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Weak convergence of probability measures on hyperspaces with the upper Fell-topology

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Abstract

Let $\boldsymbol{E}$ be a locally compact second countable Hausdorff space and $\boldsymbol{\mathcal{F}}$ the pertaining family of all closed sets. We endow $\boldsymbol{\mathcal{F}}$ respectively with the Fell-topology, the upper Fell topology or the upper Vietoris-topology and investigate weak convergence of probability measures on the corresponding hyperspaces with a focus on the upper Fell topology. The results can be transferred to distributional convergence of random closed sets in $\boldsymbol{E}$ with applications to the asymptotic behavior of measurable selection.

Keywords: Weak convergence, hyperspaces, upper Fell topology, upper Vietoris topology, random closed sets, capacity functionals.

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1 Introduction

To begin with we introduce concepts, which are fundamental for the whole paper: the *Fell-topology*, the *upper Fell-topology* and *weak convergence on topological spaces*. To introduce the Fell-topologies let E be a non-empty set endowed with a topology $\mathcal{G}$ and the pertaining families $\mathcal{F}$ and $\mathcal{K}$ of all closed sets and all compact sets, respectively. For an arbitrary subset $A \subseteq E$ one defines

$$\mathcal{M}(A) := \{F \in \mathcal{F} : F \cap A = \emptyset\}$$

and

$$\mathcal{H}(A) := \{F \in \mathcal{F} : F \cap A \neq \emptyset\}.$$

The elements of $\mathcal{M}(A)$ or $\mathcal{H}(A)$ are called *missing sets* or *hitting sets*, respectively, of A. Put

$$\mathcal{S}_F = \{\mathcal{M}(K) : K \in \mathcal{K}\} \cup \{\mathcal{H}(G) : G \in \mathcal{G}\} \subseteq 2^{\mathcal{F}}.$$

and

$$\mathcal{S}_{uF} = \{\mathcal{M}(K) : K \in \mathcal{K}\} \subseteq 2^{\mathcal{F}}.$$

Then the topologies on $\mathcal{F}$ generated by $\mathcal{S}_F$ or $\mathcal{S}_{uF}$ are called *Fell-topology* or *upper Fell-topology* and are denoted by τ_F or τ_{uF}, respectively. The name goes back to J. Fell (1962) [4]. The topological spaces $(\mathcal{F}, \tau_F)$ and $(\mathcal{F}, \tau_{uF})$ are examples of *hyperspaces*. From now on we assume that the underlying *carrier space* $(E, \mathcal{G})$ is locally compact, second-countable and Hausdorff. Then the hyperspace $(\mathcal{F}, \tau_F)$ is compact, second-countable and Hausdorff, confer G. Beer [1] for these properties and much more information on the Fell-topology. Whereas $(\mathcal{F}, \tau_{uF})$ is also compact and second-countable, but it is not Hausdorff, confer Ferger [5].

To introduce the second concept let $(X, \mathcal{O})$ be an arbitrary topological space with induced Borel σ-algebra $\mathcal{B} \equiv \mathcal{B}(X) := \sigma(\mathcal{O})$. Let

$$\Pi \equiv \Pi(X, \mathcal{O}) := \{P : P \text{ is a probability measure on } (X, \mathcal{B})\}.$$

For each open $O \in \mathcal{O}$ consider the *evaluation map* $e_O : \Pi \to \mathbb{R}$ defined by $e_O(P) := P(O)$ for all $P \in \Pi$. If $\mathcal{O}_> := \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is the *right-order topology* on $\mathbb{R}$, then the initial topology with respect to the functions $e_O : \Pi \to (\mathbb{R}, \mathcal{O}_>)$, $O \in \mathcal{O}$, is called *weak topology* on Π and denoted by τ_{weak}. It goes back to Topsøe [13]. An introduction to the weak topology can be found in the textbook of Gänssler and Stute [7].

Next, let $(A, \leq)$ be a directed set and (P_α) be a net in Π converging to $P \in \Pi$ in the weak topology:

$$P_\alpha \to P \text{ in } (\Pi, \tau_{weak}). \tag{1}$$

By construction of the initial topology we have that (1) is equivalent to

$$e_O(P_\alpha) \to e_O(P) \text{ in } (\mathbb{R}, \mathcal{O}_>) \quad \forall\, O \in \mathcal{O},$$

which in turn by the definitions of e_O and $\mathcal{O}_>$ is equivalent to

$$\liminf_\alpha P_\alpha(O) \geq P(O) \quad \forall\, O \in \mathcal{O}. \tag{2}$$

By complementation we further obtain from (2) that (1) is equivalent to

$$\limsup_\alpha P_\alpha(F) \leq P(F) \quad \text{for all closed sets } F \text{ in } (X, \mathcal{O}). \tag{3}$$

Thus, if $(X, \mathcal{O})$ is metrizable and P_α actually is a sequence, then one can conclude from the Portmanteau-Theorem that convergence in (Π, τ_{weak}) is the same as the well-known *weak convergence* of probability measures on metric spaces, confer, e.g. Billingsley [2]. Therefore, in general we say that (P_α) *converges weakly* to P on $(X, \mathcal{O})$, if (1) holds and alternatively write for this: $P_\alpha \to_w P$ on $(X, \mathcal{O})$.

We are now in a position to explain what this paper is about. It is (mainly) about the characterisation of weak convergence $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$ and its relation to weak convergence on $(\mathcal{F}, \tau_F)$. Let $\mathcal{B}_F := \sigma(\tau_F)$ and $\mathcal{B}_{uF} = \sigma(\tau_{uF})$ be the underlying Borel-σ algebras. It follows from Lemma 2.1.1 in Schneider and Weil [12] that these coincide:

$$\mathcal{B}_F = \mathcal{B}_{uF}. \tag{4}$$

Therefore the involved probability measures have the same domain.

The paper is organized as follows: In the next section we learn about equivalent characterisations for weak convergence on $(\mathcal{F}, \tau_{uF})$. Since the Fell topology is stronger than the upper Fell topology, weak convergence on $(\mathcal{F}, \tau_F)$ entails this on $(\mathcal{F}, \tau_{uF})$ and in general the reversal is not true, but it is under an additional assumption. Moreover, we find a close relationship to weak convergence on $(\mathcal{F}, \tau_{uV})$, where τ_{uV} is the *upper Vietoris topology*. This is created when $\mathcal{K}$ is replaced by $\mathcal{F}$ in the construction of τ_{uF}. A surprising result is that every net of probability measures converges weakly on $(\mathcal{F}, \tau_{uF})$. And whenever it converges to some P it converges also to every Q, which *dominates* P. Consequently, the space $(\Pi(\mathcal{F}, \tau_{uF}), \tau_{weak})$ is compact and in general not Hausdorff. In section 3 we extend our results to *random closed sets* in E and *measurable selections*. It is shown that the distributions of these selections converge weakly to a *Choquet capacity* in the sense of Ferger [6], Definition 1.4. Under a uniqueness assumptions one obtains classical weak convergence in $(E, \mathcal{G})$. Finally, in section 4 (appendix) we present some statements that are used in our proofs.

2 Weak convergence of probability measures on the hyperspace $(\mathcal{F}, \tau_{uF})$

By construction $\mathcal{S}_{uF}$ is a subbase of τ_{uF}. This means that every basic open set has the form $\bigcap_{i=1}^m \mathcal{M}(K_i)$ for compact sets $K_1, \ldots, K_m$ and $m \in \mathbb{N}$. But since $\bigcap_{i=1}^m \mathcal{M}(K_i) = \mathcal{M}(\bigcup_{i=1}^m K_i)$ and $K := \bigcup_{i=1}^m K_i$ is compact, we see that $\mathcal{S}_{uF}$ actually is a base of τ_{uF}. Consequently each open set $\mathbf{O} \in \tau_{uF}$ has the representation

$$\mathbf{O} = \bigcup_{K \in \mathcal{K}^*} \mathcal{M}(K)$$

for some subclass $\mathcal{K}^* \subseteq \mathcal{K}$. Therefore, a general τ_{uF}-closed set $\mathbf{F}$ can be written as

$$\mathbf{F} = \bigcap_{K \in \mathcal{K}^*} \mathcal{H}(K).$$

Thus the equivalence (1) $\Leftrightarrow$ (3) immediately yields a first characterisation:

Proposition 1. *The following two statements (i) and (ii) are equivalent:*

(i) $P_\alpha \to_w P$ *on* $(\mathcal{F}, \tau_{uF})$.

(ii)

$$\limsup_{\alpha} P_\alpha(\bigcap_{K\in\mathcal{K}^*} \mathcal{H}(K)) \leq P(\bigcap_{K\in\mathcal{K}^*} \mathcal{H}(K)) \tag{5}$$

for every collection $\mathcal{K}^* \subseteq \mathcal{K}$ of compact sets in E.

We say that (P_α) is *asymptotically compact-bounded*, if for each $\epsilon > 0$ there exists a $K \in \mathcal{K}$ such that

$$\liminf_{\alpha} P_\alpha(\{F \in \mathcal{F} : F \subseteq K\}) \geq 1 - \epsilon. \tag{6}$$

Notice that $\{F \in \mathcal{F} : F \subseteq K\} = (\mathcal{H}(K^c))^c$ is τ_F-closed, because $K^c \in \mathcal{G}$. Therefore it lies in $\mathcal{B}_F = \mathcal{B}_{uF}$, the domain of the P_α. Similarly, a single probability measure P is said to be *compact-bounded*, if for each $\epsilon > 0$ there exists a $K \in \mathcal{K}$ such that

$$P(\{F \in \mathcal{F} : F \subseteq K\}) \geq 1 - \epsilon. \tag{7}$$

Corollary 1. *If $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$ and (P_α) is asymptotically compact-bounded, then*

$$\limsup_{\alpha} P_\alpha(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)) \leq P(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)) \tag{8}$$

for every collection $\mathcal{F}^* \subseteq \mathcal{F}$ of closed sets in E.

If the limit P is compact-bounded, then the reverse conclusion holds: (8) implies that $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$ and that (P_α) is asymptotically compact-bounded.

Proof. In Remark 1 below we will show that the intersections in (8) are elements of $\mathcal{B}_{uF}$, so that all probabilities are well-defined. Let $\epsilon > 0$ and $\mathbf{F} := \{F \in \mathcal{F} : F \subseteq K\}$ with K as in (6). Then using the partition $\{\mathbf{F}, \mathbf{F}^c\}$ one finds that $\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F) \subseteq \bigcap_{F\in\mathcal{F}^*}(\mathcal{H}(F) \cap \mathbf{F}) \cup \mathbf{F}^c$. Since $\mathcal{H}(F) \cap \mathbf{F} \subseteq \mathcal{H}(F \cap K)$ for all $F \in \mathcal{F}^*$, we obtain:

$$\limsup_{\alpha} P_\alpha(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)) \leq \limsup_{\alpha} P_\alpha(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F \cap K)) + \limsup_{\alpha} P_\alpha(\mathbf{F}^c). \tag{9}$$

Now, $F \cap K \in \mathcal{K}$, whence by Proposition 1

$$\limsup_{\alpha} P_\alpha(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F \cap K)) \leq P(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F \cap K)) \leq P(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)), \tag{10}$$

where the last equality is trivial, because $F \cap K \subseteq F$ and so $\mathcal{H}(F \cap K) \subseteq \mathcal{H}(F)$ for all F. By complementation the condition (6) is equivalent to

$$\limsup_{\alpha} P_\alpha(\mathbf{F}^c) \leq \epsilon. \tag{11}$$

Combining (9)-(11) we arrive at

$$\limsup_{\alpha} P_\alpha(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)) \leq P(\bigcap_{F\in\mathcal{F}^*} \mathcal{H}(F)) + \epsilon \quad \forall\, \epsilon > 0.$$

Taking the limit $\epsilon \to 0$ yields the assertion (8).

As to the reverse implication let $\epsilon > 0$ and $K \in \mathcal{K}$ as in (7). Fix some $r > 0$ and consider the open r-neighborhood K^{r-} and the closed r-neighborhood K^r of K, see in the appendix for their definitions. Notice that $K \subseteq K^{r-} \subseteq K^r$. It follows:

$$\begin{aligned}
&\limsup_\alpha P_\alpha(F \in \mathcal{F} : F \nsubseteq K^r) = \limsup_\alpha P_\alpha(F \in \mathcal{F} : F \cap (K^r)^c \neq \emptyset) \\
&= \limsup_\alpha P_\alpha(\mathcal{H}((K^r)^c)) \\
&\leq \limsup_\alpha P_\alpha(\mathcal{H}((K^{r-})^c)) \quad \text{because } K^r \supseteq K^{r-} \\
&\leq P(\mathcal{H}((K^{r-})^c)) \quad \text{by (8) with } \mathcal{F}^* = \{(K^{r-})^c\} \subseteq \mathcal{F}, \text{ because } K^{r-} \in \mathcal{G} \\
&\leq P(\mathcal{H}(K^c)) \quad \text{because } K^{r-} \supseteq K \\
&= 1 - P(\mathcal{M}(K^c)) = 1 - P(\{F \in \mathcal{F} : F \subseteq K\}) \leq \epsilon \quad \text{by (7)}.
\end{aligned}$$

After complementation this shows asymptotic compact-boundedness of (P_α) upon noticing that K^r is compact by Lemma 4. Since $\mathcal{K} \subseteq \mathcal{F}$, condition (8) entails condition (5). Therefore an application of Proposition 1 yields that $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$. □

Remark 1. *Let τ_{uV} be the* upper Vietoris topology*. This is generated by $\mathcal{S}_{uV} := \{\mathcal{M}(F) : F \in \mathcal{F}\}$. Notice that*

$$\mathcal{H}(B) \in \mathcal{B}_F \text{ for all Borel-sets } B \in \mathcal{B}(E), \tag{12}$$

confer Matheron [9], p.30. Thus by Lemma 2.1.1 in Schneider and Weil [12] and (12) the Borel-σ algebra $\mathcal{B}_{uV} := \sigma(\tau_{uV})$, like $\mathcal{B}_{uF}$, is the same as $\mathcal{B}_F$. Similarly as for $\mathcal{S}_{uF}$ one shows that $\mathcal{S}_{uV}$ is a base for τ_{uV}, whence the family $\{\bigcap_{F\in\mathcal{F}^*}\mathcal{H}(F)), \mathcal{F}^* \subseteq \mathcal{F}\}$ is exactly the family of all τ_{uV}-closed sets. In particular, these intersections are Borel-sets, i.e., they are elements of $\mathcal{B}_{uV} = \mathcal{B}_F = \mathcal{B}_{uF}$. As a further consequence we obtain that (8) is equivalent to $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uV})$.

Recall that the family of sets occurring in (5) coincides with the family of all τ_{uF}-closed sets. Our next result shows that one can reduce this family significantly. For its formulation we need the following denotation: For a set $\mathbf{A} \subseteq \mathcal{F}$ the boundary of $\mathbf{A}$ with respect to the Fell-topology is denoted by $\partial_F \mathbf{A}$.

Theorem 2. *The following statements (i)-(iii) are equivalent:*

(i) $P_\alpha \to_w P$ *on* $(\mathcal{F}, \tau_{uF})$.

(ii)

$$\limsup_\alpha P_\alpha\Big(\bigcap_{i=1}^m \mathcal{H}(K_i)\Big) \leq P\Big(\bigcap_{i=1}^m \mathcal{H}(K_i)\Big) \tag{13}$$

for every $m \in \mathbb{N}$ and every finite collection $K_1, \ldots, K_m$ of non-empty compact sets in E.

(iii) The inequality (13) holds for every $m \in \mathbb{N}$ and every finite collection $K_1, \ldots, K_m$ of non-empty compact sets in E such that $P(\partial_F \mathcal{H}(K_i)) = 0$ for all $1 \leq i \leq m$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Proposition 1 with $\mathcal{K}^* = \{K_1, \ldots, K_m\}$. For the reverse direction $(ii) \Rightarrow (i)$ one has to show (3), i.e.

$$\limsup_{\alpha} P_\alpha(\mathbf{F}) \leq P(\mathbf{F}) \quad \text{for all } \tau_{uF} - \text{closed sets } \mathbf{F}. \tag{14}$$

We prove this by contradiction. So, assume that there exists a τ_{uF}-closed set $\mathbf{F}$ such that

$$\limsup_{\alpha} P_\alpha(\mathbf{F}) > P(\mathbf{F}). \tag{15}$$

The complement $\mathbf{F}^c$ of $\mathbf{F}$ in $\mathcal{F}$ lies in τ_{uF} and τ_{uF} has a countable base $\{\mathcal{M}(K) : K \in \mathcal{K}_0\}$, where $\mathcal{K}_0$ is a certain countable family of non-empty compact sets, confer, e.g., Ferger [5] or Gersch [8]. Thus one can find a sequence $(K_i)_{i\in\mathbb{N}}$ of non-empty compact sets such that $\mathbf{F}^c = \bigcup_{i\in\mathbb{N}} \mathcal{M}(K_i)$, whence $\mathbf{F} = \bigcap_{i\in\mathbb{N}} \mathcal{H}(K_i)$. For every $k \in \mathbb{N}$ put $\mathbf{F}_k := \bigcap_{i=1}^k \mathcal{H}(K_i)$. Then $\mathbf{F}_k \downarrow \mathbf{F}, k \to \infty$ and therefore

$$P(\mathbf{F}_k) \downarrow P(\mathbf{F}), k \to \infty. \tag{16}$$

Let $a := \frac{1}{2}(\limsup_\alpha P_\alpha(\mathbf{F}) - P(\mathbf{F})$. By assumption (15) the real number a is positive and by definition satisfies

$$\limsup_{\alpha} P_\alpha(\mathbf{F}) = P(\mathbf{F}) + 2a > P(\mathbf{F}) + a. \tag{17}$$

Recall that $(A, \leq)$ is the directed set, where α is at home. By (16) and $a > 0$ there exists a $k \in \mathbb{N}$ such that

$$P(\mathbf{F}_k) - P(\mathbf{F}) < a/2. \tag{18}$$

By definition $\limsup_\alpha P_\alpha(\mathbf{F}) = \inf_{\beta\in A} \sup_{\alpha\geq\beta} P_\alpha(\mathbf{F})$ and thus (17) ensures that

$$\sup_{\alpha\geq\beta} P_\alpha(\mathbf{F}) > P(\mathbf{F}) + a \quad \forall\, \beta \in A. \tag{19}$$

It follows that

$$\sup_{\alpha\geq\beta} P_\alpha(\mathbf{F}_k) \geq \sup_{\alpha\geq\beta} P_\alpha(\mathbf{F}) > P(\mathbf{F}) + a/2 + a/2 > P(\mathbf{F}_k) + a/2 \quad \forall\, \beta \in A. \tag{20}$$

Here, the first inequality holds, because $\mathbf{F}_k \supset \mathbf{F}$, the second equality is (19) and the last equality follows from (18). Taking the infimum over all $\beta \in A$ we obtain

$$\begin{aligned} P(\mathbf{F}_k) + a/2 &\leq \inf_{\beta\in A} \sup_{\alpha\geq\beta} P_\alpha(\mathbf{F}_k) = \limsup_{\alpha} P_\alpha(\mathbf{F}_k) = \limsup_{\alpha} P_\alpha\Big(\bigcap_{i=1}^k \mathcal{H}(K_i)\Big) \\ &\leq P\Big(\bigcap_{i=1}^k \mathcal{H}(K_i)\Big) = P(\mathbf{F}_k) \quad \text{ by assumption (ii).} \end{aligned}$$

Consequently, $P(\mathbf{F}_k) + a/2 \leq P(\mathbf{F}_k)$ and thus $a \leq 0$ in contradiction to $a > 0$. Since the implication $(ii) \Rightarrow (iii)$ is trivial, it remains to prove that (iii) implies (ii).

For this purpose consider $R_i = R(K_i) := \{r > 0 : P(\partial\mathcal{H}(K_i^r)) = 0\}, 1 \le i \le m$. Here, K_i^r is the closed r-neighborhood of K_i, which by Lemma 4 in the appendix is compact. We know from Lemma 7 in the appendix that for each index i the complement R_i^c of R_i is denumerable, whence $\bigcup_{i=1}^m R_i^c$ is denumerable as well. As a consequence $R(K_1, \ldots, K_m) := \bigcap_{i=1}^m R(K_i) = (\bigcup_{i=1}^m R_i^c)^c$ lies dense in $[0, \infty)$. Thus there exists a sequence $(r_j)_{j\in\mathbb{N}}$ in $R(K_1, \ldots, K_m)$ such that $r_j \downarrow 0, j \to \infty$. Conclude that

$$\limsup_\alpha P_\alpha(\bigcap_{i=1}^m \mathcal{H}(K_i)) \le \limsup_\alpha P_\alpha(\bigcap_{i=1}^m \mathcal{H}(K_i^{r_j})) \le P(\bigcap_{i=1}^m \mathcal{H}(K_i^{r_j})) \quad \forall\, j \in \mathbb{N}. \tag{21}$$

Here, the first inequality holds, because $K_i \subseteq K_i^{r_j}$ and therefore $\mathcal{H}(K_i) \subseteq \mathcal{H}(K_i^{r_j})$. As to the second inequality observe that $r_j \in R(K_i)$ for every $1 \le i \le m$, which means that $P(\partial\mathcal{H}(K_i^{r_j}) = 0$ for all i and so we can use assumption (iii) taking into account that the $K_i^{r_j}$ are all compact. Finally, consider $\mathbf{E}_j := \bigcap_{i=1}^m \mathcal{H}(K_i^{r_j}), j \in \mathbb{N}$, which by monotonicity of (r_j) are monotone decreasing. So σ-continuity of P from above yields:

$$\lim_{j\to\infty} P(\bigcap_{i=1}^m \mathcal{H}(K_i^{r_j})) = \lim_{j\to\infty} P(\mathbf{E}_j) = P(\bigcap_{j\in\mathbb{N}} \mathbf{E}_j) = P(\bigcap_{i=1}^m \bigcap_{j\in\mathbb{N}} \mathcal{H}(K_i^{r_j})). \tag{22}$$

Put $C_{ij} := K_i^{r_j}$. Observe that for each fixed $1 \le i \le m$ we have that $C_{ij} \downarrow K_i, j \to \infty$ and that $(C_{ij})_{j\in\mathbb{N}} \subseteq \mathcal{K}$. Thus $\bigcap_{j\in\mathbb{N}} \mathcal{H}(C_{ij}) = \mathcal{H}(K_i)$ for every $1 \le i \le m$ by Lemma 8 in the appendix. Infer that

$$P(\bigcap_{i=1}^m \bigcap_{j\in\mathbb{N}} \mathcal{H}(K_i^{r_j})) = P(\bigcap_{i=1}^m \mathcal{H}(K_i)). \tag{23}$$

Thus taking the limit $j \to \infty$ in (21) yields (ii) by (22) and (23). □

Next we relate $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$ with $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_F)$. A first simple relation is given in:

Proposition 3. *If $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_F)$, then $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$.*

Proof. This follows from the equivalent characterization (2) taking into account that $\tau_F \supseteq \tau_{uF}$. □

Weak convergence on $(\mathcal{F}, \tau_F)$ is well-studied in contrast to that on $(\mathcal{F}, \tau_{uF})$. For example $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_F)$ if and only if $P_\alpha(\mathcal{H}(K)) \to P(\mathcal{H}(K))$ for all compact K with $P(\partial_F \mathcal{H}(K)) = 0$. This and other characterisations can be found in in Molchanov [10]. Here, only sequences of probability measures are considered. However, in Ferger [5] we carry over the theory to nets of probability measures.

Our next result is really astonishing and has interesting consequences. Here we say that Q *dominates* P (in symbol: $P \le Q$) if $P(\bigcap_{i=1}^m \mathcal{H}(K_i)) \le Q(\bigcap_{i=1}^m \mathcal{H}(K_i))$ for every $m \in \mathbb{N}$ and for every collection $K_1, \ldots, K_m$ of non-empty compact sets.

Lemma 1. *Every net (P_α) is weakly convergent on $(\mathcal{F}, \tau_{uF})$ with limit δ_E, the Dirac-measure at point E. Moreover, if $P_\alpha \to_w P$ on $(\mathcal{F}, \tau_{uF})$, then $P_\alpha \to_w Q$ on $(\mathcal{F}, \tau_{uF})$ for each Q that dominates P.*

Proof. If $K_1, \ldots, K_m$ are non-empty compact sets, then $\delta_E(\bigcap_{i=1}^m \mathcal{H}(K_i)) = 1$, whence (ii) of Theorem 2 is fulfilled and therefore $P_\alpha \to_w \delta_E$ on $(\mathcal{F}, \tau_{uF})$. Each Q that dominates P satisfies (ii) of Theorem 2, which shows the second claim by another application of Theorem 2. $\square$

As immediate consequences we obtain:

Corollary 2. *The topological space $(\Pi(\mathcal{F}, \tau_{uF}), \tau_{weak})$ is compact. In general it is not Hausdorff and therefore not metrizable.*

Corollary 3. *In general the reverse conclusion in Proposition 3 is not true.*

In view of the last result, the question arises under which additional conditions the reversal applies. The answer involves the family $\mathbf{F}_{0,1}$ of all sets with at most one element, i.e., $\mathbf{F}_{0,1} = \{\emptyset\} \cup \{\{x\} : x \in E\}$. Since the empty set and all singletons are closed, $\mathbf{F}_{0,1} \subseteq \mathcal{F}$. According to Lemma 9 in the appendix $\mathbf{F}_{0,1}$ is τ_F-closed and thus in particular is a Borel-set: $\mathbf{F}_{0,1} \in \mathcal{B}_F$.

Theorem 4. *Suppose that*

(i) $P_\alpha \to_w P$ *on* $(\mathcal{F}, \tau_{uF})$,

(ii) For each $\epsilon > 0$ *there exists a* $K \in \mathcal{K}$ *such that*

$$\liminf_\alpha P_\alpha(\{F \in \mathcal{F} : \emptyset \neq F \subseteq K\}) \geq 1 - \epsilon, \tag{24}$$

(This is a bit more than asymptotic compact-boundedness (6).)

(iii) $P(\boldsymbol{F}_{0,1}) = 1$.

Then

$$P_\alpha \to_w P \text{ on } (\mathcal{F}, \tau_F). \tag{25}$$

Proof. Let $\mathbf{F}$ be τ_F-closed and for each $\epsilon > 0$ let $K \in \mathcal{K}$ as in (ii). Put $\mathbf{B} := \{F \in \mathcal{F} : \emptyset \neq F \subseteq K\}$. Since $\mathbf{B} = \mathcal{H}(E) \cap \mathcal{M}(K^c)$ Lemma 2.1.1 in Schneider and Weil [12] ensures that $\mathbf{B} \in \mathcal{B}_F$ is a Borel-set and hence the probabilities in (24) are well-defined. The decomposition $\mathbf{F} = (\mathbf{F} \cap \mathbf{B}) \cup (\mathbf{F} \cap \mathbf{B}^c)$ yields that $\mathbf{F} \subseteq \mathbf{C} \cup \mathbf{B}^c$ with $\mathbf{C} = \mathbf{F} \cap \mathbf{B} \in \mathcal{B}_F$. According to (ii) $\liminf_\alpha P_\alpha(\mathbf{B}) \geq 1 - \epsilon$, whence by complementation $\limsup_\alpha P_\alpha(\mathbf{B}^c) \leq \epsilon$. It follows that

$$\limsup_\alpha P_\alpha(\mathbf{F}) \leq \limsup_\alpha P_\alpha(\mathbf{C}) + \epsilon \quad \text{ for every } \epsilon > 0. \tag{26}$$

In the sequel cl_{uF} and cl_F refer to the closure with respect to the upper Fell-topology and the Fell-topology, respectively. The next relation is the key of our proof.

$$\mathbf{C} \subseteq cl_{uF}(\mathbf{C}) \subseteq cl_F(\mathbf{C}) \cup \mathbf{F}_{0,1}^c. \tag{27}$$

The first $\subseteq$ holds by definition of the closure. As to the second one let $F \in cl_{uF}(\mathbf{C})$. Then there exists a net (F_α) in $\mathbf{C}$ with $F_\alpha \to F$ in $(\mathcal{F}, \tau_{uF})$. If $F \notin \mathbf{F}_{0,1}$, then it lies in the set on the right side of (27) as desired. So it remains to consider $F \in \mathbf{F}_{0,1}$. If $F = \emptyset$, then $F_\alpha \to F$ in $(\mathcal{F}, \tau_F)$ by Lemma 10 (b) in the appendix and consequently $F \in cl_F(\mathbf{C})$. If $F = \{x\}$ is a singleton, then we use that $\emptyset \neq F_\alpha \subseteq K$ for all $\alpha \in A$, because $F_\alpha \in \mathbf{C} \subseteq \mathbf{B}$. Thus we can apply Lemma 10 (a) in the appendix, which yields that $F_\alpha \to F$ in $(\mathcal{F}, \tau_F)$, whence $F \in cl_F(\mathbf{C})$ also in that last case. This finally shows that (27) is true.

Next, observe that

$$\limsup_\alpha P_\alpha(\mathbf{C}) \leq \limsup_\alpha P_\alpha(cl_F(\mathbf{C})) \leq P(cl_{uF}(\mathbf{C}), \tag{28}$$

by the first part of (27) and by assumption (i), because $cl_{uF}(\mathbf{C})$ is τ_{uF}-closed. The second part of (27) gives

$$P(cl_{uF}(\mathbf{C})) \leq P(cl_F(\mathbf{C})) + P(\mathbf{F}_{0,1}^c) = P(cl_F(\mathbf{C})) \leq P(cl_F(\mathbf{F})) = P(\mathbf{F}). \tag{29}$$

Here, the first inequality holds by the second part of (27), the equality holds by assumption (iii), the second inequality holds by $cl_F(\mathbf{C}) \subseteq cl_F(\mathbf{F})$, because $\mathbf{C} \subseteq \mathbf{F}$ and the last equality holds, since $\mathbf{F}$ is τ_F-closed. Combining (26), (28) and (29) results in

$$\limsup_\alpha P_\alpha(\mathbf{F}) \leq P(\mathbf{F}) + \epsilon \quad \forall\, \epsilon > 0.$$

Taking the limit $\epsilon \to 0$ yields the convergence in (25). □

3 Distributional convergence of random closed sets in $(\mathcal{F}, \tau_{uF})$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A map $C : \Omega \to \mathcal{F}$ is called *random closed set (in E on $(\Omega, \mathcal{A}, \mathbb{P})$)*, if it is $\mathcal{A} - \mathcal{B}_F$ measurable. Its distribution $\mathbb{P} \circ C^{-1}$ is a probability measure on $(\mathcal{F}, \tau_F)$, but also on $(\mathcal{F}, \tau_{uF})$ by (4). Conversely, by the canonical construction every probability measure P on $(\mathcal{F}, \tau_F)$ or $(\mathcal{F}, \tau_{uF})$, respectively, is the distribution of a random closed set on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Indeed, one can take $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathcal{F}, \mathcal{B}_F, P) = (\mathcal{F}, \mathcal{B}_{uF}, P)$ and C is equal to the identity map.

If $(C_\alpha)_{\alpha \in A}$ is a net of random closed sets C_α in E on $(\Omega_\alpha, \mathcal{A}_\alpha, \mathbb{P}_\alpha)$, then as usual we define distributional convergence by weak convergence of the distributions. More precisely, $C_\alpha \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uF})$, if $\mathbb{P}_\alpha \circ C_\alpha^{-1} \to_w \mathbb{P} \circ C^{-1}$ on $(\mathcal{F}, \tau_{uF})$ and $C_\alpha \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_F)$, if $\mathbb{P}_\alpha \circ C_\alpha^{-1} \to_w \mathbb{P} \circ C^{-1}$ on $(\mathcal{F}, \tau_F)$.

The short discussion above shows that every result in the last section (except for Corollary 2) can be formulated in terms of random closed sets. For example Theorem 2 takes the following form:

Theorem 5. *The following statements are equivalent:*

(i) $C_\alpha \xrightarrow{\mathcal{D}} C$ *in* $(\mathcal{F}, \tau_{uF})$.

(ii)

$$\limsup_\alpha \mathbb{P}_\alpha(C_\alpha \cap K_1 \neq \emptyset, \dots, C_\alpha \cap K_m \neq \emptyset) \leq \mathbb{P}(C \cap K_1 \neq \emptyset, \dots, C \cap K_m \neq \emptyset) \quad (30)$$

for every $m \in \mathbb{N}$ *and every finite collection* $K_1, \dots, K_m$ *of non-empty compact sets in* E.

(iii) The inequality (30) holds for every $m \in \mathbb{N}$ *and every finite collection* $K_1, \dots, K_m$ *of non-empty compact sets in* E *such that* $\mathbb{P}(C \in \partial_F \mathcal{H}(K_i)) = 0$ *for all* $1 \leq i \leq m$.

Notice that by Lemma 6 (i) $\mathbb{P}(C \in \partial_F \mathcal{H}(K_i)) = \mathbb{P}(C \cap K_i \neq \emptyset, C \cap K_i^0 = \emptyset)$. Thus we see that Theorem 5 is a generalization of Vogel's [15] Lemma 2.1, where only sequences rather than nets of random closed sets are considered and furthermore E is required to be the euclidian space $\mathbb{R}^d$.

Assume C and D are random closed sets with $\mathbb{P}^*(C \nsubseteq D) = 0$, where $\mathbb{P}^*$ is the outer measure of $\mathbb{P}$. This means that $\{C \nsubseteq D\}$ is a $\mathbb{P}$-null set and by completion of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we can achieve that $\{C \nsubseteq D\} \in \mathcal{A}$. Now we can say that $C \subseteq D$ $\mathbb{P}$-almost surely (a.s.). In this case $\mathbb{P} \circ D^{-1}$ dominates $\mathbb{P} \circ C^{-1}$ and from Lemma 1 we can deduce:

Lemma 2. $C_\alpha \xrightarrow{\mathcal{D}} E$ *in* $(\mathcal{F}, \tau_{uF})$ *for all nets* (C_α) *of random closed sets in* E. *Moreover, if* $C_\alpha \xrightarrow{\mathcal{D}} C$ *in* $(\mathcal{F}, \tau_{uF})$ *and* D *is a random closed set with* $C \subseteq D$ $\mathbb{P}$*-a.s., then* $C_\alpha \xrightarrow{\mathcal{D}} D$ *in* $(\mathcal{F}, \tau_{uF})$.

In short, every superset of a limit set is also a limit set. Conversely, every net (D_α) of subsets, i.e., $D_\alpha \subseteq C_\alpha$ $\mathbb{P}_\alpha$-a.s., also converges to C. In fact, a somewhat more general result applies:

Lemma 3. *Let* (C_α) *and* (D_α) *be nets of random closed sets in* E *on* $(\Omega_\alpha, \mathcal{A}_\alpha, \mathbb{P}_\alpha)$ *such that*

$$\limsup_\alpha \mathbb{P}_\alpha(D_\alpha \nsubseteq C_\alpha) = 0. \quad (31)$$

Then $C_\alpha \xrightarrow{\mathcal{D}} C$ *in* $(\mathcal{F}, \tau_{uF})$ *entails* $D_\alpha \xrightarrow{\mathcal{D}} C$ *in* $(\mathcal{F}, \tau_{uF})$.

Proof. Let $K_1, \dots, K_m$ be non-empty compact sets in E. Then

$$\begin{aligned}
&\bigcap_{i=1}^m \{D_\alpha \cap K_i \neq \emptyset\} \\
&= \bigcap_{i=1}^m (\{D_\alpha \cap K_i \neq \emptyset\} \cap \{D_\alpha \subseteq C_\alpha\}) \cup (\bigcap_{i=1}^m \{D_\alpha \cap K_i \neq \emptyset\}) \cap \{D_\alpha \nsubseteq C_\alpha\} \\
&\subseteq \bigcap_{i=1}^m \{C_\alpha \cap K_i \neq \emptyset\} \cup \{D_\alpha \nsubseteq C_\alpha\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_{\alpha} \mathbb{P}_{\alpha}(\bigcap_{i=1}^{m}\{D_{\alpha} \cap K_i \neq \emptyset\}) \\
&\leq \limsup_{\alpha} \mathbb{P}_{\alpha}(\bigcap_{i=1}^{m}\{C_{\alpha} \cap K_i \neq \emptyset\}) + \limsup_{\alpha} \mathbb{P}_{\alpha}(D_{\alpha} \nsubseteq C_{\alpha}) \\
&= \limsup_{\alpha} \mathbb{P}_{\alpha}(\bigcap_{i=1}^{m}\{C_{\alpha} \cap K_i \neq \emptyset\}) \quad \text{by (31)} \\
&\leq \mathbb{P}(C \cap K_1 \neq \emptyset, \ldots, C \cap K_m \neq \emptyset) \quad \text{by Theorem 5.}
\end{aligned}$$

Another application of Theorem 5 yields the assertion. □

With our next result we can give an answer to the following question: If $C_\alpha \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uF})$, then what can we say about the asymptotic behaviour of random variables $\xi_\alpha \in C_\alpha$?

Theorem 6. *For each $\alpha \in A$ let $\xi_\alpha : (\Omega_\alpha, \mathcal{A}_\alpha, \mathbb{P}_\alpha) \to (E, \mathcal{B}(E))$ be a measurable map (random variable in E). Suppose that:*

(i) $C_\alpha \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uF})$.
(ii) $\limsup_\alpha \mathbb{P}_\alpha(\xi_\alpha \notin C_\alpha) = 0$.
(iii) For every $\epsilon > 0$ there exists a $K \in \mathcal{K}$ such that

$$\liminf_{\alpha} \mathbb{P}_{\alpha}(\xi_{\alpha} \in K) \geq 1 - \epsilon.$$

Then

$$\limsup_{\alpha} \mathbb{P}_{\alpha}(\xi_{\alpha} \in F) \leq T_C(F) \quad \forall\, F \in \mathcal{F}, \tag{32}$$

where T_C is a Choquet-capacity, namely the capacity functional of C given by $T_C(B) = \mathbb{P}(C \cap B \neq \emptyset), B \in \mathcal{B}(E)$.
If in addition $C \subseteq \{\xi\}$ $\mathbb{P}$-a.s. for some random variable ξ in E on $(\Omega, \mathcal{A}, \mathbb{P})$, then

$$\xi_{\alpha} \xrightarrow{\mathcal{D}} \xi \quad \textit{in } (E, \mathcal{G}). \tag{33}$$

Proof. By Lemma 2.1.1 in Schneider and Weil [12] $\mathcal{B}_F = \sigma(\{\mathcal{H}(G) : G \in \mathcal{G}\})$. Thus $D_\alpha := \{\xi_\alpha\}$ are random closed sets, because $\{D_\alpha \in \mathcal{H}(G)\} = \{\xi_\alpha \in G\} \in \mathcal{A}_\alpha$. From (i) and (ii) it follows by Lemma 3 that $\{\xi_\alpha\} \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uF})$. Conclude with (iii) and Corollary 1 (in the formulation for random closed sets and with $\mathcal{F}^* = \{F\}$ a singleton) that

$$\limsup_{\alpha} \mathbb{P}_{\alpha}(\xi_{\alpha} \in F) = \limsup_{\alpha} \mathbb{P}_{\alpha}(\{\xi_{\alpha}\} \cap F \neq \emptyset) \leq \mathbb{P}(C \cap F \neq \emptyset) = T_C(F) \quad \forall\, F \in \mathcal{F}. \tag{34}$$

This shows (32). Under the additional assumption $C \subseteq \{\xi\}$ a.s. it follows that $T_C(F) \leq \mathbb{P}(\xi \in F)$, whence we can infer from (34) that $\limsup_\alpha \mathbb{P}_\alpha(\xi_\alpha \in F) \leq \mathbb{P}(\xi \in F)$ for all closed F, which by (3) yields the distributional convergence (33). $\square$

If a net (ξ_α) satisfies (32), then we say that it *converges in distribution to (the random closed set) C* and denote this by

$$\xi_\alpha \xrightarrow{\mathcal{D}} C.$$

This new type of distributional convergence has been introduced and analyzed in Ferger [6]. Here, for instance we show in Theorem 4.2 that $\xi_\alpha \xrightarrow{\mathcal{D}} C$ is equivalent to $\{\xi_\alpha\} \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uV})$.

Remark 2. *Suppose that $C \subseteq \{\xi\}$ $\mathbb{P}$-a.s. Under the assumptions (i)-(iii) of the last theorem it follows that actually $C = \{\xi\}$ $\mathbb{P}$-a.s. To see this notice that $E \in \mathcal{F}$ and $\mathbb{P}_\alpha(\xi_\alpha \in E) = 1$ for all $\alpha \in A$. Therefore (32) with $F = E$ yields that $\mathbb{P}(C \cap E \neq \emptyset) = \mathbb{P}(C \neq \emptyset) = 1$, whence $\mathbb{P}(C = \emptyset) = 0$, which in turn by the assumption on C gives the a.s. equality.*

Remark 3. *Since $T_C(B) = \mathbb{P}(C \in \mathcal{H}(B))$, the capacity functional T_C is by (12) well-defined on the Borel-σ algebra $\mathcal{B}(E)$ on $(E, \mathcal{G})$. In general, T_C is not a probability measure. In fact, it is a probability measure if and only if there exists a random variable $\{\xi\}$ such that $C \stackrel{\mathcal{D}}{=} \{\xi\}$, see Ferger [5]. For further properties of T_C we refer to Molchanov [10].*

If C_α is a random closed set and ξ_α is a random variable in E with $\xi_\alpha \in C_\alpha$ $\mathbb{P}_\alpha$-a.s., then ξ_α is called a *measurable selection of* C_α. By the *Fundamental selection theorem*, confer Molchanov [10] on p.77, the existence of ξ_α is guaranteed. A net (ξ_α) satisfying condition (iii) of the above Theorem 6 is called *asymptotically tight*. This condition is much weaker than the classical (uniform) tightness, which requires that $\mathbb{P}_\alpha(\xi_\alpha \notin K) \leq \epsilon$ for all $\alpha \in A$ and not only in the limit.

The following corollary provides an answer to the question posed above.

Corollary 4. *Assume that $C_\alpha \xrightarrow{\mathcal{D}} C$ in $(\mathcal{F}, \tau_{uF})$ and that (ξ_α) is a net of measurable selections ξ_α of C_α. If (ξ_α) is asymptotically tight, then $\xi_\alpha \xrightarrow{\mathcal{D}} C$. In case that $C \subseteq \{\xi\}$ a.s. for some random variable ξ we obtain: $\xi_\alpha \xrightarrow{\mathcal{D}} \xi$ in $(E, \mathcal{G})$.*

Proof. Conditions (i) and (iii) of Theorem 6 are fulfilled by assumption. Since each ξ_α is a measurable selection, we have that $\mathbb{P}(\xi_\alpha \notin C_\alpha) = 0$ for all $\alpha \in A$, whence condition (ii) is trivially fulfilled and thus Theorem 6 yields the assertion. $\square$

4 Appendix

In this section we present several results, which we use in our proofs above. For some of these, the statements are known in case E is a finite-dimensional linear space with a metric d such as for example $E = \mathbb{R}^d$. More details are given in our notes at the end of the appendix.

Since $(E, \mathcal{G})$ is locally compact, second-countable and Hausdorff it is metrizable. By Theorem 2 of Vaughan [14] the underlying metric d can be chosen such that:

$$\text{Every bounded set is relatively compact.} \tag{35}$$

In addition (E, d) is complete and thus a polish metric space. For the extremely useful result (35) confer also Engelking [3], Exercise 4.2C on p. 265. Although Vaughan's theorem was published in 1937, it does not seem to be so well known.

Given a point $x \in E$ and a non-empty subset $A \subseteq E$ let $d(x, A) := \inf\{d(x, a) : a \in A\}$ denote the distance of x from A. As usual $B(x, r) := \{y \in E : d(y, x) < r\}$ denotes the (open) ball with center x and radius $r > 0$. Moreover, $A^r := \{x \in E : d(x, A) \leq r\}$ and $A^{r-} := \{x \in E : d(x, A) < r\}$ are respectively the *closed* and *open r-neighborhoods* of A, where $r > 0$.

We use the usual notation $Int(A) \equiv A^o, cl(A) \equiv \overline{A}$ and ∂A for the interior, the closure and boundary, respectively, of A in $(E, \mathcal{G})$.

Lemma 4. *If $K \neq \emptyset$ is compact, then K^r is compact for all $r > 0$.*

Proof. Firstly, observe that $K^r = d(\cdot, K)^{-1}((-\infty, r])$ is closed as the pre-image of the closed half-line $(-\infty, x]$ under the continuous function $x \mapsto d(x, K)$. Therefore by (35) it suffices to show that K^r is bounded, i.e., there exist some $x_0 \in E$ and some $s > 0$ such that $K^r \subseteq B(x_0, s)$. We prove this by contradiction. So, assume that $K^r \nsubseteq B(x_0, s)$ for each $x_0 \in E$ and for all $s > 0$, that means there exists some $y \in E$ with $d(y, K) \leq r$, but $d(y, x_0) \geq s$. Since $K \in \mathcal{K}$, there exists some $z \in K$ such that $d(y, K) = d(y, z)$. By the triangle-inequality we know that $d(x_0, y) \leq d(x_0, z) + d(z, y)$, which implies that $s - r \leq d(x_0, y) - d(z, y) \leq d(x_0, z)$. It follows that $s - r \leq d(x_0, z)$ for all $x_0 \in E$ and for all $s > 0$. Choosing $x_0 = z$ leads to $s \leq r$ for all $s > 0$, which is a contradiction. Now, by closedness $K^r = \overline{K^r}$ and $\overline{K^r} \in \mathcal{K}$ by (35), whence K^r is compact. □

For $\mathbf{A} \subseteq \mathcal{F}$ let $Int_F(\mathbf{A})$ and $cl_F(\mathbf{A})$ denote the interior and closure, respectively, of $\mathbf{A}$ in $(\mathcal{F}, \tau_F)$. In the following we will use that by construction of τ_F the basic open sets $\mathbf{B}$ are all of the type

$$\mathbf{B} = \mathcal{M}(K) \cap \mathcal{H}(G_1) \cap \ldots \cap \mathcal{H}(G_l) \tag{36}$$

with $K \in \mathcal{K}, G_1, \ldots, G_l \in \mathcal{G}$ and $l \in \mathbb{N}_0$. For $l = 0$ we obtain $\mathbf{B} = \mathcal{M}(K)$.

Lemma 5. *If $A \subseteq E$ is an arbitrary subset, then*

$$Int_F(\mathcal{H}(A)) = \mathcal{H}(Int(A)). \tag{37}$$

Proof. W.l.o.g. A is nonempty, because otherwise equation (37) is trivially fulfilled. We first prove the relation $\subseteq$. So, let $F \in Int_F(\mathcal{H}(A))$. By definition of the interior there exists an $\mathbf{O} \in \tau_F$ such that $F \in \mathbf{O}$ and $\mathbf{O} \subseteq \mathcal{H}(A)$. Since every open set is the union of basic open sets we find a basic open set as in (36) such that

$$F \in \mathcal{M}(K) \cap \mathcal{H}(G_1) \cap \ldots \cap \mathcal{H}(G_l) \subseteq \mathcal{H}(A). \tag{38}$$

In particular, $F \in \mathcal{H}(A)$, i.e., $F \cap A \neq \emptyset$, whence $F \neq \emptyset$, which is the same as $F \in \mathcal{H}(E)$. Now, $E \in \mathcal{G}$ and therefore we can assume that $l \geq 1$. It follows that:

$$\exists\, i \in \{1, \ldots, l\} : G_i \cap K^c \subseteq A. \tag{39}$$

We prove (39) by contradiction. For that purpose recall that, if U and V are subsets of E, then the following equivalence holds: $U \subseteq V \Leftrightarrow U \cap V^c = \emptyset$. So let us assume that (39) is not true, which means that $G_i \cap K^c \cap A^c \neq \emptyset \quad \forall\, i \in \{1, \ldots, l\}$. Consequently, for each $i \in \{1, \ldots, l\}$ there exists a point x_i with $x_i \in G_i, x_i \notin K$ and $x_i \notin A$. Introduce $H := \{x_1, \ldots, x_l\}$. Then H has the following properties: $H \in \mathcal{F}, H \cap K = \emptyset$ (because all x_i are not K) and $H \cap G_i \neq \emptyset \; \forall\, 1 \leq i \leq l$ (because $x_i \in H \cap G_i$ for each i). Thus

$$H \in \mathcal{M}(K) \cap \mathcal{H}(G_1) \cap \ldots \mathcal{H}(G_l) \subseteq \mathcal{H}(A),$$

where the last relation $\subseteq$ holds by (38). It follows that $H \in \mathcal{H}(A)$, i.e., $H \cap A \neq \emptyset$ in contradiction to $x_i \notin A$ for all i. This is the proof of (39).

Put $G := G_i \cap K^c$ with G_i from (39). Then $G \in \mathcal{G}$ and $G \subseteq A$. Moreover:

$$\emptyset \neq F \cap G_i = (F \cap G_i \cap K^c) \cup (F \cap G_i \cap K) = F \cap G_i \cap K^c = F \cap G. \tag{40}$$

Here, the first relation $\neq$ follows from $F \in \mathcal{H}(G_i)$ by (38). The subsequent equality follows from the decomposition $E = K^c \cup K$. The next equality also follows from (38), which implies that $F \in \mathcal{M}(K)$. Hence $F \cap K = \emptyset$ and so $F \cap G_i \cap K = \emptyset$ a fortiori. Since $G \in \mathcal{G}$ and $G \subseteq A$, we have that $G \subseteq Int(A)$. Deduce from (40) that $\emptyset \neq F \cap G \subseteq F \cap Int(A)$, whence $F \cap Int(A) \neq \emptyset$, which is the same as $F \in \mathcal{H}(Int(A))$.

To see the reverse relation $\supseteq$ in (37) assume that $F \in \mathcal{H}(Int(A))$. In the sequel we use the following property of $\mathcal{H}(\cdot)$: $\mathcal{H}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} \mathcal{H}(A)$ for every family $\mathcal{A}$ of subsets of E. Apply this property to the family $\mathcal{A} = \{G \in \mathcal{G} : G \subseteq A\}$. It yields that

$$\mathcal{H}(Int(A)) = \mathcal{H}(\bigcup_{G \in \mathcal{A}} G) = \bigcup_{G \in \mathcal{A}} \mathcal{H}(G).$$

Thus one finds an open G with $G \subseteq A$ such that $F \in \mathcal{H}(G)$. Infer from $\mathcal{H}(G) \in \tau_F$ and $\mathcal{H}(G) \subseteq \mathcal{H}(A)$ that $\mathcal{H}(G) \subseteq Int_F(\mathcal{H}(A))$. Since $F \in \mathcal{H}(G)$ we finally obtain that $F \in Int_F(\mathcal{H}(A))$. ☐

We use Lemma 5 to describe the boundary of the hitting-sets of a compact set in the Fell-topology τ_F.

Lemma 6. *If K is compact in $(E, \mathcal{G})$, then:*

(i) $\partial_F \mathcal{H}(K) = \mathcal{H}(K) \setminus \mathcal{H}(K^0)$.

(ii) $\partial_F \mathcal{H}(K) = \{F \in \mathcal{F} : \emptyset \neq F \cap K \subseteq \partial K\}$.

Proof. (i) By definition $\partial_F \mathcal{H}(K) = cl_F(\mathcal{H}(K)) \setminus Int_F(\mathcal{H}(K))$. Here, $cl_F(\mathcal{H}(K)) = \mathcal{H}(K)$, because $\mathcal{H}(K)$ is τ_F-closed as $\mathcal{H}(K)^c = \mathcal{M}(K) \in \tau_F$. Moreover, $Int_F(\mathcal{H}(K)) = \mathcal{H}(K^0)$ by Lemma 5, which now results in the equality (i).

(ii) By (i)

$$\partial_F \mathcal{H}(K) = \mathcal{H}(K) \cap \mathcal{M}(K^0) = \{F \in \mathcal{F} : F \cap K \neq \emptyset, F \cap K^0 = \emptyset\}.$$

Consequently, we have to show: If $F \in \mathcal{F}$ satisfies $F \cap K \neq \emptyset$, then the following equivalence holds:

$$F \cap K^0 = \emptyset \quad \Leftrightarrow \quad F \cap K \subseteq \partial F$$

To see the if-part let $x \in F \cap K$, i.e., $x \in F$ and $x \in K$. We have to prove that $x \in \partial F = \overline{K} \setminus K^0 = K \cap (K^0)^c$. Since $x \in K$, it suffices to show that $x \notin K^0$. Assume that $x \in K^0$. Then $x \in F \cap K^0$ in contradiction to $F \cap K^0 = \emptyset$. For the only-if-part recall the equivalence $U \subseteq V \Leftrightarrow U \cap V^c = \emptyset$. By assumption $F \cap K \subseteq \partial F$. Since $\partial F = K \cap (K^0)^c$ the equivalence yields

$$\begin{aligned}\emptyset &= F \cap K \cap (K \cap (K^0)^c)^c = F \cap K \cap (K^c \cup K^0) = (F \cap K \cap K^c) \cup (F \cap K \cap K^0) \\ &= F \cap K \cap K^0 = F \cap K^0\end{aligned}$$

□

In the proof of Theorem 2 the set $R(K) = \{r > 0 : P(\partial_F \mathcal{H}(K^r)) = 0\}$ plays an important role.

Lemma 7. *If K is compact, then the complement $R(K)^c$ of $R(K)$ is at most countable. As a consequence $R(K)$ lies dense in $[0, \infty)$.*

Proof. The key argument for the proof is to show that the sets $\partial_F \mathcal{H}(K^r), r > 0$, are pairwise disjoint. Indeed, assume that this is not true. Then there exist two reals $0 < r < s$ such that $\partial_F \mathcal{H}(K^r) \cap \partial_F \mathcal{H}(K^s) \neq \emptyset$. Therefore we find a set $F \in \mathcal{F}$ with $F \in \partial_F \mathcal{H}(K^r)$ and $F \in \partial_F \mathcal{H}(K^s)$. By Lemma 6 (ii) this means that F satisfies the following two relations: (a) $\emptyset \neq F \cap K^r \subseteq \partial K^r$ and (b) $\emptyset \neq F \cap K^s \subseteq \partial K^s$.

Now, $\partial K^r \subseteq \{x \in E : d(x, K) = r\}$ for all $r > 0$. To see this first observe that $\partial K^r = K^r \cap \overline{(K^r)^c}$ by Lemma 4. So, if $x \in \partial K^r$, then $d(x, K) \leq r$ and there exists a net (x_α) converging to x with $d(x_\alpha, K) > r$ for all indices α. But $d(\cdot, K)$ is continuous, whence $d(x, K) \geq r$. Thus (a) and (b) imply (c) $\emptyset \neq F \cap K^r \subseteq \{x \in E : d(x, K) = r\}$ and (d) $\emptyset \neq F \cap K^s \subseteq \{x \in E : d(x, K) = s\}$.

By (c) there exists a point $x \in F \cap K^r$ with $d(x, K) = r$. Since $r < s$ and therefore $K^r \subseteq K^s$, x a fortiori lies in $F \cap K^s$, so that from (d) we can conclude that $d(x, K) = s$. It follows that $r = s$ in contradiction to $r < s$.

Next, observe that $R(K)^c = \{r > 0 : P(\partial_F \mathcal{H}(K^r)) > 0\} = \bigcup_{m \in \mathbb{N}} E_m$ with $E_m = \{r > 0 : P(\partial_F \mathcal{H}(K^r)) \geq 1/m\}$. Here, E_m contains at most m elements, because otherwise we find at least $m+1$ positive numbers $r_1, \dots, r_{m+1}$ with $P(\partial_F \mathcal{H}(K^{r_j})) \geq 1/m$ for all $1 \leq j \leq m+1$. Herewith we arrive at

$$1 \geq P(\bigcup_{j=1}^{m+1} \partial_F \mathcal{H}(K^{r_j})) = \sum_{j=1}^{m+1} P(\partial_F \mathcal{H}(K^{r_j}) \geq (m+1)\frac{1}{m} > 1,$$

a contradiction. (Note that here the pairwise disjointness is essential, because it ensures the equality.) Thus $R(K)^c = \bigcup_{m \in \mathbb{N}} E_m$ is denumerable. As to the second assertion of the lemma assume that $R(K)^c$ is not dense in $[0, \infty)$. Then there exists a point $x \in [0, \infty)$ and a non-degenerate interval I containing x with $I \cap R(K) = \emptyset$, which is the same as $I \subseteq R(K)^c$. It follows that I is denumerable, a contradiction. $\square$

If $\mathcal{A}$ is a family of subsets of E, then

$$\mathcal{H}(\bigcap_{A \in \mathcal{A}} A) \subseteq \bigcap_{A \in \mathcal{A}} \mathcal{H}(A). \tag{41}$$

Our next lemma gives a condition which ensures equality.

Lemma 8. *If $(K_j)_{j \in \mathbb{N}}$ is a sequence of compact sets with $K_j \downarrow K \in \mathcal{K}$, then $\mathcal{H}(K) = \bigcap_{j \in \mathbb{N}} \mathcal{H}(K_j)$.*

Proof. By (41) it remains to show that $\bigcap_{j \in \mathbb{N}} \mathcal{H}(K_j) \subseteq \mathcal{H}(K)$. So, let $F \in \bigcap_{j \in \mathbb{N}} \mathcal{H}(K_j)$. Then for every $j \in \mathbb{N}$ there exists a point $y_j \in F \cap K_j \neq \emptyset$. In particular, $(y_j)_{j \in \mathbb{N}}$ is a sequence in K_1, because (*) $K_1 \supseteq K_2 \supseteq \dots$ by assumption. Since K_1 is compact, $(y_j)_{j \in \mathbb{N}}$ has a convergent subsequence. For notational simplicity we assume that $y_j \to y \in K_1$. It follows from (*) that $y_j \in F \cap K_j \subseteq F \cap K_n$ for all $j \geq n$ and all $n \in \mathbb{N}$. Thus $(y_j)_{j \geq n}$ is a sequence in $F \cap K_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. By closedness the limit y lies in $F \cap K_n$ for all $n \in \mathbb{N}$, which in turn means that $y \in \bigcap_{n \in \mathbb{N}}(F \cap K_n) = F \cap \bigcap_{n \in \mathbb{N}} K_n = F \cap K$ by assumption. Consequently, $F \cap K$ is non-empty as it contains y and therefore $F \in \mathcal{H}(K)$. $\square$

Since $\mathcal{H}(\cdot)$ is monotone increasing with respect to $\subseteq$, the assertion in Lemma 8 can be rewritten as $\mathcal{H}(K_j) \downarrow \mathcal{H}(K)$.

Recall the family $\mathbf{F}_{0,1}$ of all singletons inclusive the empty set. It is a Borel-set: $\mathbf{F}_{0,1} \in \mathcal{B}_F = \sigma(\tau_F)$. This follows from the following lemma.

Lemma 9. *The set $\mathbf{F}_{0,1}$ is closed in $(\mathcal{F}, \tau_F)$.*

Proof. Since $(\mathcal{F}, \tau_F)$ is metrizable, we can argue with sequences. So, let (F_n) be a sequence in $\mathbf{F}_{0,1}$ with $F_n \to F$ in $(\mathcal{F}, \tau_F)$. If $F = \emptyset$, then $F \in \mathbf{F}_{0,1}$ and we are ready. Assume that $F \neq \emptyset$. We have to prove that F is a singleton. Let $G \in \mathcal{G}$ with $F \cap G \neq \emptyset$ (as for instance $G = E$). Then there exists a natural number n_0 such that $F_n \cap G \neq \emptyset$ for all $n \geq n_0$. Since all F_n are either empty or a singleton, we now know that these F_n are singletons. Consequently, for every $n \geq n_0$ there exists a point $x_n \in E$ with $F_n = \{x_n\}$. Since convergence in the Fell-topology is equivalent with convergence in sense of Painlevé-Kuratowski, it follows that $F = \liminf_{n\to\infty}\{x_n\}$, where $\liminf_{n\to\infty} A_n$ denotes the lower limit of a sequence $(A_n)_{n\in\mathbb{N}}$ of sets, confer, e.g., Theorem C.7 in Molchanov [10]. It follows from the definition of the lower limit that F is the set of all limit points of the sequence (x_n). Now, E is Hausdorff and therefore F is a singleton. □

Since $\tau_F \supseteq \tau_{uF}$, convergence in the Fell-topology entails that in the upper Fell-topology. As to the reverse we have:

Lemma 10. *Let (F_α) be a net in $\mathcal{F}$.*

(a) Assume that

(i) $F_\alpha \to F = \{x\}$ in $(\mathcal{F}, \tau_{uF})$ with $x \in E$.

(ii) There exists a $K \in \mathcal{K}$ and $\alpha_0 \in A$ such that $\emptyset \neq F_\alpha \subseteq K$ for all $\alpha \geq \alpha_0$.

Then $F_\alpha \to F$ in $(\mathcal{F}, \tau_F)$.

(b)

$$F_\alpha \to \emptyset \text{ in } (\mathcal{F}, \tau_{uF}) \quad \Leftrightarrow \quad F_\alpha \to \emptyset \text{ in } (\mathcal{F}, \tau_F)$$

Proof. (a) Let $\mathbf{S} \in \mathcal{S}_F = \{\mathcal{M}(K) : K \in \mathcal{K}\} \cup \{\mathcal{H}(G) : G \in \mathcal{G}\}$ be a subbase-neighborhood of F. If $\mathbf{S} = \mathcal{M}(K_0)$ with $K_0 \in \mathcal{K}$, then $\mathbf{S} \in \tau_{uF}$, whence by (i) there exists an $\alpha_1 \in A$ such that $F_\alpha \in \mathbf{S}$ for all $\alpha \geq \alpha_1$. If $\mathbf{S} = \mathcal{H}(G)$ with $G \in \mathcal{G}$, then $x \in G$. For $K_1 := K \setminus G$ with K as in (ii) we know that it is compact and that $x \notin K_1$. Consequently $F = \{x\} \in \mathcal{M}(K_1) \in \tau_{uF}$. By (i) there exists an $\alpha_2 \in A$ such that $F_\alpha \in \mathcal{M}(K_1)$ for all $\alpha \geq \alpha_2$. Conclude that

$$\emptyset = F_\alpha \cap K_1 = F_\alpha \cap (K \setminus G) = F_\alpha \cap K \cap G^c = F_\alpha \cap G^c \quad \forall\, \alpha \geq \alpha_0, \alpha_2,$$

because $F_\alpha \cap K = F_\alpha$ by (ii). Herewith it follows that

$$\emptyset \neq F_\alpha = (F_\alpha \cap G) \cup (F_\alpha \cap G^c) = F_\alpha \cap G \quad \forall\, \alpha \geq \alpha_0, \alpha_2$$

and thus $F_\alpha \in \mathcal{H}(G) = \mathbf{S}$ for all $\alpha \geq \alpha_3$ with some $\alpha_3 \geq \alpha_0, \alpha_2$. Summing up we arrive at $F_\alpha \to F$ in $(\mathcal{F}, \tau_F)$.

(b) It remains to prove the implication $\Rightarrow$. But this follows immediately, because every subase-neighborhhod lies in $\mathcal{S}_{uF}$ as $\emptyset \notin \mathcal{H}(G)$ for every open G. □

Notes

If E is a linear space with a metric d, then the statements in Lemmas 4-7 can be found in Salinetti and Wets [11]. More precisely, Lemma 5 is presented in (1.9) on p. 389 in the special case that A is compact, whereas we allow A to be an arbitrary subset of E. Lemma 6 is given in (1.10) on the same page, but without proof. Similarly, the statement of Lemma 4 is a little hidden in the line directly before Corollary 1.13 on p.390, again without proof. Furthermore, our Lemma 7 coincides with the just mentioned Corollary 1.13. Here, Salinetti and Wets [11] use a completely different technique to prove it, however the argument only works if E is actually a normed linear space. In addition, Lemma 8 is used in Molchanov [10] on p.7 without any justification. Finally, as far as Lemmata 9 and 10 are concerned, due to their special character, we assume that these are new findings.

Declarations

Compliance with Ethical Standards: I have read and I understand the provided information.

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