CORRIGENDUM TO "A NEW UNIQUENESS THEOREM FOR THE TIGHT C*-ALGEBRA OF AN INVERSE SEMIGROUP" [C. R. MATH. ACAD. SCI. SOC. R. CAN. 44 (2022), NO. 4, 88–112]

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Abstract. We correct the proof of Theorem 4.1 from [C. R. Math. Acad. Sci. Soc. R. Can. **44** (2022), no. 4, 88–112].

1. Introduction

There is a flaw in the proof of [St22, Theorem 4.1]. To explain this, let us recall the setup for [St22, Theorem 4.1]. Let P be a right LCM monoid, i.e., a left cancellative monoid such that for all $p, q \in P$, the intersection $pP \cap qP$ is either empty or of the form rP for some $r \in P$. Let $S = \{[p, q] : p, q \in P\} \cup \{0\}$ be the inverse semigroup associated with P in [St15, Proposition 3.2]. (If P is not left reversible, then S is isomorphic to the left inverse hull of P via the map that sends 0 to 0 and sends $[p, q]$ to the partial bijection $qP \to pP$ given by $qx \mapsto px$.) Let

$$S^{\mathrm{Iso}} := \{[p, q] : paP \cap qaP \neq \emptyset \text{ for all } a \in P\}$$

be the inverse semigroup from [St22, Section 4], and denote by $\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}})$ and $\mathcal{G}_{\mathrm{tight}}(S)$ the tight groupoids of S^{Iso} and S, respectively (see [Ex08] and [EP16]). The C*-algebra $C_r^*(\mathcal{G}_{\mathrm{tight}}(S))$ is the reduced boundary quotient C*-algebra of P. Since $\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}})$ is identified with an open subgroupoid of $\mathcal{G}_{\mathrm{tight}}(S)$, there is a canonical inclusion of reduced groupoid C*-algebras $C_r^*(\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}})) \subseteq C_r^*(\mathcal{G}_{\mathrm{tight}}(S))$. Explicitly, we have

$$C_r^*(\mathcal{G}_{\mathrm{tight}}(S)) = \overline{\mathrm{span}}(\{T_{[p,q]} : [p, q] \in S\}),$$

where $T_{[p,q]}$ is the characteristic function of the compact open bisection $\Theta([p, q], D_{qP})$ (see [St22, Section 3] for this notation), and $C_r^*(\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}}))$ is identified with the C*-subalgebra generated by the partial isometries $T_{[p,q]}$ for $[p, q] \in S^{\mathrm{Iso}}$.

Assume that S satisfies condition (H) from [St22, Definition 3.1], and suppose $\pi\colon C_r^*(\mathcal{G}_{\mathrm{tight}}(S)) \to B$ is a representation in a C*-algebra B. Then, [St22, Theorem 3.4] says that π is injective if and only if its restriction to $C_r^*(\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}}))$ is injective. It is asserted in the proof of [St22, Theorem 4.1] that to prove this latter claim, it suffices to prove that π is injective on the dense *-subalgebra $A_0 := \mathrm{span}(\{T_{[p,q]} : [p, q] \in S^{\mathrm{Iso}}\})$. However, this assertion is false: If we take $P = \mathbb{Z}$, then $C_r^*(\mathcal{G}_{\mathrm{tight}}(S^{\mathrm{Iso}})) = C_r^*(\mathcal{G}_{\mathrm{tight}}(S)) \cong C^*(\mathbb{Z})$, and under the canonical isomorphism $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, A_0 is carried onto the *-subalgebra of Laurent polynomials in $C(\mathbb{T})$. Given any infinite, proper compact subset $K \subseteq \mathbb{T}$,

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the map $C(\mathbb{T}) \to C(K)$ given by $f \mapsto f|_K$ is a non-injective *-homomorphism that is injective on A_0.

2. The proof of [St22, Theorem 4.1]

We give a proof of [St22, Theorem 4.1]. We shall use the notation from [St22] freely. The core submonoid of P is

$$P_c := \{p \in P : pP \cap qP \neq \emptyset \text{ for all } q \in P\},$$

with associated inverse semigroup

$$S_c := \{[p,q] : p, q \in P_c\}.$$

[St22, Theorem 4.1] is stated with the assumption that the full and reduced groupoid C*-algebras of the tight groupoid $\mathcal{G}_{\text{tight}}(S)$ coincide. It is clear that the following version stated for reduced groupoid C*-algebras implies [St22, Theorem 4.1].

Theorem 2.1. *Let P be a right LCM monoid and S the associated inverse semigroup as in* [St15, Proposition 3.2]*, let $\mathcal{Q}_r(P) = C_r^*(\mathcal{G}_{tight}(S))$ denote its reduced boundary quotient C^*-algebra, and let $\mathcal{Q}_{r,c}(P) = C^*(T_{[p,q]} : p, q \in P_c) \subseteq \mathcal{Q}_r(P)$ be the C^*-subalgebra generated by the core submonoid. Suppose that S satisfies condition (H) from* [St22, Definition 3.1]. *Then, a $*$-homomorphism $\pi : \mathcal{Q}_r(P) \to B$ to a C^*-algebra B is injective if and only if it is injective on $\mathcal{Q}_{r,c}(P)$.*

Proof. Let $\pi : \mathcal{Q}_r(P) \to B$ be a $*$-homomorphism that is injective on $\mathcal{Q}_{r,c}(P)$. We wish to show that π is injective, and by [St22, Theorem 3.4] it is enough to show that π is injective on $A := C^*(T_{[p,q]} : [p,q] \in S^{\text{Iso}})$. If P is left reversible, then $P = P_c$ and there is nothing to prove, so assume P is not left reversible. Since S satisfies (H) from [St22, Definition 3.1], the groupoid $\mathcal{G}_{\text{tight}}(S)$ is Hausdorff by [EP16, Theorem 3.16]. Thus, we have a canonical faithful conditional expectation $E\colon \mathcal{Q}_r(P) \to C(\widehat{E}_{\text{tight}})$. We follow the strategy of the proof of [LS22, Theorem 5.1] and will show that there is a linear map φ defined on $\pi(A)$ such that $\varphi \circ \pi(a) = \pi \circ E(a)$ for every $a \in A$. One can see that this amounts to showing that

$$\pi(a) \mapsto \pi(E(a)), \qquad a \in A$$

is well-defined. We will be done if we show that $\|\pi(a)\| \geq \|\pi(E(a))\|$ for all a in the canonical dense subalgebra $A_0 := \operatorname{span}(T_{[p,q]} : [p,q] \in S^{\text{Iso}})$ of A.

Let $a = \sum_{f \in F} \lambda_f T_{[p_f, q_f]} \in A_0$ be a finite linear combination of the generators of A, where F is a finite index set, $[p_f, q_f] \in S^{\text{Iso}}$, and $\lambda_f \in \mathbb{C}$. For each $f \in F$, let r_f be an element of P such that $p_f P \cap q_f P = r_f P$.

As a function on $\mathcal{G}_{\text{tight}}(S)$, the element a is a linear combination of characteristic functions on the compact open bisections $\Theta([p_f, q_f], D_{r_f P})$ for $f \in F$. Here, we used that $D_{q_f P} = D_{r_f P}$ by [St22, Lemma 4.2]. By [EP16, Proposition 3.14], we have $E(a) = \sum_{f \in F} \lambda_f 1_{\mathcal{F}_{[p_f,q_f]}}$, where $\mathcal{F}_{[p_f,q_f]}$ is a certain compact open subset of $D_{r_f P}$ (we shall not need the precise definition of $\mathcal{F}_{[p_f,q_f]}$ here; that $\mathcal{F}_{[p_f,q_f]}$ is compact open suffices for our purposes).

Each nonempty subset $F' \subseteq F$ determines a compact open subset of $\widehat{E}_{\text{tight}}$ given by

$$U_{F'} := \bigcap_{f \in F'} D_{r_f P} \setminus \left(\bigcup_{g \in F \setminus F'} D_{r_g P} \right).$$

Since $\bigcup_{f\in F} D_{r_fP} = \bigsqcup_{\emptyset\neq F'\subseteq F} U_{F'}$ and $\mathcal{F}_{[p_f,q_f]} \subseteq D_{r_fP}$ for all $f \in F$, the support of $E(a)$ is contained in $\bigsqcup_{\emptyset\neq F'\subseteq F} U_{F'}$. Thus, there exists $\emptyset \neq F' \subseteq F$ such that

$$\|E(a)\| = \max\{|a(u)| : u \in U_{F'}\}.$$

If $E(a) = 0$, then clearly $\|\pi(E(a))\| \leq \|\pi(a)\|$, so we may assume $E(a) \neq 0$, in which case $U_{F'}$ is nonempty. Since the ultrafilters are dense in $\widehat{E}_{\text{tight}}$ and $E(a) = a|_{\widehat{E}_{\text{tight}}}$ takes on only finitely many values on $\widehat{E}_{\text{tight}}$, we can find an ultrafilter $\xi \in U_{F'}$ such that $\|E(a)\| = |a(\xi)|$.

Note that $r_fP \in \xi$ for all $f \in F'$ and $r_gP \notin \xi$ for all $g \in F \setminus F'$. Since P is not left reversible and ξ is an ultrafilter, for each $g \in F \setminus F'$ we can find $k_g \in P$ such that $k_gP \in \xi$ and $k_gP \cap r_gP = \emptyset$ (see [Ex08, Lemma 12.3]). Since ξ is a filter, we have

$$\bigcap_{f\in F'} r_fP \cap \bigcap_{g\in F\setminus F'} k_gP \neq \emptyset,$$

and this intersection must be of the form bP for some $b \in P$ with $bP \in \xi$. Moreover, we have $bP \subseteq r_fP$ for all $f \in F'$ and $bP \cap r_gP = \emptyset$ for all $g \in F\setminus F'$, so that $D_{bP} \subseteq U_{F'}$. Since $\|E(a)\| = |E(a)(\xi)|$, $\xi \in D_{bP}$, and $T_{[b,b]} = 1_{D_{bP}}$, we have $\|E(a)\| = \|T_{[b,b]}E(a)\|$. Moreover, since $bP \cap r_gP = \emptyset$ for all $g \in F \setminus F'$, we have for $\eta \in D_{bP}$ that $E(T_{[p_f,q_f]})(\eta) = 0$ unless $f \in F'$ (note that $E(T_{[p_f,q_f]})$ has support in D_{r_fP} by [EP16, Proposition 3.14]). Hence,

$$\|E(a)\| = \|T_{[b,b]}E(a)\| = \sup_{\eta\in D_{bP}} |T_{[b,b]}(\eta)E(a)(\eta)| = \left\| T_{[b,b]}E\left(\sum_{f\in F'} \lambda_f T_{[p_f,q_f]}\right) T_{[b,b]} \right\|. \tag{2.1}$$

Now [St22, Lemma 4.2] implies $T^*_{[b,1]}T_{[p_g,q_g]}T_{[b,1]} = 0$ for all $g \in F \setminus F'$, while [St22, Lemma 4.3] implies that $T^*_{[b,1]}T_{[p_f,q_f]}T_{[b,1]} \in \mathcal{Q}_{r,c}(P)$ for all $f \in F'$. We have $\ker(\pi) \cap C(\widehat{E}_{\text{tight}}) = C_0(U)$, where $U \subseteq \widehat{E}_{\text{tight}}$ is an open invariant subset. Since $1_{\widehat{E}_{\text{tight}}} \in \mathcal{Q}_{r,c}(P)$, $\pi(1_{\widehat{E}_{\text{tight}}}) \neq 0$, so that U is a proper subset of $\widehat{E}_{\text{tight}}$. The groupoid $\mathcal{G}_{\text{tight}}(S)$ is minimal by [St15, Lemma 4.2], so U must be empty. Thus, π is injective – and hence isometric – on $C(\widehat{E}_{\text{tight}})$, so that $\|E(a)\| = \|\pi(E(a))\|$. Thus, we can make the following estimate:

$$\begin{aligned}
\|\pi(a)\| &= \left\| \pi\left(\sum_{f\in F} \lambda_f T_{[p_f,q_f]}\right) \right\| \\
&\geq \left\| \pi(T^*_{[b,1]})\pi\left(\sum_{f\in F} \lambda_f T_{[p_f,q_f]}\right)\pi(T_{[b,1]}) \right\| && \text{submultiplicativity, } \pi(T_{[b,1]}) \text{ an isometry} \\
&= \left\| \pi\left(\sum_{f\in F'} \lambda_f T^*_{[b,1]}T_{[p_f,q_f]}T_{[b,1]}\right) \right\| && \text{by choice of } b \\
&= \left\| \sum_{f\in F'} \lambda_f T^*_{[b,1]}T_{[p_f,q_f]}T_{[b,1]} \right\| && \pi \text{ is isometric on } \mathcal{Q}_{r,c}(P) \\
&= \left\| \sum_{f\in F'} \lambda_f T_{[b,b]}T_{[p_f,q_f]}T_{[b,b]} \right\| && \text{submultiplicativity, } T_{[b,1]} \text{ an isometry}
\end{aligned}$$

$$
\begin{aligned}
&\geq \left\| E\left(\sum_{f\in F'} \lambda_f T_{[b,b]} T_{[p_f,q_f]} T_{[b,b]}\right)\right\| && E \text{ is contractive}\\
&= \left\| T_{[b,b]} E\left(\sum_{f\in F'} \lambda_f T_{[p_f,q_f]})\right) T_{[b,b]}\right\| && T_{[b,b]} \text{ is in the multiplicative domain of } E\\
&= \|E(a)\| && \text{by (2.1)}\\
&= \|\pi(E(a))\| && \pi \text{ is isometric on } C(\widehat{E}_{\text{tight}}).
\end{aligned}
$$

Thus, the map $\pi(a) \mapsto \pi(E(a))$ is a well-defined linear idempotent contraction on the dense *-subalgebra $\pi(A_0)$ of $\pi(A)$, so it extends to a linear map on $\pi(A)$.

To complete the proof, suppose that $\pi(x) = 0$ for some $x \in A$. Then, $\pi(x^*x) = 0$, implying $\pi(E(x^*x)) = 0$. Since π is faithful on the image of E we must have $E(x^*x) = 0$, and since E is faithful we get $x^*x = 0$, implying $x = 0$. $\square$

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