SOBOLEV SPACE THEORY FOR POISSON'S EQUATION IN NON-SMOOTH DOMAINS VIA SUPERHARMONIC FUNCTIONS AND HARDY'S INEQUALITY

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ABSTRACT. We introduce a general L_p-solvability result for the Poisson equation in non-smooth domains $\Omega \subset \mathbb{R}^d$, with the zero Dirichlet boundary condition. Our sole assumption for the domain Ω is the Hardy inequality: There exists a constant $N > 0$ such that

$$\int_\Omega \Big|\frac{f(x)}{d(x,\partial\Omega)}\Big|^2 \,\mathrm{d}x \leq N \int_\Omega |\nabla f|^2 \,\mathrm{d}x \quad \text{for any} \quad f \in C_c^\infty(\Omega)\,.$$

To describe the boundary behavior of solutions in a general framework, we propose a weight system composed of a superharmonic function and the distance function to the boundary. Additionally, we explore applications across a variety of non-smooth domains, including convex domains, domains with exterior cone condition, totally vanishing exterior Reifenberg domains, and domains $\Omega \subset \mathbb{R}^d$ for which the Aikawa dimension of Ω^c is less than $d-2$. Using superharmonic functions tailored to the geometric conditions of the domain, we derive weighted L_p-solvability results for various non-smooth domains and specific weight ranges that differ for each domain condition. Furthermore, we provide an application for the Hölder continuity of solutions.

CONTENTS

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1. Introduction

The Poisson equation is among the most classical partial differential equations. L_p-theory for this equation in $\mathbb{R}^d$ and C^2-domains has been developed long before, alongside Schauder theory and L_2-theory. In particular, there are extensions in various directions, including variable coefficients [17, 36], nonlocal or nonlinear operators [21, 22], and non-smooth domains.

Our primary focus is the Poisson equation on *non-smooth* domains Ω, with the zero-Dirichlet boundary condition:

$$\Delta u = f \quad \text{in } \Omega \quad ; \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{1.1}$$

Unweighted or weighted L_p-theories for this equation have been developed for various types of domains, including C^1-domains [27, 29], Reifenberg domains [11], convex domains [1, 19], Lipschitz domains [23], domains with Ahlfors regular boundary [46], domains with point singularities [43, 44], and piecewise smooth domains [10, 45]. Despite the extensive analyses of the Poisson equation across these domains, a comprehensive theorem for L_p-solvability for various types of non-smooth domains remains elusive. Moreover, L_p-theory has been primarily developed on domains with sufficient regularity, such as those mentioned above.

This paper presents a general result on weighted L_p-solvability for (1.1) in non-smooth domains. We consider domains $\Omega \subsetneq \mathbb{R}^d$ admitting the Hardy inequality: There exists a constant $\mathrm{C}_0(\Omega) > 0$ such that

$$\int_\Omega \Big|\frac{f(x)}{d(x,\partial\Omega)}\Big|^2 \,\mathrm{d}x \leq \mathrm{C}_0(\Omega)\int_\Omega |\nabla f(x)|^2 \,\mathrm{d}x \quad \text{for all} \quad f \in C_c^\infty(\Omega)\,. \tag{1.2}$$

One of the notable sufficient conditions for (1.2) is the volume density condition:

$$\inf_{\substack{p\in\partial\Omega\\ r>0}} \frac{\big|\Omega^c \cap B_r(p)\big|}{\big|B_r(p)\big|} > 0 \tag{1.3}$$

(see Remark 4.11). We also use a class of superharmonic functions, called superharmonic Harnack functions, as a weight function in our L_p-estimate. Consequently, roughly speaking, we establish that for equation (1.1) in a domain Ω with (1.2), each *superharmonic Harnack function* ψ immediately leads to a weighted L_p-solvability result associated with ψ, for general $p \in (1,\infty)$, where ψ describes the boundary behavior of solutions. We apply our result to various types of non-smooth domains, constructing appropriate superharmonic functions. A detailed discussion of our main result and its applications can be found in the last part of Subsection 1.1 and Subsection 1.2, respectively.

1.1. Historical remarks and overview of the main results.

Historical remarks on the L_p-solvability in non-smooth domains. Studies of L_p-theory for non-smooth domains have mainly focused on the individual analysis of specific domain classes. One of the most significant contributions to the study of non-smooth domains is the work of Jerison and Kenig [23] for Lipschitz domains. The authors provided the following results for domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$ (resp. $d = 2$):

(1) If $p \in [3/2, 3]$ (resp. $p \in [4/3, 4]$), then for any bounded Lipschitz domains Ω, the Poisson equation (1.1) has a unique solution in $\mathring{L}_1^p(\Omega)$ whenever $f \in L_{-1}^p(\Omega)$.

(2) For each $p > 3$ (resp. $p > 4$), there exists a bounded Lipschitz domain Ω and $f \in C^\infty(\overline{\Omega})$ such that (1.1) has no solution in $\mathring{L}_1^p(\Omega)$.

(for the definition of function spaces $\mathring{L}_1^p(\Omega)$ and $L_{-1}^p(\Omega)$, see Remark 1.4). The first result provides a universal range of p that assures the unique solvability in unweighted Sobolev spaces. However, as shown in the second result, the Poisson equation is *not uniquely solvable* in unweighted Sobolev spaces $\mathring{L}_1^p(\Omega)$, for general non-smooth domains Ω and values of $p \in (1, \infty)$. Given these limitations in unweighted Sobolev spaces, we turn our attention to theories in weighted Sobolev spaces.

Elliptic equations in smooth or polygonal cones have been extensively studied in the literature, as indicated in monographs [10, 44, 45]. Here,

$$\Omega := \{r\sigma \,:\, r > 0 \;\text{ and }\; \sigma \in \mathcal{M}\} \quad (\mathcal{M} \subset \mathbb{S}^{d-1}), \tag{1.4}$$

is called a smooth cone if $\mathcal{M}$ is a smooth subdomain of $\mathbb{S}^{d-1}$, and Ω is called a polygonal cone if $\mathcal{M}$ is a spherical polygon. For these domains, scholars have investigated the unique solvability of elliptic equations in specific types of weighted L_p-Sobolev spaces for general $p \in (1, \infty)$. The weight system in these spaces is composed of the distance functions for each vertex and edge of the domain; the range of weights for the unique solvability is closely related to the *eigenvalues* of the spherical Laplacian on $\mathcal{M}$. For example, consider the case of $\mathcal{M} = \{(\cos\theta, \sin\theta) : 0 < \theta < \kappa\} \subset \mathbb{S}^1$, $\kappa \in (0, 2\pi)$, and $\Omega \subset \mathbb{R}^2$ defined by (1.4). For any $p \in (1, \infty)$ and $\frac{2}{p} - \frac{\pi}{\kappa} < \mu < \frac{2}{p} + \frac{\pi}{\kappa}$, we have the estimate

$$\big\||x|^{-\mu}u\big\|_p + \big\||x|^{-\mu+1}Du\big\|_p + \big\||x|^{-\mu+2}D^2u\big\|_p \lesssim \big\||x|^{-\mu+2}\Delta u\big\|_p$$

for $u \in C_c^\infty(\Omega)$ (see [45, 2.6.6. Example]). The value of μ describes the behavior of solutions near the vertex, and the term $\frac{\pi}{\kappa}$ in the range of μ is directly related to the first eigenvalue of $\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$ on $\mathcal{M}$.

The aforementioned studies on Lipschitz domains and smooth cones indicate that, in order to develop a general framework for the L_p-solvability of the Poisson equation in various non-smooth domains, we need to adopt a weight system associated with the Laplace operator and the geometric features of each domain. Furthermore, this weight system enables us to describe the *boundary behavior* of solutions.

There are many other notable studies for various non-smooth domains. Subsection 1.2 summarizes prior works relevant to several types of non-smooth domains and introduces our result in each situation. Before introducing our result, we leave some comments on one of the primary methods of this paper.

Remark on the localization argument. One of our primary methods is the localization argument developed by Krylov [34]. Krylov investigated the Poisson equation in the half space $\mathbb{R}_+^d$, and one of the main results is as follows: If $\frac{1}{p} < \mu < 1 + \frac{1}{p}$, then for any $u \in C_c^\infty(\mathbb{R}_+^d)$ and $f_0, f_1, \dots, f_d$ such that $\Delta u = f_0 + \sum_{i\geq 1} D_i f_i$, we have

$$\|\rho^{-\mu}u\|_p + \|\rho^{1-\mu}Du\|_p \lesssim \|\rho^{-\mu}u\|_p + \|\rho^{2-\mu}f_0\|_p + \sum_{i\geq 1}\|\rho^{1-\mu}f_i\|_p \tag{1.5}$$

$$\lesssim \|\rho^{2-\mu}f_0\|_p + \sum_{i\geq 1}\|\rho^{1-\mu}f_i\|_p\,, \tag{1.6}$$

where $\rho(x) := d(x, \partial\mathbb{R}^d_+)$ is the boundary distance function on $\mathbb{R}^d_+$. Here, the parameter μ describes the boundary behaviors of solutions and their derivatives (consider the case of $\mu = 1$). The range $\frac{1}{p} < \mu < 1 + \frac{1}{p}$ is sharp as mentioned in [34, Remark 4.3]. From a technical point of view, this range follows from the proof of (1.6) in which the weighted Hardy inequalities for $\mathbb{R}_+$ and its sharp constants play crucial roles. On the other hand, to derive estimate (1.5), the author applied a localization argument based on the Poisson equation's results in the whole space $\mathbb{R}^d$. We note that this argument is applicable to *any domain* Ω *and any* $\mu \in \mathbb{R}$, not just to $\mathbb{R}^d_+$ and specific μ, as shown in [30].

While Krylov [34] dealt with only the half space because of estimate (1.6), the work of Kim [28] reveals a connection between the approach in [34] and the classical Hardy inequality (1.2) for non-smooth domains. Kim [28] studied stochastic parabolic equations in non-smooth domains, obtaining (1.5) and (1.6) type estimates for bounded domains Ω admitting the Hardy inequality, instead of $\mathbb{R}^d_+$. However, it should be noted that in [28, Theorem 2.12], the range of μ for the solvability is restricted to around $\frac{2}{p}$, and this range is not specified; briefly speaking, the boundary behavior of solutions is not adequately described sufficiently well (cf. Krylov's work on $\mathbb{R}^d_+$ mentioned above).

Overview of the main result. Following [28], we concentrate on the class of domains admitting the Hardy inequality. This concentration stems from the fact that the Hardy inequality holds on various non-smooth domains (see (1.3)).

A key distinguishing feature of the present paper from earlier studies is the utilization of *superharmonic functions*. We employ superharmonic functions in conjunction with the Hardy inequality. This combination allows us to effectively capture the boundary behavior of solutions (see (1.7) or Theorem 2.7). Furthermore, we introduce the concepts of *Harnack functions* and *regular Harnack functions*, extending the localization argument used in [34] to a broader class of weight functions. Consequently, as weight functions, we utilize superharmonic Harnack functions ψ, which are locally integrable functions that satisfy the following conditions:

(1) $\Delta\psi \leq 0$ in the sense of distribution.
(2) $\psi > 0$ and that there exists a constant $N > 0$ such that

$$\operatorname*{ess\,sup}_{B(x,\rho(x)/2)} \psi \leq N \operatorname*{ess\,inf}_{B(x,\rho(x)/2)} \psi \quad \text{for all} \quad x \in \Omega\,,$$

where $\rho(x) := \mathrm{dist}(x, \partial\Omega)$.

Our main result (Theorem 3.14) contains the following estimate:

Let Ω admit the Hardy inequality (1.2) and ψ be a superharmonic Harnack function on Ω. For any $1 < p < \infty$ and $-\frac{1}{p} < \mu < 1 - \frac{1}{p}$, it holds that for any $u \in C_c^\infty(\Omega)$ and $f_0, f_1, \ldots, f_d$ such that $\Delta u = f_0 + \sum_{i\geq 1} D_i f_i$, we have

$$\|\psi^{-\mu}\rho^{-2/p}u\|_p + \|\psi^{-\mu}\rho^{-2/p+1}Du\|_p \lesssim \|\psi^{-\mu}\rho^{-2/p+2}f_0\|_p + \sum_{i\geq 1}\|\psi^{-\mu}\rho^{-2/p+1}f_i\|_p\,. \tag{1.7}$$

Here, the superharmonic Harnack function ψ describes the boundary behavior of solutions; by applying a Sobolev-Hölder embedding theorem, we also derive pointwise estimates for solutions (see Theorem 1.8 and Proposition 3.13).

Our main result does not specify a particular superharmonic Harnack function ψ. The flexibility in choosing ψ is the primary advantage of our theorem, enabling applications in a wide range of non-smooth domains. We offer a non-trivial general example of ψ related to the Green functions in Example 3.17. Additionally, throughout Sections 4 and 5, we explore the construction of suitable ψ for various geometric domain conditions. The domain conditions we investigate include the following:

(1) Domains satisfying the exterior cone condition, and planar domains satisfying the exterior line segment condition;
(2) Convex domains;
(3) Domains satisfying the totally vanishing exterior Reifenberg condition;
(4) Domains Ω satisfying the volume density condition (1.3);
(5) Domains $\Omega \subset \mathbb{R}^d$ for which the Aikawa dimension of Ω^c is less than $d-2$.

For a domain Ω under each condition above, we construct suitable superharmonic functions ψ such that $\psi \simeq d(\cdot,\partial\Omega)^\alpha$ for some $\alpha \in \mathbb{R}$. Notably, the range of α is different for each domain condition. We sequentially introduce simplified versions of our results for the aforementioned conditions in Subsubsections 1.2.1 - 1.2.5, together with earlier works for each domain condition.

Finally, we mention that the approach presented in this paper is applicable not only to the Poisson equation but also to linear evolution equations based on the Laplace operator, such as the classical heat or time-fractional heat equations and the stochastic heat equation. The localization argument in Section 3 and the superharmonic functions in Sections 4 and 5 can readily be employed for these equations. Applications to the mentioned three equations are considered future work.

Since the table of contents is provided at the beginning of the paper, we omit the summary of the organization of this paper.

1.2. **Summary of applications to various domain conditions.** This subsection considers a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. We denote $\rho(x) := d(x,\partial\Omega)$, and introduce weighted Sobolev spaces. For $p \in (1,\infty)$, $\theta \in \mathbb{R}$, and $n \in \{0,1,2,\ldots\}$, we denote

$$\|f\|_{W^n_{p,\theta}(\Omega)} := \sum_{k=0}^{n}\|\rho^k D^k f\|_{L_{p,\theta}(\Omega)} := \sum_{k=0}^{n}\left(\int_\Omega \big|\rho(x)^k D^k f(x)\big|^p \rho(x)^\theta\,\mathrm{d}x\right)^{1/p},$$

$$\|f\|_{W^{-n}_{p,\theta}(\Omega)} := \inf\left\{\sum_{|\alpha|\leq n}\|\rho^{-|\alpha|}f_\alpha\|_{L_{p,\theta}(\Omega)}\,:\, f = \sum_{|\alpha|\leq n}D^\alpha f_\alpha\right\}.$$

For $n \in \mathbb{Z}$, $W^n_{p,\theta}(\Omega)$ denotes the set of all $f \in \mathcal{D}'(\Omega)$ such that $\|f\|_{W^n_{p,\theta}(\Omega)} < \infty$.

Remark 1.1. The spaces $W^n_{p,\theta}(\Omega)$ appears only in this subsection. However, this space has the equivalent relation, $W^n_{p,\theta}(\Omega) = H^n_{p,\theta+d}(\Omega)$ (see Lemma 3.12), where $H^n_{p,\theta+d}$ is a function space introduced in Subsections 3.2.

For convenience, we define the following statement:

Statement 1.2 (Ω, p, θ)**.** *Let $\lambda \geq 0$. For any $n \in \mathbb{Z}$, if $f \in W^n_{p,\theta}(\Omega)$, then the equation $\Delta u - \lambda u = f$ has a unique solution u in $W^{n+2}_{p,\theta+2p}(\Omega)$. Moreover, we have*

$$\|u\|_{W^{n+2}_{p,\theta}(\Omega)} + \lambda \|u\|_{W^{n}_{p,\theta+2p}(\Omega)} \leq N \|f\|_{W^{n}_{p,\theta+2p}(\Omega)}\,, \tag{1.8}$$

where N is independent of f, u, and λ.

1.2.1. (Subsection 5.1) Domains with exterior cone condition. For $\delta \in [0, \pi/2)$ and $R > 0$, Ω is said to satisfy the *exterior (δ, R)-cone condition* if for every $p \in \partial\Omega$, there exists a unit vector $e_p \in \mathbb{R}^d$ such that

$$\{x \in \mathbb{R}^d \,:\, (x - p)\cdot e_p \geq |x - p| \cos\delta\ ,\ |x - p| < R\} \subset \Omega^c\,;$$

when $\delta = 0$, this condition is often called the exterior R-line segment condition. Examples of this condition are given in Example 5.2 and illustrated in Figure 5.1.

Given $\delta > 0$, we denote

$$\lambda_\delta := -\frac{d-2}{2} + \sqrt{\Big(\frac{d-2}{2}\Big)^2 + \Lambda_\delta}\,,$$

where $\Lambda_\delta > 0$ is the first eigenvalue for Dirichlet spherical Laplacian on

$$\{\sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{S}^{d-1} \,:\, \sigma_1 > -\cos\delta\}\,.$$

When $d = 2$ and $\delta = 0$, we set $\lambda_\delta = 1/2$. We provide information on λ_δ in (5.4) and Proposition 5.3. Note that $\lambda_\delta > 0$ for all $\delta > 0$, and if $d = 2$, then $\lambda_\delta = \frac{\pi}{2(\pi-\delta)} \geq \frac{1}{2}$ for all $\delta \geq 0$.

Our result also covers some unbounded domains, but here, we only introduce the result regarding bounded domains.

Theorem 1.3 (see Theorem 5.6)**.** *Let $\delta \in (0, \pi)$ if $d \geq 3$, and $\delta \in [0, \pi)$ if $d = 2$. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain satisfying the (δ, R)-exterior cone condition for some $R > 0$. Then, for any $p \in (1, \infty)$ and $\theta \in \mathbb{R}$ satisfying*

$$-\lambda_\delta(p-1) - 2 < \theta < \lambda_\delta - 2\,,$$

Statement 1.2 (Ω, p, θ) holds. In addition, then N in (1.8) depends only on d, p, n, θ, δ, $\mathrm{diam}(\Omega)/R$.

The exterior cone condition is more general than the Lipschitz boundary condition. It should be noted that, however, Theorem 5.6 and the work of Jerison and Kenig [23, Theorems 1.1, 1.3] (on Lipschitz domains) cannot be directly compared because they address different aspects of the Poisson equation in non-smooth domains. While Theorem 5.6 covers a *broader* domain class than [23], if our focus is restricted only to Lipschitz domains, the results in [23] are more general than Theorem 5.6 in terms of unweighted estimates for higher regularity. To compare [23] with Theorem 5.6, we refer the reader to the following remark on the relations between the function spaces $H^\gamma_{p,\theta+d}(\Omega)$ (see Remark 1.1) and the Sobolev spaces presented in [23]:

Remark 1.4. Let Ω be a bounded Lipschitz domain. We refer to the function space $L^p_s(\Omega)$ and $L^p_{s,\mathrm{o}}$ as introduced in [23, Section 2], where $p \in (1, \infty)$ is the integrability parameter, and $s \in \mathbb{R}$ is the regularity parameter. For clarity, we use the notation $\mathring{L}^p_s(\Omega)$ to denote the space $L^p_{s,\mathrm{o}}$. It is noted that for $k \in \mathbb{N}_0$, we have $L^p_k(\Omega) = W^k_p(\Omega)$.

The space $\mathring{L}_k^p(\Omega)$ is defined by the closure of $C_c^\infty(\Omega)$ in $L_k^p(\Omega)$, and $L_{-k}^p(\Omega)$ is defined by the dual space of $\mathring{L}_k^{p/(p-1)}(\Omega)$.

It is directly follows from the definition that $H_{p,d}^0(\Omega) = L_0^p(\Omega) = L_p(\Omega)$. For $k \in \mathbb{N}$, it is implied by the weighted Hardy inequality for Lipschitz domains (see, *e.g.*, [38]) and the boundedness of Ω that $\|u\|_{H^k_{p,d-kp}(\Omega)} \simeq \|u\|_{L_k^p(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in both of $H^k_{p,d-kp}(\Omega)$ and $\mathring{L}_k^p(\Omega)$, separately (see Lemma 3.10.(1) with $\Psi \equiv 1$), $H^k_{p,d-kp}(\Omega)$ coincides with $\mathring{L}_k^p(\Omega)$. The interpolation properties for $\mathring{L}_s^p(\Omega)$ and $H^\gamma_{p,\theta}(\Omega)$ (see [23, Corollary 2.10] and [40, Proposition 2.4], respectively) implies that $H^s_{p,d-sp}(\Omega) = \mathring{L}_s^p(\Omega)$ for all $s > 0$. By considering these dual spaces (see Lemma 3.10.(2)), we also have $H^{-s}_{p,d+sp}(\Omega) = L^p_{-s}(\Omega)$ for all $s > 0$.

1.2.2. (Subsection 5.2) Convex domain.

Theorem 1.5 (see Theorem 5.10)**.** *Let $d \geq 2$ and $1 < p < \infty$. Suppose that Ω is a convex domain (not necessarily bounded). For any $p \in (1, \infty)$ and $\theta \in \mathbb{R}$ satisfying*

$$-p - 1 < \theta < -1 \,,$$

Statement 1.2 (Ω, p, θ) holds. In addition, N in (1.8) depends only on d, p, n, θ. In particular, N is independent of Ω.

Adolfsson [1] and Fromm [19] have established the solvability of the Poisson equation in bounded convex domains. Regarding unweighted estimates for higher regularity, their results is more general than Theorem 5.10. However, Theorem 5.10 deals with convex domains that are not necessarily bounded, and this theorem also provides solvability results in *weighted* Sobolev spaces. When comparing these results with Theorem 5.10, it is helpful to note Remark 1.4 and that bounded convex domains are Lipschitz domains (see, *e.g.*, [20, Corollary 1.2.2.3]).

Combining the results of Theorem 5.10 with [20, Theorem 3.2.1.2] may yield results similar to [19, Corollary 1]. However, we do not pursue this direction in this paper.

1.2.3. (Subsection 5.3) Totally vanishing exterior Reifenberg condition. This subsubsection introduces the *totally vanishing exterior Reifenberg* condition (abbreviated to '$\langle$TVER$\rangle$'), which is a generalization of the concept of bounded vanishing Reifenberg domains introduced below (5.10).

To clarify the main point of $\langle$TVER$\rangle$ presented in Definition 5.11.(3), we provide a simplified version of this concept in Definition 1.6. Note that $\langle$TVER$\rangle$ in Definition 1.6 is a sufficient condition for the totally vanishing exterior Reifenberg condition in Definition 5.11.(3). In Figure 5.3, we describe the difference between the vanishing Reifenberg condition, $\langle$TVER$\rangle$ in Definition 1.6, and the totally vanishing exterior Reifenberg condition in Definition 5.11.(3).

Definition 1.6. We say that Ω satisfies the *totally vanishing exterior Reifenberg condition* (abbreviate to '$\langle$TVER$\rangle$') if for any $\delta \in (0,1)$, there exist $R_{0,\delta}$, $R_{\infty,\delta} > 0$ satisfying the following: For every $p \in \partial\Omega$ and $r > 0$ with $r \leq R_{0,\delta}$ or $r \geq R_{\infty,\delta}$, there exists a unit vector $e_{p,r} \in \mathbb{R}^d$ such that

$$\Omega \cap B_r(p) \subset \{x \in B_r(p) \,:\, (x-p)\cdot e_{p,r} < \delta r\} \,. \tag{1.9}$$

As shown in Example 5.13, $\langle$TVER$\rangle$ is fulfilled by bounded domains of the following types: the vanishing Reifenberg domains, C^1-domains, domains with the exterior ball condition, and finite intersections of them. Furthermore, several unbounded domains also satisfy $\langle\text{TVER}\rangle$ (see Proposition 5.14).

Theorem 1.7 (see Theorem 5.18)**.** *Suppose that Ω satisfies $\langle\text{TVER}\rangle$. For any $p \in (1,\infty)$ and $\theta \in \mathbb{R}$ satisfying*

$$-p-1<\theta<-1\,,$$

Statement 1.2 (Ω,p,θ) holds. In addition, N in (1.8) depends only on d, p, n, θ, and $\left\{R_{0,\delta}/R_{\infty,\delta}\right\}_{\delta\in(0,1]}$.

The Poisson equation in bounded vanishing Reifenberg domains has been investigated in the literature, such as the works of Byun and Wang [11], Choi and Kim [13], and Dong and Kim [17]. These studies focus on the elliptic equations with variable coefficients, and provide weighted L_p-estimates for Muckenhoupt A_p-weight functions. However, these studies mostly dealt with bounded vanishing Reifenberg domains. Differing from these, Theorem 1.7 considers domains satisfying $\langle\text{TVER}\rangle$, thereby including bounded vanishing Reifenberg domains.

1.2.4. (Subsection 4.1) Domains with fat exterior. Consider a domain Ω satisfying the *capacity density condition*:

$$\inf_{\substack{p\in\partial\Omega\\ r>0}}\frac{\mathrm{Cap}\left(\Omega^c\cap\overline{B}_r(p),B_{2r}(p)\right)}{\mathrm{Cap}\left(\overline{B}_r(p),B_{2r}(p)\right)}\geq\epsilon_0>0\,, \tag{1.10}$$

where $\mathrm{Cap}(K,U)$ denotes the L_2-capacity of K relative to U (for the definition, see (4.11)). Condition (1.10) has been studied in the literature, including [4, 5, 6, 26, 31, 39]. It is worth noting that the volume density condition (1.3) is a sufficient condition for (1.10) (see Remark 4.11).

In Subsection 4.1, we consider another condition equivalent to condition (1.10), called the local harmonic measure decay condition. To clarify, we introduce some corollaries instead of the main result (Theorem 4.13).

Theorem 1.8 (see Corollary 4.15 with Lemma 4.10)**.** *Let Ω be a bounded domain satisfy (1.10). There exists $\alpha_0>0$ depending only on d, N_0, and ϵ_0 (in (1.10)) such that for any $\alpha\in(0,\alpha_0]$, the following holds: Let $\lambda\geq 0$, and f_0, f_1, …, f_d be measurable functions such that $|f_0|\lesssim\rho^{-2+\alpha}$ and $|f_1|$, …, $|f_d|\lesssim\rho^{-1+\alpha}$. For any $\beta<\alpha$, the equation*

$$\Delta u-\lambda u=f_0+\sum_{i\geq 1}D_if_i\quad\text{in}\quad\Omega\quad;\quad u=0\quad\text{on}\quad\partial\Omega \tag{1.11}$$

has a unique solution u in $C^{0,\beta}(\Omega)$. In addition, we have

$$\sup_{\Omega}\rho^{-\beta}|u|+\|u\|_{C^{0,\beta}(\Omega)}\leq N\sup_{\Omega}\Big(\rho^{-2+\alpha}|f_0|+\sum_{i\geq 1}\rho^{-1+\alpha}|f_i|\Big)\,,$$

where N depends only on d, $|\Omega|$, ϵ_0 (in (1.10)), α, β.

Remark 1.9 (see Remark 4.8). Theorem 1.8 still holds for bounded domains Ω satisfying the following, instead of (1.10):

For any $F\in C(\partial\Omega)$, the Laplace equation

$$\Delta u=0\quad\text{in }\Omega\quad;\quad u=F\quad\text{on }\partial\Omega$$

has a (unique) classical solution $u \in C(\overline{\Omega})$. Additionally, there is $\alpha_1 \in (0,1)$ such that $\|u\|_{C^{0,\alpha_1}(\Omega)} \leq N\|F\|_{C^{0,\alpha_1}(\partial\Omega)}$, where N is a constant independent of u and F.

Under this revised assumption, α_0 in Theorem 1.8 can be chosen as α_1 in the revised assumption.

We also provide an unweighted L_p-solvability result for (1.11), where p is close to 2. Although similar results are provided in the literature, as detailed in the above Corollary 4.16, we present the following theorem to emphasize the applicability of our main result:

Theorem 1.10 (see Corollary 4.16)**.** *Let Ω satisfies* (1.10)*, and let*

$$\lambda \geq 0 \quad \text{if} \quad D_\Omega := \sup_{x\in\Omega} d(x,\partial\Omega) < \infty \ , \quad \text{and} \quad \lambda > 0 \quad \text{if} \quad D_\Omega = \infty\,.$$

Then there exists $\epsilon \in (0,1)$ depending only on d, ϵ_0 (in (1.10)*) such that for any $p \in (2-\epsilon, 2+\epsilon)$, the following holds: For any $f_0, f_1, \ldots, f_d \in L_p(\Omega)$, equation* (1.11) *has a unique solution u in $\mathring{W}^1_p(\Omega)$ ($:=$ the closure of $C_c^\infty(\Omega)$ in $W^1_p(\Omega)$). Moreover, we have*

$$\|\nabla u\|_p + \big(\lambda^{1/2} + D_\Omega^{-1}\big)\|u\|_p \lesssim_{d,p,\epsilon_0} \min\big(\lambda^{-1/2}, D_\Omega\big)\|f\|_p + \sum_{i\geq 1}\|f^i\|_p\,.$$

1.2.5. (Subsection 4.2) Domains with thin exterior. For a closed set $E \subset \mathbb{R}^d$, the *Aikawa dimension* of E, denoted by $\dim_{\mathcal{A}} E$, is defined as the infimum of $\beta \geq 0$ such that

$$\sup_{p\in\Omega^c, r>0} \frac{1}{r^\beta}\int_{B(p,r)} \frac{1}{d(x,E)^{d-\beta}}\,\mathrm{d}x \leq A_\beta < \infty\,,$$

with considering $0^{-1} = \infty$. We consider a domain Ω for which $\dim_{\mathcal{A}}(\Omega^c) < d-2$. A relation between the Aikawa dimension, the Hausdorff dimension, and the Assouad dimension is mentioned in Remark 4.1.

Theorem 1.11 (see Theorem 4.19)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, satisfy $\dim_{\mathcal{A}}(\Omega^c) =: \beta_0 < d-2$. For any $p \in (1,\infty)$ and $\theta \in \mathbb{R}$ satisfying*

$$-d+\beta_0 < \theta < (p-1)(d-\beta_0) - 2p\,,$$

Statement 1.2 (Ω, p, θ) *holds. In addition, N in* (1.8) *depends only on d, p, n, θ, β_0, $\{A_\beta\}_{\beta>\beta_0}$.*

1.3. **Notation.**

- We use $:=$ to denote a definition.
- The letter N denotes a finite positive constant which may have different values along the argument while the dependence will be informed; $N = N(a,b,\cdots)$ means that this N depends only on the parameters inside the parentheses.
- For a list of parameters L, $A \lesssim_L B$ means that $A \leq N(L)B$, and $A \simeq_L B$ means that $A \lesssim_L B$ and $B \lesssim_L A$.
- $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$.
- For a Lebesgue measurable set $E \subset \mathbb{R}^d$, $|E|$ denotes the Lebesgue measure of E.

- $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{R}^d_+ := \{(x^1,\dots,x^d)\in\mathbb{R}^d : x^1>0\}$, $\mathbb{R}_+ := \mathbb{R}^1_+$, and $\mathbb{S}^{d-1} := \{x\in\mathbb{R}^d : |x|=1\}$. In adition, for $p\in\mathbb{R}^d$ and $r>0$, $B_r(p) := B(p,r) := \{x\in\mathbb{R}^d : |x|<r\}$, and $B_r := B_r(0)$.
- A non-empty connected open set is called a domain.
- For sets E, $F\subset\mathbb{R}^d$, $d(x,E) := \inf_{y\in E}|x-y|$ and $d(E,F) := \inf_{x\in E} d(x,F)$. For a fixed open set $\mathcal{O}\subset\mathbb{R}^d$, we usually denote $\rho(x) := d(x,\partial\mathcal{O})$ when there is no confusion.
- For a set $E\subset\mathbb{R}^d$, 1_E denotes the function defined by $1_E(x)=1$ for $x\in E$, and $1_E(x)=0$ for $x\notin E$. For a function f defined in E, $f1_E$ denotes the function defined as $\big(f1_E\big)(x)=f(x)$ if $x\in E$, and $\big(f1_E\big)(x)=0$ if $x\neq E$.
- $\mathrm{supp}(f)$ denotes the support of the function f defined as the closure of $\{x : f(x)\neq 0\}$.
- For an open set $\mathcal{O}\subseteq\mathbb{R}^d$, $C_c^\infty(\mathcal{O})$ is the the space of infinitely differentiable functions f for which $\mathrm{supp}(f)$ is a compact subset of $\mathcal{O}$. Also, $C^\infty(\mathcal{O})$ denotes the the space of infinitely differentiable functions in $\mathcal{O}$.
- For an open set $\mathcal{O}\subseteq\mathbb{R}^d$, $\mathcal{D}'(\mathcal{O})$ denotes the set of all distributions on $\mathcal{O}$, which is the dual of $C_c^\infty(\Omega)$. For $f\in\mathcal{D}'(\mathcal{O})$, the expression $\langle f,\varphi\rangle$, $\varphi\in C_c^\infty(\mathcal{O})$ denote the evaluation of f with the test function φ.
- For any multi-index $\alpha=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\{0\}\cup\mathbb{N}$, we denote $|\alpha| := \sum_{i=1}^d\alpha_i$. For a function f defined on an open set $\mathcal{O}\subset\mathbb{R}^d$, $f_{x^i} := D_if := \frac{\partial f}{\partial x^i}$, and $D^\alpha f(x) := D_d^{\alpha_d}\cdots D_1^{\alpha_1}f(x)$. For the second order derivatives we denote D_jD_if by $D_{ij}f$. We often use the notation $|gf_x|$ for $\sum_{i=1}^d|gD_if|$, $|gf_{xx}|$ for $\sum_{i,j=1}^d|gD_{ij}f|$, and $\big|gD^mf\big|$ for $\sum_{|\alpha|=k}|gD^\alpha f|$. We extend these notations to a sublinear function $\|\cdot\| : \mathcal{D}'(\Omega)\to[0,+\infty]$; for example, $\|gf_x\| := \sum_{i=1}^d\|gD_if\|$.
- $\Delta f := \sum_{i=1}^d D_{ii}f$ denotes the Laplacian for a function f defined on $\mathcal{O}$.
- For an open set $\mathcal{O}\subseteq\mathbb{R}^d$, $C(\mathcal{O})$ denotes the set of all continuous functions f in $\mathcal{O}$ such that $|f|_{C(\mathcal{O})} := \sup_{\mathcal{O}}|f|<\infty$. For $n\in\mathbb{N}_0$, $C^n(\mathcal{O})$ denotes the set of all strongly n-times continuously differentiable function f on $\mathcal{O}$ such that $\|f\|_{C^n(\mathcal{O})} := \sum_{k=0}^n|D^kf|_{C(\mathcal{O})}<\infty$. For $\alpha\in(0,1]$, $C^{n,\alpha}(\mathcal{O})$ denotes the set of all $f\in C^n(\mathcal{O})$ such that $\|f\|_{C^{n,\alpha}(\mathcal{O})} := \|f\|_{C^n(\mathcal{O})}+[f]_{C^{n,\alpha}(\mathcal{O})}<\infty$, where $[f]_{C^{n,\alpha}(\mathcal{O})} := \sup_{x\neq y\in\mathcal{O}}\frac{|D^nf(x)-D^nf(y)|}{|x-y|^\alpha}$. For any set $E\subset\mathbb{R}^d$, we define the space $C^{0,\alpha}(E)$ in the same way.
- Let $(A,\mathcal{A},\mu)$ be a measure space. For a a measurable function $f : A\to[-\infty,\infty]$, $\operatorname*{ess\,sup}_A f$ is defined by the infimum of $a\in[-\infty,\infty]$ for which $\mu\big(\{x\in A : f(x)>a\}\big)=0$, and $\operatorname*{ess\,inf}_A f := -\operatorname*{ess\,sup}_A(-f)$.
- Let $\mathcal{O}\subseteq\mathbb{R}^d$ be an open set. For $p\in[1,\infty]$, $L_p(\mathcal{O})$ is the set of all measurable functions f on $\mathcal{O}$ such that $\|f\|_p := \big(\int_{\mathcal{O}}|f|^p\,\mathrm{d}x\big)^{1/p}<\infty$ if $p<\infty$, and $\|f\|_\infty := \operatorname*{ess\,sup}_A|f|<\infty$ if $p=\infty$. For $n\in\mathbb{N}_0$, $W_p^n(\mathcal{O}) := \{f : \sum_{|\alpha|\leq n}\|D^\alpha f\|_p<\infty\}$, the Sobolev space.
- Let $\mathcal{O}\subseteq\mathbb{R}^d$ be an open set. For $X(\mathcal{O})=L_p(\mathcal{O})$ or $C^n(\mathcal{O})$ or $C^{n,\alpha}(\mathcal{O})$, $X_{\mathrm{loc}}(\mathcal{O})$ denotes the set of all function f on $\mathcal{O}$ such that $f\zeta\in X(\mathcal{O})$ for all $\zeta\in C_c^\infty(\mathcal{O})$. Especially, if $f\in L_{1,\mathrm{loc}}(\Omega)$, then f is said to be locally integrable in Ω.

2. Key estimates for the Poisson equation

This section aims to obtain an estimate for the zeroth-order derivatives (the function itself) of solutions to the Poisson equation (1.1) on a domain admitting the Hardy inequality (1.2). In the main theorem, Theorem 2.7, superharmonic functions are used as weight functions. We begin with the definition and elementary properties of superharmonic functions.

Definition 2.1.

(1) A function $\phi \in L_{1,\mathrm{loc}}(\Omega)$ is said to be *superharmonic* if $\Delta\phi \leq 0$ in the sense of distribution on Ω, *i.e.*, for any nonnegative $\zeta \in C_c^\infty(\Omega)$,

$$\int_\Omega \phi\,\Delta\zeta\,\mathrm{d}x \leq 0\,.$$

(2) A function $\phi : \Omega \to (-\infty, +\infty]$ is called a *classical superharmonic function* if the following conditions are satisfied:

(a) ϕ is lower semi-continuous on Ω.

(b) For any $x \in \Omega$ and $r > 0$ satisfying $\overline{B}_r(x) \subset \Omega$,

$$\phi(x) \geq \frac{1}{\big|B_r(x)\big|}\int_{B_r(x)} \phi(y)\,\mathrm{d}y\,.$$

(c) $\phi \not\equiv +\infty$ on each connected component of Ω.

Recall that ϕ is said to be *harmonic* if both ϕ and $-\phi$ are classical superharmonic functions.

Remark 2.2. Equivalent definitions of classical superharmonic functions are introduced in [7, Definition 3.1.2, Theorem 3.2.2]. Especially, if ϕ is a classical superharmonic function on a neighborhood of every $x \in \Omega$, then ϕ is a classical superharmonic function on Ω.

Lemma 2.3. *A function $\phi : \Omega \to [-\infty, +\infty]$ is superharmonic if and only if there exists a classical superharmonic function ϕ_0 on Ω such that $\phi = \phi_0$ almost everywhere on Ω.*

The proof of this lemma can be found in [7, Theorem 4.3.2] and [51, Proposition 30.6] for the 'if' part and the 'only if' part, respectively.

Lemma 2.4. *Let ϕ be a classical superharmonic function on Ω.*

(1) If ϕ is twice continuously differentiable, then $\Delta\phi \leq 0$.

(2) ϕ is locally integrable on Ω.

(3) For any compact set $K \subset \Omega$, ϕ has the minimum value on K.

(4) For $\epsilon > 0$, put

$$\phi^{(\epsilon)}(x) = \int_{B_1(0)} \big(\phi 1_\Omega\big)(x - \epsilon y) \cdot N_0\,\mathrm{e}^{-1/(1-|y|^2)}\,\mathrm{d}y\,, \tag{2.1}$$

where $N_0 := \big(\int_{B_1} \mathrm{e}^{-1/(1-|y|^2)}\,\mathrm{d}y\big)^{-1}$. Then for any compact set $K \subset \Omega$ and $0 < \epsilon < d(K, \Omega^c)$, the following hold:

(a) $\phi^{(\epsilon)}$ is infinitely smooth on $\mathbb{R}^d$.

(b) $\phi^{(\epsilon)}$ is a classical superharmonic function on K°.

(c) For any $x \in K$, $\phi^{(\epsilon)}(x) \nearrow \phi(x)$ as $\epsilon \searrow 0$.

For this lemma, (1) - (3) follow from Definition 2.1 and Lemma 2.3, and (4) can be found in [7, Theorem 3.3.3].

Lemma 2.5. *Let ϕ be a positive superharonic function on Ω and denote $\phi^{(\epsilon)}$ the function defined in (2.1).*

(1) For any $c \leq 1$, ϕ^c is locally integrable in Ω.

(2) If $f \in L_1(\Omega)$ and $\operatorname{supp}(f)$ is a compact subset of Ω, then for any $c \in \mathbb{R}$,

$$\lim_{\epsilon\to 0}\int_\Omega |f|\big(\phi^{(\epsilon)}\big)^c\,\mathrm{d}x = \int_\Omega |f|\phi^c\,\mathrm{d}x\,. \tag{2.2}$$

(3) If $f \in L_\infty(\Omega)$ and $\operatorname{supp}(f)$ is a compact subset of Ω, then for any $c \leq 1$,

$$\lim_{\epsilon\to 0}\int_\Omega f\big(\phi^{(\epsilon)}\big)^c\,\mathrm{d}x = \int_\Omega f\phi^c\,\mathrm{d}x\,.$$

Proof. (1) Let K be a compact subset of Ω. If $c \in (0,1]$, then by Lemma 2.4.(2),

$$\int_K \phi^c\,\mathrm{d}x \leq |K|^{1-c}\Big(\int_K \phi\,\mathrm{d}x\Big)^c < \infty\,.$$

In addition, if $c \leq 0$, then by Lemma 2.4.(3), $\max_K(\phi^c) = \big(\min_K \phi\big)^c < \infty$.

(2) Take a bounded open set U such that $\operatorname{supp}(f) \subset U$ and $\overline{U} \subset \Omega$. Consider only $\epsilon \in (0, d(\operatorname{supp}(f), U^c)$. If $c \geq 0$, then due to Lemma 2.4.(4), (2.2) follows from the monotone convergence theorem. If $c < 0$, then $|f|\big(\phi^{(\epsilon)}\big)^c \leq \big(\min_{\overline{U}}\phi\big)^c|f|$, and therefore (2.2) follows from the Lebesgue dominated convergence theorem.

(3) Since $f \in L_\infty(\Omega)$, (1) of this lemma implies that $f\phi^c \in L_1(\Omega)$. The proof is completed by applying (2) of this lemma for $\max(f,0)$ and $\max(-f,0)$ instead of f. □

We present the key lemma of this section.

Lemma 2.6. *Let $p \in (1,\infty)$ and $c \in (-p+1,1)$, and suppose that $u \in C(\Omega)$ satisfies that*

$$\begin{gathered}\operatorname{supp}(u)\ \textit{is a compact subset of}\ \Omega\,,\\ u \in C^2_{\mathrm{loc}}\big(\{x\in\Omega\,:\,u(x)\neq 0\}\big)\ \ ,\ \ \textit{and}\quad \int_{\{u\neq 0\}}|u|^{p-1}|D^2u|\,\mathrm{d}x < \infty\,,\end{gathered} \tag{2.3}$$

and ϕ is a positive superharmonic function on a neighborhood of $\operatorname{supp}(u)$.

(1) If ϕ is twice continuously differentiable, then

$$\int_\Omega |u|^p\phi^{c-2}|\nabla\phi|^2\,\mathrm{d}x \leq \Big(\frac{p}{1-c}\Big)^2\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\,. \tag{2.4}$$

(2) If $(\Delta u)1_{\{u\neq 0\}}$ is bounded, then

$$\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x \leq N\int_{\Omega\cap\{u\neq 0\}}(-\Delta u)\cdot u|u|^{p-2}\phi^c\,\mathrm{d}x\,, \tag{2.5}$$

where $N = N(p,c) > 0$.

(3) If the Hardy inequality (1.2) holds for Ω, then

$$\int_\Omega |u|^p\phi^c\rho^{-2}\,\mathrm{d}x \leq N\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\,, \tag{2.6}$$

where $N = N(p,c,\mathrm{C}_0(\Omega)) > 0$.

Lemma 2.6 is mainly used for $u \in C_c^\infty(\Omega)$. However, we employ condition (2.3) to establish Lemma 2.8, which is a crucial lemma for the existence of solutions in the proof of the main theorem (Theorem 3.14). To handle condition (2.3), we prove the following results in Lemma A.1: If $u \in C(\mathbb{R}^d)$ satisfies (2.3), then $|u|^{p/2-1}u \in W_2^1(\mathbb{R}^d)$ and $|u|^p \in W_1^2(\mathbb{R}^d)$ with

$$\begin{aligned} D_i(|u|^{p/2-1}u) = \frac{p}{2}|u|^{p/2-1}(D_iu)1_{\{u\neq 0\}}\ , \quad D_i\big(|u|^p\big) = p|u|^{p-2}uD_iu1_{\{u\neq 0\}}\ , \\ D_{ij}\big(|u|^p\big) = \big(p|u|^{p-2}uD_{ij}u + p(p-1)|u|^{p-2}D_iuD_ju\big)\, 1_{\{u\neq 0\}}\,. \end{aligned} \tag{2.7}$$

Proof of Lemma 2.6. By Lemma 2.3, we may assume that ϕ is a classical superharmonic function on a neighborhood of $\mathrm{supp}(u)$. In this proof, all of the integrations by parts are based on (2.7).

(1) Recall that ϕ is twice continuously differentiable on a neighborhood of $\mathrm{supp}(u)$. Integrate by parts to obtain

$$\begin{aligned} &(1-c)\int_\Omega |u|^p\phi^{c-2}|\nabla\phi|^2\,\mathrm{d}x \\ =& -\int_\Omega |u|^p\nabla\phi\cdot\nabla(\phi^{c-1})\,\mathrm{d}x \\ =& \,p\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}u\,\phi^{c-1}(\nabla u\cdot\nabla\phi)\,\mathrm{d}x + \int_\Omega |u|^p\phi^{c-1}\Delta\phi\,\mathrm{d}x \\ \leq& \,p\Big(\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |u|^p\phi^{c-2}|\nabla\phi|^2\,\mathrm{d}x\Big)^{1/2}, \end{aligned} \tag{2.8}$$

where the last inequality follows from the Hölder inequality and that $\Delta\phi \leq 0$ on $\{u \neq 0\}$. Since the first term of (2.8) is finite, we obtain (2.4). The proof of (1) is completed.

Although we do not assume that ϕ is infinitely smooth in (2) and (3), we only need to consider the case where ϕ is additionally assumed to be smooth on its domain. This is because if (2.5) and (2.6) hold for $\phi^{(\epsilon)}$ instead of ϕ, for all sufficiently small $\epsilon > 0$, then by Lemma 2.5, (2.5) and (2.6) also hold for ϕ. Note that if $0 < \epsilon < d\big(\mathrm{supp}(u), \partial\Omega\big)$, then $\phi^{(\epsilon)}$ is a positive superharmonic function on $\mathrm{supp}(u)$ (see Lemma 2.4). In addition, $|u|^{p-2}|\nabla u|^2 1_{\{u\neq 0\}}$ and $|u|^p\rho^{-2}$ are integrable (see Lemma A.1), and $-\Delta u\cdot u|u|^{p-2}1_{\{u\neq 0\}}$ in (2.5) is bounded. Therefore, in the proof of (2) and (3), we additionally assume that ϕ is infinitely smooth.

(2) **Case 1:** $0 \leq c < 1$**.** Integrate by parts to obtain

$$\int_\Omega -\Delta u\cdot u|u|^{p-2}\phi^c\,\mathrm{d}x = (p-1)\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\Delta(\phi^c)\,\mathrm{d}x\,.$$

Since

$$\Delta(\phi^c) = c\,\phi^{c-1}\Delta\phi + c(c-1)\phi^{c-2}|\nabla\phi|^2 \leq 0 \qquad \text{on} \quad \mathrm{supp}(u)\,,$$

(2.5) is obtained.

Case 2: $-p+1 < c < 0$**.** Due to integration by parts, the Hölder inequality, and (2.4), we have

$$\begin{aligned} &\int_\Omega -\Delta u\cdot u|u|^{p-2}\phi^c\,\mathrm{d}x \\ =& \,(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x + c\int_\Omega (\nabla u)\cdot(\nabla\phi)u|u|^{p-2}\phi^{c-1}\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\geq (p-1)\int_\Omega |u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\\
&\quad + c\left(\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\cdot\int_\Omega |u|^p\phi^{c-2}|\nabla\phi|^2\,\mathrm{d}x\right)^{1/2}\\
&\geq \frac{p+c-1}{1-c}\int_\Omega |u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x\,.
\end{aligned}
$$

(3) Note that our assumption of the Hardy inequality (1.2) implies that the inequality in (1.2) also holds for $f\in W_2^1(\Omega)$ whose support is a compact subset of Ω.

Since ϕ is assumed to be positive and smooth on a neighborhood of $\mathrm{supp}(u)$, it follows from Lemma A.1 that $|u|^{p/2-1}u\phi^{c/2}$ belongs to $\mathring{W}_2^1(\Omega)$, and

$$
\nabla\big(|u|^{p/2-1}u\phi^{c/2}\big)=\frac{p}{2}|u|^{p/2-1}(\nabla u)1_{\{u\neq 0\}}\phi^{c/2}+\frac{c}{2}|u|^{p/2}\phi^{c/2-1}\nabla\phi\,.
$$

Therefore, due to the Hardy inequality and (2.4), we have

$$
\begin{aligned}
&\int_\Omega \big||u|^{p/2-1}u\phi^{c/2}\big|^2\rho^{-2}\,\mathrm{d}x\\
&\lesssim_{p,c}\mathrm{C}_0(\Omega)\int_\Omega\Big(|u|^{p-2}|\nabla u|^2\phi^c 1_{\{u\neq 0\}}+|u|^p\phi^{c-2}|\nabla\phi|^2\Big)\,\mathrm{d}x\\
&\lesssim_{p,c}\mathrm{C}_0(\Omega)\int_{\Omega\cap\{u\neq 0\}}|u|^{p-2}|\nabla u|^2\phi^c\,\mathrm{d}x.
\end{aligned}
$$

□

Theorem 2.7. *Let Ω admit the Hardy inequality (1.2). For any $p\in(1,\infty)$, $c\in(-p+1,1)$, and positive superharmonic function ϕ on Ω, the following holds: If $u\in C(\Omega)$ satisfies (2.3) and $(\Delta u)1_{\{u\neq 0\}}$ is bounded, then for any $\lambda\geq 0$,*

$$
\int_\Omega |u|^p\phi^c\rho^{-2}\,\mathrm{d}x\leq N\int_\Omega|\Delta u-\lambda u|^p\phi^c\rho^{2p-2}\,\mathrm{d}x\,,
$$

where $N=N(p,c,\mathrm{C}_0(\Omega))$.

Proof. Since $\lambda\geq 0$, Lemma 2.6 implies

$$
\begin{aligned}
\int_\Omega |u|^p\phi^c\rho^{-2}\,\mathrm{d}x&\leq N\int_\Omega(-\Delta u)\cdot u|u|^{p-2}1_{\{u\neq 0\}}\phi^c\,\mathrm{d}x\\
&\leq N\int_\Omega(-\Delta u+\lambda u)\cdot u|u|^{p-2}1_{\{u\neq 0\}}\phi^c\,\mathrm{d}x\,,
\end{aligned}
\tag{2.9}
$$

where $N=N(p,c,\mathrm{C}_0(\Omega))>0$. Since $\phi^c\rho^{-2}$ is locally integrable on Ω (see Lemma 2.5.(1)), the first term in (2.9) is finite. By the Hölder inequality, the proof is completed. □

Lemma 2.8 (Existence of a weak solution)**.** *Suppose that (1.2) holds for Ω. Then for any $\lambda\geq 0$ and $f\in C_c^\infty(\Omega)$, there exists a measurable function $u:\Omega\to\mathbb{R}$ satisfying the following:*

(1) $u\in L_{1,\mathrm{loc}}(\Omega)$.
(2) $\Delta u-\lambda u=f$ in the sense of distribution on Ω, i.e., for any $\zeta\in C_c^\infty(\Omega)$,

$$
\int_\Omega u\big(\Delta\zeta-\lambda\zeta\big)\,\mathrm{d}x=\int_\Omega f\zeta\,\mathrm{d}x\,.\tag{2.10}
$$

(3) *For any* $p \in (1,\infty)$, $c \in (-p+1, 1)$, *and positive superharmonic function* ϕ *on* Ω,

$$\int_\Omega |u|^p \phi^c \rho^{-2}\,\mathrm{d}x \leq N \int_\Omega |f|^p \phi^c \rho^{2p-2}\,\mathrm{d}x \tag{2.11}$$

where $N = N(p, c, \mathrm{C}_0(\Omega)) > 0$.

Proof. Take infinitely smooth bounded open sets Ω_n, $n \in \mathbb{N}$, such that

$$\mathrm{supp}(f) \subset \Omega_1 \ , \quad \overline{\Omega_n} \subset \Omega_{n+1} \ , \quad \bigcup_n \Omega_n = \Omega$$

(see, *e.g.*, [15, Proposition 8.2.1]). For arbitrary $h \in C_c^\infty(\Omega_1)$ and $n \in \mathbb{N}$, by $R_{\lambda,n}h$ we denote the classical solution $H \in C^\infty(\overline{\Omega_n})$ of the equation

$$\Delta H - \lambda H = h 1_{\Omega_1} \quad \text{on } \Omega_n \quad ; \quad H|_{\partial\Omega_n} \equiv 0\,.$$

Note that $\overline{\Omega_n}$ is a compact subset of Ω, $R_{\lambda,n}h \in C^\infty(\overline{\Omega_n})$, and $R_{\lambda,n}h|_{\partial\Omega_n} \equiv 0$. Therefore $\big(R_{\lambda,n}h\big)1_{\Omega_n}$ is continuous on Ω and satisfies (2.3). By Theorem 2.7, for any $p \in (1,\infty)$, $c \in (-p+1,1)$ and positive superharmonic functions ϕ on Ω, we have

$$\int_\Omega \big|\big(R_{\lambda,n}h\big)1_{\Omega_n}\big|^p \phi^c \rho^{-2}\,\mathrm{d}x \leq N(p,c,\mathrm{C}_0(\Omega)) \int_\Omega |h|^p \phi^c \rho^{2p-2}\,\mathrm{d}x\,. \tag{2.12}$$

Note that N in (2.12) is independent of n.

Take $F \in C_c^\infty(\Omega_1)$ such that $F \geq |f|$, and put $f_1 := f - F$ and $f_2 := -F$ so that f_1, $f_2 \leq 0$, and $f_1 - f_2 = f$.

For $v_n := \big(R_{\lambda,n}f_1\big)1_{\Omega_n}$, the maximum principle implies that $0 \leq v_n \leq v_{n+1}$ on Ω. We define $v(x) := \lim_{n\to\infty} v_n(x)$. By applying the monotone convergence theorem to (2.12) with $(h,\phi,p,c) := (f_1, 1_\Omega, 2, 0)$, we obtain that $\int_\Omega |v|^2\rho^{-2}\,\mathrm{d}x \lesssim \int_\Omega |f_1|^2\rho^2\,\mathrm{d}x$, which implies that $v \in L_{1,\mathrm{loc}}(\Omega)$.

We next claim that for any $\zeta \in C_c^\infty(\Omega)$,

$$\int_\Omega v\big(\Delta\zeta - \lambda\zeta\big)\,\mathrm{d}x = \int_\Omega f_1\zeta\,\mathrm{d}x\,. \tag{2.13}$$

Fix $\zeta \in C_c^\infty(\Omega)$, and take large enough $N \in \mathbb{N}$ such that $\mathrm{supp}(\zeta) \subset \Omega_N$. Then for any $n \geq N$, the definition of $v_n = R_{\lambda,n}f_1$ implies that (2.13) holds for v_n instead of v. Since $0 \leq v_n \leq v$ and $v \in L_{1,\mathrm{loc}}(\Omega)$, the Lebesgue dominated convergence theorem yields (2.13).

By the same argument, $w := \lim_{n\to\infty}\big(R_{\lambda,n}f_2\big)1_{\Omega_n}$ belongs to $L_{1,\mathrm{loc}}(\Omega)$, and (2.13) holds for (w, f_2) instead of (v, f_1).

Put $u := v - w = \lim_{n\to\infty}\big(R_{\lambda,n}f\big)1_{\Omega_n}$ (the limit exists almost everywhere on Ω). Then $u \in L_{1,\mathrm{loc}}(\Omega)$, and u satisfies (2.10). In addition, by applying Fatou's lemma to (2.12) with $h := f$, (2.11) is obtained. □

3. Weighted Sobolev spaces and solvability of the Poisson equation

In this section, we focus on the Poisson equation

$$\Delta u - \lambda u = f \quad (\lambda \geq 0)$$

in an open set $\Omega \subset \mathbb{R}^d$ admitting the Hardy inequality, in terms of the weighted Sobolev $\Psi H_{p,\theta}^\gamma(\Omega)$ introduced in Definition 3.7. It is worth noting that the zero Dirichlet condition ($u|_{\partial\Omega} = 0$) is implicitly considered in $\Psi H_{p,\theta}^\gamma(\Omega)$, as $C_c^\infty(\Omega)$ is dense in $\Psi H_{p,\theta}^\gamma(\Omega)$ (see Lemma 3.10).

The organization of this section is as follows: In Subsection 3.1, we present the notions of *Harnack function* and *regular Harnack function*. Subsection 3.2 introduces the weighted Sobolev space $\Psi H_{p,\theta}^{\gamma}(\Omega)$. In Subsection 3.3, we prove the main theorem of this section (Theorem 3.14), using the results in Section 2 and extending the localization argument employed in [34] to the version of $\Psi H_{p,\theta}^{\gamma}(\Omega)$.

3.1. **Harnack function and regular Harnack function.**

Definition 3.1.

(1) We call a measurable function $\psi : \Omega \to \mathbb{R}_+$ a *Harnack function*, if there exists a constant $C =: \mathrm{C}_1(\psi) > 0$ such that

$$\operatorname*{ess\,sup}_{B(x,\rho(x)/2)} \psi \leq C \operatorname*{ess\,inf}_{B(x,\rho(x)/2)} \psi \quad \text{for all } x \in \Omega\,.$$

(2) We call a function $\Psi \in C^{\infty}(\Omega)$ a *regular Harnack function*, if $\Psi > 0$ and there exists a sequence of constants $\{C^{(k)}\}_{k\in\mathbb{N}} =: \mathrm{C}_2(\Psi)$ such that for each $k \in \mathbb{N}$,

$$|D^k\Psi| \leq C^{(k)}\,\rho^{-k}\Psi \quad \text{on} \quad \Omega\,.$$

(3) Let ψ be a measurable function and Ψ be a regular Harnack function on Ω. We say that Ψ is a *regularization* of ψ, if there exists a constant $C =: \mathrm{C}_3(\psi,\Psi) > 0$ such that

$$C^{-1}\Psi \leq \psi \leq C\,\Psi \quad \text{almost everywhere on } \Omega.$$

A relation between the notions of Harnack functions and regular Harnack functions is provided in Lemma 3.6.

Remark 3.2. We introduced the notion of the Harnack function to facilitate a localization argument (see Lemma 3.18). Separately, there is an earlier work [52] for the relation between the boundary behavior of continuous Harnack functions and the quasihyperbolic distance; note that in [52], the term 'Harnack function' is defined as a continuous Harnack function, distinct from the definition provided in Definition 3.1.

Example 3.3.

(1) For any $E \subset \Omega^c$, the function $x \mapsto d(x,E)$ is a Harnack function on Ω. Additionally, $\mathrm{C}_1\big(d(\,\cdot\,,E)\big)$ can be chosen as 3.
(2) Let $\Psi \in C^{\infty}(\Omega)$ satisfy $\Psi > 0$ and $\Delta\Psi = -\Lambda\Psi$ for some constant $\Lambda \geq 0$. We claim that Ψ is a regular Harnack function on Ω, and $\mathrm{C}_2(\Psi)$ can be chosen to depend only on d. To observe this, for a fixed $x_0 \in \Omega$, put

$$u(t,x) := \mathrm{e}^{-\Lambda\rho(x_0)^2 t}\,\Psi\big(x_0 + \rho(x_0)x\big)$$

so that $u_t = \Delta u$ on $\mathbb{R}\times B_1(0)$. The interior estimates (see, *e.g.*, [35, Theorem 2.3.9]) and the parabolic Harnack inequality imply that for any $k \in \mathbb{R}$,

$$\rho(x_0)^k|D^k\Psi(x_0)| = |D_x^k u(0,0)| \lesssim_{k,d} \|u\|_{L_2((-1/4,0]\times B_{1/2}(0))} \lesssim_d u(1,0) \leq \Psi(x_0)\,.$$

(3) The multivariate Faá di Bruno's formula (see, *e.g.*, [14, Theorem 2.1]) implies the following:

Let $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}$ be open sets and $f : U \to V$ and $l : V \to \mathbb{R}$ be smooth functions. For any multi-index α,

$$\big|D^\alpha(l \circ f)\big| \leq N(d,\alpha)\sum_{k=1}^{|\alpha|}\Big(\big|(D^k l)\circ f\big|\sum_{\substack{\beta_1+\ldots+\beta_k=\alpha\\ |\beta_i|\geq 1}}\prod_{i=1}^{k}|D^{\beta_k}f|\Big).$$

This inequality implies that for any regular Harnack function Ψ on Ω, and $\sigma \in \mathbb{R}$, Ψ^σ is also a regular Harnack function on Ω, and $\mathrm{C}_2(\Psi^\sigma)$ can be chosen to depend only on d, σ, $\mathrm{C}_2(\Psi)$.

(4) If Ψ and Φ are regularizations of ψ and ϕ, respectively, then $\Psi\Phi$, $\Psi+\Phi$, and $\frac{\Phi\Psi}{\Phi+\Psi}$ are regularizations of $\psi\phi$, $\max(\psi,\phi)$, and $\min(\psi,\phi)$, respectively.

Lemma 3.4. *A measurable function $\psi : \Omega \to \mathbb{R}_+$ is a Harnack function if and only if there exists $r \in (0,1)$ and $N_r > 0$ such that*

$$\operatorname*{ess\,sup}_{B(x,r\rho(x))}\psi \leq N_r \operatorname*{ess\,inf}_{B(x,r\rho(x))}\psi \quad \textit{for all } x \in \Omega.$$

In this case, $\mathrm{C}_1(\psi)$ and N_r depend only on each other and r.

Proof. We only need to prove that for fixed constants r_0, $r \in (0,1)$ and $\widetilde{N} \geq 1$, if

$$\begin{aligned} &\text{if} \quad \operatorname*{ess\,sup}_{B(x,r_0\rho(x))}\psi \leq \widetilde{N}\operatorname*{ess\,inf}_{B(x,r_0\rho(x))}\psi \quad \forall\, x\in\Omega\,,\\ &\text{then} \quad \operatorname*{ess\,sup}_{B(x,r\rho(x))}\psi \leq \widetilde{N}^{2K+1}\operatorname*{ess\,inf}_{B(x,r\rho(x))}\psi \quad \forall\, x\in\Omega\,, \end{aligned} \tag{3.1}$$

where K is the smallest integer such that $K \geq \frac{r}{(1-r)r_0}$.

If $r \leq r_0$, then there is nothing to prove. Consider the case $r > r_0$. For $x \in \Omega$, we denote $B(x) = B\big(x, r_0\rho(x)\big)$. For fixed $x_0 \in \Omega$ and $y \in \overline{B}\big(x_0, r\rho(x_0)\big)$, put $x_i := (1-\frac{i}{K})x_0 + \frac{i}{K}y$, $i = 1, \ldots, M$. One can observe that $|x_{i-1}-x_i| \leq r_0\rho(x_i)$, and therefore $x_{i-1} \in B(x_i)$. This implies that $B(x_{i-1}) \cap B(x_i) \neq \emptyset$, and hence

$$\operatorname*{ess\,sup}_{B(x_i)}\psi \leq \widetilde{N}\operatorname*{ess\,inf}_{B(x_i)}\psi \leq \widetilde{N}\operatorname*{ess\,inf}_{B(x_{i-1})\cap B(x_i)}\psi \leq \widetilde{N}\operatorname*{ess\,sup}_{B(x_{i-1})}\psi\,. \tag{3.2}$$

By applying (3.2) for $i = 1, \ldots, K$, we obtain that $\operatorname*{ess\,sup}_{B(y)}\psi \leq \widetilde{N}^k\operatorname*{ess\,sup}_{B(x)}\psi$. Since $B(x_0, r\rho(x_0))$ is contained in a finite union of elements in $\big\{B(y) : y \in \overline{B}(x_0, r\rho(x_0))\big\}$, we have

$$\operatorname*{ess\,sup}_{B(x_0,r\rho(x_0))}\psi \leq \widetilde{N}^K\operatorname*{ess\,sup}_{B(x)}\psi = \widetilde{N}^K\operatorname*{ess\,sup}_{B(x_0,r_0\rho(x_0))}\psi\,.$$

The same argument implies that $\operatorname*{ess\,inf}_{B(x_0,r_0\rho(x_0))}\psi \leq \widetilde{N}^k\operatorname*{ess\,inf}_{B(x_0,r\rho(x_0))}\psi$. Consequently, we have

$$\operatorname*{ess\,sup}_{B(x_0,r\rho(x_0))}\psi \leq \widetilde{N}^K\operatorname*{ess\,sup}_{B(x_0,r_0\rho(x_0))}\psi \leq \widetilde{N}^{K+1}\operatorname*{ess\,inf}_{B(x_0,r_0\rho(x_0))}\psi \leq \widetilde{N}^{2K+1}\operatorname*{ess\,inf}_{B(x_0,r\rho(x_0))}\psi\,,$$

where the second inequality is implied by the assumption in (3.1). $\square$

Remark 3.5. Let ψ be a Harnack function on Ω. Since $\psi \in L_{1,\mathrm{loc}}(\Omega)$, almost every point in Ω is a Lebesgue point of ψ. If $x \in \Omega$ is a Lebesgue point of ψ, then for any $r \in (0,1)$,

$$\operatorname*{ess\,inf}_{B(x,r\rho(x))}\psi \leq \psi(x) \leq \operatorname*{ess\,sup}_{B(x,r\rho(x))}\psi\,.$$

By Lemma 3.4, we obtain that for almost every $x \in \Omega$ and for any $r \in (0,1)$, there exists $N_r > 0$ depending only on $\mathrm{C}_1(\psi)$ and r such that

$$N_r^{-1} \operatorname*{ess\,sup}_{B(x,r\rho(x))} \psi \leq \psi(x) \leq N_r \operatorname*{ess\,inf}_{B(x,r\rho(x))} \psi \,.$$

Lemma 3.6.

(1) If ψ is a Harnack function, then there exists a regularization of ψ. For this regularization of ψ, denoted by $\widetilde{\psi}$, $\mathrm{C}_2(\widetilde{\psi})$ and $\mathrm{C}_3(\psi, \widetilde{\psi})$ can be chosen to depend only on d and $\mathrm{C}_1(\psi)$.

(2) If Ψ is a regular Harnack function, then it is also a Harnack function and $\mathrm{C}_1(\Psi)$ can be chosen to depend only on d and $\mathrm{C}_2(\Psi)$.

This lemma implies that a measurable function is a Harnack function if and only if it has a regularization.

Proof of Lemma 3.6.

(1) Let ψ be a Harnack function on Ω. Take $\zeta \in C_c^\infty(B_1)$ such that $\zeta \geq 0$ and $\int_{B_1} \zeta dx = 1$. For $i = 1, 2, 3$ and $k \in \mathbb{Z}$, put

$$U_{i,k} = \{x \in \Omega \,:\, 2^{k-i} < \rho(x) < 2^{k+i}\} \quad \text{and} \quad \zeta_k(x) = \frac{1}{2^{(k-4)d}} \zeta\Big(\frac{x}{2^{k-4}}\Big) \,.$$

Note that for each i,

$$\big\{U_{i,k}\big\}_{k\in\mathbb{Z}} \text{ is a locally finite cover of } \Omega, \text{ and } \sum_{k\in\mathbb{Z}} 1_{U_{i,k}} \leq 2i \,. \tag{3.3}$$

For each $k \in \mathbb{Z}$, put

$$\Psi_k(x) := \big(\psi 1_{U_{2,k}}\big) * \zeta_k(x) := \int_{B(x,2^{k-4})} \big(\psi 1_{U_{2,k}}\big)(y) \zeta_k(x-y) \,\mathrm{d}y \,,$$

so that $\Psi_k \in C^\infty(\Omega)$.

If $x \in U_{1,k}$, then $B(x, 2^{k-4}) \subset B(x, \rho(x)/2) \cap U_{2,k}$. Therefore we have

$$\Psi_k \geq \Big(\operatorname*{ess\,inf}_{B(x,\rho(x)/2)} \psi\Big) 1_{U_{1,k}}(x) \,. \tag{3.4}$$

If $x \in U_{3,k}$, then $B(x, 2^{k-4}) \subset B(x, \rho(x)/2)$, and if $x \notin U_{3,k}$, then $B(x, 2^{k-4}) \cap U_{2,k} = \emptyset$. Therefore we have

$$\Psi_k(x) \leq \Big(\operatorname*{ess\,sup}_{B(x,\rho(x)/2)} \psi\Big) 1_{U_{3,k}}(x) \,. \tag{3.5}$$

By (3.4), (3.5), and Remark 3.5, we obtain that

$$N^{-1} \psi(x) 1_{U_{1,k}}(x) \leq \Psi_k(x) \leq N \psi(x) 1_{U_{3,k}}(x) \tag{3.6}$$

for almost every $x \in \Omega$, where $N = N(\mathrm{C}_1(\psi))$. Moreover,

$$\begin{aligned} |D^\alpha \Psi_k(x)| &\leq \|D^\alpha \zeta_k\|_\infty \int_{B(x,2^{k-4})} \psi 1_{U_{2,k}} \,\mathrm{d}y \\ &\leq 2^{-|\alpha|k} \Bigg(\operatorname*{ess\,sup}_{B(x,\rho(x)/2)} \psi\Bigg) 1_{U_{3,k}}(x) \;\lesssim_N \rho(x)^{-|\alpha|} \psi(x) 1_{U_{3,k}}(x) \end{aligned} \tag{3.7}$$

for almost every $x \in \Omega$, where $N = N(d, \alpha, \mathrm{C}_1(\psi))$. Due to (3.3), (3.6), and (3.7), we obtain that $\Psi := \sum_{k\in\mathbb{Z}} \Psi_k$ belongs to $C^\infty(\Omega)$, and

$$\Psi \simeq_{\mathrm{C}_1(\psi)} \psi \quad \text{and} \quad |D^\alpha \Psi| \leq \sum_{k\in\mathbb{Z}} |D^\alpha \Psi_k| \lesssim_N \rho^{-|\alpha|}\psi \tag{3.8}$$

almost everywhere on Ω, where $N = N(d, \alpha, \mathrm{C}_1(\psi))$. By (3.8), the proof is completed.

(2) Let $x, y \in \Omega$ satisfy $|x - y| < \rho(x)/2$. For $r \in [0, 1]$, put $x_r = (1 - r)x + ry$, so that $x_r \in B\big(x, \rho(x)/2\big)$ and $\rho(x_r) \geq \rho(x) - |x - x_r| \geq |x - y|$. Then we have

$$\begin{aligned} \Psi(x_r) &\leq \Psi(x_0) + |x-y| \int_0^r \big|(\nabla\Psi)(x_t)\big|\,\mathrm{d}t \\ &\leq \Psi(x_0) + N_0|x-y| \int_0^r \rho(x_t)^{-1}\Psi(x_t)\,\mathrm{d}t \leq \Psi(x_0) + N_0 \int_0^r \Psi(x_t)\,\mathrm{d}t\,, \end{aligned}$$

where $N_0 = N(d, \mathrm{C}_2(\Psi)) > 0$. By Grönwall's inequality, we obtain

$$\Psi(y) = \Psi(x_1) \leq \mathrm{e}^{N_0}\Psi(x_0) = \mathrm{e}^{N_0}\Psi(x).$$

For any $x \in \Omega$, if $y \in B(x, \rho(x)/3)$, then $|x - y| < \min\big(\rho(x), \rho(y)\big)/2$. Therefore we have

$$\mathrm{e}^{-N_0} \operatorname*{ess\,sup}_{B(x,\rho(x)/3)} \Psi(y) \leq \Psi(x) \leq \mathrm{e}^{N_0} \operatorname*{ess\,inf}_{B(x,\rho(x)/3)} \Psi(y)\,,$$

and by Lemma 3.4, the proof is completed. □

3.2. **Weighted Sobolev spaces and regular Harnack functions.** In this subsection, we introduce the weighted Sobolev spaces $H^\gamma_{p,\theta}(\Omega)$ and $\Psi H^\gamma_{p,\theta}(\Omega)$. The space $H^\gamma_{p,\theta}(\Omega)$ was first introduced by Krylov [34] for $\Omega = \mathbb{R}^d_+$, and later generalized by Lototsky [40] for arbitrary domains $\Omega \subsetneq \mathbb{R}^d$. We introduce the weighted Sobolev spaces $\Psi H^\gamma_{p,\theta}(\Omega)$ which is a generalization of the Krylov type weighted Sobolev spaces through regular Harnack functions Ψ.

We first recall the definition of the Bessel potential space on $\mathbb{R}^d$. For $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$, $H^\gamma_p = H^\gamma_p(\mathbb{R}^d)$ denotes the space of Bessel potential with the norm

$$\|f\|_{H^\gamma_p} := \big\|(1-\Delta)^{\gamma/2}f\big\|_{L_p(\mathbb{R}^d)} := \Big\|\mathcal{F}^{-1}\big[(1+|\xi|^2)^{\gamma/2}\mathcal{F}(f)(\xi)\big]\Big\|_p\,,$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform. If $\gamma \in \mathbb{N}_0$, then H^γ_p coincides with the Sobolev space

$$W^\gamma_p(\mathbb{R}^d) := \left\{ f \in \mathcal{D}'(\mathbb{R}^d) \,:\, \sum_{k=0}^{\gamma} \int_{\mathbb{R}^d} |D^k f|^p \,\mathrm{d}x < \infty \right\}.$$

We next recall $H^\gamma_{p,\theta}(\Omega)$ and introduce $\Psi H^\gamma_{p,\theta}(\Omega)$. It is worth mentioning in advance that for $\gamma \in \mathbb{N}_0$, the space $\Psi H^\gamma_{p,\theta}(\Omega)$ coincides with the space

$$\left\{ f \in \mathcal{D}'(\Omega) \,:\, \sum_{k=0}^{\gamma} \int_\Omega \big|\rho^k D^k f\big|^p \Psi^p \rho^{\theta-d} \,\mathrm{d}x < \infty \right\},$$

where $\rho(x) := d(x, \partial\Omega)$ (see Lemma 3.12). In the remainder of this subsection, we assume that

$$p \in (1, \infty)\,, \quad \gamma, \theta \in \mathbb{R}\,, \quad \Psi \text{ is a regular Harnack function on } \Omega\,.$$

By $\widetilde{\rho}$ we denote the regularization of $\rho(\,\cdot\,) := d(\,\cdot\,, \partial\Omega)$ constructed in Lemma 3.6.(1). Recall that for each $k \in \mathbb{N}_0$, there exists a constant $N_k = N(d,k) > 0$ such that

$$\widetilde{\rho} \simeq_{N_0} \rho \quad \text{and} \quad |D^k \widetilde{\rho}| \leq N_k \rho^{1-k} \quad \text{on} \quad \Omega\,.$$

To define the weighted Sobolev spaces, fix $\zeta_0 \in C_c^\infty(\mathbb{R}_+)$ such that

$$\mathrm{supp}(\zeta_0) \subset [\mathrm{e}^{-1}, \mathrm{e}] \quad , \quad \zeta_0 \geq 0 \quad , \quad \sum_{n\in\mathbb{Z}} \zeta_0(\mathrm{e}^n t) = 1 \quad \text{for all } t \in \mathbb{R}_+\,.$$

For $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$, put

$$\zeta_{0,(n)}(x) := \zeta_0\big(\mathrm{e}^{-n}\widetilde{\rho}(x)\big) 1_\Omega(x)\,, \tag{3.9}$$

so that

$$\begin{gathered}\sum_{n\in\mathbb{Z}} \zeta_{0,(n)} \equiv 1 \quad \text{on } \Omega\,, \quad \mathrm{supp}(\zeta_{0,(n)}) \subset \{x \in \Omega \,:\, \mathrm{e}^{n-1} \leq \widetilde{\rho}(x) \leq \mathrm{e}^{n+1}\}\,, \\ \zeta_{0,(n)} \in C^\infty(\mathbb{R}^d)\,, \quad \text{and} \quad |D^\alpha \zeta_{0,(n)}| \leq N(d,\alpha,\zeta)\, \mathrm{e}^{-n|\alpha|}\,.\end{gathered} \tag{3.10}$$

Definition 3.7.

(1) By $H_p^\gamma(\Omega)$ we denote the class of all distributions $f \in \mathcal{D}'(\Omega)$ such that

$$\|f\|^p_{H^\gamma_{p,\theta}(\Omega)} := \sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta} \big\| \big(\zeta_{0,(n)} f\big)(\mathrm{e}^n \cdot)\big\|^p_{H^\gamma_p(\mathbb{R}^d)} < \infty\,.$$

(2) By $\Psi H^\gamma_{p,\theta}(\Omega)$ we denote the class of all distributions $f \in \mathcal{D}'(\Omega)$ such that $f = \Psi g$ for some $g \in H^\gamma_{p,\theta}(\Omega)$. The norm in $\Psi H^\gamma_{p,\theta}(\Omega)$ is defined by

$$\|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)} := \|\Psi^{-1} f\|_{H^\gamma_{p,\theta}(\Omega)}\,.$$

We also denote

$$L_{p,\theta}(\Omega) := H^0_{p,\theta}(\Omega) \quad \text{and} \quad \Psi L_{p,\theta}(\Omega) := \Psi H^0_{p,\theta}(\Omega)\,.$$

In the rest of this subsection, we collect properties of $H^\gamma_{p,\theta}(\Omega)$ and $\Psi H^\gamma_{p,\theta}(\Omega)$. As $\Psi H^\gamma_{p,\theta}(\Omega)$ is a variant of $H^\gamma_{p,\theta}(\Omega)$, we drive properties of $\Psi H^\gamma_{p,\theta}(\Omega)$ based on those of $H^\gamma_{p,\theta}(\Omega)$. Note that we cite the properties of $H^\gamma_{p,\theta}(\Omega)$ from [40] as refined versions. Specifically, in Lemma 3.8 and the proof of Lemma 3.10.(2), the constants in their estimates are independent of Ω. The validity of these refined estimates is supported by the proof in [40], with complete details provided in [49, Appendix A.1].

The spaces $H^\gamma_{p,\theta}(\Omega)$ and $\Psi H^\gamma_{p,\theta}(\Omega)$ are independent of the choice of ζ_0 (see Lemma 3.8.(2) of this paper). Therefore, we ignore the dependence on ζ_0. We denote

$$\mathcal{I} = \{d,\, p,\, \gamma,\, \theta\} \quad \text{and} \quad \mathcal{I}' = \{d,\, p,\, \gamma,\, \theta,\, \mathrm{C}_2(\Psi)\}\,,$$

where $\mathrm{C}_2(\Psi)$ is the sequence of constants in Definition 3.1.(2).

Lemma 3.8 (see [40] or Proposition A.3 in [49])**.**

(1) For any $s < \gamma$,

$$\|f\|_{H^s_{p,\theta}(\Omega)} \lesssim_{\mathcal{I},s} \|f\|_{H^\gamma_{p,\theta}(\Omega)}\,.$$

(2) For any $\eta \in C_c^\infty(\mathbb{R}_+)$,

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta} \|\eta\big(\mathrm{e}^{-n}\widetilde{\rho}(\mathrm{e}^n \cdot)\big) f(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p} \lesssim_{\mathcal{I},\eta} \|f\|^p_{H^\gamma_{p,\theta}(\Omega)}.$$

If η additionally satisfies

$$\inf_{t\in\mathbb{R}_+}\left[\sum_{n\in\mathbb{Z}}\eta(\mathrm{e}^n t)\right]>0\,,$$

then

$$\|f\|^p_{H^\gamma_{p,\theta}(\Omega)}\lesssim_{\mathcal{I},\eta}\sum_{n\in\mathbb{Z}}\mathrm{e}^{n\theta}\big\|\eta\big(\mathrm{e}^{-n}\widetilde{\rho}(\mathrm{e}^n\cdot)\big)f(\mathrm{e}^n\cdot)\big\|^p_{H^\gamma_p}\,.$$

(3) *For any $s\in\mathbb{R}$,*

$$\|\widetilde{\rho}^{\,s}f\|_{H^\gamma_{p,\theta}(\Omega)}\simeq_{\mathcal{I},s}\|f\|_{H^\gamma_{p,\theta+sp}(\Omega)}\,.$$

(4) *For any multi-index $k\in\mathbb{N}$,*

$$\|f\|_{H^\gamma_{p,\theta}(\Omega)}\simeq_{\mathcal{I},k}\sum_{i=0}^{k}\|D^if\|_{H^{\gamma-k}_{p,\theta+ip}(\Omega)}. \tag{3.11}$$

In particular, $\|D^kf\|_{H^{\gamma-k}_{p,\theta+kp}(\Omega)}\lesssim_{\mathcal{I},k}\|f\|_{H^\gamma_{p,\theta}(\Omega)}$.

(5) *Let $k\in\mathbb{N}_0$ such that $|\gamma|\leq k$. If $a\in C^k_{\mathrm{loc}}(\Omega)$ satisfies*

$$|a|^{(0)}_k:=\sup_\Omega\sum_{|\alpha|\leq k}\rho^{|\alpha|}|D^\alpha a|<\infty\ ,$$

then

$$\|af\|_{H^\gamma_{p,\theta}(\Omega)}\lesssim_{\mathcal{I}}|a|^{(0)}_k\|f\|_{H^\gamma_{p,\theta}(\Omega)}.$$

Remark 3.9. Lemma 3.8 also holds if f is replaced by $\Psi^{-1}f$. Therefore, all of the assertions in Lemma 3.8, except Lemma 3.8.(4), remain valid when $H^*_{*,*}(\Omega)$ is replaced by $\Psi H^*_{*,*}(\Omega)$.

Lemma 3.10.

(1) *$C^\infty_c(\Omega)$ is dense in $\Psi H^\gamma_{p,\theta}(\Omega)$.*

(2) *$\Psi H^\gamma_{p,\theta}$ is a reflexive Banach space with the dual $\Psi^{-1}H^{-\gamma}_{p',\theta'}(\Omega)$, where*

$$\frac{1}{p}+\frac{1}{p'}=1\quad\textit{and}\quad\frac{\theta}{p}+\frac{\theta'}{p'}=d\,. \tag{3.12}$$

Moreover, for any $f\in\mathcal{D}'(\Omega)$, we have

$$\|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)}\simeq_{\mathcal{I}'}\sup_{g\in C^\infty_c(\Omega),g\neq0}\frac{\langle f,g\rangle}{\|g\|_{\Psi^{-1}H^{-\gamma}_{p',\theta'}(\Omega)}}\,.$$

(3) *For any $k,\,l\in\mathbb{N}_0$,*

$$\|\big(D^k\Psi\big)D^lf\|_{H^\gamma_{p,\theta}(\Omega)}\leq_{\mathcal{I}',l,k}\|\Psi f\|_{H^{\gamma+l}_{p,\theta-(k+l)p}(\Omega)}$$

(4) *Let Φ be a regular Harnack function on Ω, and there exist a constant $N_0>0$ such that $\Psi\leq N_0\Phi$ on Ω. Then*

$$\|\Psi f\|_{H^\gamma_{p,\theta}(\Omega)}\leq N\|\Phi f\|_{H^\gamma_{p,\theta}(\Omega)}.$$

where $N=N(\mathcal{I}',\mathrm{C}_2(\Phi),N_0)$.

(5) *Let $p'\in(1,\infty)$, γ', $\theta'\in\mathbb{R}$, and Ψ' be a regular Harnack function on Ω, if $f\in\Psi H^\gamma_{p,\theta}(\Omega)\cap\Psi'H^{\gamma'}_{p',\theta'}(\Omega)$, then there exists $\{f_n\}_{n\in\mathbb{N}}\subset C^\infty_c(\Omega)$ such that*

$$\|f-f_n\|_{\Psi H^\gamma_{p,\theta}(\Omega)}+\|f-f_n\|_{\Psi'H^{\gamma'}_{p',\theta'}(\Omega)}\to0\quad\textit{as }n\to\infty\,.$$

Proof. (1), (2) When $\Psi \equiv 1$, the results can be found in [40] (or see [49, Proposition A.2]). Since the map $f \mapsto \Psi^{-1} f$ is an isometric isomorphism from $\Psi H^{\gamma}_{p,\theta}(\Omega)$ to $H^{\gamma}_{p,\theta}(\Omega)$, there is nothing to prove.

(3) Since Ψ and $\widetilde{\rho}$ are regular Harnack functions, we obtain that for any k, $m \in \mathbb{N}_0$,

$$\Big|\frac{D^k\Psi}{\widetilde{\rho}^{-k}\Psi}\Big|^{(0)}_m \leq N(d,k,m,\mathrm{C}_2(\Psi))\,.$$

By Lemma 3.8.(5) and (3), we have

$$\|\big(D^k\Psi\big)f\|_{H^{\gamma}_{p,\theta}(\Omega)} \lesssim_{\mathcal{I}',k} \|\widetilde{\rho}^{-k}\Psi f\|_{H^{\gamma}_{p,\theta}(\Omega)} \lesssim_{\mathcal{I}',k} \|\Psi f\|_{H^{\gamma}_{p,\theta-kp}(\Omega)}\,. \tag{3.13}$$

Therefore, we only need to prove that for any $l \in \mathbb{N}$,

$$\|\Psi D^l f\|_{H^{\gamma}_{p,\theta}(\Omega)} \lesssim_{\mathcal{I}',l} \|\Psi f\|_{H^{\gamma+l}_{p,\theta-lp}(\Omega)}\,.$$

Recall that Ψ^{-1} is a regular Harnack function, and $\mathrm{C}_2(\Psi^{-1})$ can be chosen to depend only on $\mathrm{C}_2(\Psi)$ and d. It follows from Leibniz's rule, (3.13), and Lemma 3.8.(4) and (1) that

$$\begin{aligned}\|\Psi D^l(\Psi^{-1}\Psi f)\|_{H^{\gamma}_{p,\theta}(\Omega)} &\lesssim_{d,l} \sum_{n=0}^{l}\|\Psi D^{l-n}(\Psi^{-1})\cdot D^n(\Psi f)\|_{H^{\gamma}_{p,\theta}(\Omega)}\\ &\lesssim_N \sum_{n=0}^{l}\|D^n(\Psi f)\|_{H^{\gamma}_{p,\theta-(l-n)p}(\Omega)} \lesssim_N \|\Psi f\|_{H^{\gamma+l}_{p,\theta-lp}(\Omega)}\,.\end{aligned}$$

(4) For any $k \in \mathbb{N}_0$,

$$|\Psi\Phi^{-1}|^{(0)}_k \leq N(d,k,\mathrm{C}_2(\Psi),\mathrm{C}_2(\Phi),N_0)\,.$$

Therefore, it follows from Lemma 3.8.(5). that

$$\|\Psi f\|_{H^{\gamma}_{p,\theta}(\Omega)} = \|\Psi\Phi^{-1}(\Phi f)\|_{H^{\gamma}_{p,\theta}(\Omega)} \lesssim_N \|\Phi f\|_{H^{\gamma}_{p,\theta}(\Omega)}\,.$$

(5) It directly follows from Lemma A.3. □

Remark 3.11. It follows from Lemma 3.10.(4) that for regular Harnack functions Ψ and Φ, if $N^{-1}\Phi \leq \Psi \leq N\Phi$ for some constant $N > 0$, then $\Psi H^{\gamma}_{p,\theta}(\Omega)$ coincides with $\Phi H^{\gamma}_{p,\theta}(\Omega)$. Therefore, applying Lemma 3.8.(3), we obtain that if Ψ is a regularization of ρ^{σ} ($\sigma \in \mathbb{R}$), then $\Psi H^{\gamma}_{p,\theta}(\Omega) = H^{\gamma}_{p,\theta-\sigma p}(\Omega)$.

Lemma 3.12. *Let $f \in \mathcal{D}'(\Omega)$.*

(1) If $\gamma \in \mathbb{N}_0$, then

$$\|f\|^p_{\Psi H^{\gamma}_{p,\theta}(\Omega)} \simeq_{\mathcal{I}'} \sum_{|\alpha|\leq\gamma}\int_{\Omega}\big|\rho^{|\alpha|}D^{\alpha}f\big|^p\Psi^{-p}\rho^{\theta-d}\,\mathrm{d}x\,.$$

(2) For any $k \in \mathbb{N}$,

$$\|f\|_{\Psi H^{\gamma}_{p,\theta}(\Omega)} \simeq_{\mathcal{I}',k} \inf\bigg\{\sum_{|\alpha|\leq k}\|f_{\alpha}\|_{\Psi H^{\gamma+k}_{p,\theta-|\alpha|p}(\Omega)} : f = \sum_{|\alpha|\leq k}D^{\alpha}f_{\alpha}\bigg\}. \tag{3.14}$$

In particular, if $\gamma = -1, -2, \ldots$, then

$$\|f\|_{\Psi H^{\gamma}_{p,\theta}(\Omega)} \simeq_{\mathcal{I}'} \inf\bigg\{\sum_{|\alpha|\leq-\gamma}\|f_{\alpha}\|_{\Psi L_{p,\theta-|\alpha|p}(\Omega)} : f = \sum_{|\alpha|\leq-\gamma}D^{\alpha}f_{\alpha}\bigg\}.$$

Proof. (1) Due to (3.11), we only need to prove the case of $\gamma = 0$. This case is proved by the following:

$$\begin{aligned}\|f\|^p_{\Psi L_{p,\theta}(\Omega)} &:= \sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta}\int_\Omega \big|\big(\zeta_{0,(n)}\Psi^{-1}f\big)(\mathrm{e}^n x)\big|^p\,\mathrm{d}x\\ &= \int_\Omega\bigg(\sum_{n\in\mathbb{Z}}\mathrm{e}^{n(\theta-d)}|\zeta_{0,(n)}|^p\bigg)|f|^p\Psi^{-p}\,\mathrm{d}x \simeq_{d,p,\theta}\int_\Omega \rho^{\theta-d}|f|^p\Psi^{-p}\,\mathrm{d}x\,,\end{aligned}$$

where the last similarity is implied by properties of $\zeta_{0,(n)}$ (see (3.9) and (3.10)).

(2) Repeatedly applying Lemma A.2, we obtain $\{f_\alpha\}_{|\alpha|\leq k}\subset\mathcal{D}'(\Omega)$ such that

$$f = \sum_{|\alpha|\leq k} D^\alpha f_\alpha \quad\text{and}\quad \sum_{|\alpha|\leq k}\|f_\alpha\|_{\Psi H^{\gamma+k}_{p,\theta-|\alpha|p}(\Omega)} \lesssim_{\mathcal{I}',k} \|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)}\,.$$

Therefore we obtain (3.14) where '$\simeq_{\mathcal{I},k}$' is replaced by '$\gtrsim_{\mathcal{I},k}$'.

For the inverse inequality, let $f=\sum_{|\alpha|\leq n}D^\alpha f_\alpha$ where $f_\alpha\in H^{\gamma+n}_{p,\theta-|\alpha|p}(\Omega)$. It follows from Lemma 3.10.(2) and Lemmas 3.10.(3) and 3.8.(1) that for any $g\in C_c^\infty(\Omega)$,

$$\begin{aligned}|\langle f,g\rangle| =&\quad \bigg|\sum_{|\alpha|\leq n}\big\langle\Psi^{-1}f_\alpha,\Psi D^\alpha g\big\rangle\bigg|\\ \lesssim_{\mathcal{I}',n}&\quad\sum_{|\alpha|\leq n}\Big(\|\Psi^{-1}f_\alpha\|_{H^{\gamma+n}_{\theta-|\alpha|p}(\Omega)}\|\Psi D^\alpha g\|_{H^{-\gamma-n}_{p',\theta'+|\alpha|p'}(\Omega)}\Big)\\ \lesssim_{\mathcal{I}',n}&\quad\Big(\sum_{|\alpha|\leq n}\|\Psi^{-1}f_\alpha\|_{H^{\gamma+n}_{\theta-|\alpha|p}(\Omega)}\Big)\|\Psi g\|_{H^{-\gamma}_{p',\theta'}(\Omega)}\,,\end{aligned}$$

where p' and θ' are constants in (3.12). By applying Lemma 3.10.(2), we have

$$\|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)}\lesssim_{\mathcal{I}',n}\inf\bigg\{\sum_{|\alpha|\leq n}\|f_\alpha\|_{\Psi H^{\gamma+n}_{p,\theta-|\alpha|p}(\Omega)} : f=\sum_{|\alpha|\leq n}D^\alpha f_\alpha\bigg\}.$$

Therefore, the proof is completed. □

We end this subsection with a Sobolev-Hölder embedding theorem for the spaces $\Psi H^\gamma_{p,\theta}(\Omega)$. For $k\in\mathbb{N}_0$, $\alpha\in(0,1]$ and $\delta\in\mathbb{R}$, we define the weighted Hölder norm

$$|f|^{(\delta)}_{k,\alpha} := \sum_{i=0}^k\sup_\Omega\big|\rho^{\delta+i}D^i f\big| + \sup_{x,y\in\Omega}\frac{\big|\big(\widetilde{\rho}^{\delta+k+\alpha}D^k f\big)(x)-\big(\widetilde{\rho}^{\delta+k+\alpha}D^k f\big)(y)\big|}{|x-y|^\alpha}\,.$$

Proposition 3.13. *Let $k\in\mathbb{N}_0$, $\alpha\in(0,1]$.*

(1) For any $\delta\in\mathbb{R}$,

$$\begin{aligned}\big|\Psi^{-1}f\big|^{(\delta)}_{k,\alpha}\simeq_N&\sum_{i=0}^k\sup_{x\in\Omega}\big|\Psi(x)^{-1}\rho(x)^{\delta+i}D^i f(x)\big|\\ &+\sup_{x\in\Omega}\bigg(\Psi^{-1}(x)\rho^{\delta+k+\alpha}(x)\sup_{y:|y-x|<\frac{\rho(x)}{2}}\frac{\big|D^k f(x)-D^k f(y)\big|}{|x-y|^\alpha}\bigg)\,,\end{aligned}$$

where $N=N(d,k,\alpha,\delta,\mathrm{C}_2(\Psi))$.

(2) If $\alpha\in(0,1)$ and $k+\alpha\leq\gamma-d/p$, then for any $f\in\Psi H^\gamma_{p,\theta}(\Omega)$,

$$\big|\Psi^{-1}f\big|^{(\theta/p)}_{k,\alpha}\lesssim_{\mathcal{I}',k,\alpha}\|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)}\,.$$

Proof. (1) This result follows from the direct calculation and the definition of regular Harnack functions. Therefore, we leave the proof to the reader.

(2) We only need to prove for $\Psi \equiv 1$, and the result for this case is stated in [40, Theorem 4.3]. We give proof for the convenience of the reader.

For $f \in H^{\gamma}_{p,\theta}(\Omega)$, the Sobolev embedding theorem implies

$$\left\|\big(f\zeta_{0,(n)}\big)(\mathrm{e}^{n}\,\cdot)\right\|_{C^{k,\alpha}} \leq N\left\|\big(f\zeta_{0,(n)}\big)(\mathrm{e}^{n}\,\cdot)\right\|_{H^{\gamma}_{p}} < \infty\,, \tag{3.15}$$

where $N = N(d,p,\gamma,k,\delta)$. Hence f belongs to $C^{k}_{\mathrm{loc}}(\Omega)$. For $x \in \Omega$, take $n_0 \in \mathbb{Z}$ such that $\mathrm{e}^{n_0-1} \leq \rho(x) \leq \mathrm{e}^{n_0}$. If $|x-y| < \frac{\rho(x)}{2}$, then $\mathrm{e}^{n_0-2} \leq \rho(y) \leq \mathrm{e}^{n_0+2}$. Take constants A and B depending only on d such that $A^{-1}\rho \leq \widetilde{\rho} \leq A\rho$, and $\sum_{|n|\leq B}\zeta_0\big(\mathrm{e}^{n}t\big) \equiv 1$ for all $\frac{1}{A\mathrm{e}^2} \leq t \leq A\mathrm{e}^2$. Then we have

$$\sum_{|n-n_0|\leq B}\zeta_{0,(n)} \equiv 1 \quad \text{on} \quad U_{n_0} := \big\{y\,:\,\mathrm{e}^{n_0-2} \leq \rho(y) \leq \mathrm{e}^{n_0+2}\big\}\,.$$

Due to $B(x,\rho(x)/2) \subset U_{n_0}$ and (3.15), we have

$$\begin{aligned}
&\sum_{i=0}^{k}\Big(\rho(x)^{\theta/p+i}\,\big|D^{i}f(x)\big|\Big) + \rho(x)^{\theta/p+k+\alpha}\sup_{y:|y-x|<\frac{\rho(x)}{2}}\frac{\big|D^{k}f(x)-D^{k}f(y)\big|}{|x-y|^{\alpha}}\\
&\lesssim_N \mathrm{e}^{n_0\theta/p}\Bigg(\sum_{i=0}^{k}\big|D^{i}\big(f(\mathrm{e}^{n_0}\,\cdot)\big)(x)\big|\\
&\qquad\qquad + \sup_{\mathrm{e}^{-n_0}y\in U_{n_0}}\frac{\big|D^{k}\big(f(\mathrm{e}^{n_0}\,\cdot)\big)(x)-D^{k}\big(f(\mathrm{e}^{n_0}\,\cdot)\big)(y)\big|}{|x-y|^{\alpha}}\Bigg)\\
&\leq \sum_{|n-n_0|\leq B}\mathrm{e}^{n_0\theta/p}\,\big\|(f\zeta_{0,(n)})(\mathrm{e}^{n}\,\cdot)\big\|_{C^{k,\alpha}}\\
&\lesssim_N \left(\sum_{n\in\mathbb{Z}}\mathrm{e}^{n\theta}\,\big\|(f\zeta_{0,(n)})(\mathrm{e}^{n}\,\cdot)\big\|^{p}_{H^{\gamma}_{p}}\right)^{1/p},
\end{aligned}$$

where $N = N(d,p,\gamma,\theta,k,\delta)$. By (1) of this proposition, the proof is completed. $\square$

3.3. **Solvability of the Poisson equation.** The goal of this subsection is to prove the following theorem:

Theorem 3.14. *Let Ω be an open set admitting the Hardy inequality (1.2) and ψ be a superharmonic Harnack function on Ω, with its regularization Ψ. Then for any $p \in (1,\infty)$, $\mu \in (-1/p, 1-1/p)$, and $\gamma \in \mathbb{R}$, the following assertion holds: For any $\lambda \geq 0$ and $f \in \Psi^{\mu}H^{\gamma}_{p,d+2p-2}(\Omega)$, the equation*

$$\Delta u - \lambda u = f \tag{3.16}$$

has a unique solution u in $\Psi^{\mu}H^{\gamma+2}_{p,d-2}(\Omega)$. Moreover, we have

$$\|u\|_{\Psi^{\mu}H^{\gamma+2}_{p,d-2}(\Omega)} + \lambda\|u\|_{\Psi^{\mu}H^{\gamma}_{p,d+2p-2}(\Omega)} \leq N\|f\|_{\Psi^{\mu}H^{\gamma}_{p,d+2p-2}(\Omega)}, \tag{3.17}$$

where $N = N(d,p,\gamma,\mu,\mathrm{C}_0(\Omega),\mathrm{C}_2(\Psi),\mathrm{C}_3(\psi,\Psi))$.

Recall that $\mathrm{C}_0(\Omega)$ is the constant in (1.2), and $\mathrm{C}_2(\Psi)$ and $\mathrm{C}_3(\psi,\Psi)$ are the constants in Definition 3.1.

In Theorem 3.14, one can take $\psi = \Psi = 1_\Omega$. Another example of ψ is introduced in Example 3.17, which is associated with the Green function and valid for any domain admitting the Hardy inequality.

Remark 3.15. In Theorem 3.14, the spaces $\Psi^\mu H^{\gamma+2}_{p,d-2}(\Omega)$ and $\Psi^\mu H^{\gamma}_{p,d+2p-2}(\Omega)$ do not depend on the specific choice of Ψ among regularizations of ψ (see Remark 3.11). If we take Ψ as $\widetilde{\psi}$ which is the regularization of ψ provided in Lemma 3.6.(1), then Theorem 3.14 can be reformulated in terms of ψ. Indeed, $\mathrm{C}_2(\widetilde{\psi})$ and $\mathrm{C}_3(\psi,\widetilde{\psi})$ depend only on d and $\mathrm{C}_1(\psi)$, and therefore the constant N in (3.17) depends only on d, p, γ, μ, $\mathrm{C}_0(\Omega)$, and $\mathrm{C}_1(\psi)$. Additionally, for the case $\gamma\in\mathbb{Z}$, equivalent norms of $\widetilde{\psi}^\mu H^{\gamma+2}_{p,d-2}(\Omega)$ and $\widetilde{\psi}^\mu H^{\gamma}_{p,d+2p-2}(\Omega)$ are provided in Lemma 3.12, and they also can be reformulated in terms of ψ.

Remark 3.16. If $\mu\notin(-1/p,1-1/p)$, then Theorem 3.14 does not hold in general, as pointed out in [34, Remark 4.3]. To observe this, consider the equation

$$\Delta u = f \quad \text{in} \quad \Omega := (0,\pi) \tag{3.18}$$

and put $\psi(x)=\Psi(x)=\sin x$, and $\gamma=0$.

Let $\mu\geq 1-1/p$, and let $f\in C^\infty_c(\Omega)$ with $f\leq 0$, so that $f\in\Psi^\mu L_{p,d+2p-2}(\Omega)$. We assume that there exists a solution $u_1\in\Psi^\mu H^2_{p,d-2}(\Omega)$ of (3.18). Then this u_1 belongs to $H^2_{p,d-2}(\Omega)$. Let u_0 be the classical solution of (3.18) with the boundary condition $u(0)=u(\pi)=0$. Then $u_0\in H^2_{p,d-2}(\Omega)$. Due to Theorem 3.14, (3.18) has a unique solution, and therefore $u_0\equiv u_1$. However $u_0\notin\Psi^\mu H^2_{p,d-2}(\Omega)$ for all $\mu\geq 1-1/p$ (observe that $u_0\simeq\sin x$). It is contradiction. Therefore there exists no solution $u\in\Psi^\mu H^2_{p,d-2}(\Omega)$ of (3.18).

If $\mu<-1/p$, then $0\cdot 1_\Omega$ and 1_Ω belong to $\Psi^\mu H^2_{p,d-2}(\Omega)$ (see Lemma 3.12). Therefore (3.18) with $f:=0$ has at least two solutions in $\Psi^\mu H^2_{p,d-2}(\Omega)$.

Consider the case $\mu=-1/p$. For $n\in\mathbb{N}$, take $\zeta_n\in C^\infty_c(\Omega)$ such that

$$1_{\left[\frac{2}{n},\pi-\frac{2}{n}\right]}\leq\zeta_n\leq 1_{\left[\frac{1}{n},\pi-\frac{1}{n}\right]}\quad\text{and}\quad\left|D^k\zeta_n\right|\leq N(k)n^k\,.$$

By putting $u:=\zeta_n$, one can observe that there is no constant N satisfying (3.17).

Example 3.17. Let $\Omega\subset\mathbb{R}^d$ be a domain admitting the Hardy inequality. We denote $G_\Omega:\Omega\times\Omega\to[0,+\infty]$ the Green function of the Poisson equation (for the definition and the existence of G_Ω, see [7, Definition 4.1.3], and [7, Theorems 4.1.2 and 5.3.8] and [6, Theorem 2], respectively). We claim that for any fixed $x_0\in\Omega$, $\phi_0:=G_\Omega(x_0,\,\cdot\,)\wedge 1$ is a superharmonic Harnack function on Ω. It is worth noting that ϕ_0 is the smallest positive classical superharmonic function, up to constant multiples (see [7, Lemma 4.1.8]), *i.e.*, if ϕ is a positive classical superharmonic function on Ω, then there exists $N_0=N(\phi,\Omega,x_0)>0$ such that $\phi_0\leq N_0\phi$ on Ω.

Note that $G_\Omega(x_0,\,\cdot\,)$ is a positive classical superharmonic function on Ω, and $G_\Omega(x_0,\,\cdot\,)$ is harmonic on $\Omega\setminus\{x_0\}$. This implies that ϕ_0 is a classical superharmonic function on Ω (see Lemma 4.5.(1)).

For $x\in\Omega$, denote $B(x):=B\big(x,\rho(x)/8\big)$. If $|x-x_0|>\rho(x)/4$, then $G_\Omega(x_0,\,\cdot\,)$ is harmonic on $B\big(x,\rho(x)/4\big)$. By the Harnack inequality, we have

$$\sup_{B(x)}\phi_0=\Big(\sup_{y\in B(x)}G_\Omega(x_0,y)\Big)\wedge 1\lesssim_d\Big(\inf_{y\in B(x)}G_\Omega(x_0,y)\Big)\wedge 1=\inf_{B(x)}\phi_0\,.$$

If $|x-x_0| \le \rho(x)/4$, then $\rho(x) \le \frac{4}{3}\rho(x_0)$, which implies that $B(x) \subset B\big(x_0, \rho(x_0)/2\big)$. By Lemma 2.4.(3), there exists $\epsilon_0 \in (0,1]$ such that $G(x_0, \,\cdot\,) \ge \epsilon_0$ on $B\big(x_0, \rho(x_0)/2\big)$. Therefore we have

$$\sup_{B(x)} \phi_0 \le 1 \le \epsilon_0^{-1} \inf_{B(x)} \phi_0 \,.$$

Consequently, ϕ_0 is a superharmonic Harnack function on Ω.

To prove Theorem 3.14, we need the help of Lemmas 3.18 and 3.19. These lemmas are based on a localization argument, wherein Ω is an arbitrary domain and Ψ is an arbitrary regular Harnack function. The proof of Theorem 3.14 is provided after the proof of Lemma 3.19:

Lemma 3.18 (Higher order estimates)**.** *Let $p \in (1,\infty)$, γ, $s \in \mathbb{R}$, $\theta \in \mathbb{R}$, and Ψ be a regular. Then there exists a constant $N = N(d,p,\theta,\gamma,C_2(\Psi),s) > 0$ such that the following assertion holds: Let $\lambda \ge 0$, and suppose that u, $f \in \mathcal{D}'(\Omega)$ satisfy* (3.16)*. Then*

$$\|u\|_{\Psi H^{\gamma+2}_{p,\theta}(\Omega)} + \lambda\|u\|_{\Psi H^{\gamma}_{p,\theta+2p}(\Omega)} \le N\left(\|u\|_{\Psi H^{s}_{p,\theta}(\Omega)} + \|f\|_{\Psi H^{\gamma}_{p,\theta+2p}(\Omega)}\right) . \tag{3.19}$$

Proof. We denote $\Phi = \Psi^{-1}$ so that $\mathrm{C}_2(\Phi)$ depends only on d and $\mathrm{C}_2(\Psi)$.

Step 1. First, we consider the case $s \ge \gamma + 1$. One can certainly assume that

$$\|\Phi u\|_{H^s_{p,\theta}(\Omega)} + \|\Phi f\|_{H^{\gamma}_{p,\theta+2p}(\Omega)} < \infty \,,$$

for if not, there is nothing to prove. Since

$$\|\Phi u\|_{H^{\gamma+1}_{p,\theta}(\Omega)} \lesssim_{d,p,s,\gamma} \|\Phi u\|_{H^s_{p,\theta}(\Omega)}$$

(see Lemma 3.8.(1)), we only need to prove for $s = \gamma + 1$. Put

$$v_n(x) = \zeta_0\big(\mathrm{e}^{-n}\widetilde{\rho}(\mathrm{e}^n x)\big)\Phi(\mathrm{e}^n x)u(\mathrm{e}^n x)\,.$$

Since

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta}\,\|v_n\|^p_{H^{\gamma+1}_p(\mathbb{R}^d)} = \|\Phi u\|^p_{H^{\gamma+1}_{p,\theta}(\Omega)} < \infty\,,$$

we have $v_n \in H^{\gamma+1}_p(\mathbb{R}^d)$. Observe that

$$\Delta v_n - \mathrm{e}^{2n}\lambda v_n = \widetilde{f}_n \quad \text{in} \quad \mathbb{R}^d\,, \tag{3.20}$$

where

$$\begin{aligned}
\widetilde{f}_n(x) :=&\, \mathrm{e}^{2n}\zeta_{0,(n)}(\mathrm{e}^n x)\big(\Phi f\big)(\mathrm{e}^n x) - \mathrm{e}^{2n}\zeta_{0,(n)}(\mathrm{e}^n x)\big(\Phi\Delta u\big)(\mathrm{e}^n x) + \Delta v_n(x)\\
=&\Big[\,\mathrm{e}^{2n}\zeta_{0,(n)}\Big(\Phi f + 2\big(\nabla u\cdot\nabla\Phi\big) + (\Delta\Phi)u\Big)\\
&\quad + \mathrm{e}^n\big(\zeta_0'\big)_{(n)}\Big(2\big(\nabla\widetilde{\rho}\cdot\nabla(\Phi u)\big) + (\Delta\widetilde{\rho})\Phi u\Big) + \big(\zeta_0''\big)_{(n)}|\nabla\widetilde{\rho}|^2\Phi u\,\Big](\mathrm{e}^n x)
\end{aligned}$$

(see (3.9) for the definition of $\zeta_{(n)}$). Make use of Lemmas 3.8.(1) - (3) and 3.10.(3) to obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta}\big\|\widetilde{f}_n\big\|^p_{H^{\gamma}_p(\mathbb{R}^d)}\\
\lesssim_N&\, \|\Phi f\|^p_{H^{\gamma}_{p,\theta+2p}(\Omega)} + \big\|2\big(\nabla u\cdot\nabla\Phi\big) + (\Delta\Phi)u\big\|^p_{H^{\gamma}_{p,\theta+2p}(\Omega)}\\
&+ \big\|2\big(\nabla\widetilde{\rho}\cdot\nabla(\Phi u)\big) + (\Delta\widetilde{\rho})\Phi u\big\|^p_{H^{\gamma}_{p,\theta+p}(\Omega)} + \big\||\nabla\widetilde{\rho}|^2\Phi u\big\|^p_{H^{\gamma}_{p,\theta}(\Omega)}\\
\lesssim_N&\|\Phi f\|^p_{H^{\gamma}_{p,\theta+2p}(\Omega)} + \|\Phi u\|^p_{H^{\gamma+1}_{p,\theta}(\Omega)} < \infty\,,
\end{aligned} \tag{3.21}$$

where $N=N(d,p,\gamma,\theta,\mathrm{C}_2(\Psi))$. This implies that for any $n\in\mathbb{Z}$, $\tilde{f}_n\in H_p^\gamma(\mathbb{R}^d)$.

Due to (3.20) and that $v_n\in H_p^{\gamma+1}(\mathbb{R}^d)$ and $\widetilde{f}_n\in H_p^\gamma(\mathbb{R}^d)$, we have

$$V_n:=(1-\Delta)^{\gamma/2}v_n\in H_p^1(\mathbb{R}^d)\quad,\quad F_n:=(1-\Delta)^{\gamma/2}\widetilde{f}_n\in L_p(\mathbb{R}^d)\,,$$
$$\Delta V_n-(\mathrm{e}^{2n}\lambda+1)V_n=F_n-V_n\,.$$

It is implied by classical results for the Poisson equation in $\mathbb{R}^d$ (see, *e.g.*, [35, Theorem 4.3.8, Theorem 4.3.9]) that

$$\begin{aligned}\|v_n\|_{H_p^{\gamma+2}(\mathbb{R}^d)}+\mathrm{e}^{2n}\lambda\|v_n\|_{H_p^\gamma(\mathbb{R}^d)}=&\quad\|V_n\|_{H_p^2(\mathbb{R}^d)}+\mathrm{e}^{2n}\lambda\,\|V_n\|_{L_p(\mathbb{R}^d)}\\ \leq&\quad\|\Delta V_n\|_{L_p(\mathbb{R}^d)}+(\mathrm{e}^{2n}\lambda+1)\,\|V_n\|_{L_p(\mathbb{R}^d)}\\ \lesssim_{d,p}&\|F_n-V_n\|_{L_p(\mathbb{R}^d)}\\ \leq&\quad\|\widetilde{f}_n\|_{H_p^\gamma(\mathbb{R}^d)}+\|v_n\|_{H_p^\gamma(\mathbb{R}^d)}\,.\end{aligned}$$

Combine this with (3.21) to obtain that

$$\begin{aligned}&\|\Phi u\|^p_{H^{\gamma+2}_{p,\theta}(\Omega)}+\lambda^p\|\Phi u\|^p_{H^\gamma_{p,\theta+2p}(\Omega)}\\ =&\quad\sum_{n\in\mathbb{Z}}\mathrm{e}^{n\theta}\Big(\|v_n\|^p_{H_p^{\gamma+2}(\mathbb{R}^d)}+(\mathrm{e}^{2n}\lambda)^p\|v_n\|^p_{H_p^\gamma(\mathbb{R}^d)}\Big)\\ \lesssim_N&\sum_{n\in\mathbb{Z}}\mathrm{e}^{n\theta}\left(\|v_n\|^p_{H_p^\gamma(\mathbb{R}^d)}+\|\tilde{f}_n\|^p_{H_p^\gamma(\mathbb{R}^d)}\right)\\ \lesssim_N&\|\Phi u\|^p_{H^{\gamma+1}_{p,\theta}(\Omega)}+\|\Phi f\|^p_{H^\gamma_{p,\theta+2p}(\Omega)}\,.\end{aligned}$$

Therefore the case $s=\gamma+1$ is proved. Consequently, (3.19) holds for all $s\geq\gamma+1$.

Step 2. For $s<\gamma+1$, take $k\in\mathbb{N}$ such that $\gamma+1-k\leq s<\gamma+2-k$. Due to the result in Step 1, (3.19) holds for (γ,s) replaced by $(\gamma,\gamma+1)$, $(\gamma-1,\gamma)$, ..., $(\gamma-k,\gamma+1-k)$. Therefore we have

$$\begin{aligned}\|\Phi u\|_{H^{\gamma+2}_{p,\theta}(\Omega)}+\lambda\|\Phi u\|_{H^\gamma_{p,\theta+2p}(\Omega)}\lesssim_N&\|\Phi u\|_{H^{\gamma+1}_{p,\theta}(\Omega)}+\|\Phi f\|_{H^\gamma_{p,\theta+2p}(\Omega)}\\ \lesssim_N&\quad\cdots\\ \lesssim_N&\|\Phi u\|_{H^{\gamma-k+1}_{p,\theta}(\Omega)}+\|\Phi f\|_{H^\gamma_{p,\theta+2p}(\Omega)}\,.\end{aligned}$$

Since $\|\Phi u\|_{H^{\gamma-k+1}_{p,\theta}(\Omega)}\lesssim\|\Phi u\|_{H^s_{p,\theta}(\Omega)}$ (see Lemma 3.8.(1)), the proof is completed. $\square$

Lemma 3.19. *Let $p\in(1,\infty)$, $\theta\in\mathbb{R}$, and regular Harnack function Ψ, and let $\lambda\geq0$. Suppose that there exists $\gamma\in\mathbb{R}$ such that the following holds:*

> *For any $f\in\Psi H^\gamma_{p,\theta+2p}(\Omega)$, in $\Psi H^{\gamma+2}_{p,\theta}(\Omega)$ there exists a unique solution u of equation (3.16). For this solution, we have*
> $$\|u\|_{\Psi H^{\gamma+2}_{p,\theta}(\Omega)}+\lambda\|u\|_{\Psi H^\gamma_{p,\theta+2p}(\Omega)}\leq N_\gamma\|f\|_{\Psi H^\gamma_{p,\theta+2p}(\Omega)}\,,\tag{3.22}$$
> *where N_γ is a constant independent of f and u.*

Then for all $s\in\mathbb{R}$, the following holds:

> *For any $f\in\Psi H^s_{p,\theta+2p}(\Omega)$, in $\Psi H^{s+2}_{p,\theta}(\Omega)$ there exists a unique solution u of equation (3.16). For this solution, we have*
> $$\|u\|_{\Psi H^{s+2}_{p,\theta}(\Omega)}+\lambda\|u\|_{\Psi H^s_{p,\theta+2p}(\Omega)}\leq N_s\|f\|_{\Psi H^s_{p,\theta+2p}(\Omega)}\,,\tag{3.23}$$
> *where $N_s=N(d,p,\gamma,\theta,\mathrm{C}_2(\Psi),N_\gamma,s)$.*

Proof. To prove the uniqueness of solutions, let us assume that $\overline{u} \in \Psi H^{s+2}_{p,\theta}(\Omega)$ satisfies $\Delta\overline{u} - \lambda\overline{u} = 0$. By Lemma 3.18, $\overline{u}$ belongs to $\Psi H^{\gamma+2}_{p,\theta}(\Omega)$. Due to the assumption of this lemma, in $\Psi H^{\gamma+2}_{p,\theta}(\Omega)$, the zero distribution is the unique solution for the equation $\Delta u - \lambda u = 0$. Consequently, $\overline{u}$ is also the zero distribution, and the uniqueness of solutions is proved. Thus, it remains to show the existence of solutions and estimate (3.23).

Step 1. We first consider the case $s > \gamma$. Let $f \in \Psi H^{s}_{p,\theta+2p}(\Omega)$. Due to $\Psi H^{s}_{p,\theta+2p}(\Omega) \subset \Psi H^{\gamma}_{p,\theta+2p}(\Omega)$, f belongs to $\Psi H^{\gamma}_{p,\theta+2p}(\Omega)$, and hence there exists a solution $u \in \Psi H^{\gamma+2}_{p,\theta}(\Omega)$ of equation (3.16). It follows from Lemma 3.18, (3.22), and Lemma 3.8.(1) that

$$\begin{aligned}
\|u\|_{\Psi H^{s+2}_{p,\theta}(\Omega)} + \lambda \|u\|_{\Psi H^{s}_{p,\theta+2p}(\Omega)} &\lesssim_N \|u\|_{\Psi H^{\gamma+2}_{p,\theta}(\Omega)} + \|f\|_{\Psi H^{s}_{p,\theta+2p}(\Omega)} \\
&\leq N_\gamma \|f\|_{\Psi H^{\gamma}_{p,\theta+2p}(\Omega)} + \|f\|_{\Psi H^{s}_{p,\theta+2p}(\Omega)} \\
&\lesssim_N (N_\gamma + 1)\, \|f\|_{\Psi H^{s}_{p,\theta+2p}(\Omega)}\ ,
\end{aligned}$$

where $N = N(d,p,\theta,\gamma,\mathrm{C}_2(\Psi),s)$. Therefore u belongs to $\Psi H^{s+2}_{p,\theta}(\Omega)$, and the proof is completed.

Step 2. Consider the case $s < \gamma$. Since the case $s \geq \gamma$ is proved in Step 1, by mathematical induction, it is sufficient to show that if this lemma holds for $s = s_0 + 1$, then this also holds for $s = s_0$.

Let us assume that this lemma holds for $s = s_0 + 1$. For $f \in \Psi H^{s_0}_{p,\theta+2p}(\Omega)$, by Lemma A.2, there exists $f^0 \in \Psi H^{s_0+1}_{p,\theta+2p}(\Omega)$ and $f^1, \ldots, f^d \in \Psi H^{s_0+1}_{p,\theta+p}(\Omega)$ such that $f = f^0 + \sum_{i=1}^d D_i f^i$ and

$$\|f^0\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)} + \sum_{i=1}^d \|\widetilde{\rho}^{-1} f^i\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)} \leq N \|f\|_{\Psi H^{s_0}_{p,\theta+2p}(\Omega)}\,, \tag{3.24}$$

where $N = N(d,p,\theta,s_0,\mathrm{C}_2(\Psi))$. Due to the assumption that this lemma holds for $s = s_0 + 1$, there exist $v^0, \cdots, v^d \in \Psi H^{s_0+3}_{p,d-2}(\Omega)$ such that

$$\Delta v^0 - \lambda v^0 = f^0 \quad \text{and} \quad \Delta v^i - \lambda v^i = \widetilde{\rho}^{-1} f^i \quad \text{for } i = 1, \ldots, d\,,$$

and

$$\begin{aligned}
&\sum_{i=0}^d \Big(\|v^i\|_{\Psi H^{s_0+3}_{p,\theta}(\Omega)} + \lambda \|v^i\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)} \Big) \\
\leq\ & N_{s_0+1} \bigg(\|f^0\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)} + \sum_{i=1}^d \|\widetilde{\rho}^{-1} f^i\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)} \bigg) \\
\lesssim_N\ & N_{s_0+1} \|f\|_{\Psi H^{s_0}_{p,\theta+2p}(\Omega)}\,,
\end{aligned} \tag{3.25}$$

where the last inequality follows from (3.24). Put $v = v^0 + \sum_{i=1}^d D_i(\widetilde{\rho} v^i)$, and observe that

$$\Delta v - \lambda v = f + \sum_{i=1}^d D_i\big(\Delta(\widetilde{\rho} v^i) - \widetilde{\rho}\Delta v^i\big)\,.$$

By Lemmas 3.8 and 3.10.(3), we have

$$\|D_i\left(\Delta(\widetilde{\rho} v^i) - \widetilde{\rho}\Delta v^i\right)\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)}$$

$$
\begin{aligned}
&\lesssim_N \|\Delta(\widetilde{\rho}v^i)-\widetilde{\rho}\Delta v^i\|_{\Psi H^{s_0+2}_{p,\theta+p}(\Omega)}\\
&\leq \ \big\|\big(D^2\widetilde{\rho}\big)v^i\big\|_{\Psi H^{s_0+2}_{p,\theta+p}(\Omega)}+\big\|\big(D\widetilde{\rho}\big)Dv^i\big\|_{\Psi H^{s_0+2}_{p,\theta+p}(\Omega)}\\
&\lesssim_N \|v^i\|_{\Psi H^{s_0+3}_{p,\theta}(\Omega)}<\infty\,,
\end{aligned}
$$

where $N=N(d,p,\theta,s_0,\mathrm{C}_2(\Psi))$. Due to the assumption that this lemma holds for $s=s_0+1$, there exists $w\in\Psi H^{s_0+3}_{p,\theta}(\Omega)$ such that

$$
\Delta w-\lambda w=\sum_{i=1}^d D_i\big(\Delta(\widetilde{\rho}v^i)-\widetilde{\rho}\Delta v^i\big)\quad(=\Delta v-\lambda v-f)\,.
$$

This w satisfies

$$
\begin{aligned}
\|w\|_{\Psi H^{s_0+3}_{p,\theta}(\Omega)}+\lambda\|w\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)}\leq\ & N_{s_0+1}\sum_{i=1}^d\big\|D_i\left(\Delta(\widetilde{\rho}v^i)-\widetilde{\rho}\Delta v^i\right)\big\|_{\Psi H^{s_0+1}_{p,\theta+2p}(\Omega)}\\
\lesssim_N & N_{s_0+1}\sum_{i=1}^d\|v^i\|_{\Psi H^{s_0+3}_{p,\theta}(\Omega)}\,.
\end{aligned}\tag{3.26}
$$

Put $u=v-w=v^0+\sum_{i=1}^d D_i(\widetilde{\rho}v^i)-w$. Then u satisfies $\Delta u-\lambda u=f$. Moreover, by (3.25) and (3.26), we obtain (3.23) for $s=s_0$. □

Proof of Theorem 3.14. By Lemma 3.19, we only need to prove for $\gamma=0$.

***A priori* estimates.** Let $u\in\Psi^\mu H^2_{p,d-2}(\Omega)$ and $\Delta u-\lambda u\in\Psi^\mu L_{p,d+2p-2}(\Omega)$. By Lemma 3.18, we obtain

$$
\begin{aligned}
&\|u\|_{\Psi^\mu H^2_{p,d-2}(\Omega)}+\lambda\|u\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\\
\lesssim_N&\|u\|_{\Psi^\mu L_{p,d-2}(\Omega)}+\|\Delta u-\lambda u\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}<\infty\,,
\end{aligned}\tag{3.27}
$$

where $N=N(d,p,\mu,\mathrm{C}_2(\Psi))$. Due to (3.27) and Lemma 3.10.(5), whether $\lambda=0$ or $\lambda>0$, there exists $u_n\in C_c^\infty(\Omega)$ such that

$$
\lim_{n\to\infty}\Big(\|u-u_n\|_{\Psi^\mu H^2_{p,d-2}(\Omega)}+\lambda\|u-u_n\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\Big)=0\,.
$$

This implies that

$$
\lim_{n\to\infty}\big\|\big(\Delta-\lambda\big)(u-u_n)\big\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}=0\,.
$$

Since Ψ is a regularization of the superharmonic Harnack function ψ, Theorem 2.7 and Lemma 3.12 imply

$$
\begin{aligned}
\|u_n\|_{\Psi^\mu L_{p,d-2}(\Omega)}\ &\simeq_N\int_\Omega|u_n|^p\psi^{-\mu p}\rho^{-2}\,\mathrm{d}x\\
&\lesssim_N\int_\Omega|\Delta u_n-\lambda u_n|^p\psi^{-\mu p}\rho^{2p-2}\,\mathrm{d}x\\
&\simeq_N\ \|\Delta u_n-\lambda u_n\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\,,
\end{aligned}\tag{3.28}
$$

where $N=N(d,p,\mu,\mathrm{C}_0(\Omega),\mathrm{C}_2(\Psi),\mathrm{C}_3(\psi,\Psi))$. By letting $n\to\infty$, we obtain (3.28) for u instead of u_n. By combining this with (3.27), we have

$$
\begin{aligned}
&\|u\|_{\Psi^\mu H^2_{p,d-2}(\Omega)}+\lambda\|u\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\\
\lesssim_N&\ \|u\|_{\Psi^\mu L_{p,d-2}(\Omega)}+\|\Delta u-\lambda u\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\\
\lesssim_N&\ \|\Delta u-\lambda u\|_{\Psi^\mu L_{p,d+2p-2}(\Omega)}\,.
\end{aligned}\tag{3.29}
$$

Note that estimate (3.29) also implies the uniqueness of solutions.

Existence of solutions. Let $f \in \Psi^{\mu} L_{p,d+2p-2}(\Omega)$. Since $C_c^{\infty}(\Omega)$ is dense in $\Psi^{\mu} L_{p,d+2p-2}(\Omega)$, there exists $f_n \in C_c^{\infty}(\Omega)$ such that $f_n \to f$ in $\Psi^{\mu} L_{p,d+2p-2}(\Omega)$. Lemmas 2.8 and 3.12 yield that for each $n \in \mathbb{N}$, there exists $u_n \in \Psi^{\mu} L^2_{p,d-2}(\Omega)$ such that $\Delta u_n - \lambda u_n = f_n$. Due to Lemma 3.18, $u_n \in \Psi^{\mu} H^2_{p,d-2}(\Omega)$. Since $f_n \to f$ in $\Psi^{\mu} L_{p,d+2p-2}(\Omega)$, it follows from (3.29) that

$$\|u_n - u_m\|_{\Psi^{\mu} H^2_{p,d-2}(\Omega)} \leq N \|f_n - f_m\|_{\Psi^{\mu} L_{p,d+2p-2}} \to 0$$

as n, $m \to \infty$. Therefore there exists $u \in \Psi^{\mu} H^2_{p,d-2}(\Omega)$ such that u_n converges to u in $\Psi^{\mu} H^2_{p,d-2}(\Omega)$. Since u_n and f_n converge to u and f in the sense of distribution, respectively (see Lemma 3.10.(2)), u is a solution of equation (3.16). $\square$

We end this subsection with a global uniqueness of solutions.

Theorem 3.20 (Global uniqueness)**.** *Suppose that (1.2) holds for Ω, and that for each $i = 1, 2$, Ψ_i is a regularization of a superharmonic Harnack function, $p_i \in (1, \infty)$, $\gamma_i \in \mathbb{R}$, and $\mu_i \in (-1/p_i, 1 - 1/p_i)$. Let $f \in \bigcap_{i=1,2} \Psi_i^{\mu_i} H^{\gamma_i}_{p_i,d+2p_i-2}(\Omega)$, and let for each $i = 1, 2$, $u^{(i)} \in \Psi_i^{\mu_i} H^{\gamma_i+2}_{p_i,d-2}(\Omega)$ be solutions of the equation $\Delta u = f$. Then $u^{(1)} = u^{(2)}$ in $\mathcal{D}'(\Omega)$.*

Proof. By Lemma 3.10.(5), there exist $\{f_n\} \subset C_c^{\infty}(\Omega)$ such that $f_n \to f$ in $\bigcap_{i=1,2} \Psi_i^{\mu_i} H^{\gamma_i}_{p_i,d+2p_i-2}(\Omega)$. By Lemmas 2.8 and 3.12, for each $n \in \mathbb{N}$, there exists $u_n \in \bigcap_{i=1,2} \Psi_i^{\mu_i} L_{p_i,d-2}(\Omega)$ such that $\Delta u_n - \lambda u_n = f_n$. Lemma 3.18 yields that $u_n \in \bigcap_{i=1,2} \Psi_i^{\mu_i} H^{\gamma_i+2}_{p_i,d-2}(\Omega)$. Since

$$(\Delta - \lambda)\big(u_n - u^{(1)}\big) = (\Delta - \lambda)\big(u_n - u^{(2)}\big) = f_n - f\,,$$

For each $i = 1, 2$, Theorem 3.14 implies that $u_n \to u^{(i)}$ in $\Psi_i^{\mu_i} H^{\gamma_i+2}_{p_i,d-2}(\Omega)$, and by Lemma 3.10.(2), this convergences also holds in $\mathcal{D}'(\Omega)$. Therefore $u^{(1)} = u^{(2)} = \lim_{n\to\infty} u_n$ in $\mathcal{D}'(\Omega)$. $\square$

4. Application I - Domain with fat exterior or thin exterior

In this section, we introduce applications of the results in Sections 3 to domains satisfying fat exterior or thin exterior conditions. The notions of the fat exterior and thin exterior are closely related to the geometry of a domain Ω, namely the Hausdorff dimension and the Aikawa dimension of Ω^c.

For a set $E \subset \mathbb{R}^d$, the Hausdorff dimension of E is defined by

$$\dim_{\mathcal{H}}(E) := \inf\big\{\lambda \geq 0 \,:\, H^{\lambda}_{\infty}(E) = 0\big\}\,,$$

where

$$\mathcal{H}^{\lambda}_{\infty}(E) := \inf\Big\{\sum_{i\in\mathbb{N}} r_i^{\lambda} \,:\, E \subset \bigcup_{i\in\mathbb{N}} B(x_i, r_i) \quad \text{where } x_i \in E \text{ and } r_i > 0\Big\}\,.$$

The Aikawa dimension of E, denoted by $\dim_{\mathcal{A}}(E)$, is defined by the infimum of $\beta \geq 0$ for which

$$\sup_{p\in E,\, r>0} \frac{1}{r^{\beta}} \int_{B_r(p)} \frac{1}{d(x, E)^{d-\beta}}\,\mathrm{d}x < \infty\,,$$

with considering $\frac{1}{0} = +\infty$.

Remark 4.1.

(1) While the Aikawa dimension is defined in terms of integration, it is equivalent to a dimension defined in terms of a covering property, the so-called Assouad dimension.
(2) For any $E \subset \mathbb{R}^d$, $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{A}}(E)$, and the equality does not hold in general (see [37, Section 2.2]). However, if E is Alfors regular, for example, if E has a self-similar property such as Cantor set or Koch snowflake set, then $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{A}}(E)$; see [37, Lemma 2.1] and [42, Theorem 4.14].

Koskela and Zhong [33] established the dimensional dichotomy results for domains admitting the Hardy inequality, using the Hausdorff and Minkowski dimension. Their result can be expressed through the Hausdorff and Aikawa dimension, as shown in [37, Theorem 5.3].

Proposition 4.2 (Theorem 5.3 of [37])**.** *Suppose a domain $\Omega \subset \mathbb{R}^d$ admits the Hardy inequality. Then there is a constant $\epsilon > 0$ such that for each $p \in \partial\Omega$ and $r > 0$, either*

$$\dim_{\mathcal{H}}\big(\Omega^c \cap \overline{B}(p, 4r)\big) \geq d - 2 + \epsilon \quad \textit{or} \quad \dim_{\mathcal{A}}\big(\Omega^c \cap \overline{B}(p, r)\big) \leq d - 2 - \epsilon\,.$$

We refer the reader to [31, 53] for a deeper discussion of the dimensional dichotomy.

In virtue of Proposition 4.2, we consider domains $\Omega \subset \mathbb{R}^d$ which satisfy one of the following situations:

(1) (Fat exterior) There exists $\epsilon \in (0, 1)$ and $c > 0$ such that

$$\mathcal{H}_\infty^{d-2+\epsilon}\big(\Omega^c \cap \overline{B}(p, r)\big) \geq c\, r^{d-2+\epsilon} \quad \text{for all } p \in \partial\Omega\ ,\quad r > 0\,. \tag{4.1}$$

(2) (Thin exterior) $\dim_{\mathcal{A}}(\Omega^c) < d - 2$.

These two conditions have been studied extensively; we discuss previous works on these conditions, specifically those related to the Hardy inequality, in Subsections 4.1 and 4.2.

In this section and Section 5, for various domains $\Omega \subset \mathbb{R}^d$, we construct superharmonic functions equivalent to the function $d(\,\cdot\,, \partial\Omega)^\alpha$, for some α. This type of superharmonic function ensures the validity of the following statement for all $p \in (1, \infty)$ and suitable θ (see Lemma 4.4):

Statement 4.3 (Ω, p, θ)**.** *For any $\lambda \geq 0$ and $\gamma \in \mathbb{R}$, if $f \in H^\gamma_{p,\theta+2p}(\Omega)$, then the equation*

$$\Delta u - \lambda u = f \tag{4.2}$$

has a unique solution u in $H^{\gamma+2}_{p,\theta}(\Omega)$. Moreover, we have

$$\|u\|_{H^{\gamma+2}_{p,\theta}(\Omega)} + \lambda\|u\|_{H^\gamma_{p,\theta+2p}(\Omega)} \leq N_1\|f\|_{H^\gamma_{p,\theta+2p}(\Omega)}\,, \tag{4.3}$$

where N_1 is a constant independent of u, f, and λ.

Lemma 4.4. *Let Ω admit the Hardy inequality (1.2), and suppose that for a fixed $\alpha \in \mathbb{R} \setminus \{0\}$, there exists a superharmonic function ψ and a constant $M > 0$ such that*

$$M^{-1}\rho^\alpha \leq \psi \leq M\rho^\alpha\,. \tag{4.4}$$

Then Statement 4.3 (Ω, p, θ) holds for all $p \in (1, \infty)$ and $\theta \in \mathbb{R}$ with

$$d - 2 - (p-1)\alpha < \theta < \quad d - 2 + \alpha \qquad \textit{if} \quad \alpha > 0\,;$$

$$d-2+\alpha \quad < \theta < d-2-(p-1)\alpha \quad \textit{if} \quad \alpha < 0\,.$$

Moreover, N_1 in (4.3) depends only d, p, γ, θ, $\mathrm{C}_0(\Omega)$, α and M (in (4.4)).

Proof. Observe that ψ is a superharmonic Harnack function, and $\Psi := \widetilde{\rho}^{\,\alpha}$ is a regularization of ψ. For this Ψ, the constants $\mathrm{C}_2(\Psi)$ and $\mathrm{C}_3(\Psi,\psi)$ can be chosen to depend only on d, α and M. In addition, Lemmas 3.8.(3) implies that for any $p \in (1,\infty)$ and γ, $\theta \in \mathbb{R}$, there exists $N = N(d,p,\gamma,\alpha,\mu,M)$ such that $\|f\|_{\Psi^\mu H^\gamma_{p,\theta}(\Omega)} \simeq_N \|f\|_{H^\gamma_{p,\theta-\alpha\mu}(\Omega)}$ for all $f \in \mathcal{D}'(\Omega)$. Therefore the proof is completed by applying Theorem 3.14 with $\Psi := \widetilde{\rho}^{\,\alpha}$. □

We collect basic properties of classical superharmonic functions, which are used in this section and Section 5.

Lemma 4.5. *Let Ω be an open set in $\mathbb{R}^d$.*

(1) Let ϕ_1, ϕ_2 be classical superharmonic functions on Ω. Then $\phi_1 \wedge \phi_2$ is also a classical superharmonic function on Ω.

(2) Let $\{\phi_\alpha\}$ be a family of positive classical superharmonic functions on Ω. Then $\phi := \inf_\alpha \phi_\alpha$ is a superharmonic function on Ω.

(3) Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^d$ and ϕ_i be a classical superharmonic function on Ω_i, for $i=1,2$. Suppose that

$$\liminf_{x\to x_1, x\in\Omega_2} \phi_2(x) \geq \phi_1(x_1) \quad \textit{for all} \quad x_1 \in \Omega_1 \cap \partial\Omega_2\,;$$
$$\liminf_{x\to x_2, x\in\Omega_1} \phi_1(x) \geq \phi_2(x_2) \quad \textit{for all} \quad x_2 \in \Omega_2 \cap \partial\Omega_1\,.$$

Then the function

$$\phi(x) := \begin{cases} \phi_1(x) & x \in \Omega_1 \setminus \Omega_2 \\ \phi_1(x) \wedge \phi_2(x) & x \in \Omega_1 \cap \Omega_2 \\ \phi_2(x) & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is also a classical superharmonic function on Ω.

For the proof of Lemma 4.5, (1) follows from the definition of classical superharmonic functions, (2) can be found in [7, Theorem 3.7.5], and (3) is implied by [7, Corollary 3.2.4].

4.1. **Domain with fat exterior : Harmonic measure decay property.** This subsection begins by introducing a relation among the condition (4.1), classical potential theory, and the Hardy inequality (see Lemma 4.10 and Remark 4.11).

We first recall notions in classical potential theory. For a bounded open set $U \subset \mathbb{R}^d$, $d \geq 2$, and a bounded Borel function f on ∂U, the Perron-Wiener-Brelot solution (abbreviated to 'PWB solution') of the equation

$$\Delta u = 0 \quad \text{in } U \quad ; \quad u = F \quad \text{on } \partial U \tag{4.5}$$

is defined by

$$\begin{aligned} u(x) := \inf\big\{\phi(x)\, :\, &\phi \text{ is a superharmonic function on } U \text{ and} \\ &\liminf_{y\to z, y\in U} \phi(y) \geq F(z) \text{ for all } z \in \partial U \big\}\,. \end{aligned} \tag{4.6}$$

This u is harmonic on U. However, $\lim_{y\to z} u(y) = F(z)$ does not hold, in general, for $z \in \partial U$ and $F \in C(\partial U)$. For basic properties of PWB solutions, we refer the reader to [7].

For a Borel set $E \subset \partial U$, $w(\,\cdot\,, U, E)$ denotes the PWB solution u of equation (4.5) with $F := 1_E$. This w is called the *harmonic measure* of E over U.

We fix an arbitrary open set $\Omega \subset \mathbb{R}^d$ (not necessarily bounded), $d \geq 2$. For $p \in \partial\Omega$ and $r > 0$, we denote

$$w(\,\cdot\,, p, r) = w\big(\,\cdot\,, \Omega \cap B_r(p), \Omega \cap \partial B_r(p)\big)$$

(see Figure 4.1 below); note that $\Omega \cap \partial B_r(p)$ is a relatively open subset of $\partial\big(\Omega \cap B_r(p)\big)$.

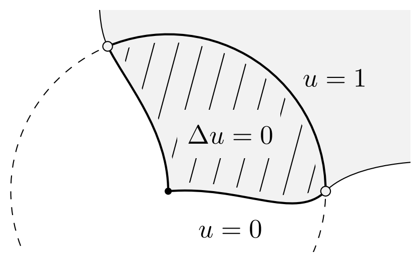


FIGURE 4.1. $u := w(\,\cdot\,, p, r)$

For convenience, based on Lemma 4.6, we consider $w(\,\cdot\,, p, r)$ to be continuous on $\Omega \cap \overline{B}(p, r)$ with $w(x, p, r) = 1$ for $x \in \Omega \cap \partial B(p, r)$.

Lemma 4.6.
 (1) *$w(\,\cdot\,, p, r)$ is harmonic on $\Omega \cap B_r(p)$ with values in $[0, 1]$.*
 (2) *For any $x_0 \in \Omega \cap \partial B_r(p)$, $w(x, p, r) \to 1$ as $x \to x_0$ with $x \in \Omega \cap B_r(p)$.*
 (3) *For any $0 < r < R$ and $N_0 \geq 0$, if $w(\cdot, p, R) \leq N_0$ on $\Omega \cap \partial B_r(p)$, then $w(\cdot, p, R) \leq N_0 w(\cdot, p, r)$ on $\Omega \cap B_r(p)$.*

Proof. (1) and (2) are the basic properties of $w(\,\cdot\,, p, r)$ which can be found in [7, Chapter 6]. Therefore we only prove (3).

For convenience, denote $U_R := \Omega \cap B_R(p)$ and $U_r := \Omega \cap B_r(p)$, and consider $w(\cdot, p, R) := 1_{\Omega \cap \partial B_R(p)}$ on ∂U_R. It follows from [7, Theorem 6.3.6] that $w(x, p, R)|_{U_r}$ is the PWB solution of (4.5) for $U := U_r$ and $F := w(\cdot, p, R)|_{\partial U_r}$. One can observe that

$$\partial U_r \setminus \big(\Omega \cap \partial B_r(p)\big) \subset (\partial\Omega) \cap B_R(p) \subset \partial U_R\,,$$

which implies that $w(x, p, R) = 1_{\Omega \cap \partial B_R(p)}(x) = 0$ for $x \in \partial U_r \setminus \big(\Omega \cap \partial B_r(p)\big)$. Since $w(x, p, R) \leq N_0$ on $\Omega \cap \partial B_r(p)$, we have $w(\cdot, p, R)|_{\partial U_r} \leq N_0 1_{\Omega \cap \partial B_r(p)}$. Due to the definition of PWB solution (4.6), $w(\cdot, p, R) \leq N_0 w(\cdot, p, r)$ on $U_r := \Omega \cap B_r(p)$. $\square$

Definition 4.7. A domain Ω is said to satisfy the *local harmonic measure decay property* with exponent $\alpha > 0$ (abbreviated to '**LHMD**(α)'), if there exists a constant $M_\alpha > 0$ depending only on Ω and α such that

$$w(x, p, r) \leq M_\alpha \left(\frac{|x - p|}{r}\right)^\alpha \quad \text{for all } x \in \Omega \cap B(p, r) \tag{4.7}$$

whenever $p \in \partial\Omega$ and $r > 0$.

Remark 4.8. The notion of **LHMD** is closely related to the Hölder continuity of the PWB solutions. Let Ω be a bounded domain. For $F \in C(\partial\Omega)$, by $H_\Omega F$ we denote the PWB solution u of equation (4.6) with $U := \Omega$. $H_\Omega F$ is called the classical solution if $\lim_{y\to z} H_\Omega F(y) = F(z)$ for all $z \in \partial\Omega$. Aikawa [4, Theorem 2, Theorem 3] provides the following results: Let $0 < \alpha < 1$.

(1) If $H_\Omega F$ is the classical solution for any $F \in C(\partial\Omega)$, and

$$\sup_{F\in C^{0,\beta}(\partial\Omega), F\not\equiv 0} \frac{\|H_\Omega F\|_{C^{0,\alpha}(\Omega)}}{\|F\|_{C^{0,\alpha}(\partial\Omega)}} < \infty\,, \tag{4.8}$$

then Ω satisfies $\mathbf{LHMD}(\alpha)$.

(2) Conversely, if Ω satisfies $\mathbf{LHMD}(\beta)$ for some $\beta > \alpha$, then $H_\Omega F$ is the classical solution for any $F \in C(\Omega)$, and (4.8) holds.

Lemma 4.9. *Let Ω be a bounded domain, and suppose that for a constant $\alpha > 0$, there exist constants r_0, $\widetilde{M} \in (0,\infty)$ such that*

$$w(x,p,r) \le \widetilde{M}\left(\frac{|x-p|}{r}\right)^\alpha \quad \text{for all} \quad x \in \Omega \cap B(p,r) \tag{4.9}$$

whenever $p \in \partial\Omega$ and $0 < r \le r_0$. Then Ω satisfies $\mathbf{LHMD}(\alpha)$, where M_α in (4.9) depends only on α, $\widetilde{M}$ and $\mathrm{diam}(\Omega)/r_0$.

Proof. Let $p \in \partial\Omega$. If $r > \mathrm{diam}(\Omega)$, then $\Omega \cap \partial B(p,r) = \emptyset$, which implies that $w(\,\cdot\,,p,r) \equiv 0$. In addition, due to the assumption of this lemma, we do not need to pay attention to the case of $r \le r_0$. Therefore, we only consider the case of $r_0 < r \le \mathrm{diam}(\Omega)$.

For $r_0 < r \le \mathrm{diam}(\Omega)$, it follows from Lemmas 4.6.(1) and (3) that $w(x,p,r) \le 1$ in general, and $w(x,p,r) \le w(x,p,r_0)$ if $|x-p| < r_0$. Due to (4.9) and that $r_0 < r \le \mathrm{diam}(\Omega)$, we have

$$w(x,p,r) \le \max\left(\widetilde{M},1\right)\left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^\alpha \left(\frac{|x-p|}{r}\right)^\alpha \qquad \text{for all} \quad x \in \omega \cap B(p,r)\,.$$

The proof is completed. □

We finally introduce the relation between (4.1) and the local harmonic measure decay property.

Lemma 4.10. *Let Ω be a domain in $\mathbb{R}^d$.*

(1) The following conditions are equivalent:
   - *(a) There exists $\epsilon > 0$ such that the fat exterior condition (4.1) holds.*
   - *(b) There exists $\alpha > 0$ such that $\mathbf{LHMD}(\alpha)$ holds.*
   - *(c) There exists $\epsilon_0 > 0$ such that*

$$\inf_{p\in\partial\Omega, r>0} \frac{\mathrm{Cap}\big(\Omega^c \cap \overline{B}(p,r), B(p,2r)\big)}{\mathrm{Cap}\big(\overline{B}(p,r), B(p,2r)\big)} \ge \epsilon_0 > 0\,. \tag{4.10}$$

Here, $\mathrm{Cap}\big(K,B\big)$ is the capacity of a compact set $K \subset B$ relative to an open ball B, defined as follows:

$$\mathrm{Cap}(K,B) := \inf\left\{\|\nabla f\|_2^2 \,:\, f \in C_c^\infty(B)\ ,\ \ f \ge 1 \text{ on } K\right\}\,. \tag{4.11}$$

In particular, constants (c,ϵ) in (4.1), (α, M_α) in (4.7), and ϵ_0 in (4.10) depend only on each other and d.

(2) If (4.10) holds, then Ω admits the Hardy inequality (1.2), where $\mathrm{C}_0(\Omega)$ depends only on d and ϵ_0.

For this lemma, the equivalence between conditions (a) and (c) is established by Lewis [39, Theorem 1] and Aikawa [3, Theorem B] (see, *e.g.*, [31, Theorem 7.22] for a simplified version). Additionally, the equivalence between (b) and (c), and Lemma 4.10.(2) is provided by Ancona [6, Lemma 3, Theorem 1].

Remark 4.11.

(1) (4.10) is called the capacity density condition. For domains Ω in $\mathbb{R}^2$, (4.10) holds if and only if Ω admits the Hardy inequality (1.2) (see Ancona [6, Theorem 2]).
(2) A well-known sufficient condition to satisfy (4.10) is the volume density condition:
$$\inf_{p\in\partial\Omega,r>0}\frac{|\Omega^c\cap\overline{B}(p,r)|}{|\overline{B}(p,r)|}\geq\epsilon_1>0\,,$$
(see, *e.g.*, [31, Example 6.18]). For a deeper discussion of the capacity density condition, we refer the reader to [31, 32, 39] and the references given therein.

Based on this discussion, we consider domains satisfying $\mathbf{LHMD}(\alpha)$ for some $\alpha>0$, instead of (4.1). This condition is implied by geometric conditions introduced in Section 5, and the value of α reflects each geometric condition; see Theorem 5.5. In the rest of this subsection, we construct appropriate superharmonic functions related to α (see Lemma 4.4). The results in this subsection are crucially used in Section 5.

Theorem 4.12. *Let Ω satisfy $\mathbf{LHMD}(\alpha)$, $\alpha>0$. Then for any $\beta\in(0,\alpha)$, there exists a superharmonic function ϕ on Ω such that*
$$N^{-1}\rho(x)^{\beta}\leq\phi(x)\leq N\rho(x)^{\beta}$$
for all $x\in\Omega$, where $N=N(\alpha,\beta,M_\alpha)>0$.

Before proving Theorem 4.12, we look at the following corollaries:

Theorem 4.13. *Let $\Omega\subset\mathbb{R}^d$ satisfy $\mathbf{LHMD}(\alpha)$, $\alpha>0$. For any $p\in(1,\infty)$ and $\theta\in\mathbb{R}$ satisfying*
$$d-2-(p-1)\alpha<\theta<d-2+\alpha\,,\tag{4.12}$$
Statement 4.3 (Ω,p,θ) holds. In addition, N_1 (in (4.3)) depends only on d, p, γ, θ, α, M_α.

Remark 4.14. The Poisson equation (4.2) is not explicitly equipped with specific boundary conditions. Nonetheless, one can interpret Theorem 4.13 to include the zero-Dirichlet boundary condition, $u|_{\partial\Omega}\equiv 0$. This interpretation is supported by the fact that $C_c^\infty(\Omega)$ is dense in $H^\gamma_{p,\theta+2p}(\Omega)$, and for $f\in C_c^\infty(\Omega)$, the solution u implied by Theorem 4.13 satisfies that $u\in H^{\gamma+2}_{p,\theta}(\Omega)$ for any $p\in(1,\infty)$, θ in (4.12), and $\gamma\in\mathbb{R}$ (see Theorem 3.20). In addition, by taking appropriate p, θ, and $\gamma>0$, it follows from Proposition 3.13 that this u is continuous on Ω and $u\to 0$ as $\rho(x)\to 0$.

Proof of Theorem 4.13. Take $\beta\in(0,\alpha)$ such that
$$d-2-(p-1)\beta<\theta<d-2+\beta.$$
It follows from Theorem 4.12 that there exists a superharmonic function ϕ such that $\phi\simeq_N\rho^{\beta}$, where $N=N(\alpha,\,\beta,\,M_\alpha)$. Lemma 4.10 yields that Ω admits the

Hardy inequality (1.2), where $\mathrm{C}_0(\Omega)$ can be chosen to depend only on d, α and M_α (in (4.7)). Therefore, the proof is completed by Lemma 4.4. $\square$

Proof of Theorem 4.12. The following construction is a combination of [6, Theorem 1] and [25, Lemma 2.1]. Recall that M_α is the constant in (4.7), and $\beta < \alpha$. Take $r_0 \in (0,1)$ small enough to satisfy $M_\alpha r_0^\alpha < r_0^\beta$, and take $\eta \in (0,1)$ small enough to satisfy

$$(1-\eta)M_\alpha r_0^\alpha + \eta \leq r_0^\beta\,.$$

For $w(x,p,r)$, we shall need only the following properties (see Lemma 4.6 and Definition 4.7):

$$\begin{aligned}
&w(\cdot,p,r) \text{ is a classical superharmonic function on } \Omega\cap B(p,r)\,;\\
&w(\cdot,p,r) = 1 \text{ on } \Omega\cap\partial B(p,r)\,;\\
&0 \leq w(\cdot,p,r) \leq M_\alpha r_0^\alpha \text{ on } \Omega\cap B(p,r_0 r)\,.
\end{aligned}$$

For $p \in \partial\Omega$ and $k \in \mathbb{Z}$, put

$$\phi_{p,k}(x) = r_0^{k\beta}\big((1-\eta)\,w(x,p,r_0^k) + \eta\big).$$

Then $\phi_{p,k}$ is a classical superharmonic function on $\Omega \cap B(p, r_0^k)$,

$$\begin{aligned}
\phi_{p,k} &\leq r_0^{(k+1)\beta} &&\text{on}\quad \Omega\cap\overline{B}(p,r_0^{k+1})\,,\\
\phi_{p,k} &= r_0^{k\beta} &&\text{on}\quad \Omega\cap\partial B(p,r_0^{k})\,,\\
\eta\cdot r_0^{k\beta} \leq \phi_{p,k} &\leq r_0^{k\beta} &&\text{on}\quad \Omega\cap B(p,r_0^{k})\,.
\end{aligned}$$

For $p \in \partial\Omega$ and $x \in \Omega$, we denote

$$\phi_p(x) = \inf\{\phi_{p,k}(x)\,:\,|x-p| < r_0^k\}.$$

If we prove the following:

$$\phi_p \text{ is a classical superharmonic function on } \Omega\;; \tag{4.13}$$

$$\eta|x-p|^\beta \leq \phi_p(x) \leq r_0^{-\beta}|x-p|^\beta\,, \tag{4.14}$$

then $\phi := \inf_{p\in\partial\Omega}\phi_p$ is superharmonic on Ω (see Lemma 4.5.(2)) and satisfies

$$\eta\rho(x)^\beta \leq \phi(x) \leq r_0^{-\beta}\rho(x)^\beta\,.$$

Therefore the proof is completed.

- (4.13) **:** We only need to prove that for each $k_0 \in \mathbb{Z}$, ϕ_p is a classical superharmonic function on $U_{k_0} := \{x \in \Omega\,:\, r_0^{k_0+2} < |x-p| < r_0^{k_0}\}$ (see Remark 2.2). For $x \in U_{k_0}$, put

$$v_{p,k_0}(x) = \begin{cases}\phi_{p,k_0}(x) & \text{if}\quad r_0^{k_0+1} \leq |x-p| < r_0^{k_0}\\ \phi_{p,k_0}(x)\wedge\phi_{p,k_0+1}(x) & \text{if}\quad r_0^{k_0+2} < |x-p| < r_0^{k_0+1}\,.\end{cases}$$

Since $\phi_{p,k_0} \leq \phi_{p,k_0+1}$ on $\Omega\cap\partial B(p,r_0^{k_0+1})$, Lemma 4.5.(4) implies that v_{p,k_0} is a classical superharmonic function on U_{k_0}. Observe that

$$\phi_p(x) = v_{p,k_0}(x)\wedge\inf\{\phi_{p,k}(x)\,:\,k\leq k_0-1\}.$$

Moreover, if $\eta\, r_0^{k\beta} \geq r_0^{k_0\beta}$ then

$$v_{p,k_0}(x) \leq \phi_{p,k_0}(x) \leq r_0^{k_0\beta} \leq \eta\, r_0^{k\beta} \leq \phi_{p,k}(x)\,.$$

Therefore

$$\phi_p(x) = v_{p,k_0}(x) \wedge \inf\{\phi_{p,k}(x) \,:\, k \leq k_0 - 1 \quad \text{and} \quad \eta\, r_0^{k\beta} \leq r_0^{k_0\beta}\}\,,$$

which implies that on U_{k_0}, ϕ_p is the minimum of finitely many classical superharmonic functions. Consequently, by Lemma 4.5.(1), ϕ_p is a classical superharmonic function on U_{k_0}.

- (4.14) **:** Let $x \in \Omega$ satisfy $r_0^{k_0+1} \leq |x-p| < r_0^{k_0}$, $k_0 \in \mathbb{Z}$. Since

$$\phi_{p,k_0}(x) \leq r_0^{k_0\beta}\ , \quad \text{and} \quad \phi_{p,k}(x) \geq \eta r_0^{k\beta} \geq \eta r_0^{k_0\beta} \quad \text{for all} \quad k \leq k_0\,,$$

we obtain that $\eta\, r_0^{k_0\beta} \leq \phi_p(x) \leq r_0^{k_0\beta}$. This implies (4.14). □

We end this subsection providing two corollaries of Theorem 4.13.

Corollary 4.15. *Let Ω satisfy* $\mathbf{LHMD}(\alpha)$*, $\alpha \in (0,1]$, and there exists $M \geq 0$ such that $\int_\Omega \rho(x)^M\,\mathrm{d}x < \infty$. Consider the equation*

$$\Delta u - \lambda u = f_0 + \sum_{i=1}^{d} D_i f_i \quad \textit{in} \;\; \Omega \quad ; \quad u = 0 \quad \textit{on} \;\; \partial\Omega\,, \tag{4.15}$$

where $f_0, f_1, \ldots, f_d$ are measurable functions on Ω such that

$$|f_0| \lesssim \rho^{-2+\alpha}\ , \quad |f_1| + \cdots + |f_d| \lesssim \rho^{-1+\alpha}\,.$$

Then for any $0 < \beta < \alpha$, equation (4.15) has a unique solution u in $C^{0,\beta}(\overline{\Omega})$. In addition, we have

$$\sup_\Omega \rho^{-\beta}|u| + [u]_{C^{0,\beta}(\Omega)} \lesssim_N \sup_\Omega \Big(\rho^{2-\alpha}|f_0| + \rho^{1-\alpha}|f_1| + \cdots + \rho^{1-\alpha}|f_d|\Big) =: N_F\,, \tag{4.16}$$

where N depends only on d, α, M_α, β, and $\int_\Omega \rho(x)^M$.

Proof. We first mention that the assumption $\int_\Omega \rho(x)^M\,\mathrm{d}x < \infty$ implies that the function ρ is bounded; moreover, $\lim_{|x|\to\infty}\rho(x) = 0$. This implies that if LHS in (4.16) is finite, then $u \in C^{0,\beta}(\overline{\Omega})$.

- Uniqueness of solutions. If Ω is bounded, then the uniqueness of solutions directly follows from the maximum principle. Consider the case of when Ω is unbounded, and let $u \in C^{0,\beta}(\overline{\Omega})$ satisfies (4.15) for $f_0 = \ldots = f_d \equiv 0$. Since $\lim_{|x|\to\infty}\rho(x) = 0$, the conditions for u implies that $\lim_{|x|\to\infty}u(x) = 0$. Combining this with the Maximum principle, we have

$$\sup_\Omega |u| = \lim_{R\to\infty}\sup_{\Omega\cap B_R}|u| = \lim_{R\to\infty}\sup_{\partial(\Omega\cap B_R)}|u| = 0\,.$$

Therefore, the uniqueness of solutions is proved.

- Existence of solutions and (4.16)**.** For $\beta \in (0,\alpha)$, put $p := \frac{d+M}{\alpha-\beta}$ so that $\beta \leq 1 - \frac{d}{p}$ and $\theta := -p\beta$ satisfies (4.12). Observe that

$$\begin{aligned}\|F\|^p_{H^{-1}_{p,\theta+2p}(\Omega)} &\lesssim_{p,d,\beta} \int_\Omega \Big(|\rho^{2-\beta}f_0|^p + \sum_{i=1}^{d}|\rho^{1-\beta}f_i|^p\Big)\rho^{-d}\,\mathrm{d}x \\ &\leq \big(N_F\big)^p\int_\Omega \rho(x)^M\,\mathrm{d}x < \infty\,,\end{aligned}$$

where the last inequality follows from that $p(\alpha-\beta)=d+M$. Theorem 4.13 provide a solution $u\in H^1_{p,\theta}(\Omega)$ of equation (4.15) with

$$\|u\|_{H^1_{p,\theta}(\Omega)}\lesssim_N \|F\|^p_{H^{-1}_{p,\theta+2p}(\Omega)}\lesssim_N \Big(\int_\Omega \rho^M\,\mathrm{d}x\Big)^{1/p}N_F\,,$$

where $N=N(d,\alpha,M_\alpha,\beta)$. Proposition 3.13 implies $|u|^{(-\beta)}_{0,\beta}\lesssim_{d,\alpha,\beta}\|u\|_{H^1_{p,\theta}(\Omega)}$, and therefore we obtain (4.16). By the comment at the first in this proof, we have $u\in C^{0,\beta}(\overline{\Omega})$. □

The following corollary is an unweighted L_p-solvability result when p is close to 2. It is worth noting that similar results for various equations are introduced in the literature, such as [26], utilizing the reverse Hölder inequality. The reason to provide Corollary 4.16 is that its proof is independent of the reverse Hölder inequality; instead, this proof relies on the weighted solvability result (Theorem 4.13). This theorem also provides the L_p estimate unaffected by dilation (see (4.18)).

We denote $\mathring{W}^1_p(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $W^1_p(\Omega)$.

Corollary 4.16. *Let Ω satisfy (4.10) and*

$$\lambda\geq 0\quad \textit{if}\quad D_\Omega<\infty\quad \textit{and}\quad \lambda>0\quad \textit{if}\quad D_\Omega=\infty\,,\tag{4.17}$$

where $d_\Omega:=\sup_{x\in\Omega}d(x,\partial\Omega)$. Then there exists $\epsilon\in(0,1)$ depending only on d, ϵ_0 (in (4.10)) such that for any $p\in(2-\epsilon,2+\epsilon)$, the following holds: For any $f^0,\,\dots,\,f^d\in L_p(\Omega)$, equation (4.15) has a unique solution u in $\mathring{W}^1_p(\Omega)$. Moreover, we have

$$\|\nabla u\|_p+\big(\lambda^{1/2}+D_\Omega^{-1}\big)\|u\|_p\lesssim_{d,p,\epsilon_0}\min\big(\lambda^{-1/2},D_\Omega\big)\|f^0\|_p+\sum_{i=1}^d\|f^i\|_p\,.\tag{4.18}$$

Proof. We first note the following two results for the capacity density condition (4.10):

(a) By Lemma 4.10.(1), there exists $\alpha\in(0,1)$ such that Ω satisfies **LHMD**(α). Due to Theorem 4.13, Statement 4.3 $(\Omega,p,d-p)$ holds for $p\in(2-\alpha_1,2+\alpha_1)$, and N_1 (in (4.3)) depends only on d, p, γ, ϵ_1.

(b) It is implied by [39, Theorem 1, Theorem 2] (or see [32, Theorem 3.7, Corollary 3.11]) that there exists $\alpha_2\in(0,1)$ depending only on d and ϵ_0 such that for any $p>2-\alpha_2$,

$$\int_\Omega\Big|\frac{u(x)}{\rho(x)}\Big|^p\,\mathrm{d}x\leq N(d,p,\epsilon_0)\int_\Omega|\nabla u|^p\,\mathrm{d}x\quad\forall\ u\in C^\infty_c(\Omega)\,.\tag{4.19}$$

Put $0<\epsilon<\min(\alpha_1,\alpha_2)$ and consider $p\in(2-\epsilon,2+\epsilon)$.

Step 1. Uniqueness of solutions. Since Statement 4.3 $(\Omega,p,d-p)$ holds, it suffices to show that $\mathring{W}^1_p(\Omega)\subset H^1_{p,d-p}(\Omega)$. For any $u\in\mathring{W}^1_p(\Omega)$, there exists $\{u_n\}_{n\in\mathbb{N}}\subset C^\infty_c$ such that $u_n\to u$ in $W^1_p(\Omega)$. One can choose this $\{u_n\}$ to converge to u almost everywhere on Ω. Consider (4.19) for $u_n\in C^\infty_c(\Omega)$, and apply Fatou's lemma, to obtain that (4.19) holds for $u\in\mathring{W}^1_p(\Omega)$. This implies that $u\in H^1_{p,d-p}(\Omega)$.

Step 2. Existence of solutions and estimate (4.18). In this Step, we use Lemma 3.12.(1), $D_\Omega^{-1}\|u\|_p\leq\|\rho^{-1}u\|_p$, and $\|\rho f\|_p\leq D_\Omega\|f\|_p$, without mentioning. Additionally, we also use the fact that

$$\|u\|_{H^\gamma_{p,d}(\Omega)}\lesssim_{p,d}\|u\|^{1/2}_{H^{\gamma+1}_{p,d-p}(\Omega)}\|u\|^{1/2}_{H^{\gamma-1}_{p,d+p}(\Omega)}\,,\tag{4.20}$$

which follows from that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \mathrm{e}^{nd} \big\| \big(\zeta_{0,(n)}u\big)(\mathrm{e}^{n}\cdot)\big\|_{H_p^{\gamma}}^p \\
&\lesssim_{p,d} \sum_{n\in\mathbb{Z}} \mathrm{e}^{nd} \big\| \big(\zeta_{0,(n)}u\big)(\mathrm{e}^{n}\cdot)\big\|_{H_p^{\gamma+1}}^{p/2} \big\| \big(\zeta_{0,(n)}u\big)(\mathrm{e}^{n}\cdot)\big\|_{H_p^{\gamma-1}}^{p/2} \\
&\leq \left(\sum_{n\in\mathbb{Z}} \mathrm{e}^{n(d-p)} \big\| \big(\zeta_{0,(n)}u\big)(\mathrm{e}^{n}\cdot)\big\|_{H_p^{\gamma+1}}^p\right)^{1/2} \left(\sum_{n\in\mathbb{Z}} \mathrm{e}^{n(d+p)} \big\| \big(\zeta_{0,(n)}u\big)(\mathrm{e}^{n}\cdot)\big\|_{H_p^{\gamma-1}}^p\right)^{1/2} .
\end{aligned}$$

To prove the existence of solutions, it is enough to find a solution in $L_{p,d}(\Omega)\cap H^1_{p,d-p}(\Omega)$. Indeed, $L_{p,d}(\Omega)\cap H^1_{p,d-p}(\Omega)$ is continuouly embedded in W^1_p, and $C_c^\infty(\Omega)$ is dense in $L_{p,d}(\Omega)\cap H^1_{p,d-p}(\Omega)$ (see Lemma 3.10.(5)).

Without loss of generality, we assume that $\lambda=0$ or $\lambda=1$ by dilation. Note that ϵ_0 in (4.10) is invariant even if Ω is replaced by $r\Omega=\{rx : x\in\Omega\}$, for any $r>0$.

Step 2.1) Consider the case $\lambda=1$. Since Statement 4.3 $(\Omega,p,d-p)$ holds, there exists $v\in H^2_{p,d-p}(\Omega)$ such that $\Delta v - v = \widetilde{\rho}^{-1}f^0$ and

$$\|v\|_{H^2_{p,d-p}(\Omega)} + \|v\|_{L_{p,d+p}(\Omega)} \lesssim_{d,p,\epsilon_0} \big\|\widetilde{\rho}^{-1}f^0\big\|_{L_{p,d+p}(\Omega)} \simeq_{p,d} \|f^0\|_p \,. \tag{4.21}$$

Due to (4.20) and (4.21), we have

$$\|v\|_{L_{p,d+p}(\Omega)} + \|v\|_{H^1_{p,d}(\Omega)} \lesssim_{d,p} \|v\|_{H^2_{p,d-p}(\Omega)} + \|v\|_{L_{p,d+p}(\Omega)} \lesssim_{d,p,\epsilon_0} \|f^0\|_p \,. \tag{4.22}$$

Put

$$\widetilde{f} := f^0 - \Delta(\widetilde{\rho}v) + \widetilde{\rho}v = -2\Big[\sum_{i=1}^d D_i\big(vD_i\widetilde{\rho}\big)\Big] + v\Delta\widetilde{\rho}\,,$$

and observe that

$$\big\|\widetilde{f}\big\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim_{d,p} \|v\|_{L_{p,d}(\Omega)} \lesssim_{d,p} \|v\|_{H^1_{p,d}(\Omega)} \lesssim_{d,p,\epsilon_0} \big\|f^0\big\|_p \ ,$$

where the first and third inequalities follow from Lemma 3.12.(2) and (4.22), respectively. Since Statement 4.3 $(\Omega,p,d-p)$ holds, there exists $w\in H^1_{p,d-p}(\Omega)$ such that

$$\Delta w - w = \sum_{i=1}^d D_if^i + \widetilde{f}$$

and

$$\|w\|_{H^1_{p,d-p}(\Omega)} + \|w\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim_{d,p,\epsilon_0} \sum_{i=1}^d \|f^i\|_{L_{p,d}(\Omega)} + \big\|\widetilde{f}\big\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim \sum_{i=0}^d \|f^i\|_p \,. \tag{4.23}$$

Therefore, by (4.20) and (4.23), we have

$$\|w\|_{L_{p,d}(\Omega)} + \|w\|_{H^1_{p,d-p}(\Omega)} \lesssim_{d,p} \|w\|_{H^1_{p,d-p}(\Omega)} + \|w\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim_{d,p,\epsilon_0} \sum_{i\geq 0} \big\|f^i\big\|_p \ . \tag{4.24}$$

Put $u = v\widetilde{\rho} + w$. Then u is a solution of equation (4.15) and satisfies

$$\begin{aligned}
&\|u_x\|_p + (1+D_\Omega^{-1})\|u\|_p \lesssim_{d,p} \|u\|_{L_{p,d}(\Omega)} + \|u\|_{H^1_{p,d-p}(\Omega)} \qquad (4.25)\\
&\lesssim_{d,p} \|w\|_{L_{p,d}(\Omega)} + \|w\|_{H^1_{p,d-p}(\Omega)} + \|v\|_{L_{p,d+p}(\Omega)} + \|v\|_{H^1_{p,d}(\Omega)} \lesssim_{d,p,\epsilon_0} \sum_{i\geq 0} \|f^i\|_p \,,
\end{aligned}$$

where the last inequality follows from (4.22) and (4.24); note that (4.25) also implies that $u \in L_{p,d}(\Omega) \cap H^1_{p,d-p}(\Omega)$.

Step 2.2) Consider the case $D_\Omega < \infty$, and observe that

$$\begin{aligned} \Big\| f^0 + \sum_{i\geq 1} D_i f^i \Big\|_{H^{-1}_{p,d+p}(\Omega)} &\lesssim_{d,p} \|f^0\|_{L_{p,d+p}(\Omega)} + \sum_{i\geq 1} \|f^i\|_{L_{p,d}(\Omega)} \\ &\leq \quad D_\Omega \|f^0\|_p + \sum_{i\geq 1} \|f^i\|_p < \infty\,. \end{aligned} \tag{4.26}$$

Since Statement 4.3 $(\Omega, p, d-p)$ holds, there exists $\widetilde{u} \in H^1_{p,d-p}(\Omega)$ such that

$$\Delta \widetilde{u} - \lambda \widetilde{u} = f^0 + \sum_{i\geq 1} D_i f^i\,,$$

and

$$\|\widetilde{u}\|_{H^1_{p,d-p}(\Omega)} + \lambda \|\widetilde{u}\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim \|f^0 + \sum_{i\geq 1} D_i f^i\|_{H^{-1}_{p,d+p}(\Omega)}\,. \tag{4.27}$$

By (4.20), (4.26), and (4.27), we obtain that

$$\begin{aligned} &\|\nabla \widetilde{u}\|_{L_p(\Omega)} + D_\Omega^{-1} \|\widetilde{u}\|_{L_p(\Omega)} + \lambda^{1/2} \|\widetilde{u}\|_{L_p(\Omega)} \\ \lesssim_{d,p} &\|\widetilde{u}\|_{H^1_{p,d-p}(\Omega)} + \lambda \|\widetilde{u}\|_{H^{-1}_{p,d+p}(\Omega)} \lesssim_{d,p,\epsilon_0} D_\Omega \|f^0\|_p + \sum_{i\geq 1} \|f^i\|_p\,. \end{aligned} \tag{4.28}$$

Due to (4.28), we have $\widetilde{u} \in L_{p,d}(\Omega) \cap H^1_{p,d-p}(\Omega)$.

Step 2.3) The existence of solutions is proved in Steps 2.1 and 2.2, for all λ and D_Ω satisfying (4.17). For the cases where $D_\Omega = \infty$ and $\lambda = 1$, and $D_\Omega < \infty$ and $\lambda = 0$, estimate (4.18) is proved in (4.25) and (4.28), respectively. Therefore, we only need prove estimate (4.18) in the remaining case where $D_\Omega < \infty$ and $\lambda = 1$. Since u in Step 2.1 and $\widetilde{u}$ in Step 2.2 are the same (due to the result in Step 1), (4.18) follows from (4.25) and (4.28). □

4.2. **Domain with thin exterior : Aikawa dimension.** The notion of the Aikawa dimension was first introduced by Aikawa [2]. We recall the definition of the Aikawa dimension. For a set $E \subset \mathbb{R}^d$, the Aikawa dimension of E, denoted by $\dim_{\mathcal{A}}(E)$, is defined by

$$\dim_{\mathcal{A}}(E) = \inf \Big\{ \beta \geq 0 \,:\, \sup_{p\in E,\, r>0} \frac{1}{r^\beta} \int_{B_r(p)} \frac{1}{d(y,E)^{d-\beta}}\,\mathrm{d}y < \infty \Big\}$$

with considering $\frac{1}{0} = \infty$.

In this subsection, we assume that $d \geq 3$, and Ω satisfies

$$\beta_0 := \dim_{\mathcal{A}} \Omega^c < d-2\,.$$

Theorem 4.17. *For a constant $\beta < d-2$, if there exists a constant A_β such that*

$$\sup_{p\in\Omega^c,\, r>0} \frac{1}{r^\beta} \int_{B_r(p)} \frac{1}{d(y,\Omega^c)^{d-\beta}}\,\mathrm{d}y \leq A_\beta < \infty\,, \tag{4.29}$$

then the function

$$\phi(x) := \int_{\mathbb{R}^d} |x-y|^{-d+2} \rho(y)^{-d+\beta}\,\mathrm{d}y$$

is a superharmonic function on $\mathbb{R}^d$ with $-\Delta\phi = N(d)\rho^{-d+\beta}$. Moreover, we have

$$N^{-1}\rho(x)^{-d+2+\beta} \leq \phi(x) \leq N\rho(x)^{-d+2+\beta} \tag{4.30}$$

for all $x \in \Omega$, *where* $N = N(d, \beta, A_\beta)$.

Before proving Theorem 4.17, we first look at the corollaries of this theorem.

Corollary 4.18. *The Hardy inequality* (1.2) *holds on* Ω, *where* $\mathrm{C}_0(\Omega)$ *depends only on* d, β_0, $\{A_\beta\}_{\beta>\beta_0}$.

Actually, this corollary follows from the more general result [2, Theorem 3], and its proof is based on Muckenhoupt's A_p weight theory. Considering only Corollary 4.18, this result can be proved differently, as the following:

Proof of Corollary 4.18. We first note the following inequality provided in [8, Lemma 3.5.1]: If $f \in C_c^\infty(\mathbb{R}^d)$ and $s > 0$ is a smooth superharmonic function on a neighborhood of $\mathrm{supp}(f)$, then

$$\int_{\mathbb{R}^d} \frac{-\Delta s}{s} |f|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}x \quad \text{for all} \quad f \in C_c^\infty(\mathbb{R}^d) \tag{4.31}$$

(the proof of this inequality is based on integrating $\big|\nabla f - (f/s)\nabla s\big|^2$ and performing integration by parts). Take any $\beta \in (\beta_0, d-2)$, and let ϕ be the function in Theorem 4.17, so that

$$-\Delta\phi \geq N_1 \rho^{-2}\phi > 0 \tag{4.32}$$

where $N_1 = N(d, \beta, A_\beta) > 0$. Fix $f \in C_c^\infty(\Omega)$. For $0 < \epsilon < d\big(\mathrm{supp}(f), \partial\Omega\big)$, let $\phi^{(\epsilon)}$ be the mollification of ϕ in (2.1). Observe that

$$-\Delta\big(\phi^{(\epsilon)}\big) \geq N_1^{-1}\big(\rho^{-2}\phi\big)^{(\epsilon)} \geq N_1^{-1}(\rho+\epsilon)^{-2}\phi^{(\epsilon)} \quad \text{on} \quad \mathbb{R}^d\,,$$

where N_1 is in (4.32). By appling the monotone convergence theorem to (4.31) with $s = \phi^{(\epsilon)}$(see Lemma 2.5.(2)), we obtain (1.2) with $\mathrm{C}_0(\Omega) = N_1$. □

Theorem 4.19. *For any* $p \in (1, \infty)$ *and* $\theta \in \mathbb{R}$ *satisfying*

$$\beta_0 < \theta < (d-2-\beta_0)p + \beta_0\,,$$

Statement 4.3 (Ω, p, θ) *holds. In addition,* N_1 *in* (4.3) *depends only on* d, p, γ, θ, β_0, $\{\mathcal{A}_\beta\}_{\beta>\beta_0}$.

Remark 4.20. Theorem 4.19 deals with the Poisson equation in $\Omega \subset \mathbb{R}^d$, $d \geq 3$. Moreover, this theorem can also be interpreted as establishing the solvability of the Poisson equation $\Delta u - \lambda u = f$ in $\mathbb{R}^d$, particularly when f blows up near a set E with $\dim_{\mathcal{A}}(E) < d-2$. In other words, if $u \in H^2_{p,\theta}(\Omega)$ and $f \in L_{p,\theta+2p}(\Omega)$ satisfy equation (4.2), then

$$\int_{\mathbb{R}^d} u(\Delta\phi - \lambda\phi)\,\mathrm{d}x = \int_{\mathbb{R}^d} f\phi\,\mathrm{d}x \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}^d)\,;$$

we leave the proof to the reader, with a comment to utilize the test functions $\phi_k := \phi \sum_{|n|\leq k} \zeta_{0,(n)} \in C_c^\infty(\Omega)$, where $\zeta_{0,(n)}$ is the function in (3.9).

Proof of Theorem 4.19. Takd $\beta \in (\beta_0, d-2)$ satisfying

$$\beta < \theta < (d-2-\beta)p + \beta\,.$$

By Corollary 4.18 and Theorem 4.17, Ω admits the Hardy inequality (1.2), and there exists a superharmonic function ϕ satisfying $\phi \simeq \rho^{-d+2+\beta}$. Therefore by Lemma 4.4, the proof is completed. □

Proof of Theorem 4.17. We first prove (4.30). For a fixed $x \in \mathbb{R}^d$, put

$$I_j = \int_{E_j} |x-y|^{-d+2}\rho(y)^{-d+\beta}dy \quad \text{for} \quad j = 0,\, 1,\, \dots\,,$$

where $E_0 := B\big(x, 2^{-1}\rho(x)\big)$ and $E_j := B\big(x, 2^{j-1}\rho(x)\big) \setminus B\big(x, 2^{j-2}\rho(x)\big)$ for $j = 1,\, 2,\, \dots$. Then $\phi(x) = \sum_{j\in\mathbb{N}_0} I_j$. If $y \in E_0$ then $\frac{1}{2}\rho(x) \le \rho(y) \le 2\rho(x)$, which implies

$$I_0 \simeq_{d,\beta} \rho(x)^{-d+\beta} \int_{B(x,\rho(x)/2)} |x-y|^{-d+2}\,\mathrm{d}y \simeq_d \rho(x)^{-d+2+\beta}\,. \tag{4.33}$$

For I_j, $j \ge 1$, take $p_x \in \partial\Omega$ such that $|x - p_x| = \rho(x)$, and observe that

$$I_j \lesssim_d \big(2^j\rho(x)\big)^{-d+2} \int_{B(p_x,2^j\rho(x))} \rho(y)^{-d+\beta}\,\mathrm{d}y \le N\big(2^j\rho(x)\big)^{-d+2+\beta}\,, \tag{4.34}$$

where $N = N(d,\beta,A_\beta)$. (4.33) and (4.34) imply (4.30).

To prove that $-\Delta\phi = N(d)\,\phi$ in the sense of distribution, recall that

$$-\Delta_x\Big(|x-y|^{-d+2}\Big) = N(d)\,\delta_0(x-y)$$

in the sense of distribution, where $\delta_0(\cdot)$ is the Dirac delta distribution. Due to (4.29) and $\phi \simeq \rho^{-d+2+\beta}$, ϕ is locally integrable in $\mathbb{R}^d$. Therefore we obtain that for any $\zeta \in C_c^\infty(\mathbb{R}^d)$, by the Fubini theorem,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x)\big(-\Delta\zeta\big)(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |x-y|^{-d+2}(-\Delta\zeta)(x)\,\mathrm{d}x\Big)\rho(y)^{-d+\beta}\,\mathrm{d}y\\ &= N(d)\int_{\mathbb{R}^d}\zeta(y)\rho(y)^{-d+\beta}\,\mathrm{d}y\,.\end{aligned}$$

□

5. Application II - Various domains with fat exterior

This section presents results for the exterior cone condition, convex domains, the exterior Reifenberg condition, and Lipschitz cones. These domains and conditions imply the fat exterior condition.

Throughout this section, we consider a domain $\Omega \subsetneq \mathbb{R}^d$, $d \ge 2$.

5.1. **Exterior cone condition and exterior line segment condition.**

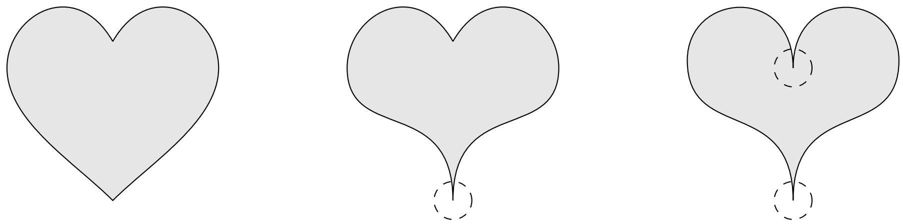
Definition 5.1 (Exterior cone condition)**.** For $\delta \in [0, \frac{\pi}{2})$ and $R \in (0,\infty]$, a domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the *exterior* (δ, R)*-cone condition* if for every $p \in \partial\Omega$, there exists a unit vector $e_p \in \mathbb{R}^d$ such that

$$\{x \in B_R(p)\,:\,(x-p)\cdot e_p \ge |x-p|\cos\delta\} \subset \Omega^c\,. \tag{5.1}$$

Note that the left hand side of (5.1) is the result of translating and rotating the set

$$\{x = (x_1,\dots,x_d) \in B_R(0)\,:\,x_1 \ge |x|\cos\delta\}\,.$$

The exterior $(0,R)$-cone condition can be called the *exterior R-line segment condition*, since if $\delta = 0$, then LHS in (5.1) equals $\{p + re_p\,:\,r \in [0,R)\}$. For examples of the exterior cone condition and exterior line segment condition, see Figure 5.1 below.



A. Lipschitz boundary condition

B. Exterior $(\frac{\pi}{3}, \infty)$-cone condition

(doesn't satisfy Lipschitz boundary condition)

C. Exterior ∞-line segment condition

(doesn't satisfy (δ, R)-cone condition, $\forall\, \delta, R > 0$)

FIGURE 5.1. Examples for exterior cone condition

Example 5.2. Suppose that there exists K, $R \in (0, \infty]$ such that for any $p \in \partial\Omega$, there exists a function $f_p \in C(\mathbb{R}^{d-1})$ such that

$$|f_p(y') - f_p(z')| \le K|y' - z'| \quad \text{for all} \quad y',\, z' \in \mathbb{R}^{d-1} \quad , \quad \text{and} \tag{5.2}$$

$$\Omega \cap B_R(p) = \big\{y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, y_d > f_p(y') \quad \text{and} \quad |y| < R\big\}\,, \tag{5.3}$$

where $(y', y_d) = (y_1, \cdots, y_d)$ in (5.3) is an orthonormal coordinate system centered at p. Then Ω satisfies the exterior (δ, R)-cone condition, where $\delta = \arctan(1/K) \in [0, \pi/2)$.

In addition, if $f \in C(\mathbb{R}^{d-1})$ satisfies (5.2) for f instead of f_p, then the domain

$$\big\{(x', x_n) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, x_n > f(x')\big\}$$

satisfies the exterior (δ, ∞)-cone condition, where $\delta = \arctan(1/K)$.

For $\delta \in (0, \pi)$, let $E_\delta := \{\sigma \in \partial B_1(0) \,:\, \sigma_1 > -\cos\delta\}$ (see Figure 5.2 below). By Λ_δ, we denote the first Dirichlet eigenvalue of the spherical Laplacian on E_δ. Alternatively, Λ_δ is expressed by

$$\Lambda_\delta = \inf_{f \in F_{\pi-\delta}} \frac{\int_0^{\pi-\delta} |f'(t)|^2 (\sin t)^{d-2}\,\mathrm{d}t}{\int_0^{\pi-\delta} |f(t)|^2 (\sin t)^{d-2}\,\mathrm{d}t}\,, \tag{5.4}$$

where $F_{\pi-\delta}$ is the set of all non-zero Lipschitz continuous function $f : [0, \pi-\delta] \to \mathbb{R}$ such that $f(\pi - \delta) = 0$ (see [18]). We also define

$$\lambda_\delta := -\frac{d-2}{2} + \sqrt{\Big(\frac{d-2}{2}\Big)^2 + \Lambda_\delta}\,,$$

and when $d = 2$, we define $\lambda_0 = \frac{1}{2}$.

The following quantitative information of Λ_δ and λ_δ is provided in [9]:

Proposition 5.3. *Let* $\delta \in (0, \pi)$.

(1) If $d = 2$ *then* $\lambda_\delta = \sqrt{\Lambda_\delta} = \frac{\pi}{2(\pi-\delta)} > \frac{1}{2}$.

(2) If $d = 4$ *then* $\lambda_\delta = -1 + \sqrt{1 + \Lambda_\delta} = \frac{\delta}{\pi-\delta}$.

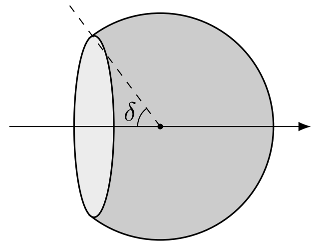


FIGURE 5.2. E_δ

(3) For $d \geq 3$,

$$\Lambda_\delta \geq \left(\int_0^{\pi-\delta} (\sin t)^{-d+2} \Big(\int_0^t (\sin r)^{d-2}\, \mathrm{d}r \Big)\, \mathrm{d}t \right)^{-1} .$$

Moreover, $\Lambda_{\pi/2} = d-1$, $\lim_{\delta \searrow 0} \Lambda_\delta = 0$, *and* $\lim_{\delta \nearrow \pi} \Lambda_\delta = +\infty$.

Note that when $d = 3$, $\Lambda_\delta \geq \frac{1}{2} |\log \sin \frac{\delta}{2}|^{-1}$.

Remark 5.4. For each $\delta > 0$, there is a function $F \in C\big(\overline{E_\delta}\big) \cap C^\infty(E_\delta)$ such that

$$F > 0 \quad \text{and} \quad \Delta_{\mathbb{S}} F + \Lambda_\delta F = 0 \quad \text{in} \quad E_\delta \quad ; \quad F = 0 \quad \text{on} \quad \overline{E_\delta} \setminus E_\delta$$

(see, *e.g.*, [18, Section 5]), where $\Delta_{\mathbb{S}}$ is the spherical Laplacian. Due to

$$\Delta = D_{rr} + \frac{d-1}{r} D_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{d-1}}$$

(the representation of the Laplacian operator on $\mathbb{R}^d$ by the spherical coordinate), the function $v_\delta(x) := |x|^{\lambda_\delta} F(x/|x|)$ is harmonic in

$$U_\delta := \big\{ y \in B_1(0) \,:\, y_1 > -|y| \cos \delta \big\} ,$$

and vanishes on $\partial U_\delta \cap B_1(0)$.

With the help of λ_δ, we state the main results of this subsection.

Theorem 5.5. *Let*

$$\delta \in [0, \pi/2) \quad \textit{if} \;\; d = 2 \quad ; \quad \delta \in (0, \pi/2) \quad \textit{if} \;\; d \geq 3 ,$$

and let $\Omega \subset \mathbb{R}^d$ satisfy the exterior (δ, R)-cone condition, where

$$R \in (0, \infty] \quad \textit{if} \;\; \Omega \;\; \textit{is bounded} , \quad \textit{and} \quad R = \infty \quad \textit{if} \;\; \Omega \;\; \textit{is unbounded.}$$

Then Ω satisfies $\mathbf{LHMD}(\lambda_\delta)$*, where M_{λ_δ} in (4.7) depends only on d, δ, and additionally* $\mathrm{diam}(\Omega)/R$ *if Ω is bounded.*

Before proving Theorem 5.5, we present a corollary that directly follows from Theorems 5.5 and 4.13.

Theorem 5.6. *Let $p \in (1, \infty)$. Under the same assumption of Theorem 5.5, if $\theta \in \mathbb{R}$ satisfies*

$$-2 - (p-1)\lambda_\delta < \theta - d < -2 + \lambda_\delta ,$$

then Statement 4.3 (Ω, p, θ) holds. In addition, N_1 in (4.3) depends only on d, p, θ, γ, δ, and additionally $\mathrm{diam}(\Omega)/R$ *if Ω is bounded.*

To prove Theorem 5.5, we use the boundary Harnack principle on Lipschitz domains.

Lemma 5.7 (see Theorem 1 of [54])**.** *Let D be a bounded Lipschitz domain, A be a relatively open subset of ∂D, and U be a subdomain of D with $\partial U \cap \partial D \subset A$.Then there exists $N = N(D, A, U) > 0$ such that if u, v are positive harmonic funtion on D, and vanish on E, then*

$$\frac{u(x)}{v(x)} \leq N \frac{u(x_0)}{v(x_0)} \quad \textit{for any} \quad x_0,\, x \in U\,.$$

Proof of Theorem 5.5. By Lemma 4.9, it is sufficient to prove that there exists a constant $M > 0$ such that

$$w(x, p, r) \leq M \left(\frac{|x-p|}{r}\right)^{\lambda_\delta} \quad \text{for all } \ x \in \Omega \cap B(p, r)$$

whenever $p \in \partial\Omega$ and $r \in (0, R)$. For any $p \in \partial\Omega$, there exists a unit vector $e_p \in \mathbb{R}^d$ such that

$$C_p := \{y \in B_R(p) \,:\, (y-p)\cdot e_p \geq |y-p|\cos\delta\} \subset \Omega^c\,.$$

Since

$$\Omega \cap B_r(p) \subset B_r(p) \setminus C_p \quad \text{and} \quad \Omega \cap \partial B_r(p) \subset \partial B_r(p) \setminus C_p\,,$$

we have

$$w(x, p, r) \leq w\big(x,\, B_r(p) \setminus C_p\,,\, \partial B_r(p) \setminus C_p\,\big), \tag{5.5}$$

by directly applying the definition of $w(\cdot, p, r)$ (see (4.6)). Consider a rotation map T such that $T(e_p) = (-1, 0, \ldots, 0)$, and put $T_0(x) = r^{-1}T(x - p)$. Then

$$w\big(x,\, B_r(p) \setminus C_p\,,\, \partial B_r(p) \setminus C_p\,\big) = w\big(T_0(x), U_\delta, E_\delta\big), \tag{5.6}$$

where

$$U_\delta = \{y \in B_1(0) \,:\, y_1 > -|y|\cos\delta\} \ \text{ and } \ E_\delta = \{y \in \partial B_1(0) \,:\, y_1 > -|y|\cos\delta\}\,.$$

Due to (5.5) and (5.6), it is sufficient to show that there exists a constant $M > 0$ depending only on d and δ such that

$$w(x, U_\delta, E_\delta) \leq M|x|^{\lambda_\delta} \quad \text{for all} \quad x \in U_\delta\,, \tag{5.7}$$

Case 1: $\delta > 0$**.** Put $v(x) = |x|^{\lambda_\delta} F_0(x/|x|)$ where F_0 is the first Dirichlet eigenfunction of spherical laplacian on $E_\delta \subset \partial B_1(0)$, with $\sup_{E_\delta} F_0 = 1$ (see Remark 5.4). Note that U_δ is a bounded Lipschitz domain, and $w(\,\cdot\,, U_\delta, E_\delta)$ and v are positive harmonic functions on U_δ, and vanish on $\partial U_\delta \cap B_1$. By applying Lemma 5.7 for $D = U_\delta$, $A = (\partial U_\delta) \cap B_1(0)$, and $U = U_\delta \cap B_{1/2}(0)$, we obtain that there exists a constant $N_0 = N_0(d, \delta) > 0$ such that

$$w(x, U_\delta, E_\delta) \leq N_0 v(x) \leq N_0|x|^{\lambda_\delta} \quad \text{for} \quad x \in U_\delta \cap B_{1/2}(0).$$

Therefore (5.7) is obtained, where $M_0 = \max\big(N_0, 2^{\lambda_0}\big)$.

Case 2: $\delta = 0$ and $d = 2$**.** We consider $\mathbb{R}^2$ as $\mathbb{C}$. Note

$$U_0 = \{r\mathrm{e}^{i\theta} \,:\, r \in (0,1),\, \theta \in (-\pi, \pi)\}\ , \quad E_0 = \{\mathrm{e}^{i\theta} \,:\, \theta \in (-\pi, \pi)\}\,.$$

Observe that a function s is a classical superharmonic function on U_0 if and only if $s(z^2)$ is a classical superharmonic function on $B_1(0) \cap \mathbb{R}^2_+$ (use Lemma 2.4). It is implied by the definition of PWB solutions (see (4.6)) that

$$w(z^2, U_0, E_0) = w\big(z, B_1(0) \cap \mathbb{R}^2_+, \partial B_1(0) \cap \mathbb{R}^2_+\big)\,.$$

Since the map $z = (z_1, z_2) \mapsto z_1$ is harmonic on $B_1(0) \cap \mathbb{R}^2_+$, by Lemma 5.7 with $D = B_1(0) \cap \mathbb{R}^2_+$, we obtain that

$$w\big(z, B_1(0) \cap \mathbb{R}^2_+, \big(\partial B_1(0)\big) \cap \mathbb{R}^2_+\big) \leq N|z| \quad \text{for } z \in B_{1/2}(0) \cap \mathbb{R}^2_+ \,, \tag{5.8}$$

where N depends on nothing. Therefore the proof is completed. □

5.2. **Convex domains.** Recall that a set $E \subset \mathbb{R}^d$ is said to be *convex* if $(1-t)x + ty \in E$ for any x, $y \in E$ and $t \in [0,1]$.

Lemma 5.8. *For an open set $\Omega \subset \mathbb{R}^d$, Ω is convex if and only if for any $p \in \partial\Omega$, there exists a unit vector $e_p \in \mathbb{R}^d$ such that*

$$\Omega \subset \{x \,:\, (x-p)\cdot e_p < 0\} =: U_p \,. \tag{5.9}$$

Proof. Let Ω be a convex domain, and fix $p \in \partial\Omega$. Since the set $\{p\}$ is convex and disjoint from Ω, the hyperplane separation theorem (see, *e.g.*, [48, Theorem 3.4.(a)]) implies that there exists a unit vector $e_p \in \mathbb{R}^d$ such that (5.9) holds. Conversely, suppose that for any $p \in \partial\Omega$, there exists a unit vector e_p satisfying (5.9). Then $E := \bigcap_{p\in\partial\Omega} U_p$ is convex, $\Omega \subset E$, and $E \cap \partial\Omega = \emptyset$. These imply $E = \Omega$; if not, $E \cap \partial\Omega \neq \emptyset$ which is a contradiction. Therefore our claim is proven. □

Theorem 5.9. *Let $\Omega \subset \mathbb{R}^d$ be a convex domain. Then Ω satisfies* **LHMD**(1) *where M_1 in (4.7) depends only on d.*

Proof. The argument to obtain (5.8) also implies that for any $d \in \mathbb{N}$,

$$w\big(x, B_1(0) \cap \mathbb{R}^d_+, (\partial B_1(0)) \cap \mathbb{R}^d_+\big) \leq N(d)|x| \quad \text{for all } x \in B_1(0) \cap \mathbb{R}^d_+ \,.$$

By translation, dilation, and rotation, we obtain that for a convex domain Ω and $p \in \partial\Omega$,

$$w(x,p,r) \leq w(x, B_r(p) \cap U_p, \big(\partial B_r(p)\big) \cap U_p) \leq N(d)\frac{|x-p|}{r}$$

for all $x \in B_r(p) \cap \Omega$, where U_p is the set on the right-hand side of (5.9). Therefore, the proof is completed. □

This result also implies that the Hardy inequality (1.2) holds on Ω, where $\mathrm{C}_0(\Omega)$ depends only on d (see Lemma 4.10); it is worth noting that Marcus, Mizel, and Pinchover [41, Theorem 11] provided that for a convex domain Ω, (1.2) holds where $\mathrm{C}_0(\Omega) = 4$, and $\mathrm{C}_0(\Omega)$ cannot be chosen less than 4.

By combining Theorems 4.13 and 5.9, we obtain the following result:

Theorem 5.10. *Let $\Omega \subset \mathbb{R}^d$ be a convex domain. For any $p \in (1,\infty)$ and $\theta \in \mathbb{R}$ with*

$$-p-1 < \theta - d < -1 \,,$$

Statement 4.3 (Ω, p, θ) holds. In addition, N_1 in (4.3) depends only on d, p, γ, θ. In particular, Ω is not necessarily bounded, and N_1 is independent of Ω.

5.3. **Exterior Reifenberg condition.** The notion of the vanishing Reifenberg condition was introduced by Reifenberg [47] and has been extensively studied in the literature (see, *e.g.*, [11, 12, 24, 50] and Subsubsection 1.2.3 of this paper). The following definition can be found in [11, 24]: For $\delta \in (0,1)$ and $R > 0$, a domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the (δ, R)-*Reifenberg condition*, if for every $p \in \partial\Omega$ and $r \in (0, R]$, there exists a unit vector $e_{p,r} \in \mathbb{R}^d$ such that

$$\begin{aligned} &\Omega \cap B_r(p) \subset \{x \in B_r(p) \,:\, (x-p)\cdot e_{p,r} < \delta r\} \quad \text{and} \\ &\Omega \cap B_r(p) \supset \{x \in B_r(p) \,:\, (x-p)\cdot e_{p,r} > -\delta r\}\,. \end{aligned} \tag{5.10}$$

In addition, Ω is said to satisfy the *vanishing Reifenberg condition* if for any $\delta \in (0,1)$, there exists $R_\delta > 0$ such that Ω satisfies the (δ, R_δ)-Reifenberg condition. Note that the vanishing Reifenberg condition is weaker than the C^1-boundary condition (see Example 5.13.(2) and (3)).

In this subsection, we present the totally vanishing exterior Reifenberg condition, which is a generalization of the vanishing Reifenberg condition. We also obtain a result for the Poisson equation on domains satisfying the totally vanishing exterior Reifenberg condition (see Theorem 5.18).

Definition 5.11 (Exterior Reifenberg condition)**.**

(1) By $\mathbf{ER}_\Omega$ we denote the set of all $(\delta, R) \in [0,1]\times\mathbb{R}_+$ satisfying the following: For each $p \in \partial\Omega$, and each connected component $\Omega^{(i)}_{p,R}$ of $\Omega\cap B(p,R)$, there exists a unit vector $e^{(i)}_{p,R} \in \mathbb{R}^d$ such that
$$\Omega^{(i)}_{p,R} \subset \{x \in B_R(p) \,:\, (x-p)\cdot e^{(i)}_{p,R} < \delta R\}\,. \tag{5.11}$$
By $\delta(R) := \delta_\Omega(R)$ we denote the infimum of δ such that $(\delta, R) \in \mathbf{ER}_\Omega$.

(2) For $\delta \in [0,1]$, we say that Ω satisfies the *totally δ-exterior Reifenberg condition* (abbreviate to '$\langle \mathrm{TER}\rangle_\delta$'), if there exist constants $0 < R_0 \le R_\infty < \infty$ such that
$$\delta_\Omega(R) \le \delta \quad \text{whenever} \quad R \le R_0 \;\text{ or }\; R \ge R_\infty\,. \tag{5.12}$$

(3) We say that Ω satisfies the *totally vanishing exterior Reifenberg condition* (abbreviate to '$\langle\mathrm{TVER}\rangle$'), if Ω satisfies $\langle\mathrm{TER}\rangle_\delta$ for all $\delta \in (0,1]$. In other word,
$$\lim_{R\to 0}\delta_\Omega(R) = \lim_{R\to\infty}\delta_\Omega(R) = 0\,.$$

The main theorem in this subsection concerns domains satisfying $\langle\mathrm{TER}\rangle_\delta$ for sufficiently small $\delta > 0$. However, our main interest is the condition $\langle\mathrm{TVER}\rangle$. For a comparison between the Refenberg condition and $\langle\mathrm{TVER}\rangle$, see Figure 5.3 and Example 5.13 below.

Lemma 5.12. *For any $R > 0$, $\big(\delta(R), R\big) \in \mathbf{ER}_\Omega$.*

Proof. Take a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that $(\delta_n, R) \in \mathbf{ER}_\Omega$ and $\delta_n \to \delta(R)$ as $n \to \infty$. Since $(\delta_n, R) \in \mathbf{ER}_\Omega$, for any $p \in \partial\Omega$ and any connected component of $\Omega \cap B(p,R)$, denoted by $\Omega_{p,R}$, there exists a unit vector e_n such that
$$\Omega_{p,R} \subset \{x \in B_R(p) \,:\, (x-p)\cdot e_n < \delta_n R\}\,. \tag{5.13}$$
Since $\{e_n\}_{n\in\mathbb{N}} \subset \partial B(0,1)$, there exists a subsequence $\{e_{n_k}\}_{k\in\mathbb{N}}$ such that $e_p := \lim_{k\to\infty} e_{n_k}$ exists in $\partial B(0,1)$. It is impliled by (5.13) that
$$\Omega_{p,R} \subset \{x \in B_R(p) \,:\, (x-p)\cdot e_p < \delta(R)R\}\,.$$

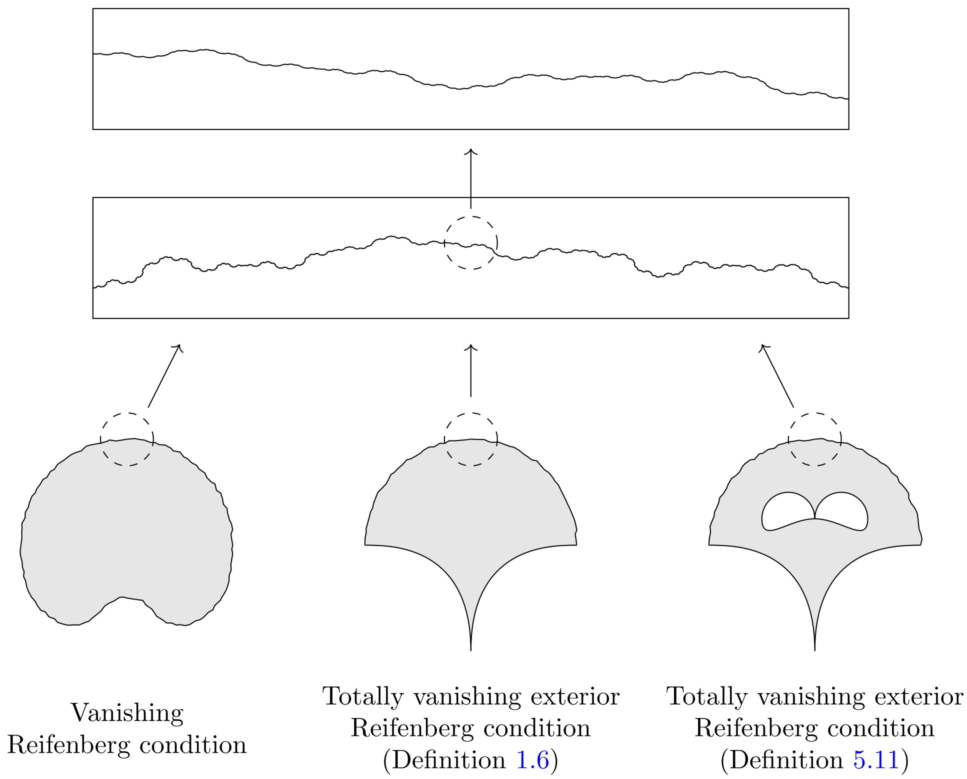


FIGURE 5.3. Totally vanishing exterior Reifenberg condition

Therefore $\big(\delta(R), R\big) \in \mathbf{ER}_\Omega$. □

Example 5.13.

(1) If Ω satisfies the (δ, R_1)-Reifenberg condition, then $\delta(R) \le \delta$ for all $R \le R_1$, indeed the first line of (5.10) implies (5.11) with $e^{(i)}_{p,r} = e_{p,r}$. Moreover, if Ω is bounded, then Proposition 5.14 implies $\delta(R) \le \mathrm{diam}(\Omega)/R$. Therefore, if Ω is a bounded domain satisfying the vanishing Reifenberg condition, then Ω also satisfies $\langle$TVER$\rangle$.

(2) By $\lambda_*(\mathbb{R}^{d-1})$, we denote the little Zygmund class, which is the set of all $f \in C(\mathbb{R}^{d-1})$ such that

$$\lim_{h\to 0} \sup_{x\in\mathbb{R}^{d-1}} \frac{|f(x+h) - 2f(x) + f(x-h)|}{|h|} = 0\,.$$

For $f \in \lambda_*(\mathbb{R}^{d-1})$, put

$$\Omega = \{(x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \,:\, x_d > f(x')\}\,.$$

Then, as mentioned in [12, Example 1.4.3] (see also [16, Theorem 6.3]), Ω satisfies the vanishing Reifenberg condition, which implies $\lim_{R\to 0}\delta_\Omega(R) = 0$. Moreover, since $A := \|f\|_{C(\mathbb{R}^{d-1})} < \infty$, Proposition 5.14 implies that $\delta(R) \le \frac{2\|f\|_{C(\mathbb{R}^{d-1})}}{R}$. Therefore Ω satisfies $\langle$TVER$\rangle$.

(3) Suppose that Ω is bounded, and for any $p \in \partial\Omega$ there exists $R > 0$ and $f \in \lambda_*(\mathbb{R}^{d-1})$ such that

$$\Omega \cap B(p, R) = \left\{y = (y', y_n) \in \mathbb{R}^{d-1} \times \mathbb{R} \, : \, |y| < R \ \text{and} \ y_n > f(y')\right\},$$

where $(y', y_n) = (y_1, \ldots, y_n)$ is an orthonormal coordinate system centered at p. Then Ω satisfies the vanishing Reifenberg condition, and therefore Ω satisfies $\langle \text{TVER} \rangle$.

(4) Let Ω satisfy the exterior R_0-ball condition, *i.e.*, there exists $R_0 > 0$ such that for any $p \in \partial\Omega$, there exists $q \in \mathbb{R}^d$ satisfying $|p - q| = R_0$ and $B(q, R_0) \subset \Omega^c$. Then $\delta(R) \leq \frac{R}{2R_0}$, and therefore $\lim_{R\to 0} \delta(R) = 0$.

(5) If a domain Ω is an intersection of domains satisfying the totally vanishing Reifenberg condition, then Ω satisfies $\langle \text{TVER} \rangle$.

All of the following examples are valid even if $\langle \text{TVER} \rangle$ is defined by (1.9) instead of (5.11).

A sufficient condition for $\lim_{R\to\infty} \delta_\Omega(R) = 0$ is that $\delta_\Omega(R) \lesssim 1/R$. We provide an equivalent condition for Ω to satisfy $\delta_\Omega(R) \lesssim 1/R$.

Proposition 5.14.

$$\sup_{R>0} R\, \delta_\Omega(R) = \sup_{p\in\partial\Omega} d\big(p, \partial(\Omega_{\text{c.h.}})\big)\,,$$

where $\Omega_{\text{c.h.}}$ *is the convex hull of* Ω, *i.e.,*

$$\Omega_{\text{c.h.}} := \left\{(1-t)x + ty \, : \, x, y \in \Omega \ , \ t \in [0,1]\right\}.$$

Remark 5.15. It follows from the definition of $\delta_\Omega(R)$ that $R\delta_\Omega(R)$ increases as $R \to \infty$. Therefore if $\delta_\Omega(r_0) > 0$ for some $r_0 > 0$, then $\delta_\Omega(R) \gtrsim 1/R$ as $R \to \infty$. As a result, due to Proposition (5.14), an equivalent condition for $\delta(R)$ to have minimal nontrivial decay (*i.e.*, $\delta_\Omega(R) \simeq 1/R$) is that $\sup_{p\in\partial\Omega} d\big(p, \partial(\Omega_{\text{c.h.}})\big) < \infty$.

Proof of Proposition 5.14. We only need to prove that for any $N_0 > 0$,

$$\sup_{R>0} R\, \delta_\Omega(R) \leq N_0 \quad \Longleftrightarrow \quad \sup_{p\in\partial\Omega} d\big(p, \partial(\Omega_{\text{c.h.}})\big) \leq N_0\,. \tag{5.14}$$

Step 1. We first claim that LHS of (5.14) holds if and only if for any $p \in \partial\Omega$, there exists a unit vector e_p such that

$$\Omega \subset \{x \in \mathbb{R}^d \, : \, (x - p) \cdot e_p < N_0\}\,. \tag{5.15}$$

The 'if' part is obvious. Therefore, we only need to prove the 'only if' part. Assume that LHS of (5.14) holds. Fix $p \in \partial\Omega$, and take $\{\widetilde{\Omega}_n\}_{n\in\mathbb{N}}$ such that $\widetilde{\Omega}_n$ is a connceted component of $\Omega \cap B_n(p)$, and $\widetilde{\Omega}_1 \subset \widetilde{\Omega}_2 \subset \widetilde{\Omega}_3 \subset \cdots$. Since Ω is a domain, Ω is path connected, which implies

$$\bigcup_{n\in\mathbb{N}} \widetilde{\Omega}_n = \Omega\,. \tag{5.16}$$

Since $R\delta(R) \leq N_0$, for each $n \in \mathbb{N}$, there exists $e_n \in \partial B_1(0)$ such that

$$\widetilde{\Omega}_n \subset \{x \in \mathbb{R}^d \, : \, (x - p) \cdot e_n < N_0\} \tag{5.17}$$

(see Lemma 5.12). Since $\partial B_1(0)$ is compact, there exists a subsequence $\{e_{n_k}\}$ which converges to a certain point, $e_p \in \partial B_1(0)$. Due to (5.16) and (5.17), we obtain that (5.15) holds for this e_p.

Step 2. Due to (5.14), we only need to prove the following: For $p \in \partial\Omega$, (5.15) holds for some $e_p \in \partial B_1(0)$ if and only if $d\big(p, \partial(\Omega_{\text{c.h.}})\big) \leq N_0$.

To prove the 'only if'part, suppose (5.15) and observe that

$$p \in \partial\Omega \subset \overline{\Omega_{\text{c.h.}}} \subset \{x \in \mathbb{R}^d \,:\, (x-p)\cdot e_p \leq N_0\}\,.$$

Put $\alpha_0 := \sup\{\alpha \geq 0 \,:\, p + \alpha e_p \in \overline{\Omega_{\text{c.h.}}}\}$. Then $p + \alpha_0 e_p \in \partial(\Omega_{\text{c.h.}})$, and therefore $d\big(p, \partial(\Omega_{\text{c.h.}})\big) \leq \alpha_0 \leq N_0$.

To prove the 'if' part, suppose that there exists $q \in \partial(\Omega_{\text{c.h.}})$ such that

$$|p-q| = d\big(p, \partial(\Omega_{\text{c.h.}})\big) \leq N_0\,.$$

Due to Lemma 5.8 and that $\Omega_{\text{c.h.}}$ is a convex domain, there is a unit vector $\widetilde{e}_q$ such that

$$\Omega_{\text{c.h.}} \subset \{x \in \mathbb{R}^d \,:\, (x-q)\cdot \widetilde{e}_q < 0\}\,.$$

This implies that for any $x \in \Omega \subset \Omega_{\text{c.h.}}$,

$$(x-p)\cdot\widetilde{e}_q < (q-p)\cdot\widetilde{e}_q \leq |p-q| \leq N_0\,.$$

Therefore (5.15) holds for $e_p := \widetilde{e}_q$. □

Remark 5.16. From Step 1 in the proof of Proposition 5.14, one can observe that this proposition remains valid even if the definition of $\delta_\Omega(R)$ is replaced by the infimum of $\delta > 0$ such that, for any $p \in \partial\Omega$, there exists a unit vector $e_{p,R}$ satisfying (1.9) for $r = R$.

Now, we state the main result of this subsection. We temporarily assume Theorem 5.17 (they are proved at the end of this subsection) and prove Theorem 5.18.

Theorem 5.17. *For any $\epsilon \in (0,1)$, there exists $\delta > 0$ depending only on d, ϵ such that if Ω satisfies $\langle \mathrm{TER}\rangle_\delta$, then Ω satisfies* $\mathbf{LHMD}(1-\epsilon)$ *where $M_{1-\epsilon}$ in (4.7) depends only on d, ϵ, δ, and R_0/R_∞, where R_0 and R_∞ are constants in (5.12).*

Theorem 5.18. *For any $p \in (1,\infty)$ and $\theta \in \mathbb{R}$ with $-p-1 < \theta - d < -1$, there exists $\delta > 0$ depending only on d, p, ϵ such that if Ω satisfies $\langle \mathrm{TER}\rangle_\delta$, then Statement 4.3 (Ω, p, θ) holds. In addition, N_1 in (4.3) depends only on d, p, γ, θ, and R_0/R_∞, where R_0 and R_∞ are constants in (5.12). In particular, if Ω satisfying $\langle \mathrm{TVER}\rangle$, then Statement 4.3 (Ω, p, θ) holds for all $p \in (1,\infty)$ and $\theta \in \mathbb{R}$ with $-p-1 < \theta - d < -1$.*

Proof. Take $\epsilon \in (0,1)$ such that

$$-p-1+(p-1)\epsilon < \theta - d < -1-\epsilon\,, \tag{5.18}$$

and put δ as the constant in Theorem 5.17 for this ϵ. Consider a domain Ω satisfying $\langle \mathrm{TER}\rangle_\delta$. By Theorem 5.17, this Ω satisfies $\mathbf{LHMD}(1-\epsilon)$. Therefore Theorem 4.13 and (5.18) imply that Statement 4.3 (Ω, p, θ) holds with $N_1 = N(d, p, \gamma, \theta, R_0/R_\infty)$. □

Remark 5.19. Kenig and Toro [25, Lemma 2.1] established that if a bounded domain satisfies the vanishing Reifenberg condition, then this domain also satisfies $\mathbf{LHMD}(1-\epsilon)$ for all $\epsilon \in (0,1)$.

To prove Theorem 5.17, we need the following lemma:

Lemma 5.20. *If $(\delta, R) \in \mathbf{ER}_\Omega$, then there exists a continuous function $w_{p,R} : \Omega \to (0,1]$ satisfying the following:*

(1) $w_{p,R}$ *is a classical superharmonic function on* Ω *.*
(2) $w_{p,R} = 1$ *on* $\{x \in \Omega : |x - p| > (1 - \delta)R\}$ *.*
(3) $w_{p,R} \leq M\delta$ *on* $\Omega \cap B(p, \delta R)$ *.*

Here, M *is a constant depending only on* d*. In particular,* M *is independent of* δ*.*

Proof of Lemma 5.20. If $\delta > 1/8$, then by putting $w_{p,R} \equiv 1$ and $M = 8$, this lemma is proved. Therefore we only need to consider the case $\delta \leq 1/8$. For a fixed $p \in \partial\Omega$, let $\{\Omega^{(i)}_{p,R}\}$ be the set of all connected components of $\Omega \cap B(p, R)$. For each i, take a unit vector $e^{(i)}_{p,R}$ satisfying (5.11). Put

$$q = p + R(\delta + 1/4)e^{(i)}_{p,R} , \tag{5.19}$$

so that

$$|p - q| = R(\delta + 1/4) \quad \text{and} \quad \Omega^{(i)}_{p,R} \cap B(q, R/4) \neq \emptyset \tag{5.20}$$

(see Figure 5.4 below).

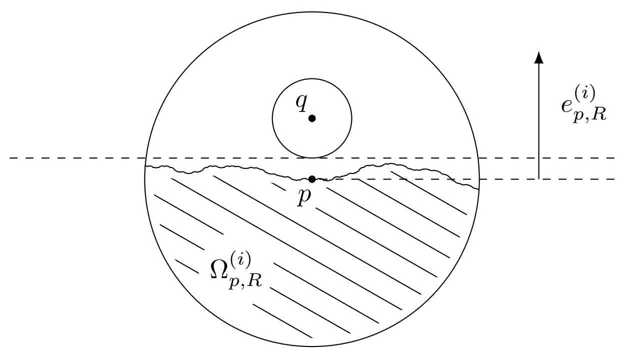


FIGURE 5.4. q and $B(q, R/4)$ in (5.19), (5.20)

Put $W^{(i)}(x) = F_0\big(4R^{-1}|x - q|\big)/F_0(2)$, where

$$F_0(t) = \log(t) \quad \text{if} \quad d = 2 \quad ; \quad F_0(t) = 1 - t^{2-d} \quad \text{if} \quad d \geq 3 , \tag{5.21}$$

so that $\Delta W^{(i)} = 0$ on $\mathbb{R}^d \setminus \{q\}$. Observe that

$$\begin{aligned} 0 \leq W^{(i)}(x) &\leq M_0\big(4R^{-1}|x - q| - 1\big) && \text{if } |x - q| \geq R/4 ; \\ W^{(i)}(x) &\geq 1 && \text{if } |x - q| \geq R/2 , \end{aligned}$$

where M_0 is a constant depends only on d. Due to (5.20) and that $\delta < \frac{1}{8}$, for $x \in \Omega^{(i)}_{p,R}$,

$$\begin{aligned} &\text{if} \quad |x - p| \leq \delta R , && \text{then} \quad \frac{R}{4} \leq |x - q| \leq \frac{R}{4} + 2\delta R ; \\ &\text{if} \quad |x - p| \geq (1 - \delta)R , && \text{then} \quad |x - q| \geq \frac{(3 - 8\delta)R}{4} \geq \frac{R}{2} . \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} 0 \leq W^{(i)}(x) \leq 8M_0\delta \quad & \text{if } |x - p| \leq \delta R ; \\ W^{(i)}(x) \geq 1 \quad & \text{if } |x - p| \geq (1 - \delta)R . \end{aligned}$$

Put

$$w_{p,R}(x) = \begin{cases} W^{(i)}(x) \wedge 1 & \text{if } x \in \Omega^{(i)}_{p,R} \\ 1 & \text{if } x \in \Omega \setminus B(p,R)\,. \end{cases}$$

Then $w_{p,R}$ is continuous on Ω, and satisfies (2) and (3) of this lemma. (1) of this lemma follows from (5.21) and Lemma 4.5. □

Proof of Theorem 5.17. Let $M > 0$ be the constant in Lemma 5.20. For given $\epsilon \in (0,1)$, take small enough $\delta \in (0,1)$ such that $M\delta < \delta^{1-\epsilon}$. We assume that Ω satisfies (5.12) for this δ. By using dilation and Lemma 5.12, without loss of generality, we assume that $(\delta, R) \in \mathbf{ER}_\Omega$ whenever $R \leq \widetilde{R}_0 := R_0/R_\infty$ (≤ 1) or $R \geq 1$.

Note that for $(\delta, R) \in \mathbf{ER}_\Omega$, due to Lemma 5.20 and the definition of PWB solutions (4.6), $w(\cdot, p, R) \leq M\delta \leq \delta^{1-\epsilon}$ on $\Omega \cap \partial B_{\delta R}(p)$. Therefore, by Lemma 4.6.(3),

$$w(\cdot, p, R) \leq \delta^{1-\epsilon} w(\cdot, p, \delta R) \quad \text{on} \quad \Omega \cap B_{\delta R}(p)\,. \tag{5.22}$$

The proof is completed by establishing (4.7) for $\alpha := 1-\epsilon$ and $M_{1-\epsilon}$ depending only on δ and $\widetilde{R}_0$. We prove (4.7) by dividing r and $|x-p|$ into the following five cases:

Case 1: $r \leq \widetilde{R}_0$. Take $n_0 \in \mathbb{N}_0$ such that $\delta^{n_0+1} r \leq |x-p| < \delta^{n_0} r$. Since $(\delta, \delta^k r) \in \mathbf{ER}_\Omega$ for all $k \geq 0$, it follows from (5.22) that

$$w(x,p,r) \leq \delta^{n_0(1-\epsilon)} w(x,p,\delta^{n_0} r) \leq \delta^{n_0(1-\epsilon)} \leq \left(\frac{|x-p|}{\delta\, r}\right)^{1-\epsilon}.$$

Case 2: $|x-p| < \widetilde{R}_0 < r \leq 1$. By Lemmas 4.6.(1) and (3) and the result in Case 1, we have

$$w(x,p,r) \leq w(x,p,\widetilde{R}_0) \lesssim_{\delta,\widetilde{R}_0} |x-p|^{1-\epsilon} \leq \left(\frac{|x-p|}{r}\right)^{1-\epsilon}$$

Case 3: $\widetilde{R}_0 \leq |x-p| < r \leq 1$. It directly follows that

$$w(x,p,r) \leq 1 \leq \left(\frac{|x-p|}{\widetilde{R}_0\, r}\right)^{1-\epsilon}.$$

Case 4: $|x-p| < 1 < r$. Take $n_0 \in \mathbb{N}_0$ such that $\delta^{n_0+1} r \leq 1 < \delta^{n_0} r$. Then $(\delta, \delta^k r) \in \mathbf{ER}_\Omega$ for all $k = 0, 1, \ldots, n_0 - 1$. Therefore we have

$$w(x,p,r) \leq \delta^{n_0(1-\epsilon)} w(x,p,\delta^{n_0} r) \leq \delta^{n_0(1-\epsilon)} w(x,p,1) \lesssim_{\delta,\widetilde{R}_0} \delta^{n_0(1-\epsilon)} |x-p|^{1-\epsilon}\,,$$

where the first inequality follows from (5.22), the second follows from Lemmas 4.6.(1) and (3), and the last follows from the result in Cases 2 and 3. Since $\delta^{n_0} \leq 1/(\delta r)$, we have $w(x,p,r) \lesssim_{\delta,\widetilde{R}_0} \big(|x-p|/r\big)^{1-\epsilon}$.

Case 5: $1 \leq |x-p| < r$. Take $n_0 \in \mathbb{N}_0$ such that $\delta^{n_0+1} r \leq |x-p| < \delta^{n_0} r$. Since $1 < |x-p|$, we have $(\delta, \delta^k r) \in \mathbf{ER}_\Omega$ for all $k = 0, 1, \ldots, n_0 - 1$. This implies that

$$w(x,p,r) \leq \delta^{n_0(1-\epsilon)} w(x,p,\delta^{n_0} r) \leq \delta^{n_0(1-\epsilon)} \leq \left(\frac{|x-p|}{\delta\, r}\right)^{1-\epsilon}.$$

□

Appendix A. Auxiliary results

Lemma A.1. *Let $p \in (1,\infty)$ and $u \in C(\mathbb{R}^d)$ satisfy (2.3).*

(1) $|u|^{p/2-1}u \in W_2^1(\mathbb{R}^d)$ and $D_i(|u|^{p/2-1}u) = \frac{p}{2}|u|^{p/2-1}(D_iu)1_{\{u\neq 0\}}$.

(2) $|u|^p \in W_1^2(\mathbb{R}^d)$ and

$$\begin{aligned} D_i\big(|u|^p\big) &= p|u|^{p-2}uD_iu1_{\{u\neq 0\}}\ ; \\ D_{ij}\big(|u|^p\big) &= \big(p|u|^{p-2}uD_{ij}u + p(p-1)|u|^{p-2}D_iuD_ju\big)\,1_{\{u\neq 0\}}\,. \end{aligned} \tag{A.1}$$

Proof. This proof is a variant of [34, Lemma 2.17]. Take a sequence of nonnegative functions $\{g_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ such that $g_n = 0$ on a neighborhood of 0 for each $n \in \mathbb{N}$, and $g_n(s) \nearrow |s|^{p/2-1}1_{s\neq 0}$ for all $s \in \mathbb{R}$. Put

$$F_n(t) := \int_0^t g_n(s)\,\mathrm{d}s \quad , \quad G_n(t) := \int_0^t \big(g_n(s)\big)^2\,\mathrm{d}s\,.$$

Recall the assumption (2.3), and denote $A = \sup|u|$. Since $0 \leq g_n(s) \leq |s|^{p/2-1}$, the Lebesgue dominated convergence theorem implies that $F_n(t) \to \frac{2}{p}|t|^{p/2-1}t$ and $G_n(t) \to \frac{1}{p-1}|t|^{p-2}t$ uniformly for $t \in [-A,A]$. Furthermore, there absolute values increase as $n \to \infty$. Since $F_n\big(u(\,\cdot\,)\big)$ and $G_n\big(u(\,\cdot\,)\big)$ vanish on a neighborhood of $\{u = 0\}$, they are supported on a compact subset of $\{u \neq 0\}$, and continuously differentiable with

$$D_i\big(F_n(u)\big) = g_n(u)D_iu\,1_{\{u\neq 0\}} \quad \text{and} \quad D_i\big(G_n(u)\big) = \big(g_n(u)\big)^2D_iu\,1_{\{u\neq 0\}}\,.$$

(1) Integrate by parts to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |g_n(u)\nabla u\,1_{\{u\neq 0\}}|^2\,\mathrm{d}x &= -\int_{\mathbb{R}^d} G_n(u)\Delta u\,1_{\{u\neq 0\}}\,\mathrm{d}x \\ &\leq \frac{1}{p-1}\int_{\{u\neq 0\}} |u|^{p-1}|\Delta u|\,\mathrm{d}x\,. \end{aligned}$$

By the monotone convergence theorem, we have $|u|^{p/2-1}|\nabla u| \in L_2(\mathbb{R}^d)$. We denote $v = \frac{2}{p}|u|^{p/2-1}u$. For any $\zeta \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} -\int_{\mathbb{R}^d} v\cdot D_i\zeta\,\mathrm{d}x &= -\lim_{n\to\infty}\int_{\mathbb{R}^d} F_n(u)\cdot D_i\zeta\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_{\{u\neq 0\}} g_n(u)D_iu\cdot\zeta\,\mathrm{d}x = \int_{\{u\neq 0\}} |u|^{p/2-1}D_iu\cdot\zeta\,\mathrm{d}x\,. \end{aligned}$$

Here, the first and the last equalities follow from the Lebesgue dominated convergence theorem, because $|F_n(u)| \leq |v|$ and $|g_n(u)D_iu| \leq |u|^{p/2-1}|\nabla u| \in L_2(\mathbb{R}^d)$. Therefore $v \in W_2^1(\mathbb{R}^d)$ and $D_iv = |u|^{p/2-1}D_iu\,1_{\{u\neq 0\}}$.

(2) It follows from (1) of this lemma that $|u|^p \in W_1^1(\mathbb{R}^d)$ with $D_i\big(|u|^p\big) = p|u|^{p-2}uD_iu1_{u\neq 0}$. For any $\zeta \in C_c^\infty$, we have

$$\begin{aligned} &\frac{1}{p-1}\int_{\{u\neq 0\}} |u|^{p-2}uD_iu\cdot D_j\zeta\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} G_n(u)D_iu\cdot D_j\zeta\,\mathrm{d}x \\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^d} \Big(|g_n(u)|^2D_iuD_ju + G_n(u)D_{ij}u\Big)\zeta\,\mathrm{d}x \end{aligned}$$

$$= -\int_{\{u\neq 0\}} \Big(|u|^{p-2}D_iuD_ju + \frac{1}{p-1}|u|^{p-2}uD_{ij}u1_{\{u\neq 0\}}\Big)\zeta\,\mathrm{d}x\,.$$

Here, the first and last inequalities follow from the Lebesgue dominated convergence theorem, because $|G_n(u)| \le \frac{1}{p-1}|u|^{p-1}$ and $|g_n(u)| \le |u|^{p/2-1}$ (recall the assumption for u, and (1) of this lemma). Therefore $|u|^p \in W_1^2(\mathbb{R}^d)$ with (A.1). □

Lemma A.2. *There exist linear maps*

$$\Lambda_0 \,:\, \Psi H_{p,\theta}^{\gamma} \to \Psi H_{p,\theta}^{\gamma+1}(\Omega) \quad \textit{and} \quad \Lambda_1,\,\ldots,\,\Lambda_d : \Psi H_{p,\theta}^{\gamma} \to \Psi H_{p,\theta-p}^{\gamma+1}(\Omega)$$

such that for any $f \in \Psi H_{p,\theta}^{\gamma}(\Omega)$, $f = \Lambda_0 f + \sum_{i=1}^d D_i(\Lambda_i f)$ *and*

$$\|\Lambda_0 f\|_{\Psi H_{p,\theta}^{\gamma+1}(\Omega)} + \sum_{i=1}^d \|\Lambda_i f\|_{\Psi H_{p,\theta-p}^{\gamma+1}(\Omega)} \le N\|f\|_{\Psi H_{p,\theta}^{\gamma}(\Omega)}\,, \tag{A.2}$$

where $N = N(d,p,\gamma,\theta,\mathrm{C}_2(\Psi))$.

Proof. **Step 1.** We first prove the case $\Psi \equiv 1$. Consider linear operators from H_p^{γ} to $H_p^{\gamma+1}$ defined by $L_0 := (1-\Delta)^{-1}$ and $L_i := -D_i(1-\Delta)^{-1}$ for $i = 1,\,\ldots,\,d$. They satisfy that for any $g \in H_p^{\gamma}$,

$$L_0 g + \sum_{i=1}^d D_i L_i g = g \quad \text{and} \quad \sum_{i=0}^d \|L_i g\|_{H_p^{\gamma+1}} \lesssim_{d,p,\gamma} \|g\|_{H_p^{\gamma}}\,. \tag{A.3}$$

We denote $\zeta_1(t) = \zeta_0(\mathrm{e}^{-1}t) + \zeta_0(t) + \zeta_0(\mathrm{e}t)$ and $\zeta_{1,(n)}(x) := \zeta_1\big(\mathrm{e}^{-n}\widetilde{\rho}(x)\big)$. Put

$$\begin{aligned}
\widetilde{\Lambda}_0 f(x) &:= \sum_{n\in\mathbb{Z}} \zeta_{1,(n)}(x)\,L_0\Big[\big(\zeta_{0,(n)}f\big)(\mathrm{e}^n\cdot)\Big](\mathrm{e}^{-n}x)\\
&\qquad -\sum_{k=1}^d\sum_{n\in\mathbb{Z}} \mathrm{e}^n\big(D_k\zeta_{1,(n)}\big)(x)\,L_k\Big[\big(\zeta_{0,(n)}f\big)(\mathrm{e}^n\cdot)\Big](\mathrm{e}^{-n}x)\,,\\
\widetilde{\Lambda}_i f(x) &:= \sum_{n\in\mathbb{Z}} \mathrm{e}^n\zeta_{1,(n)}(x)\,L_i\Big[\big(\zeta_{0,(n)}f\big)(\mathrm{e}^n\cdot)\Big](\mathrm{e}^{-n}x)\,,
\end{aligned}$$

for $i = 1,\,\ldots,\,d$. Due to (A.3), we have

$$\begin{aligned}
\widetilde{\Lambda}_0 f + \sum_{i=1}^d D_i\widetilde{\Lambda}_i f &= \sum_{n\in\mathbb{Z}}\left(\zeta_{1,(n)}(\,\cdot\,)\times\Big[\big(L_0 + \sum_{i=1}^d D_iL_i\big)\big[(\zeta_{0,(n)}f)(\mathrm{e}^n\cdot)\big]\Big](\mathrm{e}^{-n}\,\cdot\,)\right)\\
&= \sum_{n\in\mathbb{Z}}\Big[\zeta_{1,(n)}\zeta_{0,(n)}f\Big] = \sum_{n\in\mathbb{Z}}\zeta_{0,(n)}f = f\,.
\end{aligned}$$

In addition, we also obtain

$$\begin{aligned}
&\big\|\big(\zeta_{0,(n)}\widetilde{\Lambda}_0 f\big)(\mathrm{e}^n\cdot)\big\|_{H_p^{\gamma+1}}^p + \sum_{i=1}^d \mathrm{e}^{-np}\big\|\big(\zeta_{0,(n)}\widetilde{\Lambda}_i f\big)(\mathrm{e}^n\cdot)\big\|_{H_p^{\gamma+1}}^p \qquad\qquad (\mathrm{A.4})\\
&\lesssim_N \sum_{i=0}^d\sum_{|k|\le 2}\Big\|\big(\zeta_{0,(n)}\zeta_{1,(n+k)}\big)(\mathrm{e}^n\cdot)\times L_i\Big[\big(\zeta_{0,(n+k)}f\big)(\mathrm{e}^{n+k}\cdot)\Big](\mathrm{e}^{-k}\cdot)\Big\|_{H_p^{\gamma+1}}^p\\
&\qquad + \mathrm{e}^{np}\sum_{i=0}^d\sum_{|k|\le 2}\Big\|\big(\zeta_{0,(n)}D\zeta_{1,(n+k)}\big)(\mathrm{e}^n\cdot)\times L_i\Big[\big(\zeta_{0,(n+k)}f\big)(\mathrm{e}^{n+k}\cdot)\Big](\mathrm{e}^{-k}\cdot)\Big\|_{H_p^{\gamma+1}}^p
\end{aligned}$$

$$\lesssim_N \sum_{|k|\leq 2}\sum_{i=0}^{d}\Big\|L_i\Big[\big(\zeta_{0,(n+k)}f\big)(\mathrm{e}^{n+k}\cdot)\Big]\Big\|^p_{H_p^{\gamma+1}} \lesssim_{d,p,\gamma} \sum_{|k|\leq 2}\big\|\big(\zeta_{0,(n+k)}f\big)(\mathrm{e}^{n+k}\cdot)\big\|^p_{H_p^\gamma}\,,$$

Here, the first and second inequalities follow from that

$$\big\|\big(\zeta_{0,(n)}\zeta_{1,(n+k)}\big)(\mathrm{e}^n\cdot)\big\|_{C^m(\mathbb{R}^d)} + \mathrm{e}^n\big\|\big(\zeta_{0,(n)}\cdot D\zeta_{1,(n+k)}\big)(\mathrm{e}^n\cdot)\big\|_{C^m(\mathbb{R}^d)} \leq N(d,k,m)\,,$$

where $N(d,k,l)=0$ if $|k|\geq 3$ (note (3.10) and that $\mathrm{supp}(\zeta_1)\subset[\mathrm{e}^{-2},\mathrm{e}^2]$). (A.4) implies (A.2) for $\Psi\equiv 1$ and $\widetilde{\Lambda}_i$ instead of Λ_i.

Step 2. For $f\in \Psi H_{p,\theta}^\gamma$ ($\Leftrightarrow \Psi^{-1}f\in H_{p,\theta}^\gamma(\Omega)$), put

$$\Lambda_0 f = \Psi\widetilde{\Lambda}_0(\Psi^{-1}f) - \sum_{k=1}^{d}\big(D_k\Psi\big)\cdot\widetilde{\Lambda}_k(\Psi^{-1}f)\quad;\quad \Lambda_i f = \Psi\widetilde{\Lambda}_i(\Psi^{-1}f)\,.$$

for $i=1,\cdots,d$. Then we have

$$\Big(\Lambda_0+\sum_{i=1}^{d}D_i\Lambda_i\Big)f = \Psi\Big(\widetilde{\Lambda}_0+\sum_{i=1}^{d}D_i\widetilde{\Lambda}_i\Big)\big(\Psi^{-1}f\big) = f\,.$$

Moreover, Lemma 3.10.(3) and (A.4) imply that

$$\begin{aligned}&\|\Psi^{-1}\Lambda_0 f\|_{H_{p,\theta}^{\gamma+1}(\Omega)} + \sum_{i=1}^{d}\|\Psi^{-1}\Lambda_i f\|_{H_{p,\theta-p}^{\gamma+1}(\Omega)}\\ \lesssim\;& \|\widetilde{\Lambda}_0(\Psi^{-1}f)\|_{H_{p,\theta}^{\gamma+1}(\Omega)} + \sum_{i=1}^{d}\|\widetilde{\Lambda}_i(\Psi^{-1}f)\|_{H_{p,\theta-p}^{\gamma+1}(\Omega)} \lesssim \|\Psi^{-1}f\|_{H_{p,\theta}^\gamma(\Omega)}\,.\end{aligned}$$

Therefore, the proof is completed. □

Lemma A.3. *Let $\eta\in C_c^\infty(\mathbb{R}^d)$ satisfy $\eta=1$ on $B_1(0)$ and $\mathrm{supp}(\eta)\subset B_2(0)$. For each $i\in\mathbb{N}$, let $N(i)\in\mathbb{N}$ be a constant satisfying*

$$\mathrm{supp}\Big(\sum_{|n|\leq i}\zeta_{0,(n)}\Big)\subset\big\{x\in\Omega\,:\,\big(N(i)/2\big)^{-1}\leq\rho(x)\leq N(i)/2\big\}\,.$$

Let Λ_i, $\Lambda_{i,j}$, $\Lambda_{i,j,k}$ are linear functionals on $\mathcal{D}'(\Omega)$ defined as

$$\Lambda_i f := \Big(\sum_{|n|\leq i}\zeta_{0,(n)}\Big)f\ \ ,\ \ \Lambda_{i,j}f=\eta(j^{-1}\cdot)\Lambda_i f\ \ ,\ \ \Lambda_{i,j,k}f=\big(\Lambda_{i,j}f\big)^{(N(i)^{-1}k^{-1})}\,,$$

where $\big(\Lambda_{i,j}f\big)^{(\epsilon)}$ is defined in the same way as in (2.1). Then for any $p\in(1,\infty)$, $\gamma,\theta\in\mathbb{R}$, and regular Harnack function Ψ, the following hold:

1. *For any $f\in\mathcal{D}'(\Omega)$, $\Lambda_{i,j,k}f\in C_c^\infty(\Omega)$.*
2. *For any $f\in\Psi H_{p,\theta}^\gamma(\Omega)$,*

$$\begin{aligned}&\sup_i\|\Lambda_i f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\leq N_1\|f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\\&\sup_j\|\Lambda_{i,j} f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\leq N_2\|f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\\&\sup_k\|\Lambda_{i,j,k} f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\leq N_3\|f\|_{\Psi H_{p,\theta}^\gamma(\Omega)}\,,\end{aligned}\tag{A.5}$$

   where N_1, N_2, N_3 are constants independent of f.
3. *For any $f\in\Psi H_{p,\theta}^\gamma(\Omega)$,*

$$\lim_{k\to\infty}\Lambda_{i,j,k}f=\Lambda_{i,j}f\ \ ,\ \ \lim_{j\to\infty}\Lambda_{i,j}f=\Lambda_i f\ \ ,\ \ \lim_{i\to\infty}\Lambda_i f=f\quad in\quad \Psi H_{p,\theta}^\gamma(\Omega)\,.\tag{A.6}$$

Proof. (1) It follows directly from the properties of distributions.

(2), (3) **Step 1:** Λ_i**.** Let $f \in H^\gamma_{p,\theta}(\Omega)$. From (3.10), one can observe that

$$\|f - \Lambda_i f\|^p_{\Psi H^\gamma_{p,\theta}(\Omega)} \lesssim_N \sum_{|n| \geq i-1} \mathrm{e}^{n\theta} \big\| \big(\Psi^{-1} f \zeta_{0,(n)}\big)(\mathrm{e}^n \cdot)\big\|^p_{H^\gamma_p} \leq \|f\|^p_{\Psi H^\gamma_{p,\theta}(\Omega)}\,,$$

where $N = N(d,p,\gamma,\theta)$. Therefore we have

$$\sup_i \|\Lambda_i f\|_{\Psi H^\gamma_{p,\theta}(\Omega)} \leq N \|f\|_{\Psi H^\gamma_{p,\theta}(\Omega)} \quad \text{and} \quad \lim_{i\to\infty} \|f - \Lambda_i f\|_{\Psi H^\gamma_{p,\theta}(\Omega)} = 0\,.$$

Step 2: $\Lambda_{i,j}$**.** The definition of $H^\gamma_{p,\theta}(\Omega)$ implies that for any $A > 1$, if $F \in \mathcal{D}'(\Omega)$ or $F \in \mathcal{D}'(\mathbb{R}^d)$, and F is supported on $\{x \in \Omega \,:\, A^{-1} \leq \rho(x) \leq A\}$, then

$$\|F\|_{H^\gamma_{p,\theta}(\Omega)} \simeq_N \|F\|_{H^\gamma_p}\,, \tag{A.7}$$

where $N = N(d,p,\theta,\gamma,A)$. For each $i \in \mathbb{N}$, $\Psi^{-1}\Lambda_i f$ and $\Psi^{-1}\Lambda_{i,j} f$ are supported on

$$\big\{x \in \Omega \,:\, N(i)^{-1} \leq \rho(x) \leq N(i)\big\}\,.$$

Therefore $\Psi^{-1}\Lambda_i f \in H^\gamma_p$. Since $\Psi^{-1}\Lambda_{i,j} f = \eta(j^{-1}\,\cdot)\Psi^{-1}\Lambda_i f$, we obtain that

$$\lim_{j\to\infty} \big\|\Psi^{-1}\Lambda_i f - \Psi^{-1}\Lambda_{i,j} f\big\|_{H^\gamma_p} = 0 \quad \text{and} \quad \big\|\Psi^{-1}\Lambda_{i,j} f\big\|_{H^\gamma_p} \lesssim_{N_2} \big\|\Psi^{-1}\Lambda_i f\big\|_{H^\gamma_p}\,,$$

where $N_2 = N(d,p,\gamma,\theta,i,\eta)$. Due to (A.7), (A.5) and (A.6) for $\Lambda_{i,j}$ are proved.

Step 3: $\Lambda_{i,j,k}$**.** Put

$$K_{i,j} = \{x \in \Omega \,:\, N(i)^{-1} \leq \rho(x) \leq N(i)\,,\ |x| \leq 2j\}\,,$$

which is a compact subset of Ω, and $\Lambda_{i,j} f$ and $\Lambda_{i,j,k} f$ are supported on there. Since Ψ and Ψ^{-1} belong to $C^\infty(\Omega)$, we obtain that

$$\|\Lambda_{i,j} f\|_{\Psi H^\gamma_{p,\theta}(\Omega)} := \|\Psi^{-1}\Lambda_{i,j} f\|_{H^\gamma_{p,\theta}(\Omega)} \simeq_N \|\Psi^{-1}\Lambda_{i,j} f\|_{H^\gamma_p} \simeq_N \|\Lambda_{i,j} f\|_{H^\gamma_p}, \tag{A.8}$$

where $N = N(d,p,\gamma,\theta,i,j,\Psi)$; it also holds for $\Lambda_{i,j,k} f$ and $\Lambda_{i,j} f - \Lambda_{i,j,k} f$, instead of $\Lambda_{i,j} f$.

Since $\Lambda_{i,j,k} f$ is a mollification of $\Lambda_{i,j} f$, we have

$$\|\Lambda_{i,j,k} f\|_{H^\gamma_p} \lesssim_{N_3} \|\Lambda_{i,j} f\|_{H^\gamma_p} \quad \text{and} \quad \lim_{k\to\infty} \big\|\Lambda_{i,j} f - \Lambda_{i,j,k} f\big\|_{H^\gamma_p} = 0\,,$$

where $N_3 = N(d,p,\theta,\gamma,\Psi,i,j,\eta)$. □

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