SOLID LINES IN AXIAL ALGEBRAS OF JORDAN TYPE $\frac{1}{2}$ AND JORDAN ALGEBRAS

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ABSTRACT. We show that a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ is a Jordan algebra if and only if every 2-generated subalgebra is *solid*, a notion introduced recently by Ilya Gorshkov, Sergey Shpectorov and Alexei Staroletov. As a byproduct, we show that a subalgebra generated by axes a, b is solid if and only if the associator $[L_a, L_b]$ is a derivation. Moreover, we show that 2-generated subalgebras that are not solid contain precisely 3 axes.

1. INTRODUCTION

Primitive axial algebras of Jordan type η were introduced in 2015 by John Hall, Felix Rehren and Sergey Shpectorov [8, 9]. A primitive axial algebra (A, X) of Jordan type η is a non-associative algebra A required to be generated by a set of primitive idempotents X the multiplication of which is diagonalizable with eigenvalues $0, 1, \eta$, and the eigenvectors of which multiply following a specific fusion law (see Section 2).

In [14, Conjecture 4.3], Justin McInroy and Sergey Shpectorov conjecture that the connected (in the sense of [14, Section 3.2]) primitive axial algebras of Jordan type $\eta = \frac{1}{2}$ are either Jordan algebras, or quotients of Matsuo algebras, certain algebras that are constructed from 3-transposition groups (see Definition 2.6). For 2-generated and 3-generated primitive axial algebras of Jordan type, this has been shown to be true ([7]), and 3-generated algebras are in fact all Jordan algebras. Very recently, Tom De Medts, Louis Rowen and Yoav Segev studied the 4-generated case in [4], though it is still unclear whether these 4-generated axial algebras are either Jordan algebras or quotients of Matsuo algebras.

To further advance the structure theory of primitive axial algebras of Jordan type $\eta = \frac{1}{2}$, Gorshkov, Staroletov and Shpectorov introduced the idea of *solid subalgebras* (also called *solid lines*) in [6]. These are subalgebras of an axial algebra A generated by two axes for which all primitive idempotents of this subalgebra are axes of the algebra A. The intuition behind this notion is that in a Jordan algebra, any 2-generated subalgebra is automatically solid by the Peirce decomposition [13, Section III.1, Lemma 1], while for Matsuo algebras which are not Jordan, most 2-generated subalgebras contain at most 3 axes. However, examples of non-Jordan Matsuo algebras containing both solid lines and non-solid lines do exist, and an example was found by Gorshkov and Staroletov (see [6, Example 7.2]), namely $M(3^3 : S_4)$, the Matsuo algebra constructed from the 3-transposition group $3^3 : S_4$ as in Definition 2.6.

In this paper, we prove the following three results, the first of which extends the results in [6].

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Theorem 1.1. *Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with* $\operatorname{char}\mathbb{F} \neq 2$. *Write $(\cdot,\cdot)$ for the unique Frobenius form on A. Given $a, b \in X$, the subalgebra $\langle\langle a, b\rangle\rangle$ is solid whenever $(a, b) \neq \frac{1}{4}$ or $\langle\langle a, b\rangle\rangle$ is not 3-dimensional.*

Theorem 1.2. *Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with* $\operatorname{char}\mathbb{F} \neq 2$. *The subalgebra $\langle\langle a, b\rangle\rangle$ is solid if and only if the associator $[L_a, L_b]$ is a derivation of A.*

Theorem 1.3. *Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with* $\operatorname{char}\mathbb{F} \neq 2$. *The subalgebra $\langle\langle a, b\rangle\rangle$ is solid for all $a, b \in X$ if and only if A is a Jordan algebra.*

We give an overview of the new ideas that we needed to prove these results.

The crucial step in proving Theorem 1.1, which is done in Section 4, is finding polynomials $P_{x,y}, Q_x$ (defined in Lemma 4.1) of small degree such that axes in $\langle\langle a, b\rangle\rangle$ and roots of these polynomials correspond to each other in a subtle way. Using this, we can prove that once we have enough axes in $\langle\langle a, b\rangle\rangle$, all primitive idempotents have to be axes, since then the defined polynomials would have to be identically zero. The proof given here extends the results of [6] to all 2-generated subalgebras in arbitrary characteristic, and the result is sharp in the sense that counterexamples exist in every characteristic when one of the conditions in the theorem does not hold. The method of proof in [6] is computational in nature, and requires the classification of 3-generated primitive axial algebras of Jordan type. Moreover, in [6], the techniques used for different types (see Section 3) of 2-generated subalgebras varies from case to case. We present a more conceptual and uniform method of proof, relying only on the classification of 2-generated subalgebras.

Intuitively, a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ containing solid lines is equivalent to its automorphism group scheme having positive dimension. Theorem 1.2 makes this more concrete, as a subalgebra $\langle\langle a, b\rangle\rangle \subseteq A$ is solid if and only if the associator defined by $D_{a,b}(x) := [L_a, L_b](x) = a(bx) - b(ax)$ is a derivation of A. To prove this statement (see Section 5) we continue using the class of polynomials $P_{x,y}, Q_x$ from the previous paragraph, but it is necessary to use base change techniques to obtain that $[L_a, L_b]$ is a derivation if $\langle\langle a, b\rangle\rangle$ is solid.

In characteristic zero, the converse direction is obtained by taking the exponential of the derivation, but in positive characteristic other techniques are required. By studying the roots of the polynomials $P_{x,y}, Q_x$ over a bigger base ring (corresponding to axes in the base changed algebra), we obtain more information about the multiplicity of roots over the ground field. Using this, we can prove Theorem 1.2.

Rings A with $2 \in A^\times$ such that $[L_x, L_y]$ is a derivation for all $x, y \in A$ are called *almost Jordan rings*, a less well known concept that was first studied by Marshall Osborn [12, 16, 17]. Now, it turns out that an almost Jordan algebra spanned by primitive idempotents will always be a Jordan algebra. We then combine Theorems 1.1 and 1.2 to finally obtain Theorem 1.3 by an induction argument, see Section 6.

The methods used to prove Theorem 1.3 allow us to interpret Theorem 1.2 in an interesting way. It tells us that the subspace of $D_{x,y}$ with $x, y \in A$ such that $D_{x,y}$ is a derivation, quantifies in a concrete way how close the algebra is to being Jordan. In contrast, subalgebras $\langle\langle a, b\rangle\rangle$ for which $D_{a,b}$ is not a derivation always correspond to two involutions the product of which has order 3. This shows in a very concrete way the dichotomy between Jordan and Matsuo algebras. We believe these results give a better idea of the general structure of primitive axial algebras of Jordan type $\eta = \frac{1}{2}$, and will lead to more examples of primitive axial algebras of Jordan type $\eta = \frac{1}{2}$.

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2. Preliminaries

The first ingredient we need to define axial algebras is the concept of a *fusion law*.

Definition 2.1. A fusion law $(\mathcal{F}, \star)$ is a set $\mathcal{F}$ with a map $\star\colon \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$. Here $2^{\mathcal{F}}$ denotes the set of all subsets of $\mathcal{F}$.

A fusion law can be denoted using a multiplication table, in the same way any binary operation can be. As a convention, if the product of $a, b \in \mathcal{F}$ is the empty set, we leave the entry empty. We also omit the set brackets $\{\}$ in the entries. The fusion law we will be interested in is the Jordan type fusion law $\mathcal{J}(\eta)$ (Table 1) with parameter $\eta = \frac{1}{2}$. Using

$\star$	1	0	η
1	1		η
0		0	η
η	η	η	$0, 1$

Table 1. The Jordan fusion law $\mathcal{J}(\eta)$

these fusion laws, we can now define *axial algebras*.

Definition 2.2. Let $\mathcal{F}$ be a fusion law and A a commutative non-associative R-algebra, where R is a unital, associative, commutative ring.

(i) For $a \in A$, let L_a denote the endomorphism of A defined by $L_a(x) = ax$ for all $x \in A$. If $\lambda \in R$ is an eigenvalue of L_a, the λ-eigenspace will be denoted by $A_\lambda(a)$.

(ii) An idempotent $a \in A$ is an $\mathcal{F}$-axis if L_a is semisimple, $\mathrm{Spec}(L_a) \subseteq \mathcal{F}$ and for all $\lambda, \mu \in \mathrm{Spec}(L_a)$:
$$A_\lambda(a)A_\mu(a) \subseteq \bigoplus_{\nu \in \lambda \star \mu} A_\nu(a).$$
An $\mathcal{F}$-axis is *primitive* if $A_1(a) = \langle a \rangle$.

(iii) (A, X) is a (primitive) $\mathcal{F}$-axial algebra if $X \subset A$ is a set of (primitive) $\mathcal{F}$-axes that generate A. If $|X| = k \in \mathbb{N}$, we will call (A, X) a k-generated $\mathcal{F}$-axial algebra.

We will often also write A instead of (A, X) for convenience. In what follows, we are interested in primitive $\mathcal{J}(\frac{1}{2})$-axial algebras. We will also call these algebras *primitive axial algebras of Jordan type* $\eta = \frac{1}{2}$.

Note that we defined axial algebras over rings instead of over fields. We only do this because we want to make some arguments which require base change, while still retaining the same language, see for example Proposition 5.1.

Properties of fusion laws can lead to properties of axial algebras, as is the case for the Seress property.

Definition 2.3 (Seress property)**.** A fusion law $\mathcal{F}$ is called Seress if both $0, 1 \in \mathcal{F}$ and for every $\lambda \in \mathcal{F}$ we have $1 \star \lambda, 0 \star \lambda \subseteq \{\lambda\}$.

Lemma 2.4 (Seress lemma)**.** *Let A be an axial algebra with fusion law $\mathcal{F}$. If $\mathcal{F}$ is Seress, then every axis a associates with $A_1(a) \oplus A_0(a)$.*

Proof. See [9, Proposition 3.9]. □

What makes the Jordan fusion law special and interesting to study is its connection to group theory, as we will see now.

Definition 2.5 ([5, Definition 2.5])**.**

(i) A fusion law $(\mathcal{F}, \star)$ is called $\mathbb{Z}/2\mathbb{Z}$-*graded* if $\mathcal{F}$ can be partitioned into two subsets $\mathcal{F}_+$ and $\mathcal{F}_-$ such that

$$\lambda \star \mu \subseteq \mathcal{F}_+ \text{ whenever } \lambda, \mu \in \mathcal{F}_+,$$
$$\lambda \star \mu \subseteq \mathcal{F}_+ \text{ whenever } \lambda, \mu \in \mathcal{F}_-,$$
$$\lambda \star \mu \subseteq \mathcal{F}_- \text{ whenever } \lambda \in \mathcal{F}_+ \text{ and } \mu \in \mathcal{F}_- \text{ or } \lambda \in \mathcal{F}_- \text{ and } \mu \in \mathcal{F}_+.$$

(ii) Let (A, X) be a $(\mathcal{F}, \star)$-axial algebra for some $\mathbb{Z}/2\mathbb{Z}$-graded fusion law $(\mathcal{F}, \star)$. We associate to each $(\mathcal{F}, \star)$-axis a of A a *Miyamoto involution* $\tau_a \in \operatorname{Aut}(A)$ defined by linearly extending

$$x^{\tau_a} = \begin{cases} x & \text{if } x \in A_{\mathcal{F}_+}(a), \\ -x & \text{if } x \in A_{\mathcal{F}_-}(a). \end{cases}$$

Because of the $\mathbb{Z}/2\mathbb{Z}$-grading of the fusion rule, these maps define automorphisms of A.

(iii) We call the subgroup $\langle \tau_e \mid e \in X \rangle \leq \operatorname{Aut}(A)$ the *Miyamoto group* of the axial algebra (A, X), and we denote it by $\operatorname{Miy}(A, X)$.

Other than Jordan algebras, the most basic example of Jordan type axial algebras are so-called *Matsuo algebras*. They arise from 3-transposition groups, objects that have been studied extensively by Jonathan Hall and Hans Cuypers among others. For more information on 3-transposition groups, see [2, 11].

Definition 2.6. (i) A 3-*transposition group* (G, D) is a group G generated by a conjugacy class of involutions D such that the product of any two elements in D has order at most 3.

(ii) Given a field $\mathbb{F}$, a 3-transposition group (G, D) and $\eta \in \mathbb{F}$, we write $M_\eta(\mathbb{F}, (G, D))$ for the algebra $\mathbb{F}D$ with multiplication

$$a \cdot b := \begin{cases} a & \text{if } a = b, \\ 0 & \text{if } o(ab) = 2, \\ \frac{\eta}{2}(a + b - a^b) & \text{if } o(ab) = 3, \end{cases}$$

where $a^b := bab \in D$ is the conjugate of a by b in the group G. When all parameters are clear from context, we also write $M(G)$ for $M_\eta(\mathbb{F}, (G, D))$.

(iii) We call direct sums of the algebras above *Matsuo algebras.*

It is easy to check that Matsuo algebras (M, D) with parameter η are primitive axial algebras with respect to the fusion law $\mathcal{J}(\eta)$. In fact, whenever $\eta \neq \frac{1}{2}$, these are essentially the only examples of primitive $\mathcal{J}(\eta)$-axial algebras [8, Theorem 1.3]. In [5, Section 4], it was shown that the Miyamoto group of a Matsuo algebra returns the group you started with, quotiented by a central subgroup.

When $\eta = \frac{1}{2}$, the situation is much less clear. Shpectorov, Gorshkov and Staroletov introduced the notion of solidness of 2-generated subalgebras of a primitive $\mathcal{J}(\frac{1}{2})$-axial algebra as a tool to help classify these algebras. For convenience, we will sometimes call 2-generated subalgebras *lines.*

Definition 2.7. Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over a ring R with $2 \in R^{\times}$. For any $a \neq b \in X$ we will call the 2-generated subalgebra $B = \langle\langle a, b \rangle\rangle$ *solid* if every primitive idempotent in B is a $\mathcal{J}(\frac{1}{2})$-axis for A.

Proposition 2.8. *Any 2-generated subalgebra in a Jordan algebra is solid.*

Proof. Any idempotent in a Jordan algebra satisfies the Pierce decomposition [13, Section III.1, Lemma 1]. This implies that any idempotent satisfies the $\mathcal{J}(\frac{1}{2})$ fusion law, and thus any 2-generated subalgebra is automatically solid. □

The last tool that we will need is the existence of a *Frobenius form*, i.e. a bilinear form on A that associates with the product.

Definition 2.9. A bilinear form $(\cdot, \cdot)$ on an algebra A is a Frobenius form if

$$(a, b \cdot c) = (a \cdot b, c)$$

for all $a, b, c \in A$.

Lemma 2.10. *Given a primitive axial algebra (A, X) of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$, there always exists a unique Frobenius form $(\cdot, \cdot)$ on A such that $(a, a) = 1$ for every $a \in X$. Moreover, $(\cdot, \cdot)$ is invariant under $\operatorname{Aut}(A)$.*

Proof. This is [10, Theorem 4.1 and Lemma 4.3]. □

From here on, we will use $(\cdot, \cdot)$ as notation for the Frobenius form (normalized as above) of a given axial algebra (A, X) of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$. We immediately have some basic computations which will be used throughout.

Lemma 2.11. *Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over a ring R with $2 \in R^{\times}$, with Frobenius form $(\cdot, \cdot)$ such that $(a, a) = 1$ for all $a \in X$. Let $x \in A$ and $a \in X$ an axis. Then we have*

(i) $ax = \frac{1}{4}(x - x^{\tau_a}) + (a, x)a$,
(ii) $x^{\tau_a} = x + 4(a, x)a - 4ax$,
(iii) $a(ax) = \frac{1}{2}(ax + (a, x)a)$.

Proof. As a is an axis, we can write $x = x_0 + x_1 + x_{1/2}$ in a unique way as a sum of eigenvectors of L_a, where $x_\lambda \in A_\lambda(a)$. Then $x^{\tau_a} = x_0 + x_1 - x_{1/2}$, so $x_{1/2} = \frac{1}{2}(x - x^{\tau_a})$. Moreover since a is primitive, we have $x_1 = \lambda a$ for some $\lambda \in R$, and $(a, x) = (a, x_1) = \lambda$. Thus $x = x_0 + (a, x)a + \frac{1}{2}(x - x^{\tau_a})$. This implies $ax = \frac{1}{4}(x - x^{\tau_a}) + (a, x)a$. Clearly (i),(ii) and (iii) now follow. □

Lastly, we will need the associators alluded to in the introduction.

Definition 2.12. Let A be a commutative algebra over a field $\mathbb{F}$, and $a, b \in A$. We define the map $D_{a,b} \colon A \to A$ by $D_{a,b}(x) = a(bx) - b(ax)$. Clearly we have $D_{a,b} = [L_a, L_b]$, the associator of a and b.

3. 2-generated subalgebras

We will give a quick review of 2-generated axial algebras of Jordan type $\eta = \frac{1}{2}$. For more information on this topic, we refer to [6, 8]. Any 2-generated primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ is a quotient of a 3-dimensional algebra, and we can classify them by the value of the Frobenius form (a, b), where a and b are generating axes of the 2-generated axial algebra B [8, Theorem 4.6]. They come in three different flavors: toric, baric and flat lines. We discuss each of them separately, and review some facts we will use.

Definition 3.1. Let $B = \langle\!\langle a, b \rangle\!\rangle$ be a 2-generated primitive axial algebra of Jordan type $\frac{1}{2}$, and denote by $(,)$ its unique Frobenius form from Lemma 2.10.

(i) If $(a, b) \neq 0, 1$ then we call B *toric*,
(ii) if $(a, b) = 0$ then we call B *flat*,
(iii) if $(a, b) = 1$ then we call B *baric*.

Moreover, we collectively call the flat and baric algebras *unipotent*.

One major distinction between toric lines and flat or baric lines is the structure of their automorphism groups. In the toric case, the identity component of the automorphism group will be a multiplicative group. In contrast, for baric or flat algebras, the identity component of their automorphism group is unipotent, hence the naming conventions.

3.1. **Toric algebras.** In this subsection, we describe toric algebras, the variety of primitive idempotents in toric algebras and the action of the Miyamoto involutions.

Lemma 3.2. *Let $B = \langle\!\langle a, b \rangle\!\rangle$ be a toric algebra over an algebraically closed field $\mathbb{F}$. Then B is a 3-dimensional simple unital Jordan algebra (with unit u), and there exists a basis $\{e, u, f\}$ of B with $e^2 = f^2 = 0$, $ef = \frac{1}{8}u$, $a = e + \frac{1}{2}u + f$ and $b = \mu e + \frac{1}{2}u + \mu^{-1} f$ for a certain $\mu \in \mathbb{F}^\times$.*

Proof. See [6, Lemma 3.7, Lemma 3.8]. Note that we can always rescale e, f such that $a = e + \frac{1}{2}u + f$. □

Definition 3.3. For $\lambda \in \mathbb{F}^\times$, we write $a_\lambda = \lambda e + \frac{1}{2}u + \lambda^{-1} f$.

Lemma 3.4. *Any primitive idempotent in the toric algebra $\langle\!\langle a, b \rangle\!\rangle$ is of the form a_λ.*

Proof. See [6, Lemma 3.5]. □

Lastly, we need to know a bit about the action of the Miyamoto involutions on the primitive idempotents. Note that every primitive idempotent satisfies the Jordan fusion law, since $\langle\!\langle a, b \rangle\!\rangle$ is Jordan.

Lemma 3.5. *Let τ_{a_μ} be the Miyamoto involution corresponding to the axis a_μ. Then*

$$a_\lambda^{\tau_{a_\mu}} = a_{\lambda^{-1}\mu^2} \text{ for all } \lambda, \mu \in \mathbb{F}^\times.$$

Proof. This follows from [6, Lemma 3.9]. □

Corollary 3.6. *Let $\langle\!\langle a, b \rangle\!\rangle$ be toric. Write $G = \langle \tau_a, \tau_b \rangle$. The union of orbits $O = a^G \cup b^G$ has size $n < \infty$ if and only if μ is an n-th root of unity, where μ is as in Lemma 3.2.*

Proof. Note that $a = a_1$ and $b = a_\mu$. Given $k \in \mathbb{Z}$, we get that $a^{(\tau_a \tau_b)^k} = a_{\mu^{2k}} \in O$ and $b^{(\tau_a \tau_b)^k} = a_{\mu^{2k+1}} \in O$. It is then easy to see that $O = \{a_{\mu^k} \in \langle\!\langle a, b \rangle\!\rangle \mid k \in \mathbb{Z}\}$. For this set to be finite, it is both sufficient and necessary that $\mu^n = 1$ for a certain $n \in \mathbb{N}$, and if n is minimal with respect to this property, then $|O| = n$. □

Corollary 3.7. *Let $\langle\!\langle a, b \rangle\!\rangle$ be toric. Write $G = \langle \tau_a, \tau_b \rangle$. If $O = a^G \cup b^G$ has size 3, then $(a, b) = \frac{1}{4}$.*

Proof. We know from Corollary 3.6 that in this case, $a = e + \frac{1}{2}u + f, b = \mu e + \frac{1}{2}u + \mu^2 f$, where μ is a primitive third root of unity. Using the description of the Frobenius form in [6, p.12-13], we get that $(a, b) = \frac{1}{4}(\mu + \mu^2) + \frac{1}{2}$. Since μ is a primitive third root of unity, we have $\mu^2 + \mu = -1$, and thus $(a, b) = \frac{1}{4}$. □

3.2. **Flat algebras.** In this subsection, we describe flat algebras, the variety of primitive idempotents in flat algebras and the action of the Miyamoto involutions.

Lemma 3.8. *Let $B = \langle\langle a, b\rangle\rangle$ be a flat algebra over a field $\mathbb{F}$. Then B is Jordan and either*

(i) *a 3-dimensional algebra, denoted $\mathfrak{J}(0)$, with basis $\{a, b, v = ab\}$,*
(ii) *or a 2-dimensional quotient of this algebra, isomorphic to $\mathbb{F} \oplus \mathbb{F}$.*

Proof. See [6, Theorem 3.2(a)]. □

Definition 3.9. For $\lambda \in \mathbb{F}$, we write $a_\lambda = a + \lambda v$ and $b_\lambda = b + \lambda v$ with $v = ab$.

Lemma 3.10. *Any primitive idempotent in the flat algebra $\langle\langle a, b\rangle\rangle$ is of the form a_λ or b_λ.*

Proof. See [6, Proposition 6.6]. The calculations were made for $\operatorname{char} \mathbb{F} = 0$, but remain true for $\operatorname{char} \mathbb{F} \neq 2$. □

Again, we describe the action of the Miyamoto involution on the primitive idempotents.

Lemma 3.11. *Let $\tau_{a_\mu}, \tau_{b_\mu}$ be the Miyamoto involution corresponding to the axis a_μ, b_μ respectively. Then*

$$a_\lambda^{\tau_{a_\mu}} = a_{2\mu-\lambda} \text{ and } a_\lambda^{\tau_{b_\mu}} = a_{-4-2\mu-\lambda} \text{ for all } \lambda, \mu \in \mathbb{F}.$$

Proof. Note that $v = ab$ is a $\frac{1}{2}$-eigenvector for a and b and $v^2 = 0$ by [6, Theorem 3.1]. We can then compute using Lemma 2.11 that

$$a_\lambda^{\tau_{a_\mu}} = a_\lambda + 4(a_\lambda, a_\mu)a_\mu - 4a_\lambda a_\mu = a_{\lambda+4\mu-2\lambda-2\mu}$$

and

$$a_\lambda^{\tau_{b_\mu}} = a_\lambda + 4(a_\lambda, b_\mu)b_\mu - 4a_\lambda b_\mu = a_{\lambda-4-2\lambda-2\mu}.$$

□

Corollary 3.12. *Suppose $\langle\langle a, b\rangle\rangle$ is flat and not 2-dimensional. Write $G = \langle \tau_a, \tau_b\rangle$. The orbit $O = a^G$ has size $n < \infty$ if and only if $2 \neq \operatorname{char} \mathbb{F} = p > 0$, and then $|O| = p$.*

Proof. Note that $a = a_0$ and $b = b_0$. Given $k \in \mathbb{Z}$, we get that $a^{(\tau_a\tau_b)^k} = a_{-4k} \in O$. It is then easy to see that $O = \{a_{4k} \in \langle\langle a, b\rangle\rangle \mid k \in \mathbb{Z}\}$. For this set to be finite, it is both sufficient and necessary that the characteristic of the field is positive, and since $\operatorname{char} \mathbb{F} = p \neq 2$, we then have that $|\{a_k | k \in \{0, \dots, p-1\}\}| = p$. □

3.3. **Baric algebras.** In this subsection, we describe baric algebras, the variety of primitive idempotents in baric algebras and the action of the Miyamoto involutions.

Lemma 3.13. *Let $B = \langle\langle a, b\rangle\rangle$ be a baric algebra over a field $\mathbb{F}$. Then B is Jordan and either*

(i) *a 3-dimensional algebra, denoted $\mathfrak{J}(1)$, with basis $\{a, v = 2(ab - a), v^2\}$,*
(ii) *a 2-dimensional quotient of this algebra, denoted $\overline{\mathfrak{J}(1)}$, with basis $\{a, v = 2(ab-a)\}$,*
(iii) *or a 1-dimensional quotient.*

Proof. See [6, Theorem 3.2(b)]. □

Definition 3.14. For $\lambda \in \mathbb{F}$, we write $a_\lambda = a + \lambda v + \lambda^2 v^2$ with $v = 2(ab - a)$.

Lemma 3.15. *Any primitive idempotent in the flat algebra $\langle\langle a, b\rangle\rangle$ is of the form a_λ.*

Proof. See [6, Proposition 6.4]. The calculations were made for $\operatorname{char} \mathbb{F} = 0$, but remain true for $\operatorname{char} \mathbb{F} \neq 2$. □

We describe the action of the Miyamoto involution on the primitive idempotents one final time.

Lemma 3.16. *Let τ_{a_μ} be the Miyamoto involution corresponding to the axis a_μ. Then*

$$a_\lambda^{\tau_{a_\mu}} = a_{2\mu-\lambda} \text{ for all } \lambda, \mu \in \mathbb{F}.$$

Proof. Note that $v = 2(ab - a)$ is a $\frac{1}{2}$-eigenvector for a and v^2 is a 0-eigenvector for a by [6, Theorem 3.1]. We can then compute using Lemma 2.11 that

$$a_\lambda^{\tau_{a_\mu}} = a_\lambda + 4(a_\lambda, a_\mu)a_\mu - 4a_\lambda a_\mu = a_{\lambda+4\mu-2\lambda-2\mu}. \qquad \square$$

Corollary 3.17. *Suppose $\langle\langle a, b\rangle\rangle$ is baric and not 1-dimensional. Write $G = \langle \tau_a, \tau_b\rangle$. The orbit $O = a^G$ has size $n < \infty$ if and only if $2 \neq \operatorname{char}\mathbb{F} = p > 0$, and then $|O| = p$.*

Proof. Note that $a = a_0$ and $b = a_1$. Given $k \in \mathbb{Z}$, we get that $a^{(\tau_a\tau_b)^k} = a_{2k} \in O$. It is then easy to see that $O = \{a_{2k} \in \langle\langle a, b\rangle\rangle \mid k \in \mathbb{Z}\}$. For this set to be finite, it is both sufficient and necessary that the characteristic of the field is positive, and since $\operatorname{char}\mathbb{F} = p \neq 2$, we then have that $|\{a_k | k \in \{0, \ldots, p-1\}\}| = p$. $\square$

4. Proof of Theorem 1.1

Even though a large portion of this theorem was already proved in [6], we give an alternative proof here. This proof uses the same techniques for all types of lines, and it does not use the classification of 3-generated primitive axial algebras of Jordan type [7]. To prove this fact, we will try to find polynomials of small degree which tell us when a subalgebra is solid (Lemma 4.1), and then prove they have many zeroes in most cases, which forces the polynomials to be identically zero. For the remainder of this section, (A, X) is a primitive axial algebra of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$ (assumed algebraically closed, without loss of generality) with $\operatorname{char}\mathbb{F} \neq 2$, and $a, b \in X$ are axes.

Lemma 4.1. *Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$, and $a, b \in X$. The subalgebra $\langle\langle a, b\rangle\rangle$ is solid if and only if*

$$Q_x(c) := c(cx) - \tfrac{1}{2}(cx + (c, x)c)$$

and

$$P_{x,y}(c) := 4(cx)(cy) - (c, y)cx - (cy)x - (c, x)cy - (cx)y - (c, xy)c + c(xy)$$

are zero for all primitive idempotents $c \in \langle\langle a, b\rangle\rangle$ and $x, y \in A$. Moreover, if $c \in \langle\langle a, b\rangle\rangle$ satisfies the Jordan fusion law, then $Q_x(c) = P_{x,y}(c) = 0$ for all $x, y \in A$.

Proof. Given a primitive idempotent $c \in \langle\langle a, b\rangle\rangle$, let $\phi_c \colon A \to A$ be the map defined by $x^{\phi_c} = x + 4(c, x)c - 4cx$.

Suppose $P_{x,y}(c) = 0 = Q_x(c)$ for all $x, y \in A, c \in \langle\langle a, b\rangle\rangle$. We will show ϕ_c is an automorphism of A for every primitive idempotent $c \in \langle\langle a, b\rangle\rangle$. For every $x, y \in A$ we have

$$\begin{aligned} x^{\phi_c}y^{\phi_c} - (xy)^{\phi_c} &= (x + 4(c, x)c - 4cx)(y + 4(c, y)c - 4cy) - (xy + 4(c, xy)c - 4c(xy)) \\ &= 4(c, y)cx - 4(cy)x + 4(c, x)cy + 16(c, x)(c, y)c - 16(c, x)c(cy) \\ &\quad - 4(cx)y - 16(c, y)c(cx) + 16(cx)(cy) - (xy + 4(c, xy)c - 4c(xy)) \end{aligned} \tag{1}$$

Since $Q_x(c) = Q_y(c) = 0$ this last expression is equal to $4P_{x,y}(c)$, which is equal to zero. This means ϕ_c is an automorphism for every primitive idempotent $c \in \langle\langle a, b\rangle\rangle$. But $\{a, b\}^{\langle \phi_c | c \in \mathbb{F}\rangle}$ is equal to the variety of all primitive idempotents in $\langle\langle a, b\rangle\rangle$ by Lemmas 3.4, 3.5, 3.10, 3.11, 3.15 and 3.16, which means the line is solid.

Conversely, if c is an axis, then $Q_x(c) = 0$ for every $x \in A$ by Lemma 2.11, and then (1) shows $P_{x,y}(c) = 0$ for every $x, y \in A$, since $\phi_c = \tau_c$ is an automorphism of A. □

4.1. **The toric case.** We will write $a_\lambda = \lambda e + \frac{1}{2}u + \lambda^{-1} f$, where $\lambda \in \mathbb{F}^\times$ and e, u, f as in Lemma 3.2, chosen with respect to the generating axes a, b of $\langle\langle a, b \rangle\rangle$.

Definition 4.2. Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over $\mathbb{F}$, and $a, b \in X$ such that $\langle\langle a, b \rangle\rangle$ is toric. Let R be a unital commutative associative $\mathbb{F}$-algebra, and $\lambda \in R^\times$ an invertible element.

(i) We define $a_\lambda := \lambda e + \frac{1}{2}u + \lambda^{-1} f \in \langle\langle a, b \rangle\rangle_R$.
(ii) Given $x, y \in A_R$, we define $P^t_{x,y}$ and Q^t_x as the maps

$$P^t_{x,y} \colon R^\times \to A \colon \lambda \mapsto P_{x,y}(a_\lambda) \text{ and } Q^t_x \colon R^\times \to A \colon \lambda \mapsto P_{x,y}(a_\lambda).$$

Note that the maps $P^t_{x,y}, Q^t_{x,y}$ are rational maps, but $\lambda^2 P^t_{x,y}, \lambda^2 Q^t_{x,y} \in A[\lambda]$ are polynomial maps, since $a_\lambda = \lambda^{-1}(\lambda^2 e + \lambda \frac{1}{2} u + f)$.

Proposition 4.3. *Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ and $a, b \in X$ with $(a, b) \neq 0, 1, \frac{1}{4}$. Then $\langle\langle a, b \rangle\rangle$ is solid.*

Proof. Since $(a, b) \neq 0, 1$, we know $\langle\langle a, b \rangle\rangle$ is toric, and $\langle\langle a, b \rangle\rangle$ contains at least 4 axes by Corollary 3.7. The polynomial maps $\lambda^2 P^t_{x,y}, \lambda^2 Q^t_x$ have degree less than or equal to 4 for all $x, y \in A$. If $\langle\langle a, b \rangle\rangle$ has at least 5 different axes, then the polynomials $\lambda^2 P^t_{x,y}, \lambda^2 Q^t_x$ have at least 5 different zeroes, and hence are zero. So for every $\lambda \in \mathbb{F}^\times$ and $x, y \in A$, we have that $P^t_{x,y}(\lambda) = Q^t_x(\lambda) = 0$, and by Lemma 4.1, the subalgebra $\langle\langle a, b \rangle\rangle$ is solid.

Now suppose $\langle\langle a, b \rangle\rangle$ contains only 4 axes. Then these have to be of the form a_μ, with $\mu^4 = 1$ by Corollary 3.6. We write

$$\lambda^2 P^t_{x,y}(\lambda) = a_0(x, y) + a_1(x, y)\lambda + a_2(x, y)\lambda^2 + a_3(x, y)\lambda^3 + a_4(x, y)\lambda^4$$

and

$$\lambda^2 Q^t_x(\lambda) = b_0(x) + b_1(x)\lambda + b_2(x)\lambda^2 + b_3(x)\lambda^3 + b_4(x)\lambda^4$$

for every $x, y \in A$. Then one can compute

$$\begin{aligned}
a_4(x, y) &= 4(ex)(ey) - (e, x)ey - (e, y)ex - (e, xy)e, \\
a_3(x, y) &= 2(ex)(uy) + 2(ux)(ey) - \tfrac{1}{2}(e, y)ux - \tfrac{1}{2}(u, y)ex \\
&\quad - (ey)x - \tfrac{1}{2}(e, x)uy - \tfrac{1}{2}(u, x)ey - (ex)y - \tfrac{1}{2}(e, xy)u - \tfrac{1}{2}(u, xy)e + e(xy), \\
a_0(x, y) &= 4(fx)(fy) - (f, x)fy - (f, y)fx - (f, xy)f, \\
b_4(x) &= e(ex) - \tfrac{1}{2}(e, x)e, \\
b_3(x) &= \tfrac{1}{2}e(ux) + \tfrac{1}{2}u(ex) - \tfrac{1}{2}ex - \tfrac{1}{4}(e, x)u - \tfrac{1}{4}(u, x)e.
\end{aligned}$$

Note that $a_3(x, e) = 2b_4(x)$ and $a_3(x, u) = 4b_3(x) + e(ux) - u(ex)$. For all $x, y \in A$, the polynomial maps $\lambda^2 P^t_{x,y}, \lambda^2 Q^t_x$ have $\pm 1, \pm i$ as zeroes, where i is a primitive fourth root of unity by Corollary 3.6 and Lemma 4.1. Any polynomial of degree at most 4 with fourth roots of unity as roots is a scalar multiple of $x^4 - 1$. This implies that $a_3(x, y) = 0$ for all $x, y \in A$, and thus $b_4(x) = 0$ for all $x \in A$. So $Q^t_x(\lambda) = 0$ for all $\lambda \in \mathbb{F}^\times$.

We still need to prove that $a_4(x, y) = 0$ for all $x, y \in A$. Since $a_3(x, u) = b_3(x) = 0$ for all $x \in A$, we also have $e(ux) = u(ex) = \frac{1}{2}ex + \frac{1}{4}((e, x)u + (u, x)e)$ for all $x \in A$. Given

$x \in A$, we compute

$$\begin{aligned}
(2) \quad e(x^{\tau_a}) &= e(x + 4(a,x)a - 4ax) \\
&= ex + 2(a,x)e + \tfrac{1}{2}(a,x)u - 4e(ex) - 2e(ux) - 4e(fx) \\
&= ex + 2(e,x)e + (u,x)e + 2(f,x)e + \tfrac{1}{2}(e,x)u + \tfrac{1}{4}(u,x)u \\
&\quad + \tfrac{1}{2}(f,x)u - 2(e,x)e - ex - \tfrac{1}{2}(e,x)u - \tfrac{1}{2}(u,x)e - 4e(fx) \\
&= -4e(fx) + \tfrac{1}{2}(u,x)e + 2(f,x)e + \tfrac{1}{4}(u,x)u + \tfrac{1}{2}(f,x)u,
\end{aligned}$$

proving that $e(fx) = -\frac{1}{4}e(x^{\tau_a}) + \frac{1}{2}(f,x)e + \frac{1}{8}(u,x)e + \frac{1}{8}(f,x)u + \frac{1}{16}(u,x)u$.

Next, we show that $a_4(fx,y) = 0$ for all $x, y \in A$:

$$\begin{aligned}
a_4(fx,y) = -a_0(fx,y) &= -4(f(fx))(fy) + (f,fx)fy + (f,y)f(fx) + (f,(fx)y)f \\
&= -2(f,x)f(fy) + \tfrac{1}{2}(f,y)(f,x)f + \tfrac{1}{2}(f,y)(f,x)f \\
&= -(f,y)(f,x)f + (f,y)(f,x)f = 0.
\end{aligned}$$

On the other hand, using Equation (2), we get

$$\begin{aligned}
a_4(fx,y) &= 4(e(fx))(ey) - (e,fx)ey - (e,y)e(fx) - (e,(fx)y)e \\
&= -(e(x^{\tau_a}))(ey) + \tfrac{1}{2}(u,x)e(ey) + 2(f,x)e(ey) + \tfrac{1}{4}(u,x)u(ey) + \tfrac{1}{2}(f,x)u(ey) - \tfrac{1}{8}(u,x)ey \\
&\quad + \tfrac{1}{4}(e,y)e(x^{\tau_a}) - \tfrac{1}{8}(e,y)(u,x)e - \tfrac{1}{2}(e,y)(f,x)e - \tfrac{1}{16}(e,y)(u,x)u - \tfrac{1}{8}(e,y)(f,x)u \\
&\quad + \tfrac{1}{4}(e(x^{\tau_a}),y)e - \tfrac{1}{8}(u,x)(e,y)e - \tfrac{1}{2}(f,x)(y,e)e - \tfrac{1}{16}(u,x)(u,y)e - \tfrac{1}{8}(f,x)(u,y)e \\
&= -(e(x^{\tau_a}))(ey) + \tfrac{1}{4}(e,y)e(x^{\tau_a}) + \tfrac{1}{4}(e,x^{\tau_a}y)e + \tfrac{1}{4}(f,x)ey \\
&\qquad + \left(\tfrac{1}{2}(u,x) + 2(f,x)\right)\left(e(ey) - \tfrac{1}{2}(e,y)e\right) \\
&\qquad + \left(\tfrac{1}{4}(u,x) + \tfrac{1}{2}(f,x)\right)\left(u(ey) - \tfrac{1}{2}ey - \tfrac{1}{4}((e,y)u + (u,y)e)\right) \\
&= -(e(x^{\tau_a}))(ey) + \tfrac{1}{4}(e,y)e(x^{\tau_a}) + \tfrac{1}{4}(e,x^{\tau_a}y)e + \tfrac{1}{4}(e,x^{\tau_a})ey \\
&= -\tfrac{1}{4}a_4(x^{\tau_a},y).
\end{aligned}$$

This implies $a_4(x^{\tau_a},y) = 0$ for all $x, y \in A$, and since τ_a is a bijection, we get $a_4(x,y) = 0$ for all $x, y \in A$. This in turn implies that $\lambda^2 P^t_{x,y}$ is the zero polynomial, so $P^t_{x,y}(\lambda) = 0$ for all $\lambda \in \mathbb{F}^\times$. We can now use Lemma 4.1 to show that $\langle\langle a,b\rangle\rangle$ is solid. $\square$

Remark 4.4. When $\operatorname{char}\mathbb{F} \neq 3$, subalgebras $\langle\langle a,b\rangle\rangle$ with $(a,b) = \frac{1}{4}$ are not always solid. If that were the case, any Matsuo algebra would be a Jordan algebra by Theorem 1.3 and Proposition 4.6, since in a Matsuo algebra $M(G,D)$, the value of the Frobenius form (a,b) is either 0 or $\frac{1}{4}$ for all $a, b \in D$. But this would contradict [3]. The smallest example of an axial algebra containing non-solid lines in every characteristic is $M(W(D_4))$, the 12-dimensional Matsuo algebra coming from the Weyl group of a root system of type D_4.

4.2. **The unipotent case.** In [6, Section 6] it was proven that $\langle\langle a,b\rangle\rangle$ is solid for $(a,b) = 0, 1$ when $\operatorname{char}\mathbb{F} = 0$. We will extend this result to positive characteristic.

Definition 4.5. Let (A,X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over $\mathbb{F}$, and $a, b \in X$ such that $\langle\langle a,b\rangle\rangle$ is unipotent. Let R be a unital commutative associative $\mathbb{F}$-algebra, and $\lambda \in R$.

(i) We define $a_\lambda := a + \lambda v + \lambda^2 v^2 \in B_R$, where $v = ab, 2(ab - a)$ when $(a,b) = 0, 1$ respectively.
(ii) Given $x, y \in A_R$, we define $P^u_{x,y}$ and Q^u_x as the maps

$$P^u_{x,y} \colon R \to A \colon \lambda \mapsto P_{x,y}(a_\lambda) \text{ and } Q^u_x \colon R \to A \colon \lambda \mapsto P_{x,y}(a_\lambda).$$

Before we state the following proposition, recall the notations for the different possible 2-generated algebras from Lemmas 3.8 and 3.13.

Proposition 4.6. *Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$ and $a, b \in X$ with $\langle\!\langle a, b\rangle\!\rangle \not\cong \mathfrak{J}(1)$, or $\operatorname{char} \mathbb{F} \neq 3$ and $\langle\!\langle a, b\rangle\!\rangle \cong \mathfrak{J}(1)$. Then $\langle\!\langle a, b\rangle\!\rangle$ is solid.*

Proof. Suppose first that $\langle\!\langle a, b\rangle\!\rangle \not\cong \mathfrak{J}(1)$. If $\langle\!\langle a, b\rangle\!\rangle$ is isomorphic to $\mathbb{F}\oplus\mathbb{F}$, the proposition is true since the only primitive idempotents in $\langle\!\langle a, b\rangle\!\rangle$ are a and b. Suppose $\langle\!\langle a, b\rangle\!\rangle \cong \mathfrak{J}(0)$ or $\overline{\mathfrak{J}(1)}$. Then $\langle\!\langle a, b\rangle\!\rangle$ contains at least 3 different axes a_λ by Corollaries 3.12 and 3.17, so the polynomial equations $P^u_{x,y}(\lambda)$ and $Q^u_x(\lambda)$ from Lemma 4.1 have at least 3 different roots for $x, y \in A$. However, $\deg P^u_{x,y}, \deg Q^u_x \leq 2$, so this is only possible when $P^u_{x,y}, Q^u_x$ are identically zero for every $x, y \in A$. For flat lines, we also have that $P_{x,y}(b-\lambda v), Q_x(b-\lambda v)$ are zero for all $x, y \in A, \lambda \in \mathbb{F}$ by symmetry. The result now follows from Lemma 4.1.

If $\langle\!\langle a, b\rangle\!\rangle \cong \mathfrak{J}(1)$ and $\operatorname{char} \mathbb{F} \neq 3$, then the polynomial equations $P^u_{x,y}(\lambda)$ and $Q^u_x(\lambda)$ from Lemma 4.1 have at least 5 different zeroes for $x, y \in A$ by Lemma 4.1 and Corollary 3.17, and $\deg P^u_{x,y}, \deg Q^u_x \leq 4$. Then the result again follows from Lemma 4.1. $\square$

Remark 4.7. Over fields of characteristic 3, subalgebras isomorphic to $\mathfrak{J}(1)$ are not always solid! If that were the case, any Matsuo algebra in characteristic 3 would be a Jordan algebra by Theorem 1.3. But this would contradict [3, 19], just as in Remark 4.4.

Propositions 4.3 and 4.6 together show that Theorem 1.1 holds.

5. Proof of Theorem 1.2

In the previous section we have proved more than just the fact that most lines are solid. Because we proved certain polynomials are identically zero, we can exploit some base change techniques. From here onwards, ideas from the theory of affine group schemes are used, but since we only use basic techniques, we do not use the technical language. For an introduction to this theory, see [15, 18].

Proposition 5.1. *Given $a, b \in X$ such that $\langle\!\langle a, b\rangle\!\rangle$ is solid and $\lambda \in R$ (resp. $R^\times$ when $\langle\!\langle a, b\rangle\!\rangle$ is toric), where R is a (unital, commutative, associative) $\mathbb{F}$-algebra. Then a_λ is an axis of $R \otimes_{\mathbb{F}} A$.*

Proof. Let R be a unital, commutative associative $\mathbb{F}$-algebra. The polynomials or rational maps $P^t_{x,y}, Q^t_x, P^u_{x,y}, Q^u_x$ are all zero maps, and remain so after base change. We can then use Lemma 4.1 to show a_λ is an axis. $\square$

In the next proposition, we let $\mathbb{F}[\varepsilon]$ be the algebra of dual numbers, i.e. $\mathbb{F}[\varepsilon] := \{a + \varepsilon b \mid a, b \in \mathbb{F}\}$ with $\varepsilon^2 = 0$. The following proposition is again well known, and a common way to construct the Lie algebra of the automorphism group scheme of an algebra.

Proposition 5.2. *Let A be an $\mathbb{F}$-algebra, and ϕ an endomorphism of $A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ such that there exists $f\colon A \to A$ with $\phi(x) = x + \varepsilon f(x)$ for all $x \in A$. Then f is a derivation of A if and only if ϕ is an automorphism of $A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$.*

Proof. For any $x, y \in A \subset A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ we have $(xy)^\phi = x^\phi y^\phi = xy + \varepsilon(f(x)y + xf(y))$ if and only if $f(x)y + xf(y) = f(xy)$. Every element of $A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ is an $\mathbb{F}[\varepsilon]$-linear combination of elements in A, so f is a derivation if and only if ϕ is an automorphism of $A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$. $\square$

Corollary 5.3. *If $\langle\!\langle a, b\rangle\!\rangle$ is solid, then $D_{a,b}$ is a derivation.*

Proof. Given $a, b \in X$ such that $B = \langle\langle a, b \rangle\rangle$ is solid. By Proposition 5.1, we know τ_{a_λ} is an automorphism of $A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ for $\lambda \in \mathbb{F}[\varepsilon]$ or $\mathbb{F}[\varepsilon]^\times$, where $\mathbb{F}[\varepsilon]$ is the algebra of dual numbers. If $\langle\langle a, b \rangle\rangle$ is toric, then we consider $\lambda = 1 + \varepsilon$, and $a_\lambda = a + \varepsilon(e - f)$. If $\langle\langle a, b \rangle\rangle$ is unipotent, then we consider $\lambda = \varepsilon$. If $\langle\langle a, b \rangle\rangle$ is flat, then $a_\lambda = a + \varepsilon ab$ and if $\langle\langle a, b \rangle\rangle$ is baric, then $a_\lambda = a + 2\varepsilon(ab - a)$. In all three cases, we have $a_\lambda = a + \varepsilon v$ where v is a $\frac{1}{2}$-eigenvector of L_a. We compute

$$\begin{aligned} a(vx) = \tfrac{1}{4}(vx + vx^{\tau_a}) + \tfrac{1}{2}(v, x)a &= \tfrac{1}{4}(vx + vx + 2(a, x)v - 4v(ax)) + \tfrac{1}{2}(v, x)a \\ &= -v(ax) + \tfrac{1}{2}(vx + (v, x)a + (a, x)v), \end{aligned}$$

so $[L_a, L_v](x) = -2v(ax) + \frac{1}{2}(vx + (v, x)a + (a, x)v)$. We will compute the action of $\rho = \tau_a \tau_{a_\lambda}$ on A, where $\lambda \in \mathbb{F}[\varepsilon]$ is chosen as above. Given $x \in A \subset A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$, we obtain

$$\begin{aligned} x^\rho &= x^{\tau_a} + 4(a + \varepsilon v, x^{\tau_a})(a + \varepsilon v) - 4(a + \varepsilon v)x^{\tau_a} \\ &= x^{\tau_a} + 4(a, x^{\tau_a})a - 4ax^{\tau_a} \\ &\quad + 4\varepsilon(-(v, x)a + (a, x)v - v(x + 4(a, x)a - 4ax)) \\ &= x + 4\varepsilon(-(v, x)a - (a, x)v - vx - 4v(ax))) \\ &= x + 8\varepsilon[L_a, L_v](x). \end{aligned} \tag{3}$$

Now Proposition 5.2 shows $[L_a, L_v]$ is a derivation. For each of these cases, $\langle\langle a, b \rangle\rangle_{\frac{1}{2}}(a) = \langle v \rangle$. This implies, by the Seress Lemma (Lemma 2.4), that $[L_a, L_b] = \mu[L_a, L_v]$ for a certain $\mu \in \mathbb{F}$. So $D_{a,b} = [L_a, L_b]$ is also a derivation. □

We now prove a converse to Corollary 5.3. When the characteristic is zero, this is easily done by taking the exponential of the derivations. In positive characteristic however, this is harder. For that reason, we will prove this using a different technique.

Lemma 5.4. *Suppose $P \in A[\lambda]$ is a polynomial such that $P(\mu + \nu\varepsilon) = 0 \in A \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ for certain $\mu \in \mathbb{F}, \nu \in \mathbb{F}^\times$. Then μ is a root of multiplicity at least 2 of P.*

Proof. Write $P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} \cdots + a_0$. Since $\varepsilon^2 = 0$, we can compute that $P(\mu + \nu\varepsilon) = P(\mu) + \nu\varepsilon(na_n\mu^{n-1} + (n-1)a_{n-1}\mu^{n-2} + \ldots a_1) = P(\mu) + \nu\varepsilon P'(\mu)$, where P' denotes the formal derivative of P. Since $P(\mu + \nu\varepsilon)$ has to be zero, this implies μ is a root for P and its formal derivative. This implies μ is a root of P of multiplicity at least 2. □

Proposition 5.5. *Let (A, X) be a primitive axial algebra of Jordan type $\frac{1}{2}$ over a field $\mathbb{F}$. If $a, b \in X$ such that $D_{a,b}$ is a derivation, then $\langle\langle a, b \rangle\rangle$ is solid.*

Proof. We have to prove this when $(a, b) = \frac{1}{4}$ and $\langle\langle a, b \rangle\rangle$ is 3-dimensional. We may assume $\mathbb{F}$ is algebraically closed. Note that $\rho = \tau_a\tau_b$ has order 3, since $(\tau_a\tau_b)^3 = \tau_{a^{\tau_b}}\tau_{b^{\tau_a}} = \mathrm{id}_A$.

Clearly $\langle\langle a, b \rangle\rangle$ is toric unless $\operatorname{char}\mathbb{F} = 3$, in which case $\langle\langle a, b \rangle\rangle$ is baric. We will set $\mathcal{P}_{x,y} = \lambda^2 P^t_{x,y}$, $\mathcal{Q}_x = \lambda^2 Q^t_x \in A[\lambda]$ if $\operatorname{char}\mathbb{F} \neq 3$ and $\mathcal{P}_{x,y} = P^u_{x,y}$, $\mathcal{Q}_x = \lambda^2 Q^u_x \in A[\lambda]$ if $\operatorname{char}\mathbb{F} = 3$ for all $x, y \in A$.

Note that $\chi = \tau_a\tau_{a+\varepsilon v} = \mathrm{id}_{A_{\mathbb{F}[\varepsilon]}} + 8\varepsilon[L_a, L_v]$ is an automorphism of $A_{\mathbb{F}[\varepsilon]}$ by Proposition 5.2 and Equation (3), where $v = e - f$ if $\operatorname{char}\mathbb{F} \neq 3$ and $v = 2(ab - a)$ if $\operatorname{char}\mathbb{F} = 3$. This means that if a_λ is an axis of $A_{\mathbb{F}[\varepsilon]}$, then a_λ^χ is also an axis of $A_{\mathbb{F}[\varepsilon]}$. Computing this, we have

$$a_\lambda^\chi = \begin{cases} a_{\lambda(1-2\varepsilon)} & \text{if } \operatorname{char}\mathbb{F} \neq 3, \\ a_{\lambda-4\varepsilon} & \text{if } \operatorname{char}\mathbb{F} = 3. \end{cases}$$

Since a_λ is an axis for at least 3 different values of $\lambda \in \mathbb{F}$, the polynomials $\mathcal{P}_{x,y}, \mathcal{Q}_x \in A[\lambda]$ have at least 6 roots (with multiplicity) for every $x, y \in A$ by Lemmas 4.1 and 5.4.

Since $\deg_\lambda \mathcal{P}_{x,y}, \deg_\lambda \mathcal{Q}_x \leq 4$, this means $\mathcal{P}_{x,y} = \mathcal{Q}_x = 0$ for all $x, y \in A$. This implies that $Q_x(c) = P_{x,y}(c) = 0$ for every $x, y \in A$ and every primitive idempotent $c \in \langle\langle a, b\rangle\rangle$, so Lemma 4.1 in turn shows $\langle\langle a, b\rangle\rangle$ is solid. □

Now Corollary 5.3 and Proposition 5.5 together show Theorem 1.2.

6. Proof of Theorem 1.3

First we need the definition of an almost Jordan ring.

Definition 6.1. A non-associative ring R is *almost Jordan* if for all $x, y \in R$ the identity

$$2(yx \cdot x) \cdot x + yx^3 = 3(yx^2)x \tag{4}$$

holds.

Such algebras and their relation to Jordan algebras were first studied by Osborn in [16].

Lemma 6.2. *A non-associative commutative ring R with $2 \in R^\times$ is almost Jordan if and only if $D_{x,y}$ is a derivation of R for all $x, y \in R$.*

Proof. This is proven in [16, p. 1115, Equations (4) and (5)]. □

Corollary 6.3. *Suppose (A, X) is a primitive axial algebra of Jordan type $\frac{1}{2}$, and X spans A linearly. If for every $a, b \in X$ the line $\langle\langle a, b\rangle\rangle$ is solid, then A is almost Jordan.*

Proof. For any $a, b \in X$, the map $[L_a, L_b]$ is a derivation by Theorem 1.2. Since X spans A, the map $[L_x, L_y]$ is a derivation for any $x, y \in A$. Lemma 6.2 shows A is almost Jordan. □

As was noted in [16], the existence of idempotents very often forces almost Jordan algebras to be Jordan. We prove a similar result that is applicable to our situation.

Proposition 6.4. *An almost Jordan algebra A over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ that is linearly spanned by primitive idempotents is a Jordan algebra.*

Proof. By [16, Lemma 1], if $a \in A$ is any idempotent, it is a Jordan type $\frac{1}{2}$ axis. If we linearize the Jordan identity $x^2(ax) - x(ax^2)$ and divide the resulting identity by 2, we get

$$(yz)(ax) + (xy)(az) + (xz)(ay) - ((yz)a)x - ((xy)a)z - ((xz)a)y. \tag{5}$$

Because A is linearly spanned by axes, we may assume a is an axis, and x, y, z are eigenvectors of L_a with eigenvalue $\lambda_x, \lambda_y, \lambda_z \in \{1, 0, \frac{1}{2}\}$. Then Equation (5) becomes

$$\lambda_x(yz)x + \lambda_z(xy)z + \lambda_y(xz)y - ((yz)a)x - ((xy)a)z - ((xz)a)y. \tag{6}$$

Because of symmetry, we need to check this for $(\lambda_x, \lambda_y, \lambda_z) = (1, 1, 1)$, $(1, 1, \frac{1}{2})$, $(1, 1, 0)$, $(1, \frac{1}{2}, \frac{1}{2})$, $(1, \frac{1}{2}, 0)$, $(1, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, 0)$ and $(0, 0, 0)$.

If $\lambda_x = 1$ we can assume $x = a$, and Equation (6) becomes $a(yz) - a(a(yz)) + (\lambda_y - \lambda_z)^2 yz$. If moreover $y = a$, then this becomes $-(\lambda_z^3 - 3\lambda_z^2 + \lambda_z)z$, which is zero for $\lambda_z = 0, 1$ or $\frac{1}{2}$. If $x = a, \lambda_y = \frac{1}{2}$, then $a(yz) - a(a(yz)) = (\frac{1}{2} - \lambda_z)^2 yz$, and thus Equation (6) is zero as well for $\lambda_z = \frac{1}{2}, 0$.

On the other hand, if $\lambda_z = 0$, then xz (respectively yz) is a λ_x-eigenvector (respectively λ_y) for L_a, and Equation (6) becomes $\lambda_x(yz)x + \lambda_y(xz)y - \lambda_y(yz)x - ((xy)a)z - \lambda_x(xz)y$. If then also $\lambda_y = 0$, this becomes $\lambda_x(yz)x - \lambda_x(xy)z - \lambda_x(xz)y$, which is zero if $x = a$ or $\lambda_x = 0$. If $\lambda_x = \lambda_y = \frac{1}{2}$, we also get zero, since in this case $((xy)a)z = 0$.

That leaves us with only two cases left to check, namely $(\lambda_x, \lambda_y, \lambda_z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\lambda_x, \lambda_y, \lambda_z) = (\frac{1}{2}, 0, 0)$. These are the only cases where we need that A is almost Jordan.

If $(\lambda_x, \lambda_y, \lambda_z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then we have $D_{a,x}(yz) = D_{a,x}(y)z + D_{a,x}(z)y$, since A is almost Jordan. When written out, this becomes

$$(a(yz))x - \tfrac{1}{2}(yz)x = \tfrac{1}{2}(xy)z + \tfrac{1}{2}(xz)y - (a(xy))z - (a(xz))y.$$

But this is precisely saying that Equation (6) is zero for $(\lambda_x, \lambda_y, \lambda_z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

If $(\lambda_x, \lambda_y, \lambda_z) = (\frac{1}{2}, 0, 0)$, then Equation (6) becomes $\frac{1}{2}(yz)x - \frac{1}{2}(xy)z - \frac{1}{2}(xz)y$ which is zero by [16, Lemma 2, Equation (9)].

The above shows that A satisfies the linearized Jordan identity (5). If $\operatorname{char} \mathbb{F} \neq 3$, then this suffices for A to be Jordan. If $\operatorname{char} \mathbb{F} = 3$, then [19, Lemma 3.1] shows A is Jordan. □

Remark 6.5. When (A, X) is a primitive axial algebra of Jordan type $\frac{1}{2}$ with nondegenerate Frobenius form and the characteristic of the field $\mathbb{F}$ is not equal to $2, 3, 5$, there is a less computational method to prove this, see [1, Proposition A.8].

This is enough to show that a primitive axial algebra (A, X) of Jordan type $\frac{1}{2}$ such that $\langle\!\langle a, b \rangle\!\rangle$ is solid for all $a, b \in X$ and X spans A linearly, is actually a Jordan algebra. By using Theorem 1.2, we can prove more. The ideas in the proof of the following lemma are adapted from a conversation with Sergey Shpectorov.

Lemma 6.6. *Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$. If $a, b, c \in X$ are such that $\langle\!\langle a, b \rangle\!\rangle$ and $\langle\!\langle a, c \rangle\!\rangle$ are solid, then $\langle\!\langle a, c^{\tau_b} \rangle\!\rangle$ is also solid. Moreover, if not $(a, b) = (a, c) = (a, bc) = 0$, then $\langle\!\langle b, c \rangle\!\rangle$ is also solid.*

Proof. We may assume $\mathbb{F}$ is algebraically closed. Suppose that $\langle\!\langle a, b \rangle\!\rangle$ is not flat. Then we can choose $p_1, p_2 \in \langle\!\langle a, c \rangle\!\rangle$ such that $(a, p_1), (a, p_2) \neq \frac{1}{4}$ and a, p_1, p_2 span $\langle\!\langle a, c \rangle\!\rangle$ linearly. Since $\langle\!\langle a, b \rangle\!\rangle$ is not flat, the variety of axes V in $\langle\!\langle a, b \rangle\!\rangle$ is irreducible. This implies that the maps $\pi_i \colon V \to \mathbb{F} \colon q \mapsto (p_i, q)$ are either constant or dominant, so there are only finitely many axes $q \in V$ with $(q, p_1) = \frac{1}{4}$ or $(q, p_2) = \frac{1}{4}$. But then we can find $q_1, q_2 \in V$ such that a, q_1, q_2 span $\langle\!\langle a, b \rangle\!\rangle$ linearly and $(p_i, q_j) \neq \frac{1}{4}$. This implies $[L_{p_i}, L_{q_j}]$ is a derivation of the algebra A for $i, j \in \{1, 2\}$, and so are $[L_a, L_{q_i}], [L_a, L_{p_i}]$. This implies $[L_x, L_y]$ is a derivation for every $x \in \langle\!\langle a, b \rangle\!\rangle, y \in \langle\!\langle a, c \rangle\!\rangle$, and thus also $[L_b, L_c]$. Thus $\langle\!\langle b, c \rangle\!\rangle$ is solid.

If $(a, b) = (a, c) = 0$ but $(a, bc) = (c, ab) \neq 0$, then by a similar reasing there exist $\lambda, \mu \in \mathbb{F}^\times$ such that $\langle\!\langle a + \lambda ab, c \rangle\!\rangle, \langle\!\langle b + \mu ab, c \rangle\!\rangle$ are solid, and thus $[L_a, L_c], [L_{a+\lambda a \cdot b}, L_c], [L_{b+\mu a \cdot b}, L_c]$ are derivations. But then $[L_b, L_c]$ is a derivation, thus $\langle\!\langle b, c \rangle\!\rangle$ is solid by Theorem 1.2.

In either one of these two cases, we have for any $x \in \langle\!\langle a, b \rangle\!\rangle$ that both $D_{x,a}$ and $D_{x,c}$ are derivations. This implies that $D_{a^{\tau_b}, c}$ is a derivation, so $D_{a, c^{\tau_b}}$ is as well. Now suppose $(a, b) = (a, c) = (a, bc) = 0$ and $(b, c) = \frac{1}{4}$. Then $(a, c^b) = (a, b + c - bc) = 0$, and thus $\langle\!\langle a, b^c \rangle\!\rangle$ is solid by Theorem 1.2. □

We are now finally able to prove Theorem 1.3.

Theorem 6.7. *A primitive axial algebra (A, X) of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$ such that for all $a, b \in X$ the line $\langle\!\langle a, b \rangle\!\rangle$ is solid, is Jordan.*

Proof. By [8, Corollary 1.2], the set $\overline{X} = X^{\operatorname{Miy}(X)}$ spans the axial algebra A linearly. If we prove that for all $a, b \in \overline{X}$ that the subalgebra $\langle\!\langle a, b \rangle\!\rangle$ is solid, the theorem follows from Corollary 6.3 and Proposition 6.4. We prove that for $a, b \in X$ and $g, h \in \operatorname{Miy}(X)$ that $\langle\!\langle a^g, b^h \rangle\!\rangle$ is solid by induction on the length of gh^{-1} in terms of the generators $\tau_a, a \in X$. Indeed, let $a, b, c \in X$. We need to prove that the line $\langle\!\langle a, b^{\tau_c} \rangle\!\rangle$ is solid. But this follows immediately from Lemma 6.6. Now, the line $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_k}}, b^{\tau_{c_n}\cdots\tau_{c_{k+1}}}\rangle\rangle$ is solid if and only if $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_n}}, b\rangle\rangle$ is. By induction, $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_{n-1}}}, b\rangle\rangle$ and $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_{n-1}}}, c_n\rangle\rangle$ are solid, so by Lemma 6.6, so is $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_{n-1}}}, b^{\tau_{c_n}}\rangle\rangle$. But this is equivalent to $\langle\langle a^{\tau_{c_1}\cdots\tau_{c_n}}, b\rangle\rangle$ being solid. $\square$

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