

CLASSIFICATION OF GLOBAL SOLUTIONS TO THE OBSTACLE PROBLEM IN THE PLANE

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ABSTRACT. Global solutions to the obstacle problem were first completely classified in two dimensions by Sakai using complex analysis techniques. Although the complex analysis approach produced a very succinct proof in two dimensions, it left the higher dimensional cases, and even closely related problems in two dimensions, unresolved. A complete classification in dimensions $n \geq 3$ was recently given by Eberle, Figalli and Weiss, forty years after Sakai published his proof. In this paper we give a proof of Sakai's classification result for unbounded coincidence sets in the spirit of the recent proof by Eberle, Figalli and Weiss. Our approach, in particular, avoids the need for complex analysis techniques and offers new perspectives on two-dimensional problems that complex analysis cannot address.

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1. INTRODUCTION

The classification of global solutions to the obstacle problem

$$(1.1) \quad \Delta u = \chi_{\{u>0\}}, \quad u \geq 0, \quad \text{in } \mathbb{R}^n,$$

was first achieved for $n = 2$ by Sakai using the Riemann mapping theorem [18]. (In what follows we will restrict ourselves to the classification of solutions such that the *coincidence set*, that is the set $\{u = 0\}$, has non-empty interior (see Remark 2.2)).

Theorem 1.1 (Sakai, 1981). *Let $n = 2$ and let u be a solution of (1.1) such that $\{u = 0\}$ has non-empty interior. Then $\{u = 0\}$ is either a half-plane, ellipse, parabola or a strip.*

After Sakai's result, the classification of solutions to (1.1) with bounded coincidence set was achieved using tools from potential theory for all dimensions $n \geq 2$ in [15] (see also [5, 6, 16] for earlier partial classifications as well as [9] for a short proof). On the other hand, the case when the coincidence set is unbounded remained an open problem until very recently. A complete classification for dimensions $n \geq 6$ was given in [8] while the restriction on the dimension was later removed in [7], thus completing the classification for dimensions $n \geq 3$.

Theorem 1.2 (Eberle, Figalli and Weiss (2022)). *Let $n \geq 3$ and let u be a solution of (1.1) such that $\{u = 0\}$ has non-empty interior. Then $\{u = 0\}$ is either a half-space, ellipsoid, paraboloid or a cylinder with ellipsoid or paraboloid as base.*

In both [8] and [7], the focus is showing that solutions with unbounded, non-cylindrical coincidence sets are paraboloids as this implies Theorem 1.2 (see for instance [8, Section 3.2]). Broadly speaking, the general method in both proofs is to match the asymptotic behaviour of u at infinity with that of a solution of (1.1) with prescribed coincidence set, and then show that these two solutions are indeed identical.

In the case $n \geq 4$, the method in [7] is to first obtain an expansion of a solution u to (1.1) of the form

$$(1.2) \quad u = p(x) + b \cdot x + o(|x|),$$

where $p(x) = \lim_{r \rightarrow \infty} r^{-2} u(rx)$ is the blow-down and $b \in \mathbb{R}^n$ is some vector that depends on the solution. Then a solution u_{matched} is constructed so that $\{u_{\text{matched}} = 0\}$ is a paraboloid and $u_{\text{matched}} = p(x) + b \cdot x + o(|x|)$. This choice of matching meant that $u - u_{\text{matched}}$ has sublinear growth at infinity so that an application of the Alt-Caffarelli-Friedmann (ACF) functional (see [1]) yields that $u = u_{\text{matched}}$.

For $n = 3$ however, an expansion as in (1.2) is not possible. In fact, it can be shown that $\int_{B_R} |u - p| \simeq R \log(R)$ when $\{u = 0\}$ is a paraboloid and hence in this sense, the

case $n = 3$ is critical for the method. Nevertheless, a two stage version of the method where the $R \log(R)$ behaviour is carefully removed succeeds in constructing a solution u_{matched} such that $\{u_{\text{matched}} = 0\}$ is a paraboloid and $u - u_{\text{matched}}$ has sublinear growth at infinity.

When $n = 2$ it can be shown that $\int_{B_R} |u - p| \simeq R^{\frac{3}{2}}$ and so this corresponds to a supercritical case for the method. The purpose of this paper is to classify solutions u to (1.1) with unbounded coincidence sets when $n = 2$ in the spirit of [7]. In this way, we give a proof of Theorem 1.1 that avoids complex analysis techniques.

Apart from unifying the proofs of Theorems 1.1 and 1.2, it is important to give a proof of Theorem 1.1 that does not rely on complex analysis since there are closely related problems in two dimensions that complex analysis cannot solve. For instance, we believe that global solutions to the obstacle problem in two dimensional exterior domains cannot be addressed using complex analysis techniques, while the techniques developed in this paper could shed light on how to approach this related problem.

As was noticed in both [7, 8], and as will be explained later for the case $n = 2$, in order to prove Theorem 1.1, we only need to classify a certain class of solutions to (1.1). Precisely, we will show the following:

Theorem 1.3. *Let $N = 2$ and u be an x_2 -monotone global solution to the obstacle problem as in Definition 2.7 below. Then $\{u = 0\}$ is a paraboloid.*

Although the general strategy is shared with that in [7], in the supercritical setting, we must overcome some new difficulties.

Similar to the case when $n = 3$, obtaining an expansion such as (1.2) is not possible. However, it is also not possible to remove the $R^{\frac{3}{2}}$ behaviour from the difference $u - p$. Instead, we can only prove an expansion of the form

$$u = p(x) + \alpha v + o(|x|^{\frac{3}{2}}),$$

where v is the $\frac{3}{2}$ -homogeneous solution to the thin obstacle problem. Unlike in dimensions $n \geq 3$, our proof of the expansion does not rely on a decomposition of the generalised Newtonian potential of the set $\{u = 0\}$, but rather on the classification of blow-down limits of the sequence $r^{-\frac{3}{2}}(u - p)(rx)$ as $r \rightarrow \infty$. In order to do this, we must first prove the almost monotonicity of the Almgren frequency functional and from this conclude sharp growth estimates of $u - p$.

The second difficulty we must overcome is that $u - u_{\text{matched}}$ is no longer sublinear at infinity and so the standard ACF functional cannot be used to conclude that $u = u_{\text{matched}}$. In our setting, the difference $u - u_{\text{matched}}$ is only $o(|x|^{\frac{3}{2}})$ and so we must develop a modified ACF functional in two dimensions that converges to zero for functions that are $o(|x|^{\frac{3}{2}})$.

1.1. Structure of the paper. In Section 2 we recall known results regarding solutions to the obstacle problem. In particular, we recall the existence of paraboloid

solutions of (1.1) and give a generalised Newtonian potential expansion for these solutions. We conclude Section 2 by recalling important results on solutions to the thin obstacle problem which play an important role in our analysis. In Section 3 the modified ACF functional is constructed. Section 4 is dedicated to the classification of the blow-down limits of $u - p$ with respect to the frequency scaling. Section 5 uses the blow-down analysis to obtain estimates on the coincidence set while Section 6 is dedicated to obtaining sharp growth estimates of $u - p$ by establishing the almost monotonicity of the Almgren frequency for $u - p$. In Section 7 we identify paraboloid solutions satisfying $u - u_{\text{matched}} = o(|x|^{\frac{3}{2}})$, and in Section 8, using the modified ACF functional we conclude.

2. NOTATION AND PRELIMINARIES

2.1. Notation. Throughout this work, a point $x \in \mathbb{R}^n$ will be denoted by $(x_1, \dots, x_n)$ while $\{e_i\}_{1 \leq i \leq n}$ represents the canonical basis vectors of $\mathbb{R}^n$. A ball with centre x and radius $r > 0$ will be denoted as $B_r(x)$ and when x is the origin it will be omitted.

We will denote by $\mathbb{R}_+^n$ the upper half space $\{x \in \mathbb{R}^n : x_1 > 0\}$ and similarly $\mathbb{R}_-^n$ represents $\{x \in \mathbb{R}^n : x_1 < 0\}$. The sets $B_r(x) \cap \mathbb{R}_+^n$ and $B_r(x) \cap \mathbb{R}_-^n$ will be denoted by $B_r(x)^-$ and $B_r(x)^+$ respectively. We will denote the outward unit normal on B_1 as ν .

When $A \subset \mathbb{R}^n$, the n -dimensional Lebesgue measure of A will be denoted by $|A|$, while $\mathcal{H}^m(A)$ will denote the m -dimensional Hausdorff measure. The interior of A will be represented by A° . Given $B \subset \mathbb{R}^n$ the Hausdorff distance between A and B will be denoted by $\text{dist}(A, B)$.

Given any $w : \mathbb{R}^n \rightarrow \mathbb{R}$ we will denote by $w_r(x)$ the rescaled function $w(rx)$. We will denote f_+ the function $\max\{f, 0\}$ and f_- the function $\max\{-f, 0\}$. Functions of the form $\frac{1}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$ will be called *half-space solutions*.

We will denote the *coincidence set* as $\mathcal{C} = \{u = 0\}$ and we will refer to $\partial\mathcal{C}$ as the *free boundary*. We recall the following properties for solutions of (1.1) (see for instance [17]). Any constant that depends only on the dimension will be called universal.

2.2. Basic properties of global solutions. We recall the following properties for solutions of (1.1) (see for instance [17]).

Lemma 2.1 (Basic Properties). *Let u be a solution to (1.1). Then the following hold:*

- i. *if $0 \in \partial\mathcal{C}$ then u is uniformly $C^{1,1}$, that is there exists a universal constant C such that $\|D^2u\|_{L^\infty(\mathbb{R}^2)} \leq C$,*
- ii. *u is convex and hence the coincidence set $\mathcal{C}$ is convex,*
- iii. *and if $\mathcal{C}$ has non-empty interior then $\partial\mathcal{C}$ is analytic.*

Throughout the rest of this work we will assume that $0 \in \partial\mathcal{C}$.

Remark 2.2. *If $\{u = 0\}$ has empty interior then $\Delta u \equiv 1$ (since $u \in C^{1,1}$) and so Liouville's theorem implies that u must be a quadratic polynomial.*

We will now recall the subharmonic properties of the difference of two solutions to the obstacle problem. Again these results can be found in [7, Lemma 2.13].

Lemma 2.3. *Let $u_1, u_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be global solutions to the obstacle problem. Then $(u_1 - u_2)_+$, $(u_1 - u_2)_-$ and $|u_1 - u_2|$ are subharmonic and we have that*

$$\int_{B_r} |\nabla(u_1 - u_2)|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}} (u_1 - u_2)^2 dx.$$

The blow-down limits with respect to quadratic rescaling have been classified in [2] and this is the content of the following Lemma.

Lemma 2.4 (Classification of blow-downs). *Let u be a solution to (1.1). Then*

$$\frac{u(rx)}{r^2} \rightarrow u_0 \text{ in } C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2)$$

where u_0 is a solution of (1.1). Moreover, either u_0 is a non-negative quadratic polynomial or $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$.

Remark 2.5. *Observe that if the blow-down limit of a global solution u is a half-space solution, then u must be a half-space solution. Indeed, supposing $u(0) = 0$, [17, Proposition 5.4] implies that u is a monotone function of one variable, that is, there exists some $U \in C^{1,1}(\mathbb{R})$ and a direction $e \in \mathbb{S}^{n-1}$ such that $u(x) = U(x \cdot e)$ where $U''(t) = \chi_{\{U>0\}}$, $U(0) = U'(0) = 0$ and $U'(t) \geq 0$. This implies that $U(t) = \frac{1}{2}t_+^2$ and so u is a half-space solution.*

Since we are interested in classifying global solutions with unbounded coincidence sets, we will recall the following Lemma which establishes the monotonicity of global solutions whose coincidence set contains a line that goes to infinity in some direction [3]. Observe that by convexity, if $\mathcal{C}$ is unbounded then it must contain an infinite ray in some direction.

Lemma 2.6 ([3]). *Let u be a solution to (1.1) such that $\mathcal{C}$ contains a ray in the e -direction. Then either $(\partial_e u)_+ \equiv 0$ or $(\partial_e u)_- \equiv 0$.*

We will now restrict our attention to solutions of (1.1) for the case $n = 2$. If $\mathcal{C}$ contained an infinite line, say $\{x_1 = 0\}$, then u is independent of the x_2 -direction (see for instance [7, Lemma 2.7]). Therefore, if $\mathcal{C}$ has non-empty interior and contained $\{x_1 = 0\}$, then $\mathcal{C}$ must be a strip in $\mathbb{R}^2$ and the coincidence set is trivially classified. If on the other hand $\mathcal{C}$ has empty interior and contained $\{x_1 = 0\}$ then $\Delta u \equiv 1$ and we have that $u(x) = \frac{1}{2}x_1^2$ so that the coincidence set is just the line $\{x_1 = 0\}$. Therefore we do not need to consider the case when $\mathcal{C}$ contains an infinite line.

Now if u is monotone, and $\mathcal{C}$ is convex and does not contain an infinite line, we can assume, up to a rotation and translation, that $\mathcal{C} \cap \{x_2 < 0\} = \emptyset$. Moreover, since the free boundary is analytic when $\mathcal{C}$ has non-empty interior, we can assume that $\mathcal{C} \cap \{x_2 \leq 0\} = \{0\}$. Indeed, if $\mathcal{C} \cap \{x_2 \leq 0\}$ contained a line segment, then the free boundary would have to be $\{x_2 = 0\}$ by analyticity, in which case u would be the half-space solution.

Along with Remark 2.5, the above considerations imply that we only need to classify a certain class of solutions to (1.1). The focus of this work will therefore be on classifying x_2 -monotone solutions as in the following definition.

Definition 2.7 (x_2 -monotone solutions). *A global solution u to the obstacle problem (1.1) is an x_2 -monotone solution if*

- (i) *the blow-down limit is $p(x) = \frac{1}{2}x_1^2$,*
- (ii) *$\partial_2 u \leq 0$,*
- (iii) *$\mathcal{C} \cap \{x_2 \leq 0\} = \{0\}$,*
- (iv) *and $\mathcal{C}$ has non-empty interior.*

In order to classify the second order blow-down we will need the following Hölder bound for the difference $u - p$ when $n = 2$ [14, Lemma A.5]. It is important to note that this Lemma is only true in higher dimensions if $p(x)$ is kept as $\frac{1}{2}x_1^2$. In particular, this is not true for the difference between u and its blow-down if $n \geq 3$.

Lemma 2.8. *There exists some $\beta \in (0, 1)$ and a universal constant C so that*

$$(2.1) \quad \|(u - p)_r\|_{C^{0,\beta}(B_{1/2})} \leq C \|(u - p)_r\|_{L^2(B_1)}.$$

We provide the proof of Lemma 2.8 in Appendix A.

Remark 2.9. *Two key properties we will use is that $(u - p)\Delta(u - p) \geq 0$ and that $\partial_{11}(u - p) \leq 0$. Indeed,*

$$(u - p)\Delta(u - p) = p\chi_{\{u=0\}} \geq 0$$

and

$$\partial_{11}(u - p) = \Delta(u - p) - \partial_{22}(u - p) = -\chi_{\{u=0\}} - \partial_{22}u \leq 0,$$

using the convexity of u .

Finally, we recall that for $w \in C^{1,1}$ the Almgren frequency functional is given by

$$\phi(r, w) = r \frac{\int_{B_r} |\nabla w|^2}{\int_{\partial B_r} w^2 d\sigma}$$

and we observe that we have the following upper bound (see [8, Proposition 4.1]).

Proposition 2.10 (Frequency Estimate). *For all $r > 0$ we have that*

$$(2.2) \quad \phi(r, u - p) \leq 2$$

2.3. Paraboloid solutions. For each $\gamma > 0$ we define the paraboloid $P_\gamma = \{-\gamma^{1/2}\sqrt{y_2} \leq y_1 \leq \gamma^{1/2}\sqrt{y_2}, y_2 \geq 0\} = \gamma\{-\sqrt{y_2} \leq y_1 \leq \sqrt{y_2}, y_2 \geq 0\}$. We will denote $P = P_1$ and so $P_\gamma = \gamma P$. The following Proposition has already been shown with complex analysis techniques in [18] as well as a potential theory approach in [19].

Proposition 2.11 (Existence of Paraboloid Solutions). *Given $\gamma > 0$ there exists a solution to (1.1) denoted by $u_{\gamma P}$ such that $\{u_{\gamma P} = 0\} = P_\gamma$.*

Proposition 2.12. *For each $\gamma > 0$ we have that*

$$(2.3) \quad \lim_{r \rightarrow \infty} \frac{u_{\gamma P}(rx)}{r^2} = \frac{1}{2}x_1^2.$$

Proof. It is clear by Remark 2.5 that $u_0 := \lim_{r \rightarrow \infty} r^{-2}u_{\gamma P}(r \cdot)$ is a quadratic polynomial. Since $\{x_1 = 0, x_2 \geq 0\} \subset \{u_0 = 0\}$ we must have that u_0 is monotone in the x_2 -direction by Lemma 2.6 and hence u_0 cannot contain multiples of x_1x_2 or x_2^2 . Since $n = 2$ this means that $u_0(x) = \frac{1}{2}x_1^2$. $\square$

It is shown in [7] that solutions to (1.1) with $n \geq 3$ can be expressed in terms of a generalised Newtonian potential. For our purposes it is enough to establish this result for paraboloid solutions.

Defining

$$G(x, y) = \log(|x - y|) - \log(|y|) + \frac{x \cdot y}{|y|^2},$$

the generalised Newtonian potential for a set $M \subset \mathbb{R}^2$ is then

$$V_M(x) = -\frac{1}{2\pi} \int_M G(x, y) dy.$$

We have the following important scaling property.

Lemma 2.13. *For each $\gamma > 0$ we have that*

$$V_M(\gamma x) = \gamma^2 V_{\frac{1}{\gamma}M}(x)$$

Proof. We first observe that

$$\begin{aligned} G(\gamma x, y) &= \log(|\gamma x - y|) - \log(|y|) + \frac{\gamma x \cdot y}{|y|^2} \\ &= \log\left(\left|x - \frac{y}{\gamma}\right|\right) + \log(\gamma) - \log\left(\left|\frac{y}{\gamma}\right|\right) - \log(\gamma) + \frac{x \cdot \frac{y}{\gamma}}{\left|\frac{y}{\gamma}\right|^2} \\ &= G\left(x, \frac{y}{\gamma}\right). \end{aligned}$$

Using this scaling we have that

$$\begin{aligned} V_M(\gamma x) &= -\frac{1}{2\pi} \int_M G\left(x, \frac{y}{\gamma}\right) dy \\ &= -\frac{1}{2\pi} \int_{\frac{1}{\gamma}M} G(x, z) \gamma^2 dz \\ &= \gamma^2 V_{\frac{1}{\gamma}M}(x). \end{aligned}$$

$\square$

We can now state the expansion for paraboloid solutions.

Proposition 2.14. *Let $u_{\gamma P}$ be a paraboloid solution of (1.1) in $\mathbb{R}^2$. Then*

$$u_{\gamma P}(x) = p(x) + V_{\gamma P}(x)$$

for all $x \in \mathbb{R}^2$.

The proof of Proposition 2.14 is deferred to Appendix B.

2.4. The thin obstacle problem. Finally, an important role will be played in our analysis by homogeneous global solutions to the thin obstacle problem as these appear as the second blow-down limit. We recall here that for $R > 0$, v is a solution to the thin obstacle problem with zero obstacle in $B_R \subset \mathbb{R}^2$ if

$$(2.4) \quad \begin{aligned} v\Delta v &= 0 \text{ and } \Delta v \leq 0 \text{ in } B_R \\ v &\geq 0 \text{ on } \{x_1 = 0\} \cap B_R \text{ and,} \\ \Delta v &= 0 \text{ on } B_R \setminus \{x_1 = 0\}. \end{aligned}$$

If v solves (2.4) in $\mathbb{R}^2$ we will call v a global solution to the thin obstacle problem.

We now recall the following optimal regularity for solutions to the thin obstacle problem [17, Theorem 9.13]

Theorem 2.15. *Let v be a solution of (2.4). There exists some constant $C = C(R)$ such that*

$$\|v\|_{C^{1, \frac{1}{2}}(\overline{B_{\frac{R}{2}}^+})} \leq C \|v\|_{L^2(B_R^+)}.$$

We have the monotonicity of the Almgren frequency for solutions v of (2.4) [17, Theorem 9.4].

Proposition 2.16. *If v is a solution of (2.4) then $\phi(r, v)$ is monotone non-decreasing for $0 < r < R$. Moreover $\phi(r, v) \equiv \kappa$ for $0 < r < R$ if and only if v is homogeneous of degree κ in B_R .*

The homogeneous global solutions to (2.4) are classified and we summarise these results from [17, Section 9.4.1] in the following Lemma.

Lemma 2.17. *Let v be a κ -homogeneous global solution of (2.4). Then the following hold.*

- i. *The possible values of κ are $1, 2m, 2m - \frac{1}{2}$ or $2m + 1, m \in \mathbb{N}$;*
- ii. *if $\kappa = 2m$ then v must be a $2m$ -homogeneous harmonic polynomial;*
- iii. *if $\kappa = 2m - \frac{1}{2}$ then $v = \operatorname{Re}(x_1 + |x_2|i)^{2m - \frac{1}{2}}$ up to a constant,*
- iv. *and if $\kappa = 1$ or $\kappa = 2m + 1$ then $v = \operatorname{Im}(x_1 + |x_2|i)^{2m + 1}$ up to a constant.*

3. MODIFIED ACF FUNCTIONAL

In this section we establish a modified ACF functional in 2-dimensions.

Theorem 3.1. *Suppose for $i = 1, 2, 3$, $v_i \in C^{1,1}(B_2)$ satisfy $v_i \Delta v_i \geq 0$ and have supports in Ω_i pairwise disjoint. Moreover suppose that $v_i = 0$ on $\partial(\Omega_i \cap B_1)$. Then for each $\beta \leq 9$, the quantity*

$$\Phi(r) = \frac{1}{r^\beta} \prod_{i=1}^3 \int_{B_r} |\nabla v_i|^2 dx$$

is monotone non-decreasing in r , $0 < r < 1$.

Proof. Denoting $\frac{d}{dr} \Phi(r)$ as $\Phi'(r)$, we have

$$\Phi'(r) = -\beta r^{-1} \Phi(r) + r^\beta \sum_{i=1}^3 \int_{\partial B_r} |\nabla v_i|^2 d\sigma \prod_{j \neq i} \int_{B_r} |\nabla v_j|^2.$$

Dividing through by $r^{-1} \Phi(r)$ we have

$$r \frac{\Phi'(r)}{\Phi(r)} = -\beta + r \sum_{i=1}^3 \frac{\int_{\partial B_r} |\nabla v_i|^2 d\sigma}{\int_{B_r} |\nabla v_i|^2},$$

so that by rescaling, we just need to show that the quantity

$$P(\beta) := -\beta + \sum_{i=1}^3 \frac{\int_{\partial B_1} |\nabla v_i|^2 d\sigma}{\int_{B_1} |\nabla v_i|^2}$$

is non-negative. Since $v_i \Delta v_i \geq 0$, integrating by parts yields

$$\int_{B_1} |\nabla v_i|^2 = - \int_{B_1} v_i \Delta v_i + \int_{\partial B_1} v_i (v_i)_\rho d\sigma \leq \int_{\partial B_1} v_i (v_i)_\rho d\sigma,$$

where $(v_i)_\rho$ denotes the exterior radial derivative along ∂B_1 . Applying then Hölder's inequality we have

$$\int_{\partial B_1} v_i (v_i)_\rho d\sigma \leq \left(\int_{\partial B_1} v_i^2 d\sigma \right)^{1/2} \left(\int_{\partial B_1} (v_i)_\rho^2 d\sigma \right)^{1/2}.$$

On the other hand, Young's inequality yields

$$\int_{\partial B_1} |\nabla v_i|^2 \geq 2 \left(\int_{\partial B_1} (v_i)_\rho^2 d\sigma \right)^{1/2} \left(\int_{\partial B_1} (v_i)_\theta^2 d\sigma \right)^{1/2},$$

where $(v_i)_\theta$ is the tangential derivative of v_i along ∂B_1 . We can therefore estimate $P(\beta)$ from below and obtain

$$(3.1) \quad P(\beta) \geq 2 \left(\sum_{i=1}^3 \frac{\left(\int_{\partial B_1} |(v_i)_\theta|^2 d\sigma \right)^{\frac{1}{2}}}{\left(\int_{\partial B_1} |v_i|^2 d\sigma \right)^{\frac{1}{2}}} - \frac{\beta}{2} \right).$$

We must now minimise the sum appearing in the right-hand of (3.1). To this end we let $\Gamma_i = \partial B_1 \cap \Omega_i$ and we recall some remarks from [4, Proof of Theorem 12.1]. Observe that

$$\frac{\int_{\partial B_1} |(v_i)_\theta|^2 d\sigma}{\int_{\partial B_1} |v_i|^2 d\sigma} \geq \inf_{v \in H_0^1(\Gamma_i)} \frac{\int_{\partial B_1} |(v)_\theta|^2 d\sigma}{\int_{\partial B_1} |v|^2 d\sigma}$$

where the right-hand side is the first eigenvalue of the domain Γ_i . Among all possible domains with measure $|\Gamma_i|$, this infimum is attained when the domain is a connected arc¹. Since $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$, and since larger arcs have smaller eigenvalues, we have that the sum in (3.1) will be minimised if the Γ_i are three disjoint arcs whose union is the circle.

Supposing these three arcs have lengths $2\pi\alpha$, $2\pi\gamma$ and $2\pi(1-\alpha-\gamma)$ for some $\alpha, \gamma \in (0, 1)$ satisfying $0 < \alpha + \gamma < 1$, we find that the three principle eigenfunctions are $\sin\left(\frac{\theta}{2\alpha}\right)$, $\sin\left(\frac{\theta}{2\gamma}\right)$ and $\sin\left(\frac{\theta}{2(1-\alpha-\gamma)}\right)$. We therefore have that

$$\sum_{i=1}^3 \frac{\left(\int_{\partial B_1} |(v_i)_\theta|^2 d\sigma\right)^{\frac{1}{2}}}{\left(\int_{\partial B_1} |v_i|^2 d\sigma\right)^{\frac{1}{2}}} \geq \frac{1}{2\alpha} + \frac{1}{2\gamma} + \frac{1}{2(1-\alpha-\gamma)} \geq \frac{9}{2}$$

where in the last step we minimised $\frac{1}{2\alpha} + \frac{1}{2\gamma} + \frac{1}{2(1-\alpha-\gamma)}$ over the domain $\{0 < \alpha < 1\} \cap \{0 < \gamma < 1\} \cap \{0 < \alpha + \gamma < 1\}$. This completes the proof as we then have

$$P(\beta) \geq 9 - \beta,$$

which is non-negative if $\beta \leq 9$. □

4. BLOW-DOWN ANALYSIS

We begin by defining the family of rescalings

$$\tilde{w}_r = \frac{w_r}{\|w_r\|_{L^2(\partial B_1)}}.$$

Thanks to the frequency estimate in Proposition 2.2 there exists a universal constant C such that

$$(4.1) \quad \|\tilde{w}_r\|_{W^{1,2}(B_1)} \leq C \quad \text{for all } r > 0.$$

The following Proposition shows that this can be improved to uniform $W^{1,2}(B_R)$ -bounds for each $R > 1$.

Proposition 4.1. *For each $R > 1$ we have*

$$\|\tilde{w}_r\|_{W^{1,2}(B_R)} \leq C(R)$$

for all $r > 1$.

Proof. We follow [8, Proposition 4.2]. First observe that from the frequency estimate (2.2) for w at radius Rr for every $r > 0$, we obtain

$$\int_{B_R} |\nabla w_r|^2 \leq \frac{2}{R} \int_{\partial B_R} w_r^2 d\sigma.$$

¹Indeed, if Γ_i was not connected then symmetrising v cannot increase the quotient by the Pólya-Szegő inequality.

Consequently for every $R > 1$ and every $r > 0$ we have that

$$\|\tilde{w}_r\|_{W^{1,2}(B_R)} \leq C \frac{\int_{\partial B_R} w_r^2 d\sigma}{\int_{\partial B_1} w_r^2 d\sigma}.$$

Now suppose that for some $R > 1$ there exists a sequence $(r_k)_{k \in \mathbb{N}}$ such that $r_k \rightarrow \infty$ and

$$(4.2) \quad \frac{\int_{\partial B_R} w_{r_k}^2 d\sigma}{\int_{\partial B_1} w_{r_k}^2 d\sigma} \rightarrow +\infty \text{ as } k \rightarrow \infty$$

Defining the sequence

$$\bar{w}_{r_k} = \frac{w_{r_k}}{\|w_{r_k}\|_{L^2(\partial B_R)}},$$

we have that

$$\int_{\partial B_R} |\bar{w}_{r_k}|^2 d\sigma = 1,$$

so that by the frequency estimate (2.2) we obtain

$$(4.3) \quad \int_{B_R} |\nabla \bar{w}_{r_k}|^2 \leq 2R^{-1}.$$

Along with Lemma 2.8, (4.3) implies that there exists some $\bar{w} \in W^{1,2}(B_R)$ such that, up to taking a subsequence which we do not relabel, we have

$$\begin{aligned} \bar{w}_{r_k} &\rightarrow \bar{w} \text{ weakly in } W^{1,2}(B_R), \\ \bar{w}_{r_k} &\rightarrow \bar{w} \text{ in } L^2(\partial B_R), \\ \bar{w}_{r_k} &\rightarrow \bar{w} \text{ in } C_{\text{loc}}^0(B_R). \end{aligned}$$

Moreover, since $\Delta w_{r_k} = -\chi_{\{u_{r_k}=0\}} < 0$ we have for each $\alpha \in (0, 1)$ and each $\eta \in C_c^\infty(B_R)$ satisfying $\eta \geq 0$ and $\eta \equiv 1$ on $B_{\alpha R}$, that

$$\int_{B_{\alpha R}} |\Delta \bar{w}_{r_k}| \leq \int_{B_R} \eta |\Delta \bar{w}_{r_k}| = - \int_{B_R} \eta \Delta \bar{w}_{r_k} \leq C(\alpha) \|\nabla \bar{w}_{r_k}\|_{L^2(B_R)}.$$

Hence $\Delta \bar{w}_{r_k}$ converges locally to $\Delta \bar{w}$ as measures and in particular, $\Delta \bar{w}$ is a non-positive measure supported on $\{te^2 : t \geq 0\} \cap B_{\theta R}$ for each $\theta \in (0, 1)$.

We will now show that $\bar{w} \equiv 0$ on B_1 . Firstly, the locally uniform convergence implies that $\bar{w} = 0$ on $\{te^2 : 0 \leq t \leq 1\}$. Moreover, by (4.2), we also have that

$$\int_{\partial B_1} \bar{w}_{r_k}^2 d\sigma = \frac{\int_{\partial B_1} w_{r_k}^2 d\sigma}{\int_{\partial B_R} w_{r_k}^2 d\sigma} \rightarrow 0,$$

and so $\bar{w} \equiv 0$ on ∂B_1 . Since $\bar{w}$ is harmonic in the set $\Omega = B_1 \setminus \{te^2 : 0 \leq t \leq 1\}$ while being zero on $\partial\Omega$, we must have that $\bar{w} \equiv 0$ in B_1 .

Now, since $\bar{w}$ is analytic in $B_R \setminus \{te^2 : 0 \leq t \leq R\}$ and zero on B_1 , we conclude that $\bar{w} \equiv 0$ in B_R . This contradicts the fact that

$$\int_{\partial B_R} \bar{w}^2 d\sigma = \lim_{k \rightarrow \infty} \int_{\partial B_R} \bar{w}_{r_k}^2 d\sigma = 1.$$

□

Proposition 4.2. *For each sequence $(r_k)_{k \in \mathbb{N}}$ such that $r_k \rightarrow \infty$ as $k \rightarrow \infty$, there exists a subsequence, which we do not relabel, such that*

$$\tilde{w}_{r_k} \rightarrow v_0 \text{ in } W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2),$$

where v_0 is a global solution to the thin obstacle problem (2.4) (which may depend on the sequence) satisfying

$$(4.4) \quad \|v_0\|_{L^2(\partial B_1)} = 1,$$

$$(4.5) \quad \text{supp}(\Delta v_0) \subset \{x_1 = 0\} \cap \{x_2 \geq 0\},$$

$$(4.6) \quad \Delta v_0 \leq 0,$$

$$(4.7) \quad \partial_2 v_0 \leq 0,$$

and

$$(4.8) \quad v_0 \leq 0 \text{ in } \{x_2 \geq 0\}.$$

Moreover,

$$(4.9) \quad \phi(R, v_0) \leq 2$$

for every $R > 0$.

Proof. Step 1: Convergence to a global solution of the thin obstacle problem:

Fixing $R > 1$ we have by Proposition 4.1 and Lemma 2.8, up to a subsequence which we do not relabel, that

$$\begin{aligned} \tilde{w}_{r_k} &\rightarrow v_0 \text{ weakly in } W^{1,2}(B_R), \\ \tilde{w}_{r_k} &\rightarrow v_0 \text{ in } L^2(\partial B_R), \\ \tilde{w}_{r_k} &\rightarrow v_0 \text{ in } C_{\text{loc}}^0(B_R), \end{aligned}$$

for some $v_0 \in W^{1,2}(B_R)$.

As in the proof of Proposition 4.1 we have that Δv_0 is a non-positive measure supported on $\{te^2 : 0 \leq t \leq \frac{R}{2}\}$ while $v_0 = 0$ on $\{te^2 : 0 \leq t \leq \frac{R}{2}\}$. Moreover, $v_0 \geq 0$ on $\{x_1 = 0\} \cap B_{\frac{R}{2}}$. We therefore conclude that

$$\begin{aligned} v_0 \Delta v_0 &= 0 \text{ and } \Delta v_0 \leq 0 \text{ in } B_{\frac{R}{2}} \\ v_0 &\geq 0 \text{ on } \{x_1 = 0\} \cap B_{\frac{R}{2}} \\ \Delta v_0 &= 0 \text{ on } B_{R/2} \setminus (\{x_1 = 0\} \cap \{x_2 \geq 0\}), \end{aligned}$$

that is, v_0 is a solution to the thin-obstacle problem in $B_{\frac{R}{2}}$ with thin obstacle $\{x_1 = 0\} \cap B_{\frac{R}{2}}$.

This establishes weak $W_{\text{loc}}^{1,2}(\mathbb{R}^2)$ and $C_{\text{loc}}^0(\mathbb{R}^2)$ convergence to some v_0 that is a global solution of (2.4). Furthermore, (4.4), (4.5) and (4.6) are immediate, while (4.9) follows from Proposition 2.2. Moreover, since $\partial_2 w = \partial_2 u \leq 0$ we immediately obtain (4.7) by the weak convergence in $W_{\text{loc}}^{1,2}$. Finally we recall that $\partial_{11} w \leq 0$ by Remark 2.9, and

hence $w \leq 0$ in $\{x_2 \geq 0\}$ (since $w = -\frac{1}{2}x_1^2$ on $\{u = 0\}$ and $\{u = 0\}$ has non-empty interior). This together with the C_{loc}^0 convergence proves (4.8).

Step 2: Strong $W_{\text{loc}}^{1,2}$ convergence:

In this step we will improve the weak $W^{1,2}(B_R)$ convergence to strong $W^{1,2}(B_{\frac{R}{2}})$ convergence. To this end we let $(\eta_j)_{j \in \mathbb{N}}$ be a smooth approximation of $\chi_{B_{\frac{R}{2}}}$ in B_R , i.e. $\eta_j \in C_c^\infty(B_R)$, $\eta_j \equiv 1$ in $B_{\frac{R}{2}}$ and $\eta_j \rightarrow \chi_{B_{\frac{R}{2}}}$ pointwise. We then observe that since $w\Delta w \geq 0$ we obtain

$$\begin{aligned} \int_{B_{\frac{R}{2}}} |\nabla \tilde{w}_{r_k}|^2 &\leq \int_{B_R} \eta_j |\nabla \tilde{w}_{r_k}|^2 \\ &= - \int_{B_R} (\tilde{w}_{r_k} \nabla \eta_j \cdot \nabla \tilde{w}_{r_k} + \eta_j \tilde{w}_{r_k} \Delta \tilde{w}_{r_k}) \\ &\leq - \int_{B_R} \tilde{w}_{r_k} \nabla \eta_j \cdot \nabla \tilde{w}_{r_k}. \end{aligned}$$

Now passing the limit as $k \rightarrow \infty$ we find that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{B_{\frac{R}{2}}} |\nabla \tilde{w}_{r_k}|^2 &\leq - \int_{B_R} v_0 \nabla \eta_j \cdot \nabla v_0 \\ &= \int_{B_R} (\eta_j |\nabla v_0|^2 + \eta_j v_0 \Delta v_0) \\ &= \int_{B_R} \eta_j |\nabla v_0|^2. \end{aligned}$$

Now passing $j \rightarrow \infty$ we obtain

$$\limsup_{k \rightarrow \infty} \int_{B_{\frac{R}{2}}} |\nabla \tilde{w}_{r_k}|^2 \leq \int_{B_{\frac{R}{2}}} |\nabla v_0|^2,$$

so that by lower semicontinuity of the Dirichlet energy we have

$$\lim_{k \rightarrow \infty} \int_{B_{\frac{R}{2}}} |\nabla \tilde{w}_{r_k}|^2 = \int_{B_{\frac{R}{2}}} |\nabla v_0|^2.$$

Since for each $B_R \subset \mathbb{R}^2$ we have that $W^{1,2}(B_R) \subset W^{1,\frac{3}{2}}(B_R)$ by Hölder's inequality, and $W^{1,\frac{3}{2}}(B_R)$ embeds compactly in $L^2(B_R)$ [10, Section 5.7, Theorem 1], the strong $W_{\text{loc}}^{1,2}$ convergence follows. $\square$

At this point it is difficult to conclude precisely what the blow-down is, or even whether the blow-down obtained depends on the chosen blow-down sequence or not. To overcome this issue, we will now blow-down the blow-down limits obtained in Proposition 4.2 and classify these first. To this end we define, in polar coordinates, the function

$$(4.10) \quad \hat{v}_{3/2}(r, \theta) = -r^{3/2} \cos(\theta) \left\| r^{3/2} \cos(\theta) \right\|_{L^2(\partial B_1)}^{-1}$$

Note that $\hat{v}_{3/2}$ is the $3/2$ -homogeneous global solution to the thin obstacle problem with coincidence set $\{x_1 = 0\} \cap \{x_2 \geq 0\}$ and $\|\hat{v}_{3/2}\|_{L^2(\partial B_1)} = 1$.

Proposition 4.3. *Given a sequence $(r_k)_{k \in \mathbb{N}}$ going to infinity and v_0 the corresponding blow-down limit obtained in Proposition 4.2, we define the family of rescalings*

$$(\tilde{v}_0)_s = \frac{v_0(s)}{\|(v_0)_s\|_{L^2(\partial B_1)}}.$$

For each sequence $(s_k)_{k \in \mathbb{N}}$ such that $s_k \rightarrow \infty$ as $k \rightarrow \infty$, we have up to a subsequence which we do not relabel, that

$$(\tilde{v}_0)_{s_k} \rightarrow \hat{v}_{3/2}$$

weakly in $W_{\text{loc}}^{1,2}(\mathbb{R}^2)$ and strongly in $C_{\text{loc}}^1(\overline{\mathbb{R}_+^2})$.

Proof. Since the proof is carried out along the same lines as Propositions 4.1 and 4.2 we will only point out the differences.

Step 1: Uniform $W^{1,2}(B_R)$ bounds for each $R > 1$:

This step is the same as Proposition 4.1 except we do not obtain C_{loc}^0 convergence from Lemma 2.8. Rather, from Theorem 2.15 applied in $\overline{B_R^\pm}$, we obtain convergence in $C_{\text{loc}}^1(\overline{\mathbb{R}_+^2})$.

Step 2: Convergence to some blow-down limit:

Arguing as in Step 1 of Proposition 4.2 (again with Lemma 2.8 replaced by Theorem 2.15) we obtain weak $W_{\text{loc}}^{1,2}(\mathbb{R}^2)$, strong $L^2(\partial B_1)$ and strong $C_{\text{loc}}^1(\overline{\mathbb{R}_+^2})$ convergence to some limit $v_{00} \in W_{\text{loc}}^{1,2}(\mathbb{R}^2)$. Moreover,

$$(4.11) \quad \|v_{00}\|_{L^2(\partial B_1)} = 1,$$

$$(4.12) \quad \text{supp}(\Delta v_{00}) \subset \{x_1 = 0\} \cap \{x_2 \geq 0\},$$

$$(4.13) \quad \Delta v_{00} \leq 0,$$

$$(4.14) \quad \partial_2 v_{00} \leq 0,$$

$$(4.15) \quad v_{00} \leq 0 \quad \text{in } \{x_2 \geq 0\},$$

and

$$(4.16) \quad \phi(R, v_{00}) \leq 2 \quad \text{for all } R > 0,$$

are direct consequences of (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9).

Step 3: Classification of the limits v_{00} :

The first observation is that v_{00} is homogeneous. Indeed using the monotonicity of the Almgren frequency, Proposition 2.16, as well as the convergence in $C_{\text{loc}}^1(\overline{\mathbb{R}_+^2})$, we see that for each $R > 0$,

$$\phi(R, v_{00}) = \lim_{k \rightarrow \infty} \phi(Rs_k, v_0) = \phi(+\infty, v_0).$$

In this last step we used the monotonicity of $\phi(\rho, v_0)$ in ρ (Proposition 2.16) and (4.16) to see that the limit $\phi(+\infty, v_0)$ is well defined. Using Proposition 2.16 and

(4.16) again, we see that v_{00} is a homogeneous function and $\phi(R, v_{00}) \leq 2$ for all $R > 0$. By Lemma 2.17, we then have that the frequency is either $2, \frac{3}{2}$ or 1 .

If the frequency was 2 then, by Lemma 2.17, v_{00} would be a 2 -homogeneous harmonic polynomial vanishing along $\{x_1 = 0\}$. However, in two dimensions the only 2 -homogeneous harmonic polynomials satisfying this condition are multiples of $x_1 x_2$. Clearly, (4.14) excludes v_{00} from being a multiple of $x_1 x_2$ and so the frequency cannot be 2 .

If on the other hand the frequency was 1 , then again by Lemma 2.17, v_{00} would be a 1 -homogeneous harmonic polynomial in both $\{x_1 \geq 0\}$ and $\{x_1 \leq 0\}$ vanishing along $\{x_1 = 0\}$. In this case, taking into account (4.13), the options are $\pm x_1$ or $-|x_1|$ which both cannot occur. Indeed $\pm x_1$ violates (4.15), while $-|x_1|$ does not satisfy (4.12).

Since the blow-down cannot be identically zero by (4.11), we conclude that v_{00} is the $3/2$ homogeneous solution $\hat{v}_{3/2}$ defined in (4.10). $\square$

Proposition 4.4. *For each $\varepsilon > 0$ there exists an $r_1(\varepsilon)$ such that*

$$(4.17) \quad \frac{3}{2} - \varepsilon \leq \phi(r, w) \leq \frac{3}{2} + \varepsilon$$

for $r \geq r_1(\varepsilon)$.

Proof. For each $\varepsilon > 0$ we first fix $s_0 = s_0(\varepsilon) \geq 1$ large enough so that

$$(4.18) \quad \left| \phi(s, v_0) - \frac{3}{2} \right| < \frac{\varepsilon}{2}$$

for every $s \geq s_0$. We will show the following claim.

Claim : There exists an $r_0 = r_0(\varepsilon) \geq 1$ such that

$$(4.19) \quad \frac{3}{2} - \varepsilon < \phi(s_0 r, w) < \frac{3}{2} + \varepsilon$$

for every $r \geq r_0$.

The Proposition then follows from the claim by taking $r_1(\varepsilon) = s_0 r_0$.

Proof of Claim: We will prove that the lower bound in (4.19) must hold, and then an identical argument can be used to show that the upper bound must hold too. Suppose that for some $\varepsilon > 0$ the lower bound in (4.19) does not hold. Then for every $k \in \mathbb{N}$ there exists $r_k \geq k$ such that

$$\phi(s_0 r_k, w) < \frac{3}{2} - \varepsilon.$$

Then letting $k \rightarrow \infty$ we find that

$$\phi(s_0, v_0) \leq \frac{3}{2} - \varepsilon,$$

which is a contradiction to (4.18). $\square$

$\square$

Proposition 4.4 tells us that $\lim_{r \rightarrow \infty} \phi(r, w) = \frac{3}{2}$ and so we can now use this to classify the blow-down limits obtained in Proposition 4.2.

Corollary 4.5. *Given a sequence $(r_k)_{k \in \mathbb{N}}$ going to infinity and v_0 the corresponding blow-down limit obtained in Proposition 4.2 we have that $v_0 = \hat{v}_{3/2}$.*

Proof. By Proposition 4.4 we now have for each $R > 0$ that

$$\phi(R, v_0) = \lim_{k \rightarrow \infty} \phi(Rr_k, w) = \frac{3}{2},$$

so that v_0 is $\frac{3}{2}$ -homogeneous. Arguing as in Step 3 of Proposition 4.3 we conclude that $v_0 = \hat{v}_{3/2}$. $\square$

5. GROWTH ESTIMATES FOR THE COINCIDENCE SET

We will begin with the following $3/2$ -order doubling.

Lemma 5.1. *Given $\lambda \in (0, 1)$ there exists an $\bar{r} = \bar{r}(\lambda)$ so that*

$$(5.1) \quad \|w_r\|_{L^2(\partial B_1)} \leq C(\lambda) r^{\frac{3}{2} + \lambda}$$

for all $r \geq \bar{r}(\lambda)$. As a consequence,

$$(5.2) \quad C \cap \{y_2 \geq \bar{r}(\lambda)\} \subset \{|y_1|^2 \leq y_2^{\frac{3}{2} + \lambda}\}$$

Proof. Firstly (5.2) follows from (5.1) as in [8, Proposition 5.1]. To obtain (5.1) we observe that as $r \rightarrow \infty$ we have

$$\begin{aligned} \frac{\int_{\partial B_1} w_{2r}^2 d\sigma}{\int_{\partial B_1} w_r^2 d\sigma} &= \frac{1}{2} \frac{\int_{\partial B_2} w_r^2 d\sigma}{\int_{\partial B_1} w_r^2 d\sigma} \\ &= \frac{1}{2} \int_{\partial B_2} \tilde{w}_r^2 d\sigma \\ &\rightarrow \frac{1}{2} \int_{\partial B_2} v^2 d\sigma \\ &= \frac{1}{2} \frac{\int_{\partial B_2} (2)^3 \sin^2(\frac{3}{2}\theta) d\sigma}{\int_{\partial B_1} (1)^3 \sin^2(\frac{3}{2}\theta) d\sigma} \\ &= 8. \end{aligned}$$

Then denoting $f(r) := \|w_r\|_{L^2(\partial B_1)}$ we have that for all $\lambda > 0$ there exists an $r(\lambda) < +\infty$ so that

$$f(2r) \leq 2^{\frac{3}{2} + \lambda} f(r), \text{ for all } r \geq r(\lambda).$$

Iterating this we find that

$$f(r) \leq C(\lambda) r^{\frac{3}{2} + \lambda} \text{ for all } r \geq r(\lambda).$$

$\square$

We can now show the following almost optimal growth estimate on the coincidence set.

Proposition 5.2. *Given $\delta \in (0, 1)$ there exists an $r_2(\delta) > 1$ so that*

$$(5.3) \quad C \cap \{y_2 \geq r_2(\delta)\} \subset \{|y_1|^2 \leq y_2^{1+\delta}\}$$

and

$$(5.4) \quad C \cap \{y_2 \leq r_2(\delta)\} \subset \{-r_2(\delta)^{\frac{1+\delta}{2}} \leq y_1 \leq r_2(\delta)^{\frac{1+\delta}{2}}\} \cap \{0 \leq y_2 \leq r_2(\delta)\}$$

Proof. We first note that (5.4) follows immediately from (5.3) and the fact that $\partial_2 u \leq 0$. We will now prove (5.3) by contradiction. If (5.3) did not hold then there exists some $\delta \in (0, 1)$ such that for every $k \in \mathbb{N}$ there exists some point $x^k \in C$ such that $x_2^k \geq k$ and $|x_1^k| \geq (x_2^k)^{\frac{1+\delta}{2}}$. With no loss of generality we can take a subsequence, that we do not relabel, such that $x_1^k \geq (x_2^k)^{\frac{1+\delta}{2}}$.

We define $\rho_k = x_2^k$ and observe that since $w(x) = -\frac{1}{2}x_1^2$ on C we have that

$$w(x^k) = -\frac{1}{2}(x_1^k)^2 \leq -\frac{1}{2}\rho_k^{1+\delta}$$

and

$$\partial_1 w(x^k) = -x_1^k \leq -\rho_k^{\frac{1+\delta}{2}}.$$

Now since $\partial_{11} w \leq 0$ (c.f. Remark 2.9) we have using the above estimates, as well as (5.2) for k large enough, that

$$|w(\rho_k, \rho_k)| \geq c(\delta)\rho_k^{\frac{3+\delta}{2}}.$$

Now using (5.1) with $\lambda = \frac{\delta}{4}$ we have for k large enough that

$$|\tilde{w}(\rho_k, \rho_k)| = \frac{|w(\rho_k, \rho_k)|}{\|w_{\rho_k}\|_{L^2(\partial B_1)}} \geq c(\delta)\rho_k^{\frac{\delta}{4}},$$

which contradicts the pointwise convergence of $|\tilde{w}(\rho_k, \rho_k)| \rightarrow |v(1, 1)| < +\infty$. $\square$

6. SHARP GROWTH ESTIMATES OF $u - p$

We will define

$$H(r, w) = r^{1-n} \int_{\partial B_r} w^2 d\sigma = \int_{\partial B_1} w_r^2 d\sigma,$$

and

$$D(r, w) = r^{2-n} \int_{B_r} |\nabla w|^2 = \int_{B_1} |\nabla w_r|^2,$$

so that the Almgren frequency is

$$\phi(r, w) = \frac{D(r, w)}{H(r, w)}.$$

We will now collect some known differential identities on the quantities $H(r, w)$ and $\phi(r, w)$.

Lemma 6.1. *For each $r > 0$ we have that*

$$(6.1) \quad \int_{\partial B_1} w_r(w_r)_\nu d\sigma = \int_{B_1} |\nabla w_r|^2 + \int_{B_1} w_r \Delta w_r.$$

Consequently,

$$(6.2) \quad \frac{d}{dr} H(r, w) = \frac{2}{r} \int_{\partial B_1} w_r(w_r)_\nu d\sigma \geq 0$$

and

$$(6.3) \quad \frac{d}{dr} (\log(H(r, w))) = \frac{2}{r} \left(\phi(r, w) + \frac{\int_{B_1} w_r \Delta w_r}{H(r, w)} \right).$$

Proof. Integrating by parts the quantity $\int_{B_1} |\nabla w_r|^2$ we immediately obtain (6.1). A direct computation yields the identity in (6.2) while (6.1) together with Remark 2.9 establishes the non-negativity of $\frac{d}{dr} H(r, w)$. Finally, (6.3) follows from both the identity in (6.2) and (6.1) after dividing through by $H(r, w)$. $\square$

Lemma 6.2. *For each $r > 0$ we have*

$$(6.4) \quad \frac{d}{dr} (\phi(r, w)) \geq -2 \frac{r^3}{H(r, w)} \int_{B_1 \cap \{u_r=0\}} x_1^2.$$

Proof. The proof is also a direct computation. Using the notation $\partial_r = \frac{d}{dr}$, we first observe that

$$\begin{aligned} \frac{1}{2} H(r, w)^2 \partial_r \left(\frac{D(r, w)}{H(r, w)} \right) &= \int_{B_1} \nabla w_r \cdot \nabla (\partial_r w_r) \int_{\partial B_1} w_r^2 d\sigma - \int_{B_1} |\nabla w_r|^2 \int_{\partial B_1} w_r \partial_r w_r d\sigma \\ &= \int_{\partial B_1} w_r^2 d\sigma \left\{ - \int_{B_1} \partial_r w_r \Delta w_r + \int_{\partial B_1} \partial_r w_r (x \cdot \nabla w_r) d\sigma \right\} \\ &\quad - \int_{\partial B_1} w_r \partial_r w_r d\sigma \left\{ - \int_{B_1} w_r \Delta w_r + \int_{\partial B_1} w_r (x \cdot \nabla w_r) d\sigma \right\}. \end{aligned}$$

Now since $x \cdot \nabla w_r = r \partial_r w_r$, we have that

$$\begin{aligned} \frac{1}{2} H(r, w)^2 \partial_r \left(\frac{D(r, w)}{H(r, w)} \right) &= r \left\{ \int_{\partial B_1} w_r^2 d\sigma \int_{\partial B_1} (\partial_r w_r)^2 d\sigma - \left(\int_{\partial B_1} w_r \partial_r w_r d\sigma \right)^2 \right\} \\ &\quad + \int_{B_1} w_r \Delta w_r \int_{\partial B_1} w_r \partial_r w_r d\sigma - \int_{\partial B_1} w_r^2 d\sigma \int_{B_1} \partial_r w_r \Delta w_r. \end{aligned}$$

The bracket on the right is non-negative due to the Cauchy-Schwarz inequality while the second term is non-negative thanks to (6.2) and Remark 2.9. Finally, since $\Delta w_r = -r^2 \chi_{\{u_r=0\}}$, we have that $\partial_r w_r \Delta w_r = r^3 x_1^2 \chi_{\{u_r=0\}}$. Thanks to these considerations we find that

$$\frac{1}{2} H(r, w)^2 \partial_r \left(\frac{D(r, w)}{H(r, w)} \right) \geq -r^3 H(r, w) \int_{B_1 \cap \{u_r=0\}} x_1^2$$

which establishes 6.4. $\square$

We can now give the almost monotonicity of the Almgren frequency. We recall that given $\varepsilon > 0$ and $\delta > 0$ the quantities $r_1(\varepsilon)$ and $r_2(\delta)$ are obtained from Proposition 4.4 and Proposition 5.2 respectively. From this point onwards we will use the fact that $n = 2$.

Lemma 6.3. *Let $\varepsilon = \delta = \frac{1}{16}$ and define $r_0 = \max\{r_1(\varepsilon), r_2(\delta)\}$. There exists a constant $C = C(r_0)$ so that*

$$\frac{d}{dr}(\phi(r, w) - Cr^{-\frac{1}{4}}) \geq 0$$

for $r \geq r_0$.

Proof. The result will follow from Lemma 6.2 after estimating $H(r, w)$ from below and $\int_{B_1 \cap \{u_r=0\}} x_1^2$ from above.

Step 1: Control of $H(r, w)$ from below:

Since $w\Delta w \geq 0$ (c.f. Remark 2.9) we find using (6.3) and (4.17) that

$$\frac{d}{dr}(\log(H(r, w))) \geq \frac{2}{r} \left(\frac{3}{2} - \varepsilon \right)$$

for $r \geq r_0$. Integrating this identity from r_0 to r we find

$$\int_{r_0}^r \frac{d}{d\rho}(\log(H(\rho, w))) d\rho \geq \int_{r_0}^r \frac{2}{\rho} \left(\frac{3}{2} - \varepsilon \right) d\rho$$

so that

$$(6.5) \quad H(r, w) \geq Cr^{3-2\varepsilon}.$$

Step 2: Controlling $\int_{B_1 \cap \{u_r=0\}} x_1^2$ from above:

By Proposition 5.2 we have

$$\begin{aligned} \int_{B_1 \cap \{u_r=0\}} x_1^2 &\leq \int_{\{-r_0^{\frac{1+\delta}{2}} r^{-1} \leq x_1 \leq r_0^{\frac{1+\delta}{2}} r^{-1}\} \cap \{0 \leq x_2 \leq \frac{r_0}{r}\}} x_1^2 dx \\ &\quad + \int_{B_1 \cap \{|x_1|^2 \leq r^{\delta-1} x_2^{1+\delta}\} \cap \{\frac{r_0}{r} \leq x_2 \leq 1\}} x_1^2 dx \\ &\leq Cr^{-4} + \int_{\frac{r_0}{r}}^1 \int_{-r^{\frac{\delta-1}{2}} x_2^{\frac{1+\delta}{2}}}^{r^{\frac{\delta-1}{2}} x_2^{\frac{1+\delta}{2}}} x_1^2 dx_1 dx_2 \\ &\leq Cr^{\frac{3}{2}(\delta-1)}. \end{aligned}$$

Step 3: Controlling $\phi(r, w)$ from below:

By (6.4), Step 1, and Step 2, we have that

$$\frac{d}{dr}(\phi(r, w)) \geq -Cr^{\frac{3}{2}(\delta-1)+2\varepsilon},$$

and since $\varepsilon = \delta = \frac{1}{16}$ we have that $\frac{3}{2}\delta + 2\varepsilon \leq \frac{1}{4}$ which concludes the proof. $\square$

The main result of this section is the following sharp growth estimate on $w = u - p$.

Proposition 6.4. *Let ε, δ and r_0 be as in Lemma 6.3. There exists a constant $C = C(r_0)$ such that*

$$(6.6) \quad H(r, w) \leq Cr^3$$

for $r \geq r_0$.

Proof. We first observe that using (6.5) we obtain (as in Step 2 of the previous proof) that

$$\frac{1}{r} \frac{\int_{B_1} w_r \Delta w_r}{H(r, w)} \leq Cr^{2\varepsilon} \int_{B_1 \cap \{u_r=0\}} x_1^2 \leq Cr^{-\frac{5}{4}}$$

for $r \geq r_0$. Using now (6.3) and Lemma 6.3 we find

$$\begin{aligned} \frac{d}{dr}(\log(H(r, w))) &\leq \frac{2}{r} \left(\phi(r, w) - Cr^{-\frac{1}{4}} \right) + Cr^{-\frac{5}{4}} \\ &\leq \frac{3}{r} + Cr^{-\frac{5}{4}} \end{aligned}$$

where in the last step we used that $\lim_{r \rightarrow \infty} \phi(r, w) = \frac{3}{2}$. Integrating this we obtain the result. $\square$

We have the following immediate consequence.

Corollary 6.5. *There exists a constant $C = C(r_0)$ such that for $r \geq r_0$ we have*

$$(6.7) \quad \left(\int_{B_r} w^2 \right)^{\frac{1}{2}} \leq Cr^{\frac{3}{2}}.$$

Proof. We observe by integrating that

$$\begin{aligned} \int_{B_r} w^2 &= \frac{1}{|B_r|} \int_0^r \int_{\partial B_\rho} w^2 d\sigma d\rho \\ &= \frac{1}{|B_r|} \int_0^{r_0} \int_{\partial B_\rho} w^2 d\sigma d\rho + \frac{1}{|B_r|} \int_{r_0}^r \int_{\partial B_\rho} w^2 d\sigma d\rho \\ &\leq \frac{1}{|B_r|} \int_{B_{r_0}} w^2 + \frac{C}{|B_r|} \int_{r_0}^r \rho^{n-1+3} d\rho \\ &\leq Cr^3 \end{aligned}$$

which establishes (6.7). $\square$

7. REFINED BLOW-DOWN AND MATCHING

In this section we use the sharp growth estimates obtained for a solution u of (1.1) to show that the blow-down sequence, $r^{-\frac{3}{2}}(u - p)(r \cdot)$, has a limit as $r \rightarrow \infty$. We will then find a paraboloid solution such that its blow-down limit coincides with that of u . As before we set $w = u - p$ and we recall that $\hat{v}_{3/2}$ is defined in (4.10) and is the $\frac{3}{2}$ -homogeneous solution to the thin obstacle problem

Proposition 7.1. *Given a sequence $(r_k)_{k \in \mathbb{N}}$ such that $r_k \rightarrow \infty$ there exists a subsequence and some constant (that may depend on the sequence) $\alpha_u > 0$ such that*

$$(7.1) \quad \frac{w(r_k x)}{r_k^{\frac{3}{2}}} \rightarrow \alpha_u v \quad \text{in } W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2).$$

Proof. The proof of (7.1) follows the same lines as that of Proposition 4.2, however, in this case the uniform $W^{1,2}$ bounds follow from the sharp growth estimates on $u - p$. Precisely, since p is itself a global solution to the obstacle problem we can apply Lemma 2.3 and then (6.7) to find for each $R > 1$ and each $r > 0$ that, setting $\hat{w}_r(x) := r^{-3/2}w(rx)$,

$$\int_{B_R} |\nabla \hat{w}_r|^2 \leq \frac{C}{R^2} \int_{B_{2R}} (r^{-3/2}w(r \cdot))^2 = \frac{C}{r^3 R^2} \int_{B_{2Rr}} w^2 \leq CR.$$

Moreover, from (6.7) we have for each $R > 1$ that

$$(7.2) \quad \int_{B_R} \hat{w}_r^2 \leq CR^3 \quad \text{for all } r > 0,$$

so that

$$\|\hat{w}_r\|_{W^{1,2}(B_R)} \leq C(R) \quad \text{for all } r > 0.$$

Arguing then as in the proof of Proposition 4.2 we find, up to a subsequence, that $\hat{w}_{r_k} \rightarrow v_0$ in $W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2)$, where v_0 is a global solution to the thin obstacle problem (2.4), that may depend on the sequence chosen. However, since $\lim_{r \rightarrow \infty} \phi(r, w) = \frac{3}{2}$ by Proposition 4.4, we can argue as in the derivation of Corollary 4.5 to find that $\hat{w}_{r_k} \rightarrow \alpha_u \hat{v}_{3/2}$ in $W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2)$ where $\alpha_u := \lim_{k \rightarrow \infty} \|\hat{w}_{r_k}\|_{L^2(\partial B_1)}$. $\square$

In order to complete the matching we must show uniqueness of the blow-down limits obtained in Proposition 7.1. This will be achieved using the following monotonicity formula which appears in [12, Lemma 5.3 and Lemma B.1] for odd integer homogeneous solutions to the thin obstacle problem.

Lemma 7.2. *For each $r > 0$ we have that*

$$\frac{d}{dr} \left(\frac{1}{r^{\frac{3}{2}}} \int_{\partial B_1} \hat{v}_{3/2} w_r d\sigma \right) \geq 0.$$

Proof. We first observe that by integrating by parts twice $\int_{B_1} w_r \Delta v$, we have

$$\int_{\partial B_1} \hat{v}_{3/2} \partial_\nu w_r d\sigma = - \int_{B_1} w_r \Delta \hat{v}_{3/2} + \int_{\partial B_1} w_r \partial_\nu \hat{v}_{3/2} d\sigma + \int_{B_1} \hat{v}_{3/2} \Delta w_r.$$

Now since $\hat{v}_{3/2}$ is $\frac{3}{2}$ -homogeneous we have that $\partial_\nu \hat{v}_{3/2} = \frac{3}{2} \hat{v}_{3/2}$ on ∂B_1 . Moreover, since $\hat{v}_{3/2} \leq 0$ for $x_2 \geq 0$ we have that $\hat{v}_{3/2} \chi_{\{u_r=0\}} \leq 0$ for every $r > 0$. Finally, $w_r \Delta \hat{v}_{3/2} = 0$

since $w_r = 0$ on $\text{supp}(\Delta \hat{v}_{3/2}) = \{x_1 = 0, x_2 \geq 0\}$. Altogether we obtain

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{r^{\frac{3}{2}}} \int_{\partial B_1} \hat{v}_{3/2} w_r d\sigma \right) &= -\frac{3}{2} r^{-\frac{5}{2}} \int_{\partial B_1} \hat{v}_{3/2} w_r d\sigma \\ &\quad + r^{-\frac{5}{2}} \left(\frac{3}{2} \int_{\partial B_1} \hat{v}_{3/2} w_r d\sigma - \int_{B_1} w_r \Delta \hat{v}_{3/2} + \int_{B_1} \hat{v}_{3/2} \Delta w_r \right) \\ &= -r^{-\frac{1}{2}} \int_{B_1} \hat{v}_{3/2} \chi_{\{u_r=0\}} \geq 0. \end{aligned}$$

□

Proposition 7.3. *There exists a unique constant α_u such that*

$$\lim_{r \rightarrow \infty} \frac{w(rx)}{r^{\frac{3}{2}}} \rightarrow \alpha_u \hat{v}_{3/2} \quad \text{in } W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2).$$

Proof. Suppose there exists two different accumulation points along sequences $r_k^{(i)}$ for $i = 1, 2$. That is, there exists constants $\alpha^{(i)}$ such that

$$\alpha^{(i)} \hat{v}_{3/2} = \lim_{k \rightarrow \infty} \frac{w(r_k^{(i)} x)}{(r_k^{(i)})^{\frac{3}{2}}}$$

for $i = 1, 2$. Supposing that $\alpha_u^{(1)} > \alpha_u^{(2)}$ we have that $r \mapsto r^{-\frac{3}{2}} \int_{\partial B_1} w_r (\alpha_u^{(1)} - \alpha_u^{(2)}) \hat{v}_{3/2}$ is monotone non-decreasing by Lemma 7.2. Moreover by Hölder's inequality and (7.2) it is bounded from above and hence it has a unique limit as $r \rightarrow \infty$. This implies that

$$\int_{\partial B_1} \alpha_u^{(1)} (\alpha_u^{(1)} - \alpha_u^{(2)}) \hat{v}_{3/2}^2 d\sigma = \int_{\partial B_1} \alpha_u^{(2)} (\alpha_u^{(1)} - \alpha_u^{(2)}) \hat{v}_{3/2}^2 d\sigma.$$

Re-arranging this we obtain

$$\int_{\partial B_1} (\alpha_u^{(1)} - \alpha_u^{(2)})^2 \hat{v}_{3/2}^2 d\sigma = 0$$

and so $\alpha_u^{(1)} = \alpha_u^{(2)}$. □

When $u_{\gamma P}$ is a paraboloid solution as constructed in Proposition 2.11 we will denote the blow-down limit as $\alpha_\gamma v$. As a consequence of the uniqueness of the blow-down limit, we have the following identity for α_γ .

Corollary 7.4. *Let $u_{\gamma P}$ be a paraboloid solution and $\alpha_\gamma v$ its blow-down. Then*

$$\alpha_\gamma = \gamma^{\frac{1}{2}} \alpha_1.$$

Proof. We use Proposition 2.14 and Lemma 2.13 to see that

$$r^{-\frac{3}{2}} (u_{\gamma P} - p)(rx) = r^{-\frac{3}{2}} V_{\gamma P}(rx) = \gamma^2 r^{-\frac{3}{2}} V_P\left(\frac{1}{\gamma} rx\right).$$

Using Proposition 8.1 we pass to the limit as $r \rightarrow \infty$ in the above and find that

$$\alpha_\gamma \hat{v}_{3/2}(x) = \gamma^2 \alpha_1 \hat{v}_{3/2}\left(\frac{1}{\gamma} x\right) = \gamma^{\frac{1}{2}} \alpha_1 \hat{v}_{3/2}(x),$$

where in the last step we used the fact that $\hat{v}_{3/2}$ is $3/2$ -homogeneous. $\square$

We can now match the second order blow-down with translations of paraboloid solutions. To this end, given $\sigma \in \mathbb{R}$ denote by u_σ the global solution that has $\gamma P - \sigma e_1$ as its coincidence set. Observe that $u_\sigma(x_1, x_2) = u_{\gamma P}(x_1 + \sigma, x_2)$.

Proposition 7.5. *Given a x_2 -monotone solution u to the obstacle problem as in Definition 2.7 there exists a paraboloid solution $u_{\gamma P}$ such that*

$$(7.3) \quad \lim_{r \rightarrow \infty} \frac{(u - u_{\gamma P})(r \cdot)}{r^{\frac{3}{2}}} = 0 \text{ in } W_{\text{loc}}^{1,2} \cap C_{\text{loc}}^0(\mathbb{R}^2)$$

Moreover given $\sigma \in [-1, 1]$ we have that

$$(7.4) \quad \lim_{r \rightarrow \infty} \frac{(u - u_\sigma)(r \cdot)}{r^{\frac{3}{2}}} = 0 \text{ in } C_{\text{loc}}^0(\mathbb{R}^2).$$

Proof. Given α_u from Proposition 7.3 let $\gamma = \left(\frac{\alpha_u}{\alpha_1}\right)^2$ so that $\alpha_\gamma = \alpha_u$ (by Corollary 7.4) and hence (7.3) follows directly from Proposition 7.1.

Now to obtain (7.4) we use a Taylor expansion with remainder along with the $C^{1,1}$ regularity of the solution $u_{\gamma P}$ to see that for each $R > 0$,

$$\sup_{B_R} \frac{|u_{\gamma P}(rx_1 + \sigma, rx_2) - u_{\gamma P}(rx_1, rx_2)|}{r^{\frac{3}{2}}} \leq C(R)r^{-\frac{1}{2}}.$$

This along with (7.3) establishes the C_{loc}^0 convergence. $\square$

8. PROOF OF THEOREM 1.3

We begin this section with the following Proposition.

Proposition 8.1. *Let u_1, u_2 be two x_2 -monotone solutions. If there exists some non-empty open set $\Omega \subset \{u_1 > 0\} \cap \{u_2 > 0\}$ such that $(u_1 - u_2)|_\Omega \equiv 0$ then $u_1 \equiv u_2$.*

Proof. We first observe that the set $\Sigma := \{u_1 > 0\} \cap \{u_2 > 0\}$ is connected since it is the subgraph of a function of the x_1 -variable. Since $\Delta(u_1 - u_2) \equiv 0$ in the connected set Σ , and $(u_1 - u_2) \equiv 0$ on the open set Ω , we must have that $u_1 \equiv u_2$ in Σ . It follows that $u_1 = u_2 = 0$ on $\partial\Sigma$ which completes the proof. $\square$

The following is a useful consequence.

Corollary 8.2. *Let u_1, u_2 be two x_2 -monotone solutions. If there exists $r > 0$ such that $\{u_1 = 0\} = \{u_2 = 0\}$ in B_r^+ or in B_r^- , then $u_1 \equiv u_2$.*

Proof. If the coincidence sets of u_1 and u_2 coincide in B_r^+ then there exists a ball B such that $(B \cap (\{u_1 = 0\} \cap \{u_2 = 0\}))^\circ \neq \emptyset$ and $(B \cap (\{u_1 > 0\} \cap \{u_2 > 0\}))^\circ \neq \emptyset$ (see Figure 1). In B we have that $\Delta(u_1 - u_2) = 0$ and so we can conclude that $u_1 - u_2 \equiv 0$ in B . On the other hand, $(B \cap (\{u_1 > 0\} \cap \{u_2 > 0\}))^\circ \neq \emptyset$ and so there exists some open set $\Omega \subset B \cap (\{u_1 > 0\} \cap \{u_2 > 0\})$ such that $u_1 - u_2 \equiv 0$ on Ω . Applying Proposition 8.1 we conclude that $u_1 \equiv u_2$. $\square$

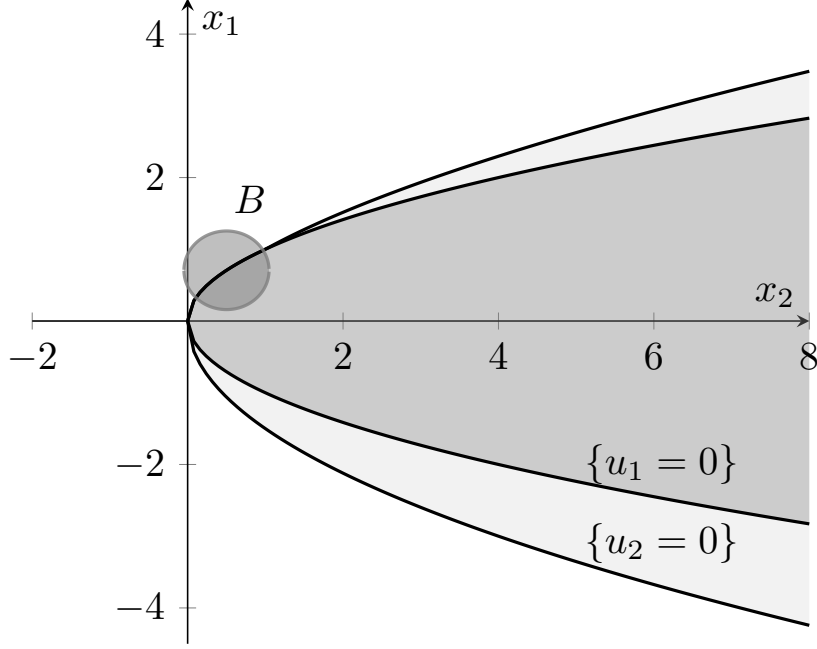


FIGURE 1. Setting of the proof of Corollary 8.2

Finally, we will require the following comparison result for any two solutions of the obstacle problem.

Proposition 8.3. *Suppose that u_1, u_2 are two solutions of the obstacle problem. Then each connected component of the set $\{u_1 > u_2\}$ must be unbounded.*

Proof. Suppose that there is a connected component Ω of $\{u_1 > u_2\}$ that is bounded. Then $|u_1 - u_2| = 0$ on $\partial\Omega$ but it is subharmonic by Lemma 2.3. Hence, $u_1 \equiv u_2$ in Ω , a contradiction. $\square$

We can now give the

Proof of Theorem 1.3: Given any global solution u as in Definition 2.7 we let $u_{\gamma P}$ be the paraboloid solution given by Proposition 7.5. Moreover we recall the notation $u_\sigma = u_{\gamma P}(x_1 + \sigma, x_2)$ for $\sigma \in \mathbb{R}$ and we will denote the coincidence set of u_σ as P_σ . We will show that $u = u_{\gamma P}$ by a contradiction argument. If u and $u_{\gamma P}$ were not identical, we would have that C and γP touch at the origin but cannot be identical in any half ball around the origin by Corollary 8.2. Since ∂C is analytic, there exists some small $\rho > 0$ such that the free boundaries of u and $u_{\gamma P}$ do not intersect in B_ρ except at the origin (recall that $\partial\{u_{\gamma P} > 0\}$ is a paraboloid).

Claim 1: There exists some $\varepsilon_0 > 0$ such that the boundary of P_σ intersects ∂C at least twice in B_ρ for $0 < \sigma \leq \varepsilon_0$ or $-\varepsilon_0 \leq \sigma < 0$.

Proof of Claim 1: We define

$$\varepsilon_0 = \frac{1}{2} \min\{\text{dist}(\gamma P \cap B_\rho^+, C \cap B_\rho^+), \text{dist}(\gamma P \cap B_\rho^-, C \cap B_\rho^-)\}.$$

Now, there are three possible cases, either

- (1) $C \cap B_\rho \subset \gamma P$, or,
- (2) $\gamma P \cap B_\rho \subset C$, or,
- (3) $\gamma P \cap B_\rho^+ \subset C$ while $C \cap B_\rho^- \subset \gamma P$ (or vice versa).

We examine the first case. Since $\partial\{u_{\gamma P} > 0\}$ and ∂C touch at the origin, passing from $u_{\gamma P}$ to u_σ for $0 \neq |\sigma| \leq \varepsilon_0$ will produce at least two points of intersection (see Figure 2). Case 2 is handled in an identical manner.

In the third case, passing from $u_{\gamma P}$ to u_σ for $-\varepsilon_0 \leq \sigma < 0$ will produce at least three points of intersection. Indeed, since C and γP touch at the origin there will be at least two points of intersection between C and P_σ in B_ρ^+ and at least one in B_ρ^- . In the vice versa case one would take $0 < \sigma \leq \varepsilon_0$, which concludes the proof of the claim. $\square$

Now for each σ given by Claim 1, there must exist at least two curves $L_1^\sigma, L_2^\sigma \subset \mathbb{R}^2$ along which $u - u_\sigma = 0$ and $u - u_\sigma$ changes sign across them (see Figure 2). Moreover, by Proposition 8.3 the curves L_1^σ and L_2^σ must be unbounded, and so there are three unbounded sets Ω_i^σ for $i = 1, 2, 3$ such that $u - u_\sigma = 0$ on $\partial\Omega_i^\sigma$ (see Figure 2). For $i = 1, 2$, Ω_i^σ will have boundary given by L_i^σ and the unbounded component of $\partial C \cup \partial P_\sigma$ that intersects L_i^σ only and along which $u - u_\sigma = 0$. The set Ω_3^σ will have boundary given by L_1^σ, L_2^σ and the component of $\partial C \cup \partial P_\sigma$ along which $u - u_\sigma = 0$ and that lies between the points of intersection with L_1^σ and L_2^σ .

Observe that for $i = 1, 2, 3$ we have that $(\Omega_i^\sigma \cap (\{u > 0\} \cap \{u_\sigma > 0\}))^\circ \neq \emptyset$ and $\partial\Omega_i^\sigma \cap \partial B_r \neq \emptyset$ for every $r \geq \rho$.

Since $|u - u_\sigma|$ is a non-negative subharmonic function by Lemma 2.3, we can apply Theorem 3.1 to the functions $v_i^\sigma = |u - u_\sigma| \chi_{\Omega_i^\sigma}$ to see that the functional

$$\Phi(r; \sigma) := \frac{1}{r^9} \prod_{i=1}^3 \int_{B_r} |\nabla v_i^\sigma|^2$$

is monotone non-decreasing in r for $r \geq \rho$. We then find using Lemma 2.3 again that

$$\Phi(r; \sigma) \leq C \prod_{i=1}^3 \int_{B_2} (r^{-3/2} (v_i^\sigma)_r)^2.$$

By Proposition 7.5 we have that the right hand side above converges to 0 as $r \rightarrow \infty$. Since $\Phi(r; \sigma)$ is non-negative and monotone non-decreasing for $r \geq \rho$ we have that $\Phi(r; \sigma) = 0$ for all $r \geq \rho$. Since $\Phi(\rho; \sigma) = 0$ implies that $\Phi(r; \sigma) = 0$ for all $r \leq \rho$ we have that

$$\Phi(r; \sigma) = 0 \text{ for all } r \geq 0.$$

So for each σ given by Claim 1, one of the $v_i^\sigma \equiv 0$ which means there exists some ball $B^\sigma \subset \{u > 0\} \cap \{u_\sigma > 0\}$ such that $(u - u_\sigma)|_{B^\sigma} \equiv 0$. Proposition 8.1 then implies that

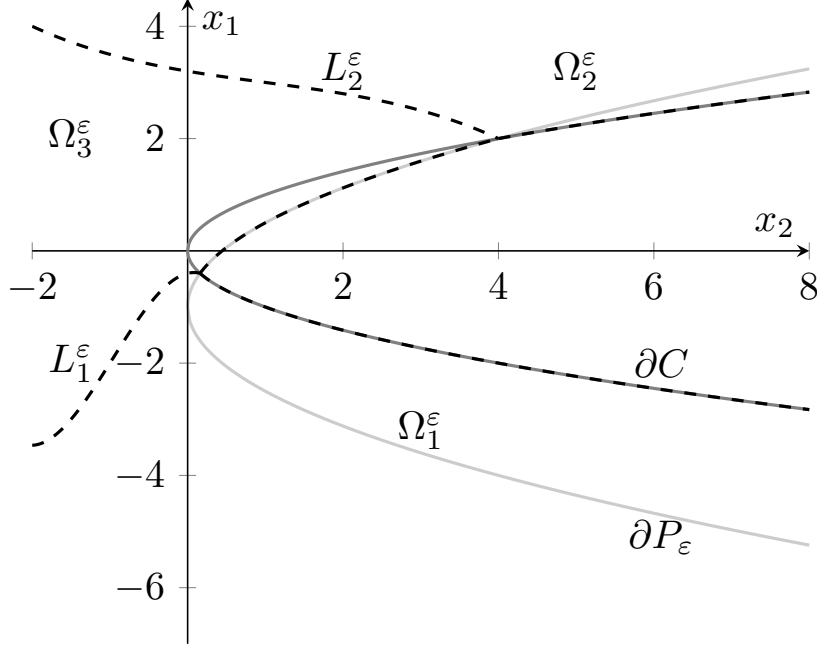


FIGURE 2. The sliding of γP downwards produces at least three unbounded regions. The dashed lines represent their boundaries.

$u \equiv u_\sigma$. This is clearly not possible for each σ as in Claim 1, and so we must have instead that $C = \gamma P$. $\square$

A. PROOF OF LEMMA 2.8

In this appendix we provide the proof of Lemma 2.8 that is in [14, Lemma A5]. We note that a Lipschitz estimate can be obtained in our setting (see for instance [13]), however we will not do that here. For simplicity we will work in B_8 and given $w : B_2 \rightarrow \mathbb{R}$ Lipschitz we introduce the notation

$$[\delta_j w]_\sigma = \sup_{x \in B_1, |t| \leq 1} \frac{|u(x + te_j) - u(x)|}{t^\sigma}$$

for $j \in \{1, \dots, n\}$ and $\sigma \in (0, 1]$. We first note the following estimate on the Hölder semi-norms of $w \in Lip(B_2)$ (see [14, Lemma C.1]).

Lemma A.1. *Let $w \in Lip(B_2)$. For each $p > 1$ and each $\sigma \in (0, 1]$ there exists a constant $C = C(n, p, \sigma) > 0$ such that*

$$[w]_{C^\beta(B_1)} \leq C \left(\sum_{j=1}^{n-1} [\delta_j w]_\sigma + \|\partial_n w\|_{L^p(B_2)} \right)$$

where $\beta = \frac{\sigma(p-1)}{\sigma p + n - 1}$.

We can now give the

Proof of Lemma 2.8. The aim is to apply Lemma A.1 when $n = 2$ to the difference $w_r = (u - p)_r$ with $p = 2$ and $\sigma = 1$. We first note that as a consequence of Lemma 2.3 there exists a universal constant C such that

$$(A.1) \quad \|w_r\|_{L^\infty(B_4)} \leq C \|w_r\|_{L^2(B_8)}.$$

and

$$(A.2) \quad \|\nabla w_r\|_{L^2(B_4)} \leq C \|w_r\|_{L^2(B_8)}.$$

Indeed, $|w_r|$ is subharmonic, and so for each $x \in B_4$ we have that

$$|w_r|(x) \leq \int_{B_4(x)} |w_r| \leq C \|w_r\|_{L^2(B_8)},$$

which establishes (A.1). Moreover, (A.2) follows directly from Lemma 2.3. We will now show that there exists a universal constant C such that

$$(A.3) \quad [\delta_2 w_r]_1 \leq C \|w_r\|_{L^2(B_8)}.$$

Given $0 < t < 1$ we define the function

$$(\delta_2^\pm w_r)(x) = \frac{w_r(x \pm t e_2) - w_r(x)}{t}$$

and we observe that

$$(A.4) \quad \|\delta_2^\pm w_r\|_{L^2(B_2)} \leq C \|w_r\|_{L^2(B_8)}$$

as a consequence of (A.2). Now since $p(x) = \frac{1}{2}x_1^2$ we have that $\delta_2^\pm w_r = \delta_2^\pm u_r$, and so we find that $\Delta(\delta_2^\pm w_r) \leq 0$ in $\{u_r > 0\} \cap B_4$ and $\delta_2^\pm w_r \geq 0$ in $\{u_r = 0\} \cap B_4$. This implies that $\min\{\delta_2^\pm w_r, 0\}$ is superharmonic and we obtain using (A.4) that

$$\min_{B_2} \delta_2^\pm w_r \geq -C \|w_r\|_{L^2(B_8)}.$$

Now since $\delta_2^\pm w_r(x \mp t e_j) = -(\delta_2^\mp w_r)(x)$ we find

$$\max_{B_1} \delta_2^\pm w_r \leq C \|w_r\|_{L^2(B_8)}$$

which concludes Step 3. Now Lemma A.1 can be applied and we obtain, using (A.1), the desired estimate. $\square$

B. PROOF OF PROPOSITION 2.14

In this appendix we prove Proposition 2.14. However, we note that using the estimates on the coincidence set obtained in Proposition 5.2, the following computations (and hence the expansion) can be established for any x_2 -monotone global solution as in Definition 2.7. Recall that

$$G(x, y) = \log(|x - y|) - \log(|y|) + \frac{x \cdot y}{|y|^2}$$

and note that

$$(B.1) \quad -\frac{1}{2\pi} \Delta_x G(x, y) = -\delta_x$$

in the sense of distributions while

$$(B.2) \quad G(0, y) = |\nabla_x G(0, y)| = 0.$$

The main step in the proof of Proposition 2.14 consists in showing that $V_{\gamma P}(x)$ is subquadratic. Once this is achieved the proof can be carried out using a Liouville argument exactly as in the proof of [7, Proposition 3.4].

Lemma B.1. *There exists a constant $C = C(\gamma)$ such that*

$$|V_{\gamma P}(x)| \leq C(1 + |x|^{\frac{7}{4}}) \text{ for all } x \in \mathbb{R}^2.$$

Proof. Throughout this proof we will use C to denote a constant that depends on δ and which may change from line to line. The first observation to make is that, by a Taylor expansion with remainder, we have

$$(B.3) \quad |G(x, y)| \leq C|x|^2 \int_0^1 \frac{1}{|y - \tau x|^2} d\tau.$$

Indeed, setting $f(\tau) = \ln(|\tau x - y|)$ we have

$$\begin{aligned} f'(\tau) &= \frac{1}{|\tau x - y|^2} (\tau x - y) \cdot x \\ f''(\tau) &= \frac{|x|^2}{|\tau x - y|^2} - 2 \frac{1}{|\tau x - y|^4} [(\tau x - y) \cdot x]^2, \end{aligned}$$

from which (B.3) easily follows. In particular if $|y| \geq 2|x|$ we have that $|y - \tau x| \geq \frac{1}{2}|y|$ and we have

$$(B.4) \quad |G(x, y)| \leq C \frac{|x|^2}{|y|^2}.$$

Fixing now $\hat{r} \geq 1$ such that $\log(r) \leq r^{\frac{1}{4}}$ for $r \geq \hat{r}$, we observe that for each $x \in \mathbb{R}^2$ we can split $V_{\gamma P}(x)$ up as

$$V_{\gamma P}(x) = \int_{\gamma P \cap B_{2\hat{r}}} G(x, y) dy + \int_{\gamma P \cap (B_{2|x|} \setminus B_{2\hat{r}})} G(x, y) dy + \int_{\gamma P \setminus B_{2|x|}} G(x, y) dy.$$

In the third integral we have $|y| \geq 2|x|$ and so we can use (B.4) to obtain

$$(B.5) \quad \left| \int_{\gamma P \setminus B_{2|x|}} G(x, y) dy \right| \leq C|x|^2 \int_{2|x|}^{\infty} \int_{-(\gamma y_2)^{\frac{1}{2}}}^{(\gamma y_2)^{\frac{1}{2}}} \frac{1}{y_2^2} dy_1 dy_2 \leq C\gamma^{\frac{1}{2}} |x|^{\frac{3}{2}},$$

For the first integral we first estimate from above,

$$\begin{aligned} \int_{\gamma P \cap B_{2\hat{r}}} G(x, y) &= \int_{\gamma P \cap B_{2\hat{r}}} \log(|x - y|) - \int_{\gamma P \cap B_{2\hat{r}}} \log(|y|) + \int_{\gamma P \cap B_{2\hat{r}}} \frac{x \cdot y}{|y|^2} \\ &\leq \int_{(\gamma P \cap B_{2\hat{r}}) \setminus B_1(x)} \log(|x - y|) - \int_{(\gamma P \cap B_{2\hat{r}}) \cap B_1} \log(|y|) + \int_{B_{2\hat{r}}} \frac{|x|}{|y|} \\ &\leq C(1 + |x|), \end{aligned}$$

where we used that in $(\gamma P \cap B_{2\hat{r}}) \setminus B_1(x)$ we have that $1 \leq |x - y| \leq C\hat{r} + |x|$. Similarly we obtain the lower bound

$$\begin{aligned} \int_{\gamma P \cap B_{2\hat{r}}} G(x, y) &\geq - \int_{B_{2\hat{r}}} \frac{|x|}{|y|} + \int_{B_1(x)} \log(|x - y|) - \int_{B_{2\hat{r}} \setminus B_1} \log(|y|) \\ &\geq -C(1 + |x|), \end{aligned}$$

and we conclude that

$$(B.6) \quad \left| \int_{C \cap B_{2\hat{r}}} G(x, y) dy \right| \leq C(1 + |x|).$$

Finally the middle integral is non-zero only if $|x| \geq \hat{r} > 0$ and so we can write $x = |x|z$ for some $z \in \partial B_1$ and we find using the scaling properties of $G(x, y)$ that

$$\begin{aligned} \left| \int_{\gamma P \cap (B_{2|x|} \setminus B_{2\hat{r}})} G(x, y) dy \right| &\leq \int_{\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}} |G(|x|z, y)| dy \\ &= \int_{\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}} \left| G\left(z, \frac{y}{|x|}\right) \right| dy \\ &\leq |x|^2 \int_{\{|y_1| \leq |x|^{-\frac{1}{2}} (\gamma y_2)^{\frac{1}{2}}\} \cap \{\frac{\hat{r}}{|x|} \leq y_2 \leq 2\}} |\log(|z - y|)| + |\log(|y|)| + \frac{|z|}{|y|} dy. \end{aligned}$$

Since we are integrating now over a bounded domain (and $\frac{1}{|y|}$ and $\ln(|y|)$ are integrable around zero) we can separate this integral. Dealing with the second and third terms first, we find

$$\begin{aligned} |x|^2 \int_{\{|y_1| \leq |x|^{-\frac{1}{2}} (\gamma y_2)^{\frac{1}{2}}\} \cap \{\frac{\hat{r}}{|x|} \leq y_2 \leq 2\}} |\log(|y|)| + \frac{|z|}{|y|} dy &\leq |x|^2 \int_{\frac{\hat{r}}{|x|}}^2 \int_{-|x|^{-\frac{1}{2}} (\gamma y_2)^{\frac{1}{2}}}^{|x|^{-\frac{1}{2}} (\gamma y_2)^{\frac{1}{2}}} \log(\hat{r}|x|) + \frac{1}{y_2} dy \\ &\leq C|x|^{\frac{3}{2}} \log(|x|). \end{aligned}$$

Now for the first component of the integral we observe

$$\begin{aligned} |x|^2 \int_{\{|y_1| \leq |x|^{-\frac{1}{2}} (\gamma y_2)^{\frac{1}{2}}\} \cap \{\frac{\hat{r}}{|x|} \leq y_2 \leq 2\}} |\log(|z - y|)| dy &= \int_{\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}} \left| \log\left(\frac{1}{|x|}\right) + \log(|x - y|) \right| dy \\ &\leq \int_{\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}} \log(|x|) - \int_{B_1(x)} \log(|x - y|) dy \\ &\quad + \int_{(\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}) \setminus B_1(x)} \log(|x - y|) dy \\ &\leq C|x|^{\frac{3}{2}} \log(|x|), \end{aligned}$$

where in the last step we used that $1 \leq |x - y| \leq C|x|$ in the region $(\{|y_1| \leq (\gamma y_2)^{\frac{1}{2}}\} \cap \{\hat{r} \leq y_2 \leq 2|x|\}) \setminus B_1(x)$. Since $\log(|x|) \leq |x|^{\frac{1}{4}}$ for $|x| \geq \hat{r}$, we obtain the estimate

$$(B.7) \quad \left| \int_{C \cap (B_{2|x|} \setminus B_{2\hat{r}})} G(x, y) dy \right| \leq C|x|^{\frac{7}{4}}.$$

Putting (B.5), (B.6) and (B.7) together we obtain the result. $\square$

Proof of Proposition 2.14. For each $\rho > 0$ we have that $\Delta V_{\gamma P \cap B_\rho}(x) = -\chi_{\gamma P \cap B_\rho} \in L^\infty(\mathbb{R}^n)$ and so by [11, Proposition 2.18] applied in each ball B_R , we have that $V_{\gamma P \cap B_\rho} \rightarrow V_{\gamma P}$ and $\nabla V_{\gamma P \cap B_\rho} \rightarrow \nabla V_{\gamma P}$ locally uniformly in $\mathbb{R}^n$ as $\rho \rightarrow \infty$. By (B.2) we therefore have that

$$(B.8) \quad V_{\gamma P}(0) = 0 \text{ and } |\nabla V_{\gamma P}(0)| = 0.$$

Now, since $\Delta(u_\gamma - p) = -\chi_{\gamma P}$ we have that $\Delta(u - p - V_{\gamma P}) = 0$ and since both $u - p$ and $V_{\gamma P}$ grow subquadratically as $|x| \rightarrow \infty$, we have by Liouville's theorem that

$$u = p + V_{\gamma P}(x) + \ell(x) + c$$

where $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear function and $c \in \mathbb{R}$. By (B.8), $\ell = c = 0$ and this concludes the proof. $\square$

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