_k X$  is the normalization of the closure of  $Z$ .

For  $(X, D)$  any modulus pair with  $U = X \setminus D$  and  $(\overline{X}, D' + X_\infty)$  a compactification of  $(X, D)$  (see 3.1) we set

$$\underline{\omega}^{\mathbf{CI}} F(X, D) = \{a \in F(U) \mid a \text{ has modulus } (\overline{X}, D' + N \cdot X_\infty) \text{ for some } N \gg 0\}.$$

In fact the assignment  $(X, D) \mapsto \underline{\omega}^{\mathbf{CI}} F(X, D)$  defines a cube invariant semipure modulus presheaf with M-reciprocity  $\underline{\omega}^{\mathbf{CI}} F$  as defined in [KS22] and [Sai20] and with the notation from there we have  $\underline{\omega}^{\mathbf{CI}} F = \tau_1 h_0^\square \omega^* F$ .

A presheaf with transfers  $F$  is a *reciprocity presheaf* if for any  $U \in \mathbf{Sm}$  any section  $a \in F(U)$  has a modulus, i.e., if  $F(U) = \bigcup \underline{\omega}^{\mathbf{CI}} F(X, D)$ , where the union is over all proper modulus pairs  $(X, D)$  with  $X \setminus D = U$ . A *reciprocity sheaf* is a Nisnevich sheaf on  $\mathbf{Sm}$  which is also a reciprocity presheaf. The category of reciprocity sheaves is denoted by  $\mathbf{RSC}_{\text{Nis}}$ .

If  $F \in \mathbf{RSC}_{\text{Nis}}$  and  $(X, D)$  is a modulus pair, then the assignment

$$(\text{étale } X\text{-schemes}) \ni (u : V \rightarrow X) \mapsto \underline{\omega}^{\mathbf{CI}} F(V, u^* D) =: \Gamma(V, \underline{\omega}^{\mathbf{CI}} F_{(X, D)})$$

is a sheaf on  $X_{\text{Nis}}$  by [RS21, Corollary 4.16] and is denoted by  $\underline{\omega}^{\mathbf{CI}} F_{(X, D)}$ .

**5.2.** Let  $F \in \mathbf{NST}$  and let  $c = \{c_L : F(L) \rightarrow \mathbb{N}_0\}_{L \in \Phi}$  be a semi-continuous conductor (see 3.2). Let  $(X, D)$  be a *proper* modulus pair with  $U = X \setminus D$ . For  $a \in F(U)$  we write

$$c_X(a) \leq D$$

as a shorthand for

$$c_L(\rho^* a) \leq v_L(\rho^* D), \quad \text{for all } L \in \Phi \text{ and all } \rho : \text{Spec } L \rightarrow U.$$

Following [RS21, 4.8] we define for any modulus pair  $(X, D)$  with a compactification  $(\overline{X}, D' + X_\infty)$

$$\tilde{F}_c(X, D) := \{a \in F(U) \mid c_{\overline{X}}(a) \leq D' + N \cdot X_\infty, \text{ for some } N \gg 0\}.$$

By [RS21, (4.8.1)] the assignment

$$(\text{étale } X\text{-schemes}) \ni (u : V \rightarrow X) \mapsto \tilde{F}_c(V, u^* D) =: \Gamma(V, \tilde{F}_{c, (X, D)})$$

defines a Nisnevich sheaf  $\tilde{F}_{c, (X, D)}$  on  $X_{\text{Nis}}$ . By [RS21, Theorem 4.15(4)] we have

$$(5.2.1) \quad \tilde{F}_c \subset \underline{\omega}^{\mathbf{CI}} F, \quad \text{for } F \in \mathbf{RSC}_{\text{Nis}}.$$

For  $L \in \Phi$  with ring of integers  $\mathcal{O}_L$ , maximal ideal  $\mathfrak{m}_L$ , and  $r \geq 1$  set

$$\tilde{F}_c(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \varinjlim_{(U, Z)} \tilde{F}_c(U, rZ),$$

where the filtered colimit is over all modulus pairs  $(U, Z)$  with  $U$  and  $Z$  smooth and connected such that  $\mathcal{O}_L = \mathcal{O}_{U, z}^h$ , with  $z \in Z$  is the generic point. As  $c$  is assumed to be semi-continuous we have

$$(5.2.2) \quad \tilde{F}_c(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \{a \in F(L) \mid c_L(a) \leq r\},$$

see [RS21, Lemma 4.23]. Finally we recall that if the conductor has level  $m \leq \infty$  and  $(X, D)$  is a modulus pair with  $U = X \setminus D$ , then

$$\tilde{F}_c(X, D) = \{a \in F(U) \mid f^* a \in \tilde{F}_c(S, f^* D), \text{ for all } f : S \rightarrow X \text{ with } S \in \mathbf{Sm}, \dim S \leq m\},$$

see [RS21, Corollary 4.18].

**5.3.** The following notion of modulus using dilatations is motivated by work of Abbes and (Takeshi) Saito, who used this approach to study the ramification of Galois torsors, see [AS11] and [Sai17]. In the framework of reciprocity sheaves this notion was studied in [RS]. Let  $(X, D)$  be a modulus pair with  $X \in \mathbf{Sm}$  and assume that the reduced divisor  $D_{\text{red}}$  underlying  $D$  has only simple normal crossings, i.e.,  $D_{\text{red}}$  is an SNCD divisor. Denote by  $\text{Bl}_D(X \times X)$  the blow-up in  $D$  diagonally embedded into  $X \times X$  and denote by  $\widetilde{X \times D}$  and  $\widetilde{D \times X}$  the strict transforms of  $X \times D$  and  $D \times X$ , respectively. Set

$$P_X^{(D)} = \text{Bl}_D(X \times X) \setminus (\widetilde{X \times D} \cup \widetilde{D \times X}).$$

We note that  $P_X^{(D)}$  is smooth and that the open embedding  $U \times U \hookrightarrow X \times X$ , with  $U = X \setminus D$ , induces an embedding  $U \times U \hookrightarrow P_X^{(D)}$ , see e.g. [RS, Lemma 2.3]. We denote by  $p_1, p_2 : P_X^{(D)} \rightarrow X$  the maps induced by the projection  $X \times X \rightarrow X$  to the first and second factor, respectively; they restrict to the projection maps on  $U \times U$ . For  $F \in \mathbf{RSC}_{\text{Nis}}$  we define

$$(5.3.1) \quad F^{\text{AS}}(X, D) = \{a \in F(U) \mid p_1^* a - p_2^* a \in F(P_X^{(D)})\},$$

for this definition to make sense we use that  $F(U \times U)$  is a subgroup of  $F(P_X^{(D)})$  by [KSY16, Theorem 6] together with [KSY22, Corollary 3.2.3]. As an étale map  $u : V \rightarrow X$  induces a morphism  $P_V^{(u^* D)} \rightarrow P_X^{(D)}$  we obtain a Nisnevich sheaf  $F_{(X, D)}^{\text{AS}}$  on  $X_{\text{Nis}}$  given by

$$(u : V \rightarrow X) \mapsto F^{\text{AS}}(V, u^* D) = \Gamma(V, F_{(X, D)}^{\text{AS}}).$$

By [RS, Theorem 2.6]

$$(5.3.2) \quad \underline{\omega}^{\text{CI}} F_{(X, D)} \subset F_{(X, D)}^{\text{AS}},$$

for  $X \in \mathbf{Sm}$  and  $D_{\text{red}}$  SNCD. By Theorem 2.10 in *loc. cit.* this is an equality if  $(X, D)$  additionally admits a projective SNC-compactification.

**Theorem 5.4.** *Let  $c = \{c_L : W_n \Omega_L^q \rightarrow \mathbb{N}_0\}_{L \in \Phi}$  with*

$$(5.4.1) \quad c_L(a) = \min \{r \geq 0 \mid a \in \text{Fil}_r^p W_n \Omega_L^q\}, \quad a \in W_n \Omega_L^q,$$

*be the semi-continuous conductor of level  $q+1$  from Theorem 3.3. Let  $(X, D)$  be any modulus pair and let  $n, q \geq 1$ . Then*

$$\underline{\omega}^{\text{CI}} W_n \Omega_{(X, D)}^q = \widetilde{W_n \Omega_{c, (X, D)}^q}.$$

*In particular*

$$\underline{\omega}^{\text{CI}} W_n \Omega^q(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \text{Fil}_r^p W_n \Omega_L^q,$$

*for all  $L \in \Phi$  and  $r \geq 0$ . If furthermore  $X$  is smooth and  $D_{\text{red}}$  is an SNCD, then*

$$\underline{\omega}^{\text{CI}} W_n \Omega_{(X, D)}^q = \widetilde{W_n \Omega_{c, (X, D)}^q} = (W_n \Omega^q)_{(X, D)}^{\text{AS}}.$$

The proof will be given after the next remark.

**Remark 5.5.** (1) Note that the statement is not true for  $q = 0$ . As the absolute Frobenius on  $X$  induces an endomorphism of  $W_n \mathcal{O}_X$  one has to consider the Frobenius saturation of  $\text{fil}_r W_n(L)$ , see [RS21, Theorem 7.20] and [RS, Proposition 5.3].

(2) The case  $n = 1$  and  $q \geq 1$  follows from [RS, Theorem 6.6 and Corollary 6.8]. In particular this says that if  $X$  and  $D$  are smooth and  $r \geq 0$  then

$$(5.5.1) \quad (\Omega^q)^{\text{AS}}(X, rD) = \begin{cases} \Omega_X^q(\log D)((r-1)D) & \text{if } p \nmid r \\ \Omega_X^q(rD) & \text{if } p \mid r, \end{cases}$$

which at the generic point  $\eta$  of  $D$  coincides with  $\mathrm{Fil}_r^p W_1 \Omega_{L_\eta}^q$ , where  $L_\eta = \mathrm{Frac}(\mathcal{O}_{X,\eta}^h)$ .

(3) In the case  $D_{\mathrm{red}}$  is an SNCD, Theorem 5.4 and Lemma 2.8 imply

$$(5.5.2) \quad \underline{\omega}^{\mathrm{CI}} W_n \Omega_{(X, D_{\mathrm{red}})}^q = W_n \Omega_X^q(\log D),$$

where the right hand side is the logarithmic de Rham-Witt sheaf, see, e.g., [Mat17, Proposition-Definition 3.10]. (This  $\supset$  inclusion is immediate, this  $\subset$  inclusion can be, for example, deduced from the isomorphism [Ill79, I, (2.14.8)] together with [Mat17, Corollary 4.4].) Note that the above equality is also true for  $q = 0$  as in this case both sides are equal to  $W_n \mathcal{O}_X$ . As a consequence we obtain the equality

$$\mathcal{L}og(\underline{\omega}^{\mathrm{CI}} W_n \Omega^q)(X, D_{\mathrm{red}}) = W_n \Omega_X^q(\log D),$$

where  $\mathcal{L}og : \underline{\mathbf{MNST}}_{\log} \rightarrow \mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}$  is the functor defined in [Sai23, (6.0.2)]. This gives a new proof of [Mer, Theorem 4.4].

*Proof.* First of all we note that by [RS21, Theorem 4.15] and (5.2.2) the first statement of Theorem 5.4 is equivalent to show

$$\underline{\omega}^{\mathrm{CI}} W_n \Omega^q(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \mathrm{Fil}_r^p W_n \Omega_L^q, \quad \text{for all } L \in \Phi.$$

Furthermore, if  $X$  is smooth and  $D_{\mathrm{red}}$  is an SNCD, then  $P_X^{(D)}$  (see 5.3) is smooth and  $W_n \Omega_{P_X^{(D)}}^q$  is a successive extension of locally free sheaves (see [Ill79, I, Corollaire 3.9]) and hence the question whether an element  $a \in W_n \Omega^q(U)$  lies in  $(W_n \Omega^q)^{\mathrm{AS}}(X, D)$  is Nisnevich local around the 1-codimensional points of  $D$ . Thus we are reduced to the following situation:

Let  $X = \mathrm{Spec} A$ , with  $A$  a smooth  $k$ -algebra, let  $D = \mathrm{div}(z)$  be a smooth connected divisor on  $X$  with generic point  $\eta \in D^{(0)}$ , and set  $\mathcal{O}_L = \mathcal{O}_{X,\eta}^h$  and  $L = \mathrm{Frac}(\mathcal{O}_{X,\eta}^h)$ . Then we have to show

$$\mathrm{Fil}_r^p W_n \Omega_L^q = \underline{\omega}^{\mathrm{CI}} W_n \Omega^q(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \left( (W_n \Omega^q)_{(X, rD)}^{\mathrm{AS}} \right)_\eta^h, \quad \text{for all } r \geq 0,$$

where the right hand side denotes the Nisnevich stalk of  $(W_n \Omega^q)_{(X, rD)}^{\mathrm{AS}}$  in  $\eta$ . By (5.2.1) and (5.3.2) it remains to show

$$\left( (W_n \Omega^q)_{(X, rD)}^{\mathrm{AS}} \right)_\eta^h \subset \mathrm{Fil}_r^p W_n \Omega_L^q.$$

By (5.5.1) this is true for  $n = 1$ . Thus we assume  $n \geq 2$  in the following. As  $\{\mathrm{Fil}_r^p W_n \Omega_L^q\}_{r \geq 0}$  is an exhaustive filtration of  $W_n \Omega_L^q$  it remains to show

$$(5.5.3) \quad \mathrm{Fil}_r^p W_n \Omega_L^q \cap (W_n \Omega^q)^{\mathrm{AS}}(X, (r-1)D) \subset \mathrm{Fil}_{r-1}^p W_n \Omega_L^q,$$

where we identify  $(W_n \Omega^q)^{\mathrm{AS}}(X, (r-1)D)$  with its image in  $W_n \Omega_L^q$ . By Lemma 2.8 this is true for  $r = 1$  and we can assume  $r \geq 2$ . For  $\phi$  in the left hand side of (5.5.3) we claim

$$(5.5.4) \quad \phi = 0 \quad \text{in} \quad \frac{\mathrm{Fil}_r^p W_n \Omega_L^q}{\mathrm{Fil}_{r-1}^p W_n \Omega_L^q + \underline{p}(\mathrm{Fil}_r^p W_{n-1} \Omega_L^q)}.$$

Indeed, by [RS, Theorem 6.6] (see also Remark 5.5) we have

$$(\Omega^q)^{\mathrm{AS}}(X, (r-1)D) \cap \Omega_L^q = \mathrm{Fil}_{r-1}^p \Omega_L^q,$$

and since the formation  $E \mapsto E^{\mathrm{AS}}$  is functorial with respect to any morphism of sheaves of abelian groups we find

$$F^{n-1}((W_n \Omega^q)^{\mathrm{AS}}(X, (r-1)D)) \subset \mathrm{Fil}_{r-1}^p \Omega_L^q$$

and

$$F^{n-1}d((W_n \Omega^q)^{\mathrm{AS}}(X, (r-1)D)) \subset \mathrm{Fil}_{r-1}^p \Omega_L^{q+1}.$$

Thus the claim (5.5.4) follows from Lemma 4.1.

Therefore (after possibly shrinking  $X$  around  $\eta$ ) we can write  $\phi = \psi + \underline{p}(\phi')$  with

$$\psi \in \mathrm{Fil}_{r-1}^p W_n \Omega_L^q \cap W_n \Omega_{A[\frac{1}{z}]}^q \quad \text{and} \quad \phi' \in \mathrm{Fil}_r^p W_{n-1} \Omega_L^q \cap W_{n-1} \Omega_{A[\frac{1}{z}]}^q.$$

By Theorem 3.3, (5.2.1), and (5.3.2) the form  $\psi$  lies in  $(W_n \Omega^q)^{\mathrm{AS}}(X, (r-1)D)$  and hence so does  $\underline{p}(\phi')$ . Since  $P_X^{((r-1)D)} \setminus U \times U$  is supported on a smooth divisor (e.g. [RS, Lemma 2.3]), the equality (5.3.1) together with Lemma 5.6 below and the injectivity of  $\underline{p}$  imply that  $\phi' \in (W_{n-1} \Omega^q)^{\mathrm{AS}}(X, (r-1)D)$ . By induction on  $n$  we thus have  $\phi' \in \mathrm{Fil}_{r-1}^p W_{n-1} \Omega_L^q$ . This proves the containment (5.5.3) and hence the theorem.  $\square$

**Lemma 5.6** ([GK, Lemma 6.7]). *Let  $C$  be a smooth  $k$ -algebra and  $t \in C$  an element such that  $C/tC$  is smooth. Then (see (2.1.1) for the notation  $\underline{p}$ )*

$$\underline{p}(W_{n-1} \Omega_{C[\frac{1}{t}]}^q) \cap W_n \Omega_C^q = \underline{p}(W_{n-1} \Omega_C^q) \quad \text{in } W_n \Omega_{C[\frac{1}{t}]}^q.$$

## 6. THE STRUCTURE OF HODGE-WITT SHEAVES WITH MODULUS

In this section we investigate the structure of Hodge-Witt sheaves with modulus. The main result is Theorem 6.4, whose proof will occupy all of this section and which will be essential for the proof of the main statement in section 9.

Throughout this section we assume  $X \in \mathbf{Sm}$  and we let  $D$  be an effective Cartier divisor on  $X$  such that  $D_{\mathrm{red}}$  is an SNCD.

**6.1.** If  $E$  is any divisor on  $X$  and  $m \in \mathbb{Z} \setminus \{0\}$ , then we write

$$m \mid E :\iff m \text{ divides the multiplicity of every irreducible component of } E$$

and

$$m \nmid E :\iff m \text{ does not divide the multiplicity of any irreducible component of } E.$$

Given an increasing sequence of natural numbers  $1 \leq r_1 < \dots < r_s$  we say

$$E = E' + p^{r_1} E_1 + \dots + p^{r_s} E_s$$

is a *p-divisibility decomposition of  $E$  (with respect to  $r_1 < \dots < r_s$ )*, if  $p^{r_1} \nmid E'$  and  $p^{r_i} \nmid p^{r_{i-1}} E_{i-1}$ , for  $i = 2, \dots, s$ , and  $E'_{\mathrm{red}} + \sum_{i=1}^s E_{i,\mathrm{red}}$  is a reduced divisor. Note that  $p \mid E_s$  is allowed and that a *p-divisibility decomposition of  $E$*  always exists and is uniquely determined by the sequence  $r_1 < \dots < r_s$ . For example if  $E = \sum_i n_i \mathcal{E}_i$  with  $\mathcal{E}_i$  the irreducible components of  $E$ ,  $\mathcal{E}_i \neq \mathcal{E}_j$ , for  $i \neq j$ , then to say that

$$E = E' + p^r E_r$$

is a *p-divisibility decomposition of  $E$*  means that

$$E' = \sum_{i, p^r \nmid n_i} n_i \mathcal{E}_i \quad \text{and} \quad E_r = \sum_{i, p^r \mid n_i} \frac{n_i}{p^r} \mathcal{E}_i.$$

Finally for  $E$  and  $m$  as above we set

$$\lceil E/m \rceil := \sum_i \lceil n_i/m \rceil \mathcal{E}_i \quad \text{and} \quad \lfloor E/m \rfloor := \sum_i \lfloor n_i/m \rfloor \mathcal{E}_i,$$

where  $\lceil - \rceil$  (resp.  $\lfloor - \rfloor$ ) denotes the round-up (resp. round-down).

**6.2.** Under the assumptions of this section all three ways to define Hodge-Witt sheaves with modulus discussed in Section 5 coincide by Theorem 5.4. We therefore set

$$(6.2.1) \quad W_n \Omega_{(X,D)}^q := \underline{\omega}^{\mathbf{CI}} W_n \Omega_{(X,D)}^q, \quad \text{for } q \geq 1,$$

which coincides with  $\widetilde{W_n \Omega_{c,(X,D)}^q}$  and  $(W_n \Omega_{(X,D)}^q)^{\text{AS}}$ . For  $q = 0$  we set

$$(6.2.2) \quad W_n \Omega_{(X,D)}^0 := W_n \mathcal{O}_{(X,D)} := (\widetilde{W_n})_{c,(X,D)}$$

in the notation of 5.2 and where  $c$  is the conductor defined in (5.4.1) for  $q = 0$ , i.e., it is defined by the  $p$ -saturated Kato-Matsuda filtration  $\text{fil}_*^p W_n(L)$ , see 2.5 and Definition 2.6. By [RS21, Theorem 7.20] we have

$$(6.2.3) \quad \underline{\omega}^{\mathbf{CI}} W_n \mathcal{O}_{(X,D)} = W_n \mathcal{O}_{(X,D)}^F,$$

where the upper index  $F$  denotes the saturation under the absolute Frobenius acting on  $W_n \mathcal{O}_U$ , with  $U = X \setminus D$ . Note that  $\underline{\omega}^{\mathbf{CI}} W_n \mathcal{O}_{(X,D)}$  is not a  $W_n \mathcal{O}_X$ -module.

Let  $D = D_0 + pD_1$  be a  $p$ -divisibility decomposition, see 6.1. Recall from [RS, Theorem 6.6] (see also Theorem 5.4) that with the above notation

$$(6.2.4) \quad \Omega_{(X,D)}^q = \begin{cases} \Omega_X^q(\log D_0)(D_0 - D_{0,\text{red}} + pD_1), & q \geq 1; \\ \mathcal{O}_X(D_0 - D_{0,\text{red}} + pD_1), & q = 0. \end{cases}$$

For us the notation  $\Omega_X^q(\log E)$  for a possibly non-reduced effective Cartier divisor with simple normal crossing support  $E$  will always mean  $\Omega_X^q(\log E_{\text{red}})$ .

We record:

**Proposition 6.3.** *The sheaves  $W_n \Omega_{(X,D)}^q$  are coherent  $W_n \mathcal{O}_X$ -modules, for all  $q \geq 0$ . Furthermore, let  $a \in W_n \Omega^q(U)$ , with  $U = X \setminus D$ . Then*

$$a \in \Gamma(X, W_n \Omega_{(X,D)}^q) \iff a_\eta \in \text{Fil}_{m_\eta}^p W_n \Omega_{L_\eta}^q, \quad \text{for all } \eta \in |D|^{(0)},$$

where  $L_\eta = \text{Frac}(\mathcal{O}_{X,\eta}^h)$ ,  $a_\eta \in W_n \Omega_{L_\eta}^q$  denotes the image of  $a$ , and  $m_\eta$  is the multiplicity of  $\eta$  in  $D$ .

*Proof.* Denote by  $W_n \mathcal{O}_X(D)$  the invertible  $W_n \mathcal{O}_X$ -submodule of  $j_* W_n \mathcal{O}_U$ , with  $j : U \hookrightarrow X$  the open immersion, which is uniquely characterized by the equality

$$W_n \mathcal{O}_X(D)|_{U_0} = W_n \mathcal{O}_{U_0} \cdot \frac{1}{[f]},$$

for any open  $U_0 \subset X$  with  $D|_{U_0} = \text{Div}(f)$ ,  $f \in \mathcal{O}(U_0)$ . By Corollary 2.12 we have the following inclusions of  $W_n \mathcal{O}_X$ -modules

$$(6.3.1) \quad W_n \Omega_X^q \subset W_n \Omega_{(X,D)}^q \subset W_n \Omega_X^q \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}_X(p^{n-1}D), \quad \text{for all } q \geq 0.$$

As  $W_n X = (|X|, W_n \mathcal{O}_X)$  is a scheme of finite type over  $\text{Spec } W_n(k)$  and as  $W_n \Omega_X^q$  is a coherent  $W_n \mathcal{O}_X$ -module, we see that  $W_n \Omega_{(X,D)}^q$  is a coherent  $W_n \mathcal{O}_X$ -module for all  $q \geq 0$ .

The second statement follows for  $q \geq 1$  from the equality  $W_n \Omega_{(X,D)}^q = (W_n \Omega_{(X,D)}^q)^{\text{AS}}$ , see the beginning of the proof of Theorem 5.4. For  $q = 0$ , it is direct to check from the definition.  $\square$

The aim of this section is to prove the following theorem:

**Theorem 6.4.** *For  $n, q \geq 0$  set*

$$(6.4.1) \quad Z_n \Omega_{(X,D)}^q := F^n(W_{n+1} \Omega_{(X,D)}^q) \quad \text{and} \quad B_n \Omega_{(X,D)}^q := \begin{cases} F^{n-1} d(W_n \Omega_{(X,D)}^{q-1}), & n \geq 1; \\ 0, & n = 0. \end{cases}$$

*Then  $Z_n \Omega_{(X,D)}^q$  and  $B_n \Omega_{(X,D)}^q$  are locally free  $\mathcal{O}_X$ -submodules of  $F_{X*}^n \Omega_{(X,D)}^q$ , where  $F_X : X \rightarrow X$  denotes the absolute Frobenius. Furthermore, there is a short exact sequence of coherent  $W_{n+1} \mathcal{O}_X$ -modules*

$$0 \longrightarrow B_n \Omega_{(X,D)}^{q+1} \longrightarrow \frac{W_{n+1} \Omega_{(X,D)}^q}{p W_n \Omega_{(X,D)}^q} \xrightarrow{F^n} Z_n \Omega_{(X,D)}^q \longrightarrow 0,$$

*where the first map on the left is given by  $F^{n-1} d(\beta) \mapsto V(\beta)$  and the  $W_{n+1} \mathcal{O}_X$ -module structure on  $Z_n$  and  $B_n$  is induced by  $F^n : W_{n+1} \mathcal{O}_X \rightarrow \mathcal{O}_X$ .*

The above theorem follows from Proposition 6.15 and Proposition 6.18 proven below. In Proposition 6.15 and Lemma 6.16 we also give alternative descriptions of  $B_n \Omega_{(X,D)}^q$  and  $Z_n \Omega_{(X,D)}^q$ , in the former using certain twisted Cartier operators, in the latter using the functors  $\underline{\omega}^{\text{CI}}$  and  $(-)^{\text{AS}}$  introduced in section 5.

**Remark 6.5.** For  $D = \emptyset$ , the first part of the above theorem holds by [Ill79, I, Proposition 3.11] and [Ill79, 0, Proposition 2.2.8], and the second part, the exactness of the sequence, holds by [Eke84, (0.6.2)]. Note that in this case there is also a second short exact sequence which is dual (under Grothendieck duality) to the one above, it describes  $\text{Ker}(R : W_{n+1} \Omega_X^q \rightarrow W_n \Omega_X^q)$  as an extension of locally free  $\mathcal{O}_X$ -modules, see [Ill79, I, (3.9.1)]. For  $D \neq \emptyset$  this latter sequence does not seem to work, cf. Remark 6.12.

**Lemma 6.6.** *Let  $A$  be an SNCD on  $X$ . Let  $B$  be a (not necessarily effective) Cartier divisor whose support is contained in  $A$  and let  $A_0$  be an irreducible component of  $A$ , which is not a component of  $|B|$ . Let  $s \in \mathbb{Z}$  and  $\rho \in \mathbb{N}$  with  $0 \leq \rho \leq p-1$ . Then the natural map*

$$\Omega_{X/k}^\bullet(\log A)(B + psA_0) \longrightarrow \Omega_{X/k}^\bullet(\log A)(B + (ps + \rho)A_0)$$

*is a quasi-isomorphism. Here  $\Omega_X^q(\log E)(D) := \Omega_X^q(\log E) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ .*

*Proof.* By [MS97, Lemma 4.1] (applied to  $A = D_1 + D_2$ ,  $A_0 = D_1$  and  $D = B + A_0, \dots, B + \rho A_0$ ) we have a quasi-isomorphism as in the statement for  $s = 0$ . Applying  $F_{X*}$  we obtain a quasi-isomorphism of  $\mathcal{O}_X$ -modules; twisting with  $\mathcal{O}_X(sA_0)$  therefore yields the statement.  $\square$

**Lemma 6.7.** *Let  $E$  be an effective Cartier divisor on  $X$  such that  $D_{\text{red}} + E_{\text{red}}$  is a reduced SNCD. Then the inverse Cartier operator*

$$(6.7.1) \quad C^{-1} : \Omega_X^q(\log D) \xrightarrow{\sim} \mathcal{H}^q(F_{X*} \Omega_X^\bullet(\log D)),$$

*see, e.g., [Kat70, Theorem (7.2)], induces an isomorphism*

$$(6.7.2) \quad C^{-1} : \Omega_X^q(\log D)([D/p] - [D/p]_{\text{red}} + E) \xrightarrow{\sim} \mathcal{H}^q(F_{X*} \Omega_X^\bullet(\log D)(D - D_{\text{red}} + pE)).$$

*Proof.* Observe

$$D - D_{\text{red}} = p([D/p] - [D/p]_{\text{red}}) + \sum_i \rho_i D_i,$$

for certain  $\rho_i \in \mathbb{N}$  with  $0 \leq \rho_i \leq p-1$ , where the  $D_i$  are the irreducible components of  $D$ . Indeed, for  $r \geq 1$  we have  $r-1 = p(\lceil r/p \rceil - 1) + \rho$ , for some  $0 \leq \rho \leq p-1$ . Thus tensoring the isomorphism of  $\mathcal{O}_X$ -modules (6.7.1) with  $\mathcal{O}_X([D/p] - [D/p]_{\text{red}})$  and applying Lemma 6.6 yields the isomorphism (6.7.2) in the case  $E = 0$ . Tensoring further with  $\mathcal{O}_X(E)$  yields the general case.  $\square$

**6.8.** Let  $E$  be an effective Cartier divisor on  $X$  such that  $D_{\text{red}} + E_{\text{red}}$  is a reduced SNCD. We will use the following notation

$$(6.8.1) \quad \Omega_n^q(D, E) := \Omega_X^q(\log D)([D/p^n] - [D/p^n]_{\text{red}} + E), \quad q \geq 0, n \geq 0.$$

This is a locally free coherent  $\mathcal{O}_X$ -module. We observe:

- (a) Let  $n \geq 1$ . If  $D = D' + p^n D_n$  is a  $p$ -divisibility decomposition (see 6.1), then

$$\Omega_n^q(D, E) = \Omega_X^q(\log(D' + D_n))([D'/p^n] - [D'/p^n]_{\text{red}} + D_n - D_{n,\text{red}} + E).$$

Hence we have an inclusion

$$(6.8.2) \quad \Omega_n^q(D, E) \subset \Omega_n^q(D', D_n + E).$$

Note that this inclusion is strict if  $D_n \neq 0$  and  $q < \dim X$  and that the cokernel is annihilated by  $\mathcal{O}_X(-D_{n,\text{red}})$ . In particular it is not an inclusion of vector bundles.

- (b) Let  $n \geq 0$ . There is a well-defined map  $d : \Omega_n^q(D, pE) \rightarrow \Omega_n^{q+1}(D, pE)$  induced by the usual differential of the de Rham complex. (Note that we need to have  $pE$  instead of  $E$  for this.) We therefore get a subcomplex  $\Omega_n^\bullet(D, pE) \subset j_* \Omega_U^\bullet$ , where  $j : U = X \setminus |D + E| \hookrightarrow X$  denotes the open immersion. Given a  $p$ -divisibility decomposition  $D = D' + p^{n+1} D_{n+1}$  we obtain an inclusion of complexes

$$(6.8.3) \quad \Omega_n^\bullet(D, pE) \subset \Omega_n^\bullet(D', p(D_{n+1} + E)).$$

- (c) Let  $n \geq 1$ . Using the equality

$$[D/p^n] = \lceil [D/p^{n-1}]/p \rceil$$

we find that the inverse Cartier operator (6.7.2) induces an isomorphism of  $\mathcal{O}_X$ -modules

$$(6.8.4) \quad C^{-1} : \Omega_n^q(D, E) \xrightarrow{\sim} \mathcal{H}^q(F_{X*} \Omega_{n-1}^\bullet(D, pE)).$$

Let  $n \geq 1$ . If  $D = D' + p^n D_n$  is a  $p$ -divisibility decomposition, then we obtain a commutative diagram of locally free  $\mathcal{O}_X$ -modules

$$(6.8.5) \quad \begin{array}{ccc} \Omega_n^q(D, E) & \xrightarrow[\simeq]{C^{-1}} & \mathcal{H}^q(F_{X*} \Omega_{n-1}^\bullet(D, pE)) \\ \downarrow & & \downarrow \\ \Omega_n^q(D', D_n + E) & \xrightarrow[\simeq]{C^{-1}} & \mathcal{H}^q(F_{X*} \Omega_{n-1}^\bullet(D', p(D_n + E))) \end{array}$$

Note that the injectivity of the vertical map on the right follows from the injectivity of the vertical map on the left.

**6.9.** For a complex  $C^\bullet$  of  $\mathcal{O}_X$ -modules on a scheme  $X$ , we denote

$$\mathcal{Z}^q(C^\bullet) = \text{Ker}(d : C^q \rightarrow C^{q+1}), \quad \text{and} \quad \mathcal{B}^q(C^\bullet) = \text{Im}(d : C^{q-1} \rightarrow C^q),$$

which we view as  $\mathcal{O}_X$ -modules.

**6.10.** Let  $X$ ,  $D$ , and  $E$  be as in 6.8 above. The Cartier isomorphism defines the following exact sequence of locally free  $\mathcal{O}_X$ -modules

$$(6.10.1) \quad 0 \rightarrow \mathcal{B}^q(F_{X*} \Omega_{n-1}^\bullet(D, pE)) \rightarrow \mathcal{Z}^q(F_{X*} \Omega_{n-1}^\bullet(D, pE)) \xrightarrow{C} \Omega_n^q(D, E) \rightarrow 0.$$

The fact that these modules are locally free follows via descending induction on  $q$  by considering as well the exact sequence

$$0 \rightarrow \mathcal{Z}^q \rightarrow F_{X*} \Omega_{n-1}^q(D, pE) \xrightarrow{d} \mathcal{B}^{q+1} \rightarrow 0.$$

We obtain a class in the Ext-group which we denote by the symbol

$$\epsilon_n^q(D, E) \in \text{Ext}^1(\Omega_n^q(D, E), \mathcal{B}^q(F_{X*}\Omega_{n-1}^\bullet(D, pE))).$$

**6.11.** Let  $X$ ,  $D$ , and  $E$  be as in 6.8 above. Let  $n \geq 0$  and let

$$D = D_0 + pD_1 + \dots + p^n D_n$$

be a  $p$ -divisibility decomposition. Set

$$(6.11.1) \quad \underline{D}_j := D_0 + \dots + p^j D_j, \quad j = 0, \dots, n,$$

and

$$(6.11.2) \quad \underline{D}^0 := 0 \quad \text{and} \quad \underline{D}^j := pD_{n-j+1} + p^2 D_{n-j+2} + \dots + p^j D_n, \quad j = 1, \dots, n.$$

Thus we have  $p \mid \underline{D}^j$ , for all  $j$ , and

$$D = \underline{D}_{n-j} + p^{n-j+1} \left( \frac{1}{p} \underline{D}^j \right)$$

is a  $p$ -divisibility decomposition in the sense of 6.1. By abuse of notation we will also say that  $D = \underline{D}_{n-j} + p^{n-j} \underline{D}^j$  is a  $p$ -divisibility decomposition, keeping in mind that there is an extra  $p$  hidden in  $\underline{D}^j$ . For  $j = 1, \dots, n$ , we will use the following notations repeatedly

$$\Omega_{j,n}^q := F_{X*}^j \Omega_{n-j}^q(\underline{D}_{n-j}, \underline{D}^j + p^j E), \quad \mathcal{B}_{j,n}^q := \mathcal{B}^q(\Omega_{j,n}^\bullet), \quad \mathcal{Z}_{j,n}^q := \mathcal{Z}^q(\Omega_{j,n}^\bullet).$$

We define  $\mathcal{O}_X$ -submodules

$$B_{j,n}^q(D, E) \subset Z_{j,n}^q(D, E) \subset \Omega_{j,n}^q$$

by setting

$$B_{0,n}^q(D, E) := 0 \subset \Omega_{0,n}^q := \Omega_n^q(\underline{D}_n, \underline{D}^0 + E) =: Z_{0,n}^q(D, E),$$

and for  $j \geq 1$  recursively by the condition that the squares on the right in the following two diagrams are pullback squares

$$(6.11.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{B}_{j,n}^q & \longrightarrow & B_{j,n}^q(D, E) & \longrightarrow & B_{j-1,n}^q(D, E) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}_{j,n}^q & \longrightarrow & Z_{j,n}^q(D, E) & \longrightarrow & Z_{j-1,n}^q(D, E) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}_{j,n}^q & \longrightarrow & \mathcal{Z}_{j,n}^q & \xrightarrow{C} & F_{X*}^{j-1} \Omega_{n-j+1}^q(\underline{D}_{n-j}, D_{n-j+1} + \underline{D}^{j-1} + p^{j-1} E) \longrightarrow 0, \end{array}$$

where the vertical inclusion in the lower right corner is induced by the natural inclusion

$$\Omega_{j-1,n}^q \hookrightarrow F_{X*}^{j-1} \Omega_{n-j+1}^q(\underline{D}_{n-j}, D_{n-j+1} + \underline{D}^{j-1} + p^{j-1} E).$$

Phrased differently  $B_{j,n}^q(D, E)$ ,  $Z_{j,n}^q(D, E)$  are obtained by pulling back the exact sequence

$$F_{X*}^{j-1} \epsilon_{n-j+1}^q(\underline{D}_{n-j}, D_{n-j+1} + \underline{D}^{j-1} + p^{j-1} E),$$

see (6.10.1), along the respective inclusions on the right in the diagram (6.11.3).

Some consequences from this definition:

(a) We have for  $n \geq 1$

$$B_{1,n}^q(D, E) = \mathcal{B}_{1,n}^q = \mathcal{B}^q(F_{X*}\Omega_{n-1}^\bullet(\underline{D}_{n-1}, \underline{D}^1 + pE))$$

and

$$Z_{1,n}^q(D, E) = \mathcal{Z}^q(F_{X*}\Omega_{n-1}^\bullet(D, pE)) + B_{1,n}^q(D, E).$$

This follows directly from the definition and (6.8.5).

(b) The sheaves  $B_{j,n}^q(D, E)$  and  $Z_{j,n}^q(D, E)$  are locally free coherent  $\mathcal{O}_X$ -modules, which we can identify with submodules of  $\sigma_* F_{X*}^j \Omega_U^q$ , where  $\sigma : U = X \setminus D + E \hookrightarrow X$  is the open immersion. Furthermore the  $j$ -fold iteration of the inverse Cartier operator induces an isomorphism

$$\Omega_n^q(D, E) = \frac{Z_{0,n}^q(D, E)}{B_{0,n}^q(D, E)} \xrightarrow{\simeq} \frac{Z_{j,n}^q(D, E)}{B_{j,n}^q(D, E)}.$$

(c) For  $j \geq 1$ , there are inclusions of  $\mathcal{O}_X$ -submodules of  $\Omega_{j,n}^q$

$$F_{X*}B_{j-1,n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE) \subset B_{j,n}^q(D, E)$$

and

$$Z_{j,n}^q(D, E) \subset F_{X*}Z_{j-1,n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE).$$

This follows from the inclusions in the case  $j = 1$  and the fact that for  $j \geq 2$  the bottom sequence in (6.11.3) constructed from  $D, E, n, j$  is equal to  $F_{X*}$  applied to the same sequence constructed from  $\underline{D}_{n-1}, \underline{D}^1 + pE, n-1, j-1$ . Furthermore, the  $(j-1)$ -fold iteration of the inverse Cartier operator induces an isomorphism

$$B_{1,n}^q(D, E) = \frac{B_{1,n}^q(D, E)}{F_{X*}B_{0,n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE)} \xrightarrow{\simeq} \frac{B_{j,n}^q(D, E)}{F_{X*}B_{j-1,n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE)}$$

as follows from the above and the snake lemma.

**Remark 6.12.** Classically one defines  $B_j\Omega_X^q \subset Z_j\Omega_X^q \subset F_{X*}^j\Omega_X^q$  as above in the case  $D = E = 0$ . More precisely, we have  $B_j\Omega_X^q = B_{j,n}^q(0, 0)$ , for all  $n \geq j$ , and similarly for  $Z_j\Omega_X^q$ . In this case the quotients  $F_{X*}^j\Omega_X^q/Z_j\Omega_X^q$  are locally free as well, e.g. [Ill79, I, Corollaire 3.9]. We warn the reader that this is in general not the case for

$$(6.12.1) \quad \Omega_{j,n}^q/Z_{j,n}^q(D, E).$$

For example, for  $j = 1$  it follows from (6.11.3) that we have an exact sequence

$$0 \rightarrow \frac{\Omega_n^q(\underline{D}_{n-1}, D_n + E)}{\Omega_n^q(D, E)} \rightarrow \frac{F_{X*}\Omega_{n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE)}{Z_{1,n}^q(D, E)} \rightarrow \frac{F_{X*}\Omega_{n-1}^q(\underline{D}_{n-1}, \underline{D}^1 + pE)}{Z_{1,n}^q} \rightarrow 0.$$

It is easy to see that the quotient on the right is locally free, but the quotient on the left is non-zero if  $D_n \neq 0$  and  $q < \dim X$  and is annihilated by  $\mathcal{O}_X(-D_n)$  and hence is not locally free. Thus the quotient in the middle is not locally free as well.

**Lemma 6.13.** *Let  $L \in \Phi$  (see Notation 2.2). Then for  $q, r, n \geq 0$*

$$R^n(\text{Fil}_r^p W_{n+1}\Omega_L^q) = \begin{cases} \Omega_{\mathcal{O}_L}^q(\log) \cdot \mathfrak{m}_L^{-[r/p^n]+1}, & \text{if } v_p(r) \leq n, \\ \Omega_{\mathcal{O}_L}^q \cdot \mathfrak{m}_L^{-r/p^n}, & \text{if } v_p(r) \geq n+1, \end{cases}$$

where  $\text{Fil}_r^p W_{n+1}\Omega_L^q$  is the filtration from Definition 2.6 and  $R^n : W_{n+1}\Omega_L^q \rightarrow \Omega_L^q$  is the restriction.

*Proof.* By Definition of  $\text{Fil}^p$  we have

$$R^n(\text{Fil}_r^p W_{n+1} \Omega_L^q) = R^n(\text{fil}_r W_{n+1} \Omega_L^q) + dR^n(\text{fil}_r W_{n+1} \Omega_L^{q-1}).$$

Assume  $v_p(r) \leq n$ . In this case we have by Definition 2.4

$$R^n(\text{fil}_r W_{n+1} \Omega_L^q) = R^n(\text{fil}_{r-1}^{\log} W_{n+1}(L)) \cdot \text{dlog}(K_q^M(L)) = \mathfrak{m}_L^{\lceil -\frac{r-1}{p^n} \rceil} \cdot \Omega_{\mathcal{O}_L}^q(\log).$$

It is direct to check that  $\lceil -\frac{r-1}{p^n} \rceil = -\lceil \frac{r}{p^n} \rceil + 1$ , which yields the statement in this case. Assume  $v_p(r) \geq n+1$ . In this case we have by Definition 2.4 and the above

$$\begin{aligned} R^n(\text{fil}_r W_{n+1} \Omega_L^q) &= \mathfrak{m}_L^{\lceil -\frac{r-1}{p^n} \rceil} \cdot \Omega_{\mathcal{O}_L}^q(\log) + R^n(\text{fil}_r^{\log} W_{n+1}(L)) \cdot \text{dlog}(K_q^M(\mathcal{O}_L)) \\ &= \mathfrak{m}_L^{-\frac{r}{p^n}+1} \cdot \Omega_{\mathcal{O}_L}^q(\log) + \mathfrak{m}_L^{-\frac{r}{p^n}} \cdot \Omega_{\mathcal{O}_L}^q = \mathfrak{m}_L^{-\frac{r}{p^n}} \cdot \Omega_{\mathcal{O}_L}^q. \end{aligned}$$

As  $r/p^n$  is divisible by  $p$ , we have  $d(\mathfrak{m}_L^{-r/p^n} \cdot \Omega_{\mathcal{O}_L}^{q-1}) \subset \mathfrak{m}_L^{-r/p^n} \cdot \Omega_{\mathcal{O}_L}^q$ . Hence the statement.  $\square$

**Corollary 6.14.** *Let  $D = D' + p^{n+1}D_{n+1}$  be a  $p$ -divisibility decomposition. Then for  $q, n \geq 0$  we have with the notation from 6.8*

$$R^n(W_{n+1} \Omega_{(X,D)}^q) = \Omega_n^q(D', pD_{n+1}).$$

*Proof.* As  $\Omega_n^q(D', pD_{n+1})$  is locally free it suffices to show this “ $\subset$ ” inclusion for the Nisnevich stalks in the generic points of  $D$ . In this case the statement follows from Lemma 6.13 and the notation (6.8.1), see 6.2, 5.2, and Theorem 5.4. To show the other inclusion we can argue locally around the *closed* points of  $D$ . Let  $x \in D$  be a point and set  $A = \mathcal{O}_{X,x}$ . Then we find an étale map  $k[t_1, \dots, t_d] \rightarrow A$ , such that  $t_1, \dots, t_d$  form a regular sequence of parameters for  $A$  and on  $\text{Spec } A$  we have

$$D' = \text{Div}(t_1^{m_1} \dots t_r^{m_r}) \quad \text{and} \quad D_{n+1} = \text{Div}(t_{r+1}^{m_{r+1}} \dots t_s^{m_s}),$$

with  $0 \leq r \leq s \leq d$  and  $p^{n+1} \nmid m_i$ , for  $i = 1, \dots, r$  (with the convention that  $r = 0$  means that  $D' = \emptyset$  and  $r = s$  means that  $D_{n+1} = \emptyset$ ). Thus

$$D = \text{Div}(t_1^{m_1} \dots t_r^{m_r} \cdot t_{r+1}^{p^{n+1}m_{r+1}} \dots t_s^{p^{n+1}m_s}).$$

A basis of the free  $A$ -module  $\Omega_n^q(D', pD_{n+1})_x$  (Zariski stalk at  $x$ ) is given by

$$e_{I,J} = t_1^{-\lceil \frac{m_1}{p^n} \rceil + 1} \dots t_r^{-\lceil \frac{m_r}{p^n} \rceil + 1} t_{r+1}^{-pm_{r+1}} \dots t_s^{-pm_s} \cdot \text{dlog } \underline{t}_I \text{dlog}(\underline{1+t}_J),$$

where  $I$  and  $J$  run through the tuples  $I = (1 \leq i_1 < \dots < i_{q_1} \leq r)$  and  $J = (r+1 \leq j_1 < \dots < j_{q_2} \leq d)$  with  $q_1 + q_2 = q$  and where we use the notations

$$\underline{t}_I = \{t_{i_1}, \dots, t_{i_{q_1}}\} \in K_{q_1}^M(A[\frac{1}{t_{i_1} \dots t_{i_{q_1}}}] \quad \text{and} \quad \underline{1+t}_J = \{1+t_{j_1}, \dots, 1+t_{j_{q_2}}\} \in K_{q_2}^M(A).$$

As the Zariski stalk  $W_{n+1} \Omega_{(X,D),x}^q$  is a  $W_{n+1}(A)$ -module and  $R^n(W_{n+1}(A)) = A$ , it suffices to show that there exists an element  $\tilde{e}_{I,J} \in W_{n+1} \Omega_{(X,D),x}^q$ , for each  $I, J$ , with  $R^n(\tilde{e}_{I,J}) = e_{I,J}$ . By the second part of Proposition 6.3 it is direct to check from the definition of  $\text{Fil}_r^p W_n \Omega_L^q$  that the element

$$\tilde{e}_{I,J} := [t_1]^{-\lceil \frac{m_1}{p^n} \rceil + 1} \dots [t_r]^{-\lceil \frac{m_r}{p^n} \rceil + 1} [t_{r+1}]^{-pm_{r+1}} \dots [t_s]^{-pm_s} \cdot \text{dlog } \underline{t}_I \text{dlog}(\underline{1+t}_J),$$

which a priori is an element in  $W_{n+1} \Omega_A^q[\frac{1}{t_1 \dots t_s}]$ , has the looked for properties.  $\square$

**Proposition 6.15.** *Let  $D = D' + p^{n+1}D_{n+1}$  be a  $p$ -divisibility decomposition. Then with the notation from (6.4.1) and 6.11 we have for  $n \geq 0$*

$$B_n \Omega_{(X,D)}^q = B_{n,n}^q(D', pD_{n+1}), \quad \text{and} \quad Z_n \Omega_{(X,D)}^q = Z_{n,n}^q(D', pD_{n+1}).$$

In particular,  $B_n \Omega_{(X,D)}^q$  and  $Z_n \Omega_{(X,D)}^q$  are locally free coherent  $\mathcal{O}_X$ -modules.

*Proof.* For  $n = 0$  the statement follows from the definitions and (6.2.4). Assume now  $n \geq 1$ . Let  $D' = D'' + p^n D_n$  be a  $p$ -divisibility decomposition. By induction we have

$$(6.15.1) \quad Z_{n-1} \Omega_{(X,D)}^q = Z_{n-1,n-1}^q(D'', pE), \quad \text{where } E = D_n + pD_{n+1}.$$

and similarly with  $Z$  replaced by  $B$ . Consider the following diagram (in which we neglect the  $\mathcal{O}_X$ -module structure)

$$(6.15.2) \quad \begin{array}{ccccc} W_n \Omega_{(X,D)}^{q-1} & \xrightarrow{d} & d(W_n \Omega_{(X,D)}^{q-1}) & \xrightarrow{F^{n-1}} & B_n \Omega_{(X,D)}^q \subset Z_{n-1} \Omega_{(X,D)}^q \\ \downarrow R^{n-1} & & \downarrow R^{n-1} & & \searrow \\ & & & & \frac{Z_{n-1,n-1}^q(D'', pE)}{B_{n-1,n-1}^q(D'', pE)} \\ \Omega_{n-1}^{q-1}(D'', pE) & \xrightarrow{d} & B_{1,n}^q(D', pD_{n+1}) & \xrightarrow{C^{-(n-1)}} & \frac{B_{n,n}^q(D', pD_{n+1})}{B_{n-1,n-1}^q(D'', pE)} \end{array}$$

Here the map  $d$  on the lower left is surjective by 6.11(a) and the vertical map  $R^{n-1}$  on the left hand side is surjective by Corollary 6.14; clearly the square commutes and hence also the vertical  $R^{n-1}$  in the middle is surjective. The lower horizontal map  $C^{-(n-1)}$  is induced by the  $(n-1)$ -fold iteration of the inverse Cartier operator; it is an isomorphism, see 6.11(c). The upwards diagonal inclusion on the right hand side is induced by the natural inclusions in 6.11(c), the downwards diagonal map is induced by (6.15.1). The inclusion on the top right is given by

$$(6.15.3) \quad B_n \Omega_{(X,D)}^q = F^{n-1} d(W_n \Omega_{(X,D)}^{q-1}) = F^n dV(W_n \Omega_{(X,D)}^{q-1}) \subset F^n (W_{n+1} \Omega_{(X,D)}^q) = Z_n \Omega_{(X,D)}^q.$$

To see that the right part of the diagram commutes observe that by 6.11(b)

$$\frac{Z_{n-1,n-1}^q(D'', pE)}{B_{n-1,n-1}^q(D'', pE)} \cong \Omega_{n-1}^q(D'', pE)$$

is a locally free  $\mathcal{O}_X$ -module; hence it suffices to check the commutativity after restricting to  $U = X \setminus D$  in which case it follows from the classical fact that the Frobenius on the de Rham-Witt complex is a lift of the inverse Cartier operator, see [Ill79, I, Proposition (3.3)]. Thus the whole diagram (6.15.2) commutes and all the maps in the diagram are defined and are surjections, injections, or isomorphisms as indicated. Furthermore we have the inclusions

$$B_{n-1,n-1}^q(D'', pE) = F^{n-2} d(W_{n-1} \Omega_{(X,D)}^{q-1}) = F^{n-1} dV(W_{n-1} \Omega_{(X,D)}^{q-1}) \subset B_n \Omega_{(X,D)}^q,$$

where the first equality holds by induction. Therefore diagram (6.15.2) yields

$$\frac{B_{n,n}^q(D', pD_{n+1})}{B_{n-1,n-1}^q(D'', pE)} = \frac{B_n \Omega_{(X,D)}^q}{B_{n-1,n-1}^q(D'', pE)}$$

and hence we get the equality of subsheaves of  $\Omega_{(X,D)}^q$

$$(6.15.4) \quad B_{n,n}^q(D', pD_{n+1}) = B_n \Omega_{(X,D)}^q.$$

Next we consider the diagram

$$(6.15.5) \quad \begin{array}{ccc} W_{n+1} \Omega_{(X,D)}^q & \xrightarrow{F^n} & Z_n \Omega_{(X,D)}^q \subset Z_{n-1} \Omega_{(X,D)}^q \\ \downarrow R^n & & \searrow \\ \Omega_n^q(D', pD_{n+1}) & \xrightarrow[C^{-n}]{} & \frac{Z_{n-1,n-1}^q(D'', pE)}{B_{n,n}^q(D', pD_{n+1})} \end{array}$$

Here  $R^n$  on the left hand side is surjective by Corollary 6.14. The lower horizontal map  $C^{-n}$  is an isomorphism by 6.11(b). The upward diagonal injection is induced by 6.11(c), the downward diagonal map is induced by (6.15.1). We claim this diagram commutes. As above it suffices to show that the quotient on the right is locally free as the restriction of the diagram to  $U = X \setminus D$  is known to commute. To this end consider the isomorphisms

$$\frac{Z_{n-1,n-1}^q(D'', pE)}{B_{n,n}^q(D', pD_{n+1})} \cong \frac{Z_{n-1,n-1}^q(D'', pE)/B_{n-1,n-1}(D'', pE)}{B_{n,n}^q(D', pD_{n+1})/B_{n-1,n-1}(D'', pE)} \cong \frac{\Omega_{n-1}^q(D'', pE)}{\mathcal{B}^q(\Omega_{n-1}^\bullet(D'', pE))},$$

where the isomorphisms hold by 6.11(a)–(c). Moreover we have the exact sequence

$$0 \rightarrow \Omega_n^q(D'', E) \xrightarrow{C^{-1}} \frac{F_{X*} \Omega_{n-1}^q(D'', pE)}{\mathcal{B}^q(\Omega_{n-1}^\bullet(D'', pE))} \xrightarrow{d} \mathcal{B}^{q+1}(\Omega_{n-1}^\bullet(D'', pE)) \rightarrow 0,$$

see (6.8.4). Since the two outer terms of this exact sequence are locally free so is the middle term. Hence the quotient on the right in diagram (6.15.5) is locally free and the diagram commutes. Using the equality (6.15.4) and the inclusion (6.15.3), the commutativity of (6.15.5) yields

$$\frac{Z_n \Omega_{(X,D)}^q}{B_n \Omega_{(X,D)}^q} = \frac{Z_{n,n}(D', pD_{n+1})}{B_n \Omega_{(X,D)}^q}.$$

Hence we get the equality

$$Z_n \Omega_{(X,D)}^q = Z_{n,n}(D', pD_{n+1})$$

of subsheaves of  $\Omega_{(X,D)}^q$ , which yields the statement.  $\square$

**Lemma 6.16.** *Let  $j : U = X \setminus D \hookrightarrow X$  be the open immersion. With the notation from section 5 we have*

$$(6.16.1) \quad B_n \Omega_{(X,D)}^q = (\underline{\omega}^{\mathbf{CI}} B_n \Omega^q)_{(X,D)} = (B_n \Omega^q)_{(X,D)}^{\text{AS}} = j_*(B_n \Omega_U^q) \cap \Omega_{(X,D)}^q, \quad \text{for all } q \geq 0,$$

and

$$(6.16.2) \quad Z_n \Omega_{(X,D)}^q = (\underline{\omega}^{\mathbf{CI}} Z_n \Omega^q)_{(X,D)} = (Z_n \Omega^q)_{(X,D)}^{\text{AS}} = j_*(Z_n \Omega_U^q) \cap \Omega_{(X,D)}^q, \quad \text{for all } q \geq 1,$$

and

$$(6.16.3) \quad Z_n \Omega_{(X,D)}^0 = j_*(Z_n \Omega_U^0) \cap \Omega_{(X,D)}^0 = j_* F^n(W_{n+1} \mathcal{O}_U) \cap \mathcal{O}_{(X,D)}.$$

*Proof.* The second equality in (6.16.3) holds by definition. The first equality reduces to

$$F^n(\mathrm{fil}_r W_{n+1} \Omega_L^0) = F^n(W_{n+1}(L)) \cap \mathrm{fil}_r \Omega_L^0, \quad \text{for } L \in \Phi, r \geq 0,$$

which follows directly from Definition 2.4. As  $B_n \Omega^0 = 0$  we assume  $q \geq 1$  in the following. By the definition of  $B_n \Omega_{(X,D)}^q$  and  $Z_n \Omega_{(X,D)}^q$  in (6.4.1), by Theorem 5.4, and (5.3.2) we see that (6.16.1) and (6.16.2) hold with “=” replaced by “ $\subset$ ”. Thus it remains to show

$$j_*(B_n \Omega_U^q) \cap \Omega_{(X,D)}^q \subset B_n \Omega_{(X,D)}^q \quad \text{and} \quad j_*(Z_n \Omega_U^q) \cap \Omega_{(X,D)}^q \subset Z_n \Omega_{(X,D)}^q.$$

We first consider the  $B_n$  case. Set  $M := j_*(B_n \Omega_U^q) \cap \Omega_{(X,D)}^q$  and  $N := B_n \Omega_{(X,D)}^q$ . We note that  $M$  and  $N$  are  $\mathcal{O}_X$ -submodules of  $V := j_* B_n \Omega_U^q$ , with  $N$  locally free of finite rank, by Proposition 6.15, and with  $M$  coherent, as it is a submodule of the coherent module  $F_*^n \Omega_{(X,D)}^q$ . Moreover,  $M|_U = N|_U = V|_U$  is a finite locally free  $\mathcal{O}_U$ -module. By [Bou98, VII, §4, No. 2, Corollaire] it suffices therefore to show the inclusion of stalks  $M_\eta \subset N_\eta$ , for all generic points  $\eta$  of  $D_{\mathrm{red}}$ . As  $N_\eta^h \cap V_\eta = N_\eta$  we may consider Nisnevich stalks instead and are thus reduced to show the following claim for each henselian dvf  $L \in \Phi$ :

$$(6.16.4) \quad F^{n-1}d(\mathrm{Fil}_s^p W_n \Omega_L^{q-1}) \cap \mathrm{Fil}_r^p \Omega_L^q \subset F^{n-1}d(\mathrm{Fil}_r^p W_n \Omega_L^{q-1}), \quad \text{for all } s \geq r \geq 1.$$

Indeed, this holds trivially for  $s = r$ . Assume  $s > r$  and let  $\alpha \in \mathrm{Fil}_s^p W_n \Omega_L^{q-1}$  such that  $F^{n-1}d(\alpha) \in \mathrm{Fil}_r^p \Omega_L^q \subset \mathrm{Fil}_{s-1}^p \Omega_L^q$ . By Lemma 4.2 we find elements  $\beta \in W_{n+1} \Omega_L^q$ ,  $\alpha' \in \mathrm{Fil}_{s-1}^p W_n \Omega_L^q$ , and  $\alpha'' \in \mathrm{Fil}_s^p W_{n-1} \Omega_L^q$  such that  $\alpha = \alpha' + \underline{p}(\alpha'') + F(\beta)$ . Hence

$$F^{n-1}d(\alpha) = F^{n-1}d(\alpha') \in F^{n-1}d(\mathrm{Fil}_{s-1}^p W_n \Omega_L^q) \cap \mathrm{Fil}_r^p \Omega_L^q$$

and claim (6.16.4) holds by induction on  $s$ . This completes the proof of (6.16.1).

For  $Z_n$  we can argue similarly as above to reduce to the following claim for each  $L \in \Phi$ :

$$(6.16.5) \quad F^n(\mathrm{Fil}_s^p W_{n+1} \Omega_L^q) \cap \mathrm{Fil}_r^p \Omega_L^q \subset F^n(\mathrm{Fil}_r^p W_{n+1} \Omega_L^q).$$

Indeed, this holds trivially for  $s = r$ . We assume  $s > r$ . Let  $\alpha \in \mathrm{Fil}_s^p W_{n+1} \Omega_L^q$  such that  $F^n(\alpha) \in \mathrm{Fil}_r^p \Omega_L^q \subset \mathrm{Fil}_{s-1}^p \Omega_L^q$ . By Lemma 4.2 we find elements  $\beta \in W_n \Omega_L^q$ ,  $\alpha' \in \mathrm{Fil}_{s-1}^p W_{n+1} \Omega_L^q$  and  $\alpha'' \in \mathrm{Fil}_s^p W_n \Omega_L^q$  such that  $\alpha = \alpha' + \underline{p}(\alpha'') + V(\beta)$ . Hence

$$F^n(\alpha) = F^n(\alpha') \in F^n(\mathrm{Fil}_{s-1}^p W_{n+1} \Omega_L^q) \cap \mathrm{Fil}_r^p \Omega_L^q$$

and claim (6.16.5) holds by induction. This completes the proof of (6.16.2).  $\square$

**Lemma 6.17.** *For the statement of this lemma denote by  $F_{(X,D)}^n : W_{n+1} \Omega_{(X,D)}^q \rightarrow \Omega_{(X,D)}^q$  the map induced by the  $n$ th-power of the de Rham-Witt Frobenius  $F^n$  (which otherwise is denoted by  $F^n$  as well). Then we have the following equality of subsheaves of  $W_{n+1} \Omega_{(X,D)}^q$*

$$\mathrm{Ker}(F_{(X,D)}^n) \cap \mathrm{Ker}(F_{(X,D)}^n \circ d) = \underline{p}(j_* W_n \Omega_U^q) \cap W_{n+1} \Omega_{(X,D)}^q = \underline{p}(W_n \Omega_{(X,D)}^q),$$

where  $j : U = X \setminus D \hookrightarrow X$  is the open immersion.

*Proof.* Denote by  $F_U^n : j_* W_{n+1} \Omega_U^q \rightarrow j_* \Omega_U^q$  the Frobenius on  $U$ . As  $\Omega_{(X,D)}^q \subset j_* \Omega_U^q$  we have

$$\mathrm{Ker}(F_{(X,D)}^n) = \mathrm{Ker}(F_U^n) \cap W_{n+1} \Omega_{(X,D)}^q \quad \text{and} \quad \mathrm{Ker}(F_{(X,D)}^n d) = \mathrm{Ker}(F_U^n d) \cap W_{n+1} \Omega_{(X,D)}^q.$$

As  $\mathrm{Ker}(F_U^n) \cap \mathrm{Ker}(F_U^n d) = \underline{p}(j_* W_n \Omega_U^q)$  by [Eke84, (0.6.3)] the first equality follows.

For the second equality let  $a \in j_* W_n \Omega_U^q$  be a local section with  $\underline{p}(a) \in W_{n+1} \Omega_{(X,D)}^q$ . As  $\underline{p}$  is injective we have to show that  $a \in W_n \Omega_{(X,D)}^q$ . By Proposition 6.3 it suffices to show

$$(6.17.1) \quad \underline{p}(W_n \Omega_L^q) \cap \mathrm{Fil}_r^p W_{n+1} \Omega_L^q = \underline{p}(\mathrm{Fil}_r^p W_n \Omega_L^q),$$

for all  $q, r \geq 0$  and  $L \in \Phi$ . For  $r = 0$  this follows from Lemma 5.6 and Lemma 2.8. Assume  $r \geq 1$  and let  $a \in W_n \Omega_L^q$  with  $\underline{p}(a) \in \text{Fil}_r^p W_{n+1} \Omega_L^q$ . It follows from Lemma 4.1 that there exist elements  $b \in \text{Fil}_r^p W_n \Omega_L^q$  and  $c \in \text{Fil}_{r-1}^p W_{n+1} \Omega_L^q$  such that

$$\underline{p}(a) = c + \underline{p}(b).$$

By induction over  $r$  we find an element  $e \in \text{Fil}_{r-1}^p W_n \Omega_L^q$  such that  $c = \underline{p}(e)$ . The injectivity of  $\underline{p}$  therefore yields  $a = e + b \in \text{Fil}_r^p W_n \Omega_L^q$ . Hence the second equality holds.  $\square$

**Proposition 6.18.** *The sequence*

$$0 \longrightarrow B_n \Omega_{(X,D)}^{q+1} \longrightarrow \frac{W_{n+1} \Omega_{(X,D)}^q}{\underline{p} W_n \Omega_{(X,D)}^q} \xrightarrow{F^n} Z_n \Omega_{(X,D)}^q \longrightarrow 0$$

is a short exact sequence of  $W_{n+1} \mathcal{O}_X$ -modules, where the first map on the left is given by  $F^{n-1} d(\beta) \mapsto V(\beta)$ , and the  $W_{n+1} \mathcal{O}_X$ -module structure on  $Z_n$  and  $B_n$  is induced by  $F^n : W_{n+1} \mathcal{O}_X \rightarrow \mathcal{O}_X$ .

*Proof.* This is a modulus version of [Eke84, Lemma 0.6]. After the previous work the proof is analogous: consider the diagram (with the notation from Lemma 6.17)

$$(6.18.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(F_{(X,D)}^n) \cap \text{Ker}(F_{(X,D)}^n d) & \longrightarrow & W_{n+1} \Omega_{(X,D)}^q & \longrightarrow & \frac{W_{n+1} \Omega_{(X,D)}^q}{\underline{p} W_n \Omega_{(X,D)}^q} \longrightarrow 0 \\ & & \downarrow \iota & & \parallel & & \downarrow F^n \\ 0 & \longrightarrow & \text{Ker}(F_{(X,D)}^n) & \longrightarrow & W_{n+1} \Omega_{(X,D)}^q & \xrightarrow{F^n} & Z_n \Omega_{(X,D)}^q \longrightarrow 0. \end{array}$$

Here the top row is exact by Lemma 6.17, the bottom row is exact by definition, and  $\iota$  is the inclusion. By [III79, I, (3.11.3)] and the inclusion  $\Omega_{(X,D)}^q \subset j_* \Omega_U^q$  we have

$$\text{Ker}(F_{(X,D)}^n) = \text{Ker}(F_U^n) \cap W_{n+1} \Omega_{(X,D)}^q = j_*(V W_n \Omega_U^q) \cap W_{n+1} \Omega_{(X,D)}^q.$$

We have the following obvious chain of inclusions

$$B_n \Omega_{(X,D)}^{q+1} = F^n d V W_n \Omega_{(X,D)}^q \subset F^n d(j_*(V W_n \Omega_U^q) \cap W_{n+1} \Omega_{(X,D)}^q) \subset j_*(B_n \Omega_U^{q+1}) \cap \Omega_{(X,D)}^{q+1}$$

and by Lemma 6.16 we have equality everywhere. We obtain an exact sequence

$$0 \rightarrow \text{Ker}(F_{(X,D)}^n) \cap \text{Ker}(F_{(X,D)}^n d) \xrightarrow{\iota} \text{Ker}(F_{(X,D)}^n) \xrightarrow{F^n d} B_n \Omega_{(X,D)}^{q+1} \rightarrow 0.$$

Hence applying the Snake Lemma to (6.18.1) yields the exact sequence from the statement.  $\square$

The proof of Theorem 6.4 is now complete. The following proposition is a finite level version of the exact sequence [IR83, II, (1.2.2)] in the modulus setting.

**Proposition 6.19.** *The following sequence is exact for all  $r \geq 1$*

$$0 \rightarrow W_n \Omega_{(X,D)}^q \xrightarrow{\underline{p}^r} W_{n+r} \Omega_{(X,D)}^q \xrightarrow{(F^n, F^n d)} W_r \Omega_{(X,D)}^q \oplus W_r \Omega_{(X,D)}^{q+1} \xrightarrow{dV^n - V^n} W_{n+r} \Omega_{(X,D)}^{q+1}.$$

*Proof.* Denote by  $j : U = X \setminus D \hookrightarrow X$  the open immersion. We first consider the case  $r = 1$ . Thus we have to show that the sequence

$$0 \rightarrow W_n \Omega_{(X,D)}^q \xrightarrow{\underline{p}} W_{n+1} \Omega_{(X,D)}^q \xrightarrow{(F^n, F^n d)} \Omega_{(X,D)}^q \oplus \Omega_{(X,D)}^{q+1} \xrightarrow{dV^n - V^n} W_{n+1} \Omega_{(X,D)}^{q+1}$$

is exact. The injectivity of  $\underline{p}$  follows from the injectivity of  $\underline{p} : W_n \Omega_U^q \rightarrow W_{n+1} \Omega_U^q$ , see [III79, I, Proposition 3.4]. The exactness at  $W_{n+1} \Omega_{(X,D)}^q$  follows from Proposition 6.18 (use  $F^n dV(\beta) = F^{n-1} d\beta$ ). Let  $\alpha \in \Omega_{(X,D)}^q$  and  $\beta \in \Omega_{(X,D)}^{q+1}$  be local sections. It remains to show

$$(6.19.1) \quad dV^n(\alpha) = V^n(\beta) \implies \alpha = F^n(\gamma) \text{ and } \beta = F^n d(\gamma), \text{ for some } \gamma \in W_{n+1} \Omega_{(X,D)}^q.$$

By [III79, I, (3.10.3)] we find  $\gamma_0 \in W_{n+1} \Omega_U^q$  such that

$$dV^n(\alpha) = dV^n F^n(\gamma_0) = V^n(F^n d(\gamma_0)) = V^n(\beta).$$

By [III79, I, Remarques 3.21.1] we find  $\delta \in W_{n+2} \Omega_U^q$  and  $\epsilon \in W_n \Omega_U^q$  such that

$$\alpha = F^n(\gamma_0) + F^{n+1}(\delta) \quad \text{and} \quad \beta = F^n d(\gamma_0) + F^n dV(\epsilon).$$

Setting  $\gamma_1 := \gamma_0 + F(\delta) + V(\epsilon) \in W_{n+1} \Omega_U^q$  we have

$$\alpha = F^n(\gamma_1) \in \Omega_{(X,D)}^q \cap j_*(Z_n \Omega_U^q) \quad \text{and} \quad \beta = F^n d(\gamma_1) \in \Omega_{(X,D)}^q \cap j_*(B_{n+1} \Omega_U^q).$$

By Lemma 6.16 we find  $\gamma' \in W_{n+1} \Omega_{(X,D)}^q$  and  $\gamma'' \in W_{n+1} \Omega_{(X,D)}^q$  such that

$$\alpha = F^n(\gamma') \quad \text{and} \quad \beta = F^n d(\gamma'').$$

By [III79, I, Remarques 3.21.1] we find (new)  $\delta, \epsilon$  on  $U$  with

$$(6.19.2) \quad \gamma_1 = \gamma' + V(\delta) = \gamma'' + F(\epsilon).$$

Hence

$$F^n(\gamma' - \gamma'') = F^{n+1}(\epsilon) \in j_*(Z_{n+1} \Omega_U^q) \cap \Omega_{(X,D)}^q$$

and

$$F^n d(\gamma'' - \gamma') = F^{n-1} d(\delta) \in j_*(B_n \Omega^{q+1}) \cap \Omega_{(X,D)}^{q+1}.$$

By Lemma 6.16 and [III79, I, Remarques 3.21.1] we find  $\epsilon' \in W_{n+2} \Omega_{(X,D)}^q$ ,  $\delta' \in W_n \Omega_{(X,D)}^q$ , and  $\eta, \zeta \in W_{n+1} \Omega_U^q$  such that

$$F(\epsilon) = F(\epsilon') + p\zeta \quad \text{and} \quad V(\delta) = V(\delta') + p\eta.$$

Thus (6.19.2) yields

$$\gamma'' - \gamma' + F(\epsilon') - V(\delta') = p(\eta - \zeta) = \underline{p}(R(\eta - \zeta)) \in \underline{p}(j_* W_n \Omega_U^q) \cap W_{n+1} \Omega_{(X,D)}^q.$$

By Lemma 6.17 we find  $\theta \in W_n \Omega_{(X,D)}^q$  such that

$$\underline{p}\theta = \underline{p}(R(\eta - \zeta)).$$

Set

$$\gamma := \gamma' + V(\delta') + \underline{p}\theta = \gamma'' + F(\epsilon') \in W_{n+1} \Omega_{(X,D)}^q.$$

Then

$$\alpha = F^n(\gamma_1) = F^n(\gamma' + V(\delta')) = F^n(\gamma) \quad \text{and} \quad \beta = F^n d(\gamma_1) = F^n d(\gamma'' + F(\epsilon)) = F^n d(\gamma).$$

This shows (6.19.1) and completes the proof in the case  $r = 1$ .

Now assume  $r \geq 2$  and consider the following diagram in which we drop for readability the subscript  $(X, D)$  everywhere

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_n \Omega^q & \xrightarrow{\underline{p}^{r-1}} & W_{n+r-1} \Omega^q & \xrightarrow{(F^n, F^n d)} & W_{r-1} \Omega^q \oplus W_{r-1} \Omega^{q+1} & \xrightarrow{dV^n - V^n} & W_{n+r-1} \Omega^{q+1} \\
& \parallel & & \downarrow \underline{p} & & \downarrow \underline{p} & & \downarrow \underline{p} \\
0 & \longrightarrow & W_n \Omega^q & \xrightarrow{\underline{p}^r} & W_{n+r} \Omega^q & \xrightarrow{(F^n, F^n d)} & W_r \Omega^q \oplus W_r \Omega^{q+1} & \xrightarrow{dV^n - V^n} & W_{n+r} \Omega^{q+1} \\
& & & \downarrow F^{n+r-1} & & \downarrow (F^{r-1}, F^{r-1} d) & & \downarrow (F^{r-1}, F^{r-1} d) & \downarrow F^{n+r-1} \\
& & & \Omega^q \oplus \Omega^{q+1} & \xrightarrow{\varphi} & (\Omega^q \oplus \Omega^{q+1}) \oplus (\Omega^{q+1} \oplus \Omega^{q+2}) & \xrightarrow{\psi} & \Omega^{q+1} \oplus \Omega^{q+2} \\
& & & \downarrow dV^{n+r-1} - V^{n+r-1} & & & & & \\
& & & W_{n+r} \Omega^{q+1}, & & & & & 
\end{array}$$

where  $\varphi$  and  $\psi$  are given by

$$\varphi(a, c) = ((a, 0), (c, 0)) \quad \text{and} \quad \psi((a, b), (c, d)) = (b, -d).$$

We observe

- the diagram is commutative;
- the columns are exact by the case  $r = 1$ ;
- the first row is exact by induction over  $r$ ;
- the third row is split exact.

We want to show that the middle row is exact on  $(X, D)$ . Clearly it is a complex and  $\underline{p}^r$  is injective by the same argument as in the case  $r = 1$ . The exactness at  $W_{n+r} \Omega_{(X,D)}^q$  follows from an easy diagram chase. The exactness at  $W_r \Omega_{(X,D)}^q \oplus W_r \Omega_{(X,D)}^{q+1}$  can be checked directly by a diagram chase as well once we observed that for  $\alpha \in W_r \Omega_{(X,D)}^q$  and  $\beta \in W_r \Omega_{(X,D)}^{q+1}$  with  $dV^n(\alpha) = V^n(\beta)$  we have

$$dV^{n+r-1}(F^{r-1}(\alpha)) = p^{r-1} dV^n(\alpha) = p^{r-1} V^n(\beta) = V^{n+r-1}(F^{r-1}(\beta)).$$

This completes the proof.  $\square$

## 7. APPLICATIONS TO HODGE-WITT COHOMOLOGY WITH MODULUS

Before we proceed to study Hodge-Witt sheaves with zeros along  $D$  in section 8, we draw some consequences from the structural results in the previous section. The main result is Theorem 7.3, which makes it possible to apply general results on cube invariant sheaves with transfers to Hodge-Witt sheaves with modulus, see 7.4.

Throughout this section we assume  $X \in \mathbf{Sm}$  pure dimensional with  $\dim X = N$  and denote by  $D$  an effective Cartier divisor on  $X$  such that  $D_{\text{red}}$  is an SNCD.

**7.1.** Recall the category  $\mathbf{MCor}$  from [KMSY21a, Definition 1.3.1]: its objects are modulus pairs  $(Y, E)$  (see 3.1) and the morphisms are left proper admissible correspondences. A *modulus presheaf with transfers* is an additive contravariant functor from  $\mathbf{MCor}$  to abelian groups and the category of modulus presheaves with transfers is denoted by  $\mathbf{MPST}$ . If  $G \in \mathbf{MPST}$  and  $(Y, E)$  is a modulus pair, then the assignment

$$(\text{étale } Y\text{-schemes}) \ni (v : V \rightarrow Y) \mapsto G(V, v^* E),$$

defines a (Nisnevich) presheaf on  $Y$  denoted by  $G_{(Y,E)}$ . By definition  $G \in \mathbf{MPST}$  is a sheaf if  $G_{(Y,E)}$  is a Nisnevich sheaf on  $Y$ , for each modulus pair  $(Y, E)$ . The category of modulus sheaves with transfers is denoted by  $\mathbf{MNST}$ . By [KMSY21a, Theorem 2] there is a sheafification functor  $\mathbf{MPST} \rightarrow \mathbf{MNST}$ , which is an exact left adjoint to the inclusion  $\mathbf{MNST} \rightarrow \mathbf{MPST}$ . The full subcategory of  $\mathbf{MPST}$  with objects the  $(\mathbb{P}^1, \infty)$ -invariant modulus presheaves with transfers having semipurity and  $M$ -reciprocity is denoted by  $\mathbf{CI}^{\tau, \text{sp}}$ , see [Sai20, Definition 1.31]. We set  $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} = \mathbf{CI}^{\tau, \text{sp}} \cap \mathbf{MNST}$ . By [Sai20, Theorem 0.4] the sheafification restricts to  $\mathbf{CI}^{\tau, \text{sp}} \rightarrow \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ . Note that if  $F \in \mathbf{RSC}_{\text{Nis}}$ , then  $\widetilde{F}_c, \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ , see 5.1 and 5.2. In particular,

$$\widetilde{W_n \Omega^q}_c \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}},$$

see 5.2 and Theorem 5.4 for the notation (here  $c$  is the conductor defined by the filtration from Definition 2.6), and  $(\widetilde{W_n \Omega^q}_c)_{(X,D)}$  as defined above coincides with  $W_n \Omega^q_{(X,D)}$  as defined in 6.2.

The assignment  $(Y, E) \mapsto K_r^M(Y \setminus E)$  defines a modulus sheaf with transfers denoted by  $\underline{\omega}^* K_r^M$ . For  $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$  and  $r \geq 0$  we define

$$\gamma^r G := \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* K_r^M, G).$$

By, e.g., [BRS22, Corollary 4.5], we have  $\gamma^r G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$  and by [BRS22, Theorem 6.3] there is a canonical isomorphism for any  $s \geq r$

$$(7.1.1) \quad (\gamma^r G)_{(X,D)} \cong R^r \pi_* G_{(\mathbb{P}^s \times X, \pi^* D)},$$

where  $\pi : \mathbb{P}^s \times X \rightarrow X$  is the projection.

**Lemma 7.2.** *Let  $G' \rightarrow G \rightarrow G''$  be a sequence in  $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ . Assume that*

$$0 \rightarrow G'_{(\mathbb{P}^s \times X, \pi^* D)} \rightarrow G_{(\mathbb{P}^s \times X, \pi^* D)} \rightarrow G''_{(\mathbb{P}^s \times X, \pi^* D)} \rightarrow 0$$

*is an exact sequence of Nisnevich sheaves on  $\mathbb{P}^s \times X$  for all  $s \geq 0$ . Then the following is an exact sequence of Nisnevich sheaves on  $X$  for all  $r \geq 0$*

$$0 \rightarrow (\gamma^r G')_{(X,D)} \rightarrow (\gamma^r G)_{(X,D)} \rightarrow (\gamma^r G'')_{(X,D)} \rightarrow 0.$$

*Proof.* From (7.1.1) we get a long exact sequence for any  $s \geq 0$

$$0 \rightarrow G'_{(X,D)} \rightarrow G_{(X,D)} \rightarrow G''_{(X,D)} \rightarrow (\gamma^1 G')_{(X,D)} \rightarrow \dots \rightarrow (\gamma^s G)_{(X,D)} \rightarrow (\gamma^s G'')_{(X,D)}.$$

Thus it remains to show that  $(\gamma^r G)_{(X,D)} \rightarrow (\gamma^r G'')_{(X,D)}$  is surjective, for every  $r \leq s$ . By (7.1.1) this is equivalent to the surjectivity of

$$R^r \pi_* G_{(\mathbb{P}^r \times X, \pi^* D)} \rightarrow R^r \pi_* G''_{(\mathbb{P}^r \times X, \pi^* D)}.$$

This holds as  $R^{r+1} \pi_* G'_{(\mathbb{P}^r \times X, \pi^* D)} = 0$ , by the blow-up formula [BRS22, Theorem 6.3].  $\square$

For  $D = \emptyset$  the following theorem is [BRS22, Theorem 11.8].

**Theorem 7.3.** *Let  $\eta = \text{dlog } c_1(\mathcal{O}(1)) \in H^1(\mathbb{P}_X^r, W_n \Omega_{\mathbb{P}_X^r, \log}^1)$ , where  $c_1(\mathcal{O}(1)) \in H^1(\mathbb{P}_X^r, \mathbb{G}_m)$  is first chern class of  $\mathcal{O}(1)$ , and denote by  $\eta^r \in H^r(\mathbb{P}_X^r, W_n \Omega_{\mathbb{P}_X^r, \log}^r)$  the  $r$ -fold cup product of  $\eta$ . Then via (7.1.1) cupping with  $\eta^r$  induces an isomorphism*

$$- \cup \eta^r : W_n \Omega_{(X,D)}^{q-r} \xrightarrow{\cong} R^r \pi_* W_n \Omega_{(\mathbb{P}^r \times X, \pi^* D)}^q \cong (\gamma^r W_n \Omega^q)_{(X,D)}.$$

*Furthermore this isomorphism is compatible with  $F$ ,  $V$ ,  $R$ ,  $\underline{p}$ , and  $d$ .*

*Proof.* The last statement follows directly from the usual compatibilities of  $V$ ,  $F$ ,  $R$ ,  $\underline{p}$ , and  $d$  with the dlog-map and the naturality of the isomorphism (7.1.1). Set

$$\mathcal{X} = (X, D), \quad \text{and} \quad \mathcal{P}^r = (\mathbb{P}^r \times X, \pi^* D).$$

For  $n = 1$  observe that by (6.2.4) we have a Künneth formula

$$\Omega_{\mathcal{P}^r}^q = \bigoplus_{a+b=q} \pi^* \Omega_{\mathcal{X}}^a \otimes_{\mathcal{O}_{\mathcal{P}^r}} \rho^* \Omega_{\mathbb{P}^r}^b,$$

where  $\rho : \mathbb{P}^r \times X \rightarrow \mathbb{P}^r$  denotes the projection. Hence in this case the statement holds by the projection formula and the classical result for  $\mathbb{P}^r$ . For  $q > N + r$  both sides of the first isomorphism in the statement are zero. Set

$$G_n^q := \widetilde{W_n \Omega_c^q}, \quad H^q := \widetilde{\Omega_c^q} \oplus \widetilde{\Omega_c^{q+1}}.$$

We have  $G_n^q, H^q \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ . Denote by  $I'$  the presheaf image of  $(F^n, F^n d) : G_{n+1}^q \rightarrow H^q$  and by  $J'$  the presheaf image of  $dV^n - V^n : H^q \rightarrow G_{n+1}^{q+1}$ . It follows that  $I', J' \in \mathbf{CI}^{\tau, \text{sp}}$ , c.f. [Sai20, Lemma 1.32]. Thus by [Sai20, Theorem 0.4] the sheafifications  $I$  and  $J$  lie in  $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$  and are the sheaf-theoretic images of the morphisms  $(F^n, F^n d)$  and  $dV^n - V^n$ , respectively. By Proposition 6.19 we have short exact sequences for all  $s \geq 0$

$$0 \rightarrow G_{n, \mathcal{P}^s}^q \xrightarrow{\underline{p}} G_{n+1, \mathcal{P}^s}^q \rightarrow I_{\mathcal{P}^s} \rightarrow 0, \quad \text{and} \quad 0 \rightarrow I_{\mathcal{P}^s} \rightarrow H_{\mathcal{P}^s}^q \rightarrow J_{\mathcal{P}^s} \rightarrow 0.$$

Hence Lemma 7.2 gives a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\gamma^r G_n^q)_{\mathcal{X}} & \xrightarrow{\underline{p}} & (\gamma^r G_{n+1}^q)_{\mathcal{X}} & \xrightarrow{(F^n, F^n d)} & (\gamma^r H^q)_{\mathcal{X}} \xrightarrow{dV^n - V^n} (\gamma^r J)_{\mathcal{X}} \\ & & \uparrow \alpha_1 & & \uparrow \alpha_2 & & \uparrow \alpha_3 \\ & & 0 & \longrightarrow & G_{n, \mathcal{X}}^{q-r} & \xrightarrow{\underline{p}} & G_{n+1, \mathcal{X}}^{q-r} \xrightarrow{(F^n, F^n d)} H_{\mathcal{X}}^{q-r} \xrightarrow{dV^n - V^n} G_{n+1, \mathcal{X}}^{q+1-r} \\ & & & & & & \uparrow \alpha_4 \\ & & & & & & (\gamma^r G_{n+1}^{q+1})_{\mathcal{X}} \end{array}$$

where the upwards pointing vertical maps are all induced by cupping with  $\eta^r$  and (7.1.1). Now  $\alpha_1$  is an isomorphism by induction over  $n$ ,  $\alpha_3$  is an isomorphism by the case  $n = 1$ , and  $\alpha_4$  is an isomorphism by descending induction over  $q$ . Hence  $\alpha_2$  is an isomorphism as well.  $\square$

**7.4.** Thanks to Theorem 7.3 the projective bundle formula, the blow-up formula, and the Gysin triangle for general  $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$  from [BRS22] can be made more explicit for  $G = \widetilde{W_n \Omega_c^q} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ :

- (1) Let  $V$  be a locally free  $\mathcal{O}_X$ -module of rank  $r + 1$  and denote by  $\pi : \mathbb{P}(V) \rightarrow X$  the corresponding projective bundle. Then there is a canonical isomorphism in  $D(W_n \mathcal{O}_X)$

$$R\pi_* W_n \Omega_{(\mathbb{P}(V), \pi^* D)}^q \cong \bigoplus_{j=0}^r W_n \Omega_{(X, D)}^{q-j}[-j].$$

The morphism is induced by cupping with  $c_1(\mathcal{O}(1))^j \in \text{CH}^j(\mathbb{P}(V))$ , see [BRS22, Theorem 6.3]. In that theorem the isomorphism takes only place in the derived category of abelian sheaves but it is direct to check that in the case at hand it is  $W_n \mathcal{O}_X$ -linear. Furthermore it is compatible with  $F$ ,  $V$ ,  $R$ ,  $\underline{p}$ , and  $d$ . For  $D = \emptyset$  this isomorphism is due to Gros [Gro85, I, Theorem 4.1.11].

- (2) Let  $i : Z \hookrightarrow X$  be a closed immersion of a smooth closed subscheme of pure codimension  $r$  which intersects  $D$  transversally, i.e., the scheme-theoretic intersection  $Z \cap D_1 \cap \dots \cap D_j$  is either empty or smooth of codimension  $j$  in  $Z$ , for any number of irreducible components  $D_1, \dots, D_j$  of  $D_{\text{red}}$ . Let  $\rho : \tilde{X} \rightarrow X$  be the blow-up in  $Z$ . Then there is a canonical isomorphism in  $D(W_n \mathcal{O}_X)$

$$R\rho_* W_n \Omega_{(\tilde{X}, \rho^* D)}^q \cong W_n \Omega_{(X, D)}^q \oplus \bigoplus_{j=1}^{r-1} i_* W_n \Omega_{(Z, D|_Z)}^{q-j}[-j],$$

see [BRS22, Corollary 7.3]. For  $D = \emptyset$ , this isomorphism is due to Gros [Gro85, IV, Corollaire 1.1.11].

- (3) Let the notation be as in (2) and denote by  $E = \rho^{-1}(Z)$  the exceptional divisor. Then there is a canonical distinguished triangle in  $D(W_n \mathcal{O}_X)$  called the Gysin triangle

$$i_* W_n \Omega_{(Z, D|_Z)}^{q-r}[-r] \xrightarrow{g} W_n \Omega_{(X, D)}^q \xrightarrow{\rho^*} R\rho_* W_n \Omega_{(\tilde{X}, \rho^* D + E)}^q \xrightarrow{\partial} i_* W_n \Omega_{(Z, D|_Z)}^{q-r}[-r+1],$$

where  $g = g_{(X, D)/(Z, D|_Z)}$  denotes the Gysin morphism, see [BRS22, Theorem 7.16]. For  $D = \emptyset$ , this triangle is also spelled out in [BRS22, Corollary 11.10(2)].

The Gysin triangle gives the following Lefschetz type theorem.

**Theorem 7.5.** *Assume that  $X$  is additionally projective and let  $H \subset X$  be a smooth hyper-surface section which intersects  $D$  transversally and satisfies*

$$H^j(X, \Omega_{(X, D)}^N(H)) = 0, \quad \text{for all } j \geq 1,$$

where  $\Omega_{(X, D)}^N(H) = \Omega_{(X, D)}^N \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)$ . Then the Gysin map

$$H^{j-1}(H, W_n \Omega_{(H, D|_H)}^{N-1}) \longrightarrow H^j(X, W_n \Omega_{(X, D)}^N)$$

is an isomorphism for  $j \geq 2$  and is surjective for  $j = 1$ .

*Proof.* By the Gysin triangle from 7.4(3) we have to show the vanishing

$$(7.5.1) \quad H^j(X, W_n \Omega_{(X, D+H)}^N) = 0, \quad \text{for } j \geq 1.$$

As  $Z_n \Omega_X^N = \Omega_X^N$ , we have

$$Z_n \Omega_{(X, D+H)}^N = \Omega_{(X, D+H)}^N = \Omega_{(X, D)}^N(H),$$

where the second equality follows from  $\Omega_X^N(\log(D+H)) = \Omega_X^N(D_{\text{red}}+H)$  and the first equality holds by Lemma 6.16. Thus (7.5.1) follows by induction over  $n$  from Proposition 6.18 noting that  $B_n \Omega_{(X, D+H)}^{N+1} = 0$ .  $\square$

Finally we mention the following version of Serre type vanishing.

**Theorem 7.6.** *Let  $H \subset X$  be a smooth and ample divisor intersecting  $D$  transversally. Then for any large enough  $m$  we have*

$$H^j(X, W_n \Omega_{(X, D+p^{n+1}mH)}^q) = 0, \quad \text{for all } j \geq 1.$$

*Proof.* With the notation from 6.8 and 6.11 we have

$$\Omega_n^q(D', p(D_{n+1} + mE)) = \Omega_n^q(D', pD_{n+1}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(pmE),$$

where  $D = D' + p^{n+1}D_{n+1}$  is a  $p$ -divisibility decomposition, and from (6.11.3) we obtain inductively

$$B_{j,n}^q(D', p(D_{n+1} + mE)) = B_{j,n}^q(D', pD_{n+1}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(pmE),$$

for all  $j = 0, \dots, n$ , and similarly for  $B$  replaced by  $Z$ . Thus by Serre vanishing the cohomology of the sheaves  $B_{j,n}^q(D', p(D_{n+1} + mE))$  and  $Z_{j,n}^q(D', p(D_{n+1} + mE))$  vanishes in positive degrees, for all  $j$  and all  $q$  and for any large enough  $m$ . Hence the statement follows from Proposition 6.18 together with Proposition 6.15.  $\square$

## 8. THE STRUCTURE OF HODGE-WITT SHEAVES WITH ZEROS

By definition a Hodge-Witt form has modulus  $(X, D)$  if it is a regular form on  $X \setminus D$  with certain pole constraints along  $D$ . In this section we introduce and study a notion of forms vanishing along  $D$ . We will see in the next section that these two notions of poles and zeros match up under duality. The main result in this section is Theorem 8.7, which gives precise information about the structure of Hodge-Witt sheaves with zeros along  $D$ .

Throughout this section we assume  $X \in \mathbf{Sm}$  and we let  $D$  be an effective Cartier divisor on  $X$  such that  $D_{\text{red}}$  is an SNCD. We remind the reader of our convention  $\log(D) = \log(D_{\text{red}})$ .

**Definition 8.1.** Suppose  $D = \sum_i D_i$  with  $D_{i,\text{red}}$  smooth. Define

$$W_n \Omega_{(X, -D)}^q := \text{Ker} \left( W_n \Omega_X^q \rightarrow \bigoplus_i W_n \Omega_{D_i}^q \right),$$

where we view  $D_i$  as a closed (in general non-reduced) subscheme of  $X$ . For  $n = 1$ , we write

$$\Omega_{(X, -D)}^q := W_1 \Omega_{(X, -D)}^q.$$

**Remark 8.2.** It follows from [Mok93, Lemme 3.15.1], see also [Nak05, Corollary 6.28], that the sheaf  $W_n \Omega_{(X, -D_{\text{red}})}^q$  coincides with  $W_n \Omega_X^q(-\log D_{\text{red}})$  defined in [Hyo88, 1.]. To our knowledge the sheaf  $W_n \Omega_{(X, -D)}^q$  was not considered before for a non-reduced divisor. Moreover it seems it cannot be defined or studied by the machinery introduced in [Nak05, 6.] as the Cartier isomorphism from [DI87, (4.2.1.3)] does not extend to this situation. For example if  $D = nD_0$  with  $D_0$  smooth connected, then the inverse Cartier operator induces an isomorphism (e.g. Lemma 6.6)

$$\Omega_X^q(\log D_0)(-nD_0) \xrightarrow{\sim} H^q(F_*(\Omega^\bullet(\log D_0)(-pn + (p-1)D_0)))$$

and only for  $n = 1$  the right hand side is isomorphic to  $H^q(F_*(\Omega_X^\bullet(\log D_0)(-nD_0)))$ .

**8.3.** Note that since the  $D_i$  may be non-reduced many of the structure results for the de Rham-Witt complex given, e.g., in [Ill79] or [IR83] cannot be applied to  $W_n \Omega_{D_i}^q$ . It is however direct from the definition that  $W_n \Omega_{(X, -D)}^q$  is a  $W_n \mathcal{O}_X$ -submodule of  $W_n \Omega_X^q$  and that the maps  $F$ ,  $V$ ,  $R$ ,  $\underline{p}$ ,  $d$  restrict to give maps

$$W_{n-1} \Omega_{(X, -D)}^q \xrightleftharpoons[R, F]{\underline{p}, V} W_n \Omega_{(X, -D)}^q \xrightarrow{d} W_n \Omega_{(X, -D)}^{q+1}.$$

In fact  $F$ ,  $V$ ,  $R$ , and  $d$  are defined on  $W \cdot \Omega_T^\bullet$  for all  $\mathbb{F}_p$ -schemes  $T$ , by [Ill79, I, Théorème 1.3 and Théorème 2.17]. For the definition of  $\underline{p}$  note, that  $K_{n, D_i} := \text{Ker}(W_n \Omega_X^* \rightarrow W_n \Omega_{D_i}^*)$  is the differential graded ideal of  $W_n \Omega_X^*$  generated by  $W_n(I_{D_i})$ , where  $I_{D_i} \subset \mathcal{O}_X$  is the ideal sheaf of the closed subscheme  $D_i \subset X$ , see [Hes04, Lemma 1.2.2]. As the restriction map  $W_{n+1}(I_{D_i}) \rightarrow W_n(I_{D_i})$  is surjective, it follows directly from the definition that  $\underline{p} : W_n \Omega_X^* \rightarrow W_{n+1} \Omega_X^*$  restricts to  $K_{n, D_i} \rightarrow K_{n+1, D_i}$  and hence also to  $W_n \Omega_{(X, -D)}^* = \cap_i K_{n, D_i} \rightarrow W_{n+1} \Omega_{(X, -D)}^*$ .

**Lemma 8.4.** Let  $D = D_0 + pD_1$  be a  $p$ -divisibility decomposition, see 6.1. Then

$$\Omega_{(X, -D)}^q = \Omega_X^q(\log D_0)(-D).$$

*Proof.* Let  $A$  be a smooth  $k$ -algebra and  $t \in A$  such that  $R = A/(t)$  is smooth as well. In this case there is an isomorphism  $A/(t^m) \cong R[t]/(t^m)$ , for all  $m \geq 1$ . We find

$$(8.4.1) \quad \Omega_{A/(t^m)}^q \cong \begin{cases} (R[t]/(t^m) \otimes_R \Omega_R^q) \oplus (R[t]/(t^{m-1}) \otimes_R \Omega_R^{q-1}) dt, & \text{if } (m, p) = 1 \\ (R[t]/(t^m) \otimes_R \Omega_R^q) \oplus (R[t]/(t^m) \otimes_R \Omega_R^{q-1}) dt, & \text{if } p|m. \end{cases}$$

Hence  $\Omega_{A/(t^m)}^q$  is a free  $R$ -module and the map  $\Omega_{A/(t^m)}^q \rightarrow \Omega_{A/(t^m)}^q \otimes_R \text{Frac}(R)$  is injective.

As  $\Omega_X^q(\log D_0)(-D)$  is locally free and clearly maps into  $\Omega_{(X, -D)}^q$ , we see from the above that it suffices to prove the equality around the generic points of  $D$ . In which case it follows directly from (8.4.1).  $\square$

**Proposition 8.5.** *Suppose  $X$  has pure dimension  $N$ . For all  $n \geq 0$  the map*

$$(8.5.1) \quad F_{X*}^n \Omega_{(X, -D)}^q \simeq F_{X*}^n \mathcal{H}om_{\mathcal{O}_X}(\Omega_{(X, D)}^{N-q}, \Omega_X^N) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(F_{X*}^n \Omega_{(X, D)}^{N-q}, \Omega_X^N),$$

given by

$$\alpha \mapsto (\beta \mapsto C^n(\alpha \wedge \beta)),$$

is an isomorphism of locally free  $\mathcal{O}_X$ -modules, where  $C^n : F_{X*}^n \Omega_X^N \rightarrow \Omega_X^N$  denotes the Cartier operator and  $\Omega_{(X, D)}^q$  is defined in 6.2.

*Proof.* Let  $D = D_0 + pD_1$  be a  $p$ -divisibility decomposition. In the case  $n = 0$  the isomorphism follows from twisting the isomorphism  $\Omega_X^q(\log D_0) \cong \mathcal{H}om(\Omega_X^{N-q}(\log D_0), \Omega_X^N(\log D_0))$  by  $\mathcal{O}_X(-D)$  together with the explicit formulas given in Lemma 8.4, in (6.2.4), and the isomorphism  $\Omega_X^N(\log D_0) \cong \Omega_X^N(D_{0, \text{red}})$ . Applying  $F_{X*}^n$  yields the first isomorphism in (8.5.1). The second map is an isomorphism as well. Indeed as  $X$  is smooth over a perfect field of characteristic  $p$  we have a natural isomorphism of  $\mathcal{O}_X$ -modules  $(F_X^n)^! \Omega_X^N \cong \Omega_X^N$  and the composition

$$(8.5.2) \quad F_{X*}^n \Omega_X^N \cong F_{X*}^n (F_X^n)^! \Omega_X^N \xrightarrow{\text{Tr}_{F_X^n}} \Omega_X^N,$$

where  $\text{Tr}_{F_X^n}$  is the counit of adjunction, is equal to the  $n$ -fold iteration of the Cartier operator  $C^n$ , e.g. [Eke84, II, Lemma 2.1]. Hence the second map is an isomorphism by adjunction.  $\square$

**8.6.** By [Ill79, I, Corollaire 3.9] we have an exact sequence of  $W_{n+1}\mathcal{O}_X$ -modules

$$(8.6.1) \quad 0 \longrightarrow \frac{F_{X*}^n \Omega_X^q}{B_n \Omega_X^q} \xrightarrow{V^n} \text{Ker} \left( W_{n+1} \Omega_X^q \xrightarrow{R} W_n \Omega_X^q \right) \xrightarrow{\beta} \frac{F_{X*}^n \Omega_X^{q-1}}{Z_n \Omega_X^{q-1}} \longrightarrow 0,$$

where the  $W_{n+1}\mathcal{O}_X$ -module structure on the two outer  $\mathcal{O}_X$ -modules is induced by  $R^n : W_{n+1}\mathcal{O}_X \rightarrow \mathcal{O}_X$  and the map  $\beta$  is given by  $\beta(V^n(a) + dV^n(b)) = b$ . Moreover we have an injection of  $W_{n+1}\mathcal{O}_X$ -modules

$$(8.6.2) \quad 0 \rightarrow \frac{F_{X*}^n \Omega_X^{q-1}}{Z_n \Omega_X^{q-1}} \xrightarrow{dV^{n-1}} F_* W_n \Omega_X^q.$$

For  $n = 0$  and  $q \geq 0$ , define the  $\mathcal{O}_X$ -modules

$$(\Omega/B)_{0, (X, -D)}^q = \Omega_{(X, -D)}^q \quad \text{and} \quad (\Omega/Z)_{0, (X, -D)}^{q-1} = 0.$$

For  $n \geq 1$ ,  $q \geq 0$  we define the  $W_{n+1}\mathcal{O}_X$ -modules  $(\Omega/B)_{n,(X,-D)}^q$  and  $(\Omega/Z)_{n,(X,-D)}^{q-1}$  by requiring that the following two diagrams are cartesian in the category of  $W_{n+1}\mathcal{O}_X$ -modules

$$(8.6.3) \quad \begin{array}{ccc} (\Omega/B)_{n,(X,-D)}^q & \longrightarrow & W_{n+1}\Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ \frac{F_{X*}^n \Omega_X^q}{B_n \Omega_X^q} & \xrightarrow{V^n} & W_{n+1}\Omega_X^q, \end{array} \quad \begin{array}{ccc} (\Omega/Z)_{n,(X,-D)}^{q-1} & \longrightarrow & F_* W_n \Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ \frac{F_{X*}^n \Omega_X^{q-1}}{Z_n \Omega_X^{q-1}} & \xrightarrow{dV^{n-1}} & F_* W_n \Omega_X^q. \end{array}$$

Note that we can view  $(\Omega/B)_{n,(X,-D)}^q$  and  $(\Omega/Z)_{n,(X,-D)}^{q-1}$  as  $\mathcal{O}_X$ -submodules of  $F_{X*}^n \Omega_X^q / B_n \Omega_X^q$  and  $F_{X*}^n \Omega_X^{q-1} / Z_n \Omega_X^{q-1}$ , respectively. In the following we will denote the top horizontal maps in the two diagrams as well by  $V^n$  and  $dV^{n-1}$ , respectively.

Some consequences from this definition:

(a) It is direct from the definition that

$$(\Omega/B)_{1,(X,-D)}^q = \{a \in F_{X*} \Omega_X^q / \mathcal{B}^q(\Omega_X^\bullet) \mid V(a) \in W_2 \Omega_{(X,-D)}^q\},$$

$$(\Omega/Z)_{1,(X,-D)}^{q-1} = \frac{F_{X*} \Omega_{(X,-D)}^{q-1}}{Z_1 \Omega_X^{q-1} \cap \Omega_{(X,-D)}^{q-1}} = \frac{F_{X*} \Omega_{(X,-D)}^{q-1}}{\mathcal{Z}^{q-1}(\Omega_{(X,-D)}^\bullet)}.$$

(b) For any  $n \geq 1$ , the diagram

$$\begin{array}{ccccc} & & \xrightarrow{F d \circ V^n} & & \\ (\Omega/B)_{n,(X,-D)}^{q-1} & \dashrightarrow & (\Omega/Z)_{n,(X,-D)}^{q-1} & \longrightarrow & F_* W_n \Omega_{(X,-D)}^q \\ \downarrow & & \downarrow & & \downarrow \\ \frac{F_{X*}^n \Omega_X^{q-1}}{B_n \Omega_X^{q-1}} & \dashrightarrow & \frac{F_{X*}^n \Omega_X^{q-1}}{Z_n \Omega_X^{q-1}} & \xrightarrow{dV^{n-1}} & F_* W_n \Omega_X^q. \end{array}$$

shows that the surjection  $\Omega_X^q / B_n \Omega_X^q \rightarrow \Omega_X^q / Z_n \Omega_X^q$  induces a well-defined map

$$(8.6.4) \quad (\Omega/B)_{n,(X,-D)}^q \rightarrow (\Omega/Z)_{n,(X,-D)}^q.$$

(c) For  $n \geq 0$ , the diagram

$$\begin{array}{ccccc} & & \xrightarrow{V \circ V^n} & & \\ F_{X*}(\Omega/B)_{n,(X,-D)}^q & \dashrightarrow & (\Omega/B)_{n+1,(X,-D)}^q & \longrightarrow & W_{n+2} \Omega_{(X,-D)}^q \\ \downarrow & & \downarrow & & \downarrow \\ \frac{F_{X*}^{n+1} \Omega_X^q}{F_{X*} B_n \Omega_X^q} & \dashrightarrow & \frac{F_{X*}^{n+1} \Omega_X^q}{B_{n+1} \Omega_X^q} & \xrightarrow{V^{n+1}} & W_{n+2} \Omega_X^q, \end{array}$$

shows that the surjection  $\Omega_X^q / B_n \Omega_X^q \rightarrow \Omega_X^q / B_{n+1} \Omega_X^q$  induces a well-defined map

$$F_*(\Omega/B)_{n,(X,-D)}^{q-1} \rightarrow (\Omega/B)_{n+1,(X,-D)}^{q-1}.$$

Similarly, for  $n \geq 1$ , the diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{F \circ dV^n} & & \\
 (\Omega/Z)_{n+1,(X,-D)}^{q-1} & \cdots \twoheadrightarrow & F_{X*}(\Omega/Z)_{n,(X,-D)}^{q-1} & \longrightarrow & F_*^2 W_n \Omega_{(X,-D)}^q \\
 \downarrow & & \downarrow & & \downarrow \\
 \frac{F_{X*}^{n+1} \Omega_X^{q-1}}{Z_{n+1} \Omega_X^{q-1}} & \twoheadrightarrow & \frac{F_{X*}^{n+1} \Omega_X^{q-1}}{F_{X*} Z_n \Omega_X^{q-1}} & \xrightarrow{dV^{n-1}} & F_*^2 W_n \Omega_X^q
 \end{array}$$

shows that the surjection  $\Omega_X^q/Z_{n+1}\Omega_X^q \rightarrow \Omega_X^q/Z_n\Omega_X^q$  induces a well-defined map

$$(8.6.5) \quad (\Omega/Z)_{n+1,(X,-D)}^{q-1} \rightarrow F_*(\Omega/Z)_{n,(X,-D)}^{q-1}.$$

Here we have used the fact that  $F_* : (W_{n+1}\mathcal{O}_X\text{-mod}) \rightarrow (W_n\mathcal{O}_X\text{-mod})$  has a left adjoint and hence the diagrams in (8.6.3) stay cartesian after applying  $F_*$ .

The aim of this section is to prove the following theorem:

**Theorem 8.7.** *Let the notation be as above and assume  $X$  is of pure dimension  $N$ .*

(1) *There is an exact sequence of  $W_{n+1}\mathcal{O}_X$ -modules*

$$0 \longrightarrow (\Omega/B)_{n,(X,-D)}^q \xrightarrow{V^n} \text{Ker} \left( W_{n+1}\Omega_{(X,-D)}^q \xrightarrow{R} W_n\Omega_{(X,-D)}^q \right) \xrightarrow{\beta} (\Omega/Z)_{n,(X,-D)}^{q-1} \longrightarrow 0,$$

where the map  $\beta$  is induced from the map  $\beta$  in (8.6.1).

(2) *The restriction map  $R : W_{n+1}\Omega_{(X,-D)}^q \rightarrow W_n\Omega_{(X,-D)}^q$  is surjective.*

(3) *The isomorphism (8.5.1) induces isomorphisms of locally free coherent  $\mathcal{O}_X$ -modules*

$$(\Omega/B)_{n,(X,-D)}^q \xrightarrow{\cong} \mathcal{H}om(Z_n\Omega_{(X,D)}^{N-q}, \Omega_X^N), \quad (\Omega/Z)_{n,(X,-D)}^{q-1} \xrightarrow{\cong} \mathcal{H}om(B_n\Omega_{(X,D)}^{N-q+1}, \Omega_X^N),$$

where  $Z_n\Omega_{(X,D)}^{N-q}$  and  $B_n\Omega_{(X,D)}^{N-q+1}$  are defined in (6.4.1).

For  $D = \emptyset$  part (1) holds by [Ill79, I, Corollaire 3.9], part (2) holds by definition of the de Rham-Witt complex, and part (3) by [Eke84, II, Lemma 2.2.20]. It will take the rest of this section to prove this theorem. Part (1) is proven in Proposition 8.14 and part (2) in Lemma 8.15, finally, part (3) follows from Proposition 8.21 and Proposition 8.19 together with Proposition 6.15.

**8.8.** Let  $E$  be an effective Cartier divisor on  $X$  such that  $D_{\text{red}} + E_{\text{red}}$  is a reduced SNCD. We will use the following notation

$$(8.8.1) \quad \Omega_n^q(-D, -E) := \Omega_X^q(\log D)(-[D/p^n] - E), \quad n, q \geq 0.$$

This is a locally free coherent  $\mathcal{O}_X$ -module. Let  $D = D' + p^n D_n$  be a  $p$ -divisibility decomposition. Similar to 6.8, we observe

- (a)  $\Omega_n^q(-D', -D_n - E) \subset \Omega_n^q(-D, -E)$ . This inclusion is strict if  $D_n \neq 0$ .
- (b) There is a well-defined differential map  $d : \Omega_n^q(-D, -pE) \rightarrow \Omega_{n+1}^{q+1}(-D, -pE)$  induced from the differential map of  $\Omega_X^\bullet$ . Moreover we have an inclusion of complexes

$$\Omega_{n-1}^\bullet(-D', -pD_n - pE) \subset \Omega_{n-1}^\bullet(-D, -pE).$$

- (c) The natural map

$$\Omega_X^\bullet(\log D)(-p[D/p^n] - pE) \rightarrow \Omega_{n-1}^\bullet(-D, -pE)$$

is a quasi-isomorphism by Lemma 6.6. Hence the inverse Cartier operator induces an isomorphism of  $\mathcal{O}_X$ -modules

$$(8.8.2) \quad C^{-1} : \Omega_n^q(-D, -E) \xrightarrow{\simeq} \mathcal{H}^q(F_{X*}\Omega_{n-1}^\bullet(-D, -pE)).$$

Moreover we obtain a commutative diagram of locally free  $\mathcal{O}_X$ -modules

$$(8.8.3) \quad \begin{array}{ccc} \Omega_n^q(-D', -D_n - E) & \xrightarrow[\simeq]{C^{-1}} & \mathcal{H}^q(F_{X*}\Omega_{n-1}^\bullet(-D', -pD_n - pE)) \\ \downarrow & & \downarrow \\ \Omega_n^q(-D, -E) & \xrightarrow[\simeq]{C^{-1}} & \mathcal{H}^q(F_{X*}\Omega_{n-1}^\bullet(-D, -pE)). \end{array}$$

As a consequence the vertical map on the right is injective as well.

**8.9.** Let  $X$ ,  $D$ , and  $E$  be as in 8.8 above. Let  $n \geq 1$ . The inverse Cartier isomorphism defines the following exact sequence of locally free  $\mathcal{O}_X$ -modules

$$(8.9.1) \quad 0 \rightarrow \Omega_n^q(-D, -E) \xrightarrow{C^{-1}} \frac{F_{X*}\Omega_{n-1}^q(-D, -pE)}{\mathcal{B}^q(F_{X*}\Omega_{n-1}^\bullet(-D, -pE))} \rightarrow \frac{F_{X*}\Omega_{n-1}^q(-D, -pE)}{\mathcal{Z}^q(F_{X*}\Omega_{n-1}^\bullet(-D, -pE))} \rightarrow 0.$$

The fact that these modules are locally free follows via descending induction on  $q$  by considering as well the exact sequence

$$0 \rightarrow \frac{F_{X*}\Omega_{n-1}^q(-D, -pE)}{\mathcal{Z}^q(F_{X*}\Omega_{n-1}^\bullet(-D, -pE))} \xrightarrow{d} F_{X*}\Omega_{n-1}^{q+1}(-D, -pE) \rightarrow \frac{F_{X*}\Omega_{n-1}^{q+1}(-D, -pE)}{\mathcal{B}^{q+1}(F_{X*}\Omega_{n-1}^\bullet(-D, -pE))} \rightarrow 0.$$

**Lemma 8.10.** *Let  $X$ ,  $D$ , and  $E$  be as above. Then for  $n \geq 0$*

$$\mathcal{B}^q(\Omega_X^\bullet) \cap \Omega_n^q(-D, -pE) = \mathcal{B}^q(\Omega_n^\bullet(-D, -pE)).$$

*In particular*

$$\mathcal{B}^q(\Omega_X^\bullet) \cap \Omega_{(X, -D)}^q = \mathcal{B}^q(\Omega_{(X, -D)}^\bullet).$$

*Proof.* The “in particular” part follows from the first statement and Lemma 8.4. For the first statement we only have to show this “ $\subset$ ” inclusion. As the  $\mathcal{O}_X$ -module  $\mathcal{B}^q(F_{X*}\Omega_n^\bullet(-D, -pE))$  is locally free by 8.9 it suffices to show the inclusion around each generic point of  $D$  or  $E$ , c.f. the argument in the proof of (6.16.1). Since the question is moreover local in the Nisnevich topology, we may assume that  $X = \operatorname{Spec} R[t]$ , with  $R$  a smooth  $k$ -algebra, and  $D$  or  $E$  is equal to  $\operatorname{Div}(t^r)$ ,  $r \geq 0$ . Thus  $\Omega_{R[t]}^q = \Omega_R^q[t] \oplus \Omega_R^{q-1}[t]dt$ , where for an  $R$ -module  $M$  we denote by  $M[t]$  the free  $R$ -module  $\bigoplus_{j \geq 0} Mt^j$ . In this situation  $\Omega_n^q(-D, -pE)$  has one of the following forms

- (a)  $t^s \Omega_R^q[t] \oplus t^{s-1} \Omega_R^{q-1}[t]dt$ , with  $s = \lceil r/p^n \rceil$ , when  $D = \operatorname{Div}(t^r)$ ;
- (b)  $t^{pr} \Omega_{R[t]}^q$ , when  $E = \operatorname{Div}(t^r)$ .

We consider the case (a). Let  $a = a_0 + \sum_{j \geq 1} (a_j t^j + b_j t^{j-1} dt) \in \Omega_{R[t]}^{q-1}$ , with  $a_j \in \Omega_R^{q-1}$  and  $b_j \in \Omega_R^{q-2}$ , and assume  $da \in t^s \Omega_R^q[t] \oplus t^{s-1} \Omega_R^{q-1}[t]dt$ . Set

$$a' := a_0 + \sum_{\substack{1 \leq j \leq s-1 \\ p|j}} (a_j t^j + b_j t^{j-1} dt) + \sum_{\substack{1 \leq j \leq s-1 \\ p \nmid j}} \frac{(-1)^{q-2}}{j} d(b_j t^j) \in \Omega_{R[t]}^{q-1}.$$

Then it is direct to check that

$$a - a' \in t^s \Omega_R^{q-1}[t] \oplus t^{s-1} \Omega_R^{q-2}[t]dt \quad \text{and} \quad da = d(a - a').$$

In case (b), take  $a \in \Omega_{R[t]}^{q-1}$  with  $da = t^{pr}b$ , with  $b \in \Omega_{R[t]}^q$ . Applying the Cartier operator yields  $0 = t^r C(b)$ . Thus  $C(b) = 0$ , thus  $b = dc$ , for some  $c \in \Omega_{R[t]}^{q-1}$ . Hence  $da = d(t^{pr}c)$ .

Both cases together imply the statement.  $\square$

**Lemma 8.11.** *Let  $D = D' + p^{n+1}D_{n+1}$  be a  $p$ -divisibility decomposition. For  $\alpha \in \Omega_X^q$  we have*

$$\underline{p}^n(\alpha) \in W_{n+1}\Omega_{(X,-D)}^q \iff \alpha \in \Omega_n^q(-D', -pD_{n+1}).$$

Moreover, the restriction map

$$R^j : W_{n+j+1}\Omega_{(X,-D)}^q \cap \underline{p}^n(W_{j+1}\Omega_X^q) \rightarrow W_{n+1}\Omega_{(X,-D)}^q \cap \underline{p}^n(\Omega_X^q)$$

is surjective.

*Proof.* We first show that for any germ  $\alpha \in \Omega_n^q(-D', -pD_{n+1})_x$ , with  $x \in D$  a closed point, we find a lift  $\tilde{\alpha} \in W_{j+1}\Omega_{X,x}^q$  such that  $\underline{p}^n(\tilde{\alpha}) \in W_{n+j+1}\Omega_{(X,-D),x}^q$ . Taking  $j = 0$  yields this “ $\Leftarrow$ ” direction and after the other direction will be proven below, it will also imply the “Moreover”-part. Let  $A = \mathcal{O}_{X,x}$ . We find an étale map  $k[t_1, \dots, t_d] \rightarrow A$ , such that  $t_1, \dots, t_d$  form a regular sequence of parameters for  $A$  and on  $\text{Spec } A$  we have

$$D' = \text{Div}(t_1^{m_1} \cdots t_r^{m_r}) \quad \text{and} \quad D_{n+1} = \text{Div}(t_{r+1}^{m_{r+1}} \cdots t_s^{m_s}),$$

with  $0 \leq r \leq s \leq d$  and  $p^{n+1} \nmid m_i$ , for  $i = 1, \dots, r$  (with the convention that  $r = 0$  means that  $D' = \emptyset$  and  $r = s$  means that  $D_{n+1} = \emptyset$ ). Thus

$$D = \text{Div}(t_1^{m_1} \cdots t_r^{m_r} \cdot t_{r+1}^{p^{n+1}m_{r+1}} \cdots t_s^{p^{n+1}m_s}).$$

A basis of the free  $A$ -module  $\Omega_n^q(-D', -pD_{n+1})_x = \Omega_X^q(\log D')(-\lceil D'/p^n \rceil - pD_{n+1})_x$  is given by

$$e_{I,J} = t_1^{\lceil \frac{m_1}{p^n} \rceil} \cdots t_r^{\lceil \frac{m_r}{p^n} \rceil} t_{r+1}^{pm_{r+1}} \cdots t_s^{pm_s} \cdot \text{dlog } \underline{t}_I \text{dlog}(\underline{1+t}_J),$$

where  $I$  and  $J$  run through the tuples  $I = (1 \leq i_1 < \dots < i_{q_1} \leq r)$  and  $J = (r+1 \leq j_1 < \dots < j_{q_2} \leq d)$  with  $q_1 + q_2 = q$  and where we use the notations

$$\underline{t}_I = \{t_{i_1}, \dots, t_{i_{q_1}}\} \in K_{q_1}^M(A[\frac{1}{t_{i_1} \cdots t_{i_{q_1}}}] \quad \text{and} \quad \underline{1+t}_J = \{1+t_{j_1}, \dots, 1+t_{j_{q_2}}\} \in K_{q_2}^M(A).$$

Set

$$\tilde{e}_{I,J} := [t_1]^{\lceil \frac{m_1}{p^n} \rceil} \cdots [t_r]^{\lceil \frac{m_r}{p^n} \rceil} [t_{r+1}]^{pm_{r+1}} \cdots [t_s]^{pm_s} \cdot \text{dlog } \underline{t}_I \text{dlog}(\underline{1+t}_J) \in W_{j+1}\Omega_A^q.$$

Note that  $\underline{p}^n : \iota_* W_{j+1}\Omega_X^q \rightarrow W_{n+j+1}\Omega_X^q$  is  $W_{n+j+1}\mathcal{O}_X$ -linear, where  $\iota : W_{j+1}X \rightarrow W_{n+j+1}X$  denotes the map of schemes induced by the restriction  $R^j : W_{n+j+1}\mathcal{O}_X \rightarrow W_{j+1}\mathcal{O}_X$ . Since  $W_{n+j+1}\Omega_{(X,-D),x}^q$  is a  $W_{n+j+1}A$ -submodule of  $W_{n+j+1}\Omega_{X,x}^q$ , it suffices to show that  $\underline{p}^n(\tilde{e}_{I,J}) \in W_{n+j+1}\Omega_{(X,-D),x}^q$ . Indeed, if  $\alpha = \sum_{I,J} a_{I,J} e_{I,J} \in \Omega_n^q(-D', -pD_{n+1})_x$ ,  $a_{I,J} \in A$ , is an arbitrary element, then we can choose any lifts  $\tilde{a}_{I,J} \in W_{j+1}(A)$  of  $a_{I,J}$ , and the element  $\tilde{\alpha} := \sum_{I,J} \tilde{a}_{I,J} \tilde{e}_{I,J}$  is a lift of  $\alpha$  and satisfies  $\underline{p}^n(\tilde{\alpha}) \in W_{n+j+1}\Omega_{(X,-D),x}^q$ .

Thus it remains to show that the element

$$\underline{p}^n(\tilde{e}_{I,J}) = p^n[t_1]^{\lceil \frac{m_1}{p^n} \rceil} \cdots [t_r]^{\lceil \frac{m_r}{p^n} \rceil} [t_{r+1}]^{pm_{r+1}} \cdots [t_s]^{pm_s} \cdot \text{dlog } \underline{t}_I \text{dlog}(\underline{1+t}_J) \in W_{n+j+1}\Omega_A^q$$

lies in  $W_{n+j+1}\Omega_{(X,-D),x}^q$ . To this end we have to show that  $\underline{p}^n(\tilde{e}_{I,J})$  vanishes in  $W_{n+j+1}\Omega_{A/t_i^m}^q$ , where  $m = m_i$ , if  $1 \leq i \leq r$ , and  $m = p^{n+1}m_i$ , if  $r+1 \leq i \leq s$ . We consider the following four cases:

- If  $r+1 \leq i \leq s$ , then

$$\underline{p}^n(\tilde{e}_{I,J}) = p^n[t_i^{pm_i}] \alpha = V^n([t_i]^{p^{n+1}m_i}) \cdot \alpha, \quad \text{for some } \alpha \in W_{n+j+1}\Omega_A^q.$$

- If  $1 \leq i \leq r$  and  $i \notin I$ , then  $p^n \lceil \frac{m_i}{p^n} \rceil = m_i + e$ , for some  $e \geq 0$ , and thus

$$\underline{p}^n(\tilde{e}_{I,J}) = p^n[t_i]^{\lceil \frac{m_i}{p^n} \rceil} \alpha = V^n([t_i]^{m_i+e}) \cdot \alpha, \quad \text{for some } \alpha \in W_{n+j+1}\Omega_A^q.$$

- If  $1 \leq i \leq r$ ,  $i \in I$ , and  $p^n \nmid m_i$ , then we have  $p^n \lceil m_i/p^n \rceil = m_i + e$ , for some  $e \geq 1$ , and thus

$$\underline{p}^n(\tilde{e}_{I,J}) = p^n[t_i]^{\lceil \frac{m_i}{p^n} \rceil} \text{dlog } t_i \cdot \beta = V^n([t_i]^{m_i}[t_i]^e \text{dlog } t_i) \cdot \beta, \quad \text{for some } \beta \in W_{n+j+1}\Omega_A^{q-1}.$$

- If  $1 \leq i \leq r$ ,  $i \in I$ , and  $p^n \mid m_i$ , then  $\mu = m_i/p^n$  is prime to  $p$  and we find

$$\underline{p}^n(\tilde{e}_{I,J}) = p^n[t_i]^\mu \text{dlog } t_i \cdot \beta = \frac{1}{\mu} dV^n([t_i]^{m_i}) \cdot \beta, \quad \text{for some } \beta \in W_{n+j+1}\Omega_A^{q-1}.$$

Therefore the vanishing of  $\underline{p}^n(\tilde{e}_{I,J})$  in  $W_{n+j+1}\Omega_{A/t_i^m}^q$  holds in all four cases.

Next we prove this “ $\Rightarrow$ ” direction. Since  $\Omega_n^q(-D', -pD_{n+1})$  is locally free on a smooth scheme, it suffices to show the statement around all generic points of  $D$ . Since the statement is furthermore Nisnevich local we may assume  $X = \text{Spec } R[t]$ , with  $R$  a smooth  $k$ -algebra, and  $D = \text{Div}(t^m)$ . Let  $R_{n+1}$  be a smooth lift of  $R$  over  $W_{n+1}(k)$ . By [IRS83, III, (1.5)] and [BER12, Proposition 8.4] there is a unique injective map

$$\tilde{F}^{n+1} : W_{n+1}\Omega_{R[t]}^q \rightarrow \Omega_{R_{n+1}[t]}^q / \mathcal{B}^q(\Omega_{R_{n+1}[t]}^\bullet),$$

which makes the following diagram commutative

$$\begin{array}{ccc} W_{n+2}\Omega_{R_{n+1}[t]}^q & \xrightarrow{F^{n+1}} & \Omega_{R_{n+1}[t]}^q \\ \downarrow & & \downarrow \\ W_{n+1}\Omega_{R[t]}^q & \xrightarrow{\tilde{F}^{n+1}} & \Omega_{R_{n+1}[t]}^q / \mathcal{B}^q, \end{array}$$

where we use that the de Rham-Witt complex exists for  $R_{n+1}[t]$  as well, see [HM04]. By [Hes04, Lemma 1.2.2]

$$W_{n+1}\Omega_{(R[t], t^m)}^q := \text{Ker}(W_{n+1}\Omega_{R[t]}^q \rightarrow W_{n+1}\Omega_{R[t]/t^m}^q)$$

is the degree  $q$  part of the differential graded ideal generated by  $W_{n+1}(t^m R[t])$ , i.e.,

$$W_{n+1}\Omega_{(R[t], t^m)}^q = W_{n+1}(t^m R[t]) \cdot W_{n+1}\Omega_{R[t]}^q + d(W_{n+1}(t^m R[t])) \cdot W_{n+1}\Omega_{R[t]}^{q-1}.$$

Thus  $\tilde{F}^{n+1}(W_{n+1}\Omega_{(R[t], t^m)}^q)$  is contained in the image of the following group in  $\Omega_{R_{n+1}[t]}^q / \mathcal{B}^q$

$$t^{pm}\Omega_{R_{n+1}[t]}^q + mt^{pm} \text{dlog}(t)\Omega_{R_{n+1}}^{q-1},$$

where the second summand contains  $\tilde{F}^{n+1}(dV^n(t^m R)W_{n+1}\Omega_R^{q-1})$ . More precisely, set

$$\Omega_e^q := t^e(\Omega_{R_{n+1}}^q \oplus \Omega_{R_{n+1}}^{q-1} \text{dlog } t), \quad \text{if } p^{n+1} \nmid e \geq 1,$$

and

$$\Omega_{0,0}^q := \Omega_{R_{n+1}}^q, \quad \Omega_{e,0}^q := t^e \Omega_{R_{n+1}}^q, \quad \Omega_{e,1}^q := t^e \Omega_{R_{n+1}}^{q-1} \text{dlog } t, \quad \text{if } p^{n+1} \mid e \geq 1.$$

Then we obtain a direct sum decomposition of complexes

$$\Omega_{R_{n+1}[t]}^\bullet = \bigoplus_{e \geq 1, p^{n+1} \nmid e} \Omega_e^\bullet \oplus \bigoplus_{e \geq 0, p^{n+1} \mid e} \Omega_{e,0}^\bullet \oplus \bigoplus_{e \geq 1, p^{n+1} \mid e} \Omega_{e,1}^\bullet$$

and hence also a direct sum decomposition

$$\frac{\Omega_{R_{n+1}[t]}^q}{\mathcal{B}^q(\Omega_{R_{n+1}[t]}^\bullet)} = \bigoplus_{e \geq 1, p^{n+1} \nmid e} \frac{\Omega_e^q}{\mathcal{B}^q(\Omega_e^\bullet)} \oplus \bigoplus_{e \geq 0, p^{n+1} \mid e} \frac{\Omega_{e,0}^q}{\mathcal{B}^q(\Omega_{e,0}^\bullet)} \oplus \bigoplus_{e \geq 1, p^{n+1} \mid e} \frac{\Omega_{e,1}^q}{\mathcal{B}^q(\Omega_{e,1}^\bullet)}.$$

With this notation the discussion above shows that  $\tilde{F}^{n+1}(W_{n+1}\Omega_{(R[t],t^m)}^q)$  is contained in (8.11.1)

$$\bigoplus_{e \geq pm, p^{n+1} \nmid e} \frac{\Omega_e^q}{\mathcal{B}^q(\Omega_e^\bullet)} \oplus \bigoplus_{e \geq pm, p^{n+1} \mid e} \frac{\Omega_{e,0}^q}{\mathcal{B}^q(\Omega_{e,0}^\bullet)} \oplus \bigoplus_{e \geq pm+1, p^{n+1} \mid e} \frac{\Omega_{e,1}^q}{\mathcal{B}^q(\Omega_{e,1}^\bullet)} \oplus \frac{m\Omega_{pm,1}^q}{\mathcal{B}^q(\Omega_{pm,1}^\bullet) \cap m\Omega_{pm,1}^q},$$

where the last summand only occurs if  $p^{n+1} \mid pm$  and  $p^{n+1} \nmid m$ .

We show that

$$\underline{p}^n : \Omega_{R[t]}^q / \Omega_n^q(-D', -pD_{n+1}) \rightarrow W_{n+1}\Omega_{R[t]}^q / W_{n+1}\Omega_{(R[t],t^m)}^q$$

is injective. Note that in the above situation we have two cases, where  $s = \lceil m/p^n \rceil$ ,

$$\Omega_n^q(-D', -pD_{n+1}) = \begin{cases} t^s(\Omega_R^q[t] + \Omega_R^q[t] \operatorname{dlog} t) & \text{if } p^{n+1} \nmid m \\ t^s\Omega_{R[t]}^q & \text{if } p^{n+1} \mid m. \end{cases}$$

Thus any element in the quotient  $\Omega_{R[t]}^q / \Omega_n^q(-D', -pD_{n+1})$  can be written in the form

$$\alpha = a_0 + \sum_{j=1}^{s-1} t^j(a_j + b_j \operatorname{dlog} t) + b_s t^s \operatorname{dlog} t,$$

for  $a_i \in \Omega_R^q$  and  $b_j \in \Omega_R^{q-1}$ , where  $b_s = 0$ , if  $p^{n+1} \nmid m$ . We obtain

$$\tilde{F}^{n+1}(\underline{p}^n \alpha) \equiv \sum_{j=0}^{s-1} p^n t^j p^{n+1} \tilde{F}^{n+1}(\tilde{a}_j) + \sum_{j=1}^s p^n t^j p^{n+1} \tilde{F}^{n+1}(\tilde{b}_j) \operatorname{dlog} t \pmod{\mathcal{B}^q},$$

where  $\tilde{a}_i$  and  $\tilde{b}_j$  are lifts of  $a_i$  and  $b_j$  to  $W_{n+1}\Omega_R^q$  and  $W_{n+1}\Omega_R^{q-1}$ , respectively. As  $s-1 < m/p^n$  we obtain

$$\tilde{F}^{n+1}(\underline{p}^n \alpha) \in \bigoplus_{\substack{0 \leq e < pm \\ p^{n+1} \nmid e}} \frac{\Omega_{e,0}^q}{\mathcal{B}^q(\Omega_{e,0}^\bullet)} \oplus \bigoplus_{\substack{1 \leq e < pm \\ p^{n+1} \mid e}} \frac{\Omega_{e,1}^q}{\mathcal{B}^q(\Omega_{e,1}^\bullet)} \oplus \frac{\Omega_{pm,1}^q}{\mathcal{B}^q(\Omega_{pm,1}^\bullet)},$$

where the last summand only occurs if  $p^{n+1} \mid m$ . Now assume  $\underline{p}^n \alpha \in W_{n+1}\Omega_{(R[t],t^m)}^q$ . Then  $\tilde{F}^{n+1}(\underline{p}^n \alpha)$  lies in (8.11.1) as well. This implies  $\tilde{F}^{n+1}(\underline{p}^n \alpha) = 0$  and hence by the injectivity of  $\tilde{F}^{n+1}\underline{p}^n$ , also that  $\alpha = 0$ . This completes the proof.  $\square$

**Lemma 8.12.** *Let  $a \in \Omega_X^q$  be a local section satisfying  $dV^{n-1}(a) \in W_n\Omega_{(X,-D)}^{q+1}$ . Then there exist local sections  $a_j \in W_{j+1}\Omega_X^q$ ,  $j = 0, \dots, n-1$ , and  $b \in W_{n+1}\Omega_X^q$  such that  $p^j a_j = \underline{p}^j R^j(a_j) \in W_{j+1}\Omega_{(X,-D)}^q$  and*

$$a = \sum_{j=0}^{n-1} F^j(a_j) + F^n(b).$$

*Proof.* We show, for all  $s = -1, \dots, n-1$  we find  $a_j \in W_{j+1}\Omega_X^q$ ,  $j = 0, \dots, s$ , as in the statement such that

$$a \equiv \sum_{j=0}^s F^j(a_j) \pmod{F^{s+1}W_{s+2}\Omega_X^q}.$$

There is nothing to show for  $s = -1$ . Assume  $0 \leq s \leq n-1$  and the statement is true for  $s-1$ , i.e., we find  $a_j$ ,  $j = 0, \dots, s-1$ , as in the claim and  $b \in W_{s+1}\Omega_X^q$  such that

$$a = \sum_{j=0}^{s-1} F^j(a_j) + F^s(b) \quad \text{in } \Omega_X^q.$$

Applying  $dV^s$  yields

$$\underline{p}^s dR^s(b) = dV^s(a) - \sum_{j=0}^{s-1} dV^{s-j}(\underline{p}^j R^j(a_j)) \in W_{s+1}\Omega_{(X,-D)}^{q+1},$$

where we use  $dV^s(a) = F^{n-1-s}dV^{n-1}(a) \in W_{s+1}\Omega_{(X,-D)}^{q+1}$ . Hence by Lemma 8.10 and Lemma 8.11 we find an element  $a_s \in W_{s+1}\Omega_X^q$  with  $\underline{p}^s R^s(a_s) \in W_{s+1}\Omega_{(X,-D)}^q$  such that  $dV^s(F^s(a_s)) = dV^s(F^s(b))$ . As the kernel of  $dV^s$  is  $F^{s+1}W_{s+2}\Omega_X^{q-1}$  the statement follows.  $\square$

**Lemma 8.13.** *The following two equalities of subsheaves of  $W_{n+1}\Omega_X^q$  hold*

$$W_{n+1}\Omega_{(X,-D)}^q \cap V^n(\Omega_X^q) = \sum_{j=0}^n V^j \left( W_{n+1-j}\Omega_{(X,-D)}^q \cap \underline{p}^{n-j}(\Omega_X^q) \right)$$

and

$$W_{n+1}\Omega_{(X,-D)}^q \cap dV^n(\Omega_X^{q-1}) = \sum_{j=0}^n dV^j \left( W_{n+1-j}\Omega_{(X,-D)}^{q-1} \cap \underline{p}^{n-j}(\Omega_X^{q-1}) \right).$$

*Proof.* We show the first equality. This “ $\supset$ ” inclusion is obvious. We show the other inclusion. Let  $a \in \Omega_X^q$  be a local section with  $V^n(a) \in W_{n+1}\Omega_{(X,-D)}^q$ . As  $dV^{n-1}(a) = FdV^n(a) \in W_n\Omega_{(X,-D)}^{q+1}$ , we find elements  $a_j \in W_{j+1}\Omega_X^q$  and an element  $b \in W_{n+1}\Omega_X^q$  as in Lemma 8.12. Applying  $V^n$  to the equality in that lemma yields

$$\underline{p}^n R^n(b) = p^n b = \sum_{j=0}^{n-1} V^{n-j}(\underline{p}^j R^j(a_j)) - V^n(a) \in W_{n+1}\Omega_{(X,-D)}^q,$$

which proves the other inclusion in the first equality.

For the second equality we take  $a \in \Omega_X^{q-1}$  such that  $dV^n(a) \in W_{n+1}\Omega_{(X,-D)}^q$ . Thus Lemma 8.12 yields the existence of  $a_j \in W_{j+1}\Omega_X^{q-1}$ ,  $j = 0, \dots, n$ , and  $b \in W_{n+2}\Omega_X^{q-1}$  such that  $\underline{p}^j R^j(a_j) \in W_{j+1}\Omega_{(X,-D)}^{q-1}$  and

$$a = \sum_{j=0}^n F^j(a_j) + F^{n+1}(b).$$

As  $dV^n(F^{n+1}(b)) = p^{n+1}Fd(b) = 0$  we find

$$dV^n(a) = \sum_{j=0}^n dV^{n-j}(\underline{p}^j R^j(a_j)),$$

which proves the second equality.  $\square$

**Proposition 8.14.** *There is a short exact sequence of  $W_{n+1}\mathcal{O}_X$ -modules*

$$(8.14.1) \quad 0 \rightarrow (\Omega/B)_{n,(X,-D)}^q \xrightarrow{V^n} \mathrm{gr}_{(X,-D)}^n \xrightarrow{\beta} (\Omega/Z)_{n,(X,-D)}^{q-1} \rightarrow 0,$$

where  $\mathrm{gr}_{(X,-D)}^n := \mathrm{Ker}(R : W_{n+1}\Omega_{(X,-D)}^q \rightarrow W_n\Omega_{(X,-D)}^q)$  and the map  $\beta$  is induced by  $V^n(a) + dV^n(b) \mapsto b$ . Furthermore the following diagram of sheaves of abelian groups is cartesian

$$(8.14.2) \quad \begin{array}{ccc} dV^n(\Omega_X^{q-1}) \cap W_{n+1}\Omega_{(X,-D)}^q & \xrightarrow{F} & W_n\Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ dV^n(\Omega_X^{q-1}) & \xrightarrow{F} & W_n\Omega_X^q. \end{array}$$

*Proof.* Consider the following diagram of  $W_{n+1}\mathcal{O}_X$ -modules

$$(8.14.3) \quad \begin{array}{ccccccc} & & & & F & & \\ & & & & \curvearrowright & & \\ (\Omega/B)_{n,(X,-D)}^q & \longrightarrow & \mathrm{gr}_{(X,-D)}^n & \cdots \twoheadrightarrow & (\Omega/Z)_{n,(X,-D)}^{q-1} & \longrightarrow & F_*W_n\Omega_{(X,-D)}^q \\ & \downarrow & \downarrow & & \downarrow & & \downarrow \\ \frac{F_{X*}^n\Omega_X^q}{B_n\Omega_X^q} & \xrightarrow{V^n} & \mathrm{gr}_X^n & \xrightarrow{\beta} & \frac{F_{X*}^n\Omega_X^{q-1}}{Z_n\Omega_X^{q-1}} & \xrightarrow{dV^{n-1}} & F_*W_n\Omega_X^q \\ & & & & \curvearrowleft & & \\ & & & & F & & \end{array}$$

The two outer squares are cartesian in the category of  $W_{n+1}\mathcal{O}_X$ -modules by definition, and also in the category of sheaves of abelian groups as the forgetful functor  $(W_{n+1}\mathcal{O}_X\text{-mod}) \rightarrow (\text{abelian-sheaves})$  has a left adjoint. Clearly the square with  $F$  as horizontal maps is commutative and we have  $dV^{n-1} \circ \beta = F$ . Hence by the definition of  $(\Omega/Z)_{n,(X,-D)}^{q-1}$  as a pullback in (8.6.3) there is a unique dotted arrow which makes the middle square commutative. Thus all the maps in (8.14.1) are defined. The injectivity of  $V^n$  holds by definition, the exactness in the middle of (8.14.1) follows from the exactness of (8.6.1) and the fact that the left square is cartesian. We claim that the diagram of sheaves of abelian groups

$$(8.14.4) \quad \begin{array}{ccc} dV^n(\Omega_X^{q-1}) \cap W_{n+1}\Omega_{(X,-D)}^q & \xrightarrow{\beta_0} & (\Omega/Z)_{n,(X,-D)}^{q-1} \\ \downarrow & & \downarrow \\ dV^n(\Omega_X^{q-1}) & \xrightarrow{\beta_0} & \frac{\Omega_X^{q-1}}{Z_n\Omega_X^{q-1}} \end{array}$$

is cartesian, where  $\beta_0$  is induced by  $\beta$ , i.e.,  $\beta_0(dV^n(a)) = a$ . If the claim is true, then the top  $\beta_0$  is surjective as the bottom one is and hence also the dotted arrow in (8.14.3) is surjective. Moreover the square (8.14.2) is cartesian, as it is a composition of two cartesian squares.

We prove the claim. Let  $a \in \Omega_X^{q-1}$  be a local section which mod  $Z_n\Omega_X^{q-1}$  lies in the image  $\mathrm{Im}((\Omega/Z)_{n,(X,-D)}^{q-1} \hookrightarrow \Omega_X^{q-1}/Z_n\Omega_X^{q-1})$ . By (8.14.3) we have  $dV^{n-1}(a) \in W_n\Omega_{(X,-D)}^q$ . As  $Z_n\Omega_X^q = F^n(W_{n+1}\Omega_X^q)$ , Lemma 8.12 yields the existence of local sections  $a_j \in W_{j+1}\Omega_X^q$ ,  $j = 0, \dots, n-1$ , such that  $p^j a_j = \underline{p}^j R^j(a_j) \in W_{j+1}\Omega_{(X,-D)}^q$  and

$$a = \sum_{j=0}^{n-1} F^j(a_j) \quad \text{in } \Omega_X^q/Z_n\Omega_X^q.$$

Moreover

$$dV^n\left(\sum_{j=0}^{n-1} F^j(a_j)\right) = \sum_{j=0}^{n-1} dV^{n-j}(\underline{p}^j R^j(a_j)) \in W_{n+1}\Omega_{(X,-D)}^q$$

and it maps to  $a$  under  $\beta_0$ . Hence (8.14.4) is cartesian.  $\square$

**Lemma 8.15.** *The restriction map  $R : W_{n+1}\Omega_{(X,-D)}^q \rightarrow W_n\Omega_{(X,-D)}^q$  is surjective.*

*Proof.* We show

$$(8.15.1) \quad R^r : W_{n+r}\Omega_{(X,-D)}^q \rightarrow W_n\Omega_{(X,-D)}^q \quad \text{is surjective, for all } r \geq 1.$$

For  $n = 1$  this holds by the ‘‘Moreover’’-part of Lemma 8.11 (take  $(j, n)$  there as  $(r, 0)$  here). Assume  $n \geq 2$ . Let  $w \in W_n\Omega_{(X,-D)}^q$  be a local section. Set  $w_{n-1} := R(w) \in W_{n-1}\Omega_{(X,-D)}^q$ . By induction there exists an  $w_{n+r} \in W_{n+r}\Omega_{(X,-D)}^q$  with  $R^{r+1}(w_{n+r}) = w_{n-1}$ . Set  $w_n :=$

$R^r(w_{n+r}) \in W_n \Omega_{(X, -D)}^q$ . Since both  $w$  and  $w_n$  are lifts of  $w_{n-1}$ , we have  $R(w - w_n) = 0$ . Thus there exist elements  $a \in \Omega_X^q$  and  $b \in \Omega_X^{q-1}$  with

$$V^{n-1}(a) + dV^{n-1}(b) = w - w_n \in W_n \Omega_{(X, -D)}^q.$$

As  $F(V^{n-1}(a)) = 0$ , the cartesian diagram (8.14.2) yields  $dV^{n-1}(b) \in W_n \Omega_{(X, -D)}^q$  and hence also  $V^{n-1}(a) \in W_n \Omega_{(X, -D)}^q$ . By Lemma 8.13 we find elements  $a_j \in \Omega_X^q$  and  $b_j \in \Omega_X^{q-1}$  such that

$$V^{n-1}(a) = \sum_{j=0}^{n-1} V^j(\underline{p}^{n-1-j} a_j) \quad \text{and} \quad \underline{p}^{n-1-j} a_j \in W_{n-j} \Omega_{(X, -D)}^q, \quad \text{for all } j,$$

and

$$dV^{n-1}(b) = \sum_{j=0}^{n-1} dV^j(\underline{p}^{n-1-j} b_j) \quad \text{and} \quad \underline{p}^{n-1-j} b_j \in W_{n-j} \Omega_{(X, -D)}^{q-1}, \quad \text{for all } j.$$

By the “Moreover”-part of Lemma 8.11 we find sections  $\alpha_j \in W_{n-j+1} \Omega_{(X, -D)}^q$  and  $\beta_j \in W_{n-j+1} \Omega_{(X, -D)}^{q-1}$  with  $R(\alpha_j) = \underline{p}^{n-j-1} a_j$  and  $R(\beta_j) = \underline{p}^{n-j-1} b_j$ , for all  $j = 0, \dots, n-1$ . Altogether we find that the element

$$R^{r-1}(w_{n+r}) + \sum_{j=0}^{n-1} V^j(\alpha_j) + \sum_{j=0}^{n-1} V^j(\beta_j)$$

lies in  $W_{n+1} \Omega_{(X, -D)}^q$  and lifts  $w$ . This completes the proof.  $\square$

The proof of part (1) and (2) of Theorem 8.7 is now complete. To prove the last part of that theorem we need to study certain variants of  $(\Omega/Z)^q$  and  $(\Omega/B)^q$  which are defined using iterates of the twisted inverse Cartier operator from 8.8, similar to what we did in section 6.

**8.16.** Let  $X$ ,  $D$ , and  $E$  be as in 8.8 above. Let  $n \geq 0$  and let

$$D = D_0 + pD_1 + \dots + p^n D_n$$

be a  $p$ -divisibility decomposition. In 6.11 we have defined divisors  $\underline{D}_j$  and  $\underline{D}^j$  such that  $D = \underline{D}_{n-j} + p^{n-j} \underline{D}^j$  is a  $p$ -divisibility decomposition, for any  $0 \leq j \leq n$ .<sup>6</sup>

For  $1 \leq j \leq n$ , define  $\mathcal{O}_X$ -modules

$$\Omega_{j,n,-}^q := F_{X*}^j \Omega_{n-j}^q(-\underline{D}_{n-j}, -\underline{D}^j - p^j E), \quad \mathcal{B}_{j,n,-}^q := \mathcal{B}^q(\Omega_{j,n,-}^\bullet), \quad \mathcal{Z}_{j,n,-}^q := \mathcal{Z}^q(\Omega_{j,n,-}^\bullet),$$

we furthermore set  $\mathcal{B}_{0,n,-}^q := 0$ . Define  $\mathcal{O}_X$ -modules  $(\Omega/B)_{j,n}^q(-D, -E)$ ,  $(\Omega/Z)_{j,n}^q(-D, -E)$  and natural maps

$$(8.16.1) \quad \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q \longrightarrow (\Omega/B)_{j,n}^q(-D, -E) \longrightarrow (\Omega/Z)_{j,n}^q(-D, -E)$$

by setting

$$(\Omega/B)_{0,n}^q(-D, -E) := \Omega_n^q(-D, -E) \rightarrow 0 =: (\Omega/Z)_{0,n}^q(-D, -E),$$

---

<sup>6</sup>Recall that  $\underline{D}^j$  is divisible by  $p$  and that the actual  $p$ -divisibility decomposition is  $D = \underline{D}_{n-j} + p^{n-j+1}(\frac{1}{p}\underline{D}^j)$ .

and for  $j \geq 1$  recursively by the condition that the two squares on the left in the following diagram are pushout squares

(8.16.2)

$$\begin{array}{ccccccc}
0 \longrightarrow & F_{X*}^{j-1} \Omega_{n-j+1}^q(-\underline{D}_{n-j}, -D_{n-j+1} - \underline{D}^{j-1} - p^{j-1}E) & \xrightarrow{C^{-1}} & \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q & \longrightarrow & \Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \parallel & \\
0 \longrightarrow & (\Omega/B)_{j-1,n}^q(-D, -E) & \longrightarrow & (\Omega/B)_{j,n}^q(-D, -E) & \longrightarrow & \Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \parallel & \\
0 \longrightarrow & (\Omega/Z)_{j-1,n}^q(-D, -E) & \longrightarrow & (\Omega/Z)_{j,n}^q(-D, -E) & \longrightarrow & \Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q & \longrightarrow 0,
\end{array}$$

where the top horizontal exact sequence is induced by (8.9.1) and the top vertical map on the left is the composition of the natural map

$$F_{X*}^{j-1} \Omega_{n-j+1}^q(-\underline{D}_{n-j}, -D_{n-j+1} - \underline{D}^{j-1} - p^{j-1}E) \hookrightarrow \Omega_{j-1,n,-}^q \longrightarrow \frac{\Omega_{j-1,n,-}^q}{\mathcal{B}_{j-1,n,-}^q}$$

with the first map of (8.16.1) (with  $j$  replaced by  $j-1$ ).

Some consequences of this definition:

(a) By definition we have for  $n \geq 1$

$$(\Omega/Z)_{1,n}^q(-D, -E) = \Omega_{1,n,-}^q / \mathcal{Z}_{1,n,-}^q.$$

Moreover we have

$$(\Omega/B)_{1,n}^q(-D, -E) = \frac{\mathcal{Z}^q(F_{X*} \Omega_{n-1}^\bullet(-D, -pE)) + \Omega_{1,n,-}^q}{\mathcal{B}^q(F_{X*} \Omega_{n-1}^\bullet(-D, -pE))}.$$

For the latter equality note that the right hand side (RHS) fits into a short exact sequence

$$0 \rightarrow \Omega_n^q(-D, -E) \xrightarrow{\alpha} \text{RHS} \xrightarrow{\beta} \Omega_{1,n,-}^q / \mathcal{Z}_{1,n,-}^q \rightarrow 0,$$

where  $\alpha$  is given by  $C^{-1}$  from (8.8.2) composed with the natural inclusion and the map  $\beta$  is defined by  $\beta(a+b) = b$ , for  $a \in \mathcal{Z}^q(F_{X*} \Omega_{n-1}^\bullet(-D, -pE))$ , and  $b \in \Omega_{1,n,-}^q$ . It is direct to check that  $\beta$  is well-defined, that the sequence is exact, and that this implies the above equality.

(b) The sheaves  $(\Omega/B)_{j,n}^q(-D, -E)$ ,  $(\Omega/Z)_{j,n}^q(-D, -E)$  are locally free coherent  $\mathcal{O}_X$ -modules. For  $j = 0$  this follows from the definition and for  $j \geq 1$  by induction from (8.16.2) and the fact that the  $\mathcal{O}_X$ -modules  $\Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q$  are locally free, see 8.9. Their restrictions to  $U = X \setminus (D + E)$  are quotients of  $\Omega_U^q$ . The natural maps

$$(8.16.3) \quad (\Omega/B)_{j,n}^q(-D, -E) \twoheadrightarrow (\Omega/Z)_{j,n}^q(-D, -E)$$

are surjective. Moreover the  $j$ -fold iteration of the Cartier operator induces an isomorphism

$$(Z/B)_{j,n}^q(-D, -E) := \text{Ker} \left( (\Omega/B)_{j,n}^q(-D, -E) \rightarrow (\Omega/Z)_{j,n}^q(-D, -E) \right) \xrightarrow{C^j} \Omega_n^q(-D, -E).$$

For  $j = 0$  this holds by definition, and for  $j \geq 1$  it follows by induction from applying the Snake Lemma to the two lower rows in (8.16.2).

(c) We have natural maps of  $\mathcal{O}_X$ -modules

$$(8.16.4) \quad F_{X*}(\Omega/B)_{j-1,n-1}^q(-\underline{D}_{n-1}, -\underline{D}^1 - pE) \rightarrow (\Omega/B)_{j,n}^q(-D, -E)$$

and surjections of  $\mathcal{O}_X$ -modules

$$(8.16.5) \quad (\Omega/Z)_{j,n}^q(-D, -E) \twoheadrightarrow F_{X*}(\Omega/Z)_{j-1,n-1}^q(-\underline{D}_{n-1}, -\underline{D}^1 - pE).$$

In the case  $j = 1$  this follows from the definition and for  $j \geq 2$  it follows by induction from the fact the top sequence in (8.16.2) constructed for  $D, E, n, j$  is equal to  $F_{X*}$  applied to the sequence constructed for  $\underline{D}_{n-1}, \underline{D}^1 + pE, n-1, j-1$ . By definition the  $(j-1)$ -fold iteration of the Cartier operator induces an isomorphism

$$C^{j-1} : \text{Ker (8.16.5)} \xrightarrow{\sim} (\Omega/Z)_{1,n}^q(-D, -E).$$

**Remark 8.17.** Note that the first map in (8.16.1) is in general not surjective, which under the duality from Proposition 8.19 relates to the fact that

$$Z_{j,n}^q(D, E) \subset F_{X*}^j \Omega_{n-j}^q(\underline{D}_{n-j}, \underline{D}^j + p^j E)$$

is not a subbundle, see Remark 6.12.

**Lemma 8.18.** *In the situation of 8.16 we have natural inclusions*

$$(\Omega/B)_{j,n}^q(-D, -E) \subset F_{X*}^j \Omega_X^q/B_j \Omega_X^q, \quad (\Omega/Z)_{j,n}^q(-D, -E) \subset F_{X*}^j \Omega_X^q/Z_j \Omega_X^q.$$

*Proof.* The statements hold by definition for  $j = 0$ . The general statement follows by induction over  $j$ . We explain the case  $(\Omega/Z)$  and a similar argument also works for  $(\Omega/B)$ . We have a morphism of exact sequences (we drop the Frobenius twists)

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\Omega/Z)_{j-1,n}^q(-D, -E) & \longrightarrow & (\Omega/Z)_{j,n}^q(-D, -E) & \longrightarrow & \Omega_{j,n,-}^q/Z_{j,n,-}^q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_X^q/Z_{j-1} \Omega_X^q & \xrightarrow{C^{-1}} & \Omega_X^q/Z_j \Omega_X^q & \longrightarrow & \Omega_X^q/Z_1 \Omega_X^q \longrightarrow 0, \end{array}$$

where left vertical map is the inclusion which we have by induction, the right vertical map is the natural inclusion, and the middle map exists by the definition of  $(\Omega/Z)_{j,n}^q(-D, -E)$  as a pushout and the fact that there is a natural map  $\Omega_{j,n,-}^q/\mathcal{B}_{j,n,-}^q \rightarrow \Omega_X^q/Z_j \Omega_X^q$ . This yields the statement.  $\square$

**Proposition 8.19.** *Let  $X$  be of pure dimension  $N$ . The isomorphism (cf. Proposition 8.5)*

$$F_{X*}^j \Omega_X^q \rightarrow \mathcal{H}om(F_{X*}^j \Omega_X^{N-q}, \Omega_X^N), \quad \alpha \mapsto (\beta \mapsto C^j(\alpha \wedge \beta)),$$

*induces isomorphisms*

$$(8.19.1) \quad (\Omega/B)_{j,n}^q(-D, -E) \cong \mathcal{H}om(Z_{j,n}^{N-q}(D, E), \Omega_X^N)$$

*and*

$$(8.19.2) \quad (\Omega/Z)_{j,n}^q(-D, -E) \cong \mathcal{H}om(B_{j,n}^{N-q}(D, E), \Omega_X^N),$$

*where  $Z_{j,n}^{N-q}(D, E)$  and  $B_{j,n}^{N-q}(D, E)$  are defined in 6.11.*

First we record the following lemma whose proof is similar to the one of Proposition 8.5.

**Lemma 8.20.** *The map*

$$F_{X*}^j \Omega_n^q(-D, -E) \xrightarrow{\sim} \mathcal{H}om(F_{X*}^j \Omega_n^{N-q}(D, E), \Omega_X^N), \quad \alpha \mapsto (\beta \mapsto C^j(\alpha \wedge \beta)),$$

*is an isomorphism for all  $j, n \geq 0$ .*

*Proof of Proposition 8.19.* This is a generalization of [Eke84, II, Lemma 2.2.20]. For  $j = 0$  both sides of (8.19.2) are zero and (8.19.1) holds by Lemma 8.20. Set

$$\mathcal{D} := \mathcal{H}om(-, \Omega_X^N) : (\text{loc. free } \mathcal{O}_X\text{-mod})^{\text{op}} \rightarrow (\text{loc. free } \mathcal{O}_X\text{-mod}).$$

Note that  $\mathcal{D}$  is dualizing, in the sense that the natural map  $\mathcal{E} \rightarrow \mathcal{D}(\mathcal{D}(\mathcal{E}))$  is an isomorphism for any locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$ . Hence  $\mathcal{D}$  maps a pushout square of locally free  $\mathcal{O}_X$ -modules to a pullback square and vice versa. Thus it remains to show that  $\mathcal{D}$  applied to the bottom exact sequence in (6.11.3) (with  $N - q$  instead of  $q$ ) is isomorphic to the top sequence in (8.16.2), for all  $j \geq 1$ . Then the result follows by induction over  $j$  directly from the definitions. The looked for duality between the two exact sequences is standard, e.g. [Mil76, Lemma 1.7]. For convenience of the reader we give the proof in this twisted situation. In the following we assume  $j \geq 1$ . By Lemma 8.20 the map

$$b_j^q : \Omega_{j,n,-}^q \longrightarrow \mathcal{D}(\Omega_{j,n}^{N-q}), \quad b_j^q(\alpha)(\beta) = C^j(\alpha \wedge \beta) \quad (\alpha \in \Omega_{j,n,-}^q, \beta \in \Omega_{j,n}^{N-q}).$$

is an isomorphism. If  $\alpha \in \mathcal{B}_{j,n,-}^q$  and  $\beta \in \mathcal{Z}_{j,n}^{N-q}$ , then  $\alpha = d\alpha'$  and  $d\beta = 0$  and hence

$$b_j^q(\alpha)(\beta) = C^j(d\alpha' \wedge \beta) = C^j(d(\alpha' \wedge \beta)) = 0.$$

Similarly, the above vanishing holds if  $\alpha \in \mathcal{Z}_{j,n}^q$  and  $\beta \in \mathcal{B}_{j,n}^{N-q}$ . Therefore the map  $b_j^q$  induces well-defined maps

$$c_j^q : \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q \longrightarrow \mathcal{D}(\mathcal{Z}_{j,n}^{N-q}) \quad \text{and} \quad a_j^q : \Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q \longrightarrow \mathcal{D}(\mathcal{B}_{j,n}^{N-q}).$$

Consider the following two diagrams,

$$(8.20.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{j,n,-}^{q-1} / \mathcal{Z}_{j,n,-}^{q-1} & \xrightarrow{d} & \Omega_{j,n,-}^q & \longrightarrow & \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q \longrightarrow 0 \\ & & a_j^{q-1} \downarrow & & b_j^q \downarrow \simeq & & c_j^q \downarrow \\ 0 & \longrightarrow & \mathcal{D}(\mathcal{B}_{j,n}^{N-q+1}) & \xrightarrow{(-1)^q d^\vee} & \mathcal{D}(\Omega_{j,n}^{N-q}) & \longrightarrow & \mathcal{D}(\mathcal{Z}_{j,n}^{N-q}) \longrightarrow 0 \end{array}$$

and

$$(8.20.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_{X*}^{j-1} \Omega_{n-j+1}^q(-\underline{D}_{n-j}, -\underline{D}_{n-j+1} - \underline{D}^{j-1} - p^{j-1}E) & \xrightarrow{C^{-1}} & \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q & \longrightarrow & \Omega_{j,n,-}^q / \mathcal{Z}_{j,n,-}^q \longrightarrow 0 \\ & & e^q \downarrow \simeq & & c_j^q \downarrow & & a_j^q \downarrow \\ 0 & \longrightarrow & \mathcal{D}(F_{X*}^{j-1} \Omega_{n-j+1}^{N-q}(\underline{D}_{n-j}, \underline{D}_{n-j+1} + \underline{D}^{j-1} + p^{j-1}E)) & \xrightarrow{C^\vee} & \mathcal{D}(\mathcal{Z}_{j,n}^{N-q}) & \longrightarrow & \mathcal{D}(\mathcal{B}_{j,n}^{N-q}) \longrightarrow 0. \end{array}$$

where  $e^q$  is the isomorphism from Lemma 8.20, which is induced by  $\alpha \mapsto C^{j-1}(\alpha \wedge \beta)$ . The two diagrams clearly commute and the rows are exact. In fact the top sequence of (8.20.2) is the top sequence of (8.16.2) and the bottom sequence of (8.20.2) is  $\mathcal{D}$  applied to the bottom sequence of (6.11.3). Thus by the above it is only left to show that  $a_j^q$  and  $c_j^q$  are isomorphisms for all  $q \geq 0$  and  $j \geq 1$ . But this follows by descending induction over  $q$  from (8.20.1) together with (8.20.2). This completes the proof of the proposition.  $\square$

**Proposition 8.21.** *Let  $D = D' + p^{n+1}D_{n+1}$  be a  $p$ -divisibility decomposition. Consider the  $\mathcal{O}_X$ -modules  $(\Omega/Z)_{n,(X,-D)}^{q-1}$  and  $(\Omega/B)_{n,(X,-D)}^q$  defined in (8.6.3). There are natural isomorphisms of  $\mathcal{O}_X$ -modules for  $n \geq 0$*

$$(8.21.1) \quad (\Omega/Z)_{n,n}^{q-1}(-D', -pD_{n+1}) \xrightarrow{\simeq} (\Omega/Z)_{n,(X,-D)}^{q-1}.$$

and

$$(8.21.2) \quad (\Omega/B)_{n,n}^q(-D', -pD_{n+1}) \xrightarrow{\sim} (\Omega/B)_{n,(X,-D)}^q.$$

In particular the sheaves on the right hand side are locally free coherent  $\mathcal{O}_X$ -modules.

*Proof.* The last statement follows from the two isomorphisms and 8.16(b). We first prove (8.21.1). In case  $n = 0$  both sides are zero by definition. The case  $n = 1$  follows from 8.6(a) and 8.16(a).

Let  $n \geq 2$ . Let  $D = D_0 + pD_1 + \dots + p^{n+1}D_{n+1}$  be a  $p$ -divisibility decomposition. Assume that by induction we have a natural isomorphism

$$(\Omega/Z)_{n-1,n-1}^{q-1}(-\underline{D}_{n-1}, -pD_n - p^2D_{n+1}) \xrightarrow{a_{n-1}} (\Omega/Z)_{n-1,(X,-D)}^{q-1}.$$

**Claim 8.21.1.** *There exists a natural morphism  $a_n$  which makes the following diagram commutative*

$$(8.21.3) \quad \begin{array}{ccc} (\Omega/Z)_{n,n}^{q-1}(-\underline{D}_n, -pD_{n+1}) & \xrightarrow{(8.16.5)} & F_*(\Omega/Z)_{n-1,n-1}^{q-1}(-\underline{D}_{n-1}, -pD_n - p^2D_{n+1}) \\ \downarrow a_n & & \downarrow a_{n-1} \\ (\Omega/Z)_{n,(X,-D)}^{q-1} & \xrightarrow{(8.6.5)} & F_*(\Omega/Z)_{n-1,(X,-D)}^{q-1}. \end{array}$$

We prove the claim. The following diagram of sheaves of abelian groups clearly commutes

$$\begin{array}{ccccc} (\Omega/Z)_{n,n}^{q-1}(-\underline{D}_n, -pD_{n+1}) & \xrightarrow{a_{n-1} \circ (8.16.5)} & (\Omega/Z)_{n-1,(X,-D)}^{q-1} & \xrightarrow{dV^{n-2}} & W_{n-1}\Omega_{(X,-D)}^q \\ \downarrow & & & & \downarrow \\ \Omega_X^{q-1}/Z_n\Omega_X^{q-1} & \xrightarrow{dV^{n-1}} & dV^{n-1}(\Omega_X^q) & \xrightarrow{F} & W_{n-1}\Omega_X^q, \end{array}$$

where the left vertical map is the natural inclusion from Lemma 8.18. As (8.14.2) is cartesian we obtain a morphism  $(\Omega/Z)_{n,n}^{q-1}(-\underline{D}_n, -pD_{n+1}) \rightarrow dV^{n-1}(\Omega_X^{q-1}) \cap W_n\Omega_{(X,-D)}^q$ , composing with the inclusion  $dV^{n-1}(\Omega_X^{q-1}) \cap W_n\Omega_{(X,-D)}^q \subset W_n\Omega_{(X,-D)}^q$  yields the top horizontal morphism in the following commutative diagram of  $W_{n+1}\mathcal{O}_X$ -modules

$$\begin{array}{ccc} (\Omega/Z)_{n,n}^{q-1}(-\underline{D}_n, -pD_{n+1}) & \xrightarrow{dV^{n-1}} & F_*W_n\Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ F_{X*}^n\Omega_X^{q-1}/Z_n\Omega_X^{q-1} & \xrightarrow{dV^{n-1}} & F_*W_n\Omega_X^q. \end{array}$$

Now the map  $a_n$  in the claim exists by definition of  $(\Omega/Z)_{n,(X,-D)}^{q-1}$  in (8.6.3). The diagram (8.21.3) commutes by construction, or maybe easier, the fact that both lines of the diagram map injectively into  $F_{X*}^n\Omega_X^{q-1}/Z_n \rightarrow F_{X*}^n\Omega_X^{q-1}/F_{X*}Z_{n-1}$ .

It remains to show that  $a_n$  is an isomorphism. As  $a_{n-1}$  is an isomorphism by induction and the map (8.16.5) is surjective it remains to show that  $a_n$  induces an isomorphism between the kernels of the horizontal maps in (8.21.3). By construction

$$\text{Ker}(8.6.5) = \{a \in Z_{n-1}\Omega_X^{q-1}/Z_n\Omega_X^{q-1} \mid dV^{n-1}(a) \in W_n\Omega_{(X,-D)}^q\}.$$

We note

- $Z_{n-1}\Omega_X^{q-1} = F^{n-1}W_n\Omega_X^{q-1}$ ;

- the  $(n-1)$ -fold iterated Cartier operator  $C^{(n-1)}$  maps  $F^{n-1}(b)$  to  $R^{n-1}(b)$ , for  $b \in W_n \Omega_X^{q-1}$ ;
- $dV^{n-1}(F^{n-1}(b)) = \underline{p}^{n-1} dR^{n-1}(b)$ .

Hence  $C^{(n-1)}$  induces an isomorphism

$$C^{(n-1)} : \text{Ker (8.6.5)} \xrightarrow{\cong} \{b \in \Omega_X^{q-1}/Z_1 \Omega_X^{q-1} \mid \underline{p}^{n-1} db \in W_n \Omega_{(X,-D)}^q\}.$$

By Lemma 8.11 and Lemma 8.10 we can rewrite this as

$$C^{(n-1)} : \text{Ker (8.6.5)} \xrightarrow{\cong} \frac{\Omega_{n-1}^q(-\underline{D}_{n-1}, -pD_n - p^2 D_{n+1})}{\mathcal{Z}^{q-1}(\Omega_{n-1}^\bullet(-\underline{D}_{n-1}, -pD_n - p^2 D_{n+1}))}.$$

This together with 8.16 (a) and (c) implies that  $a_n$  induces an isomorphism on the kernels of the horizontal maps in (8.21.3). Thus we have proven the isomorphism (8.21.1) in general.

Next (8.21.2). First note that for  $a \in W_j \Omega_X^{q-1}$ , we have  $V^j(F^{j-1}da) = p^j dV(a) = 0$ . As  $B_j \Omega_X^q = F^{j-1}d(W_j \Omega_X^{q-1})$  we obtain a well-defined morphism of  $W_{n+1} \mathcal{O}_X$ -modules

$$\underline{p}^{n-j} V^j = V^j \underline{p}^{n-j} : F_{X*}^j \Omega_X^q / B_j \Omega_X^q \rightarrow W_{n+1} \Omega_X^q.$$

**Claim 8.21.2.** *For  $j = 0, \dots, n$  the above map induces well-defined morphisms of  $W_{n+1} \mathcal{O}_X$ -modules*

$$\underline{p}^{n-j} V^j : (\Omega/B)_{j,n}^q(-D', -pD_{n+1}) \rightarrow W_{n+1} \Omega_{(X,-D)}^q.$$

We prove the claim. For  $j = 0$ , this follows from Lemma 8.11 and the definition of  $(\Omega/B)_{0,n}^q$ . Let  $j \geq 1$  and consider the following diagram

$$\begin{array}{ccc} F_{X*}^{j-1} \Omega_{n-j+1}^q(-(\underline{D}')_{n-j}, -D_{n-j+1} - (\underline{D}')^{j-1} - p^j D_{n+1}) & \xrightarrow{C^{-1}} & \Omega_{j,n,-}^q / \mathcal{B}_{j,n,-}^q \\ \downarrow & & \downarrow V^j \underline{p}^{n-j} \\ (\Omega/B)_{j-1,n}^q(-D', -pD_{n+1}) & \xrightarrow{\underline{p}^{n-j+1} V^{j-1}} & W_{n+1} \Omega_{(X,-D)}^q \end{array}$$

where

$$\Omega_{j,n,-}^q = F_{X*}^j \Omega_{n-j}^q(-(\underline{D}')_{n-j}, -(\underline{D}')^j - p^{j+1} D_{n+1}) \quad \text{and} \quad \mathcal{B}_{j,n,-}^q = d\Omega_{j,n,-}^{q-1}.$$

Here the bottom horizontal map exists by induction and the existence of the right vertical map follows from Lemma 8.11, which implies  $\underline{p}^{n-j}(\Omega_{j,n,-}^q) \subset W_{j+1} \Omega_{(X,-D)}^q$ . It is direct to check that the above diagram commutes. Thus by the definition of  $(\Omega/B)_{j,n}^q(-D', -pD_{n+1})$  as a pushout we obtain the map from the statement of the claim.

Thus we obtain a commutative diagram

$$\begin{array}{ccc} (\Omega/B)_{n,n}^q(-D', -pD_{n+1}) & \xrightarrow{V^n} & W_{n+1} \Omega_{(X,-D)}^q \\ \downarrow & & \downarrow \\ F_{X*}^n \Omega_X^q / B_n \Omega_X^q & \xrightarrow{V^n} & W_{n+1} \Omega_X^q \end{array}$$

where the top map is induced from Claim 8.21.2 with  $j = n$ . The definition of  $(\Omega/B)_{n,(X,-D)}^q$  as a pullback in (8.6.3) yields therefore a natural map

$$b_n : (\Omega/B)_{n,n}^q(-D', -pD_{n+1}) \rightarrow (\Omega/B)_{n,(X,-D)}^q.$$

Altogether we obtain the following diagram of  $\mathcal{O}_X$ -modules

$$(8.21.4) \quad \begin{array}{ccc} (\Omega/B)_{n,n}^q(-D', -pD_{n+1}) & \xrightarrow{(8.16.3)} & (\Omega/Z)_{n,n}^q(-D', -pD_{n+1}) \\ \downarrow b_n & & \downarrow a_n \\ (\Omega/B)_{n,(X,-D)}^q & \xrightarrow{(8.6.4)} & (\Omega/Z)_{n,(X,-D)}^q. \end{array}$$

The diagram commutes as both lines of the diagram map injectively into  $F_{X*}^n \Omega_X^{q-1}/B_n \rightarrow F_{X*}^n \Omega_X^q/Z_n$ . As (8.16.3) is surjective and  $a_n$  is an isomorphism as shown above, it remains to show that  $b_n$  induces an isomorphism between the kernels of the horizontal maps. By construction we have

$$\text{Ker}((8.6.4)) = \{a \in Z_n \Omega_X^q / B_n \Omega_X^q \mid V^n(a) \in W_{n+1} \Omega_{(X,-D)}^q\}.$$

We note

- $Z_n \Omega_X^q / B_n \Omega_X^q = C^{-n}(\Omega_X^q)$ ;
- $V^n(C^{-n}(b)) = \underline{p}^n b$ .

Thus the  $n$ -fold iterated Cartier operator  $C^n$  induces an isomorphism

$$C^n : \text{Ker}((8.6.4)) \xrightarrow{\sim} \{b \in \Omega_X^q \mid \underline{p}^n b \in W_{n+1} \Omega_{(X,-D)}^q\}.$$

By Lemma 8.11 the right hand side is isomorphic to  $\Omega_n^q(-D', -pD_{n+1})$  and thus it follows from 8.16(b), that  $b_n$  induces an isomorphism between the kernels of the horizontal maps of the diagram (8.21.4). This completes the proof.  $\square$

The proof of Theorem 8.7 is now complete.

**Remark 8.22.** Using Proposition 8.21, we can give an explicit description for  $(\Omega/B)_{n,(X,-D)}^q$ :

$$(\Omega/B)_{n,(X,-D)}^q = \frac{\sum_{j=0}^n C^{-j}(F_{X*}^{n-j} \Omega_j^q(-\underline{D}_j, -\underline{D}^{n+1-j}))}{\mathcal{B}_n^q(\Omega_n^\bullet(-D', -pD_{n+1}))}.$$

Here  $\mathcal{B}_n^q(\Omega_n^\bullet(-D', -pD_{n+1}))$  is defined inductively so that  $\mathcal{B}_0^q(\Omega_0^\bullet(-D', -pD_{n+1})) = 0$ , and

$$C^{-1} : \mathcal{B}_j^q(\Omega_n^\bullet(-D', -pD_{n+1})) \xrightarrow{\sim} \frac{\mathcal{B}_{j+1}^q(\Omega_n^\bullet(-D', -pD_{n+1}))}{\mathcal{B}_1^q(\Omega_n^\bullet(-D', -pD_{n+1}))},$$

for every  $j \in [0, n-1]$ .

## 9. DUALITY FOR HODGE-WITT COHOMOLOGY WITH MODULUS: FINITE LEVEL

In this section we prove the duality between Hodge-Witt sheaves with modulus and their counterpart with zeros and draw some first consequences. This is one of the main results of the paper, see Theorem 9.3.

Throughout this section we assume  $X \in \mathbf{Sm}$  is of pure dimension  $\dim X = N$  and we let  $D$  be an effective Cartier divisor on  $X$  such that  $D_{\text{red}}$  is an SNCD.

**Lemma 9.1.** *Set  $U = X \setminus D$ . The multiplication map*

$$W_n \Omega_U^q \times W_n \Omega_U^{N-q} \rightarrow W_n \Omega_U^N$$

*induces a well-defined map*

$$(9.1.1) \quad W_n \Omega_{(X,-D)}^q \times W_n \Omega_{(X,D)}^{N-q} \rightarrow W_n \Omega_X^N$$

*of Nisnevich sheaves on  $X$ .*

*Proof.* Let  $\eta \in D^{(0)}$  be a geometric point. Set  $\mathcal{O}_L := \mathcal{O}_{X,\eta}^h$ , it is a henselian discrete valuation ring with maximal ideal  $\mathfrak{m}_L$  and fraction field  $L$ . Let  $r \geq 1$  be the multiplicity of  $D$  at  $\eta$  and set

$$W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q := (W_n \Omega_{(X, -D), \eta}^q)^h = \text{Ker}(W_n \Omega_{\mathcal{O}_L}^q \rightarrow W_n \Omega_{\mathcal{O}_L/\mathfrak{m}_L^r}^q).$$

In view of the definition of  $W_n \Omega_{(X, D)}^{N-q}$  in 6.2 and as  $W_n \Omega_X^N$  is a successive extension of locally free  $\mathcal{O}_X$ -modules (see [Ill79, I, Corollaire 3.9]), it suffices to show that multiplication induces a well-defined map

$$W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q \times \text{Fil}_r^p W_n \Omega_L^{N-q} \rightarrow W_n \Omega_{\mathcal{O}_L}^N,$$

with  $\text{Fil}_r^p W_n \Omega_L^{N-q}$  as in Definition 2.6. By the equality  $\underline{p}^s(\text{Fil}_r W_{n-s} \Omega_L^{N-q}) = p^s \text{Fil}_{rp^s} W_n \Omega_L^{N-q}$  (see Lemma 2.9) and the formulas

$$\alpha \cdot \underline{p}^s(\beta) = \underline{p}^s(R^s(\alpha) \cdot \beta) \quad \text{and} \quad \alpha \cdot d(\gamma) = (-1)^q(d(\alpha \cdot \gamma) + -d(\alpha) \cdot \gamma),$$

for  $\alpha \in W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q$ ,  $\beta \in \text{Fil}_r W_{n-s} \Omega_L^{N-q}$ , and  $\gamma \in \text{fil}_r W_n \Omega_L^{N-q-1}$ , it suffices to show that multiplication induces a well-defined map

$$W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q \times \text{fil}_r W_n \Omega_L^{N-q} \rightarrow W_n \Omega_{\mathcal{O}_L}^N.$$

By [Hes04, Lemma 1.2.2]  $W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q$  is the degree  $q$ -part of the differential graded ideal in  $W_n \Omega_{\mathcal{O}_L}^*$  generated by  $W_n(\mathfrak{m}_L^r)$ . Thus any element  $\alpha \in W_n \Omega_{(\mathcal{O}_L, \mathfrak{m}_L^r)}^q$  is a sum of elements

$$V^i([a])\alpha' \quad \text{and} \quad dV^i([a])\alpha'', \quad \text{with } v_L(a) \geq r, \alpha' \in W_n \Omega_{\mathcal{O}_L}^q, \alpha'' \in W_n \Omega_{\mathcal{O}_L}^{q-1},$$

where  $v_L : L \rightarrow \mathbb{Z} \cup \{\infty\}$  denotes the discrete valuation on  $L$ . Let

$$m = \min\{v_p(r), n\}.$$

By Definition 2.4 any element in  $\text{fil}_r W_n \Omega_L^{N-q}$  is a sum of elements

- (1)  $\beta = V^j([b]) \text{dlog } u$ , with  $p^{n-j-1}v_L(b) \geq -r+1$ ,  $u \in K_{N-q}^M(L)$ ,  $j \in \{0, \dots, n-1\}$ ;
- (2)  $\gamma = V^j([c]) \text{dlog } v$ , with  $p^{n-j-1}v_L(c) \geq -r$ ,  $v \in K_{N-q}^M(\mathcal{O}_L)$ ,  $j \in \{n-m, \dots, n-1\}$ .

Thus it suffices to show that for  $a \in \mathcal{O}_L$  with  $v_L(a) \geq r$  and  $\beta$  and  $\gamma$  as in (1) and (2), respectively, we have

$$V^i([a])\beta, \quad V^i([a])\gamma \in W_n \Omega_{\mathcal{O}_L}^{N-q}, \quad dV^i([a])\beta, \quad dV^i([a])\gamma \in W_n \Omega_{\mathcal{O}_L}^{N-q+1}.$$

We consider the various cases separately.

**1st case:**  $V^i([a])\beta$ . We have

$$v_L(a^{p^j} b^{p^i}) \geq p^j r + p^i \frac{-r+1}{p^{n-j-1}} \geq p^j \geq 1 \implies V^i([a])\beta = V^{i+j}([a^{p^j} b^{p^i}] \text{dlog } u) \in W_n \Omega_{\mathcal{O}_L}^N.$$

**2nd case:**  $V^i([a])\gamma$ . We have

$$v_L(a^{p^j} c^{p^i}) \geq p^j r + p^i \frac{-r}{p^{n-j-1}} \geq 0 \implies V^i([a])\gamma = V^{i+j}([a^{p^j} c^{p^i}] \text{dlog } v) \in W_n \Omega_{\mathcal{O}_L}^N.$$

**3rd case:**  $dV^i([a])\beta$ .

**1st subcase:**  $n-1 \geq j \geq i \geq 0$ . We have

$$v_L(ba^{p^{j-i}}) \geq \frac{-r+1}{p^{n-j-1}} + p^{j-i}r > 0 \implies dV^i([a])\beta = V^j([b][a]^{p^{j-i}} \text{dlog } \{a, u\}) \in W_n \Omega_{\mathcal{O}_L}^N.$$

**2nd subcase:**  $n-1 \geq i \geq j \geq 0$ . We have

$$v_L(ab^{p^{i-j}}) \geq r + p^{i-j} \frac{-r+1}{p^{n-j-1}} \geq 1 \implies$$

$$dV^i([a])\beta = V^j \left( dV^{i-j}([ab^{p^{i-j}}] \text{dlog } u) - V^{i-j}([ab^{p^{i-j}}] \text{dlog } \{b, u\}) \right) \in W_n \Omega_{\mathcal{O}_L}^N.$$

**4th case:**  $dV^i([a])\gamma$ .

**1st subcase:**  $n-1 \geq j \geq n-m$  and  $\min\{n-2, j\} \geq i$ . We have

$$v_L(ca^{p^{j-i}}) \geq \frac{-r}{p^{n-j-1}} + p^{j-i}r > 0 \implies dV^i([a])\gamma = V^j([ca^{p^{j-i}}] \operatorname{dlog}\{a, v\}) \in W_n\Omega_{\mathcal{O}_L}^N.$$

**2nd subcase:**  $n-1 = j = i \geq n-m$ . We have

$$dV^{n-1}([a])\gamma = V^{n-1}([ca] \operatorname{dlog} a \operatorname{dlog} v)$$

and  $v_L(ca) \geq 0$ . If  $v_L(ca) \geq 1$ , then  $dV^{n-1}([a])\gamma$  is clearly regular. If  $v_L(ca) = 0$ , we have  $v_L(a) = r$  and  $v_p(r) \geq m \geq 1$ . Thus in this case  $\operatorname{dlog} a \in \Omega_{\mathcal{O}_L}^1$  and hence also  $dV^{n-1}([a])\gamma \in W_n\Omega_{\mathcal{O}_L}^N$ .

**3rd subcase:**  $n-1 > i \geq j \geq n-m$ . We have

$$v_L(ac^{p^{i-j}}) \geq r - \frac{rp^{i-j}}{p^{n-j-1}} > 0 \implies dV^i([a])\gamma = V^j \left( dV^{i-j}([ac^{p^{i-j}}] \operatorname{dlog} v) - V^{i-j}([ac^{p^{i-j}}] \operatorname{dlog}\{c, v\}) \right) \in W_n\Omega_{\mathcal{O}_L}^N.$$

**4th subcase:**  $n-1 = i \geq j \geq n-m$ . We have

$$dV^{n-1}([a])\gamma = p^j dV^{n-1}([ac^{p^{n-j-1}}] \operatorname{dlog} v) - V^{n-1}([ac^{p^{n-j-1}}] \operatorname{dlog} c \operatorname{dlog} v)$$

and  $v_L(ac^{p^{n-j-1}}) \geq 0$ . If  $v_L(ac^{p^{n-j-1}}) \geq 1$ , then  $dV^{n-1}([a])\gamma$  is clearly regular. If  $v_L(ac^{p^{n-j-1}}) = 0$ , then  $p^{n-j-1}v_L(c) = -r$ . Since  $m \leq v_p(r) = n-j-1+v_p(v_L(c))$ , we see that  $p|v_L(c)$ . Thus  $\operatorname{dlog} c \in \Omega_{\mathcal{O}_L}^1$  and hence also  $dV^{n-1}([a])\gamma \in W_n\Omega_{\mathcal{O}_L}^N$ .

This completes the proof of the lemma.  $\square$

**9.2.** Consider the scheme  $W_nX = (X, W_n\mathcal{O}_X)$  and denote by  $\pi_n : W_nX \rightarrow \operatorname{Spec} W_n(k)$  the structure map, it is separated and of finite-type. Recall that  $W_n\Omega_{(X,D)}^q$  and  $W_n\Omega_{(X,-D)}^q$  can be viewed as coherent sheaves on  $W_nX$ , see Proposition 6.3 and Definition 8.1.

By [Eke84, I, Theorem 4.1] there is a canonical isomorphism

$$(9.2.1) \quad W_n\Omega_X^N[N] \xrightarrow{\sim} \pi_n^! W_n(k),$$

where  $\pi_n^!$  denotes the twisted inverse image from Grothendieck duality, see [Har66], [Con00], also [Lip09, 4.8 and 4.10]. Hence

$$\mathcal{D}_{X,n} := R\mathcal{H}om_{W_n\mathcal{O}_X}(-, W_n\Omega_X^N) : D_c^b(W_n\mathcal{O}_X) \rightarrow D_c^b(W_n\mathcal{O}_X)$$

is a dualizing functor in the sense that the canonical map  $\operatorname{id}_{D_c^b(W_n\mathcal{O}_X)} \rightarrow \mathcal{D}_{X,n} \circ \mathcal{D}_{X,n}$  is an isomorphism, see, e.g., [Har66, V, §10]. If  $f : X \rightarrow Y$  is a proper morphism between smooth  $k$ -schemes and  $Y$  is of pure dimension  $e$ , then  $f_n : W_nX \rightarrow W_nY$  is proper as well and Grothendieck duality yields an isomorphism

$$(9.2.2) \quad Rf_{n*}(\mathcal{D}_{X,n}(-)) \cong \mathcal{D}_{Y,n}(Rf_{n*}(-))[-r],$$

where  $r = N - e$  is the relative dimension of  $f$ .

The following is one of the main results of the paper. For  $D = \emptyset$  it is due to Ekedahl, see [Eke84, II], for  $D$  is reduced, it is [Nak05, Theorem 5.3(1)], cf. [Hyo91, (3.3.1)].

**Theorem 9.3.** *The multiplication map (9.1.1) induces isomorphisms, for all  $q, n$ ,*

$$(9.3.1) \quad W_n\Omega_{(X,D)}^{N-q} \xrightarrow{\sim} \mathcal{D}_{X,n}(W_n\Omega_{(X,-D)}^q)$$

and

$$(9.3.2) \quad W_n\Omega_{(X,-D)}^q \xrightarrow{\sim} \mathcal{D}_{X,n}(W_n\Omega_{(X,D)}^{N-q}).$$

*Proof.* As  $\mathcal{D}_{X,n}$  is dualizing we obtain the isomorphism (9.3.2) from (9.3.1) by applying  $\mathcal{D}_{X,n}$ . In view of Theorem 6.4 and Theorem 8.7 the proof of (9.3.1) is along the lines of Ekedahl's proof. We explain the strategy and the required modifications in the following. The case  $n = 1$  holds by Proposition 8.5. For  $m \in \{0, \dots, n\}$  denote by  $i_m : W_m X \hookrightarrow W_{n+1} X$  the closed immersion induced by the restriction  $R^{n+1-m} : W_{n+1} \mathcal{O}_X \rightarrow W_m \mathcal{O}_X$ . By Theorem 8.7, the sheaf  $W_{n+1} \Omega_{(X,-D)}^q$  is a successive extension of  $W_{n+1} \mathcal{O}_X$ -modules of the form  $i_{1*} M$ , with  $M$  a coherent locally free  $\mathcal{O}_X$ -module. Thus

$$(9.3.3) \quad \text{Ext}_{W_{n+1} \mathcal{O}_X}^i(W_{n+1} \Omega_{(X,-D)}^q, W_{n+1} \Omega_X^N) = 0, \text{ for all } i \geq 1,$$

by the same proof as in [Eke84, II, Lemma 2.2.7]. Moreover we get that the restriction map

$$(9.3.4) \quad \mathcal{D}_{n+1,X}(W_n \Omega_{(X,-D)}^q) \rightarrow j_* \mathcal{D}_{n+1,U}(W_{n+1} \Omega_U^q)$$

along the open immersion  $j : U = X \setminus D \hookrightarrow X$  is injective. Set

$$\text{gr}_{(X,-D)}^n := \text{Ker}(R : W_{n+1} \Omega_{(X,-D)}^q \rightarrow i_{n*} W_n \Omega_{(X,-D)}^q)$$

and

$$\text{gr}_{1,(X,D)}^n := \text{Coker}(\underline{p} : i_{n*} W_n \Omega_{(X,D)}^{N-q} \rightarrow W_{n+1} \Omega_{(X,D)}^{N-q}).$$

Consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & i_{n*} W_n \Omega_{(X,D)}^{N-q} & \xrightarrow{\underline{p}} & W_{n+1} \Omega_{(X,D)}^{N-q} & \longrightarrow & \text{gr}_{1,(X,D)}^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{n*} \mathcal{D}_{X,n}(W_n \Omega_{(X,-D)}^q) & \longrightarrow & \mathcal{D}_{X,n+1}(W_{n+1} \Omega_{(X,-D)}^q) & \longrightarrow & \mathcal{D}_{X,n+1}(\text{gr}_{(X,-D)}^n) \longrightarrow 0, \end{array}$$

where the vertical maps are induced by the multiplication map (9.1.1) and the horizontal map on the bottom left side is induced by the duality isomorphism  $i_{n*} \circ \mathcal{D}_{X,n} = \mathcal{D}_{X,n+1} \circ i_{n*}$  composed with the dual of the restriction  $R$ . As the restriction map (9.3.4) is injective it follows from [Eke84, II, (2.2.8), (2.2.9)] that the diagram commutes. (Note that it is a diagram of sheaves by the vanishing of the Ext-groups shown above.) As the lines are exact it remains by induction to show that the right vertical arrow is an isomorphism. In view of the exact sequences in Theorem 6.4 and Theorem 8.7(1) the same argument as in [Eke84, II, Lemma 2.2.17] (see also the explanation around (8.5.2)) reduces us to show that the maps

$$(9.3.5) \quad Z_n \Omega_{(X,D)}^{N-q} \rightarrow \mathcal{H}om((\Omega/B)_{n,(X,-D)}^q, \Omega_X^N), \quad \text{and} \quad B_n \Omega_{(X,D)}^{N-q+1} \rightarrow \mathcal{H}om((\Omega/Z)_{n,(X,-D)}^{q-1}, \Omega_X^N),$$

induced by  $\alpha \mapsto (\beta \mapsto C^n(\alpha \wedge \beta))$ , are isomorphisms. These maps are exactly  $\mathcal{D}_{X,1}$  applied to the isomorphisms in part (3) of Theorem 8.7 and hence are isomorphisms as well. This completes the proof.  $\square$

**Corollary 9.4.** *The projective bundle formula and the blow-up formula from 7.4, (1) and (2) also hold for  $D$  replaced by  $-D$ .*

Grothendieck duality (9.2.2) yields:

**Corollary 9.5.** *Assume additionally that  $X$  is proper over  $k$ . Then there is a canonical isomorphism of finite  $W_n(k)$ -modules*

$$H^j(X, W_n \Omega_{(X,-D)}^q) \cong \text{Hom}_{W_n(k)} \left( H^{N-j}(X, W_n \Omega_{(X,D)}^{N-q}), W_n(k) \right).$$

**Corollary 9.6.** *Assume  $X$  is proper and let  $(Y, E)$  be another modulus pair with  $Y \in \mathbf{Sm}$  proper of pure dimension  $N_Y$  and  $E_{\text{red}}$  an SNCD and denote by  $\underline{\mathbf{MCor}}((X, D), (Y, E))$  the group of left proper admissible correspondences, see [KMSY21a, Definition 1.3.1]. Then for any correspondence  $\alpha \in \underline{\mathbf{MCor}}((X, D), (Y, E))$  we have a natural map*

$$\alpha_* : H^N(X, W_n \Omega_{(X, -D)}^{N-q}) \rightarrow H^{N_Y}(Y, W_n \Omega_{(Y, -E)}^{N_Y-q})$$

such that:

- (1) *If  $(Z, F)$  is a modulus pair with  $Z \in \mathbf{Sm}$  proper of pure dimension  $N_Z$  and  $F_{\text{red}}$  an SNCD, and  $\beta \in \underline{\mathbf{MCor}}((Y, E), (Z, F))$  then*

$$\beta_* \alpha_* = (\beta \circ \alpha)_* : H^N(X, W_n \Omega_{(X, -D)}^{N-q}) \rightarrow H^{N_Z}(Z, W_n \Omega_{(Z, -F)}^{N_Z-q}).$$

- (2) *If  $\alpha$  is induced by the graph of a morphism  $f : X \rightarrow Y$  satisfying  $D \geq f^*E$ , then  $\alpha_* = f_*$ , the dual of  $f^*$ .*

*In particular, if  $f : X \rightarrow Y$  is a morphism which induces an isomorphism  $X \setminus f^{-1}(E) \xrightarrow{\sim} Y \setminus E$ , then  $f_*$  induces an isomorphism  $f_* : H^N(X, W_n \Omega_{(X, f^*E)}^q) \xrightarrow{\sim} H^N(Y, W_n \Omega_{(Y, E)}^q)$ .*

*Proof.* Use Corollary 9.5 and the corresponding properties of  $H^0(X, W_n \Omega_{(X, D)}^q)$ .  $\square$

Recall that following an idea of Deligne [Har66, Appendix] Hartshorne defines in [Har72, §2] cohomology with compact support for coherent sheaves on schemes which are separated and of finite type over a field  $k$ . In fact the same approach works for schemes of finite type over an artinian ring, as it is only used that the cohomology of a coherent sheaf on a proper scheme has finite length over the base ring, to guarantee the exactness of certain limits. Let  $A$  be an artinian ring,  $V \rightarrow \text{Spec } A$  be a separated morphism of finite type, and  $G$  a coherent sheaf on  $V$ , then the corresponding cohomology with compact support is denoted by  $H_c^j(V, G)$ .

**Corollary 9.7.** *With the above notation and assuming that  $X$  is proper there are canonical isomorphisms*

$$H_c^j(W_n(U), W_n \Omega_U^q) \cong \varprojlim_r H^j(X, W_n \Omega_{(X, -rD)}^q) \cong \text{Hom}_{W_n(k)} \left( H^{N-j}(U, W_n \Omega_U^{N-q}), W_n(k) \right).$$

*Proof.* In view of the definition of  $H_c^j(V, G)$  and Theorem 9.3 we only have to check that the pro-systems of coherent  $W_n \mathcal{O}_X$ -modules  $\{W_n \Omega_{(X, -rD)}^q\}_r$  and  $\{W_n(I^r) \cdot W_n \Omega_X^q\}_r$  are isomorphic, where  $I = \text{Ker}(\mathcal{O}_X \rightarrow \mathcal{O}_D)$ . To this end, note that by definition we have an inclusion  $W_n(I^r) \cdot W_n \Omega_X^q \subset W_n \Omega_{(X, -rD)}^q$ , for all  $r \geq 1$ . Furthermore we have an inclusion

$$(9.7.1) \quad W_n \Omega_{(X, -p^{n+1}rD)}^q \subset W_n(I^{pr}) \cdot W_n \Omega_X^q.$$

To check this we may assume  $X = \text{Spec } A$  and  $rD = \text{Div}(f)$ . It follows from the recursive definition in (8.16.2) that  $(\Omega/B)_{j,n}^q(0, -prD)$  is a quotient of  $f^{p^{j+1}r} \Omega_A^q$  and similarly with  $(\Omega/Z)_{j,n}^{q-1}(0, -prD)$ . Thus  $\text{gr}_{(X, -D)}^n$  is by Propositions 8.14 and 8.21 generated by elements

$$V^n([f^{p^{n+1}r}] \alpha) = [f]^{pr} V^n(\alpha) \quad \text{and} \quad dV^n([f^{p^{n+1}r}] \beta) = [f]^{pr} dV^n(\beta),$$

where  $\alpha \in \Omega_A^q$ ,  $\beta \in \Omega_A^{q-1}$ . Thus (9.7.1) holds by induction.  $\square$

The following generalizes Ekedahl's isomorphism (9.2.1) to the case of thickenings of smooth  $k$ -schemes.

**Corollary 9.8.** *Let  $H \subset X$  be smooth closed subscheme of codimension 1. Denote by  $H_r \subset X$  the closed subscheme defined by the ideal sheaf  $\mathcal{O}_X(-rH)$ , for  $r \geq 1$ . Denote by  $f : W_n(H_r) \rightarrow \text{Spec } W_n(k)$  the structure map. Then there is an isomorphism in  $D(W_n\mathcal{O}_{H_r})$*

$$f^!W_n(k) \cong \frac{W_n\Omega_{(X,rH)}^N}{W_n\Omega_X^N}[N-1].$$

*Proof.* Denote by  $i : H_r \hookrightarrow X$  the closed immersion. Ekedahl's isomorphism (9.2.1) and Grothendieck duality yield an isomorphism

$$\mathcal{D}_{X,n}(i_*W_n\mathcal{O}_{H_r})[N] \cong i_*f^!W_n(k).$$

Now apply  $\mathcal{D}_{X,n}$  to the exact sequence

$$0 \rightarrow W_n\mathcal{O}_{(X,-rH)} \rightarrow W_n\mathcal{O}_X \rightarrow i_*W_n\mathcal{O}_{H_r} \rightarrow 0$$

and use Theorem 9.3 to obtain the isomorphism from the statement.  $\square$

## 10. DUALITY FOR HODGE-WITT COHOMOLOGY WITH MODULUS: INFINITE LEVEL

Throughout this section  $X \in \mathbf{Sm}$  has pure dimension  $\dim X = N$  and  $D$  is an effective Cartier divisor on  $X$  such that  $D_{\text{red}}$  is an SNCD. We denote by

$$(10.0.1) \quad i : W_{n-1}X \hookrightarrow W_nX$$

the closed immersion induced by the restriction  $R : W_n\mathcal{O}_X \rightarrow W_{n-1}\mathcal{O}_X$  and by

$$(10.0.2) \quad \sigma : W_nX \rightarrow W_nX$$

the finite morphism induced by  $W_n(F_X^*) : W_n\mathcal{O}_X \rightarrow W_n\mathcal{O}_X$ , where  $F_X : X \rightarrow X$  denotes the absolute Frobenius on  $X$ . With this notation, e.g., the Verschiebung  $V : (\sigma i)_*W_{n-1}\Omega_X^q \rightarrow W_n\Omega_X^q$  and the differential  $d : (\sigma^n)_*W_n\Omega_X^q \rightarrow (\sigma^n)_*W_n\Omega_X^{q+1}$  are  $W_n\mathcal{O}_X$ -linear morphisms.

**10.1.** We recall some definitions from [Eke84, III, Definition 2.1]. An *inverse de Rham-Witt system* on  $X$  consists of a family of quasi-coherent graded  $W_n\mathcal{O}_X$ -modules  $M_n$ ,  $n \geq 1$ , together with morphisms of graded  $W_n\mathcal{O}_X$ -modules

$$R : M_n \rightarrow i_*M_{n-1}, \quad F : M_n \rightarrow (\sigma i)_*M_{n-1}, \quad V : (\sigma i)_*M_{n-1} \rightarrow M_n, \quad d : (\sigma^n)_*M_n \rightarrow (\sigma^n)_*M_n(1),$$

where  $M_0 = 0$  and for a graded module  $M$  we denote by  $M(1)$  the graded module with  $M(1)^q = M^{q+1}$ , satisfying the following identities

$$RF = FR, \quad RV = VR, \quad Rd = dR, \quad FV = p, \quad VF = p, \quad d^2 = 0, \quad FdV = d.$$

A *direct de Rham-Witt system* consists of a family of quasi-coherent graded  $W_n\mathcal{O}_X$ -modules  $M_n$ ,  $n \geq 1$ , together with maps  $F$ ,  $V$ ,  $d$  as above, and (instead of the maps  $R$ ) morphisms of graded  $W_n\mathcal{O}_X$ -modules

$$\underline{p} : i_*M_{n-1} \rightarrow M_n,$$

such that  $\underline{p}$ ,  $F$ ,  $V$ , and  $d$  satisfy the same identities as above with  $R$  replaced by  $\underline{p}$ . We call a direct (resp. inverse) de Rham-Witt system *coherent* if each  $M_n$  is a coherent  $W_n\mathcal{O}_X$ -module. We denote by

$$\text{inv-dRW}_X \quad \text{and} \quad \text{dir-dRW}_X$$

the categories of inverse- and direct de Rham-Witt systems, respectively, with morphisms defined in the obvious way, and by  $\text{inv-dRW}_{X,c}$  and  $\text{dir-dRW}_{X,c}$  their full subcategories of coherent objects. These are abelian categories and the forgetful functors to the product category  $\prod_n(\text{graded } W_n\mathcal{O}_X\text{-modules})$  sending  $((M_n)_n, F, V, d, R(\text{resp. } \underline{p}))$  to  $(M_n)_n$  is exact and conservative. For  $M \in \text{dir-dRW}$  we denote by  $M(j)$  the direct de Rham-Witt system

with shifted grading  $M_n(j)^q = M_n^{q+j}$ , and where  $R_{M(j)} = R_M$ ,  $F_{M(j)} = F_M$ ,  $V_{M(j)} = V_M$ , and  $d_{M(j)} = (-1)^j d_M$ .

Assume  $X$  is proper. Then the global section functor  $\Gamma(X, -)$  derives to

$$R\Gamma(X, -) : D^b(\text{inv-dRW}_{X,c}) \rightarrow D^b(\text{inv-dRW}_{k,c}), \quad D^b(\text{dir-dRW}_{X,c}) \rightarrow D^b(\text{dir-dRW}_{k,c}),$$

where  $\text{inv-dRW}_{k,c} := \text{inv-dRW}_{\text{Spec } k,c}$  and similarly with  $\text{dir-dRW}$ , see [Eke84, III, (2.5.2)].

**Example 10.2.** We have

$$W_{\infty \bullet} \Omega_{(X, \pm D)}^* := ((W_n \Omega_{(X, \pm D)}^*)_n, \underline{p}, F, V, d) \in \text{dir-dRW}_{X,c},$$

and

$$W_{\bullet} \Omega_{(X, \pm D)}^* := ((W_n \Omega_{(X, \pm D)}^*)_n, R, F, V, d) \in \text{inv-dRW}_{X,c}.$$

**10.3.** Recall from [Eke84, III, Example 2.2.1] that the top Witt forms  $W_n \Omega_X^N$ ,  $n \geq 1$ , together with the maps  $\underline{p}$ ,  $V$ , and the  $n$ -fold Cartier operators  $C^n : (\sigma^n)_* W_n \Omega_X^N \rightarrow W_n \Omega_X^N$  form a dualizing system in the sense of [Eke84, III, Definition 2.2]. Hence we obtain a well-defined functor

$$(\text{dir-dRW}_{X,c})^o \rightarrow \text{inv-dRW}_X, \quad M = (M_n)_n \mapsto (\underline{\text{Hom}}_{W_n \mathcal{O}_X}(M_n, W_n \Omega_X^N))_n.$$

We only remark that the maps  $R$ ,  $F$ ,  $V$ , and  $d$  on  $(\underline{\text{Hom}}_{W_n \mathcal{O}_X}(M_n, W_n \Omega_X^N))_n$  are induced by  $\underline{p}_M^*$ ,  $V_M^*$ ,  $F_M^*$ , and  $d_M^*$ , respectively, and refer to [Eke84, p. 205 – 206] for the details, see also [CR12, 1.6.6].

Let  $E_{n,X}$  be the Cousin complex of  $W_n \Omega_X^N$ . By [Eke84, III, Example 2.2]  $E = (E_{n,X})_n$  has the structure of a complex of dualizing systems (and is a Witt residual complex in the sense of [CR12, Definition 1.8.3]). We obtain a functor between the derived categories

$$\mathcal{D}_{X,\bullet} : D^b(\text{dir-dRW}_{X,c})^o \rightarrow D^b(\text{inv-dRW}_X), \quad M \mapsto \mathcal{D}_{X,\bullet}(M),$$

where

$$(\mathcal{D}_{X,\bullet}(M))_n = \underline{\text{Hom}}_{W_n \mathcal{O}_X}(M_n, E_{n,X}) = \mathcal{D}_{X,n}(M_n) \quad \text{in } D_c^b(W_n \mathcal{O}_X\text{-mod}),$$

with  $\mathcal{D}_{X,n}$  as in 9.2 and where the degree  $q$ -part of  $\mathcal{D}_{X,n}(M_n)$  is  $\mathcal{D}_{X,n}(M_n^{-q})$ .

**Corollary 10.4.** *The multiplication map induces isomorphisms in  $D^b(\text{inv-dRW}_{X,c})$*

$$W_{\bullet} \Omega_{(X,-D)}^* \cong \mathcal{D}_{X,\bullet}(W_{\infty \bullet} \Omega_{(X,D)}^*)(-N) \quad \text{and} \quad W_{\bullet} \Omega_{(X,D)}^* \cong \mathcal{D}_{X,\bullet}(W_{\infty \bullet} \Omega_{(X,-D)}^*)(-N).$$

*Proof.* By Theorem 9.3 we have the isomorphisms of graded  $W_n \mathcal{O}_X$ -modules

$$W_n \Omega_{(X, \pm D)}^* \cong \mathcal{H}om_{W_n \mathcal{O}_X}(W_n \Omega_{(X, \mp D)}^*, W_n \Omega_X^N)(-N),$$

for all  $n$ . Thus it suffices to show that these isomorphisms are compatible with the maps  $R, F, V, d$  defined on both sides. As before this can be checked after restriction to  $U = X \setminus D$  and hence follows from [Eke84, III, Proposition 2.4]  $\square$

**10.5.** Recall from [IR83, I, (1.1)] that the Cartier-Dieudonné-Raynaud ring  $\mathcal{R}$  is the graded (non-commutative)  $W = W(k)$ -algebra,  $\mathcal{R} = \mathcal{R}^0 \oplus \mathcal{R}^1$  generated by symbols  $F, V \in \mathcal{R}^0$  and  $d \in \mathcal{R}^1$  subject to the relations

$$\sigma(a)F = Fa, aV = V\sigma(a), da = ad, FV = p = VF, dd = 0, FdV = d,$$

for  $a \in W$ , where  $\sigma : W \rightarrow W$  denotes the Frobenius lift.

We obtain a functor  $\lim : \text{inv-dRW}_{X,c} \rightarrow \text{Sh}(X, \mathcal{R})$ , the category of sheaves of  $\mathcal{R}$ -modules on  $X$ , where the limit is taken along the restriction map  $R$ . This functor derives to

$$R\lim : D^b(\text{inv-dRW}_{X,c}) \rightarrow D^b(X, \mathcal{R}),$$

see [Eke84, III, (2.6.2)]. For  $M \in \mathrm{Sh}(X, \mathcal{R})$ , the shift  $M(j)$  is defined by  $M(j)^q = M^{j+q}$  and the action of  $F, V, d$  on  $M(j)$  is induced by the action of  $F, V, (-1)^j d$  on  $M$ .

**Proposition 10.6.** *The following equalities hold in  $D^b(X, \mathcal{R})$*

$$R \lim W_{\bullet} \Omega_{(X, -D)}^* = W \Omega_{(X, -D)}^* := \mathrm{Ker} \left( W \Omega_X^* \rightarrow \bigoplus_i W \Omega_{D_i}^* \right),$$

where we write  $D = \sum_i D_i$  with  $D_{i, \mathrm{red}}$  smooth, and

$$R \lim W_{\bullet} \Omega_{(X, D)}^* = W \Omega_X^*(\log D),$$

the log de Rham-Witt complex, see, e.g., [Mat17, Proposition-Definition 3.10]. In particular,

$$R \lim W_{\bullet} \Omega_{(X, D)}^* = R \lim W_{\bullet} \Omega_{(X, D_{\mathrm{red}})}^*.$$

*Proof.* The statement for  $W_{\bullet} \Omega_{(X, -D)}^*$  follows directly from the surjectivity of  $R$ , see Theorem 8.7(2), and the left exactness of  $\lim$ . By (5.5.2) we have a natural inclusion,

$$(10.6.1) \quad W_{\bullet} \Omega_X^*(\log D) \hookrightarrow W_{\bullet} \Omega_{(X, D)}^*.$$

Let  $M > 0$  be a positive integer such that  $p^M$  is strictly larger than all the multiplicities of  $D$ . We claim

$$R^M(W_{n+M} \Omega_{(X, D)}^*) = W_n \Omega_X^*(\log D), \quad \text{for all } n \geq 1.$$

By the above we only have to prove this “ $\subset$ ” inclusion. It suffices to prove this after pullback to henselian dvf’s  $L$  of geometric type. Thus by Theorem 5.4 it suffices to show for  $r < p^M$  and  $s = 0, \dots, n-1$ ,

$$(10.6.2) \quad R^M(p^s \mathrm{fil}_{rp^s} W_{n+M} \Omega_L^*) \subset W_n \Omega_{\mathcal{O}_L}^*(\log).$$

Under our assumptions on  $M$ , the inequality

$$\frac{-rp^s}{p^{n+M-i-1}} \leq -1$$

is only possible if  $i > n-s-1$ . But in this case  $p^s V^i([a]) = 0$  in  $W_n(L)$ , for  $a \in L$ . In view of the definition of  $\mathrm{Fil}_r^p W_n \Omega_L^q$  in Definition 2.6 this yields the inclusion (10.6.2). It follows that the inclusion (10.6.1) is an isomorphism of pro-objects and hence

$$R \lim W_{\bullet} \Omega_{(X, D)}^* = R \lim W_{\bullet} \Omega_X^*(\log D) = W \Omega_X^*(\log D),$$

where the last equality follows from the surjectivity of the restriction map  $R$  on  $W_{\bullet} \Omega_X^*(\log D)$ .  $\square$

**Theorem 10.7.** *Assume additionally that  $X$  is proper and set*

$$W_{\infty} \Omega_{(X, D)}^* := \mathrm{colim} W_{\infty \bullet} \Omega_{(X, D)}^*,$$

see Example 10.2. Then we have isomorphisms in  $D^b(\mathcal{R})$

$$R\Gamma(X, W \Omega_{(X, -D)}^*) \cong R \mathrm{Hom}_W(R\Gamma(X, W_{\infty} \Omega_{(X, D)}^*), K/W)(-N)[-N],$$

where  $K = W[1/p]$ . In particular we have isomorphisms of  $\mathcal{R}^0$ -modules, for  $i, q \geq 0$ ,

$$H^i(X, W \Omega_{(X, -D)}^q) \cong \mathrm{Hom}_W(H^{N-i}(X, W_{\infty} \Omega_{(X, D)}^{N-q}), K/W).$$

*Proof.* Applying  $R\lim \circ R\Gamma(X, -)$  to the first isomorphism in Corollary 10.4 gives

$$\begin{aligned} R\Gamma(X, W\Omega_{(X, -D)}^*) &\cong R\Gamma(X, R\lim W_\bullet \Omega_{(X, -D)}^*) \quad (\text{by Proposition 10.6}) \\ &\cong R\lim R\mathrm{Hom}_{W_n}(R\Gamma(X, W_n \Omega_{(X, D)}^*), W_n)(-N)[-N] \quad (\text{by (9.2.2)}) \\ &\cong R\lim R\mathrm{Hom}_W(R\Gamma(X, W_n \Omega_{(X, D)}^*), K/W)(-N)[-N], \end{aligned}$$

where the transition maps in the last limit are given by precomposition with  $\underline{p}$ . Here the third isomorphism holds by definition of the restriction maps on  $\mathcal{D}_{X, \bullet}(M)$  in 10.3, see [Eke84, III, (2.3.2)]. Let  $\mathcal{U}$  be an affine covering of  $X$  and denote by  $C_n = C^\bullet(\mathcal{U}, W_n \Omega_{(X, D)}^*)$  the Čech complex (it is a complex of graded  $W_n[d]$ -modules). Then  $(C_n)_n$  is a representative of  $R\Gamma(W_\infty \Omega_{(X, D)}^*)$  in  $D^b(\mathrm{dir}\text{-}\mathrm{dRW}_{\mathrm{Spec} k, c})$  and  $\mathrm{colim}_{\underline{p}} C_n$  is a representative of  $R\Gamma(W_\infty \Omega_{(X, D)}^*)$ . As  $\underline{p} : W_n \Omega_{(X, -D)}^* \rightarrow W_{n+1} \Omega_{(X, -D)}^*$  is injective, the induced map  $\underline{p} : C_n \rightarrow C_{n+1}$  is term-wise injective. As  $K/W$  is an injective  $W$ -module,  $\underline{p}^* : \mathrm{Hom}(C_{n+1}, K/W) \rightarrow \mathrm{Hom}(C_n, K/W)$  is term-wise surjective and thus (e.g. [Sta23, Tag 07KW(5)])

$$\begin{aligned} R\lim R\mathrm{Hom}_W(R\Gamma(X, W_n \Omega_{(X, D)}^*), K/W) &= \lim \mathrm{Hom}_W(C_n, K/W) \\ &= \mathrm{Hom}_W(\mathrm{colim} C_n, K/W), \end{aligned}$$

which implies the statement.  $\square$

**10.8.** We denote by  $W[F]$  the free  $W$ -algebra with generator  $F$  subject to the relation  $F \cdot a = \sigma(a) \cdot F$ , for all  $a \in W$ , where  $\sigma : W \rightarrow W$  is the Frobenius. Let

$$R\Gamma_{\mathrm{crys}}((X, -D)/W) \in D^b(W[F])$$

be the simple complex associated to the double complex  $R\Gamma(X, W\Omega_{(X, -D)}^*)$ , where  $F$  acts via the absolute Frobenius  $F_X^*$ , cf. [Eke84, III, (5.3), (5.4)]. Furthermore denote by

$$\mathcal{F} : R\mathrm{Hom}_W(R\Gamma(X, W_\infty \Omega_{(X, D)}^q), K/W) \rightarrow R\mathrm{Hom}_W(R\Gamma(X, W_\infty \Omega_{(X, D)}^q), K/W)$$

the morphism, which sends a map  $\mu$  to the composition

$$R\Gamma(X, W_\infty \Omega_{(X, D)}^q) \xrightarrow{p^{N-q} \cdot V} R\Gamma(X, W_\infty \Omega_{(X, D)}^q) \xrightarrow{\mu} K/W \xrightarrow{\sigma} K/W.$$

Representing  $R\Gamma(X, W_\infty \Omega_{(X, D)}^q)$  by a Čech complex  $C_\infty^{\bullet, q}$  we find that  $(\mathrm{Hom}_W(C_\infty^{\bullet, q}, K/W), \mathcal{F})$  is a complex of  $W[F]$ -modules. As

$$d : \mathrm{Hom}_W(C_\infty^{\bullet, q}, K/W) \rightarrow \mathrm{Hom}_W(C_\infty^{\bullet, q-1}, K/W)$$

is given by (see [CR12, 1.6.6])

$$\mu \mapsto (-1)^{q+1} \mu \circ d$$

it follows from the relation  $Vd = pdV$  that  $(\mathrm{Hom}_W(C_\infty^{\bullet, \bullet}, K/W), \mathcal{F})$  is a double complex of  $W[F]$ -modules. We denote the image of the associated simple complex in  $D^b(W[F])$  by

$$R\mathrm{Hom}_W(R\Gamma_{\mathrm{crys}}((X, D)/W_\infty), K/W).$$

**Corollary 10.9.** *Assume  $X$  is proper. Multiplication in  $W_\bullet \Omega_X^*$  induces an isomorphism in  $D^b(W[F])$*

$$R\Gamma_{\mathrm{crys}}((X, -D)/W) \cong R\mathrm{Hom}_W(R\Gamma_{\mathrm{crys}}((X, D)/W_\infty), K/W)[-2N].$$

*Proof.* The isomorphism follows by applying the functor which sends a double complex to its associated simple complex from the isomorphism in Theorem 10.7. For the compatibility with  $F_X^*$  on the left and  $\mathcal{F}$  on the right we observe:

- for  $\alpha \in W_{n+m}\Omega_{(X,-D)}^q$  and  $\beta \in W_n\Omega_{(X,D)}^{N-q}$  ( $m \geq 1$ ) we have

$$V(R^m(F_X^*(\alpha))\beta) = V(p^q F(R^{m-1}\alpha)\beta) = R^{m-1}(\alpha) \cdot p^q V(\beta);$$

- under the identification  $W_n(k) \cong \frac{1}{p^n}W/W \subset K/W$  the inverse map to  $V : W_n(k) \rightarrow W_{n+1}(k)$  is induced by  $\sigma : K/W \rightarrow K/W$ .

Now the statement follows from this and the definition of  $\mathcal{F}$ .  $\square$

**10.10.** Set  $\mathcal{R}_n = \mathcal{R}/(V^n\mathcal{R} + dV^n\mathcal{R})$ . It is a right  $\mathcal{R}$ -module and a left  $W_n[d]$ -module. By [Ill83, (2.3)]  $\mathcal{R}_n \otimes_{\mathcal{R}}$  derives to

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L : D^b(\mathcal{R}) \rightarrow D^b(W_n[d]).$$

Recall that a complex  $M \in D^b(\mathcal{R})$  (or an  $\mathcal{R}$ -module  $M$ ) is called *coherent* (in the sense of Illusie-Raynaud-Ekedahl) if the following two conditions are satisfied

- (1)  $H^i(\mathcal{R}_n \otimes_{\mathcal{R}}^L M)$  is a finitely generated  $W_n$ -module for all  $i$ , and
- (2)  $M$  is complete, i.e.,  $M = R\lim_n(\mathcal{R}_n \otimes_{\mathcal{R}}^L M)$ .

By [Eke84, p. 190] (see also [Ill83, 2.4]) a complex  $M$  is coherent if and only if  $H^i(M)$  is a coherent  $\mathcal{R}$ -module in the sense of Illusie-Raynaud [IR83, I, Définition 3.9]. The coherent subcomplexes form a triangulated subcategory of  $D^b(\mathcal{R})$ , which is denoted by  $D_c^b(\mathcal{R})$ . Ekedahl shows in [Eke84, IV, Proposition 1.1] that there is a well-defined functor

$$(10.10.1) \quad \mathcal{D} : D_c^b(\mathcal{R})^{\text{op}} \rightarrow D_c^b(\mathcal{R}), \quad M \mapsto \mathcal{D}(M) = R\lim R\text{Hom}_{W_n}(\mathcal{R}_n \otimes_{\mathcal{R}}^L M, W_n),$$

which is dualizing in the sense that it satisfies  $\mathcal{D} \circ \mathcal{D} = \text{id}$ . Moreover for any  $M \in D_c^b(\mathcal{R})$

$$(10.10.2) \quad \mathcal{R}_n \otimes_{\mathcal{R}}^L \mathcal{D}(M) = R\text{Hom}_{W_n}(\mathcal{R}_n \otimes_{\mathcal{R}}^L M, W_n) \quad \text{in } D^b(W_n[d]).$$

In case  $D = \emptyset$  the following corollary is [IR83, II, Théorème 2.2], [Eke84, III, Theorem 2.9], and [IR83, II, (1.4.3)]. The last part on the compatibility with  $\mathcal{R}_n \otimes_{\mathcal{R}}^L$  follows also from [Nak05, Theorem 6.24], but the proof is different, see also [Mok93, Lemme 1.3.3].

**Corollary 10.11.** *Assume  $X$  is proper. We have*

$$R\Gamma(X, W\Omega_{(X, \pm D_{\text{red}})}^*) \in D_c^b(\mathcal{R}),$$

and

$$(10.11.1) \quad R\Gamma(X, W\Omega_{(X, -D_{\text{red}})}^*) \cong \mathcal{D}(R\Gamma(X, W\Omega_{(X, D_{\text{red}})}^*))(-N)[-N],$$

where  $\mathcal{D}$  is the dualizing functor (10.10.1). Moreover,

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L R\Gamma(X, W\Omega_{(X, \pm D_{\text{red}})}^*) = R\Gamma(X, W_n\Omega_{(X, \pm D_{\text{red}})}^*).$$

*Proof.* Write  $D_{\text{red}} = \sum_{j=1}^r D_j$ , with  $D_j$  smooth and connected. Set

$$D^{(s)} = \coprod_{(j_1 < \dots < j_s)} D_{j_1} \cap \dots \cap D_{j_s}.$$

By [Mat17, 9.] (see also [Nak05, Proposition 9.3]) and (5.5.2) there is an increasing filtration (the weight filtration)  $\{P_s\}_{s=0, \dots, r}$  of  $\mathcal{R}$ -submodules of  $W\Omega_{(X, D_{\text{red}})}^*$  with  $P_0 = W\Omega_X^*$  and  $P_r = W\Omega_{(X, D_{\text{red}})}^*$ , which fits into exact sequences of  $\mathcal{R}$ -modules

$$(10.11.2) \quad 0 \rightarrow P_{s-1} \rightarrow P_s \xrightarrow{\rho} W\Omega_{D^{(s)}}^*(-s) \rightarrow 0.$$

As it is not stated explicitly in *loc. cit.* we remark that  $\rho$  is a morphism of  $\mathcal{R}$ -modules. Indeed, the iterated residue map  $\rho$  is induced by the inverse of the isomorphism

$$W\Omega_{D^{(s)}}^*(-s) \rightarrow P_s/P_{s-1},$$

which sends a local form  $\alpha \in W\Omega_{D_{j_1} \cap \dots \cap D_{j_s}}^{q-s}$  to  $\mathrm{dlog}\{t_{j_1}, \dots, t_{j_s}\} \cdot \tilde{\alpha}$ , where  $\tilde{\alpha} \in W\Omega_X^{q-s}$  is a lift of  $\alpha$  and  $t_j$  is a local equation for  $D_j$ . This is clearly a morphism of  $\mathcal{R}$ -modules (see 10.5 for the “ $(-s)$ ” shift of an  $\mathcal{R}$ -module). As  $R\Gamma(D^{(s)}, W\Omega_{D^{(s)}}^*) \in D_c^b(\mathcal{R})$ , for all  $s = 0, \dots, r$ , by [IR83, II, Théorème 2.2], we find  $R\Gamma(X, W\Omega_{(X, D_{\mathrm{red}})}^*) \in D_c^b(\mathcal{R})$ . Similarly it follows from [IR83, II, (1.4.3)] (applied to  $W\Omega_{D^{(s)}}^*(-s)$ ) that we have

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L R\Gamma(X, W\Omega_{(X, D_{\mathrm{red}})}^*) = R\Gamma(X, W_n\Omega_{(X, D_{\mathrm{red}})}^*).$$

Hence applying  $R\lim \circ R\Gamma$  to (9.3.2) (for  $D_{\mathrm{red}}$ ) yields (10.11.1) in view of the discussion in 10.1, 10.3, 10.5. The remaining statements for  $-D_{\mathrm{red}}$  follow from (10.10.1) and (10.10.2).  $\square$

**10.12.** We can now use the results on coherent  $\mathcal{R}$ -modules from [IR83] and [Eke84] to get many consequences, which however only work in the case  $D$  is reduced or we work up to bounded torsion<sup>7</sup>. Here is a sample of corollaries, where we assume that  $X$  is proper additionally to our standing assumptions:

- (1)  $R\Gamma(X, W\Omega_{(X, D_{\mathrm{red}})}^*)$  is a complex of  $\mathcal{R}$ -modules of level  $N$ , i.e., it is represented by a complex of  $\mathcal{R}$ -modules which vanishes in degrees outside  $[0, N]$  and on which  $F$  is invertible in degree  $N$ . (That  $F$  is invertible in degree  $N$  in the case at hand, follows by induction from the exact sequences (10.11.2) and [Ill79, I, Proposition 3.7].) Therefore by [Eke84, III, 5.] taking the simple complex of the duality isomorphism in Theorem 10.11 we get an isomorphism of  $F$ -crystals of level  $N$

$$R\Gamma(X, W\Omega_{(X, -D_{\mathrm{red}})}^\bullet) \cong R\mathrm{Hom}_W(R\Gamma(X, W\Omega_{(X, D_{\mathrm{red}})}^\bullet), W)[-2N].$$

- (2) We have

$$R\Gamma(X, W\Omega_{(X, D_{\mathrm{red}})}^\bullet) = R\Gamma(X, W\Omega_X^\bullet(\log D)) \cong R\Gamma_{\log\text{-crys}}((X, D)/W),$$

where the equality holds by (5.5.2) and the isomorphism, e.g., by [Mat17, Theorem 7.2]. It is a perfect complex of  $W$ -modules by, e.g., [Ill79, II, Théorème 2.7] and (10.11.2). Thus duality yields an isomorphism

$$R\Gamma(X, W\Omega_{(X, -D_{\mathrm{red}})}^\bullet) \cong R\mathrm{Hom}_W(R\Gamma_{\log\text{-crys}}((X, D)/W), W)[-2N].$$

Hence  $R\Gamma(X, W\Omega_{(X, -D_{\mathrm{red}})}^\bullet)$  is a perfect complex as well.

- (3) As the cokernel of the natural inclusion  $W\Omega_{(X, -D)}^* \rightarrow W\Omega_{(X, -D_{\mathrm{red}})}^*$  is annihilated by some fixed  $p$ -power (take  $p^r$  greater than all the multiplicities in  $D$ ) it follows that

$$R\Gamma(X, W\Omega_{(X, -D)}^*) \rightarrow R\Gamma(X, W\Omega_{(X, -D_{\mathrm{red}})}^*)$$

is an isomorphism up to bounded  $p$ -primary torsion. As the complex on the right is a coherent  $\mathcal{R}$ -module, it follows that the cohomology groups  $H^i(X, W\Omega_{(X, -D)}^q)$  are finitely generated  $W$ -modules up to bounded torsion, and  $R\Gamma(X, W\Omega_{(X, -D)}^\bullet)$  is a perfect complex of  $W$ -modules up to bounded (but possibly infinitely generated) torsion.

- (4) As observed in [Nak05, Remark 5.4(1)] it follows from Shiho’s comparison of log-crystalline cohomology for  $(X, D)$  with rigid cohomology, see [Shi02, Corollary 2.4.13 and Theorem 3.1.1], and Berthelot’s Poincaré duality for rigid cohomology, see [Ber97, Théorème 2.4], that the above yields an isomorphism

$$R\Gamma(X, W\Omega_{(X, -D)}^\bullet) \otimes_W K = R\Gamma_{c, \mathrm{rig}}(U),$$

<sup>7</sup>We say that a certain statement in an additive category  $\mathcal{C}$  is true up to bounded torsion, if it is true in the localized category  $\mathcal{C}_{\mathbb{Q}}$  which has the same objects but the Hom’s are tensored with  $\mathbb{Q}$ .

where  $U = X \setminus D$  and  $K = \text{Frac}(W)$ . Thus for any effective Cartier divisor  $D$  with SNC support equal to  $X \setminus U$  the complex  $R\Gamma(X, W\Omega_{(X, -D)}^\bullet)$  is perfect up to bounded torsion and is an integral model for the compactly supported rigid cohomology of  $U$ .

(5) By [III83, Corollary 2.5.4] and the above the slope spectral sequence

$$E_1^{i,j} = H^i(X, W\Omega_{(X, -D)}^j) \implies H^*(X, W\Omega_{(X, -D)}^\bullet)$$

degenerates up to bounded torsion and thus by the same argument<sup>8</sup> as in [Blo77, III, Corollary 3.4] (see also [III79, II, Corollary 3.5]) we obtain the isomorphism

$$H_{c, \text{rig}}^n(U)_{[j, j+1[} \cong H^i(X, W\Omega_{(X, -D)}^{n-j}) \otimes_W K,$$

where the left hand side denotes the part of the compactly supported rigid cohomology of  $U$  on which the Frobenius acts with slope  $\lambda$  with  $j \leq \lambda < j+1$ . Note that the above isomorphism also appears in the proof of [Nak05, Theorem 5.9] (there for  $D$  reduced). In particular  $H^i(X, W\mathcal{O}_{(X, -D)}) \otimes_W K$  is equal to the compactly supported Witt vector cohomology of  $U$  defined in [BBE07].

(6) Assume  $U$  is affine (e.g.  $D$  is ample) then, the same argument as in [BBE07, Corollary 1.2] yields

$$(10.12.1) \quad H^i(X, W\Omega_{(X, -D)}^j) \otimes_W K = 0, \quad \text{for all } i+j < \dim X.$$

As rigid cohomology of an affine  $k$ -scheme vanishes above the dimension a similar argument using the slope decomposition of  $H^*(X, W\Omega_X^\bullet(\log D))$  yields the vanishing

$$(10.12.2) \quad H^i(X, W\Omega_{(X, D)}^j) \otimes_W K = 0, \quad \text{for all } i+j > \dim X,$$

these two vanishing results can be viewed as a kind of Kodaira-Akizuki-Nakano vanishing. See also [Tan22, Theorem 1.1], where a different kind of Kodaira vanishing for Witt vector cohomology is proven. The above two vanishing results depend only on  $D_{\text{red}}$  and can be found at least implicitly in the literature and are probably not so surprising. What might be more intriguing is that by (10.12.1) and (3) we know that  $H^i(X, W\Omega_{(X, -D)}^j)$ ,  $i+j < \dim X$ , is annihilated by some fixed  $p$ -power, say  $p^M$ . Hence by Theorem 10.7

$$p^M \cdot \text{colim}_{\underline{p}} H^i(X, W_\bullet \Omega_{(X, D)}^j) = 0, \quad \text{for all } i+j > \dim X.$$

Note that  $D$  does not need to be reduced here and that  $M$  might depend on  $D$ .

**Remark 10.13.** (1) We remark that the complex  $\mathcal{R}_n \otimes_{\mathcal{R}}^L R\Gamma(X, W\Omega_{(X, -D)}^*)$  will for non-reduced  $D$  in general not be isomorphic to  $R\Gamma(X, W_n \Omega_{(X, -D)}^*)$ . For example take  $X$  smooth projective with a smooth effective and ample divisor  $H$  such that  $H^i(X, W\mathcal{O}_X)$  does not vanish for some  $i > 0$ . Choose  $r \geq 1$  such that  $H^i(X, \Omega_{(X, -rH)}^j) = 0$  for all  $i < \dim X = N$  and for all  $j$ . (This is always possible by Serre vanishing and duality.) We claim that in this situation

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L R\Gamma(X, W\Omega_{(X, -rH)}^*) \neq R\Gamma(X, W_n \Omega_{(X, -rH)}^*) \quad \text{for all } n.$$

<sup>8</sup>This argument uses that  $V$  acts topologically nilpotent on  $H^i(X, W\Omega_{(X, -D_{\text{red}})}^{\leq j})$  in the  $p$ -adic topology, which follows from [III79, II, Proposition 2.10] and the resolution  $W\Omega_{(X, -D_{\text{red}})}^{\leq j} \rightarrow W\Omega_{D_{\text{red}}}^{\leq j}$ , see [Mok93, Lemme 3.15.1] and [Nak05, Corollary 6.28]

Indeed, else it would follow from Ekedahl's Nakayama Lemma [Eke85, I, Proposition 1.1, ii)] that we have  $H^i(X, W_n \Omega_{(X, -rH)}^j) = 0$ , for all  $i < N$  and all  $j, n$ . By Corollary 9.5 we get  $H^i(X, W_n \Omega_{(X, rH)}^j) = 0$  for all  $i > 0$  and all  $j, n$ . In particular

$$H^i(X, W\mathcal{O}_X) = H^i(X, R\lim_n W_n \mathcal{O}_{(X, rH)}) = \lim_n H^i(X, W_n \mathcal{O}_{(X, rH)}) = 0, \quad \text{for all } i,$$

where the first equality holds by Proposition 10.6, and the second follows from the fact that the  $H^i(X, W_n \mathcal{O}_{(X, rH)})$  are  $W$ -modules of finite length, for all  $i$ . This contradicts our choice of  $X$ .

(2) Item (1) in particular implies that in general

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L W\Omega_{(X, -D)}^* \neq W_n \Omega_{(X, -D)}^*.$$

That Nakajima's method [Nak05, Theorem 6.24] cannot be applied is related to Remark 8.2. We analyze the tor-terms in case  $D_{\text{red}}$  is smooth: define the quotient  $Q$  by the exact sequence

$$0 \rightarrow W\Omega_{(X, -D)}^* \rightarrow W\Omega_{(X, -D_{\text{red}})}^* \rightarrow Q \rightarrow 0.$$

Definition 8.1 together with the Snake Lemma give the exact sequence

$$0 \rightarrow Q \rightarrow W\Omega_D^* \rightarrow W\Omega_{D_{\text{red}}}^* \rightarrow 0.$$

We have

$$\mathcal{R}_n \otimes_{\mathcal{R}}^L W\Omega_{D_{\text{red}}}^* = W_n \Omega_{D_{\text{red}}}^* \quad \text{and} \quad \mathcal{R}_n \otimes_{\mathcal{R}}^L W\Omega_{(X, -D_{\text{red}})}^* = W_n \Omega_{(X, -D_{\text{red}})}^*,$$

where the first equality holds by [IR83, II, Théorème (1.2)] and the second equality by [Nak05, Theorem 6.24]. Moreover

$$\text{Tor}_0^{\mathcal{R}}(\mathcal{R}_n, W\Omega_D^*) = W_n \Omega_D^*,$$

by, e.g., [HM03, Lemma 3.2.4]. Altogether we get an exact sequence

$$0 \rightarrow \text{Tor}_1^{\mathcal{R}}(\mathcal{R}_n, W\Omega_D^*) \rightarrow \text{Tor}_0^{\mathcal{R}}(\mathcal{R}_n, W\Omega_{(X, -D)}^*) \rightarrow W_n \Omega_{(X, -D_{\text{red}})}^* \rightarrow W_n \Omega_D^* \rightarrow W_n \Omega_{D_{\text{red}}}^* \rightarrow 0$$

and equalities

$$\text{Tor}_1^{\mathcal{R}}(\mathcal{R}_n, W\Omega_{(X, -D)}^*) = \text{Tor}_2^{\mathcal{R}}(\mathcal{R}_n, W\Omega_D^*), \quad \text{Tor}_j^{\mathcal{R}}(\mathcal{R}_n, W\Omega_{(X, -D)}^*) = 0, \quad j \neq 0, 1.$$

Here the vanishing follows from [IR83, I, Corollaire (3.3)], which can be also used to express  $\text{Tor}_j^{\mathcal{R}}(\mathcal{R}_n, W\Omega_D^*)$ ,  $j = 1, 2$ , more explicitly.

## 11. MILNE-KATO DUALITY WITH MODULUS

In this section we generalize some of the duality results from Milne [Mil76] and [Kat86], [Kat87] to the modulus setup. The main result is Theorem 11.15.

We continue to assume that  $X$  is smooth of pure dimension  $N$  and  $D$  is an effective Cartier divisor on  $X$  such that  $D_{\text{red}}$  is an SNCD. We denote by  $j : U = X \setminus D \hookrightarrow X$  the open immersion of the complement.

**11.1.** We note that the Nisnevich sheaves  $W_n \Omega_{(X, \pm D)}^q$  defined by  $V \mapsto \Gamma(V, W_n \Omega_{(V, \pm D|_V)}^q)$  are in fact étale sheaves of coherent  $W_n \mathcal{O}_X$ -modules. Indeed, if  $u : V \rightarrow X$  is étale, then so is  $u_n : W_n V \rightarrow W_n X$  and it follows from the étale base change for the de Rham-Witt complex (see [Ill79, I, Proposition 1.14]) and the Definition 8.1 (in case of  $-D$ ), and the Propositions 6.15 and 6.18 (in case of  $+D$ ) that we have

$$\Gamma(V, W_n \Omega_{(V, \pm D|_V)}^q) = \Gamma(V, u_n^* W_n \Omega_{(X, \pm D)}^q).$$

In particular we have

$$(11.1.1) \quad R\epsilon_* W_n \Omega_{(X, \pm D)}^q = W_n \Omega_{(X, \pm D)}^q,$$

where  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the change-of-sites morphism.

**11.2.** Recall that the Cartier operator on the de Rham-Witt complex is the morphism on  $X_{\text{ét}}$

$$(11.2.1) \quad C : F(W_{n+1} \Omega_X^q) \rightarrow W_n \Omega_X^q, \quad F(a) \mapsto R(a).$$

The so called inverse Cartier operator is given by

$$(11.2.2) \quad C^{-1} : W_n \Omega_X^q \rightarrow W_n \Omega_X^q / dV^{n-1} \Omega_X^{q-1}, \quad a \mapsto F(\tilde{a}), \quad \text{where } R(\tilde{a}) = a.$$

Note that these are morphisms of  $W_n \mathcal{O}_X$ -modules when  $F(W_{n+1} \Omega_X^q)$  is viewed as a submodule of  $F_* W_n \Omega_X^q$  and  $W_n \Omega_X^q / dV^{n-1} \Omega_X^{q-1}$  as a quotient-module of  $F_* W_n \Omega_X^q$ . Furthermore, we remark that  $W_n \Omega^N = F(W_{n+1} \Omega_X^N)$  and that the Cartier operator coincides in this case with the composition

$$(11.2.3) \quad C : F_{X*} W_n \Omega_X^N \cong F_{X*} \pi_n^! W_n(k)[-N] \cong F_{X*} \pi_n^! F_k^! W_n(k)[-N] \\ \cong F_{X*} F_X^! \pi_n^! W_n(k)[-N] \xrightarrow{\text{tr}_{F_X}} \pi_n^! W_n(k)[-N] = W_n \Omega_X^N,$$

where  $F_X : W_n(X) \rightarrow W_n(X)$  denotes the morphism induced by the absolute Frobenius (similarly for  $F_k : W_n(\text{Spec } k) \rightarrow W_n(\text{Spec } k)$ ),  $\text{tr}_{F_X} : F_{X*} F_X^! \rightarrow \text{id}$  denotes the counit of adjunction, and the other notation is as in 9.2, see [Eke84, II, Lemma 2.1]<sup>9</sup>.

**Lemma 11.3.** *Set*

$$(FW_{n+1} \Omega^q)_{(X, D)} := j_* F(W_{n+1} \Omega_U^q) \cap W_n \Omega_{(X, D)}^q,$$

and

$$(W_n \Omega^q / dV^{n-1})_{(X, -D)} := \frac{W_n \Omega_{(X, -D)}^q}{dV^{n-1} \Omega_X^{q-1} \cap W_n \Omega_{(X, -D)}^q}.$$

These extend naturally to coherent  $W_n \mathcal{O}_X$ -modules on  $X_{\text{ét}}$ . In particular

$$R\epsilon_*(FW_{n+1} \Omega^q)_{(X, D)} = (FW_{n+1} \Omega^q)_{(X, D)},$$

and

$$R\epsilon_*(W_n \Omega^q / dV^{n-1})_{(X, -D)} = (W_n \Omega^q / dV^{n-1})_{(X, -D)},$$

where  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the change-of-sites morphism. Furthermore, on  $X_{\text{ét}}$  the Cartier operator on  $j_* F(W_{n+1} \Omega_U^q)$  restricts to

$$(11.3.1) \quad C : (FW_{n+1} \Omega^q)_{(X, D)} \rightarrow W_n \Omega_{(X, D)}^q$$

and the inverse Cartier operator  $C^{-1}$  on  $W_n \Omega_X^q$  restricts to

$$(11.3.2) \quad C^{-1} : W_n \Omega_{(X, -D)}^q \rightarrow (W_n \Omega^q / dV^{n-1})_{(X, -D)}.$$

*Proof.* Since  $FW_{n+1} \Omega_U^q = \text{Ker}(F^{n-1}d : W_n \Omega_U^q \rightarrow \Omega_U^q)$ , by [Ill79, I, (3.21.1.1)], the sheaf

$$(11.3.3) \quad (FW_{n+1} \Omega^q)_{(X, D)} = \text{Ker}(F^{n-1}d : F_* W_n \Omega_{(X, D)}^q \rightarrow F_*^n \Omega_{(X, D)}^q)$$

is a kernel between coherent (étale) sheaves and hence is coherent. Similarly, the sheaf  $(W_n \Omega^q / dV^{n-1})_{(X, -D)}$  is a quotient of the coherent étale sheaf  $F_* W_n \Omega_{(X, -D)}^q$  and hence is coherent. The existence of (11.3.2) follows from the surjectivity of  $R$  onto  $W_n \Omega_{(X, -D)}^q$ , see

<sup>9</sup>In loc. cit. the statement is for  $C^n$ , the  $n$ -fold iterate of the Cartier operator, but the same proof works here.

Lemma 8.15. For the pole side, note that  $C$  defines a morphism  $C : F(W_{n+1}\Omega^q) \rightarrow W_n\Omega^q$  of Nisnevich sheaves with transfers and hence induces a well-defined map  $C : \underline{\omega}^{\text{CI}}F(W_{n+1}\Omega^q) \rightarrow \underline{\omega}^{\text{CI}}W_n\Omega^q$ . For  $q \geq 1$  we have  $\underline{\omega}^{\text{CI}}F(W_{n+1}\Omega^q)_{(X,D)} = (FW_{n+1}\Omega^q)_{(X,D)}$  by definition (see also 5.1) and Theorem 5.4, which yields (11.3.1). By the definition in (6.2.2), the case for  $q = 0$  is reduced to show the following: Let  $L$  be a henselian dvf and let  $a \in W_{n+1}(L)$  such that  $F(a) \in \text{fil}_r^p W_n(L)$ , for  $r \geq 2$ , then  $R(a) \in \text{fil}_r^p W_n(L)$  (see 2.4 and 2.6 for the notation). By definition we find  $b_s \in \text{fil}_{rp^s} W_n(L)$ ,  $s = 0, \dots, n-1$ , such that

$$F(a) = b_0 + \sum_{s \geq 1} p^s b_s = b_0 + \sum_{s \geq 1} F(p^{s-1} V(b_s)).$$

Hence there exists an  $a_0 \in W_{n+1}(L)$  with  $F(a_0) = b_0 \in \text{fil}_r W_n(L) \subset \text{fil}_r^{\log} W_n(L)$ . It follows directly from the definition that this implies  $a_0 \in \text{fil}_r^{\log} W_{n+1}(L)$  and hence  $R(a_0) \in \text{fil}_{[r/p]}^{\log} W_n(L) \subset \text{fil}_{r-1}^{\log} W_n(L) \subset \text{fil}_r W_n(L)$ , see (2.3.2). Moreover, for  $s \geq 1$ , we have  $V(b_s) \in \text{fil}_{rp^s} W_{n+1}(L)$  and hence  $R(V(b_s)) \in \text{fil}_{rp^{s-1}} W_n(L)$ . Thus

$$R(a) = R(a_0) + \sum_{s \geq 1} p^{s-1} R(V(b_s)) \in \text{fil}_r^p W_n(L).$$

This completes the proof of (11.3.1).  $\square$

**Definition 11.4.** We define the following two complexes of abelian sheaves on  $X_{\text{ét}}$

$$\mathbb{Z}/p^n(q)_{(X,D)} := \left( (FW_{n+1}\Omega^q)_{(X,D)} \xrightarrow{1-C} W_n\Omega_{(X,D)}^q \right) [-q]$$

and

$$\mathbb{Z}/p^n(q)_{(X,-D)} := \left( W_n\Omega_{(X,-D)}^q \xrightarrow{C^{-1}-1} (W_n\Omega^q/dV^{n-1})_{(X,-D)} \right) [-q].$$

Both complexes sit in degree  $[q, q+1]$ .

**11.5.** We make some comments on the complexes defined above:

- (1) If  $D = \emptyset$ , then the two complexes are isomorphic to  $W_n\Omega_{X,\log}^q[-q]$  in the derived category of abelian sheaves on  $X_{\text{ét}}$  (see, e.g., [Kat86, Lemma 4.1.5]), which is isomorphic to the motivic complex  $\mathbb{Z}/p^n(q)_X$ , by [GL00, Theorem 8.3].
- (2) Let  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  be the change-of-sites map then by Lemma 11.3

$$R\epsilon_* \mathbb{Z}/p^n(q)_{(X,D)} = \underline{\omega}^{\text{CI}} \left( (FW_{n+1}\Omega^q) \xrightarrow{1-C} W_n\Omega^q \right)_{(X,D)} [-q], \quad q \geq 1,$$

for  $q = 0$  the complex is still a two-term complex of cube invariant Nisnevich sheaves with transfers.

- (3) As the open immersion  $j : U \hookrightarrow X$  is affine we have in  $D(X_{\text{ét}})$

$$\text{colim}_r \mathbb{Z}/p^n(q)_{(X,rD)} = Rj_* \mathbb{Z}/p^n(q)_U.$$

- (4) Complexes similar to  $\mathbb{Z}/p^n(q)_{(X,-D)}$  were already considered in [JSZ18], [Mor19], and [GK]. More precisely, in [JSZ18, (3.1.3)] the analog complex with  $W_n\Omega_{(X,-D)}^q$  replaced by  $W_n\Omega_X^q(\log D) \otimes_{W_n\mathcal{O}_X} W_n\mathcal{O}_X(-D)$  is used, and in [GK, below (7.12)] (see also [Mor19, 2.4]) the sheaf  $\text{Ker}(W_n\Omega_X^q \rightarrow W_n\Omega_D^q)$  is considered instead. The pro-systems over  $(X, -rD)$ ,  $r \geq 1$ , defined by the respective complexes are isomorphic but for fixed  $D$  the complexes differ. We do not know whether they are quasi-isomorphic, but we expect them not to be.
- (5) For  $q = 0$  a complex similar to  $\mathbb{Z}/p^n(0)_{(X,D)}$  is considered in [KS14, (3-2)], where the filtration  $\text{fil}_r^{\log} W_n(L)$  was used instead of  $\text{fil}_r^p W_n(L)$  which is used here, also there the sequence with  $F = C^{-1}$  is considered instead of the one with  $C$  considered here.

**Lemma 11.6.** *Set*

$$W_n \Omega_{(X, -D), \log}^q := \mathcal{H}^q(\mathbb{Z}/p^n(q)_{(X, -D)}).$$

*Then the natural map*

$$W_n \Omega_{(X, -D), \log}^q[-q] \xrightarrow{\sim} \mathbb{Z}/p^n(q)_{(X, -D)}$$

*is an isomorphism in the derived category  $D(X_{\text{ét}})$ .*

*Proof.* We have to show that  $C^{-1} - 1 : W_n \Omega_{(X, -D)}^q \rightarrow (W_n \Omega^q / dV^{n-1})_{(X, -D)}$  is surjective in the étale topology. For  $n = 1$  this holds by the following claim, which is a version of the surjectivity statement in [JSZ18, Theorem 1.2.1]:

**Claim 11.6.1.** *Let  $A$  be an SNCD, and  $B$  be an effective Cartier divisor with  $(A + B)_{\text{red}}$  an SNCD. Then*

$$\Omega_X^q(\log A)(-B) \xrightarrow{C^{-1}-1} \Omega_X^q(\log A)(-B) / (d\Omega_X^{q-1} \cap \Omega_X^q(\log A)(-B))$$

*is surjective on  $X_{\text{ét}}$ .*

Indeed, let  $R$  be the strict henselization of a local ring at a closed point of  $X$  and choose a system  $t_1, \dots, t_d$  of regular parameters of  $R$  such that

$$A = \text{Div}(t_1 \dots t_e), \quad B = \text{Div}(t_1^{r_1} \dots t_e^{r_e} t_{e+1}^{r_{e+1}} \dots t_f^{r_f})$$

where  $e, f$  are integers such that  $0 \leq e \leq f \leq d$ , and  $r_1, \dots, r_f$  are non-negative integers. Let  $I^q$  be the set of strictly increasing functions  $\{1, \dots, q\} \rightarrow \{1, \dots, d\}$ . Let  $s \in I^q$  and let  $\theta$  be the biggest integer in the image of  $s$  such that  $s(\theta) \leq e$ . Denote

$$\omega_s = \text{dlog } t_{s(1)} \dots \text{dlog } t_{s(\theta)} \text{dlog}(1 + t_{s(\theta+1)}) \dots \text{dlog}(1 + t_{s(q)}).$$

Then the  $\omega_s$ ,  $s \in I^q$ , form an  $R$ -basis of  $\Omega_R^q(\log A)$ . Moreover,  $\Omega_R^q(\log A)(-B) = (\pi) \cdot \Omega_R^q(\log A)$  with  $\pi = t_1^{r_1} \dots t_e^{r_e} t_{e+1}^{r_{e+1}} \dots t_f^{r_f}$ . Since  $\omega_s$  is invariant under  $C^{-1}$ , it suffices to show that for any  $a \in \pi R$ , there exists an element  $b \in \pi R$  such that  $a = b^p - b$ . Such  $b$  exists by [JSZ18, Claim 1.2.2]. This shows the claim.

Now we do induction on  $n$  and assume  $n \geq 2$ . As  $R$  is surjective by Lemma 8.15, it suffices to show that  $\text{Ker}(R : W_n \Omega_{(X, -D)}^q \rightarrow W_{n-1} \Omega_{(X, -D)}^q)$  lies in the image of  $C^{-1} - 1$ . Every element of this kernel is of the form  $V^{n-1}(\alpha) + dV^{n-1}(\beta)$  with  $\alpha \in \Omega_X^q$  and  $\beta \in \Omega_X^{q-1}$ . As  $FV^{n-1}(\alpha) = 0$ , (8.14.2) yields  $dV^{n-1}(\beta) \in W_n \Omega_{(X, -D)}^q$ . Hence  $V^{n-1}(\alpha) \in W_n \Omega_{(X, -D)}^q$  as well and it suffices to show  $V^{n-1}(\alpha)$  lies in the image of  $C^{-1} - 1$ . Combining Lemma 8.13 with Lemma 8.11, we see that  $V^{n-1}(\alpha)$  can be written as a sum

$$V^{n-1}(\alpha) = \underline{p}^{n-1}(\alpha_0) + \sum_{j=1}^{n-1} V^j(\alpha_j)$$

with  $\alpha_0 \in \Omega_{n-1}^q(-D', -pD_n)$ , where  $D = D' + p^n D_n$  is a  $p$ -divisibility decomposition, and  $\alpha_j \in W_{n-j} \Omega_{(X, -D)}^q$ , for  $j \geq 1$ . By induction  $\alpha_j$  lies in the image of  $C^{-1} - 1$ , for  $j \geq 1$ , and  $\alpha_0$  lies in the image by Claim 11.6.1, hence so does  $V^{n-1}(\alpha)$ .  $\square$

**Remark 11.7.** (1) In [JSZ18, Definition 1.1.1] a sheaf  $W_n \Omega_{(X|D), \log}^q$  is defined and in [GK, (5.3)] a sheaf  $W_n \Omega_{(X, D), \log}^q$  is defined. These are related to the complexes mentioned in 11.5 (4) in the same way as  $W_n \Omega_{(X, -D)}^q$  is defined in Lemma 11.6. We don't know what is the precise relation between these three sheaves, but the pro-systems for  $(X, rD)$ ,  $r \geq 1$ , are isomorphic.

(2) A similar argument as for Lemma 11.6 shows that in  $D(X_{\text{ét}})$  we have an isomorphism

$$\mathcal{H}^q(\mathbb{Z}/p^n(q)_{(X, D_{\text{red}})})[-q] \cong \mathbb{Z}/p^n(q)_{(X, D_{\text{red}})}.$$

(This would correspond to the case  $B = 0$  in Claim 11.6.1.) However, even in the étale topology the sheaf  $\mathcal{H}^{q+1}(\mathbb{Z}/p^n(q)_{(X, D)})$  does in general not vanish if  $D$  is not reduced. The main problem when trying to adapt the proof of Claim 11.6.1 is that for  $a \in \frac{1}{\pi} \cdot A$  the Artin-Schreier covering defined by  $b^p - b = a$  will (wildly) ramify along the vanishing locus of  $\pi$  and is in particular not étale over  $A$ . Also it is direct to show that we have

$$(11.7.1) \quad \mathcal{H}^q(\mathbb{Z}/p^n(q)_{(X, D)}) = j_* W_n \Omega_U^q,$$

which only depends on the support of  $D$ . Thus all the information about the multiplicities of the components of  $D$  is stored in the étale sheaf  $\mathcal{H}^{q+1}(\mathbb{Z}/p^n(q)_{(X, D)})$ , or the Nisnevich sheaf  $R^{q+1}\epsilon_*(\mathbb{Z}/p^n(q)_{(X, D)})$ . The study of this sheaf was initiated by Kato in [Kat89] as becomes apparent from the following Lemma.

**Lemma 11.8.** *Let  $\eta \in X$  be a generic point of  $D$  and set  $L = \text{Frac}(\mathcal{O}_{X, \eta}^h)$ . Denote by  $r \geq 1$  the multiplicity of  $D$  at  $\eta$ . Set  $H^{q+1}(\text{Spec } L_{\text{ét}}, \mathbb{Z}/p^n(q)) := H_n^{q+1}(L)$ . Then*

$$\text{Im}(\text{fil}_r W_n \Omega_L^q \rightarrow H_n^{q+1}(L)) = R^{q+1}\epsilon_*(\mathbb{Z}/p^n(q)_{(X, D)})_{\eta}^h,$$

see Definition 2.4 for notation. In particular, if  $\text{fil}_r^{\text{Kato}} H_n^{q+1}(L)$  denotes the filtration defined in [Kat89, Definition 2.1], then

$$\text{fil}_{r-1}^{\text{Kato}} H_n^{q+1}(L) \subset R^{q+1}\epsilon_*(\mathbb{Z}/p^n(q)_{(X, D)})_{\eta}^h \subset \text{fil}_r^{\text{Kato}} H_n^{q+1}(L).$$

*Proof.* By definition

$$(11.8.1) \quad R^{q+1}\epsilon_*(\mathbb{Z}/p^n(q)_{(X, D)})_{\eta}^h = \frac{\text{Fil}_r^p W_n \Omega_L^q}{(1 - C)(FW_{n+1}\Omega_L^q) \cap \text{Fil}_r^p W_n \Omega_L^q}.$$

For  $a \in \text{Fil}_{p^s r} W_n \Omega_L^q$  and  $s \geq 1$  we have

$$p^s a = F^s V^s(a) \equiv V^s(R^s(a)) \pmod{(1 - C)(FW_{n+1}\Omega_L^q)}.$$

It follows from the definition (see (2.3.2)) that we have  $R^s(\text{Fil}_{p^s r} W_n \Omega_L^q) \subset \text{Fil}_r W_{n-s} \Omega_L^q$ , hence  $\text{Fil}_r W_n \Omega_L^q$  surjects onto the quotient (11.8.1). Moreover for  $b \in \text{fil}_r W_n \Omega_L^{q-1}$  we have

$$db = FdV(b) \equiv dVR(b) \pmod{(1 - C)(FW_{n+1}\Omega_L^q)}.$$

Iterating this we see that  $db$  vanishes in the quotient, which yields the first statement. By [Kat89, Theorem (3.2)] we have a surjection

$$\text{fil}_r^{\log} W_n \Omega_L^q \twoheadrightarrow \text{fil}_r^{\text{Kato}} H_n^{q+1}(L),$$

for all  $r \geq 0$ , which gives the second statement.  $\square$

**Proposition 11.9.** *There is an isomorphism in  $D(W_n \mathcal{O}_{X_{\text{ét}}})$*

$$(FW_{n+1}\Omega^q)_{(X, D)} \xrightarrow{\cong} \mathcal{D}_{X, n}((W_n \Omega^{N-q}/dV^{n-1})_{(X, -D)}),$$

induced by  $F(\alpha) \mapsto (\beta \mapsto C(F(\alpha)\beta))$ , where  $\mathcal{D}_{X, n} = R\mathcal{H}\text{om}_{W_n \mathcal{O}_X}(-, W_n \Omega_X^N)$  denotes the dualizing functor considered in 9.2 (and extended to the étale site).

*Proof.* Consider the following two short exact sequences of  $W_n\mathcal{O}_X$ -modules

$$(11.9.1) \quad 0 \rightarrow (FW_{n+1}\Omega^q)_{(X,D)} \rightarrow F_{X*}W_n\Omega^q_{(X,D)} \xrightarrow{F^{n-1}d} i_*B_n\Omega^{q+1}_{(X,D)} \rightarrow 0,$$

$$(11.9.2) \quad 0 \rightarrow i_*(\Omega/Z)_{n,(X,-D)}^{N-q-1} \xrightarrow{dV^{n-1}} F_{X*}W_n\Omega_{(X,-D)}^{N-q} \rightarrow (W_n\Omega^{N-q}/dV^{n-1})_{(X,-D)} \rightarrow 0,$$

see (11.3.3), (6.4.1), and (8.6.3), where  $i : X \hookrightarrow W_nX$  denotes the closed immersion induced by  $R^{n-1}$ . We have an isomorphism

$$a : F_{X*}W_n\Omega^q_{(X,D)} \xrightarrow{\cong} F_{X*}\mathcal{D}_{X,n}(W_n\Omega_{(X,-D)}^{N-q}) \cong \mathcal{D}_{X,n}(F_{X*}W_n\Omega_{(X,-D)}^{N-q}),$$

which by Theorem 9.3 and (11.2.3) is induced by  $\alpha \mapsto (\beta \mapsto C(\alpha\beta))$ . Moreover, there is an isomorphism

$$b : i_*B_n\Omega_{(X,D)}^{q+1} \xrightarrow{\cong} i_*D_{X,1}((\Omega/Z)_{n,(X,-D)}^{N-q-1}) \cong D_{X,n}(i_*(\Omega/Z)_{n,(X,-D)}^{N-q-1}),$$

which is a composition of the isomorphism (9.3.5) with the duality isomorphism and which is induced by

$$i_*B_n\Omega_{(X,D)}^{q+1} \rightarrow \mathcal{H}om_{W_n\mathcal{O}_X}(i_*(\Omega/Z)_{n,(X,-D)}^{N-q-1}, W_n\Omega_X^N), \quad \alpha \mapsto (\gamma \mapsto \underline{p}^{n-1}C^m(\alpha\gamma)),$$

see [Eke84, II, (2.2.5)] for the fact that the second isomorphism in the definition of  $b$  is induced by  $\underline{p}^{n-1} : i_*\Omega_X^N \rightarrow W_n\Omega_X^N$ . Finally the following diagram commutes (note that the top row is concentrated in degree zero)

$$\begin{array}{ccc} \mathcal{D}_{X,n}(F_{X*}W_n\Omega_{(X,-D)}^{N-q}) & \xrightarrow{(dV^{n-1})^\vee} & \mathcal{D}_{X,n}(i_*(\Omega/Z)_{n,(X,-D)}^{N-q-1}) \\ \uparrow a & & \uparrow (-1)^{q-1}b \\ F_{X*}W_n\Omega_{(X,D)}^q & \xrightarrow{F^{n-1}d} & i_*B_n\Omega_{(X,D)}^{q+1}. \end{array}$$

Indeed, the commutativity follows directly from the relations  $\underline{p}^{n-1}C^{n-1} = V^{n-1}$  and  $C \circ dV^{n-1} = 0$ , which follow from the definition of the Cartier operator. Taking everything together we obtain an isomorphism (11.9.1)  $\xrightarrow{\cong} \mathcal{D}_{X,n}((11.9.2))$ , which yields the isomorphism of the statement.  $\square$

**11.10.** To extend Milne's classical duality result for étale cohomology with mod  $p^n$  coefficients (see [Mil76, Theorem 5.2] and [Mil86, Theorem 1.11]) to the modulus setup we will use Kato's general dualizing formalism from [Kat86, Kat87], which we recall in the following. A morphism of  $\mathbb{F}_p$ -schemes  $T \rightarrow X$  is relatively perfect if and only if the relative Frobenius  $F_{T/X} : T \rightarrow T \times_{X,F_X} X$  is an isomorphism. If  $T \rightarrow X$  is additionally flat, then so is  $W_n(T) \rightarrow W_n(X)$  and the following diagrams are cartesian,

$$(11.10.1) \quad \begin{array}{ccc} W_n(T) & \xrightarrow{W_n(F_T)} & W_n(T) \\ \downarrow & & \downarrow \\ W_n(X) & \xrightarrow{W_n(F_X)} & W_n(X) \end{array} \quad \begin{array}{ccc} W_n(T) & \longrightarrow & W_{n+1}(T) \\ \downarrow & & \downarrow \\ W_n(X) & \longrightarrow & W_{n+1}(X) \end{array},$$

where the horizontal maps in the diagram on the right hand side are the closed immersions induced by the restriction  $R$ , see [Kat80, Lemma 2] and [Ill79, 0, Proposition 1.5.8].

Let  $X_{\text{FRP}}$  be the site with underlying category, the category of all flat and relatively perfect  $X$ -schemes, equipped with the étale topology. In particular, the underlying category of  $X_{\text{FRP}}$  contains all étale  $X$ -schemes and we get a fully faithful functor of the underlying categories

$$u : X_{\text{ét}} \rightarrow X_{\text{FRP}},$$

which is continuous, cocontinuous, and commutes with fiber products. The induced restriction functor on the topoi  $\widetilde{X_{\text{FRP}}} \rightarrow \widetilde{X_{\text{ét}}}$ ,  $S \mapsto u^*S = S|_{X_{\text{ét}}}$ , therefore is exact and has an exact left adjoint, see [SGA72a, I, Proposition 5.4, 4) and III, Proposition 2.6]. Hence for any complex  $L \in D(X_{\text{FRP}}, \mathbb{Z}/p^n)$  in the derived category of  $\mathbb{Z}/p^n$ -sheaves on  $X_{\text{FRP}}$  we get a canonical isomorphism

$$(11.10.2) \quad R\Gamma(X_{\text{FRP}}, L) \cong R\Gamma(X_{\text{ét}}, L|_{X_{\text{ét}}})$$

in the derived category of  $\mathbb{Z}/p^n$ -modules.

For any  $n \geq 1$ , the association  $T \mapsto \Gamma(T, W_n \mathcal{O}_T)$  defines a sheaf on  $X_{\text{FRP}}$ , we denote it by  $W_n \mathcal{O}_X^{\text{FRP}}$ . Now let  $\tau = \text{Zar}, \text{Nis}$  or  $\text{ét}$ . For any  $W_n \mathcal{O}_X$ -module  $M$  on  $X_\tau$ , the association

$$(g : T \rightarrow X) \mapsto \Gamma(T, (W_n g)^* M)$$

defines a presheaf of  $W_n \mathcal{O}_X^{\text{FRP}}$ -modules on  $X_{\text{FRP}}$ , we denote its sheafification by  $M^{\text{FRP}}$ . When  $M$  is (quasi-)coherent as a  $W_n \mathcal{O}_X$ -module on  $X_\tau$ , we say that  $M^{\text{FRP}}$  is (quasi-)coherent as a  $W_n \mathcal{O}_X^{\text{FRP}}$ -module on  $X_{\text{FRP}}$ . The functor  $M \mapsto M^{\text{FRP}}$  from the category of  $W_n \mathcal{O}_X$ -modules on the site  $X_\tau$  to the category of  $W_n \mathcal{O}_X^{\text{FRP}}$ -modules is exact. In particular, the association

$$(g : T \rightarrow X) \mapsto \Gamma(T, (W_n g)^* W_n \Omega_{(X, \pm D)}^q)$$

defines a sheaf of  $W_n \mathcal{O}_X^{\text{FRP}}$ -modules on  $X_{\text{FRP}}$ , which we denote by  $W_n \Omega_{(X, \pm D)}^{q, \text{FRP}}$ . The maps  $F, V, R, \underline{p}, d, C, C^{-1}$  on  $W_\bullet \Omega_X^*$  extend naturally to maps on  $W_\bullet \Omega_{(X, \pm D)}^{*, \text{FRP}}$ ; this follows from the cartesian diagrams (11.10.1) and the flatness of  $W_n(g) : W_n(T) \rightarrow W_n(X)$ , see also 11.12 below where we spell out how to extend  $F$  and  $C$ . In particular, when  $D = \emptyset$ , we write  $W_n \Omega_X^{\bullet, \text{FRP}} := W_n \Omega_{(X, \emptyset)}^{\bullet, \text{FRP}}$ . Since  $(W_n g)^* W_n \Omega_X^\bullet \simeq W_n \Omega_T^q$  for any flat and relatively perfect morphism  $g : T \rightarrow X$ , the sheaf  $W_n \Omega_X^{\bullet, \text{FRP}}$  agrees with the presheaf given by  $T \mapsto \Gamma(T, W_n \Omega_T^q)$ , cf. the discussion in [Kat86, §4] or the proof of [LZ04, Proposition 1.7].

Denote by  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$  the full triangulated subcategory of  $D(X_{\text{FRP}}, \mathbb{Z}/p^n)$  generated by coherent  $\mathcal{O}_X^{\text{FRP}}$ -modules regarded as complexes of  $\mathbb{Z}/p^n$ -modules in degree 0. Note that  $W_n \Omega_{(X, \pm D)}^{q, \text{FRP}} \in D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$ , by the Theorems 6.4 and 8.7. Set

$$\mathbb{D}_{X, n}(-) := R\mathcal{H}om_{D(X_{\text{FRP}}, \mathbb{Z}/p^n)}(-, W_n \Omega_{X, \log}^{N, \text{FRP}}),$$

where  $W_n \Omega_{X, \log}^{N, \text{FRP}}$  is defined by the short exact sequence

$$(11.10.3) \quad 0 \rightarrow W_n \Omega_{X, \log}^{N, \text{FRP}} \rightarrow W_n \Omega_X^{N, \text{FRP}} \xrightarrow{C-1} W_n \Omega_X^{N, \text{FRP}} \rightarrow 0.$$

This functor agrees with Kato's functor defined in [Kat86, §6] and differs from the one defined in [Kat87, §3] by a shift  $[N]$ . Kato proves the following results:

- (1) If  $M \in D_c^b(X_{\text{Zar}}, W_n \mathcal{O}_X)$ , then the natural map  $W_n \Omega_X^{N, \text{FRP}} \rightarrow W_n \Omega_{X, \log}^{N, \text{FRP}}[1]$  from (11.10.3) induces an isomorphism in  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$

$$(11.10.4) \quad \mathcal{D}_{n, X}(M)^{\text{FRP}} \simeq \mathbb{D}_{n, X}(M^{\text{FRP}})[1],$$

where  $\mathcal{D}_{n, X} = R\mathcal{H}om_{W_n \mathcal{O}_X}(-, W_n \Omega_X^N)$  as in 9.2, see [Kat86, Proposition 6.1].

(2) The functor  $\mathbb{D}_{X,n}$  restricts to an involutive endo-functor of  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$ , i.e.,

$$(11.10.5) \quad \mathbb{D}_{X,n} \circ \mathbb{D}_{X,n} \simeq \text{id} : D_0(X_{\text{FRP}}, \mathbb{Z}/p^n) \rightarrow D_0(X_{\text{FRP}}, \mathbb{Z}/p^n),$$

see [Kat87, Theorem (0.1) and Proposition (3.4)].

(3) Let  $f : X \rightarrow Y$  be a proper morphism of relative dimension  $r$  between two pure dimensional smooth  $k$ -schemes. There is a natural isomorphism

$$(11.10.6) \quad Rf_* \mathbb{D}_{n,X}(M) \simeq \mathbb{D}_{n,Y}(Rf_* M)[-r]$$

in  $D_0(Y_{\text{FRP}}, \mathbb{Z}/p^n)$ , for any  $M \in D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$ , see [Kat87, Theorem (0.2)].

The following Lemma is implicitly in [Kat86] and [Mil76].

**Lemma 11.11.** *Assume  $k$  is a finite field and  $X$  is proper over  $k$ . Let  $M \in D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$ . There is a canonical isomorphism*

$$R\Gamma(X_{\text{FRP}}, \mathbb{D}_{X,n}(M)) \cong R\text{Hom}_{D(\mathbb{Z}/p^n)}(R\Gamma(X_{\text{ét}}, M|_{X_{\text{ét}}}), \mathbb{Z}/p^n)[-N-1].$$

*Proof.* We start by giving a canonical map from left to right. As  $k$  is finite, we can view  $X$  also as a smooth proper scheme of relative dimension  $N$  over  $S_0 := \text{Spec } \mathbb{F}_p$ . Note that (11.10.3) for  $S_0$  instead of  $X$  yields in view of (11.10.2) a decomposition

$$(11.11.1) \quad R\Gamma(S_{0,\text{FRP}}, W_n \Omega_{S_0,\log}^{0,\text{FRP}}) = W_n(\mathbb{F}_p)[0] \oplus W_n(\mathbb{F}_p)[-1].$$

Let  $\pi : X \rightarrow S_0 = \text{Spec } \mathbb{F}_p$  be the structure map. We define the map from the statement to be the composition

$$(11.11.2) \quad \begin{aligned} R\Gamma(X_{\text{FRP}}, \mathbb{D}_{X,n}(M)) &\xrightarrow{\simeq} R\Gamma(S_{0,\text{FRP}}, \mathbb{D}_{S_0,n}(R\pi_* M))[-N] \\ &\rightarrow R\text{Hom}_{D(\mathbb{Z}/p^n)}(R\Gamma(S_{0,\text{FRP}}, R\pi_* M), R\Gamma(S_{0,\text{FRP}}, W_n \Omega_{S_0,\log}^{0,\text{FRP}}))[-N] \\ &\cong R\text{Hom}_{D(\mathbb{Z}/p^n)}(R\Gamma(X_{\text{ét}}, M|_{X_{\text{ét}}}), W_n(\mathbb{F}_p)[0] \oplus W_n(\mathbb{F}_p)[-1])[-N] \\ &\rightarrow R\text{Hom}_{D(\mathbb{Z}/p^n)}(R\Gamma(X_{\text{ét}}, M|_{X_{\text{ét}}}), \mathbb{Z}/p^n)[-N-1], \end{aligned}$$

where the first isomorphism is induced from the duality isomorphism (11.10.6), the second map is the natural transformation  $R\Gamma(T, -) \circ R\mathcal{H}\text{om}(-, -) \rightarrow R\text{Hom}(R\Gamma(T, -), R\Gamma(T, -))$ , the third isomorphism is (11.10.2) and (11.11.1), and the last map is the projection to the second summand plus the identification  $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$ . We note that this morphism defines for  $M$  variable an exact functor of triangulated categories  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n) \rightarrow D(\mathbb{Z}/p^n)$ . Thus it suffices to show that it is an isomorphism for  $M = L^{\text{FRP}}$ , with  $L \in D_c^b(X_{\text{Zar}}, W_n \mathcal{O}_X)$ . For such  $L$  we have isomorphisms

$$(11.11.3) \quad \begin{aligned} R\Gamma(X_{\text{FRP}}, \mathbb{D}_{X,n}(L^{\text{FRP}})) &\cong R\Gamma(X_{\text{FRP}}, \mathcal{D}_{X,n}(L)^{\text{FRP}})[-1] \\ &\cong R\Gamma(X_{\text{ét}}, (\mathcal{D}_{X,n}(L)^{\text{FRP}})|_{X_{\text{ét}}})[-1] \\ &\cong R\Gamma(X_{\text{Zar}}, \mathcal{D}_{X,n}(L))[-1] \\ &\cong R\text{Hom}_{W_n(\mathbb{F}_p)}(R\Gamma(X_{\text{Zar}}, L), W_n(\mathbb{F}_p))[-N-1] \\ &\cong R\text{Hom}_{W_n(\mathbb{F}_p)}(R\Gamma(X_{\text{ét}}, (L^{\text{FRP}})|_{X_{\text{ét}}}), \mathbb{Z}/p^n)[-N-1], \end{aligned}$$

where the first isomorphism is (11.10.4), the second is (11.10.2), the third isomorphism holds since  $\mathcal{D}_{X,n}(L) \in D_c^b(X_{\text{Zar}}, W_n \mathcal{O}_X)$ , the fourth isomorphism is Ekedahl-Grothendieck duality and the last isomorphism is clear. It follows from the construction of the duality isomorphism

(11.10.6) and the isomorphism (11.10.4) that the two compositions (11.11.2) and (11.11.3) agree for  $M = L^{\text{FRP}}$ . Indeed by [Kat87, (3.2.2), (3.2.3)] we have a commutative diagram

$$\begin{array}{ccc} R\pi_* W_n \Omega_X^{N, \text{FRP}}[N] & \xrightarrow{\text{Tr}} & W_n \Omega_{S_0}^{0, \text{FRP}} \\ \downarrow & & \downarrow \\ R\pi_* W_n \Omega_{X, \log}^{N, \text{FRP}}[N+1] & \xrightarrow{\text{Tr}} & W_n \Omega_{S_0, \log}^{0, \text{FRP}}[1], \end{array}$$

where the horizontal maps are the trace maps defining the respective duality isomorphism and the vertical maps are induced by (11.10.3). With this commutativity and the formula  $R\pi_*(L^{\text{FRP}}) = (R\pi_*(L))^{\text{FRP}}$  (see [Kat87, Lemma 2.6]) it is straightforward to check that the above compositions agree. This completes the proof.  $\square$

**11.12.** We extend the complexes from Definition 11.4 to the site  $X_{\text{FRP}}$  by setting

$$\mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}} := \left( (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}} \xrightarrow{1-C} W_n \Omega_{(X,D)}^{q, \text{FRP}} \right) [-q],$$

and

$$\mathbb{Z}/p^n(q)_{(X,-D)}^{\text{FRP}} := \left( W_n \Omega_{(X,-D)}^{q, \text{FRP}} \xrightarrow{C^{-1}-1} (W_n \Omega^q / dV^{n-1})_{(X,-D)}^{\text{FRP}} \right) [-q].$$

We spell out the extension of  $1-C$  to  $X_{\text{FRP}}$ . Let  $g : T \rightarrow X$  be relatively perfect and flat. Then the Frobenius on  $W_{\bullet} \Omega_{(X, \pm D)}^{q, \text{FRP}}$  is induced by the composition

$$W_{n+1}(g)^* W_{n+1} \Omega_{(X, \pm D)}^q \xrightarrow{W_{n+1}(g)^*(F)} W_{n+1}(g)^* F_* W_n \Omega_{(X, \pm D)}^q \cong F_* W_n(g)^* W_n \Omega_{(X, \pm D)}^q,$$

which on elements is given by

$$W_{n+1} \mathcal{O}_T \otimes_{W_{n+1} \mathcal{O}_X} W_{n+1} \Omega_{(X, \pm D)}^q \ni a \otimes \alpha \mapsto F(a) \otimes F(\alpha) \in W_n \mathcal{O}_T \otimes W_n \Omega_{(X, \pm D)}^q.$$

Here the isomorphism on the right hand side comes from the cartesian diagrams (11.10.1) and the flatness of  $W_{n+1}(g)$ . Hence

$$\Gamma(T, (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}}) = \Gamma(T, F(W_{n+1}(g)^* W_{n+1} \Omega_{(X,D)}^q)).$$

The map  $(FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}} \xrightarrow{1-C} W_n \Omega_{(X,D)}^{q, \text{FRP}}$  is therefore induced by

$$F(a) \otimes F(\alpha) \mapsto F(a) \otimes F(\alpha) - R(a) \otimes R(\alpha).$$

Similarly for the map  $C^{-1}-1$  in the definition of  $\mathbb{Z}/p^n(q)_{(X,-D)}^{\text{FRP}}$ .

**Lemma 11.13.** *Let  $\alpha : A^0 \rightarrow A^1$  and  $\beta : B^0 \rightarrow B^1$  be two morphisms of sheaves of abelian groups on some site  $X$ . Let  $C^\bullet = (C^0 \xrightarrow{d} C^1)$  be a two term complex of sheaves of abelian groups on  $X$  sitting in degree  $[0, 1]$ . Assume we are given pairings*

$$\langle -, - \rangle_{0,0}^0 : A^0 \otimes_{\mathbb{Z}} B^0 \rightarrow C^0, \quad \langle -, - \rangle_{1,0}^1 : A^1 \otimes_{\mathbb{Z}} B^0 \rightarrow C^1, \quad \langle -, - \rangle_{0,1}^1 : A^0 \otimes_{\mathbb{Z}} B^1 \rightarrow C^1,$$

such that

$$\langle \alpha(-), - \rangle_{1,0}^1 + \langle -, \beta(-) \rangle_{0,1}^1 = d \circ \langle -, - \rangle_{0,0}^0 : A^0 \otimes_{\mathbb{Z}} B^0 \rightarrow C^1.$$

Then we obtain the following morphism between distinguished triangles in the homotopy category of complexes of abelian sheaves on  $X$  <sup>10</sup>

$$\begin{array}{ccccccc} A^0[0] & \xrightarrow{\alpha} & A^1[0] & \longrightarrow & \text{cone}(\alpha) & \longrightarrow & A^0[1] \\ \downarrow \pi' & & \downarrow \pi'' & & \downarrow \pi & & \downarrow \pi'[1] \\ \mathcal{H}\text{om}^\bullet(B^1, C^\bullet)[1] & \xrightarrow{\beta^*[1]} & \mathcal{H}\text{om}^\bullet(B^0, C^\bullet)[1] & \longrightarrow & \text{cone}(\beta^*[1]) & \longrightarrow & \mathcal{H}\text{om}^\bullet(B^1, C^\bullet)[2], \end{array}$$

where the non-labelled horizontal maps are the natural morphisms in the respective cone sequence and the vertical maps are defined as follows: the morphism  $\pi$  is in degree  $-1$  given by

$$A^0 \rightarrow \mathcal{H}\text{om}(B^0, C^0) \oplus \mathcal{H}\text{om}(B^1, C^1), \quad x \mapsto (-\langle x, - \rangle_{0,0}^0, -\langle x, - \rangle_{0,1}^1),$$

and in degree  $0$  by

$$A^1 \rightarrow \mathcal{H}\text{om}(B^0, C^1), \quad y \mapsto \langle y, - \rangle_{1,0}^1,$$

and  $\pi'$  and  $\pi''$  are (in degree  $0$ ) given by

$$\pi'(x) = -\langle x, - \rangle_{0,1}^1 \quad \text{and} \quad \pi''(y) = \langle y, - \rangle_{1,0}^1.$$

Furthermore, there is an isomorphism of complexes

$$(11.13.1) \quad \text{cone}(\beta^*[1]) \xrightarrow{\sim} \mathcal{H}\text{om}^\bullet(\text{cone}(\beta)[-1], C^\bullet)[1],$$

which is the identity in the degrees  $-2$  and  $0$  and which in degree  $-1$  is given by  $\text{id} \oplus (-\text{id}) \in \text{End}(\mathcal{H}\text{om}(B^0, C^0) \oplus \mathcal{H}\text{om}(B^1, C^1))$ .

*Proof.* This is all direct to check. We remark that the square on the left commutes only up to homotopy, more precisely if  $h : A^0 \rightarrow (\mathcal{H}\text{om}(B^0, C^\bullet)[1])^{-1} = \mathcal{H}\text{om}(B^0, C^0)$  is defined by  $h(x) = -\langle x, - \rangle_{0,0}^0$ , then we have  $\pi'' \circ \alpha - (\beta^*) \circ \pi' = (d[1]) \circ h$ .  $\square$

**11.14.** We define the following pairings on  $X_{\text{FRP}}$ :

$$\begin{aligned} \langle -, - \rangle_{0,0}^0 : (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}} \otimes_{\mathbb{Z}} W_n\Omega_{(X,-D)}^{N-q,\text{FRP}} &\rightarrow W_n\Omega_X^{N,\text{FRP}}, & \alpha \otimes \beta &\mapsto \alpha\beta, \\ \langle -, - \rangle_{1,0}^1 : W_n\Omega_{(X,D)}^{q,\text{FRP}} \otimes_{\mathbb{Z}} W_n\Omega_{(X,-D)}^{N-q,\text{FRP}} &\rightarrow W_n\Omega_X^{N,\text{FRP}}, & \alpha \otimes \beta &\mapsto \alpha\beta, \\ \langle -, - \rangle_{0,1}^1 : (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}} \otimes_{\mathbb{Z}} (W_n\Omega^q/dV^{n-1})_{(X,-D)}^{\text{FRP}} &\rightarrow W_n\Omega_X^{N,\text{FRP}}, & \alpha \otimes \beta &\mapsto C(\alpha\beta). \end{aligned}$$

These satisfy

$$\langle (C-1)(x), y \rangle_{1,0}^1 + \langle x, (1-C^{-1})(y) \rangle_{0,1}^1 = (C-1)\langle x, y \rangle_{0,0}^0,$$

for all local sections  $x \in (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}}$  and  $y \in W_n\Omega_{(X,-D)}^{N-q,\text{FRP}}$ . Note that

$$\mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}}[q] = \text{cone} \left( (FW_{n+1}\Omega^q)_{(X,D)}^{\text{FRP}} \xrightarrow{C-1} W_n\Omega_{(X,D)}^{q,\text{FRP}} \right) [-1],$$

$$\mathbb{Z}/p^n(N-q)_{(X,-D)}^{\text{FRP}}[N-q] = \text{cone} \left( W_n\Omega_{(X,-D)}^{q,\text{FRP}} \xrightarrow{1-C^{-1}} (W_n\Omega^q/dV^{n-1})_{(X,-D)}^{\text{FRP}} \right) [-1].$$

Hence we obtain a natural morphism in  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$

$$\begin{aligned} (11.14.1) \quad \mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}}[q] &= \text{cone}(C-1)[-1] \\ &\longrightarrow R\mathcal{H}\text{om} \left( \text{cone}(1-C^{-1})[-1], (W_n\Omega_X^{N,\text{FRP}} \xrightarrow{C-1} W_n\Omega_X^{N,\text{FRP}}) \right) \\ &\cong \mathbb{D}_{n,X} \left( \mathbb{Z}/p^n(N-q)_{(X,-D)}^{\text{FRP}}[N-q] \right), \end{aligned}$$

<sup>10</sup>We use the usual sign conventions, see, e.g., [Sta23, Tag 0FNG].

where the morphism in the second line is induced by the composition of  $\pi[-1]$  from Lemma 11.13 with the isomorphism (11.13.1) and the isomorphism in the third line is induced by the exact sequence (11.10.3).

In view of Kato's duality isomorphism (11.10.6) the following Theorem generalizes Milne's duality, see [Mil76, Theorem 5.2] (for smooth projective surfaces), [Mil86, Theorem 1.11] (for general smooth proper  $k$ -schemes), and [Kat86, Theorem 4.3], to the case of non-empty modulus.

**Theorem 11.15.** *The morphism (11.14.1) induces isomorphisms in  $D_0(X_{\text{FRP}}, \mathbb{Z}/p^n)$*

$$\begin{aligned} \mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}}[q] &\xrightarrow{\sim} \mathbb{D}_{n,X} \left( \mathbb{Z}/p^n(N-q)_{(X,-D)}^{\text{FRP}}[N-q] \right), \\ \mathbb{Z}/p^n(N-q)_{(X,-D)}^{\text{FRP}}[N-q] &\xrightarrow{\sim} \mathbb{D}_{n,X} \left( \mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}}[q] \right). \end{aligned}$$

*Proof.* By (11.10.5) it suffices to show that (11.14.1) is an isomorphism. By Lemma 11.13 we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} W_n \Omega_{(X,D)}^{q,\text{FRP}}[-1] & \longrightarrow & \mathbb{Z}/p^n(q)_{(X,D)}^{\text{FRP}}[q] & \longrightarrow & (FW_{n+1} \Omega^q)_{(X,D)}^{\text{FRP}} & \longrightarrow & \\ \downarrow \pi''[-1] & & \downarrow & & \downarrow \pi' & & \\ \mathbb{D}(W_n \Omega_{(X,-D)}^{N-q,\text{FRP}}) & \longrightarrow & \mathbb{D} \left( \mathbb{Z}/p^n(N-q)_{(X,-D)}^{\text{FRP}}[N-q] \right) & \longrightarrow & \mathbb{D}((W_n \Omega^{N-q}/dV^{n-1})_{(X,-D)}^{\text{FRP}})[1] & \longrightarrow & , \end{array}$$

where we write  $\mathbb{D}$  instead of  $\mathbb{D}_{n,X}$ . By definition the map  $\pi''[-1]$  factors as

$$\begin{aligned} W_n \Omega_{(X,D)}^{q,\text{FRP}}[-1] &\xrightarrow[\simeq]{\text{Thm. 9.3}} \mathcal{D}_{n,X}(W_n \Omega_{(X,-D)}^{N-q})^{\text{FRP}}[-1] \\ &\xrightarrow[\simeq]{(11.10.4)} \mathbb{D}_{n,X}(W_n \Omega_{(X,-D)}^{N-q,\text{FRP}}) \end{aligned}$$

and hence is an isomorphism. Similarly, the morphism  $\pi'$  factors as

$$\begin{aligned} (FW_{n+1} \Omega^q)_{(X,D)}^{\text{FRP}} &\xrightarrow[\simeq]{\text{Prop. 11.9}} \mathcal{D}_{n,X}((W_n \Omega^{N-q}/dV^{n-1})_{(X,-D)})^{\text{FRP}} \\ &\xrightarrow[\simeq]{(11.10.4)} \mathbb{D}_{n,X}((W_n \Omega^{N-q}/dV^{n-1})_{(X,-D)}^{\text{FRP}})[1] \end{aligned}$$

and hence is an isomorphism as well. This completes the proof of the theorem.  $\square$

**Corollary 11.16.** *Assume additionally that  $k$  is a finite field and  $X$  is proper over  $k$ . Then the morphism (11.14.1) induces an isomorphism in  $D(\mathbb{Z}/p^n)$*

$$R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(q)_{(X,D)}) \cong R\text{Hom}_{\mathbb{Z}/p^n} \left( R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(N-q)_{(X,-D)}), \mathbb{Z}/p^n \right) [-2N-1].$$

In particular we obtain isomorphisms of finite groups for all  $i$

$$H^{i+q}(X_{\text{ét}}, \mathbb{Z}/p^n(q)_{(X,D)}) \cong \text{Hom}_{\mathbb{Z}/p^n}(H^{2N-i-q+1}(X_{\text{ét}}, \mathbb{Z}/p^n(N-q)_{(X,-D)}), \mathbb{Z}/p^n).$$

*Proof.* As  $(\mathbb{Z}/p^n(q)_{(X,\pm D)}^{\text{FRP}})|_{X_{\text{ét}}} = \mathbb{Z}/p^n(q)_{(X,\pm D)}$ , this follows from Theorem 11.15, Lemma 11.11, and (11.10.2).  $\square$

**Remark 11.17.** By Lemma 11.6 the isomorphism in Corollary 11.16 also induces an isomorphism

$$H^{N-i+1}(X_{\text{ét}}, W_n \Omega_{(X,-D),\log}^{N-q}) \cong H^{i+q}(X_{\text{ét}}, \mathbb{Z}/p^n(q)_{(X,D)})^\vee,$$

where the  $(-)^{\vee}$  on the right hand side denotes the Pontryagin dual. Taking the limit and using 11.5(3) yields the isomorphism

$$\lim_r H^{N-i+1}(X_{\text{ét}}, W_n \Omega_{(X, -rD), \log}^{N-q}) \cong H^i(U_{\text{ét}}, W_n \Omega_{U, \log}^q)^{\vee},$$

which gives back [JSZ18, Theorem 2].

## 12. WEIGHTS ZERO AND ONE

In this section we consider the complex  $\mathbb{Z}/p^n(q)_{(X,D)}$  for  $q = 0, 1$  and apply the finite duality from the previous section. The main results are Corollaries 12.4 and 12.6, and Theorem 12.13. The assumptions on  $X$  and  $D$  are the same as in the previous sections.

### 12.1. Weight zero.

**Lemma 12.1.** *Assume  $X$  is irreducible. For all  $i$  we have natural maps*

$$(12.1.1) \quad H^i(X_{\text{ét}}, \mathbb{Z}/p^n) \rightarrow H^i(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X,D)}) \rightarrow H^i(U_{\text{ét}}, \mathbb{Z}/p^n)$$

*inducing an isomorphism*

$$\text{colim}_r H^i(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X,rD)}) \cong H^i(U_{\text{ét}}, \mathbb{Z}/p^n).$$

*Moreover, the maps (12.1.1) are injective for  $i = 1$  and identify with the inclusions*

$$H^0(X_{\text{Nis}}, R^1 \epsilon_* (\mathbb{Z}/p^n)_{X_{\text{ét}}}) \rightarrow H^0(X_{\text{Nis}}, R^1 \epsilon_* \mathbb{Z}/p^n(0)_{(X,D)}) \rightarrow H^0(X_{\text{Nis}}, j_* R^1 \epsilon_* (\mathbb{Z}/p^n)_{U_{\text{ét}}}),$$

*where  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the change of sites map.*

*Proof.* The natural maps in (12.1.1) are induced from the usual Artin-Schreier-Witt sequence on  $X_{\text{ét}}$  and  $U_{\text{ét}}$  and the isomorphism comes from 11.5(3). For the injectivity note that

$$R^0 \epsilon_* \mathbb{Z}/p^n(0)_{(X,D)} = (\mathbb{Z}/p^n)_{X_{\text{Nis}}}.$$

Hence  $H^i(X_{\text{Nis}}, R^0 \epsilon_* \mathbb{Z}/p^n(0)_{(X,D)}) = 0 = H^i(U_{\text{Nis}}, R^0 \epsilon_* \mathbb{Z}/p^n(0)_{(U_{\text{ét}}, D_{\text{ét}})})$ , for all  $i \neq 0$ , which yields the identification of (12.1.1) for  $i = 1$  with the maps in the second part of the statement. Using the usual Artin-Schreier-Witt sequence and the definition of  $\mathbb{Z}/p^n(0)_{(X,D)}$  (see Lemma 11.3 and Definition 11.4) we see that this latter maps are injective.  $\square$

**Lemma 12.2.** *There is distinguished triangle in  $D(X_{\text{ét}}, \mathbb{Z}/p^n)$*

$$\mathbb{Z}/p^n(0)_{(X, -D)} \rightarrow (\mathbb{Z}/p^n)_{X_{\text{ét}}} \rightarrow (\mathbb{Z}/p^n)_{D_{\text{ét}}} \xrightarrow{+1},$$

*where the second map is the natural restriction map. In particular, the natural map*

$$\mathbb{Z}/p^n(0)_{(X, -D)} \rightarrow \mathbb{Z}/p^n(0)_{(X, -D_{\text{red}})}$$

*is an isomorphism.*

*Proof.* By the Artin-Schreier-Witt sequence the complex of étale sheaves

$$(12.2.1) \quad \mathbb{Z}/p^n(0)_Y := \left( W_n \mathcal{O}_Y \xrightarrow{F-1} W_n \mathcal{O}_Y \right)$$

sitting in degrees  $[0, 1]$  is quasi-isomorphic to the constant sheaf  $(\mathbb{Z}/p^n)_{Y_{\text{ét}}}$ , for any  $\mathbb{F}_p$ -scheme  $Y$ . Thus the first statement follows from the exact sequence of complexes

$$0 \rightarrow \mathbb{Z}/p^n(0)_{(X, -D)} \rightarrow \mathbb{Z}/p^n(0)_X \rightarrow \mathbb{Z}/p^n(0)_D \rightarrow 0.$$

For the exactness on the left note that a local section  $a \in W_n \mathcal{O}_X$  maps to zero in  $W_n \mathcal{O}_D$  if and only if it maps to zero in all  $W_n \mathcal{O}_{D_i}$ , where  $D = \sum_i D_i$ , with  $D_{i, \text{red}} \in \mathbf{Sm}$ . The last statement follows from the equivalence of topoi  $\text{Sh}(D_{\text{ét}}) \cong \text{Sh}(D_{\text{red}, \text{ét}})$ , see [SGA72b, VIII, Théorème 1.1].  $\square$

**12.3.** Set

$$\pi_1^{\text{ab}}(X, D)^p := \lim_n \text{Hom}(H^1(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X,D)}), \mathbb{Q}/\mathbb{Z}),$$

where the transition maps in the limit are induced by  $\underline{p} : \mathbb{Z}/p^n(0)_{(X,D)} \rightarrow \mathbb{Z}/p^{n+1}(0)_{(X,D)}$ . It follows from Lemma 12.1, Lemma 11.8, and [KS14, Proposition 2.5] that this group is equal to the maximal pro- $p$  quotient of the group  $\pi_1^{\text{ab}}(X, D)$  considered in [KS14, (2-4)] (see also [KS16]). It classifies finite étale covers of  $U$  which have an abelian  $p$ -group as Galois group and whose ramification is bounded by  $D$ .

Moreover, set

$$\pi_1^{\text{ab}}(X, -D)^p := \lim_p \text{Hom}(H^1(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X,-D)}), \mathbb{Q}/\mathbb{Z}).$$

By Lemma 12.2

$$\pi_1^{\text{ab}}(X, -D)^p = \pi_1^{\text{ab}}(X, -D_{\text{red}})^p$$

and if  $X$  is proper and  $|D|$  connected we have an exact sequence of profinite groups

$$(12.3.1) \quad \pi_1^{\text{ab}}(D)^p \rightarrow \pi_1^{\text{ab}}(X)^p \rightarrow \pi_1^{\text{ab}}(X, -D)^p \rightarrow 0.$$

Note that  $\pi_1^{\text{ab}}(X, -D)$  classifies finite étale covers of  $X$ , which have an abelian  $p$ -group as Galois group and split completely when restricted to  $D$ .

Now Corollary 11.16 directly gives:

**Corollary 12.4.** *Assume  $k$  is finite and  $X$  is proper. Then*

$$\pi_1^{\text{ab}}(X, D)^p = \lim_n H^{2N}(X_{\text{ét}}, \mathbb{Z}/p^n(N)_{(X,-D)}) = H^N(X_{\text{ét}}, W\Omega_{(X,-D), \log}^N),$$

where  $W\Omega_{(X,-D), \log}^N = \lim_n W_n\Omega_{(X,-D), \log}^N$ , and

$$\pi_1^{\text{ab}}(X, -D)^p = \lim_n H^{2N}(X_{\text{ét}}, \mathbb{Z}/p^n(N)_{(X,D)}).$$

**Remark 12.5.** Taking the limit over  $\{rD\}_r$  in the first equality above gives back the isomorphism from [JSZ18, below Theorem 2].

**Corollary 12.6.** *Let  $k$  be a finite field, let  $X$  be proper and let  $H \subset X$  be a smooth ample divisor intersecting  $D$  transversally. Then the natural map*

$$H^i(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X,-D)}) \rightarrow H^i(H_{\text{ét}}, \mathbb{Z}/p^n(0)_{(H,-D|_H)})$$

- is injective, if  $N > i$ , and
- is bijective, if  $N > i + 1$ .

*Proof.* With the notation from (12.2.1) we define for a closed immersion of  $\mathbb{F}_p$ -schemes  $Z \subset Y$  the complex  $\mathbb{Z}/p^n(0)_{(Y,-Z)}$  of sheaves on  $Y_{\text{ét}}$  by the exact sequence

$$0 \rightarrow \mathbb{Z}/p^r(0)_{(Y,-Z)} \rightarrow \mathbb{Z}/p^n(0)_Y \rightarrow \mathbb{Z}/p^n(0)_Z \rightarrow 0.$$

By (the proof of) Lemma 12.2 the complex  $\mathbb{Z}/p^n(0)_{(Y,-Z)}$  coincides with the one defined in Definition 11.4 in case  $Y$  is smooth and  $Z_{\text{red}}$  is an SNCD. Furthermore, up to quasi-isomorphism, these complexes only depend on  $Y_{\text{red}}$  and  $Z_{\text{red}}$ . For  $r \geq 1$  we denote in the following by  $D + rH$  also the closed subscheme of  $X$  defined by the effective Cartier divisor. It follows directly from the definition and an application of the Snake Lemma, that we have a short exact sequence of complexes on  $X_{\text{ét}}$

$$(12.6.1) \quad 0 \rightarrow \mathbb{Z}/p^n(0)_{(X,-D-rH)} \rightarrow \mathbb{Z}/p^n(0)_{(X,-D)} \rightarrow \mathbb{Z}/p^n(0)_{(D+rH,-D)} \rightarrow 0.$$

There is a natural morphism of complexes

$$(12.6.2) \quad \mathbb{Z}/p^n(0)_{(D+rH,-D)} \rightarrow \mathbb{Z}/p^n(0)_{(rH,-D|_{rH})},$$

which is an isomorphism. Indeed the map is induced by the natural surjection  $\mathcal{O}_{D+rH}(-D) \rightarrow \mathcal{O}_{rH}(-D|_{rH})$  and it suffices to check that this latter map is an isomorphism. This is a local question and we can assume that  $X = \text{Spec } A$  with a factorial local noetherian ring  $A$  and  $D$  and  $H$  are given by elements  $d$  and  $h$  in  $A$ , which have no prime divisors in common (as  $H$  intersects  $D$  transversally). In this case the global sections of  $\mathcal{O}_{D+rH}(-D)$  are

$$\Gamma(\mathcal{O}_{D+rH}(-D)) = dA/dh^r A = dA/(dA \cap h^r A) = (dA + h^r A)/h^r A = \Gamma(\mathcal{O}_{rH}(-D|_{rH})).$$

This shows that (12.6.2) is an isomorphism. Thus (12.6.1) yields an exact sequence

$$\begin{aligned} H^i(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X, -D-rH)}) &\rightarrow H^i(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X, -D)}) \rightarrow H^i(H_{\text{ét}}, \mathbb{Z}/p^n(0)_{(rH, -D|_{rH})}) \\ &\rightarrow H^{i+1}(X_{\text{ét}}, \mathbb{Z}/p^n(0)_{(X, -D-rH)}). \end{aligned}$$

As the third term only depends on  $(rH)_{\text{red}} = H$  (see Lemma 12.2) it suffices to show for the injectivity statement that the term on the left hand side vanishes for  $r \gg 0$  and  $N > i$ , and for the bijectivity that additionally the term on the right hand side vanishes for  $r \gg 0$  and  $N > i + 1$ . By Corollary 11.16 the statement follows from the vanishing

$$(12.6.3) \quad H^{2N-i+1}(X_{\text{ét}}, \mathbb{Z}/p^n(N)_{(X, D+rH)}) = 0, \quad \text{if } N > i \text{ and } r \gg 0.$$

By definition of  $\mathbb{Z}/p^n(N)_{(X, D+rH)}$  this last cohomology group fits into an exact sequence

$$H^{N-i}(X, W_n \Omega_{(X, D+rH)}^N) \rightarrow H^{2N-i+1}(X_{\text{ét}}, \mathbb{Z}/p^n(N)_{(X, D+rH)}) \rightarrow H^{N-i+1}(X, W_n \Omega_{(X, D+rH)}^N).$$

Thus the vanishing (12.6.3) follows from the Serre-type vanishing Theorem 7.6.  $\square$

**Remark 12.7.** (1) The Lefschetz type statement that

$$\pi_1^{\text{ab}}(H, -D|_H)^p \longrightarrow \pi_1^{\text{ab}}(X, -D)^p$$

is surjective, if  $N \geq 2$ , and bijective, if  $N \geq 3$ , which is implied by Corollary 12.6, of course follows also directly from the general Lefschetz Theorem for the étale fundamental group and the sequence (12.3.1).

- (2) In [KS14, Theorem 1.1] (see also [KS22]), the authors prove a Lefschetz Theorem for  $\pi_1^{\text{ab}}(X, D)$ . Note that we cannot get a proof of this statement directly from the duality statements which we developed here so far as it would require to deal with complexes like  $\mathbb{Z}/p^n(0)_{(X, H-D)}$ , i.e., we need to allow divisors which are neither effective nor anti-effective. Note also that in the Lefschetz statement for  $\pi_1^{\text{ab}}(X, D)$  from [KS14] one needs  $H$  to be *sufficiently* ample, whereas in Corollary 12.6 any effective ample  $H$  works.

## 12.2. Weight one.

**12.8.** We will use the notation

$$(12.8.1) \quad \mathcal{O}_{(X, -D)}^\times := \text{Ker}(\mathcal{O}_X^\times \rightarrow \mathcal{O}_D^\times) \quad \text{and} \quad (\mathcal{O}^\times/p^n)_{(X, -D)} := \text{Ker}(\mathcal{O}_X^\times/p^n \rightarrow \mathcal{O}_D^\times/p^n).$$

Note that the sheaf on the left is in the literature often denoted by variants of  $\mathcal{O}_{(X, D)}^\times$ , but in order to have the notation consistent with the other parts of the paper, we put here the minus in front of the  $D$ .

**Proposition 12.9.** *There is a quasi-isomorphism induced by the dlog-map*

$$(\mathcal{O}^\times/p^n)_{(X, -D)} \xrightarrow{\sim} \mathbb{Z}/p^n(1)_{(X, -D)}[1],$$

*which gives rise to a distinguished triangle*

$$(12.9.1) \quad \mathcal{O}_{(X, -\lceil D/p^n \rceil)}^\times \xrightarrow{F^n} \mathcal{O}_{(X, -D)}^\times \longrightarrow \mathbb{Z}/p^n(1)_{(X, -D)}[1] \xrightarrow{+1}.$$

*Proof.* By definition and the surjectivity of the restriction map  $\mathcal{O}_X^\times \rightarrow \mathcal{O}_D^\times$  we have

$$(\mathcal{O}^\times/p^n)_{(X,-D)} = \frac{\mathcal{O}_{(X,-D)}^\times}{(\mathcal{O}_X^\times)^{p^n} \cap \mathcal{O}_{(X,-D)}^\times} = \text{Coker}(F^n : \mathcal{O}_{(X,-[D/p^n])}^\times \rightarrow \mathcal{O}_{(X,-D)}^\times).$$

By [CTSS83, 1.4, Lemme 2] together with [Ill79, I, Proposition 3.23.2] we have an exact sequence

$$(12.9.2) \quad 0 \rightarrow \mathcal{O}_X^\times / (\mathcal{O}_X^\times)^{p^n} \xrightarrow{\text{dlog}} W_n \Omega_X^1 \xrightarrow{C^{-1}-1} W_n \Omega_X^1 / dV^{n-1} \mathcal{O}_X.$$

In view of Lemma 11.6 it remains to show that the natural map

$$(12.9.3) \quad \text{dlog} : \mathcal{O}_{(X,-D)}^\times \rightarrow W_n \Omega_{(X,-D)}^1 \cap W_n \Omega_{X,\log}^1,$$

induced by restriction of  $\text{dlog} : \mathcal{O}_X^\times \rightarrow W_n \Omega_{X,\log}^1$ , is surjective.

We do induction on  $n$ . For  $n = 1$ , this essentially follows from [JSZ18, Theorem 1.2.1]. In fact, by *loc. cit.* the map

$$\text{dlog} : \mathcal{O}_{(X,-D)}^\times \rightarrow \Omega_X^1(\log D)(-D)$$

is surjective. As this map factors via the natural inclusion  $\Omega_X^1(\log E)(-D) \hookrightarrow \Omega_X^1(\log D)(-D)$ , for any reduced effective Cartier divisor  $E$ , satisfying  $D_{0,\text{red}} \subset E \subset D_{\text{red}}$ , where  $D = D_0 + pD_1$  is a  $p$ -divisibility decomposition, we get the surjectivity

$$(12.9.4) \quad \text{dlog} : \mathcal{O}_{(X,-D)}^\times \rightarrow \Omega_X^1(\log E)(-D) \cap \Omega_{X,\log}^1.$$

As  $\Omega_X^1(\log D_0)(-D) = \Omega_{(X,-D)}^1$ , by Lemma 8.4, we in particular obtain (12.9.3) for  $n = 1$

The induction step is adapted from the proof of [JSZ18, Theorem 2.3.1]. Denote

$$M_n := \text{Image} \left( \text{dlog} : \mathcal{O}_{(X,-D)}^\times \rightarrow W_n \Omega_{(X,-D)}^1 \cap W_n \Omega_{X,\log}^1 \right).$$

The restriction map  $R$  clearly induces a surjection  $R : M_n \rightarrow M_{n-1}$ . Consider the following commutative diagram

$$\begin{array}{ccccccc} & & M_n & \xrightarrow{R} & M_{n-1} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{X,\log}^1 & \xrightarrow{\underline{p}^{n-1}} & W_n \Omega_{X,\log}^1 & \xrightarrow{R} & W_{n-1} \Omega_{X,\log}^1 \longrightarrow 0. \end{array}$$

The second row is exact by [CTSS83, 1.4, Lemme 3]. Take  $x \in W_n \Omega_{(X,-D)}^1 \cap W_n \Omega_{X,\log}^1$ . Then  $R(x) \in W_{n-1} \Omega_{(X,-D)}^1 \cap W_{n-1} \Omega_{X,\log}^1$ , and hence by induction hypothesis,  $R(x) \in M_{n-1}$ . Since  $R : M_n \rightarrow M_{n-1}$  is surjective, there exists an element  $y \in M_n$  such that  $R(y) = R(x)$ . By the exactness of the second row,  $x - y = \underline{p}^{n-1}(z)$ , for some  $z \in \Omega_{X,\log}^1$ . In particular,  $\underline{p}^{n-1}(z) \in W_n \Omega_{(X,-D)}^1$  and thus Lemma 8.11 yields  $z \in \Omega_{n-1}^1(-D', -pD_n)$ , where  $D = D' + p^n D_n$  is a  $p$ -divisibility decomposition. By the surjectivity of (12.9.4) we find

$$z = \text{dlog } u, \quad \text{for some } u \in \mathcal{O}_{(X,-[D'/p^{n-1}]-pD_n)}^\times.$$

Hence  $\underline{p}^{n-1}(z) = \text{dlog } u^{p^{n-1}} \in M_n$  and therefore  $x = y + \underline{p}^{n-1}(z) \in M_n$ . This finishes the induction step.  $\square$

**Remark 12.10.** Let  $\mathbb{Z}_{(X|D)}(r)$  be the relative motivic cycle complex from [BS19]. By [BS19, Theorem 4.1] there is a quasi-isomorphism of complexes of étale sheaves

$$\mathbb{Z}_{(X|D)\text{ét}}(1)[1] \simeq \mathcal{O}_{(X,-D)}^\times.$$

And hence Proposition 12.9 gives rise to a distinguished triangle of complexes of étale sheaves

$$\mathbb{Z}_{(X|[D/p^n])_{\text{ét}}}(1) \xrightarrow{\mathcal{V}^n} \mathbb{Z}_{(X|D)_{\text{ét}}}(1) \rightarrow \mathbb{Z}/p^n(1)_{(X,-D)} \xrightarrow{+1},$$

where  $\mathcal{V}^n$  is induced by pullback along  $\text{id}_X \times F^n : X \times_{\mathbb{F}_p} (\mathbb{P}_{\mathbb{F}_p}^1)^r \rightarrow X \times_{\mathbb{F}_p} (\mathbb{P}_{\mathbb{F}_p}^1)^r$ , cf. [KP15, Lemma 5.7]. Thus the notation  $\mathbb{Z}/p^n(1)_{(X,-D)}$  might be a bit misleading as we rather have

$$\mathbb{Z}/p^n(1)_{(X,-D)} = “\mathbb{Z}_{(X|D)}(1)/\mathcal{V}^n \mathbb{Z}_{(X|[D/p^n])}(1)”,$$

which does not agree with  $\mathbb{Z}_{(X|D)}(1)/p^n \mathbb{Z}_{(X|D)}(1)$ . (It does, if  $D = \emptyset$ .) It is interesting to note that the difference between the  $p$ -adic and the  $V$ -adic filtration, which we have on the de Rham-Witt complex (or just on the Witt vectors of a non-perfect field), seems to be visible also on the motivic complex with modulus.

**12.11.** We define the Brauer group of  $X$  with ramification bounded by  $D$ , by

$$\text{Br}(X, D) := H^0(U, R^2 \epsilon_* \mathbb{Q}/\mathbb{Z}(1)'_U) \oplus H^0(X, R^2 \epsilon_* \mathbb{Q}_p/\mathbb{Z}_p(1)_{(X,D)}),$$

where  $U = X \setminus D$ ,  $\epsilon : X_{\text{ét}} \rightarrow X_{\text{Nis}}$  is the change of sites map, and

$$\mathbb{Q}/\mathbb{Z}(1)'_U = \text{colim}_{n'} \mu_{n',U}, \quad \mathbb{Q}_p/\mathbb{Z}_p(1)_{(X,D)} = \text{colim}_{\underline{p}} \mathbb{Z}/p^n(1)_{(X,D)}$$

with the colimit on the left over all  $n'$  which are prime to  $p$ .

We define the Brauer group with zeros along  $D$  by

$$\text{Br}(X, -D) := H^2(X_{\text{ét}}, \mathcal{O}_{(X,-D)}^\times).$$

We remark:

- (1) There are natural inclusions  $\text{Br}(X) \subset \text{Br}(X, D) \subset \text{Br}(U)$  and by 11.5(3)

$$\text{colim}_r \text{Br}(X, rD) = \text{Br}(U).$$

- (2) We have an exact sequence

$$(12.11.1) \quad \text{Pic}(X) \rightarrow \text{Pic}(D) \rightarrow \text{Br}(X, -D) \rightarrow \text{Br}(X) \rightarrow \text{Br}(D),$$

where for a singular scheme  $Z$  we use the definition  $\text{Br}(Z) = H^2(Z_{\text{ét}}, \mathbb{G}_m)$ .

- (3) Assume  $p^n$  is bigger than all the multiplicities of  $D$ . Then the  $n$ -power Frobenius induces by (12.9.1) induces a morphism  $F^n$  fitting in the following commutative diagram

$$(12.11.2) \quad \begin{array}{ccc} \text{Br}(X, -D) & \xrightarrow{p^n} & \text{Br}(X, -D) \\ & \searrow & \nearrow F^n \\ & \text{Br}(X, -D_{\text{red}}) & \end{array}$$

**Remark 12.12.** As is well-known  $\text{Br}(X)$  and  $\text{Br}(U)$  are torsion groups and hence so is  $\text{Br}(X, D)$ . But note that  $\text{Br}(X, -D)$  is in general not a torsion group as becomes apparent from the exact sequence (12.11.1). For example if  $X = \mathbb{P}^2$  and  $D = E$  is an elliptic curve we have  $E(k) \subset \text{Br}(\mathbb{P}^2, -E)$ . Even if  $k$  is a finite field the subgroup  $\text{Coker}(\text{Pic}(X) \rightarrow \text{Pic}(D))$  might not be torsion due to a contribution by  $\text{Pic}_{D/k}(k)/\text{Pic}_{D/k}^0(k)$ .

Now assume that  $k$  is a finite field,  $p \neq 2$ , and  $X$  is a smooth projective surface. In this case the Tate conjecture for divisors is equivalent to the finiteness of  $\text{Br}(X)$  which is also equivalent to the finiteness of its the  $p$ -primary torsion subgroup  $\text{Br}(X)[p^\infty]$ , see [Mil75, Theorem 4.1]. We note:

- (1)  $\text{Br}(X)$  is finite if and only if  $\text{Br}(X, -D)[p^\infty]$  is finite.

- (2) If the reduced irreducible components of  $D$  are linearly independent in  $\text{Num}(X) \otimes \mathbb{Q}$  (divisors modulo numerical equivalence) and  $\text{Br}(X)$  is finite, then  $\text{Br}(X, -D)$  is finite as well.

Indeed, first note that  $\text{Br}(D) = 0$  as follows from the Brauer-Hasse-Noether Theorem, see also [Gro68, Remarques (2.5), b)], together with [CTS21, 3.6.6]. Thus the finiteness of  $\text{Br}(X, -D)[p^\infty]$  implies the finiteness of  $\text{Br}(X)$ . On the other hand,  $\text{Pic}_{D/k}^0$  is a smooth algebraic group scheme (e.g. [BLR90, §9.2]) and hence  $\text{Pic}_{D/k}^0(k)$  is a finite group. Moreover  $(\text{Pic}_{D/k} / \text{Pic}_{D/k}^0)(k)$  is a finitely generated group and hence so is  $\text{Pic}(D) / \text{Pic}_{D/k}^0(k)$ . It follows that  $\text{Im}(\text{Pic}(D) \rightarrow \text{Br}(X, -D)) \cap \text{Br}(X, -D)[p^\infty]$  is finite, which yields the “if” direction in (1). For (2) it remains to observe that under the assumptions made there the index of  $\text{Pic}(X) \rightarrow \text{Pic}(D) / \text{Pic}_{D/k}^0(k)$  is finite.

For  $D = \emptyset$  the following duality statement is the  $p$ -part of Milne’s duality for the Brauer group of a smooth projective surface over a finite field, see [Mil75, Theorem 2.4].

**Theorem 12.13.** *Assume  $k$  is a finite field and  $X$  is a smooth proper surface (i.e.  $N = 2$ ). Then there is a canonical isomorphism of profinite groups*

$$\frac{\text{Br}(X, -D)[p^\infty]}{(\text{Br}(X, -D)[p^\infty])_{\text{div}}} \xrightarrow{\sim} \text{Hom} \left( \frac{\text{Br}(X, D)[p^\infty]}{(\text{Br}(X, D)[p^\infty])_{\text{div}}}, \mathbb{Q}/\mathbb{Z} \right),$$

where the index “div” refers to the divisible part, and  $M[p^\infty]$  denotes the  $p$ -primary torsion in  $M$ .

*Proof.* The proof follows the same strategy as the one in [Mil75, Theorem 2.4]. First of all note that the right hand side of the isomorphism in the statement is a profinite group by Pontryagin duality, hence it suffices to prove the isomorphism of abstract groups. We start by showing that we have an exact sequence

$$(12.13.1) \quad 0 \rightarrow \text{Pic}(U)/p^n \rightarrow H^2(X_{\text{ét}}, \mathbb{Z}/p^n(1)_{(X,D)}) \rightarrow H^0(X, R^2\epsilon_*(\mathbb{Z}/p^n(1)_{(X,D)})) \rightarrow 0.$$

To this end we first claim that we have

$$R^1\epsilon_*(\mathbb{Z}/p^n(1)_{(X,D)}) = j_*(\mathcal{O}_{U_{\text{Nis}}}^\times/p^n) = Rj_*(\mathcal{O}_{U_{\text{Nis}}}^\times/p^n).$$

The first equality follows directly from the definition, see also (11.7.1), and the exact sequence (12.9.2). The second equality follows from the vanishing  $R^i j_* \mathcal{O}_U^\times = 0$ ,  $i \geq 1$ , which relies on the smoothness of  $X$ . Therefore we get a distinguished triangle

$$Rj_*(\mathcal{O}_{U_{\text{Nis}}}^\times/p^n)[-1] \rightarrow R\epsilon_*(\mathbb{Z}/p^n(1)_{(X,D)}) \rightarrow R^2\epsilon_*(\mathbb{Z}/p^n(1)_{(X,D)})[-2] \xrightarrow{+1}.$$

As  $H^r(U_{\text{Nis}}, \mathcal{O}_U^\times) = 0$ , for  $r \geq 2$ , we get the exact sequence (12.13.1). Taking the colimit over  $n$  of (12.13.1) gives an exact sequence

$$(12.13.2) \quad 0 \rightarrow \text{Pic}(U) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^2(X_{\text{ét}}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \rightarrow \text{Br}(X, D)[p^\infty] \rightarrow 0.$$

Now we consider the negative divisor. The distinguished triangle (12.9.1) yields an exact sequence

$$(12.13.3) \quad 0 \rightarrow \frac{\text{Br}(X, -D)}{F^n \text{Br}(X, -\lceil D/p^n \rceil)} \rightarrow H^3(X_{\text{ét}}, \mathbb{Z}/p^n(1)_{(X, -D)}) \\ \rightarrow \text{Ker}(F^n : H^3(X_{\text{ét}}, \mathcal{O}_{(X, -\lceil D/p^n \rceil)}^\times) \rightarrow H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D)}^\times)) \rightarrow 0.$$

Assume  $n$  is large enough so that  $\lceil D/p^n \rceil = D_{\text{red}}$ . Note that the quotient sheaf  $\mathcal{Q} = \mathcal{O}_{(X, -D_{\text{red}})}^\times / \mathcal{O}_{(X, -D)}^\times$  is a successive extension of coherent  $\mathcal{O}_D$ -modules. As  $D$  is a curve we get  $H^i(X_{\text{ét}}, \mathcal{Q}) = 0$ ,  $i \geq 2$ , and hence an isomorphism

$$H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D)}^\times) \xrightarrow{\sim} H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D_{\text{red}})}^\times).$$

Thus the term on the right of (12.13.3) is equal to the  $p^n$ -torsion in  $H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D_{\text{red}})}^\times)$ . Furthermore for fixed  $n_0$  with  $\lceil D/p^{n_0} \rceil = D_{\text{red}}$  we have by (12.11.2) the following inclusions

$$p^n \text{Br}(X, -D) \subset F^n(\text{Br}(X, -D_{\text{red}})) \subset F^{n_0}(p^{n-n_0} \text{Br}(X, -D_{\text{red}})) \subset p^{n-n_0} \text{Br}(X, -D),$$

for  $n \geq n_0$ . Thus  $\{F^n \text{Br}(X, -D_{\text{red}})\}_{n \geq n_0}$  and  $\{p^n \text{Br}(X, -D)\}_n$  define the same topology on  $\text{Br}(X, -D)$ . As (12.13.3) is an exact sequence of finite groups taking the limit over  $n$  yields an exact sequence

$$0 \rightarrow \lim_n \frac{\text{Br}(X, -D)}{p^n \text{Br}(X, -D)} \rightarrow \lim_n H^3(X_{\text{ét}}, \mathbb{Z}/p^n(1)_{(X, -D)}) \rightarrow T_p H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D_{\text{red}})}^\times) \rightarrow 0,$$

where  $T_p M$  denotes the Tate module of  $M$ . As  $T_p H^3(X_{\text{ét}}, \mathcal{O}_{(X, -D_{\text{red}})}^\times)$  is torsion-free and  $\text{Pic}(U) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$  is divisible the statement follows from Corollary 11.16 by the same argument as in [Mil75, Theorem 2.4].  $\square$

**Remark 12.14.** Tate has shown that if  $X$  is an abelian surface over a finite field, then  $\text{Br}(X)[\ell^\infty]$  ( $\ell \neq p$ ) is finite, see [Tat66, Corollary]. By [Mil75, Theorem 4.1] we also have the finiteness of  $\text{Br}(X)[p^\infty]$ . Hence by Remark 12.12 and Theorem 12.13 we find that  $\text{Br}(X, D)[p^\infty]$  is the direct sum of a divisible group with a finite group, for all effective Cartier divisors  $D$  on  $X$  with  $D_{\text{red}}$  a simple normal crossing divisor.

## REFERENCES

- [AS11] Ahmed Abbes and Takeshi Saito. Ramification and cleanliness. *Tohoku Math. J. (2)*, 63(4):775–853, 2011.
- [BBE07] Pierre Berthelot, Spencer Bloch, and Hélène Esnault. On Witt vector cohomology for singular varieties. *Compos. Math.*, 143(2):363–392, 2007.
- [Ber74] Pierre Berthelot. *Cohomologie cristalline des schémas de caractéristique  $p > 0$* , volume Vol. 407 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1974.
- [Ber97] Pierre Berthelot. Dualité de Poincaré et formule de Künneth en cohomologie rigide. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):493–498, 1997.
- [BER12] Pierre Berthelot, Hélène Esnault, and Kay Rülling. Rational points over finite fields for regular models of algebraic varieties of Hodge type  $\geq 1$ . *Ann. of Math. (2)*, 176(1):413–508, 2012.
- [Blo77] Spencer Bloch. Algebraic  $K$ -theory and crystalline cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (47):187–268, 1977.
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
- [Bou98] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.
- [BRS22] Federico Binda, Kay Rülling, and Shuji Saito. On the cohomology of reciprocity sheaves. *Forum Math. Sigma*, 10:Paper No. e72, 111, 2022.
- [Bry83] Jean-Luc Brylinski. Théorie du corps de classes de Kato et revêtements abéliens de surfaces. *Ann. Inst. Fourier (Grenoble)*, 33(3):23–38, 1983.
- [BS19] Federico Binda and Shuji Saito. Relative cycles with moduli and regulator maps. *J. Inst. Math. Jussieu*, 18(6):1233–1293, 2019.
- [BT73] H. Bass and J. Tate. The Milnor ring of a global field. In *Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972)*, pages 349–446. Lecture Notes in Math., Vol. 342, 1973.

- [Con00] Brian Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
- [Cos08] Viorel Costeanu. On the 2-typical de Rham-Witt complex. *Doc. Math.*, 13:413–452, 2008.
- [CR12] Andre Chatzistamatiou and Kay Rülling. Hodge-Witt cohomology and Witt-rational singularities. *Doc. Math.*, 17:663–781, 2012.
- [CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, [2021] ©2021.
- [CTSS83] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Christophe Soulé. Torsion dans le groupe de Chow de codimension deux. *Duke Math. J.*, 50(3):763–801, 1983.
- [D'A] Marco D'Addezio. Edged crystalline cohomology. Preprint, 2024, <https://webusers.imj-prg.fr/~marco.d-addezio/papers.php>.
- [DI87] Pierre Deligne and Luc Illusie. Relèvements modulo  $p^2$  et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.
- [Eke84] Torsten Ekedahl. On the multiplicative properties of the de Rham-Witt complex. I. *Ark. Mat.*, 22(2):185–239, 1984.
- [Eke85] Torsten Ekedahl. On the multiplicative properties of the de Rham-Witt complex. II. *Ark. Mat.*, 23(1):53–102, 1985.
- [GH06] Thomas Geisser and Lars Hesselholt. On the  $K$ -theory of complete regular local  $\mathbb{F}_p$ -algebras. *Topology*, 45(3):475–493, 2006.
- [GK] Rahul Gupta and Amalendu Krishna. Ramified class field theory and duality over finite fields. Preprint 2021, <https://arxiv.org/abs/2104.03029>.
- [GL00] Thomas Geisser and Marc Levine. The  $K$ -theory of fields in characteristic  $p$ . *Invent. Math.*, 139(3):459–493, 2000.
- [Gro68] Alexander Grothendieck. Le groupe de Brauer. III. Exemples et compléments. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 88–188. North-Holland, Amsterdam, 1968.
- [Gro85] Michel Gros. Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique. *Mém. Soc. Math. France (N.S.)*, (21):87, 1985.
- [Har66] Robin Hartshorne. *Residues and duality*, volume No. 20 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.
- [Har72] Robin Hartshorne. Cohomology with compact supports for coherent sheaves on an algebraic variety. *Math. Ann.*, 195:199–207, 1972.
- [Hes04] Lars Hesselholt. Topological Hochschild homology and the de Rham-Witt complex for  $\mathbb{Z}_{(p)}$ -algebras. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 253–259. Amer. Math. Soc., Providence, RI, 2004.
- [HK94] Osamu Hyodo and Kazuya Kato. Semi-stable reduction and crystalline cohomology with logarithmic poles. *Astérisque*, (223):221–268, 1994. Périodes  $p$ -adiques (Bures-sur-Yvette, 1988).
- [HM03] Lars Hesselholt and Ib Madsen. On the  $K$ -theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.
- [HM04] Lars Hesselholt and Ib Madsen. On the De Rham-Witt complex in mixed characteristic. *Ann. Sci. École Norm. Sup. (4)*, 37(1):1–43, 2004.
- [Hyo88] Osamu Hyodo. A cohomological construction of Swan representations over the Witt ring. I. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(8):300–303, 1988.
- [Hyo91] Osamu Hyodo. On the de Rham-Witt complex attached to a semi-stable family. *Compositio Math.*, 78(3):241–260, 1991.
- [Ill79] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 12(4):501–661, 1979.
- [Ill83] Luc Illusie. Finiteness, duality, and Künneth theorems in the cohomology of the de Rham-Witt complex. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 20–72. Springer, Berlin, 1983.
- [IR83] Luc Illusie and Michel Raynaud. Les suites spectrales associées au complexe de de Rham-Witt. *Inst. Hautes Études Sci. Publ. Math.*, (57):73–212, 1983.

- [JSZ18] Uwe Jannsen, Shuji Saito, and Yigeng Zhao. Duality for relative logarithmic de Rham–Witt sheaves and wildly ramified class field theory over finite fields. *Compos. Math.*, 154(6):1306–1331, 2018.
- [Kat70] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.
- [Kat80] Kazuya Kato. A generalization of local class field theory by using  $K$ -groups. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(3):603–683, 1980.
- [Kat86] Kazuya Kato. Duality theories for the  $p$ -primary étale cohomology. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 127–148. Kinokuniya, Tokyo, 1986.
- [Kat87] Kazuya Kato. Duality theories for  $p$ -primary étale cohomology. II. *Compositio Math.*, 63(2):259–270, 1987.
- [Kat89] Kazuya Kato. Swan conductors for characters of degree one in the imperfect residue field case. In *Algebraic  $K$ -theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 101–131. Amer. Math. Soc., Providence, RI, 1989.
- [Ker10] Moritz Kerz. Milnor  $K$ -theory of local rings with finite residue fields. *J. Algebraic Geom.*, 19(1):173–191, 2010.
- [KMSY21a] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito, and Takao Yamazaki. Motives with modulus, I: Modulus sheaves with transfers for non-proper modulus pairs. *Épjournal Géom. Algébrique*, 5:Art. 1, 46, 2021.
- [KMSY21b] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito, and Takao Yamazaki. Motives with modulus, II: Modulus sheaves with transfers for proper modulus pairs. *Épjournal Géom. Algébrique*, 5:Art. 2, 31, 2021.
- [KMSY22] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito, and Takao Yamazaki. Motives with modulus, III: The categories of motives. *Ann. K-Theory*, 7(1):119–178, 2022.
- [Koi] Junnosuke Koizumi. Blow-up invariance of cohomology theories with modulus. Preprint 2023, <https://arxiv.org/abs/2306.14803>.
- [KP15] Amalendu Krishna and Jinhyun Park. DGA-structure on additive higher Chow groups. *Int. Math. Res. Not. IMRN*, (1):1–54, 2015.
- [KS14] Moritz Kerz and Shuji Saito. Lefschetz theorem for abelian fundamental group with modulus. *Algebra Number Theory*, 8(3):689–701, 2014.
- [KS16] Moritz Kerz and Shuji Saito. Chow group of 0-cycles with modulus and higher-dimensional class field theory. *Duke Math. J.*, 165(15):2811–2897, 2016.
- [KS22] Moritz Kerz and Shuji Saito. Correction to the article Lefschetz theorem for abelian fundamental group with modulus. *Algebra Number Theory*, 16(8):2001–2003, 2022.
- [KSY16] Bruno Kahn, Shuji Saito, and Takao Yamazaki. Reciprocity sheaves. *Compos. Math.*, 152(9):1851–1898, 2016. With two appendices by Kay Rülling.
- [KSY22] Bruno Kahn, Shuji Saito, and Takao Yamazaki. Reciprocity sheaves, II. *Homology Homotopy Appl.*, 24(1):71–91, 2022.
- [Lem22] Niklas Lemcke. Duality for Witt-divisorial sheaves. *Ark. Mat.*, 60(1):107–124, 2022.
- [Lip09] Joseph Lipman. Notes on derived functors and Grothendieck duality. In *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 of *Lecture Notes in Math.*, pages 1–259. Springer, Berlin, 2009.
- [LZ04] Andreas Langer and Thomas Zink. De Rham–Witt cohomology for a proper and smooth morphism. *J. Inst. Math. Jussieu*, 3(2):231–314, 2004.
- [Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
- [Mat97] Shigeki Matsuda. On the Swan conductor in positive characteristic. *Amer. J. Math.*, 119(4):705–739, 1997.
- [Mat17] Hironori Matsue. On relative and overconvergent de Rham–Witt cohomology for log schemes. *Math. Z.*, 286(1-2):19–87, 2017.
- [Mer] Alberto Merici. A motivic integral  $p$ -adic cohomology. Preprint 2022, <https://arxiv.org/abs/2211.14303>.
- [Mil75] J. S. Milne. On a conjecture of Artin and Tate. *Ann. of Math. (2)*, 102(3):517–533, 1975.
- [Mil76] J. S. Milne. Duality in the flat cohomology of a surface. *Ann. Sci. École Norm. Sup. (4)*, 9(2):171–201, 1976.

- [Mil86] J. S. Milne. Values of zeta functions of varieties over finite fields. *Amer. J. Math.*, 108(2):297–360, 1986.
- [Mok93] A. Mokrane. La suite spectrale des poids en cohomologie de Hyodo-Kato. *Duke Math. J.*, 72(2):301–337, 1993.
- [Mor19] Matthew Morrow.  $K$ -theory and logarithmic Hodge-Witt sheaves of formal schemes in characteristic  $p$ . *Ann. Sci. Éc. Norm. Supér. (4)*, 52(6):1537–1601, 2019.
- [MS97] V. B. Mehta and V. Srinivas. A characterization of rational singularities. *Asian J. Math.*, 1(2):249–271, 1997.
- [Nak05] Yuki Yoshi Nakkajima.  $p$ -adic weight spectral sequences of log varieties. *J. Math. Sci. Univ. Tokyo*, 12(4):513–661, 2005.
- [RS] Kay Rülling and Shuji Saito. Ramification theory for reciprocity sheaves, III, Abbes-Saito formula. Preprint 2022, <https://arxiv.org/abs/2204.10637>.
- [RS21] Kay Rülling and Shuji Saito. Reciprocity sheaves and their ramification filtrations. *Journal of the Institute of Mathematics of Jussieu*, page 1–74, 2021.
- [Sai17] Takeshi Saito. Wild ramification and the cotangent bundle. *J. Algebraic Geom.*, 26(3):399–473, 2017.
- [Sai20] Shuji Saito. Purity of reciprocity sheaves. *Adv. Math.*, 366:107067, 70, 2020.
- [Sai23] Shuji Saito. Reciprocity sheaves and logarithmic motives. *Compos. Math.*, 159(2):355–379, 2023.
- [SGA72a] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, volume Vol. 269 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [SGA72b] *Théorie des topos et cohomologie étale des schémas. Tome 2*, volume Vol. 270 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [Shi02] Atsushi Shiho. Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology. *J. Math. Sci. Univ. Tokyo*, 9(1):1–163, 2002.
- [SS03] Michael Spieß and Tamás Szamuely. On the Albanese map for smooth quasi-projective varieties. *Math. Ann.*, 325(1):1–17, 2003.
- [Sta23] The Stacks project authors. The stacks project. <https://stacks.math.columbia.edu>, 2023.
- [Tan22] Hiromu Tanaka. Vanishing theorems of Kodaira type for Witt canonical sheaves. *Selecta Math. (N.S.)*, 28(1):Paper No. 12, 50, 2022.
- [Tat66] John Tate. Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2:134–144, 1966.

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