ON UNITIFICATION OF $\ast$-RINGS

SANJAY MORE, ANIL KHAIRNAR AND B. N. WAPHARE

Abstract: S. K. Berberian raised the open problem "Can every weakly Rickart $\ast$-ring be embedded in a Rickart $\ast$-ring? with preservation of right projections?" Berberian has given a partial solution to this problem. Khairnar and Waphare raised a similar problem for p.q.-Baer $\ast$-rings and gave a partial solution. In this paper, we give more general partial solutions to both the problems.
Keywords: weakly Rickart $\ast$-rings, weakly p.q.-Baer $\ast$-rings, projections, central cover.

1. Introduction

Kaplansky [5] introduced Baer rings and Baer $\ast$-rings to abstract various properties of AW^*-algebras (that is a C^*-algebra which is also a Baer $\ast$-ring), von Neumann algebras and complete $\ast$-regular rings. The concept of a Baer $\ast$-ring is naturally motivated from the study of functional analysis. For example, every von Neumann algebra is a Baer $\ast$-algebra. One can refer [7, 8, 9, 10, 11, 12] for recent work on rings with involution.

Throughout this paper, R denotes an associative ring. An *ideal* of a ring R, we mean a two sided ideal. A ring R is said to be *reduced* if it does not have a nonzero nilpotent element. A ring R is said to be *abelian* if its every idempotent element is central. Let S be a nonempty subset of R. We write $r_R(S) = \{a \in R \mid sa = 0,\ \forall\ s \in S\}$, and is called the *right annihilator* of S in R, and $l_R(S) = \{a \in R \mid as = 0,\ \forall\ s \in S\}$, is the *left annihilator* of S in R. A *$\ast$-ring* R is a ring equipped with an involution $x \to x^*$, that is, an additive anti-automorphism of the period at most two. An element e of a $\ast$-ring R is called a *projection* if it is self-adjoint (i.e. $e = e^*$) and idempotent (i.e. $e^2 = e$). A $\ast$-ring R is said to be a *Rickart $\ast$-ring*, if for each $x \in R$, $r_R(\{x\}) = eR$, where e is a projection in R. For each element a in a Rickart $\ast$-ring, there is unique projection e such that $ae = a$ and $ax = 0$ if and only if $ex = 0$, called the *right projection* of a denoted by $RP(a)$. Similarly, the left projection $LP(a)$ is defined for each element a in Rickart $\ast$-ring. A $\ast$-ring R is said to be a *weakly Rickart $\ast$-ring*, if for any $x \in R$, there exists a projection e such that (1) $xe = x$, and (2) if $xy = 0$ then $ey = 0$.
Recall the following propositions and an open problem from [1].

Proposition 1.1 ([1, Proposition 2, page 13]). *If R is Rickart $\ast$-ring, then R has a unity element and the involution of R is proper.*

Proposition 1.2 ([1, Proposition 2, page 28]). *The following condition on a $\ast$-ring R are equivalent:*
(a)R is a Rickart $\ast$-ring;
(b)R is weakly Rickart $\ast$-ring with unity.

Proposition 1.1 says that the unity element exist in any Rickart $\ast$-ring and the proposition 1.2 naturally create the following problem.

2010 *Mathematics Subject Classification.* Primary 16W10; Secondary 16D25.

Problem 1: Can every weakly Rickart $*$-ring be embedded in a Rickart $*$-ring with preservation of RP's?
In [1] Berberian has given a partial solution to this problem.

According to Birkenmeier et al. [2], a $*$-ring R is said to be a *quasi-Baer $*$-ring* if the right annihilator of every ideal of R is generated, as a right ideal, by a projection in R. Birkenmeier et al. [3] introduced principally quasi-Baer (p.q.-Baer) $*$-rings as a generalization of quasi-Baer $*$-rings. A $*$-ring R is said to be a *p.q.-Baer $*$-ring*, if for every principal right ideal aR of R, $r_R(aR) = eR$, where e is a projection in R, it follows that $l_R(Ra) = Rf$ for a suitable projection f. Note that an abelian Rickart $*$-ring is a p.q.-Baer $*$-ring, and a reduced p.q.-Baer $*$-ring is a Rickart $*$-ring. We say that an element x of a $*$-ring R possesses a *central cover* if there exists a smallest central projection $h \in R$ such that $hx = x$. If such a projection h exists, then it is unique, it is called the central cover of x, denoted by $h = C(x)$. In [6] Khairnar and Waphare proved that the central cover exists for every element in any p.q.- Baer$*$-ring.

Theorem 1.3 ([6, Theorem 2.5]). *Let R be a p.q.- Baer $*$-ring and $x \in R$. Then x has a central cover $e \in R$. Further, $xRy = 0$ if and only if $yRx = 0$ if and only if $ey = 0$.*
That is $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1-e)R = R(1-e)$.

In [6] Khairnar and Waphare introduced the concept of weakly p.q.- Baer $*$-ring. A $*$-ring R is said to be a *weakly p.q.-Baer $*$-ring*, if every $x \in R$ has a central cover $e \in R$ such that, $xRy = 0$ if and only if $ey = 0$. According to [3], the involution $*$ of a $*$-ring R is *semi-proper*, if for any $a \in R$, $aRa^* = 0$ implies $a = 0$.
Recall the following results and an open problem from [6].

Proposition 1.4 ([6, Proposition 2.4]). *If R is p.q.Baer $*$-ring, then R has the unity element and the involution of R is semi-proper.*

Theorem 1.5 ([6, Theorem 3.9]). *The following conditions on a $*$-ring R are equivalent:*
(a) R is a p.q.-Baer $*$-ring.
(b) R is a weakly p.q.-Baer $*$-ring with unity.

In view of the above theorem, the following problem is raised in [6].
Problem 2: Can every weakly p.q.-Baer $*$-ring be embedded in a p.q.-Baer $*$-ring? with preservation of central covers?

In [6], Khairnar and Waphare provided a partial solution to Problem 2.

In the second section of this paper, we give a more general partial solution of problem 1 and in section 3, we give a more general partial solution of problem 2.

2. Unitification of Weakly Rickart $*$-rings

Recall the definition of unitification of a $*$-ring given by Berberian [1]. Let R be a $*$-ring. We say that R_1 is a unitification of R, if there exists a ring K, such that,
1) K is an integral domain with involution (necessarily proper), that is, K is a commutative $*$-ring with unity and without divisors of zero (the identity involution is permitted),
2) R is a $*$-algebra over K (that is, R is a left K-module such that, identically $1a = a$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$, *and* $(\lambda a)^* = \lambda^* a^*$ for $\lambda \in K$ and $a, b \in R$).
3) R is torsion free K-module (that is $\lambda a = 0$ implies $\lambda = 0$ or $a = 0$).
Define $R_1 = R \oplus K$ (the additive group direct sum), thus $(a, \lambda) = (b, \mu)$ means, by the

definition that $a = b$ and $\lambda = \mu$, and addition in R_1, is defined by the formula $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$. Define $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$, $(a, \lambda)^* = (a^*, \lambda^*)$. Evidently, R_1 is also a $*$-algebra over K, has unity element $(0, 1)$ and R is a $*$-ideal in R_1. The following lemmas are elementary facts about unitification R_1 of a $*$-ring R.

Lemma 2.1 ([1, Lemma 1, page 30]). *With notations as in the definition of unitification, if an involution on R is proper, then so is the involution of R_1.*

Lemma 2.2 ([1, Lemma 3, page 30]). *With notations as in the definition of unitification, let $x \in R$ and let e be a projection in R. Then $RP(x) = e$ in R if and only if $RP((x, 0)) = (e, 0)$ in R_1.*

Berberian has given a partial solution to Problem 1 as follows.

Theorem 2.3 ([1, Theorem 1, page 31]). *Let R be a weakly Rickart $*$-ring. If there exists an involutory integral domain K such that R is a $*$-algebra over K and it is a torsion-free K-module, then R can be embedded in a Rickart $*$-ring with preservation of RP's.*

After 1972, there was not much headway towards the solution of Problem 1. In 1996 Thakare and Waphare supplied partial solutions wherein the condition on the underlying weakly Rickart $*$-rings was weakened in two distinct ways. For the solution of this open problem, Berberian used the condition that R is torsion free left K-module, and K is an integral domain. Thakare and Waphare gave another solution in which the condition of torsion free is replaced by other condition. They establish the following.

Theorem 2.4 ([14, Theorem 2]). *A weakly Rickart $*$-ring R can be embedded into a Rickart $*$-ring, provided there exists a ring K such that*

(1) *K is an integral domain with involution,*
(2) *R is $*$-algebra over K, and*
(3) *For any $\lambda \in K - \{0\}$, there exist a projection e_λ that is an upper bound for the set of left projections of the right annihilators of λ, that is if $x \in R$ and $\lambda x = 0$ then $LP(x) \leq e_\lambda$.*

Theorem 2.5 ([14, Theorem 7]). *A weakly Rickart $*$-ring R can be embedded into Rickart $*$-ring provided the characteristic of R is non zero.*

The $*$-ring $C_\infty(T) \bigoplus M_2(\mathbb{Z}_3)$ has no embedding in the sense of Theorem 2.3 as the characteristic of R is zero though it has unitification in the sense of Theorem 2.4. This is the example that shows that Theorem 2.4 is an improvement over Theorem 2.3 of Berberian.

Now we prove the existence of largest projection corresponding to the self adjoin element by using condition (3) of the above theorem.

Lemma 2.6. *Let R be a weakly Rickart $*$-ring with condition* (3) *in Theorem 2.4. Then for any self-adjoint element a and $\lambda \neq 0$ there exists a largest projection g such that $ag = \lambda g$.*

Proof. Let $RP(a) = e'$ and e_λ be the projection as given by condition (3) of Theorem 2.4. Let $e = e' \vee e_\lambda$, then $e' \leq e$ and $e' = e'e = ee'$. Since $ae' = a$, therefore $ae'e = ae$. Hence $a = ae' = ae$. Also, $a* = a$ implies that $a = ea = eae \in eRe$. Thus $a - \lambda e \in eRe$. Let $h = RP(a - \lambda e)$ and $g = e - h$. This gives $(a - \lambda e)g = 0$, hence $ag = \lambda g$. Let k be any projection in R such that $ak = \lambda k$. Consider $\lambda(ek - k) = e\lambda k - \lambda k = eak - \lambda k = ak - \lambda k = 0$. Let $LP(ek - k) = f$. Therefore $f \leq e_\lambda \leq e$. That is $ek - k = f(ek - k) = fe(ek - k) = f(ek - ek) = 0$.

Consider $(a-\lambda e)k = ak - \lambda ek = ak - \lambda k = 0$. Therefore $RP(a-\lambda e)k = hk = 0$. Hence $kg = k(e-h) = ke - kh = k - 0 = k$. That is $k \le g$. Therefore g is largest projection such that $ag = \lambda g$. $\square$

Recall the following lemma from [1].

Lemma 2.7 ([1, Lemma 5, page 31]). *Let B be a $*$-ring with proper involution, $x \in B$ and e be a projection in B. Then e is the right projection of x if and only if e is the right projection of x^*x.*

We give a solution of Problem 1 in which the condition "K is an integral domain" is replaced by "K is a commutative ring with unity".

Let R be a $*$-ring, K be a commutative $*$-ring with unity and R be an algebra over K. Write $r = r(R,+)$ for the endomorphism ring of the additive group of R. Each $a \in R$ determines an element L_a of r via $L_a x = ax$ and each $\lambda \in K$ an element λI of r via $(\lambda I)x = \lambda x$. Let $R_1 = R \oplus K$ with the $*$-algebra operations as follows $(a,\lambda)+(b,\mu) = (a+b, \lambda+\mu)$, $\mu(a,\lambda) = (\mu a, \mu\lambda)$, $(a,\lambda)(b,\mu) = (ab+\mu a+\lambda b, \lambda\ \mu)$, $(a,\lambda)^* = (a^*,\lambda^*)$. Each $(a,\lambda) \in R_1$ determines an element $L_a + \lambda I$ of r and the mapping $(a,\lambda) \to L_a + \lambda I$ is ring homomorphism of R_1 onto a sub-ring S of r namely the subring of r generated by L_a and λI. Define $\mu(L_a + \lambda I)$ to be the ring product $(\mu I)(L_a + \lambda I)$, then S becomes an algebra over K and $(a,\lambda) \to L_a + \lambda I$ is an algebra homomorphism of R_1 onto S. Let N be the kernel of this mapping and write $\hat{R_1} = R_1/N$ for quotient algebra. Denote the coset $(a,\lambda)+N$ by $[a,\lambda]$. Hence $[a,\lambda]$ is an equivalence class of (a,λ) under equivalence relation $(a,\lambda) \equiv (b,\mu)$ if and only if $ax + \lambda x = bx + \mu x,\ \forall\ x \in R$.

The following result leads to the partial solution of Problem 1.

Theorem 2.8. *With above notations, we have the following.*

1. *The mapping $a \to\ \bar{a} = [a,0]$ is an algebra homomorphism of R into $\hat{R_1}$.*
2. *If $L(R) = \{x \in R \mid xy = 0,\ \forall\ y \in R\} = \{0\}$ (that is if the involution of R is proper) then the mapping $a \to \bar{a}$ is injective.*
3. *If the involution of R is proper then $[a,\lambda] = 0$ if and only if $[a^*,\lambda^*] = 0$ and the formula $[a,\lambda]^* = [a^*,\lambda^*]$ defines unambiguously proper involution in $\hat{R_1}$.*
4. *If R is a weakly Rickart $*$-ring $a \in R$ and e is the right projection of a in R then $\bar{e}$ is the right projection of $\bar{a}$ in $\hat{R_1}$.*

Proof. (1) and (2) are easy verification.
(3) : Observe that $[a,\lambda] = 0$ if and only if $(a,\lambda)+N = N$ if and only if $(a,\lambda) \in N$ if and only if $(L_a + \lambda I)x = 0,\ \forall\ x \in R$ if and only if $ax + \lambda x = 0,\ \forall\ x \in R$. Therefore in order to show $[a^*,\lambda^*] = 0$ whenever $[a,\lambda] = 0$ it is enough to show $a^*x + \lambda^*x = 0,\ \forall x \in R$. Consider $(a^*x+\lambda^*x)^*(a^*x+\lambda^*x) = (x^*a+\lambda x^*)(a^*x+\lambda^*x) = x^*aa^*x + x^*a\lambda^*x + \lambda x^*a^*x + \lambda x^*\lambda^*x$ $= x^*\left\{a(a^*x)+\lambda(a^*x)\right\} + x^*\left\{a(\lambda^*x)+\lambda(\lambda^*x)\right\} = x^*0 + x^*0 = 0,\ \forall x \in R$. Therefore $a^*x+\lambda^*x = 0,\ \forall x \in R$. Hence $[a,\lambda\ ]^* = [a^*,\lambda^*]$ defines an involution in $\hat{R_1}$. Also, $[a,\lambda]^*[a,\lambda] = 0$ implies that $[a^*,\lambda^*][a,\lambda] = 0$. That is $[a^*a+\lambda a^*+\lambda^*a, \lambda^*\lambda] = 0$. This gives $(a^*a+\lambda a^*+\lambda^*a)x + \lambda^*\lambda x = 0,\ \forall x \in R$. Therefore $a^*ax + \lambda a^*x + \lambda^*ax + \lambda^*\lambda x = 0,\ \forall x \in R$. Also, $(ax+\lambda x)^*(ax+\lambda x) = (x^*a^*+\lambda^*x^*)(ax+\lambda x) = x^*a^*ax + x^*a^*\lambda x + \lambda^*x^*ax + \lambda^*x^*\lambda x =$ $x^*[a^*ax + a^*\lambda x + \lambda^*ax + \lambda^*\lambda x] = x^*[a^*ax + \lambda a^*x + \lambda^*ax + \lambda^*\lambda x] = x^*0 = 0,\ \forall x \in R$. That is $ax + \lambda x = 0,\ \forall x \in R$. This gives $[a,\lambda] = 0$. Hence the involution $*$ is proper.
(4) : Let R be a weakly Rickart $*$-ring $a \in R$ and $e = RP(a)$. Then $ae = a$ and $ay = 0$

implies that $ey = 0$ for $y \in R$. We prove that $\bar{e} = RP(\bar{a})$. Consider $\bar{a}\bar{e} = [a,0][e,0] = [ae,0] = [a,0] = \bar{a}$. Let $\bar{y} = [b,\mu]$ and $\bar{a}\bar{y} = 0$. Then $[a,0][b,\mu] = [ab+\mu a,0] = 0$. This gives $(ab+\mu a)x = 0,\ \forall x \in R$. That is $a(bx+\mu x) = 0,\ \forall x \in R$. This implies that $(eb+\mu e)x = 0\ \forall x \in R$. Therefore $[eb+\mu e,0] = 0$. That is $[e,0][b,\mu] = 0$. This gives $\bar{e}\bar{y} = 0$. Therefore $\bar{e} = RP(\bar{a})$. □

The following theorem gives a more general partial solution to Problem 1, we give the solution in which we replace integral domain K by any commutative ring.

Theorem 2.9. *Let R be a weakly Rickart $*$-ring and K be a commutative $*$-ring with unity such that R is a $*$-algebra over K satisfying condition (3) of Theorem 2.4. Then R can be embedded in a Rickart $*$-ring with preservation of right projections.*

Proof. Let $\hat{R_1} = R_1/N = \{[a,\lambda] \mid (a,\lambda) \in R_1\}$ and $\hat{R_1}$ has unity element $u = [0,1]$. By Lemma 2.7, it is enough to show that every self-adjoint element of $\hat{R_1}$ has the right projection. Let $[a,\lambda] \in \hat{R_1}$ be a self-adjoint element. If $\lambda = 0$ then $e = RP(a)$ and by Theorem 2.8 $\bar{e} = RP(\bar{a})$. Suppose $\lambda \neq 0$. Then by Lemma 2.6 there exists a largest projection g such that $ag = -\lambda g$. Now we show that $RP([a,\lambda]) = [-g,1]$. Note that $[-g,1]$ is a projection. Also, $[a,\lambda][-g,1] = [-ag-\lambda g+a,\lambda] = [a,\lambda]$. Moreover $[a,\lambda][b,\mu] = 0$ if and only if $[ab+\mu a+\lambda b,\lambda\mu] = 0$ if and only if $abx+\mu ax+\lambda bx+\lambda\mu x = 0,\ \forall x \in R$ if and only if $a(bx+\mu x)+\lambda(bx+\mu x) = 0,\ \forall x \in R$ if and only if $(a+\lambda e_x)(bx+\mu x) = 0$ where $e_x = LP(bx+\mu x)$ if and only if $(a+\lambda e_x)e_x = 0,\ \forall x \in R$ if and only if $ae_x = -\lambda e_x,\ \forall x \in R$. Since g is the largest projection such that $ag = -\lambda g$. Therefore $e_x \leq g$. This gives $e_x g = g e_x = e_x$. Therefore $[a,\lambda][b,\mu] = 0$ if and only if $(e_x-g)e_x = 0,\ \forall x \in R$ if and only if $(e_x-g)(bx+\mu x) = 0,\ \forall x \in R$ if and only if $-g(bx+\mu x)+e_x(bx+\mu x) = 0,\ \forall x \in R$ if and only if $-gbx-\mu gx+bx+\mu x = 0,\ \forall x \in R$ if and only if $[-gb-\mu g+b,\mu] = 0$ if and only if $[-g,1][b,\mu] = 0$. Hence $\hat{R_1}$ is a Rickart $*$-ring. □

3. Unitification of Weakly p.q.-Baer $*$-rings

We recall the following examples of p.q.-Baer $*$-rings. This also shows how the class of p.q.-Baer $*$-rings is different than the class of Rickart $*$-rings.

Example 3.1 ([3, Exercise 10.2.24.4])**.** Let A be a domain, $A_n = A$ for all $n = 1,2,\cdots$, and B be the ring of $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n$ such that a_n is eventually constant, which is a subring of $\prod_{n=1}^{\infty} A_n$. Take $R = M_n(B)$, where n is an integer such that $n > 1$. Let $*$ be the transpose involution of R. Then R is a p.q.-Baer $*$-ring which is not quasi-Baer (hence not a quasi-Baer $*$-ring). Also, if A is commutative which is not Prüfer, then R is not a Rickart $*$-ring.

Example 3.2 ([3, Exercise 10.2.24.5])**.** Let R be a $*$-ring. If R is a right (or left) p.q.-Baer ring and $*$ is semiproper, then R is a p.q.-Baer $*$-ring. Hence, if R is biregular and $*$ is semiproper, then R is a p.q.-Baer $*$-ring.

Example 3.3 ([6, Example 1.7])**.** Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d,\ b \equiv 0, \text{ and } c \equiv 0\ (mod\ 2) \right\}$. Consider involution $*$ on R as the transpose of the matrix. In [4, Example 2(1)], it is shown that R is neither right p.p. nor left p.p. (hence not a Rickart $*$-ring) but $r_R(uR) = \{0\} = 0R$ for any nonzero element $u \in R$. Therefore R is a p.q.-Baer $*$-ring.

Recall the following result which gives the condition on m and n so that the matrix ring $M_n(\mathbb{Z}_m)$ is a Baer $*$-ring and hence a Rickart $*$-ring.

Corollary 3.4 ([13, Corollary 7]). *(i) $M_n(\mathbb{Z}_m)$ is a Baer $*$-ring for $n \geq 2$ if and only if $n = 2$ and m is a square free integer whose every prime factor is of form $4k + 3$.*
(ii) $\mathbb{Z}_m$ is a Baer $*$-ring if and only if m is a square free integer.

The following example shows that the right projections in a $*$-ring need not be central covers.

Example 3.5 ([6, Example 2.8]). Let $A = M_2(\mathbb{Z}_3)$, which is a Baer $*$-ring (hence a p.q.-Baer $*$-ring and a Rickart $*$-ring) with transpose as an involution. There is an element $x \in A$ such that $RP(x)$ is not equal to $C(y)$ for any $y \in A$.

The following is a partial solution of the Problem 2 given in [6].

Theorem 3.6 ([6, Theorem 4.6]). *A weakly p.q.-Baer $*$-ring R can be embedded in a p.q.-Baer $*$-ring, provided there exists, a ring K such that,*

(1) *K is an integral domain with involution,*
(2) *R is a $*$-algebra over K,*
(3) *For any $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in R$ that is an upper bound for the central covers of the right annihilators of λ, that is, for $t \in R$, if $\lambda\ t = 0$ then $C(t) \leq e_\lambda$.*

Let $\tilde{R}$ denote the set of all projections in a $*$-ring R. In a weakly p.q. Baer $*$-ring, following is called the condition (β): For any $0 \neq \lambda \in K, \exists\ e_\lambda \in \tilde{R}$ such that $\lambda x = 0$ implies that $C(x) \leq e_\lambda$, where K is a commutative $*$-ring with unity.

Lemma 3.7. *Let R be weakly p.q. Baer $*$-ring which is a $*$-algebra over a commutative $*$-ring K with unity satisfying condition (β). Then for any $a \in R$ and $0 \neq \lambda \in K$ there exists a largest central projection g such that $ag = \lambda g$.*

Proof. On the similar line of Lemma 2.6. □

The following result leads to the solution of Problem 2.

Theorem 3.8. *With notation as defined earlier*

(1) *The mapping $a \to \bar{a} = [a, 0]$ is an algebra homomorphism of R into $\hat{R}_1$.*
(2) *If $L(R) = \{x \in R : xy = 0,\ \forall y \in R\} = \{0\}$ then the mapping $a \to \bar{a}$ is injective and we may regard R as embedded in $\hat{R}_1$.*
(3) *If the involution $*$ is semi-proper then $[a, \lambda] = 0$ if and only if $[a^*, \lambda^*] = 0$. Hence $[a, \lambda]^* = [a^*, \lambda^*]$ defines involution in $\hat{R}_1$.*
(4) *If R is weakly p.q. Baer $*$-ring, $a \in R, C(a) = e$ then $C(\bar{a}) = \bar{e}$ in $\hat{R}_1$.*

Proof. (1) is trivial.
(2) To prove $\phi : R \to \hat{R}_1$ given by $\phi(a) = \bar{a}$ is injective. Let $\phi(a) = \phi(b)$. Then $\bar{a} = \bar{b}$, that is $[a, 0] = [b, 0]$. This gives $ax = bx, \forall x \in R$. Therefore $(a - b)x = 0,\ \forall x \in R$. This gives $a - b = 0$. Hence $a = b$.
(3) Suppose R has semi-proper involution, therefore for $a \in R$, $a^*Ra = 0$ implies that $a = 0$. Now, $[a, \lambda] = 0$ if and only if $ax + \lambda x = 0,\ \forall x \in R$. Also, for any $r \in R$, $(x^*a + \lambda x^*)r(a^*x + \lambda^*x) = x^*ara^*x + x^*ar\lambda^*x + \lambda x^*ra^*x + \lambda x^*r\lambda^*x$

$= x^*\{a(ra^*x)+\lambda(ra^*x)\}+x^*\{a(r\lambda^*x)+\lambda(r\lambda^*x)\} = x^*0+x^*0 = 0$. Therefore $[a,\lambda]=0$ if an only if $(x^*a+\lambda x^*)R(a^*x+\lambda^*x)=0$ if and only if $(a^*x+\lambda^*x)=0$ if and only if $[a^*\lambda^*]=0$. Hence $[a,\lambda]^*=[a^*\lambda^*]$ defines an involution in $\hat{R}_1$.
(4) Let R be weakly p.q. Baer $*$-ring, $a\in R$ and $C(a)=e$. Consider $\bar{a}\bar{e}=[a,e][e,0]=[ae,0]=[a,0]=\bar{a}$. Also, $\bar{a}\hat{R}_1[b,\mu]=0$ if and only if $\bar{a}\bar{e}\hat{R}_1[b,\mu]=0$ if and only if $\bar{a}\hat{R}_1\bar{e}[b,\mu]=0$ if and only if $[a,0]\hat{R}_1[eb+\mu e0]=0$ if and only if $[a,0][x,\lambda][eb+\mu e,0]=0$ if and only if $[a(x+\lambda e)(eb+\mu e),0]=0$ if and only if $a(x+\lambda e)(eb+\mu e)=0$ if and only if $aR(eb+\mu e)=0$ if and only if $e(eb+\mu e)=0$ if and only if $eb+\mu e=0$ if and only if $(eb+\mu e)x=0,\ \forall x\in R$ if and only if $[eb+\mu e,0]=0$ if and only if $[e,0][b,\mu]=0$. Therefore $C(\bar{a})=\bar{e}$. □

Now we give the more general partial solution to the Problem 2, in which we replace integral domain K by any commutative ring with unity.

Theorem 3.9. *Let R be a weakly p.q. Baer $*$-ring and K be a commutative $*$- ring with unity such that R is a $*$-algebra over K satisfying condition (β). Then R can be embedded in a p.q. Baer $*$-ring with preservation of central covers.*

Proof. Let $\hat{R}_1 = R_1/N=\{[a,\lambda]\mid(a,\lambda)\in R_1\}$. Note that $u=[0,1]$ is a unity element of $\hat{R}_1$. We show that $\hat{R}_1$ is p.q. Baer $*$-ring. It is enough to show that for every element $x\in\hat{R}_1$ there exists a central projection $e\in\hat{R}_1$ such that: (1) $xe=x$, (2) $x\hat{R}_1y=0$ if and only if $ey=0$. Let $x=[a,\lambda]\in\hat{R}_1$. If $\lambda=0$, let $C(a)=e$. By Theorem 3.8, $C(\bar{a})=\bar{e}$. Suppose $\lambda\neq0$, then by Lemma 3.7 there exists the largest central projection g such that $ag=-\lambda g$. Clearly $[-g,1]$ is a central projection. Also, $[a,\lambda][-g,1]=[-ag+a-\lambda g,\lambda]=[a,\lambda]$, that is $xe=x$ with $e=[-g,1], x=[a,\lambda]$. Suppose $[a,\lambda]\hat{R}_1[b,\mu]=0$. Therefore $[a,\lambda][r,0][b,\mu]=0$ for all $r\in R$. This gives $[arb+\lambda rb+\mu ar+\lambda\mu r,0]=0$ for all $r\in R$. This implies $arbx+\lambda rbx+\mu arx+\lambda\mu rx=0$ for all $r,x\in R$. That is $ar(bx+\mu x)+\lambda r(bx+\mu x)=0$ for all $r,x\in R$. Therefore $(ar+\lambda re_x)(bx+\mu x)=0$, where $e_x=C(bx+\mu x)$. This gives $(a+\lambda e_x)r(bx+\mu x)=0$ for all $r\in R$. That is $(a+\lambda e_x)R(bx+\mu x)=0$. Therefore $(a+\lambda e_x)e_x=0$. Hence $ae_x=-\lambda e_x$. Since g is a largest central projection such that $ag=(-\lambda)g$, therefore $e_x\le g$. Therefore $(1-g)e_x=0$. This gives $(1-g)e_x(bx+\mu x)=0$. Thus $(1-g)(bx+\mu x)=0$ for all $x\in R$. Hence $bx+\mu x-gbx-\mu gx=0$ for all $x\in R$. Therefore $[-gb-\mu g+b,\mu]=0$, that is $[-g,1][b,\mu]=0$. Hence $\hat{R}_1$ is a p.q.-Baer $*$-ring

□

Disclosure statement: The authors report there are no competing interests to declare.

References

[1] Berberian, S.K.: *Baer $*$-Rings*, Grundlehren Math. Wiss. Band 195. Vol. 296, Berlin: Springer (1972).
[2] Birkenmeier, G.F., Kim, J.Y., Park, J.K.: *Polynomial extensions of Baer and quasi-Baer rings*, J. Pure Appl. Algebra 159(1) (2001), 25-42.
[3] Birkenmeier, G.F., Park, J.K., Rizvi, S.T.: *Extensions of Rings and Modules*, Birkhäuser, New York, (2013).
[4] Hong, C. Y., Kim, N.K., Kwak, T.K.: *Ore extension of Baer and PP rings*, J. Pure Appl. Algebra 151(3) (2000), 215-226.
[5] Kaplansky, I.: *Rings of Operators*, W. A. Benjamin, Inc. New York-Amsterdam (1968).
[6] Khairnar Anil, B.N. Waphare: *Unitification of weakly p.q.-Baer $*$-rings*, Southeast Asian Bull. Math. 42 (2018), 387-400.
[7] Khairnar, Anil, Waphare, B.N.: *Order properties of generalized projections*, Linear Multilinear Algebra 65(7) (2017), 1446-1461.

[8] Khairnar Anil, Waphare B.N.: *Conrad's Partial Order on P.Q.-Baer $*$-Rings*, Discussiones Mathematicae General Algebra and Applications 38 (2018), 207-219.

[9] Khairnar Anil and B.N. Waphare, *A Sheaf Representation of Principally Quasi-Baer $*$-Rings*, Algebr. Represent. Theory 22 (2019), 79-97.

[10] Lia Vaš, *$*$-Clean Rings; Some Clean and Almost Clean Baer $*$-rings and von Neumann Algebras*, Journal of Algebra, 324 (2010) 3388-3400.

[11] Lia Vaš, *Class of Baer $*$-rings defined by a relaxed set of axioms*, J. Algebra, 297 (2006,)470-473.

[12] Roozbeh Hazrat and Lia Vaš, *Baer And Baer $*$-Ring Characterizations Of Leavitt Path Algebra*, J. Pure Appl. Algebra, 222 (2018), 39-60.

[13] Thakare, N.K., Waphare, B.N.: *Baer $*$-rings with finitely many elements*, J. Combin. Math. Combin. Comput. 26 (1998), 161-164.

[14] Thakare N.K. and Waphare B.N.: *Partial solution to the open problem of unitification of a weakly Rickart $*$-rings*, Indian J. Pure Appl. Math. 28(2) (1997), 189-195.

Department of Mathematics, Anantrao Pawar College, Pune-412115, India.
Email address: sanjaymore71@gmail.com

Department of Mathematics, Abasaheb Garware College, Pune-411004, India.
Email address: ask.agc@mespune.in; anil_maths2004@yahoo.com

Center For Advanced Studies in Mathematics, Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India.
Email address: bnwaph@math.unipune.ac.in; waphare@yahoo.com