ON THE EQUIVALENCE OF ALL NOTIONS OF GENERALIZED DERIVATIONS WHOSE DOMAIN IS A C*-ALGEBRA

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ABSTRACT. Let $\mathscr{M}$ be a Banach bimodule over an associative Banach algebra $\mathscr{A}$, and let $F : \mathscr{A} \to \mathscr{M}$ be a linear mapping. Three main uses of the term *generalized derivation* are identified in the available literature, namely,

(✓) F is a generalized derivation of the first type if there exists a derivation $d : \mathscr{A} \to \mathscr{M}$ satisfying $F(ab) = F(a)b + ad(b)$ for all $a, b \in \mathscr{A}$.

(✓) F is a generalized derivation of the second type if there exists an element $\xi \in \mathscr{M}^{**}$ satisfying $F(ab) = F(a)b + aF(b) - a\xi b$ for all $a, b \in \mathscr{A}$.

(✓) F is a generalized derivation of the third type if there exist two (non-necessarily linear) mappings $G, H : \mathscr{A} \to \mathscr{M}$ satisfying $F(ab) = G(a)b + aH(b)$ for all $a, b \in \mathscr{A}$.

There are examples showing that these three definitions are not, in general, equivalent. Despite that the first two notions are well studied when $\mathscr{A}$ is a C*-algebra, it is not known if the three notions are equivalent under these special assumptions. In this note we prove that every generalized derivation of the third type whose domain is a C*-algebra is automatically continuous. We also prove that every (continuous) generalized derivation of the third type from a C*-algebra $\mathscr{A}$ into a general Banach $\mathscr{A}$-bimodule is a generalized derivation of the first and second type. In particular, the three notions coincide in this case. We also explore the possible notions of generalized Jordan derivations on a C*-algebra and establish some continuity properties for them.

1. INTRODUCTION

Since early nineties, several notions of generalized derivations from an algebra to a bimodule have been considered in the literature, all of them built on an appropriate weak version of the proper definition of derivation. Derivations are among the most studied maps in the literature. Recall that a linear mappings d from an associative algebra $\mathscr{A}$ to an $\mathscr{A}$-bimodule $\mathscr{M}$ is called a *derivation* (respectively, a *Jordan derivation*) if it satisfies Leibniz' rule

$$d(ab) = d(a)b + ad(b) \quad \text{(respectively, } d(a^2) = d(a)a + ad(a)), \quad (\forall a, b \in \mathscr{A}).$$

A simple example of derivation can be given by fixing an element $x_0 \in \mathscr{M}$ and defining the mapping $d_{x_0} : \mathscr{A} \to \mathscr{M}$ given by the commutator associated with x_0, that is, $d_{x_0}(a) = [a, x_0] = ax_0 - x_0 a$. Such a derivation is called *inner derivation*.

The available literature contains at least three different uses of the term "*generalized derivation*". The first one, in chronological order, appears in a paper by Brešar published in 1991 [4]. Keeping the notation above, a linear mapping $G : \mathscr{A} \to \mathscr{M}$ is a *generalized derivation of the first type* if there exists a derivation $d : \mathscr{A} \to \mathscr{M}$ such that the identity

$$G(ab) = G(a)b + ad(b) \tag{1.1}$$

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holds for all $a,b \in \mathscr{A}$. In the original definition, G is only assumed to be additive. Note that (1.1) is equivalent to say that $G-d$ is a *left multiplier*, that is,

$$(G-d)(ab) = G(a)b + ad(b) - d(a)b - ad(b) = (G-d)(a)b \ \ (a,b \in \mathscr{A}).$$

In this case, the structure of a right $\mathscr{A}$-ideal on $\mathscr{M}$ is sufficient. Right multipliers can be similarly defined. This first notion is the one employed, for example, in the study by Heller, Miller, Pysiak and Sasin connecting differential geometry (connection, curvature, etc.), generalized derivations, and general relativity [18].

Keeping the chronological order, the second notion of generalized derivation was introduced by Nakajima in 1999 (cf. [36, (1.3)]). A reformulation of this notion was considered by Leger and Luks in [32]. This is the notion employed by Alaminos, Brešar, Extremera and Villena in the study of bounded linear operators preserving zero products (see [1, Definition 4.1]), and for example in [2, 11, 34, 35] and [40]. A *generalized derivation of the second type* from a Banach algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ is a linear mapping $G : \mathscr{A} \to \mathscr{M}$ for which there exists $\xi \in \mathscr{M}^{**}$ satisfying

$$G(ab) = G(a)b + aG(b) - a\xi b \ (a,b \in A). \tag{1.2}$$

Every derivation is a generalized derivation, though the class of generalized derivations is strictly wider than the set of derivations (e.g. for each $a \in \mathscr{A}$, the mapping $x \mapsto x \circ a = \frac{1}{2}(ax + xa)$ is a generalized derivation which is not a derivation). Let us observe that if $\mathscr{A}$ and $\mathscr{M}$ are unital with unit $\mathbf{1}$, module products of the from $a\xi\mathbf{1}$ and $\mathbf{1}\xi b$ lie in $\mathscr{M}$ for all $a,b \in \mathscr{A}$, and hence the left and right multiplication operators $L_{\mathbf{1}\xi}, R_{\xi\mathbf{1}} : a \mapsto \xi a, a\xi$ define two bounded linear operators from $\mathscr{A}$ to $\mathscr{M}$ (i.e., $\mathbf{1}\xi$ and $\xi\mathbf{1}$ behave like a multiplier). In this case, the mapping $d = G - L_{\mathbf{1}\xi} : \mathscr{A} \to \mathscr{M}$ is a derivation and $G(ab) = G(a)b + ad(b)$ for all $a,b \in \mathscr{A}$. Therefore, every generalized derivation of the second type is generalized derivation of the first type.

If $\mathscr{A}$ is a unital algebra, and $G : \mathscr{A} \to \mathscr{M}$ is a generalized derivation of the first type with associated derivation d. Since $G-d$ is a left multiplier, we have $(G-d)(a) = (G-d)(\mathbf{1})a$ for all $a \in \mathscr{A}$, and thus

$$G(ab) = G(a)b + aG(b) - a(G-d)(b) = G(a)b + aG(b) - a(G-d)(\mathbf{1})b \ \ (a,b \in \mathscr{A}),$$

which shows that generalized derivations of the first and second type coincide in this case.

Furthermore, every bounded left multiplier L from a general C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule $\mathscr{M}$, is of the form $L(a) = \xi a$, where $\xi \in M^{**}$ satisfies $\xi\mathscr{A} \subseteq \mathscr{A}$ (cf. [1, 2]). As before, we see that in this case, continuous generalized derivations of the first and second types from $\mathscr{A}$ to $\mathscr{M}$ coincide, thanks to the existence of bounded approximate units. It is important to note that continutiy has been assumed to establish the equivalence between generalized derivations of the first two types from a general C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule. It should be also mentioned that every generalized derivation of the second type from a C*-algebra into a Banach bimodule is automatically continuous (cf. [25, Proposition 2.1]). Hence, when the domain is a C*-algebra $\mathscr{A}$, the first two types of (continuous) generalized derivations agree.

To introduce the third notion of generalized derivation, it is necessary to revisit the definition of a *ternary derivation* or a 3-*tuple behaving like a derivation*, introduced by Jimenéz-Gestal and Pérez-Izquierdo in [26] and Shestakov in [40, 41]. However, here we relax the linearity assumptions on the last two maps in the 3-tuple.

Definition 1.1. Let $\mathscr{A}$ be an algebra, and let $\mathscr{M}$ be an $\mathscr{A}$-bimodule. A ternary derivation from $\mathscr{A}$ to $\mathscr{M}$ is a 3-tuple (F,G,H), where $G,H:\mathscr{A}\to\mathscr{M}$ are two (non-necessarily linear) mappings, and $F:\mathscr{A}\to\mathscr{M}$ is a linear map satisfying $F(ab)=G(a)b+aH(b)$ for all $a,b\in\mathscr{A}$. We shall also say that $F:\mathscr{A}\to\mathscr{M}$ is a ternary derivation with associated mappings $G,H:\mathscr{A}\to\mathscr{M}$. Since the term "ternary derivation" is also employed in another settings (like in the case of JB*-triples) with another meaning, we shall better say that the triplet (F,G,H) behaves like a derivation. In this case the mapping F is called a *generalized derivation of the third type*.

There exist examples of 3-tuples $(F,G,H):\mathscr{A}\to\mathscr{M}$ behaving like a derivation where G and H are not necessarily linear (see Example 2.1).

If D is a derivation from an algebra $\mathscr{A}$ to an $\mathscr{A}$-bimodule $\mathscr{M}$, the triplet (D,D,D) behaves like a derivation. Therefore every derivation is a generalized derivation of the third type.

Komatsu and Nakajima presented in [31] a detailed study on the relations among the notions of generalized derivations of first, second and third type and their other formal properties, mainly in the unital case and from an algebraic perspective. It is perhaps worth to recall some basic connections. Suppose $F:\mathscr{A}\to\mathscr{M}$ is a generalized derivation of the second type satisfying (1.2). The expression

$$F(ab)=(F-R_\xi)(a)b+aF(b)=F(a)b+a(F-L_\xi)(b),$$

is valid for all $a,b\in\mathscr{A}$, and hence $(F,F-R_\xi,F)$ and $(F,F,F-L_\xi)$ behave like derivations. That is, every generalized derivation of the second type is a generalized derivation of the third type. Conversely, it was already observed by Shestakov in [40, Lemma 1] that if $\mathscr{A}$ is a unital associate algebra, every generalized derivation of the third type $F:\mathscr{A}\to\mathscr{A}$ is generalized derivation of the second type (see also [31, Lemma 4.1 and Corollary 4.5]). The argument works in our general setting. Suppose (F,G,H) is a 3-tuple of mappings from a unital associative algebra $\mathscr{A}$ into a unital $\mathscr{A}$-bimodule behaving like a derivation, with F being linear. As we shall see in Lemma 2.2, both maps G and H are linear and satisfy $G(a)=F(a)-aH(\mathbf{1})$ and $H(a)=F(a)-G(\mathbf{1})a$ for all $a\in\mathscr{A}$. Therefore it follows that

$$F(ab)=F(a)b+aF(b)-a\Big(H(\mathbf{1})+G(\mathbf{1})\Big)b, \text{ for all } a,b\in\mathscr{A},$$

which shows that F is a generalized derivation of the second type.

However, if we relax the assumptions that $\mathscr{A}$ and the $\mathscr{A}$-bimodule are both unital, it is not clear whether every generalized derivation of the third type is of the second type. This naturally gives rise to a question: are these three notions of generalized derivation defined from a nonunital algebra $\mathscr{A}$ into a $\mathscr{A}$-bimodule equivalent with each other? In this paper, we shall give an affirmative answer to this question when $\mathscr{A}$ is a general C*-algebra. In fact, this problem is equivalent to a question on automatic continuity.

Bounded linear operators which are generalized derivations admit a useful algebraic characterization.

Theorem 1.2. ([1, Theorem 4.5], [7, Proposition 4.3], [2, Theorem 2.11]) *Let $T:\mathscr{A}\to X$ be a bounded linear operator from a C*-algebra to an essential Banach $\mathscr{A}$-bimodule. Then the following statements are equivalent:*

(a) T is a generalized derivation (of the second type).
(b) $aT(b)c=0$, whenever $ab=bc=0$ in $\mathscr{A}$.

(c) $aT(b)c = 0$, *whenever* $ab = bc = 0$ *in* $\mathscr{A}_{sa}$.

In case of a linear mapping acting on a von Neumann algebra $\mathscr{W}$ and behaving like a generalized derivations at certain points of the domain, continuity becomes an inherent property. More concretely, let $T : \mathscr{W} \to \mathscr{W}$ be a linear mapping on a von Neumann algebra. Suppose that for each a,b,c in any commutative von Neumann subalgebra $\mathscr{B} \subseteq \mathscr{W}$ with $ab = bc = 0$ we have $aT(b)c = 0$. Then T is automatically continuous [11, Theorem 2.12 and Corollary 2.15]. Furthermore, the above statements (a)–(c) are also equivalent to the next:

(d) $aT(b)c + cT(b)a = 0$, whenever $ab = bc = 0$ in $\mathscr{W}_{sa}$.
(e) $aT(b)a = 0$, whenever $ab = 0$ in $\mathscr{W}_{sa}$.

Every von Neumann algebra is unital, and hence every generalized derivation of the third type on a von Neuman algebra is a generalized derivation of the first and second type.

The list of studies exploring the automatic continuity of derivations and related operators is quite wide. The pioneering theorems by Sakai [43] and Ringrose [38] prove that every derivation on a C*-algebra or from a C*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is automatically continuous, respectively. Hou and Ming [24] proved that if $\mathscr{X}$ is a simple Banach space, and $\sigma, \tau : B(\mathscr{X}) \to B(\mathscr{X})$ are surjective and continuous at 0, then every (σ,τ)-derivation from $B(\mathscr{X})$ into itself is continuous. Recall that, if $\mathscr{A}$ in an algebra and $\sigma,\tau : \mathscr{A} \to \mathscr{A}$ are two mappings, a (σ,τ)*-derivation on* $\mathscr{A}$ is a linear mapping $d : \mathscr{A} \to \mathscr{A}$ satisfying

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b), \text{ for all } a,b \in \mathscr{A}.$$

More results on automatic continuity can be found in [11, 13, 15, 21, 22, 23, 29, 28, 37, 42].

Let us explain how, in the setting of C*-algebras and essential Banach bimodules, the problem of determining whether every generalized derivation of the third type is of the second type is a problem of automatic continuity. Suppose F,G,H are three mappings from a C*-algebra $\mathscr{A}$ to an essential $\mathscr{A}$-bimodule $\mathscr{M}$, such that the triplet (F,G,H) behaves like a derivation, and let us assume that F is continuous. For arbitrary $a,b,c \in \mathscr{A}_{sa}$ with $ab = bc = 0$. Choose, via functional calculus, a decomposition of b in the form $b = b^+ - b^-$, with $b^+, b^- \geq 0$, $b^+b^- = 0$, $ab^+ = ab^- = 0$, $b^+c = 0$, and $b^-c = 0$. By taking $d = (b^+)^{\frac{1}{2}} + i(b^-)^{\frac{1}{2}}$, we have $ad = dc = 0$, $d^2 = b$ and

$$aF(b)c = aF(d^2)c = aG(d)dc + adH(d)c = 0.$$

Theorem 1.2 implies that F is a generalized derivation of the second type. So it suffices to prove that every generalized of the third type from a C*-algebra into an essential Banach bimodule is continuous automatically. In Proposition 2.5 we prove that every continuous generalized derivation of the third type from a C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule is a generalized derivation of the first and second type, that is, we do not need to assume that the bimodule is essential. Theorem 2.8 completes the picture by showing that every generalized derivation of the third type from a C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ is continuous. Consequently, every generalized derivation of the third type from $\mathscr{A}$ to $\mathscr{M}$ is a generalized derivation of the second type (and of course, of the first type).

Generalized Jordan derivations of the second type from a C*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ have been already considered, for example, in [2, 6, 7]. A linear mapping $F : \mathscr{A} \to \mathscr{M}$ is a *generalized Jordan derivation of the second type* if there exists an element $\xi \in \mathscr{M}^{**}$

satisfying

$$F(a\circ b)=F(a)\circ b+a\circ F(b)-U_{a,b}(\xi),\ \text{for all } a,b\in\mathscr{A},$$

where $a\circ b=\frac{1}{2}(ab+ba)$ and $U_{a,b}(\xi)=\frac{1}{2}(a\xi b+b\xi a)$. Here, we shall say that a linear mapping $F:\mathscr{A}\to\mathscr{M}$ is a *generalized Jordan derivation of the third type* if there exist (non-necessarily linear) maps $G,H:\mathscr{A}\to\mathscr{M}$ such that

$$F(a\circ b)=G(a)\circ b+a\circ H(b),\ \text{for all } a,b\in\mathscr{A}.$$

Theorem 3.1 shows that if $\mathscr{A}$ is a C*-algebra, every generalized Jordan derivation of the third type F from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ is continuous. Moreover, if $\mathscr{M}$ is essential, F is a generalized derivation of the second type; and, in certain cases (for example, when $\mathscr{M}=\mathscr{A}^*$ or $\mathscr{M}=\mathscr{A}^*$), generalized Jordan derivations of the third type will be generalized Jordan derivation of the second type whose associated element in $\mathscr{M}^{**}$ will commute with all elements in $\mathscr{A}$ (see Proposition 3.2).

2. Automatic continuity of generalized derivations of the third type

The main goal of this section is to get a result on the automatic continuity of generalized derivation of the third type from a C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule. We begin with an example of a 3-tuple of maps behaving like a derivation in which one or two of the last two maps is not linear.

Example 2.1. Let $\mathscr{A}$ be an algebra, and let

$$\mathfrak{A}=\left\{\begin{bmatrix} a & b & c\\ 0 & 0 & 0\\ 0 & 0 & e\end{bmatrix}\ :\ a,b,c,e\in\mathscr{A}\right\}.$$

Clearly, $\mathfrak{A}$ is an algebra with respect to the natural matrix product. Let $f:\mathscr{A}\to\mathscr{A}$ be a (non-necessarily linear) mapping. Define mappings $F,G,H:\mathfrak{A}\to\mathfrak{A}$ by

$$F\left(\begin{pmatrix} a & b & c\\ 0 & 0 & 0\\ 0 & 0 & e\end{pmatrix}\right)=\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ G\left(\begin{pmatrix} a & b & c\\ 0 & 0 & 0\\ 0 & 0 & e\end{pmatrix}\right)=\begin{pmatrix} -a & f(b) & -c\\ 0 & 0 & 0\\ 0 & 0 & -e\end{pmatrix},$$

$$\text{and}\quad H\left(\begin{bmatrix} a & b & c\\ 0 & 0 & 0\\ 0 & 0 & e\end{bmatrix}\right)=\begin{bmatrix} a & a+b & b+c\\ 0 & 0 & 0\\ 0 & 0 & e\end{bmatrix}.$$

Observe that F,H is linear while G is non-necessarily linear. A straightforward verification shows that $F(AB)=G(A)B+AH(B)$, for all $A,B\in\mathfrak{A}$, which means that (F,G,H) behaves like a ternary derivation on $\mathfrak{A}$, and hence F is a generalized derivation of the third type on $\mathscr{A}$.

Let D be a derivation on $\mathscr{A}$. Consider the Banach algebra $\mathscr{B}=\left\{\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}:a\in\mathscr{A}\right\}$ and the $\mathscr{B}$-bimodule $\mathscr{M}=M_2(\mathscr{A})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}:a,b,c,d\in\mathscr{A}\right\}$, with the obvious operations. Define $F,G,H:\mathscr{B}\to M_2(\mathscr{A})$, by $F\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}:=\begin{pmatrix} D(a) & 0\\ 0 & 0\end{pmatrix}$, $G\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}:=\begin{pmatrix} D(a) & f(a)\\ 0 & 0\end{pmatrix}$, and $H\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix} := \begin{pmatrix} D(a) & 0\\ f(a) & 0\end{pmatrix}$. Clearly, G and H are non-necessarily linear, and is easy to see that (F,G,H) behaves like a derivation.

It follows from the previous counterexample that if the triplet (F,G,H) behaves like a derivation, the mappings G and H need not be, in general, linear. The existence of a unit element in $\mathscr{A}$ forces the linearity of these maps, up to an appropriate change on the module.

Lemma 2.2. *Let (F,G,H) behaves like a derivation from an algebra $\mathscr{A}$ to an $\mathscr{A}$-bimodule $\mathscr{M}$. Suppose $\mathscr{A}$ is unital with unit $\mathbf{1}$. Then the mappings $G\cdot\mathbf{1}, \mathbf{1}\cdot H : \mathscr{A}\to\mathscr{M}$, $x\mapsto G(x)\mathbf{1}$, $x\mapsto \mathbf{1}H(x)$ are linear, and the 3-tuple $(F, G\cdot\mathbf{1}, \mathbf{1}\cdot H)$ behaves like a derivation. If $\mathscr{M}$ is unital, the mappings G and H are linear.*

Proof. The conclusion is clear from the identities $(G\cdot\mathbf{1})(a) = F(a) - aH(\mathbf{1})$, $(\mathbf{1}\cdot H)(a) = F(a) - G(\mathbf{1})a$ $(a\in\mathscr{A})$ and the linearity of F. Observe that

$$F(ab) = G(a)b + aH(b) = G(a)\mathbf{1}b + a\mathbf{1}H(b) = (G\cdot\mathbf{1})(a)\, b + a\, (\mathbf{1}\cdot H)(b)\ (\forall a,b\in\mathscr{A}).$$

□

Let us recall some well known concepts. Let $\mathscr{A}$ be an algebra and let $\mathscr{M}$ be an $\mathscr{A}$-bimodule. The *modular left annihilator of* $\mathscr{A}$ is the set $lann(\mathscr{A})_{\mathscr{M}} := \{m_0\in\mathscr{M} \mid m_0\mathscr{A} = \{0\}\}$. Similarly, the *modular right annihilator of* $\mathscr{A}$ defined by $rann(\mathscr{A})_{\mathscr{M}} := \{m_0\in\mathscr{M} \mid \mathscr{A}m_0 = \{0\}\}$. The *modular annihilator of* $\mathscr{A}$ is $ann(\mathscr{A})_{\mathscr{M}} := rann(\mathscr{A})_{\mathscr{M}} \cap lann(\mathscr{A})_{\mathscr{M}}$. In particular, if $\mathscr{A} = \mathscr{M}$, then we reach the usual notions of left annihilator, right annihilator and annihilator of the algebra $\mathscr{A}$ which are denoted by $lann(\mathscr{A})$, $rann(\mathscr{A})$ and $ann(\mathscr{A})$, respectively.

Along this section, the left and right multiplication operators by an element a in an associative algebra $\mathscr{A}$ will be denoted by L_a and R_a, respectively. If $\mathscr{M}$ is an $\mathscr{A}$-bimodule, the mappings L_a and R_a will also stand for the corresponding left and right multiplication operators on $\mathscr{M}$.

The next auxiliary lemma will be used in our arguments.

Lemma 2.3. *Let $\mathscr{A}$ be an algebra, let $\mathscr{M}$ be an $\mathscr{A}$-bimodule. Suppose (F,G,H) is a 3-tuple of mappings from $\mathscr{A}$ to $\mathscr{M}$ behaving like a derivation. Then the following statements hold:*

(i) The mappings $\Psi_a : \mathscr{A}\to\mathscr{M}$, $\Psi_a(b) = aH(b) = L_aH(b)$ and $\Gamma_a : \mathscr{A}\to\mathscr{M}$, $\Gamma_a(b) = G(b)a = R_aG(b)$ are linear for every $a\in\mathscr{A}$.

(ii) If $rann(\mathscr{A})_{\mathscr{M}} = \{0\}$ (respectively, $lann(\mathscr{A})_{\mathscr{M}} = \{0\}$), then the mapping G (respectively, the mapping H) is linear.

Proof. (i) We show that the mapping Ψ_a is a linear mapping for every $a\in\mathscr{A}$. Given $a,b,c\in\mathscr{A}$ and $\lambda\in\mathbb{C}$, we have

$$\begin{aligned} F(a\lambda(b+c)) &= G(a)(\lambda b + \lambda c) + aH(\lambda(b+c))\\ &= \lambda G(a)b + \lambda G(a)c + aH(\lambda(b+c)). \end{aligned}$$

On the other hand, since F is a linear mapping, we have the following expressions:

$$\begin{aligned} F(a\lambda(b+c)) &= \lambda F(ab) + \lambda F(ac)\\ &= \lambda G(a)b + \lambda aH(b) + \lambda G(a)c + \lambda aH(c), \end{aligned}$$

for all $a,b,c \in \mathscr{A}$. Comparing the previous two equations we get that

$$aH(\lambda(b+c)) - \lambda aH(b) - \lambda aH(c) = 0, \tag{2.1}$$

which yields

$$\Psi_a(\lambda(b+c)) = \lambda\Psi_a(b) + \lambda\Psi_a(c).$$

Hence, Ψ_a is a linear mapping for any $a \in \mathscr{A}$. Similarly, one can show that the mapping $\Gamma_a : \mathscr{A} \to \mathscr{M}$, $\Gamma_a(t) = G(t)a$ is a linear mapping.

(ii) Assuming that $rann(\mathscr{A})_{\mathscr{M}} = \{0\}$, it follows from (2.1) that

$$H(\lambda(b+c)) = \lambda H(b) + \lambda H(c)$$

for all $b,c \in \mathscr{A}$ and all $\lambda \in \mathbb{C}$, and thus $H(a+b) = H(a) + H(b)$ for all $a,b \in \mathscr{A}$, $H(0) = 0$ and $H(\lambda a) = \lambda H(a)$ for all $\lambda \in \mathbb{C}$. The other statement can be similarly obtained. □

Example 2.4. Let $\mathbb{Z}$ be the set of all integers. Set

$$\mathscr{A} = \left\{ \begin{bmatrix} 2n & 0 \\ 0 & 2n \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It is evident that $\mathscr{A}$ is a nonunital ring. Let

$$\mathscr{M} = \left\{ \begin{bmatrix} i & j \\ 0 & k \end{bmatrix} : i,j,k \in \mathbb{Z} \right\}.$$

A straightforward verification shows that $lann(\mathscr{A})_{\mathscr{M}} = \{0\} = rann(\mathscr{A})_{\mathscr{M}}$.

If $\mathscr{A}$ is unital, $lann(\mathscr{A}) = \{0\} = rann(\mathscr{A})$. Also, if $\mathscr{A}$ is a semiprime algebra, then it is clear that $lann(\mathscr{A}) = rann(\mathscr{A}) = \{0\}$. If $\mathscr{A}$ satisfies *Condition (P)* in [12], then it is routine to see that $lann(\mathscr{A}) = rann(\mathscr{A}) = \{0\}$, where an algebra or a ring $\mathscr{A}$ satisfies *Condition (P)* if $aa_0a = \{0\}$ for any $a \in \mathscr{A}$ implies that $a_0 = 0$.

The dual space, $\mathscr{A}^*$, of a Banach algebra $\mathscr{A}$ is a Banach $\mathscr{A}$-bimodule with the module operation given by $(\varphi a)(b) = \varphi(ab)$ and $(a\varphi)(b) = \varphi(ba)$, for all $a,b \in \mathscr{A}$, $\varphi \in \mathscr{A}^*$. If $\mathscr{A}$ satisfies a kind of Cohen factorization property (i.e. for every $c \in \mathscr{A}$ there exists $a,b \in \mathscr{A}$ with $c = ab$), we have $lann_{\mathscr{A}^*}(\mathscr{A}) = \{0\} = rann_{\mathscr{A}^*}(\mathscr{A})$. Recall that, by Cohen's factorization theorem [19, Corollary 2.26], every Banach algebra with a bounded left approximate unit satisfies such a factorization property. To see the statement concerning modular annihilators, if $\varphi \in lann_{\mathscr{A}^*}(\mathscr{A})$, we have $\varphi(ab) = 0$ for all $a,b \in \mathscr{A}$, the factorization property implies that $\varphi = 0$.

Our next proposition shows that continuity is the essential property to conclude that a generalized derivation of the third type from a C*-algebra $\mathscr{A}$ into a general Banach $\mathscr{A}$-bimodule is a generalized derivation of the first and second type, improving in this way what we commented at the introduction.

Proposition 2.5. *Let $\mathscr{A}$ be a C*-algebra and let $F : \mathscr{A} \to \mathscr{M}$ be a continuous generalized derivation of the third type from $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule. Then F is a generalized derivation of the first and second type.*

Proof. It is known that the product of $\mathscr{A}$ and the module products on $\mathscr{M}$ can be extended to a product on $\mathscr{A}^{**}$ and $\mathscr{A}^{**}$-bimodule operations on $\mathscr{M}^{**}$ via the first Arens extensions [10, Theorem 2.6.15(iii)], respectively –the second Arens extension is also valid. Since $\mathscr{A}$ is a C*-algebra, its product is Arens regular, that is, the first and second Arens products on $\mathscr{A}^{**}$

coincide, and make the latter a von Neumann algebra (cf. [10, Corollary 3.2.37]). It is also known that the following properties holds: for each $a \in \mathscr{A}$, $\tilde{a} \in \mathscr{A}^{**}$, $x \in \mathscr{M}$ and $z \in \mathscr{M}^{**}$, the mappings $y \mapsto ay$, $y \mapsto y\tilde{a}$ (respectively, $b \mapsto xb$, $b \mapsto bz$) are weak* continuous maps on $\mathscr{M}^{**}$ (respectively, from $\mathscr{A}^{**}$ to $\mathscr{M}^{**}$) [10, Proposition A.3.52]; if (a_λ) and (x_μ) are nets in A and X, respectively, such that $a_\lambda \to a \in A^{**}$ in the weak* topology of A^{**} and $x_\mu \to x \in X^{**}$ in the weak* topology of X^{**}, then

$$ax = \lim_\lambda \lim_\mu a_\lambda x_\mu \text{ and } xa = \lim_\mu \lim_\lambda x_\mu a_\lambda \tag{2.2}$$

in the weak* topology of X^{**} (cf. [10, (2.6.26)]).

As observed in Lemma 2.3, for each $a, b \in \mathscr{A}$, the maps $\Psi_a(\cdot) = aH(\cdot) = F(a\cdot) - L_{G(a)}$ and $\Gamma_b(\cdot) = G(\cdot)b = F(\cdot b) - R_{H(b)}$ are linear, and in this case continuous by the assumptions on F. Let us consider the following sets of operators

$$\Gamma = \{G(\cdot)b : b \in \mathscr{A},\ \|b\| \leq 1\}, \text{ and } \Psi = \{aH(\cdot) : a \in \mathscr{A},\ \|a\| \leq 1\}.$$

For each $a \in \mathscr{A}$ we have

$$\|\,(G(\cdot)b)\,(a)\| = \|G(a)b\| \leq \|F\|\|a\|\|b\| + \|aH(\cdot)\|\|b\| \leq \|F\|\|a\| + \|aH(\cdot)\|,$$

and hence the uniform boundedness principle assures the existence of a positive K_1 satisfying $\|G(\cdot)b\| \leq K_1$ for all $b \in \mathscr{A}$ with $\|b\| \leq 1$. Similarly, there exists of a positive K_2 satisfying $\|aH(\cdot)\| \leq K_2$ for all $a \in \mathscr{A}$ with $\|a\| \leq 1$.

Let us take an approximate unit $(u_j)_j$ in $\mathscr{A}$. If we fix an element $a \in \mathscr{A}$ the net $(au_j)_j$ converges in norm to a, and hence by the continuity of F, $F(au_j)$ tends to $F(a)$ in norm. It is also know that $(u_j)_j \to \mathbf{1}$ in the weak*-topology of $\mathscr{A}^{**}$, where $\mathbf{1}$ stands for the unit in $\mathscr{A}^{**}$. We therefore deduce from (2.2) that $(G(a)u_j)_j \to G(a)\mathbf{1}$ in the weak*-topology of $\mathscr{M}^{**}$. Thus, the identity $F(au_j) = G(a)u_j + aH(u_j)$ implies that

$$\begin{gathered}\text{the net } (aH(u_j))_j \text{ converges in the weak}^*\text{-topology of } \mathscr{M}^{**} \\ \text{to some } R(a) \in \mathscr{M}^{**} \text{ and } F(a) = G(a)\mathbf{1} + R(a).\end{gathered} \tag{2.3}$$

It is easy to check that the mapping $R : \mathscr{A} \to \mathscr{M}^{**}$, $a \mapsto R(a)$ is linear. Moreover, since by the properties shown above $\|aH(u_j)\| \leq K_2\|a\|$ for all j, we obtain that $\|R(a)\| \leq K_2\|a\|$ for all $a \in \mathscr{A}$, and hence R is continuous.

Let us see another interesting property of the operator R. By definition and (2.2) we get

$$R(ab) = w^*\text{-}\lim_j abH(u_j) = aw^*\text{-}\lim_j bH(u_j) = aR(b),$$

which guarantees that R is a right multiplier from $\mathscr{A}$ to $\mathscr{M}^{**}$. Furthermore, $\|H(u_j)\| \leq K_2$ for all j, and hence, by the weak*-compactness of the closed unit ball of $\mathscr{M}^{**}$, there exists a subnet, denoted again by $(u_j)_j$, such that $\lim_j H(u_j) = \xi \in \mathscr{M}^{**}$ in the weak*-topology of $\mathscr{M}^{**}$. Since $R(au_j) = aR(u_j)$ for all j, and $(au_j)_j \to a$ in norm, the continuity properties of the module operations on $\mathscr{M}^{**}$ and R give $R(a) = a\xi$ for all $a \in \mathscr{A}$.

Similar arguments show the existence of a left multiplier $L : \mathscr{A} \to \mathscr{M}^{**}$ and $\eta \in \mathscr{M}^{**}$ satisfying $L(a) = \eta a$ and $F(a) = L(a) + \mathbf{1}H(a)$ for all $a \in \mathscr{A}$.

Finally, we have

$$F(ab) = G(a)b + aH(b) = G(a)\mathbf{1}b + a\mathbf{1}H(b)$$
$$= (F(a) - R(a))b + a(F(b) - L(b)) = F(a)b + bF(a) - a(\xi + \eta)b,$$

for all $a, b \in \mathscr{A}$. □

We shall next state two classical arguments in results on automatic continuity which are considered here under a more general point of view.

Lemma 2.6. *Let Z be a closed subspace of a Banach space X such that X/Z is finite dimensional, and let Y be a normed space. Suppose $F : X \to Y$ is a linear mapping whose restriction to Z is continuous. Then F is continuous.*

Proof. Since the quotient X/Z is finite dimensional, the subspace Z is topologically complemented in X, that is, there exists a continuous linear projection $P : X \to X$ whose image is Z and $Z' = (Id - P)(X)$ is finite dimensional (cf. [39, Lemma 2.21]). Clearly, by the finite dimensionality of Z', $F|_{Z'} : Z' \to Y$ is continuous. Since $F|_Z$ is continuous by hypothesis, and $F(x) = F(P(x)) + F((Id - P)(x)) = F|_Z(x) + F|_{Z'}(x)$ for all $x \in X$, the conclusion is clear. □

The next lemma is a consequence of the uniform boundedness principle.

Lemma 2.7. *Let $F : \mathscr{A} \to X$ be a linear mapping from a C*-algebra into a normed space. Suppose that for each $a \in \mathscr{A}$ the mappings $FL_a, FR_a : \mathscr{A} \to X$, $x \mapsto F(ax)$ and $x \mapsto F(xa)$ are continuous. Then F is continuous.*

Proof. The desired conclusion is clear when $\mathscr{A}$ is unital, since in that case $F(a) = FL_{\mathbf{1}}(a)$. In the general case, we observe that the bilinear mapping $(a, b) \mapsto V(a, b) := F(ab)$ is separately continuous by hypothesis, so by the uniform boundedness principle, V is jointly continuous. Hence, there exists a positive K such that $\|F(ab)\| \leq K \|a\| \|b\|$ for all $a, b \in \mathscr{A}$. For each positive element $c \in \mathscr{A}$ with $\|c\| \leq 1$ there exists a positive d with $d^2 = c$, $\|d\|^2 = \|d^2\| = \|c\| \leq 1$. Therefore $\|F(c)\| = \|F(d^2)\| \leq K\|d\|^2 = K\|c\| \leq K$. In particular, F is bounded on the closed unit ball of $\mathscr{A}$. □

A cornerstone result in the theory of C*-algebras, obtained by Cuntz in [9], asserts that a semi-norm p on a C*-algebra $\mathscr{A}$ which is bounded on each commutative self-adjoint subalgebra of $\mathscr{A}$, is bounded on the whole of $\mathscr{A}$. Cuntz' theorem has been employed in results on automatic continuity of derivations, for example, in [37], Russo and the second author of this note apply it to prove that every Jordan derivation from a C*-algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule is continuous and an associative derivation. More recently, An and He employed Cuntz theorem to prove that if $n \neq m$, then zero is the only (m, n)-Jordan derivation from a C*-algebra into a Banach bimodule (cf. [3]). We give another application next by adapting the arguments in the proof of Theorem 2.8.

We are now ready to prove one of the main results of this section, which confirm that every generalized derivation of the third type from a general C*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is continuous.

Theorem 2.8. *Let $\mathscr{A}$ be a C*-algebra, let $\mathscr{M}$ be a Banach $\mathscr{A}$-bimodule, and let F be a generalized derivation of the third type from $\mathscr{A}$ to $\mathscr{M}$. Then F is continuous. Consequently, every* *generalized derivation of the third type from $\mathscr{A}$ to $\mathscr{M}$ is a generalized derivation of the second type.*

Proof. By the previously commented theorem of Cuntz (see [9, Theorem 1.1 or Corollary 1.2]), there is no loss of generality in assuming that $\mathscr{A}$ is commutative.

The proof will be presented in several steps. Let us begin by setting

$$I = \{a \in \mathscr{A} \,:\, FL_a \text{ is continuous}\} \text{ and } J = \{a \in \mathscr{A} \,:\, L_a H \text{ is continuous}\}.$$

The identity $F(ab) = aH(b) + G(a)b$ $(a, b \in \mathscr{A})$ implies that $FL_a(\cdot)$ is continuous if and only if $L_a H(\cdot)$ is a bounded linear operator, and thus $I = J$. Since $\mathscr{A}$ is commutative we also have

$$I = \{a \in \mathscr{A} \,:\, FL_a = FR_a \text{ is continuous}\}.$$

We shall next show that I is a closed ideal of $\mathscr{A}$. Namely, take $a \in I$ and $b \in \mathscr{A}$. Clearly, the mapping $FL_aL_b : \mathscr{A} \to \mathscr{M}$, $c \mapsto F(abc)$ is continuous, and so $ab = ba \in I$. This means that I is an ideal of $\mathscr{A}$ (recall that $\mathscr{A}$ is commutative). It is well known that I must be self-adjoint [30, Corollary 4.2.10].

Next we show that I is norm-closed. Let $a \in \overline{I}$. Then there exists a sequence $(a_n)_n \subseteq I$ such that $\lim_{n\to\infty} a_n = a$. By the equality $I = J$, in order to show that $a \in I$, it suffices to show that the mapping $L_a H : \mathscr{A} \to \mathscr{M}$ is continuous. It follows from Lemma 2.3(i) and the assumptions that the mapping $L_{a_n} H =: \mathscr{A} \to \mathscr{M}$ is linear for every $c \in \mathscr{A}$. Since (a_n) is a sequence in I, the linear mapping $L_{a_n} H : \mathscr{A} \to \mathscr{M}$ is continuous for all $n \in \mathbb{N}$. It is clear that $\lim_{n\to\infty} L_{a_n} H(c) = \lim_{n\to\infty} a_n H(c) = aH(c) = L_a H(c)$, for every $c \in \mathscr{A}$, and hence, by the uniform boundedness principle, we obtain that $L_a H$ is a continuous linear mapping, and so $a \in J = I$.

It follows from the above arguments that the restricted mapping $F|_I : I \to \mathscr{M}$ satisfies the following property: for each $a \in I$, the mapping $F|_I L_a = F|_I R_a : I \to \mathscr{M}$, $x \mapsto F(ax) = F(xa)$ is continuous. Lemma 2.7 assures that $F|_I : I \to \mathscr{M}$ is continuous.

We shall next show that

$$\mathscr{A}/I \text{ is a finite-dimensional C}^*\text{-algebra.} \tag{2.4}$$

Suppose, on the contrary, that $\mathscr{A}/I$ is an infinite-dimensional C*-algebra. Another classical result in C*-algebra theory (see [30, Exercise 4.6.13]) proves that in such a case there exists an infinite sequence $(b_n + I)_n$ of mutually orthogonal non-zero positive elements in $\mathscr{A}/I$, that is, $(b_n + I)(b_k + I) = 0$, for all $n \neq k$, $b_n \geq 0$ and $b_n \neq 0$ for all n. By [30, Exercise 4.6.20] we can always lift the sequence $(b_n + I)_n$ to a sequence $(c_n)_n$ of mutually orthogonal non-zero positive elements in $\mathscr{A}$ satisfying $\pi(c_n) = b_n + I$, for all n, where $\pi : \mathscr{A} \to \mathscr{A}/I$ is the canonical projection. It follows from the fact that π is a *-homomorphism that $\pi(c_n^2) = \pi(c_n)^2 = (b_n + I)^2 \neq 0$ (observe that $\|(b_n + I)^2\| = \|b_n + I\|^2$), and thus $c_n^2 \notin I$ for all n. Therefore the mapping $FL_{c_n^2} : \mathscr{A} \to \mathscr{M}$ must be unbounded. By replacing c_n with $\frac{c_n}{\|c_n\|}$, we can assume that $\|c_n\| = 1$ for all n. By the unboundedness of the mapping $FL_{c_n^2}$, there exists $d_n \in \mathscr{A}$ satisfying $\|d_n\| \leq 2^{-n}$, and $\|F(c_n^2 d_n)\| = \|FL_{c_n^2}(d_n)\| > \|G(c_n)\| + n$ for all natural n –observe that the sequence $(\|G(c_n)\|)_n$ does not produce any obstacle here–. The elements c_n and d_n have been chosen to guarantee that the series $\sum_{n\geq 1} c_n d_n$ is (absolutely) convergent, and its limit $a_0 =$

$\sum_{n=1}^{\infty} c_n d_n$ satisfies $\|a_0\| \leq 1$ and $c_m a_0 = c_m^2 d_m$ for all $m \in \mathbb{N}$. By the hypotheses on (F,G,H) we deduce that

$$\begin{aligned}\infty > \|H(a_0)\| \geq \|c_m H(a_0)\| &= \|F(c_m a_0) - G(c_m) a_0\| \geq \|F(c_m a_0)\| - \|G(c_m) a_0\| \\ &= \|F(c_m^2 d_m)\| - \|G(c_m) a_0\| \geq m + \|G(c_m)\| - \|G(c_m)\| = m,\end{aligned}$$

for all natural m, which is impossible.

Since $F|_I$ is continuous and $\mathscr{A}/I$ is finite dimensional, we deduce from Lemma 2.6 that F is continuous.

The final statement of the theorem is a consequence of Proposition 2.5. □

The previous theorem generalizes the classical results by Ringrose [38, Theorem 2] and Sakai [43]. The following corollary summarizes some of the conclusions obtained up to now.

Corollary 2.9. *Let $F : \mathscr{A} \to \mathscr{M}$ be a linear mapping from a C^*-algebra to a Banach $\mathscr{A}$-bimodule. Then the following statements are equivalent:*

(a) *F is a generalized derivation of the first type, i.e., there is a derivation $d : \mathscr{A} \to \mathscr{M}$ such that $F(ab) = F(a)b + ad(b)\ \forall a,b \in \mathscr{A}$.*

(b) *F is a generalized derivation of the second type, i.e., there is an element $\xi \in \mathscr{M}^{**}$ such that $F(ab) = F(a)b + aF(b) - a\xi b\ \forall a,b \in \mathscr{A}$.*

(c) *F is a generalized derivation of the third type, i.e., there are two (non-necessarily linear) maps $G,H : \mathscr{A} \to \mathscr{M}$ such that $F(ab) = G(a)b + aH(b)\ \forall a,b \in \mathscr{A}$.*

Furthermore, if any of the equivalent statements holds the mapping F is automatically continuous.

Let Y and Z be Banach spaces and let $T : \mathfrak{Y} \to \mathfrak{Z}$ be a linear mapping. The *separating space* of T is the set

$$\mathfrak{S}(T) = \Big\{ z \in Z \ : \ \exists \, \{y_n\} \subseteq Y \textit{ such that } y_n \to 0, \text{ and } T(y_n) \to z \Big\}.$$

By the closed graph theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$. For additional information about separating spaces, the reader is referred to [10].

We establish next some consequences of Theorem 2.8 and deduce some properties of the mappings appearing in a triplet behaving like a derivation.

Corollary 2.10. *Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{M}$ be a Banach $\mathscr{A}$-bimodule. Suppose that $F,G,H : \mathscr{A} \to \mathscr{M}$ are three mappings, with F linear, such that (F,G,H) behaves like a derivation. The the following statements hold:*

(i) *If $rann(\mathscr{A})_{\mathscr{M}} = \{0\} = lann(\mathscr{A})_{\mathscr{M}}$, the maps F,G and H are linear and continuous.*

(ii) *If $\mathscr{M} = \mathscr{A}$, then F,G and H are bounded linear maps.*

(iii) *If $\mathscr{M} = \mathscr{A}$, then F,G and H are bounded linear maps.*

Proof. (i) According to Theorem 2.8, F is a continuous generalized derivation of the second type, and it follows from Lemma 2.3(ii) that the mappings G and H are linear. We shall show that G and H are continuous. Let $m_0 \in \mathfrak{S}(H) \subseteq \mathscr{M}$. Then there exists a sequence $\{b_n\} \subseteq \mathscr{A}$ such that $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} H(b_n) = m_0$. For any arbitrary element $a \in \mathscr{A}$, we have

$$0 = \lim_{n\to\infty} F(ab_n) = \lim_{n\to\infty} (G(a)b_n + aH(b_n)) = am_0,$$

which means that $m_0 \in rann(\mathscr{A})_{\mathscr{M}}$. By hypothesis $m_0 = 0$, and this implies that H is continuous. Similarly arguments show that G is continuous.

(ii) and (iii) are straightforward consequences of (i). $\square$

Let $\mathscr{A}$ be an algebra and let $\mathscr{M}$ be an $\mathscr{A}$-bimodule. We recall that a linear mapping $F : \mathscr{A} \to \mathscr{M}$ is called an *l-generalized derivation* if there exists a mapping $d_l : \mathscr{A} \to \mathscr{M}$ (called an associated mapping with F) such that $F(ab) = F(a)b + ad_l(b)$ for all $a,b \in \mathscr{A}$. Similarly, F is called an *r-generalized derivation* if there exists a mapping $d_r : \mathscr{A} \to \mathscr{M}$ such that $F(ab) = d_r(a)b + aF(b)$ for all $a,b \in \mathscr{A}$. In [20, Example 2.6], we can find an example of an r-generalized derivation that is not an l-generalized derivation. Clearly, l- and r-generalized derivations are generalized derivations of the third type.

The next result is therefore a straight consequence of Theorem 2.8 and Corollary 2.10.

Corollary 2.11. *Let $\mathscr{A}$ be a C*-algebra and let $\mathscr{M}$ be a Banach $\mathscr{A}$-bimodule. Then the following statements hold:*

- (i) *Every l-generalized derivation (respectively, r-generalized derivation) $F : \mathscr{A} \to \mathscr{M}$ is continuous.*
- (ii) *Suppose that $rann(\mathscr{A})_{\mathscr{M}} = \{0\} = lann(\mathscr{A})_{\mathscr{M}}$. If $F : \mathscr{A} \to \mathscr{M}$ is an l- or r-generalized derivation with an associated mapping $d : \mathscr{A} \to \mathscr{M}$, then F and d are continuous linear mappings.*

3. Automatic continuity of generalized Jordan derivation of the third type

Appropriate types of generalized Jordan derivations whose domain is a C*-algebra are studied in this section. We shall mainly focus on the automatic continuity and the relationships between the different types. Contrary to the conclusions in the previous section, generalized Jordan derivations of the third type constitute a strict subclass of the set of generalized Jordan derivations of the third type (cf. Proposition 3.2 and Remark 3.3). Generalized Jordan derivations of the second type have been considered in [6, 7, 25]. Recall that a linear mapping F from a Banach algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ is a *generalized Jordan derivation of the second type* if there exists $\xi \in \mathscr{M}^{**}$ such that

$$F(a \circ b) = F(a) \circ b + a \circ F(b) - U_{a,b}(\xi), \text{ for all } a,b \in \mathscr{A},$$

where $a \circ b = \frac{1}{2}(ab + ba)$. If $\xi = 0$ we find the usual notion of Jordan derivation. Classic results by Ringrose and Johnson assure that every Jordan derivation from a C*-algebra to a Banach bimodule is a derivation.

There are some other attempts to defined generalized Jordan derivations in the literature, According to [5], an additive (in this note we shall assume linearity) mapping T from a Banach algebra $\mathscr{A}$ into a left A-module M is called a *Jordan left derivation* if $T(a^2) = 2aT(a)$ for every $a \in \mathscr{A}$. It is shown in [5] that the existence of non-zero Jordan left derivations from a prime ring $\mathscr{R}$ into a 6-torsion free left $\mathscr{R}$-module implies that R is a commutative ring. Vukman introduced in [44] the notion of (m,n)-Jordan derivation. Let m,n be two non-negative integers with $m+n \neq 0$. An additive (along this note linear) mapping T from $\mathscr{A}$ into $\mathscr{M}$ is called an (m,n)-Jordan derivation if the identity

$$(m+n)T(a^2) = 2maT(a) + 2nT(a)a$$

holds for all $a\in\mathscr{A}$.

Jing and Lu coined for first time the term generalized Jordan derivations in [27]. We shall say that a linear mapping F from a Banach algebra $\mathscr{A}$ to a Banach $\mathscr{A}$-bimodule $\mathscr{M}$ is a *generalized Jordan derivation* in the sense of Jing and Lu if there exists a (linear) Jordan derivation $\tau:\mathscr{A}\to\mathscr{M}$ satisfying

$$F(a^2)=F(a)a+a\tau(a), \text{ for all } a\in\mathscr{A}.$$

By polarizing we get

$$F(a\circ b)=\frac{1}{2}\Big(F(a)b+F(b)a+a\tau(b)+b\tau(a)\Big), \text{ for all } a,b\in\mathscr{A}.$$

In this definition the Jordan structure is only considered on the domain algebra. It is worthwhile to mention that if $\mathscr{A}$ and $\mathscr{M}$ are both unital, by taking $b=\mathbf{1}$ in the previous identity we get $F(a)=F(\mathbf{1})a+\tau(a)$ $(a\in\mathscr{A})$, and if τ is in fact a derivation (as in the case of C*-algebras), F is a generalized derivation of the first type.

Motivated by the results in the previous section, we shall say that a linear mapping $F:\mathscr{A}\to\mathscr{M}$ is a *generalized Jordan derivation of the third type* if there exist (non-necessarily linear) maps $G,H:\mathscr{A}\to\mathscr{M}$ such that

$$F(a\circ b)=G(a)\circ b+a\circ H(b), \text{ for all } a,b\in\mathscr{A}.$$

The following theorem prove the automatic continuity of every generalized Jordan derivation of the third type when the domain is a C*-algebra by adapting the proof of Theorem 2.8.

Theorem 3.1. *Let $\mathscr{A}$ be a C*-algebra and let $\mathscr{M}$ be a Banach $\mathscr{A}$-bimodule. Then every generalized Jordan derivation of the third type $F:\mathscr{A}\to\mathscr{M}$ is continuous.*

Proof. Let $G,H:\mathscr{A}\to\mathscr{M}$ be two (non-necessarily linear) maps satisfying

$$F(a\circ b)=G(a)\circ b+a\circ H(b), \text{ for all } a,b\in\mathscr{A}.$$

Since the Jordan product is commutative, up to replacing G and H by $\frac{1}{2}(G+H)$ we can assume that $G=H$. By a new application of Cuntz' theorem [9, Theorem 1.1 or Corollary 1.2], we may also assume in the argument concerning the continuity of F that $\mathscr{A}$ is commutative.

Set

$$I=\{a\in\mathscr{A}\ :\ \text{the mapping } x\mapsto F(ax)=F(a\circ x) \text{ is continuous}\},$$
$$\text{and } \ J=\{a\in\mathscr{A}\ :\ \text{the mapping } x\mapsto a\circ G(x) \text{ is continuous}\}.$$

The identity $F(ax)=F(a\circ x)=G(a)\circ x+a\circ G(x)$, shows that $I=J$. If $a\in I$ and $b\in\mathfrak{A}$, the mapping $x\mapsto F((ab)x)=F((ba)x)=F(b(ax))=F(a(bx))$ is clearly continuous. Therefore $ab=ba\in I$, which shows that I is an ideal of $\mathscr{A}$.

To show that I is norm-closed, we take $(a_n)_n\subset I$ converging to some $a\in\mathscr{A}$ in norm. Since by assumptions $a_n\circ G(\cdot)$ is a bounded linear operator from $\mathscr{A}$ to $\mathscr{M}$ which converges pointwise to the linear mapping $a\circ G(\cdot)$, the latter must be bounded by the uniform boundedness principle, and hence $a\in I$.

We are now in a position to apply Lemma 2.7 to guarantee that $F|_I:I\to\mathscr{M}$ is continuous. The continuity of F will follow from Lemma 2.6 if we show that $\mathscr{A}/I$ is finite-dimensional. Otherwise, as in the proof of Theorem 2.8, there exists an infinite sequence $(b_n+I)_n$ of mutually orthogonal non-zero positive elements in $\mathscr{A}/I$, that is, $(b_n+I)(b_k+I)=0$, for all $n\neq k$, $b_n\geq 0$

and $b_n \neq 0$ for all n (see [30, Exercise 4.6.13]). Choose, a sequence $(c_n)_n$ of mutually orthogonal norm-one positive elements in $\mathscr{A}$ satisfying $c_n + I = b_n + I$, $\|c_n^2 + I\| = \|(b_n + I)^2\| = \|b_n + I\|^2 \neq 0$, for all n (cf. [30, Exercise 4.6.20]). Clearly, the element c_n^2 does not belong to I, and hence the mapping $x \mapsto F(c_n^2 x)$ is unbounded. We can therefore pick $d_n \in \mathscr{A}$ satisfying $\|d_n\| \leq 2^{-n}$, and $\|F(c_n^2 d_n)\| > \|G(c_n)\| + n$ for all natural n. By assumptions, the limit $a_0 = \sum_{n\geq 1} c_n d_n$ belongs to $\mathscr{A}$, $\|a_0\| \leq 1$, and $a_0 c_m = c_m a_0 = c_m^2 d_m$ for all $m \in \mathbb{N}$. Our hypotheses give

$$\begin{aligned}\infty > 2\|G(a_0)\| &\geq \|c_m \circ G(a_0)\| = \|F(c_m a_0) - G(c_m) \circ a_0\| \geq \|F(c_m a_0)\| - \|G(c_m) \circ a_0\| \\ &= \|F(c_m^2 d_m)\| - \|G(c_m) a_0\| \geq m + \|G(c_m)\| - \|G(c_m)\| = m,\end{aligned}$$

for all natural m, which is impossible. □

Under some extra hypotheses on the bimodule, we conclude that the class of generalized Jordan derivations of the third type are in fact generalized derivation of first/second/third type.

Proposition 3.2. *Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{M}$ be an essential Banach $\mathscr{A}$-bimodule. Then every generalized Jordan derivation of the third type $F : \mathscr{A} \to \mathscr{M}$ is a generalized derivation of the second type (and of course the first and third type), that is, there exist $\xi \in \mathscr{M}^{**}$ such that*

$$F(ab) = F(a)b + aF(b) - a\xi b \quad (\forall a, b \in \mathscr{A}).$$

Furthermore, if $\mathscr{M}$ coincides with $\mathscr{A}$ or $\mathscr{A}^*$, then ξ actually commutes all elements in $\mathscr{A}$.

Proof. Since $\mathscr{M}$ is assumed to be essential, we can prove that F is a generalized derivation of the second type by combining that F is continuous (cf. Theorem 3.1) and Theorem 1.2. Namely, just observe that given $a, b \in \mathscr{A}_{sa}$ with $ab = 0$, we can take $d \in \mathscr{A}$ with $d^2 = a$ and $db = bd = 0$. Then

$$bF(a)b = bF(d^2)b = b\,(2d \circ G(d))\,b = 0.$$

We can assume, without loss of generality, that there is a (non-necessarily linear) mapping $G : \mathscr{A} \to \mathscr{M}$ such that

$$F(a \circ b) = G(a) \circ b + a \circ G(b), \tag{3.1}$$

for all $a, b \in \mathscr{A}$ (compare the proof of Theorem 3.1). Then, in this case, G is actually linear and continuous. Namely, we can apply similar ideas to those in the proof of Proposition 2.5, to see that the mappings of the form $x \mapsto (G(\cdot) \circ b)(x) = G(x) \circ b = F(x \circ b) - x \circ G(b)$ are linear and continuous on $\mathscr{A}$. Since for every $a, b \in \mathscr{A}$ with $\|b\| \leq 1$ we have $\|G(a) \circ b\| = \|F(a \circ b) - a \circ G(b)\| \leq \|F\|\|a\| + \|G(\cdot) \circ a\|$, the family $\{G(\cdot) \circ b : \|b\| \leq 1\}$ is pointwise bounded, and hence uniformly bounded by the uniform boundedness principle. Thus there exists $K > 0$ such that $\|G(\cdot) \circ b\| \leq K$ for all $\|b\| \leq 1$. Let $(u_j)_j$ be a bounded approximate unit in $\mathscr{A}$. For each $a \in \mathscr{A}$, the net $F(a \circ u_j)$ converges to $F(a)$ in norm by the continuity of F, and the net $G(a) \circ u_j$ tends to $G(a)$ in norm. Consequently G is linear. We also know that $\|G(a)\| = \lim_j \|G(a) \circ u_j\| \leq K\|a\|$, note that the first equality holds because $\mathscr{M}$ is essential (see, for example, [8, Theorem 1.15]). Thus, G is continuous.

Having in mind that F is a continuous generalized derivation on $\mathscr{A}$, and by the weak*-density of $\mathscr{A}$ in $\mathscr{A}^{**}$ and the separate weak*-continuity of the product of the latter von Neumann algebra, F^{**} is a generalized derivation on $\mathscr{A}^{**}$, and thus there exists $\xi \in \mathscr{M}^{**}$ satisfying

$$F^{**}(a \circ b) = F^{**}(a) \circ b + a \circ F^{**}(b) - \frac{1}{2}(a\xi b + b\xi a), \text{ for all } a, b \in \mathscr{A}^{**}. \tag{3.2}$$

Let us prove the final statement. Assume first that $\mathscr{M}=\mathscr{A}$, then a similar argument, via weak*-density of $\mathscr{A}$ in $\mathscr{A}^{**}$ and separate weak*-continuity of the product in $\mathscr{A}^{**}$ and the identity in (3.1), shows that

$$F^{**}(a\circ b)=G^{**}(a)\circ b+a\circ G^{**}(b), \text{ for all } a,b\in\mathscr{A}^{**}. \tag{3.3}$$

Assume next that $\mathscr{M}=\mathscr{A}^*$. A consequence of the Grothendieck's inequality established by Haagerup in [16] asserts that every bounded linear mapping from a C*-algebra to the dual space of another C*-algebra factors through a Hilbert space, and thus it is weakly compact. Therefore, by Gantmacher's theorem [14], $F^{**}(\mathscr{A}^{**})\subseteq\mathscr{M}^*$ and $G^{**}(\mathscr{A}^{**})\subseteq\mathscr{M}^*$. We claim that

$$F^{**}(a\circ b)=G^{**}(a)\circ b+a\circ G^{**}(b) \tag{3.4}$$

for all $a,b\in\mathscr{A}^{**}$. We just need to extend (3.1) to $\mathscr{A}^{**}$. The left hand side can be treated by a standard argument, since $F^{**}(a\circ b)$ can be approached by a double weak*-limit via the weak*-continuity of F^{**} and the separate weak*-continuity of the product of $\mathscr{A}^{**}$. Given $b\in\mathscr{A}^{**}$, we can find a net $(b_j)_j\subseteq\mathscr{A}$ such that $G(b_j)\to G^{**}(b)\in\mathscr{A}^*$ in the weak*-topology of $\mathscr{A}^{***}$. Thanks to this, we shall handle the summands on the right hand side with the next property: if $(a_j)_j$ and (ϕ_i) are two nets in $\mathscr{A}$ and $\mathscr{A}^*$ converging to $a\in\mathscr{A}^{**}$ and $\phi\in\mathscr{A}^*$ in the $\sigma(\mathscr{A}^{**},\mathscr{A}^*)$ and the $\sigma(\mathscr{A}^{***},\mathscr{A}^{**})$ topologies, respectively, we have

$$\frac{1}{2}\lim_j\lim_i(a_j\phi_i+\phi_i a_j)=\lim_j\lim_i(a_j\circ\phi_i)=a\circ\phi=\frac{1}{2}(a\phi+\phi a) \text{ in the } \sigma(\mathscr{A}^{***},\mathscr{A}^{**}) \text{ topology.}$$

The basic properties of the module operations in $\mathscr{A}^{***}$ assure that $\lim_j\lim_i a_j\phi_i=a\phi$ in the $\sigma(\mathscr{A}^{***},\mathscr{A}^{**})$ topology, and $w^*\text{-}\lim_i\phi_i a_j=\phi a_j$ for all j. Fix now $c\in\mathscr{A}^{**}$. The net $(a_jc)_j$ converges to ac in the weak*-topology of $\mathscr{A}^{**}$. By applying that $\phi\in\mathscr{A}^*$, we deduce that

$$\lim_j(\phi a_j)(c)=\lim_j\phi(a_jc)=\phi(ac)=(\phi a)(c),$$

and thus $w^*\text{-}\lim_j\phi a_j=\phi a$, which finishes the proof of (3.4).

Summarizing, by (3.2), (3.3), and (3.4) we arrive to $G^{**}(a)=F^{**}(a)-a\circ G^{**}(\mathbf{1})$, $G^{**}(\mathbf{1})=\frac{1}{2}F^{**}(\mathbf{1})=\frac{1}{2}\xi$,

$$\begin{aligned}F^{**}(a)\circ b+a\circ F^{**}(b)-\frac{1}{2}(a\xi b+b\xi a)&=F^{**}(a\circ b)\\&=F^{**}(a)\circ b+a\circ F^{**}(b)-(a\circ G^{**}(\mathbf{1}))\circ b-(b\circ G^{**}(\mathbf{1}))\circ a,\end{aligned}$$

equivalently,

$$a\xi b+b\xi a=(a\circ\xi)\circ b+(b\circ\xi)\circ a\Leftrightarrow\xi ab+ba\xi-2b\xi a+\xi ba+ab\xi-2a\xi b=0,$$

for all $a,b\in\mathscr{A}^{**}$. In particular, $\xi p+p\xi=2p\xi p$, for every projection $p\in\mathscr{A}^{**}$, which proves that $\xi p=p\xi p=p\xi$, and hence ξ commutes with all elements in $\mathscr{A}^{**}$. □

Remark 3.3. We can conclude now that for a non-commutative C*-algebra $\mathscr{A}$, the class of generalized Jordan derivations of the third type on $\mathscr{A}$ is strictly included in the class of generalized Jordan derivation of the second type. Consider, for example, an element ζ which is not in the centre of $\mathscr{A}$ and the mapping $F:\mathscr{A}\to\mathscr{A}$, $F(a)=a\circ\zeta$ which is a generalized Jordan derivation of the second type but not of the third type.

Finally, it is worth to say a few words about what could be the Jordan version of generalized derivations of the first type. Let $\mathscr{M}$ be a Banach $\mathscr{A}$-bimodule on a Banach algebra $\mathscr{A}$. A *generalized Jordan derivation of the first type* is a linear mapping $F:\mathscr{A}\to\mathscr{M}$, for which there exists a Jordan derivation $D:\mathscr{A}\to\mathscr{M}$ satisfying $F(a\circ b)=F(a)\circ b+a\circ D(b)$ for all $a,b\in\mathscr{A}$. Indeed some algebraists defined the so-called "Jordan generalized derivation" by letting D to be just a linear map. That is the case in the paper by Li and Benkovič [33], where it is shown that any generalized Jordan derivation of the first type on a triangular algebra is a kind of generalized derivation of the third type. Every Jordan derivation of the first type is automatically a generalized Jordan derivation of the third type, and thus continuous by Theorem 3.1 when $\mathscr{A}$ is a C*-algebra. Furthermore, if $\mathscr{M}$ is essential, the sets of all generalized Jordan derivations of the first and third type, respectively, are strictly included in the set of all generalized Jordan derivation of the second type, and actually they are all generalized derivations of the first/second/third type.

Despite we have tried to clarify the relationships between the different notions of "generalized derivations", several interesting questions remain open after our study. Can we remove the hypothesis $\mathscr{M}$ being essential in Proposition 3.2? Is the final conclusion in the same proposition true for other Banach bimodules appart from $\mathscr{A}$ and $\mathscr{A}^*$?

It is also natural to expect that the notions of generalized derivations of the third type studied in this note give rise to appropriate concepts in the setting of JB*-algebras and JB*-triples. In the case of a unital JB*-algebra $\mathscr{J}$,the techniques and arguments in [17] can be appropriately modified to prove that every generalized Jordan derivation of the third type on $\mathscr{J}$ or from $\mathscr{J}$ to $\mathscr{J}^*$ is automatically continuous. However, under more general hypotheses the answer is not clear.

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