Are Colors Quanta of Light for Human Vision? A Quantum Cognition Study of Visual Perception

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Abstract

We study the phenomenon of categorical perception within the quantum measurement process. The mechanism underlying this phenomenon consists in dilating stimuli being perceived to belong to different categories and contracting stimuli being perceived to belong to the same category. We show that, due to the naturally different way in determining the distance between pure states compared to the distance between density states, the phenomenon of categorical perception is rooted in the structure of the quantum measurement process itself. We apply our findings to the situation of visual perception of colors and argue that it is possible to consider colors as light quanta for human visual perception in a similar way as photons are light quanta for physical measurements of light frequencies. In our approach we see perception as a complex encounter between the existing physical reality, the stimuli, and the reality expected by the perciever, resulting in the experience of the percepts. We investigate what that means for the situation of two colors, which we call *Light* and *Dark*, given our findings on categorical perception within the quantum measurement process.

Keywords:human vision, categorical perception, quantum measurement, Bloch sphere, quantisation, basic colors, qubits, quantum cognition

1 Introduction

We will argue that colors are quanta of light for human vision in a similar way that photons are quanta of light measured by a physical measuring device. To gather evidence for this argument, we will draw on various results obtained over the years within the research field of quantum cognition (Aerts & Aerts, 1995; Aerts & Gabora, 2005b; Busemeyer et al., 2006; Aerts, 2009a,b; Bruza & Gabora, 2009; Aerts & Sozzo, 2011; Aerts et al., 2012; Busemeyer & Bruza, 2012; Haven & Khrennikov, 2013; Khrennikov, 2014; Dalla Chiara et al., 2015; Pothos et al., 2015; Blutner & beim Graben, 2016; Moreira & Wichert, 2016; Yearsley, 2017; Aerts Arguëlles, 2018; Busemeyer et al., 2019; Surov et al., 2019; Aerts Arguëlles & Sozzo, 2020; Aerts & Beltran, 2022a), some of which also involve our own work (Aerts Arguëlles, 2018; Aerts Arguëlles & Sozzo, 2020). Nevertheless, we very deliberately want to make this article understandable to those who have not studied the investigations on which it relies, so that in addition to references to this research, and more technical portions of the article, we will also represent in an intuitive clear way the core aspects in which we frame our claim.

Our argument rests on a fundamental property of human perception, which is not limited specifically to human vision, but is present in all forms of human perception, and which has its origin in a phenomenon called 'categorical perception' (Harnad, 1987; Goldstone & Hendrickson, 2010; Aerts & Aerts Arguëlles, 2022). Categorical perception 'introduces a warping' of the 'stimulus' that generated the perception in a very specific way. Namely, stimuli belonging to the same category are perceived in a way that they are more similar for the one perceiving than an independent of human bias measurement of the stimuli reveals. Additionally, for stimuli belonging to a different category, the reverse warping takes place, they are perceived in a way that they are more different for the one perceiving than an independent of human bias measurement of the stimuli reveals. There is a long history (Bruner & Postman, 1949; Liberman et al., 1957, 1967; Lane, 1965; Eimas et al., 1971; Lawrence, 1949; Berlin & Kay, 1969; Davies et al., 1998; Davidoff, 2001; Regier & Kay, 2009; Havy & Waxman, 2016; Hess et al., 2009; Disa et al., 2011; Sidman, 1994; Schusterman et al., 2000; Rosch, 1973; Collier at al., 1973) in which the phenomenon of categorical perception was identified incrementally, and in Section 2 we give a brief description of this genesis.

In the reasoning we develop we also rely on a detailed study of the quantum measurement process, and more specifically on the way this was analyzed, using a specific measurement model, by our research group in the foundations of quantum mechanics at the Free University of Brussels (Aerts, 1986; Aerts & Sassoli de Bianchi, 2015, 2016, 2017). The mathematical model we developed is an extension of the Bloch model, where not only the pure states, the collapse probabilities and the decoherenced density states are represented, but also the measurements and the changes caused by them. We will see in the course of our analysis how this extended Bloch model is a fertile ground for identifying the phenomenon of categorical perception, and we will show that the quantum measurement model already intrinsically contains the structure that gives rise to the presence of the measurement bias of categorical perception. We will then gather the arguments that allow us to consider within the analysis we propose the basic colors as quanta of light for human visual perception in a similar way that photons are considered quanta of light for physical measurement apparatuses. For reasons of simplicity we introduce the mathematical elements for our analysis for the case of two colors that we call *Light* and *Dark*, mentioning that fundamentally the same analysis can be made for multiple colors with mathematical elements that then belong to a more than two dimensional complex Hilbert space. No other specific difficulties appear in this more dimensional situation that are not encountered in the two dimensional case, namely that of two colors and a two dimensional Hilbert space, which we consider in detail in the present article.

In Section 2 we give a brief overview of the phenomenon of categorical perception with specific attention to the case of colors. In Section 3 we introduce in a detailed and self-prescriptive way, so that prior knowledge is not necessary, the quantum measurement model, paying attention to the extension of the traditional Bloch model that also includes a representation of the measurement and the changes caused by it. In Section 4, we analyze the way the quantum measurement changes distances between states and show that this way contains exactly the warping associated with categorical perception. In other words, the bias caused by categorical perception is intrinsically contained in the structure of the quantum measurement process. In Section 5 we gather the elements of the previous sections to give expression to our claim that we can consider colors as quanta of light for human visual perception.

2 Categorical Perception, Quantization and Colors

In this section we wish to bring out the phenomenon of categorical perception, because the nature of this phenomenon, along with the details of the measurement model of quantum mechanics, which our next Section 3 deals with, plays a fundamental role in our claim that colors are quanta of light for human vision. It is interesting to reflect for a moment on the emergence of what was later commonly called categorical perception because it illustrates well the role that certain goals and hypotheses can play.

Around 1950, research on speech perception came into focus as work was being done on a speech device, the 'pattern playback machine', which was intended to make it possible to automatically convert texts into spoken form so that, for example, blind people could read with it. This required analyzing speech very thoroughly, and it is within this setting that Alvin Liberman identified the phenomenon that would become known as categorical perception. More specifically, he noticed that when generating a continuum of evenly distributed consonant-vowel syllables with endpoints reliably identified as 'b', 'd' and 'g', there is a point of rapid decrease in the probability of hearing the sound as a 'b' to hearing it as 'd'. At a later point, there is a rapid switch from 'd' to 'g' (Liberman et al., 1957). Liberman formulated an original hypothesis by which he wished to explain why people perceive an abrupt change between 'b' and 'p' in the way speech sounds are heard in contrast to what happens with a synthetic morphing device that produces the sounds with a continuous transition. His hypothesis was that this phenomenon is due to a limitation of the human speech apparatus which, due to the muscular nature of its construction, would be unable to produce continuous transitions. Because of the way people produce these sounds as they speak, people's natural vocal apparatus would be unable to pronounce anything between 'b' and 'p'. So when someone hears a sound from the synthetic morphing device, that person tries to compare that sound with what he or she would have to do with his or her voice device to produce this sound. Since a human tuning device can only produce 'b' or 'p', all continuous synthetic stimuli will be perceived as 'b' or 'p', whichever is closest.

The hypothesis was also the basis of what is now called the 'motor theory of speech perception', which assumes that people perceive spoken words by recognizing the gestures in the speech channel used to pronounce them, rather than by identifying the sound patterns that produce the speech (Liberman et al., 1967). The theory came under fire when it was found that 'identification' and 'discrimination' of stimuli not at all associated with speech behave in a similar way to stimuli associated with speech when measured in a similar manner (Lane, 1965). It was also found that children, even before they could speak, exhibited the specific categorical perception effect associated with speech perception that had been identified in adults (Eimas et al., 1971).

So, step by step it became clear that categorical perception was a much more general phenomenon than just associated with speech, and the connection was made with earlier findings having to do with the way stimuli are organized. In particular, Lawrence's experiments and his hypothesis of 'acquired distinctiveness' revealed a phenomenon that turned out to be a very basic effect of perception. The hypothesis of acquired distinctiveness states that stimuli for which one is taught to give a different response to them become more distinctive, while stimuli for which one is taught to give the same response to them become more similar (Lawrence, 1949). Both effects are at work in humans in a multitude of perceptions, stimuli that fall within the same category are perceived as more similar, while stimuli that fall into different categories are perceived as more different. What happens with 'colors' is a good example of the phenomenon of categorical perception. We see a discrete set of colors while from physical reality there is a continuum of different frequencies presented to us as stimuli. The warping effect of the categorical perception of colors consists in two stimuli that both fall within the category of, for example, green to be perceived more equally than two stimuli one of which falls within the category of green and the second within the category of blue, even if from a physical perspective both pairs of stimuli have the same difference in frequency. The cooperation of these two effects, a contraction within an existing category and a dilation between different categories, causes a clumping of colors, ultimately leading to the colors that we distinguish.

Given the research on colors that we present in this article, it is appropriate to mention Eleanor Rosch's work on colors. Indeed, it was that work that inspired her to propose the prototype theory of concepts, which remains one of the most important theories of concepts today (Rosch, 1973). The basic idea of prototype theory is that there exists a central element for a concept, which we call the prototype, relative to which the exemplars of the concept can be placed within a graded structure. Rosch suggested the idea that would develop into the primary model for concepts by studying the categorical structure among the Dani for colors and basic shapes. The Dani are a people living in Papua New Guinea, with the peculiarity that they have only two words to denote colors, one meaning *Bright* and the other *Dark* [1]. The Dani also have no words in their language for basic shapes such as *Circle*, *Square* and *Triangle*. Rosch examined whether there was a difference in learning between two groups of Dani volunteers, with one group learning colors and basic shapes, starting with stimuli that are prototype colors and prototype basic shapes, while the other group learned to start with stimuli that are different distortions of these prototypes. In a significant way, it was found that for both colors and basic shapes, learning was more qualitative for the group taught starting from the prototype stimuli. This evaluation of 'more qualitative learning' took into account the three features of how this can be measured, namely by the ease of learning sets of shape categories when a particular type was the prototype, by the ease of learning individual types within sets, and by rank order of rating types as the best example of categories, when the prototypes of both colors and shapes were the stimuli in the learning process.

Mathematical prototype models based on fuzzy sets were developed for concepts and experimentally tested, and it seemed that the way in which by warping, i.e., contraction when stimuli fall into the same category and dilation when they fall into different categories, concepts emerge and grow from stimuli, had finally found a form to understand what is taking place at a fundamental theoretical level (Collier at al., 1973; Rosch, 1975; Rosch et al., 1976; Smith & Medin, 1981; Medin at al., 1984; Geeraerts et al., 2001; Johanden & Kruschke, 2005).

A fundamentally not understood problem however, even when good mathematical models existed by which a concept and its set of exemplars and features could be modeled according to the approach of prototype theory, was the description of the combination of two concepts. This problem was noted in a first publication in which the combination of the concept *Pet* with the concept *Fish* served as an example, and therefore the problem of combining two concepts is often called the 'pet-fish problem', or also the 'guppy effect', because *Guppy* was the exemplar used to illustrate what goes wrong with prototype theory when concepts are combined (Osherson & Smith, 1981). The question posed was 'how is it that *Guppy* is not a typical example of a *Pet*, nor a typical example of a *Fish*, but a very typical example of a *Pet-Fish*'. It is starting from this 'guppy effect' that in our Brussels research group it

[1] As in other articles of our Brussels research group, we denote concepts, when they appear as subjects in the text, by writing them beginning with a capital letter and in italic.

was tried successfully whether a quantum formalism could be used to model the guppy effect mathematically, where then the sudden relevance of *Guppy* as a typical exemplar would be explained as an 'interference effect' between the concepts *Pet* and *Fish* when combined (Aerts & Gabora, 2005a,b). These first models using the quantum formalism were further developed and refined in the following years (Aerts, 2009a,b) and their relevance to quantum information science was demonstrated by also modeling data obtained from the World-Wide Web in a similar way with a quantum model (Aerts et al., 2012).

Meanwhile, it became clear that there were many more similarities between the structure and dynamics to which concepts of human language are subject and the structure and dynamics of quantum entities as described by quantum mechanics. The well-studied phenomenon of quantum entanglement, for example, also occurs with concepts, it is possible to combine concepts in a very similar way as is the case with quantum entities, such that experiments on these combinations violate Bell's inequalities, which is the experimental test for the presence of entanglement (Aerts & Sozzo, 2011, 2014). Another more recent finding consists in identifying the statistical properties of texts of stories, short stories and stories of the length of novels. It could be shown in a very convincing way that the statistics inherent in such texts is of the Bose-Einstein type, hence the same as the statistics of a class of quantum entities called bosons, such as photons. It could also be shown that it cannot be of the Maxwell-Boltzmann type – as one would expect it to be, since Maxwell-Boltzmann is the common statistics for a collection of classical entities (Aerts & Beltran, 2020). It was also investigated and shown that regarding the thermodynamic properties of texts representing stories it is the quantum mechanical von Neumann entropy that is relevant and not the classical Shannon entropy (Aerts & Beltran, 2022a,b). But the most important finding of a recent nature as far as the present article is concerned, is the one related to the phenomenon of quantization. It became clear that the phenomenon of categorical perception triggers a dynamics that leads to the emergence of quanta (Aerts & Aerts Arguëlles, 2022). In the present article, we will again focus on categorical perception and how it gives rise to the formation of quanta, and more specifically, we will show that the mechanism of categorical perception is intrinsically present in the structure of the quantum measurement itself. We will then investigate in what way colors can be considered to be quanta of light for human vision.

Rosch, during her work on colors, assumed that the way humans see colors is determined, probably genetically, at birth, also to regard the color frequencies of the basic colors. More so, that the situation with colors is thus, was actually her inspiration for proposing the prototype theory for concepts. This also means that the so-called Sapir-Whorf hypothesis, namely that there is an influence of 'how we name categories within a language' and 'the perception of stimuli belonging to these categories', was believed to be not applicable to colors. And this was generally accepted, colors were thought to be not subject to how they were named in different languages. Not only do most cultures divide the groups of colors similarly and give them separate names, but for the few cultures where this is not the case, the areas of compression and dilation were assumed to be the same. It was believed that we all see blue and green in the same way, with a blurred area in between, even if the naming is not the same (Berlin & Kay, 1969). However, this view was challenged by studies that nevertheless identified effects of the words designating colors. Comparative research on colors between speakers of Setswana, a Bantu language spoken by about 8.2 million people in South Africa, and speakers of English found many similarities, but also identified differences relevant in terms of the Sapir-Whorf hypothesis (Davies et al., 1998). Speakers of Berinmo, an indigenous language in Papua New Guinea, have only one word, 'nol', for what English speakers call green and blue. The difference they made compared to speakers of English in color discrimination tasks regarding shades between green and blue was investigated and determined (Davidoff, 2001). Later evidence was also found that linguistic categories influence categorical perception primarily in the right visual field. Since the right visual field is controlled by the left hemisphere, this finding was explained by the fact that language skills are also located in the left hemisphere (Regier & Kay, 2009).

More recent experiments, meanwhile, have demonstrated very thoroughly that language and the names given have an influence on the categorization that takes place over very primitive visual perceptions. Nine-month-old infants were shown a continua of new creature-like objects. There was a learning phase in which the infants were shown that objects from one end of the perceptual continuum moved to the left and objects from the other end moved to the right. For one group of infants, the objects were always called by the same name, while for the other group of infants, two different names were used to call the objects depending on whether they belonged to one end or the other of the continuum. The test involved showing new objects from the same continuum to all infants and then seeing if there was a difference between the two groups. What was found is that infants in the one-name condition formed one overarching category and looked at new test objects in either place. Infants in the two-name condition distinguished two categories and correctly anticipated the likely location of test objects even when they were near the poles or near the middle of the continuum (Havy & Waxman, 2016).

However, the experiment with the nine-month-old children we describe above, that demonstrates the effective existence of a Sapir-Whorf effect with colors, is not in contradiction with the findings that inspired Rosch. The colors we customarily distinguish do exist at birth, but the way these colors will play a role dynamically in an individual person's life, for this more advanced function of what colors are, that is where a Sapir-Whorf effect does play a specific role. As we shall see further in the course of this article, this is the situation that is captured if colors are considered to be quanta of light for human visual perception, indeed, it is then the quantum mechanical structures of superposition and entanglement that can account for these effects.

In her experiments with the Dani, Rosch used the basic colors as defined by Brent Berlin and Paul Kay in their authoritative work 'Basic Color Terms: Their Universality and Evolution' (Berlin & Kay, 1969), and we will use these colors as basic colors in this article as well. By the way, let us note that these are not the colors of the rainbow, Berlin & Kay (1969) suggested the following eleven colors as basic colors within the culture where English is used as a native language, *White*, *Black*, *Red*, *Yellow*, *Green*, *Blue*, *Brown*, *Purple*, *Pink*, *Orange* and *Gray*. Berlin and Kay's work studies basic color naming in different language regions and concludes that seven stages in naming these basic colors can be distinguished from an evolutionary perspective. In the first stage, only two colors are named, *White* (then also meaning *Light*) and *Black* (then also meaning *Dark*). In a second stage, where three names for basic colors exist, *Red* is systematically added. To the next two stages, *Yellow* and *Green* are introduced, sometimes *Yellow* first and sometimes *Green* first. At the next stage, *Blue* is added, and *Brown* joins the names of basic colors at the stage after that. We have then reached the seventh stage where eight to eleven names exist for basic colors, the names still added being *Purple*, *Pink*, *Orange* and *Gray*.

The color system used both by Berlin and Kay and by Rosch, and in which they thus define the eleven prototype colors belonging to the English language, is the Munsell color system, a three-dimensional system where the sizes of the three parameters, hue, chroma and value define each color. Berlin and Kay use their original nomenclature of 'focus colors' for the prototype colors, and Rosch also uses this nomenclature in her earliest articles on colors (Rosch Heider, 1971, 1972; Mervis et al., 1975). Since it was mainly her research on colors that made Rosch develop the prototype theory for concepts, she calls these focus colors in her later work the prototype colors and we will use this terminology in our writing on colors.

In the next section we examine the connection between categorical perception and quantum measurement, more specifically, we look in detail at the situation of a two-dimensional quantum entity and show how its measurement dynamics is the underlying ground for the warping mechanism at work in the phenomenon of categorical perception.

3 Quantum Measurement and the Bloch Model

We will work with the Bloch representation of a two-dimensional quantum entity that we will analyze with the intention of identifying the mechanism of categorical perception at work. Traditionally, the Bloch model is only a representation of the set of quantum states, pure states and density states, also called mixed states, of a two-dimensional quantum entity, however, in the 1980s an extension of the Bloch model was worked out in which also the measurements could explicitly be represented (Aerts, 1986). In later years the original extension was elaborated and perfected (Aerts & Sassoli de Bianchi, 2014, 2016), and since this extension contains also a specification of the change provoked by the measuring apparatus on the state of the measured upon entity, we will use it for our analysis, because it will more easily allow us to identify the mechanism of categorical perception. We introduce the extended Bloch model for a two-dimensional quantum entity step by step, explaining details as they arise, the intention being to make the content of our article accessible to readers who are not experts in quantum physics. In parallel, we also introduce the elements of the traditional complex Hilbert space formalism of a two-dimensional quantum entity. That we speak of a 'two-dimensional' quantum entity, by the way, refers to that its Hilbert space has dimension 2. Within the discipline of quantum computation a two-dimensional quantum entity is called a 'qubit'.

A two-dimensional complex Hilbert space $\mathcal{H}$ is a vector space of dimension 2 over the complex numbers with an inner product, which is a map, conjugate linear in its first variable and linear in its second, to the set $\mathbb{C}$ of complex numbers

$$\langle\ |\ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \tag{1}$$

$$\langle a\psi_1 + b\psi_2|\psi\rangle = a^*\langle\psi_1|\psi\rangle + b^*\langle\psi_2|\psi\rangle \tag{2}$$

$$\langle\psi|a\psi_1 + b\psi_2\rangle = a\langle\psi|\psi_1\rangle + b\langle\psi|\psi_2\rangle \tag{3}$$

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^* \tag{4}$$

The inner product allows to express when two states are orthogonal to each other, namely, if their inner product is equal to zero. It also introduces a norm on the Hilbert space, the inner product of a vector with itself is the square of the length of this vector.

In Figures 1 and 2 we consider the basis structure of the Bloch model for a two-dimensional quantum entity, namely the Bloch sphere, which is a sphere in three-dimensional Euclidean space with radius equal to 1. The points of the surface of this sphere represent the pure states of the considered quantum entity. These are the states that the entity can occupy independently of being measured upon or not, thus they describe the reality of the considered entity. In Hilbert space, these pure states are represented by unit vectors of the Hilbert space. We promised that we wanted to keep the content of this article understandable for those who do not know the mathematical technicalities of the quantum formalism, and we will do our best to keep this promise. Nevertheless, we are obliged to introduce some mathematical elements that are typical of the quantum formalism. The pure states of two dimensional quantum entity, hence a qubit, or the spin of a quantum particle with half integer spin, are represented in the extended Bloch model by the points of the surface of the Bloch sphere. We will denote

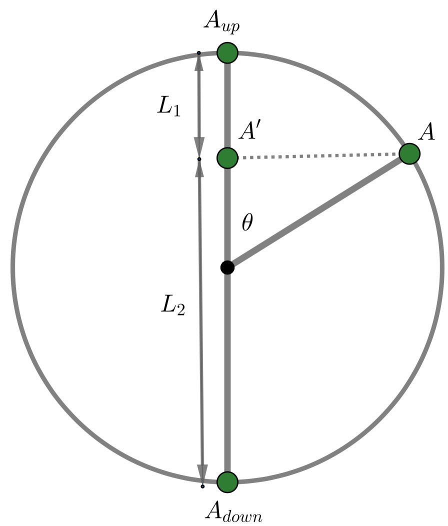

Figure 1: A representation of our extended Bloch model for a two-dimensional quantum entity. A little ball is placed in a point A on the surface of a sphere and this represents the state of the entity in A before the measurement. On a central axis of the sphere between the points A_{down} and A_{up} situated on the surface of the sphere is placed an elastic and a first phase of the measurement consists of the ball falling through the interior of the sphere orthogonally to the elastic and sticking to it in a point A' on the centerline between A_{down} and A_{up}. The second phase of the measurement consists of the elastic breaking in one of its points randomly and uniformly, hence such that the probability of breaking in a piece of the elastic is proportional to the length of this piece. A a consequence the ball is moved upwards ending up in the point A_{up} or downwards ending up in the point A_{down} depending on whether the elastic breaks in the part down to A' or the part up to A'.

these points by vectors in the three-dimensional Euclidean space, indeed, the Bloch sphere is contained in this three-dimensional Euclidean space, by the so-called spherical coordinates r, θ, and ϕ. The distance from the center of the Bloch sphere to a point on the surface of the Bloch sphere is traditionally denoted in spherical coordinates by r, and called the 'radial distance'. Since our Bloch sphere has radius equal to 1, for any point on the surface we have $r = 1$. The angle that the line connecting the center of the Bloch sphere with the considered point of the surface of the Bloch sphere with the axis of the Bloch sphere is denoted by θ, and it thus varies between values 0 and π, it is equal to 0 if the considered point coincides with the North Pole of the Bloch sphere and is called the polar angle (see Figure 2). Hence, it is equal to π if it coincides with the South Pole [2]. A second angle is still needed so that the values of r, and θ, and the value of that second angle, traditionally denoted ϕ and called the azimuthal angle, would uniquely determine each point of the three dimensional Euclidean space. This angle ϕ is chosen from a vertical plane through the axis of the Bloch sphere rotating around that axis reaching the other points not in the vertical plane, for the points in the plane ϕ is equal to zero or 2π. This means that for values of ϕ varying between 0 and 2π, all points

[2] We denote the magnitude of angles by the unit of a radian, which makes 180 degrees equal to π radians

of three-dimensional Euclidean space can be reached and hence the spherical coordinates (r, θ, ϕ) form complete coordination of the three dimensional Euclidean space. For $r = 1$, by varying the values of θ and ϕ, all points of the surface of the Bloch sphere are reached (see Figures 1 and 2). So the point A has spherical coordinates $(1, \theta, \phi)$, and cartesian coordinates (x, y, z), where

$$x = \sin\theta \sin\phi \quad (5)$$

$$y = \sin\theta \cos\phi \quad (6)$$

$$z = \cos\theta \quad (7)$$

for a cartesian coordinate system with its origin in the center of the sphere.

The coordinates, be they spherical (r, θ, ϕ) or cartesian (x, y, z), that we have considered so far, describe a three-dimensional Euclidean space, of which the Bloch sphere is a part. However, we have explicitly discussed a two-dimensional quantum entity that we want to

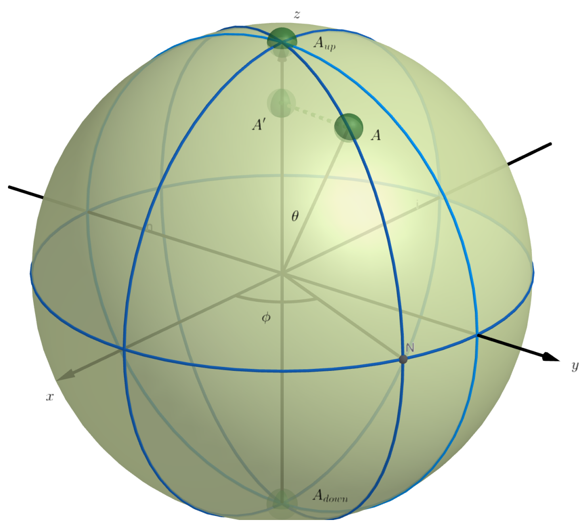

Figure 2: A three dimensional representation of the Extended Bloch Model. The little ball is in point A with spherical coordinates (ρ, θ, ϕ) and Hilbert space density operator coordinates $D_{(\rho,\theta,\phi)}$. The process of measurement proceeds like described in detail in Figure 1 by the ball falling orthogonally to the elastic and sticking to it in point A'. Then the elastic breaks uniformly in one of its points an pulls the ball upwards to end in A_{up} or downwards to end in A_{down}.

describe using its Bloch representation, while the Cartesian and spherical coordinates describe a three-dimensional space. If one remembers the representation of the complex numbers in the Euclidean plane, a mathematics subject of high school education, then it comes as no surprise that a representation of a two-dimensional complex vector space needs three dimensions to be represented in a Euclidean real space. Indeed, the set of complex numbers, if one wishes to think of it that way, is a one-dimensional complex vector space and it needs a plane, the so called complex plane, hence a two-dimensional space, if represented by points with coordinates that are real numbers, such as is the case by polar coordinates in the complex plane. Let us introduce a general vector from the two-dimensional vector space, and we immediately refer to the spherical coordinates to indicate to which point of the Bloch sphere that vector belongs.

$$|\theta,\phi\rangle = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}) \tag{8}$$

We do not yet directly explain here why precisely this vector $|\theta,\phi\rangle$ of the two-dimensional complex Hilbert space is the one we let correspond to that point $(1,\theta,\phi)$ of the three-dimensional Euclidean space, the analysis we make in the course of this article will clarify this.

The notation $|\theta,\phi\rangle$ is the one introduced by Paul Dirac, one of the founding fathers of quantum mechanics, along with Werner Heisenberg and Erwin Schrödinger (Dirac, 1939). Dirac considered the inner product as a fundamental operational element, the bracket, $\langle\theta,\phi|\theta,\phi\rangle$, and called the right side of this bracket, the ket, $|\theta,\phi\rangle$ and the left side, the bra $\langle\theta,\phi|$. Hence the notation $|\theta,\phi\rangle$ as the right side, thus as ket, was chosen as a representation of the state of the considered entity by Dirac because the inner product is taken to be 'linear' in the second variable while 'conjugate linear' in the first variable [3]. In matrix notation, in the case of a two dimensional quantum entity, the ket is written as a 'one column' to 'two row' matrix, and the bra as a 'one row' to 'two column' matrix, plus a complex conjugation of the terms of the matrix for the bra.

$$|\theta,\phi\rangle = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix} \tag{9}$$

$$\langle\theta,\phi| = \begin{pmatrix}\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\end{pmatrix} \tag{10}$$

We thus find by a direct matrix calculation that the ket is normalized as a state, indeed we have

$$\begin{aligned}\langle\theta,\phi|\theta,\phi\rangle &= \begin{pmatrix}\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix} \\ &= \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2} = 1\end{aligned} \tag{11}$$

Dirac also introduced in his bra-ket calculus the possibility of multiplying the ket by the bra in the reverse order of the bra-ket, that is, as ket-bra. Whereas the bra-ket multiplication always results in a complex number, more specifically the number 1 in the example we considered here, the reverse multiplication $|\theta,\phi\rangle\langle\theta,\phi|$ leads to a more complex quantity. We can calculate this more complex quantity directly in the case of the two-dimensional quantum entity we are considering, and by the representations of ket and bra by the two-by-one matrices, as we just explained. We then get

$$\begin{aligned}|\theta,\phi\rangle\langle\theta,\phi| &= \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\end{pmatrix} \\ &= \begin{pmatrix}\cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2}\end{pmatrix}\end{aligned} \tag{12}$$

a 'two-by-two' matrix. This two-by-two matrix represents an operator within the quantum Hilbert space formalism called an orthogonal projection, but before we continue on that, we wish to specify using the extended Bloch model what happens during a quantum measurement.

[3] This choice, by the way, is sometimes not followed by mathematicians who define the inner product in a complex Hilbert space as linear in the first variable and conjugate linear in the second.

Within the extended Bloch model, the measurement is represented by a piece of elastic stretched on the centerline between two diametrically opposed points of the sphere, A_{down} and A_{up} (see Figures 1 and 2). On the surface of the sphere is a small ball that sticks to the sphere at a point A. This ball in a point represents the state in which the quantum entity is. The measurement then occurs as follows. First, the ball falls orthogonally on the elastic and sticks to it at the point A' where this orthogonal fall brings it. Then the elastic breaks randomly in one of its points. If the breaking point is below the point A', then the unbroken part of the elastic pulls the ball up so that it ends up in the point A_{up}. However, if the breaking point is above the point A', then the ball is pulled down by the unbroken part of the elastic, and ends up in the point A_{down}. Thus, the whole of the measurement results in the ball from the point A ending up in one of two points A_{up} or A_{down}.

Now suppose that the elastic possesses a uniform breaking pattern, which means that the point at which the elastic breaks randomly will lie in a certain interval of the elastic with a probability proportional to the length of that interval. We can then calculate the probabilities $P(A \mapsto A_{up})$ and $P(A \mapsto A_{down})$ with which the measurement will carry the ball from A to A_{up} or to A_{down}, and in this way will change the state of the quantum entity, from the simple geometry of the configuration. Note indeed that we have that the length L_2 of the elastic under the point A' equals $1 + \cos\theta$ and the length L_1 of the elastic above the point A' equals $1 - \cos\theta$, which hence, taking into account the uniform nature of the breaking pattern of the elastic, gives us

$$P(A \mapsto A_{up}) = \frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{13}$$

$$P(A \mapsto A_{down}) = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{14}$$

The probabilities (13) and (14) match the quantum probabilities of the spin of a spin 1/2 quantum particle or a qubit. The ball in point A is then in a state for the spin of an angle θ with the z-axis, and the elastic lies on this z-axis.

The part of this quantum measurement that takes place at the so-called pure states is described in a complete way in the quantum Hilbert space formalism. The complex Hilbert space in this case is two-dimensional, with basis vectors $|0,0\rangle$ and $|\pi,0\rangle$, which in the Bloch representation correspond to points on the Bloch sphere located at the North and South Poles, respectively. In Hilbert space these basis vectors, and we use now their notation as one column and two row matrices, are

$$|0,0\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}_{(\theta=0,\phi=0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{15}$$

$$|\pi,0\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}_{(\theta=\pi,\phi=0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

A general pure state $|\theta,\phi\rangle$, in this Hilbert space formalism, is a normalized linear combination, called superposition in the quantum jargon, of these basis vectors

$$|\theta,\phi\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{17}$$

$$\text{with } a = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \text{ and } b = \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \text{ and } |a|^2 + |b|^2 = 1 \tag{18}$$

with superposition coefficients a and b.

We now know the quantum probabilities, namely $\cos^2 \frac{\theta}{2}$ and $\sin^2 \frac{\theta}{2}$, for the state $|\theta, \phi\rangle$ to collapse as a result of a measurement to the state $|0, 0\rangle$ or the North Pole of the Bloch sphere, or the state $|0, \pi\rangle$ or the South Pole of the Bloch sphere, respectively. That information gives us a way to determine the mathematical form in the two-dimensional complex Hilbert space that the state $|\theta, \phi\rangle$ possesses. Indeed, lets write the most general mathematical form for this state $|\theta, \phi\rangle$, namely

$$|\theta, \phi\rangle = a|0, 0\rangle + b|\pi, 0\rangle \tag{19}$$

where a and b are two complex numbers such that $|a|^2 + |b|^2 = 1$, we just apply the superposition principle here, one of the basic principles of quantum theory. And then we must have

$$|a|^2 = \cos^2 \frac{\theta}{2} \quad \text{and} \quad |b|^2 = \sin^2 \frac{\theta}{2} \tag{20}$$

We can solve these equations and should not forget that a and b are complex numbers. A straightforward solution is $a = \cos \frac{\theta}{2}$ and $b = \sin \frac{\theta}{2}$, but that is not the general solution. Note that a dependence of a solution on ϕ must vanish in $|a|^2$ and $|b|^2$, which means that this dependence on ϕ is in the form of a phase factor $e^{i\phi}$, since such a phase factor disappears if the absolute value of the complex numbers is calculated. This is how we arrive at the general solution $a = \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}}$ and $b = \sin \frac{\theta}{2} e^{i\frac{\phi}{2}}$, which we introduced in (23), noting that without loss of generality we can multiply with a phase factor such that this specific form arises [4]. It can be shown that the complex numbers as a number system on which the vector space is built are essential, it is not possible to obtain the probability structure of a qubit with a two-dimensional vector space over the real numbers (Aerts, 1986).

What we are particularly interested in for our present article is the fine structure of the mechanism in the measurement in the extended Bloch model shown in Figures 1 and 2. And before we continue, we wish to note the following. The example depicts a two-dimensional quantum entity and a measurement with two final states A_{up} and A_{down}, hence two outcomes. However, in later years, it was shown that a similar model can be built for a n-dimensional quantum entity and a measurement with an arbitrary number n of final states, and thus an arbitrary number n of outcomes (Aerts & Sassoli de Bianchi, 2014). We will not explicitly describe these higher dimensional extended Bloch models in the present article, but mention that the simple geometric properties of the configurations lead to exactly the quantum probabilities in a completely similar way than this is the case for this two dimensional quantum entity. The details of the analysis that we will now make for the two dimensional quantum entity can also be made for the higher dimensional quantum entities in a similar way. Of course, that we find exact quantum probabilities in these measurement models also depends on a certain symmetry that we introduced at the measurement level, namely that the elastic breaks with a randomness that is 'uniformly' distributed. If the elastic does not possess this symmetry property, we will still find probabilities that do not fit a classical probability model, but they will not be perfect quantum probabilities, e.g. a complex Hilbert space model will not be able to represent them, they will thus be rather quantum-like. More general quantum formalism than this of standard quantum mechanics in a complex Hilbert space, and not based on vectors representing states, can then be used (Aerts, 1982, 1992; Gudder & Zanghi, 1984).

[4] Another choice often found in textbooks is $a = \cos \frac{\theta}{2}$ and $b = \sin \frac{\theta}{2} e^{i\phi}$, which equals the solution we use in this article except a phase of $e^{-i\frac{\phi}{2}}$.

To verify that quantum measurement gives rise to the mechanism of categorical perception, we must be able to express for states how much they differ from each other. But before we get to that, we must identify which states play the role of stimuli, and which states play the role of percepts in a quantum measurement process. The stimuli are represented by the states of the quantum entity independent of any measurement that would be going on, thus they are what in the quantum jargon are called the pure states. In Hilbert space, these pure states are represented by unit vectors of the Hilbert space. In the Bloch representation, it is the points of the Bloch sphere that represent the pure states of the considered quantum entity.

In what happens during a measurement as we represent it in the extended Bloch model, not only the places corresponding to points on the surface of the Bloch sphere play a role, but also places corresponding to points of the interior of the Bloch sphere are important, for example the point A' where the ball lands and sticks to the elastic after falling orthogonally on it. The points over which the elastic is stretched are also in the interior of the Bloch sphere. Besides the vector space formalism of quantum mechanics, where the main role is played by the complex Hilbert space, which is a vector space over the complex numbers, there also exists a density matrix formalism of quantum mechanics. However, both formalism have a fundamentally different mathematical structure, the Hilbert space is a vector space, as we already mentioned, while the set of density matrices is a convex space. Convex combinations of density matrices give another density matrix, while linear combinations of vectors give another vector. Also, both structures can live together on an underlying mathematical entity, e.g. density matrices are found in the Hilbert space formalism as density operators. Both are also found in the Bloch representation, the unit vectors of Hilbert space as the points of the surface of the Bloch sphere, and the density matrices as the points of the interior of the Bloch sphere. Let us introduce more systematically these mathematical structures by writing the different vectors with respect to the canonical basis.

$$|1,0\rangle = (1,0) \tag{21}$$

$$|0,1\rangle = (0,1) \tag{22}$$

We have

$$|\theta,\phi\rangle = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}) \tag{23}$$

We already mentioned the density matrix formalism as also a quantum mechanical formalism. The points of the interior of the Bloch sphere correspond to such density matrices of this quantum formalism. We will denote a density operator corresponding to the point with spherical coordinates (r,θ,ϕ), $\rho \in [0,1]$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$ as $D_{(r,\theta,\phi)}$. Let us use some of the known properties to calculate the density matrix for some of the points of the Bloch sphere and of its interior. We already introduced the density matrix corresponding to a pure state in (12), hence in the notation we introduced this is $D_{(1,\theta,\phi)}$. Let us calculate the density matrices of the North and South Poles of the Bloch sphere, or of the points A_{up} and A_{down}. We have

$$D_{(1,0,\phi)} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix}_{\theta=0,\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{24}$$

$$D_{(1,\pi,\phi)} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix}_{\theta=0,\phi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

Let us now calculate the density operator $D_{A'}$ representing the state of the entity when it is in the point A' as in Figure 2. We remark that A' lies on the line between A_{down} and A_{up} sticking on the elastic which is stretched between A_{down} and A_{up} on this line. Making use of a general property of the set of all density operators, i.e. that it is a set closed by convex combination, we know that $D_{A'}$ is a convex combination of $D_{(1,\pi,\phi)}$ and $D_{(1,0,\phi)}$, which gives

$$D_{A'} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{26}$$

for $\lambda \in [0,1]$. From Figure 2, given that A' is obtained by projecting orthogonally to the line between A_{down} and A_{up}, we have that A' lies on the line between A and the point with spherical coordinates $(1, \theta, \phi + \pi)$, to which corresponds the density operator

$$\begin{aligned} D_{(1,\theta,\phi+\pi)} &= \begin{pmatrix} \cos^2 \frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i(\phi+\pi)} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i(\phi+\pi)} & \sin^2\frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{27}$$

This means that we have

$$D_{A'} = \mu D_{(1,\theta,\phi)} + (1-\mu) D_{(1,\theta,\phi+\pi)} \tag{28}$$

for $\mu \in [0,1]$. From (26) and (28) follows that we must have

$$\begin{aligned} &\mu \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} - (1-\mu)\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} = 0 \\ &\Leftrightarrow \mu - (1-\mu) = 0 \\ &\Leftrightarrow \mu = \frac{1}{2} \end{aligned} \tag{29}$$

and

$$\lambda = \sin^2\frac{\theta}{2} \tag{30}$$

This gives us

$$\begin{aligned} D_{A'} &= \begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix} = \cos^2\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sin^2\frac{\theta}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \cos^2\frac{\theta}{2}D_{A_{up}} + \sin^2\frac{\theta}{2}D_{A_{down}} \end{aligned} \tag{31}$$

Let us note that it follows from (28) that we also know the density matrix of each point lying on the line between the two points of the surface of the Bloch sphere, the point A with coordinates $(1, \theta, \phi)$ and the point with coordinates $(1, \theta, \phi+\pi)$, which is mirrored with respect to the North-South axis of the Bloch sphere. For a value of μ ranging from -1 to +1, the density matrix is given by

$$\begin{aligned} D_{A',\mu} &= \mu D_{(1,\theta,\phi)} + (1-\mu) D_{(1,\theta,\phi+\pi)} \\ &= \begin{pmatrix} \cos^2\frac{\theta}{2} & \mu\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i(\phi+\pi)} \\ \mu\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i(\phi+\pi)} & \sin^2\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{32}$$

the density matrix of the point corresponding to the value of μ on the line between the points $(1, \theta, \phi + \pi)$ and $(1, \theta, \phi)$. The parameter μ occurs only in the diagonal terms of the density matrix, and the orthogonal falling of the ball on the elastic takes place on the line connecting the two points $(1, \theta, \phi + \pi)$ and $(1, \theta, \phi)$, and that orthogonal falling occurs when μ goes from 1 to 0. In the jargon of quantum mechanics, the decreasing of the non-diagonal terms of the density matrix is named 'decoherence,' and is described as a phenomenon of decreasing quantum nature and increasing classical nature.

Let us briefly summarize the elements of a quantum measurement and its Bloch sphere representation. Each point (r, θ, ϕ), $r \in [0, 1]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, within the sphere shown in Figure 2, corresponds to a density operator $D_{(r,\theta,\phi)}$, and the points of the sphere surface, i.e. $r = 1$, correspond to pure states of the quantum entity. The points of the interior of the sphere correspond to density states of the quantum entity. With a measurement corresponds a convex subspace of the interior of the sphere, in the case of the measurement with possible final states after the measurement, A_{up} and A_{down}, this convex subspace is given by the line segment from $(1, \pi, \phi)$ to $(1, 0, \phi)$, where the elastic is stretched. But all lines passing through the center of the sphere in Figure 2 harbor such an equivalent measurement. The points on the line all correspond to a density state, except for the end points, which are indeed part of the surface of the sphere, and thus correspond to a pure state. When a measurement commences, on an entity that is in a pure state, such as this one in the point A, an expectation is allowed to play a role of what this state in the point A means to the measuring apparatus located on a line between two diametrical points of the sphere surface, such as the points A_{down} and A_{up}. That is the meaning of the orthogonal projection which brings the state in A, to a state in A'. Once the state has become the one corresponding to the point A', then the entity is in a density state. The transformation from the state corresponding to A to the state corresponding to A' can be read on the matrix representation, namely it is the non-diagonal terms of the matrix of the density state that disappear to arrive at the state corresponding to A'.

4 Categorical Perception and Quantum Measurement

As we already mentioned, in this article we want to show that a quantum measurement structurally incorporates the mechanism of categorical perception. At the same time, we want to propose an interpretation of 'what exactly could be imagined to happen during a quantum measurement' with the intention of representing the 'cognition aspect' of this event as clearly as possible. In this sense, our work also provides additional evidence for the 'conceptuality interpretation' of quantum mechanics an original interpretation (Aerts, 2009b) that is being further developed in our Brussels research group (Aerts et al., 2020). Indeed, the basic hypothesis of the conceptuality interpretation is that a quantum measurement has the characteristics of a cognitive process, as a consequence the quantum entities which are the subject of the measurement have a conceptual nature. When we mention that the basic hypothesis of conceptuality interpretation consists in postulating the presence of a cognitive interaction in the micro world we do not mean that this is a cognitive interaction very similar to the human cognitive interaction, perhaps and even probably so, it is of an alien nature. We do mean that there are some basic aspects present that we also find in the human cognitive interaction, for example 'the use of concepts as communication tools' (Aerts, 2009b; Aerts et al., 2020). If the phenomenon of categorical perception can be shown to be present in what is occurring during a quantum measurement then this brings extra evidence for the hypothesis that a quantum measurement is a cognitive process. Ultimately, we also want to show that it makes sense to consider colors as the quanta of light for human visual perception similar to how photons are the quanta of light for a physical measurement device during a quantum measurement.

To inspire our interpretation of what could be taking place during a quantum measurement, we present an experiment conducted by Jerome Bruner and Postman, in which elements of human perception are brought out (Bruner & Postman, 1949). Note that Bruner, considered to be one of the important psychologists of the twentieth century, and Postman worked on perception in the 1940's, an era where cognitive science was not yet defined as a sub discipline in psychology. In the experiment we will describe in some detail, Bruner and Postman's intention was to confront participants with stimuli they called 'incongruent'. To make clear what this term 'incongruent' means, we outline the view on perception in which Bruner and Postman framed the experiment. They worked within a view of 'perception' in which the hypothesis is advanced that a perception corresponding to a stimulus is an event in which there is, what they call, a 'construction-defense' balance at work. From a pattern of expectations, a structure present in the mind of the one who perceives, 'construction' takes place when a stimulus meets an expectation, while 'defense' takes place when a stimulus counters an expectation. In the experiment we will now describe, participants are confronted with stimuli that confirm but also contradict the structure of expectations with which participants' minds take part in the experiment. The purpose of the experiment is to distinguish and analyze different phases of dealing with this contradiction, which Bruner and Postman name 'incongruity'. We are particularly interested in one of the phases, which they call the 'dominance reaction'. Let us now first describe the experiment in Bruner & Postman (1949) and then further on specify which aspects we will use to craft our interpretation of a quantum measurement.

Twenty-eight participants, students at Harvard and Radcliffe, were shown successively by tachistoscopic exposure five different playing cards. From one to four of these cards were incongruous – color and suit were reversed. Order of presentation of normal and incongruous cards was randomized. The normal and 'trick' cards used were the following. Normal cards (printed in their proper color) five of hearts, ace of hearts, five of spades, seven of spades. Trick cards (printed with color reversed), three of hearts (black), four of hearts (black), two of spades (red), six of spades (red), ace of diamonds (black), six of clubs (red). Fourteen orders of presentation were worked out, and two subjects were presented the cards in each of these orders. There were three types of stimulus series (1) a single trick card embedded in a series of four normal cards, (2) a single normal card embedded in a series of four trick cards, (3) mixed series in which trick and normal cards were in the ratio of 3 to 2 or 2 to 3. Each card was presented successively until correct recognition occurred, three times each at 10 ms, 30 ms, 50 ms, 70 ms, 100 ms, 150 ms, 200 ms, 250 ms, 300 ms, 400 ms, 450 ms, 500 ms, and then in steps of 100 ms to 1000 ms. If at 1000 ms recognition did not occur, the next card was presented. In determining thresholds, correct recognition was defined as two successive correct responses. At each exposure, the subject was asked to report everything he or she saw or thought he or she saw. The cards were mounted on a medium gray cardboard and were shown in a Dodge-Gerbrands tachistoscope. The pre-exposure field was of the same gray color and consistency as the exposure field save that it contained no playing card. The light in the tachistoscope was provided by two G E. daylight fluorescent tubes.

For those readers who would like more details about the experiment and its analysis and conclusions, we refer to (Bruner & Postman, 1949), we are particularly interested in the different reactions of the participants, and especially this one that Bruner and Postman called the 'dominant reaction'. This dominant reaction to the cards shown in swift consists of providing a response that is entirely within the expectation pattern of the participant, namely that these are standard cards that have not been tricked. With respect to the tricked cards, this reaction consists, essentially, of a 'perceptual denial' of the incongruous elements in the stimulus pattern. Faced with a red six of spades, for example, a subject may report with considerable assurance, 'the six of spades' or the 'six of hearts', depending upon whether he is color or form bound. In the one case the form dominates and the color is assimilated to it, in the other the stimulus color dominates and form is assimilated to it. In both instances the perceptual resultant conforms with past expectations about the 'normal' nature of playing cards. We will not describe the other possible reactions further, as we are concerned with this one, incidentally, most common reaction. With longer times of showing the cards, and repeating the experiment with the same participants, a different pattern does develop, with the participants eventually finding out that tricked cards are being used – a real repetition of the experiment, to calculate e.g. the collapse probabilities, should, for this reason, always invite other participants.

In Bruner and Postman's work, perception is considered a phenomenon that, in addition to the influence of the stimulus, is also determined by the presence of the expectation pattern of the one who perceives. True, the model of human perception proposed by them contains elements that are probably only relevant to the human specificity of what perception and cognition are. And, if as a basic hypothesis of the conceptuality interpretation of quantum mechanics the quantum measurement is considered as a cognitive process, in which then also a perceptual process takes place, then, like we mentioned already, it is not meant that the specific properties of human cognition and perception will also be present. In this sense, we are primarily interested in the fundamental and general properties of cognitive and perceptual processes, the formation of concepts itself probably being the most important one. The evidence gathered so far in support of the conceptuality interpretation is largely based on the hypothesis that quantum entities are conceptual entities, hence that the cognitive process makes use of 'concepts'. The structure underpinning the phenomenon of categorical perception is already more specific, but still fundamental and general enough so that, given its abundance in human cognition, it makes sense to investigate its presence in a quantum measurement. Its identification in a quantum measurement will also allow us, as we will show, to distinguish precisely its most fundamental and general properties. But first, we want to point out some of these properties inspired by Bruner and Postman's view of human perception.

According to Bruner and Postman, the set of percepts is primarily a set of 'expectations related to the stimuli'. We will, for a quantum measurement, identify this set of percepts with the set of points on the centerline of the Bloch sphere, that is, where in our extended Bloch model the elastic is stretched. Where are the stimuli for a quantum measurement? The stimuli are assumed to be independent of any measurement and so, for a quantum measurement, they are the pure states of the entity in question, and within the Bloch sphere representation they are located in the points of the surface of the sphere. The orthogonal falling of the ball of a point of the sphere surface on the elastic band represents the interaction of the cognitive apparatus with the entity under consideration. This means that the transformation from a point on the surface of the Bloch sphere, say the point A, to a point on the centerline where the elastic is stretched, say the point A', represents the interaction between expectation, starting from the cognitive apparatus, on the one hand, and perception, starting from the considered entity, on the other.

If we want to give an expression to the mechanism of categorical perception, we must be able to estimate and/or measure distances between both pure states, the stimuli, located in points on the surface of the Bloch sphere, and density states, the percepts, located in the points of a centerline of the Bloch sphere. Using these distances, the differences between stimuli and the differences between percepts can be expressed quantitatively. However, we must be careful in choosing how we will measure distances in the Bloch sphere representation, since there is no unique obvious metric on the whole set of quantum states. One of the well-studied partial metrics on the set of quantum states is the trace distance

$$T(D_1, D_2) = \frac{1}{2}Tr\left[\sqrt{(D_1 - D_2)^*(D_1 - D_2)}\right] \tag{33}$$

It can be shown that for a two-dimensional quantum entity, i.e., a qubit, modeled with the Bloch sphere, the trace distance is equal to half the Euclidean distance in the three-dimensional Euclidean space of which the Bloch sphere is a part. For two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, hence represented by two points of the surface of the Bloch sphere, it can be shown that the trace distance is given by

$$T(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2} \tag{34}$$

That means, for example, that the trace distances between the quantum state where A is and the North and South Pole of the Bloch sphere are given by

$$\begin{aligned} T(D_{(1,\theta,\phi)}, D_{(1,0,0)}) &= \sqrt{1 - |\langle\theta,\phi|0,0\rangle|^2} = \sqrt{1 - |\langle(\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})|(1,0)\rangle|^2} \\ &= \sqrt{1 - \cos^2\frac{\theta}{2}} = \sin\frac{\theta}{2} \end{aligned} \tag{35}$$

$$\begin{aligned} T(D_{(1,\theta,\phi)}, D_{(1,\pi,0)}) &= \sqrt{1 - |\langle\theta,\phi|\pi,0\rangle|^2} = \sqrt{1 - |\langle(\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})|(0,1)\rangle|^2} \\ &= \sqrt{1 - \sin^2\frac{\theta}{2}} = \cos\frac{\theta}{2} \end{aligned} \tag{36}$$

We can now see interesting connections using the simple geometry of the Bloch representation, let us consider Figure 3. It represents the Bloch representation with $\phi = \frac{\pi}{2}$, hence A laying in the zy-plane. The triangle A_{up}, A, A_{down} is a right-angled triangle with right angle in A. The hypothenuse of this right-angled triangle is the centerline connecting A_{up} to A_{down} and it has length equal to 2.

The angle in A_{down} of the triangle is equal to half the angle θ, and so the length of the line segment connecting A_{up} to A is equal to $2\sin\frac{\theta}{2}$. In a similar manner, the figure shows us that the length of the line segment connecting A and A_{down} is given by $2\cos\frac{\theta}{2}$. This agrees with the calculation of the trace distance we made for the states connected to corresponding points of the Bloch sphere, as given in (35) and (36), reminding us that for a qubit the trace distance is equal to half the Euclidean distance. This reasoning and additional calculations, also the general expression for the trace distance between pure states, makes it clear that the trace distance is not the metric to be used between pure states. Indeed, any effect due to quantum coherence disappears when using this metric, since in the general expression (34) the only reference to the states is the inner product between the two states, and hence traces of quantum coherence effects are no longer present.

There is also a purely geometric way by which we can see that the trace distance is unsuitable for measuring distances between pure states. Consider any two pure states, which gives us two points of the surface of the Bloch sphere. If we connect these two points with a line, we see that the points of the line different from these two points always lie in the interior of the Bloch sphere and thus correspond to density states. It is clear that defining a distance associated with a specific geometric entity, the straight line in the three-dimensional Euclidean space to which the Bloch sphere belongs, carries limitations when, as is the case with pure states, this distance is intrinsically intended for elements of a subset with a structure in which the straight line is not contained, and we mean here the spherical surface of the Bloch sphere, to which the pure states are restricted. So we must ask ourselves at this point of our analysis

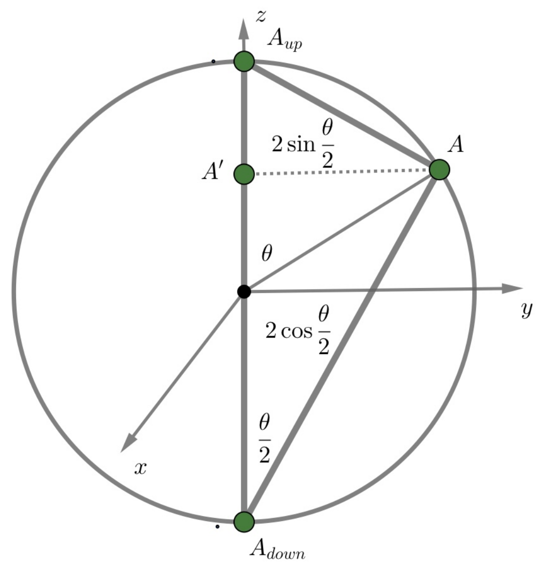

Figure 3: Using the simple geometry of the Bloch sphere, we can recover the trace distance we calculated in (35) and (36) as illustrated in this figure. The triangle A_{up}, A, A_{down} is rectangular which makes its angle in A_{down} equal to $\frac{\theta}{2}$, and consequently, knowing that the distance between A_{up} and A_{down} is equal to 2, the figure shows us that the distance between A_{up} and A is equal to $2\sin\frac{\theta}{2}$, and the distance between A_{down} and A is equal to $2\cos\frac{\theta}{2}$. Since the trace distance is equal to half the Euclidean distance for the Bloch sphere of a qubit, we have derived the results of (35) and (36) from the geometry of the Bloch sphere.

which the proper metric to be used on pure quantum states is. A natural notion of distance between pure states should only account for the pure states, that are in-between the points on the Bloch sphere representing these states, the length of the circular arc connecting these points would be such a quantity. More precisely, if θ and α are the polar angles of the pure states $|\theta, \phi\rangle$ and $|\alpha, \phi\rangle$ (see Figure 4), the distance to be used to measure how much they are separated from each other, normalized to 1, would then be

$$d_{pure}(|\theta, \phi\rangle, |\alpha, \phi\rangle) = \frac{1}{\pi}|\theta - \alpha| \tag{37}$$

Note that it is sufficient to define the distance for two pure states with equal azimuthal angle ϕ. Indeed, we can always choose a North Pole and South Pole for any two pure states such that both states lie in the plane with equal azimuthal angle, so that then the arc of the circle through both points is determined by the difference of the polar angles of both states.

This distance also results from the angle between pure states calculated from their Hilbert space inner product, which shows that it is a natural distance associated with the Bloch representation of pure states. The notion of fidelity

$$F(D_1, D_2) = \left(tr\sqrt{\sqrt{D_1}D_2\sqrt{D_1}} \right)^2 \tag{38}$$

which can also be defined on the density states, generalizes this inner product, because for pure states ψ_1 and ψ_2 we have

$$F(\psi_1, \psi_2) = |\langle\psi_1|\psi_2\rangle|^2 \tag{39}$$

but does not give rise to a metric over the entire set of quantum states. But if we consider angles, or equivalently circular arcs, as a measure of the natural distance between pure states in the Bloch model, we are actually using the notion of fidelity to evaluate differences between stimuli.

So, we will use two different notions of distance, one that considers the circular arc between two vector-states, at the surface of the Bloch sphere, and the other one which considers the Euclidean distance between density states, inside the Bloch sphere. The normalized to 1 distance between two such decohered density states

$$\begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 \\ 0 & \sin^2\frac{\alpha}{2} \end{pmatrix} \tag{40}$$

will be given by (see Figure 4)

$$d_{density}(\theta, \alpha) = \frac{1}{2}|\cos\theta - \cos\alpha| \tag{41}$$

Equipped with these two distances, let us now analyse how the phenomenon of categorical perception is naturally expressed in a quantum measurement process. As we mentioned, the points of the sphere's diameter where the elastic band is stretched, is where the percepts lie, for the measurement in question. These percepts represent an 'expected reality', which however is not to be understood as a collection of wild speculations, but rather as the best picture of what is real in relation to the meaning carried by the measured entity, but adapted to the context which the region where the elastic is stretched represents within the mathematical model of the quantum measurement. So the region where the elastic lies does not simply represent an 'expected reality', as might be misunderstood from perception theory a la Bruner and Postman, and from the experiment we described, a more correct way to describe what this region represents is the following. 'That which can be put forward from the context of the measuring apparatus and the cognitive process that takes place as the best hypothesis about reality'. And, as a form of contextual limitation, portions of expected reality creep in to this as a consequence of attempting the best hypothesis about non-contextual reality. Also some contextual biases cannot be avoided, and the warping of categorical perception is one of them. More precisely still, the one who perceives is actively engaged during a measurement process and his or her 'expected reality' plays a role in what will occur, together with the reality of the measured entity, expressed by its pure state, which is instead measurement independent. At the level of the contextual region where the elastic lies, the model is Kolmogorovian, the probabilities being an expression of the lack of knowledge of the one who perceives. This is realized in the extended Bloch model by the unpredictable point λ where the elastic breaks. Also, the connection between the reality of the considered entity (the pure states describing the stimuli) and the elements of the contextual reality (the on-elastic density states describing the percepts for the given measurement), is illustrated by the deterministic orthogonal fall of the point particle representative of the stimulus onto the elastic band, transforming it into a percept, thus reaching a stage where an outcome (an answer) is actualized, and we are back to a pure state.

To see how a quantum measurement brings about the warping effect of categorical perception, let us consider, to fix ideas, a situation where only two colors exist, *Light* and *Dark*, so we are precisely in a situation that can be described in a three-dimensional Bloch sphere. Note that Eleanor Rosch formulated the rationale for the prototype theory for concepts while teaching colors to a primitive community in Papua New Guinea, whose language, called Berinomo, has just two names for colors (Rosch 1973).

Let us locate the first color, *Light*, at the North Pole of the Bloch sphere, and the second color, *Dark*, at its South Pole. At the equator, the transition from *Light* to *Dark* will then occur (see Figure 4). Let us then introduce three different pure states, the first one located in

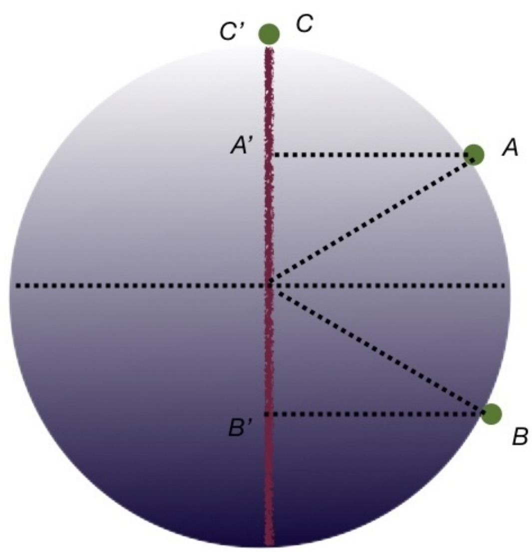

Figure 4: We consider a situation where there are two names for colors, which we call *Light* and *Dark*, and wish to show that the quantum measurement model, which in this case represents a 'qubit', incorporates the phenomenon of categorical perception. For this, we consider three pure states $|\pi/3, \phi\rangle$, $|2\pi/3, \phi\rangle$ and $|0, \phi\rangle$, respectively, in the Bloch representation located in points A, B and C, which represent stimuli associated with a quantum measurement on this qubit. With each of the three pure states corresponds a density state in which the qubit is located after the measurement, localized in the Bloch representation in respectively points A', B' and C', and described by respectively density matrices (42), (43) and (44). The pure states in A and B belong to two different colors, *Light* and *Dark*, and lie at a distance 1/3 from each other. The density states corresponding to them, located in points A' and B', are at a distance 1/2 from each other. We see here the dilation mechanism of categorical reception at work, for percepts belonging to different categories, *Light* and *Dark*. The pure states in C and A belong to the same color, *Light*, and also lie at a distance 1/3 from each other. The density states corresponding to C and A, located in points C' and A', are at a distance 1/4 from each other. We see here the contraction mechanism of categorical reception at work, for percepts belonging to the same category.

point A (see Figure 4) represented by the vector $|\pi/3, \phi\rangle$, hence with polar angle $\theta = \pi/3$, the second one located in point B (see Figure 4) represented by the vector $|2\pi/3, \phi\rangle$, hence with polar angle $\theta = 2\pi/3$, and the third one located in the North Pole (see Figure 4), represented by the vector $|0, \phi\rangle$, (hence, this is the eigenstate describing *Light*, with a polar angle 0), assuming for simplicity that they all lie on a same plane, hence have the same asimuthal angle ϕ. When a *Light-Dark* color-measurement is performed, the pre-measurement pure states deterministically transform into the fully decohered pre-collapse density states, obtained by plunging the associated point particles into the sphere, orthogonally with respect to the line subtended by the *Light* and *Dark* outcome states, which is the region of the percepts, i.e., of the 'contextual reality' relative to this specific color-measurement. This results for the pure states located in A, B and C, respectively, in density states located in points A', B' and C', and given, respectively, by the density matrices

$$\begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \tag{43}$$

$$\text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{44}$$

If we consider the pure states $|\frac{\pi}{3}, \phi\rangle$ and $|\frac{2\pi}{3}, \phi\rangle$, located respectively in points A and B, they are in the *Light* and *Dark* hemispheres of the Bloch sphere, hence they are two different colors, and we can now easily see that their transformation to the corresponding decohered density states, located respectively in points A' and B' (see Figure 4), and described respectively by the density matrices (42) and (43) exhibits a warping that is a dilation. Indeed, their distance is $1/3$, i.e., one third of the maximal distance between two stimuli, while the associated percepts, corresponding to the decohered quantum states, have a distance of $1/2$, i.e., one half of the maximal distance between two percepts. On the other hand, if we consider the pure states $|0, \phi\rangle$ and $|\frac{\pi}{3}, \phi\rangle$, located respectively in points C and A, belonging to the same color, namely the color *Light*, the opposite warping occurs. Indeed, on the stimuli side, the distance is again $1/3$, whereas on the percepts side, the corresponding density states being located in points C' and A', respectiveky, and described by the density matrices (44) and (42), we now have distance $1/4$. This means that a warping takes place which is now a contraction. This shows how the phenomenon of categorical perception is built into a the quantum measurement.

5 Visual Perception, Colors and Quanta

Let us analyze what the events are that correspond to this way of using the quantum measurement model to model the phenomenon of categorical perception of colors. And let us start from the situation where the stimulus, as measured by a physical measuring device that measures the frequency of electromagnetic radiation, is perceived by a human being, and from what we know takes place then.

Suppose the light being looked at has a frequency of 595 Terahertz, given that 1 Terahertz equals 10^{12} Hertz, which corresponds to a wavelength of about 504 nanometers. If we consult one of the many color spectra of visible light available on the World-Wide Web, we can see that 504 nanometers wavelength corresponds to a color right in between green and blue, hence light of this frequency possesses a *Blue-Green* color, but it is not obvious whether when someone is asked to classify this light among one of the basic or prototype colors, whether that person will pick either *Blue* or *Green*. Each region that makes a transition from one basic color to another basic color contains a strip where it is really not clear which of the adjacent basic colors will be chosen if such a choice is imposed, for example in an experiment.

That being said, we plan an experiment where we will show certain shades of color and on a Likert scale have participants choose to classify these shades of color with one of the basic colors. We will then investigate the extent to which we can model the shades of color in a complex Hilbert space as superpositions of the basis vectors each representing one of the basic colors. Let us note that in our example of the shade *Blue-Green* carried by light of frequency 595 Terahertz, this is a color carried by very pure light of exact frequency. We already noted that the prototype colors for the English language are 'not' the colors of the rainbow, for example the color *Brown* is not realized by light of a specific frequency. Colors of reflected light can represent very complex mixtures of different frequencies, just think of how white light is obtained by rotating a color disk rapidly, what happens there is very complex when the stimulus in question, the light shone from the rotating disk, is examined from physics. For what colors 'are', what happens in our eye on the retina, but especially in our brain, and perhaps just as importantly, in and with our mind, is fundamental. Given the eleven prototype colors considered as basic colors by Berlin & Kay (1969) and Rosch (1973), for our planned experiment we will also consider these colors as basic colors. This means that the set of all superpositions are the elements, more precisely the unit vectors, of an eleven-dimensional complex Hilbert space, with a basis of orthonormal vectors each of which represents one of the considered basic colors. Although we can also explicitly build the Bloch representation for this situation of eleven basic colors, and we plan to do so after we have collected the data from our experiment, we will not become explicit about it in this article, because the more than two-dimensional Bloch representations are more complicated, and we have focused, at least in part also for reasons of simplicity and intelligibility, but also because we wanted to turn to the fundamental in this article, to the situation of two colors, *Light* and *Dark*. After all, the fundamental, namely the way in which stimuli and expectations play a role in determining percepts, is equally and fully present in this situation of two colors.

So let us return to the situation of two colors that we have called *Light* and *Dark*. Let it also be noted here that a restriction of vision to *Light* and *Dark*, and thus shades of gray, still leads to a powerful use of the sense of 'sight'. Indeed, there are several animals that see only shades of gray, and this group is called the 'monochromatic mammals'. Several bats, rodents and the common raccoon are among them. One common denominator among these mentioned species is that they are nocturnal animals, so their monochromatic vision gives them a special advantage at night. The shade of gray that contains the most uncertainty regarding its classability as *Light* or *Dark* is the shade that lies at the equator of the Bloch sphere (see Figure 4). It is plausible to associate a probability equal to 1/2 with the point at the center of the elastic that runs between North and South Poles in the extended Bloch model.

Are the probabilities proposed by the Bloch model also obtained in the limit of the fractions obtained for the outcomes when an experiment is conducted to measure them? Regarding this question, we should mention the result of research published in Aerts & Sassoli de Bianchi (2015). If we imagine different individuals and the way they will decide to classify a particular shade of gray as *Light* rather than *Dark* or vice versa, it is very plausible to assume that there will be a fair amount of individual differences. Some will still classify grays that are above the equator as *Dark* while others will still classify grays that are below the equator as

Light. It is shown in Aerts & Sassoli de Bianchi (2015) that the average over such individual differences tends to converge to a uniform distribution, which would then correspond to the quantum probabilities proposed in the previous sections of this paper. That is, the quantum measurement model would be a kind of first-order approximation to more complex measurement situations, which would then also explain why in cases where control over such individual differences is absent, it leads to good predictive models. It is beyond the scope of our current article to explain this complex situation of 'quantum as first order approximation' in more detail, we refer the interested reader to Aerts & Sassoli de Bianchi (2017) for a very detailed account of this situation.

Let us now come to our argument in connection with colors being quanta of light for human vision, and so for a raccoon these are only two colors *Light* and *Dark*, while for English-speaking humans they are eleven, *White*, *Black*, *Red*, *Yellow*, *Green*, *Blue*, *Brown*, *Purple*, *Pink*, *Orange* and *Gray*. Let us explain why these eleven colors are quanta of light for English-speaking humans. Suppose a friends has a bicycle colored with exactly the color of light corresponding to a frequency of 595 Terahertz. And suppose for a moment that a conversation develops about this bicycle, perhaps it is lost and they are looking for it. The color might then come up. Perhaps *Green* will be specified as the color when asked, or possibly *Blue* will be named. Or, when precision is important, *Blue-Green* is said. All three possibilities bring probative value to our claim that the eleven prototype colors are quanta of light for English-speaking people. Indeed, even if *Blue-Green* is given as the color for the bicycle to be found, this word *Blue-Green* is a combination of two other words, *Blue* and *Green*. Consider the pet-fish problem we mentioned in Section 4. Indeed, in quantum cognition, such combinations of words are described by superpositions of the vectors that describe the participles of these combinations (Aerts & Gabora, 2005a,b; Aerts et al., 2012). The cultural fact that it was decided in English to include these eleven designations as separate words in the English language, but not to introduce a separate word for the color that light of 595 Terahertz shines around, is what determines their status as quanta. Indeed, it is the dynamics produced by the phenomenon of categorical perception, the mechanism that causes certain stimuli, that is, in this case of colors, certain frequencies of light, to contract so that it becomes meaningful within the cognitive interaction to classify them all as *Green*, while other stimuli are pushed apart so that it becomes meaningful within the same cognitive interaction to classify them as different colors, *Green* and *Blue*, for example, that we wish to call 'quantization'. It is our hypothesis that the same contraction and pushing apart occurs in the quanta that light shows to physical measuring devices and their cognitive activities, so that, in the case of light, the photons emerge as quanta. The process of quantization ends, or better stabilizes, as an optimum of this cognitive interaction.

This means that as far as human cognition is concerned, the interaction that takes place between the stimulus, on the one hand, and the expectation pattern of the person perceiving, on the other hand, in order to arrive at perception, is of a greater complexity than the experiment with the tricked cards described in brunerpostman1949 suggests. It is likewise of greater complexity than the dynamics of elastics breaking as we suggested in the extended Bloch model. That does not mean, however, that these models are irrelevant to making progress in the depth of what human perception and cognition are. Indeed, they both contain essential elements of this human perception and cognition and suggest that more complex interactions need to be expected. But, the result demonstrated in Aerts & Sassoli de Bianchi (2015, 2017), makes it clear that such a more complex situation of interaction between the measuring device and the entity being measured should not necessarily lead to the need for a more complex model of measurement, at least not if we are satisfied with the linear model proposed by quantum mechanics being a first order approximation of what takes place during measurement. So it is especially in comparison to a classical description that assumes that no interaction at all takes place between measuring device and the entity being measured that the first order approximation of the linear quantum model is an essential step forward.

References

Aerts, D. (1982). Description of many physical entities without the paradoxes encountered in quantum mechanics. *Foundations of Physics 12*, pp. 1131-1170. doi: 10.1007/BF00729621.

Aerts, D. (1986). A possible explanation of the probabilities of quantum mechanics. *Journal of Mathematical Physics 27*, pp. 202-210. doi: 10.1063/1.527362.

Aerts, D. (1992). The construction of reality and its influence on the understanding of quantum structures. *International Journal of Theoretical Physics 31*, pp. 1815-1837. doi: 10.1007/BF00678294.

Aerts, D. (2009). Quantum structure in cognition. *Journal of Mathematical Psychology 53*, pp. 314-348. doi: 10.1016/j.jmp.2009.04.005.

Aerts, D. (2009). Quantum particles as conceptual entities: A possible explanatory framework for quantum theory. *Foundations of Science 14*, pp. 361-411. doi: 10.1007/s10699-009-9166-y.

Aerts, D. and Aerts, S. (1995). Applications of quantum statistics in psychological studies of decision processes. *Foundations of Science 1*, pp. 85-97. doi: 10.1007/BF00208726.

Aerts, D. and Aerts Arguëlles, J. (2022). Human perception as a phenomenon of quantization. Entropy 24, 1207. doi: 10.3390/e24091207.

Aerts, D. and Beltran, L. (2020). Quantum Structure in Cognition: Human Language as a Boson Gas of Entangled Words. *Foundations of Science 25*, pp. 755-802. doi: 10.1007/s10699-019-09633-4.

Aerts, D. and Beltran, L. (2022a). Are words the quanta of human language? Extending the domain of quantum cognition. *Entropy 24*, 6. doi: 10.3390/e24010006.

Aerts, D. and Beltran, L. (2022b). A Planck Radiation and Quantization Scheme for Human Cognition and Language. Frontiers in Psychology 13, 850725. doi: 10.3389/fpsyg.2022.850725.

Aerts, D., Broekaert, J., Gabora, L. and Veloz, T. (2012). The guppy effect as interference. Quantum Interaction. *Lecture Notes in Computer Science 7620*, pp 36-47. doi: 10.1007/978-3-642-35659-9_4.

Aerts, D and Gabora, L. (2005a). A theory of concepts and their combinations I: The structure of the sets of contexts and properties.*Kybernetes 34*, pp. 167-191. doi: 10.1108/03684920510575799.

Aerts, D and Gabora, L. (2005b). A theory of concepts and their combinations II: A Hilbert space representation. *Kybernetes 34*, pp. 192-221. doi: 10.1108/03684920510575807.

Aerts, D. and Sassoli de Bianchi, M. (2014). The extended Bloch representation of quantum mechanics and the hidden-measurement solution to the measurement problem. *Annals of Physics 351*, pp. 975-1025. doi: 10.1016/j.aop.2014.09.020. See also the Erratum: *Annals of Physics 366*, pp. 197-198 (2016). doi: 10.1016/j.aop.2016.01.001.

Aerts, D. and Sassoli de Bianchi, M. (2015). The unreasonable success of quantum probability II: Quantum measurements as universal measurements. *Journal of Mathematical Psychology 67*, pp. 76-90. doi: 10.1016/j.jmp.2015.05.001.

Aerts, D. and Sassoli de Bianchi, M. (2016). The extended Bloch representation of quantum

mechanics: Explaining superposition, interference, and entanglement. *Journal of Mathematical Physics 57*. 122110. doi: 10.1063/1.4973356.
Aerts, D. and Sassoli de Bianchi, M. (2017). *Universal Measurements: How to Free Three Birds in One Move*. Singapore: World Scientific.
Aerts, D., Sassoli de Bianchi, M., Sozzo, S. and Veloz, T. (2020). On the conceptuality interpretation of quantum and relativity theories. *Foundations of Science 25*, pp. 5–54. doi: 10.1007/s10699-018-9557-z.
Aerts, D. and Sozzo, S. (2011). Quantum structure in cognition: Why and how concepts are entangled. *Quantum Interaction QI2011. Lecture Notes in Computer Science 7052*, pp. 116-127. doi: 10.1007/978-3-642-24971-6_12.
Aerts, D. and Sozzo, S. (2014). Quantum entanglement in concept combinations. *International Journal of Theoretical Physics 53*, pp. 3587–3603. doi: 10.1007/s10773-013-1946-z.
Aerts Arguëlles, J. (2018). The heart of an image: Quantum superposition and entanglement in visual perception. *Foundations of Science 23*, pp. 757-778. doi: 10.1007/s10699-018-9547-1.
Aerts Arguëlles, J. and Sozzo, S. (2020). How images combine meaning. Quantum entanglement in visual perception. *Soft Computing 24*, pp. 10277-10286. doi: 10.1007/s00500-020-04692-3.
Berlin, B. and Kay, P. (1969). *Basic Color Terms: Their Universality and Evolution*. Berkeley: University of California Press.
Blutner, R. and beim Graben, P. (2016). Quantum cognition and bounded rationality. *Synthese 193*, pp. 3239-3291. doi: 10.1007/s11229-015-0928-5.
Bruner, J. S. and Postman, L. (1949). On the perception of incongruity: A paradigm. *Journal of Personality 18*, pp. 206-223. doi: 10.1111/j.1467-6494.1949.tb01241.x.
Bruza, P. and Gabora, L. (Eds.) (2009). Special Issue: Quantum Cognition. *Journal of Mathematical Psychology 53*, pp. 303-452. doi: 10.1016/j.jmp.2009.06.002.
Busemeyer, J. and Bruza, P. (2012). *Quantum Models of Cognition and Decision*. Cambridge: Cambridge University Press.
Busemeyer, J., Kvam, P. and Pleskac, T. (2019). Markov versus quantum dynamic models of belief change during evidence monitoring. *Scientific Reports 9*, 18025. doi: 10.1038/s41598-019-54383-9.
Busemeyer, J. R., Wang, Z. and Townsend, J. T. (2006). Quantum dynamics of human decision making. *Journal of Mathematical Psychology 50*, pp. 220-241. doi: 10.1016/j.jmp.2006.01.003.
Collier, G. A., Berlin, B. and Kay, P. (1973). Basic color terms: Their universality and evolution. *Language 49*, pp. 245-248. doi: 10.2307/412128.ISSN0097-8507.
Dalla Chiara, M. L., Giuntini, R., Leporini, R., Negri, E. and Sergioli, G. (2015). Quantum information, cognition, and music. *Frontiers in Psychology 6*, 1583. doi: 10.3389/fpsyg.2015.01583.
Davidoff, J. (2001). Language and perceptual categorisation. *Trends in Cognitive Sciences 5* pp. 382-387. doi: 10.1016/s1364-6613(00)01726-5.
Davies, I. R. L., Sowden, P. T., Jerrett, D. T., Jerrett, T. and Corbett, G. G. (1998). A cross-cultural study of English and Setswana speakers on a colour triads task: A test of the Sapir-Whorf hypothesis. *British Journal of Psychology. 89*, pp. 1-15. doi: 10.1111/j.2044-8295.1998.tb02669.x.
Dirac, P. A. M. (1939). A new notation for quantum mechanics. *Mathematical Proceedings of the Cambridge Philosophical Society 35* pp. 416-418. doi: 10.1017/S0305004100021162.
Disa, S., LeGuen, O. and Haun, D. (2011). Categorical perception of emotional facial ex-

pressions does not require lexical categories. *Emotion 11*, pp. 1479-1483. doi: 10.1037/a0025336.
Eimas, P. D., Siqueland, E. R., Jusczyk, P. W. and Vigorito, J. (1971). Speech perception in infants. *Science 171*, pp. 303-306. doi: 10.1126/science.171.3968.303.
Geeraerts, D., Dirven, R., Taylor, J. R. and Langacker, R. W. (Eds.), (2001). *Applied Cognitive Linguistics, II, Language Pedagogy.* doi: 10.1515/9783110866254.
Goldstone, R.L. and Hendrickson, A.T. (2010). Categorical perception. *Wires Cognitive Science 1*, pp. 69–78. doi: 10.1002/wcs.26.
Gudder, S., Zanghí, N. Probability models. *Nuovo Cimento B 79*, pp. 291-301. doi: 10.1007/BF02748978.
Harnad, S. (Ed.). (1987). *Categorical Perception: The Groundwork of Cognition.* Cambridge, UK: Cambridge University Press.
Haven, E. and Khrennikov, A. (2013). *Quantum Social Science.* Cambridge: Cambridge University Press.
Johansen, M. K. and Kruschke, J. K. (2005). Category representation for classification and feature inference. *Journal of Experimental Psychology: Learning, Memory, and Cognition 31*, pp. 1433-1458. doi: 10.1037/0278-7393.31.6.1433.
Havy, M. and Waxman, S. R. (2016). Naming influences 9-month-olds' identification of discrete categories along a perceptual continuum. *Cognition 156*, pp. 41-51. doi: 10.1016/j.cognition.2016.07.011.
Hess, U., Adams, R. and Kleck, R (2009). The categorical perception of emotions and traits. *Social Cognition 27*, pp. 320-326. doi: 10.1521/soco.2009.27.2.320.
Khrennikov, A. (2014). *Ubiquitous Quantum Structure.* Berlin: Springer.
Lane, H. (1965). The motor theory of speech perception: A critical review. *Psychological Review 72*, pp. 275-309. doi: 10.1037/h0021986.
Lawrence, D. H. (1949). Acquired distinctiveness of cues: I. Transfer between discriminations on the basis of familiarity with the stimulus. *Journal of Experimental Psychology 39*, pp. 770-784. doi: 10.1037/h0058097.
Liberman, A. M., Cooper, F. S., Shankweiler, D. P., Studdert-Kennedy, M. (1967). Perception of the speech code. *Psychological Review 74* pp. 431-461. doi:10.1037/h0020279.
Liberman, A. M., Harris, K. S., Hoffman, H. S. and Griffith, B. C. (1957). The discrimination of speech sounds within and across phoneme boundaries. *Journal of Experimental Psychology 54*, pp. 358-368. doi:10.1037/h0044417.
Medin, D. L., Altom, M. W. and Murphy, T. D. (1984). Given versus induced category representations: Use of prototype and exemplar information in classification. *Journal of Experimental Psychology: Learning, Memory, and Cognition 10*, pp. 333-352. doi: 10.1037/0278-7393.10.3.333.
Mervis, B. C., Catlin, J. and Rosch, E. (1975). Development of the structure of color categories. *Developmental Psychology 2*, pp. 54-60. doi: 10.1037/h0076118.
Moreira, C. and Wichert, A. (2016). Quantum probabilistic models revisited: the case of disjunction effects in cognition. *Frontiers in Physics 4*. 26. doi: 10.3389/fphy.2016.00026.
Osherson, D. N. and Smith, E. E. (1981). On the adequacy of prototype theory as a theory of concepts. *Cognition 9* pp. 35-58. doi:10.1016/0010-0277(81)90013-5.
Pothos, E. M., Barque-Duran, A., Yearsley, J. M., Trueblood, J. S., Busemeyer, J. R. and Hampton, J. A. (2015). Progress and current challenges with the quantum similarity model. *Frontiers in Psychology 6*, 205. doi: 10.3389/fpsyg.2015.00205.
Regier, T. and Kay, P. (2009). Language, thought, and color: Whorf was half right. *Trends in Cognitive Sciences 13*, pp. 439-447. doi: 10.1016/j.tics.2009.07.001.
Rosch, E. H. (1973). Natural categories. *Cognitive Psychology 4*, pp. 328-350. doi: 10.1016/

0010-0285(73)90017-0.
Rosch Heider, E. (1971). "Focal" Color areas and the development of color names. *Developmental Psychology 4*, pp. 447-455. doi: 10.1037/h0030955.
Rosch Heider, E. (1972). Universals in color naming and memory. *Journal of Experimental Psychology 93*, pp. 10-20. doi: 10.1037/h0032606.
Rosch, Eleanor (1975). Cognitive representations of semantic categories. *Journal of Experimental Psychology: General 104*, pp. 192-233. doi: 10.1037//0096-3445.104.3.192. doi: 10.1037/h0032606.
Rosch, E., Mervis, C. B, Gray, W. D, Johnson, D. M and Boyes-Braem, P. (1976). Basic objects in natural categories. *Cognitive Psychology. 8*, pp. 382-439. doi: 10.1016/0010-0285(76)90013-X.
Schusterman, R. J., Reichmuth, C. J. and Kastak, D. (2000). How animals classify friends and foes. *Current Directions in Psychological Science 9*, pp. 1-6. doi: 10.1111/1467-8721.00047.
Sidman M. (1994). *Equivalence Relations and Behavior: A Research Story.* Boston, MA: Authors Cooperative.
Smith, E. E. and Medin, D. L. (1981). *Categories and Concepts.* Cambridge MA: Harvard University Press.
Surov, I. A., Pilkevich, S. V., Alodjants, A. P. and Khmelevsky, S. V. (2019). Quantum phase stability in human cognition. *Frontiers in Psychology 10*, 929. doi: 10.3389/fpsyg.2019.00929.
Yearsley, JM. (2017). Advanced tools and concepts for quantum cognition: a tutorial. *Journal of Mathematical Psychology 78*, pp. 24-39. doi: 10.1016/j.jmp.2016.07.005.