ON PARTIALLY AMPLE ULRICH BUNDLES

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Abstract. We characterize q-ample Ulrich bundles on a variety $X \subseteq \mathbb{P}^N$ with respect to $(q+1)$-dimensional linear spaces contained in X.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension $n \geq 1$. The study of positivity properties of vector bundles $\mathcal{E}$ on X is a classical one. Starting with Hartshorne's pioneering paper [H], several positivity notions have been introduced, among which, perhaps, the most important one is ampleness. The latter amounts to say that the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample. One possible weakening of this notion, so that some properties are maintained, is q-ampleness, that we now recall (see for example [To] and references therein).

Definition 1.1. Let $q \geq 0$ and let $\mathcal{L}$ be a line bundle on a scheme Y. We say that $\mathcal{L}$ is *q-ample* if for every coherent sheaf $\mathcal{F}$ on Y, there exists an integer $m_0 > 0$ such that $H^i(\mathcal{F}(m\mathcal{L})) = 0$ for $m \geq m_0$ and $i > q$. Let $\mathcal{E}$ be a vector bundle on Y. We say that $\mathcal{E}$ is *q-ample* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is q-ample.

In this paper we are interested in studying the above notion for a special class of vector bundles, namely for Ulrich bundles, that is bundles $\mathcal{E}$ such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$. The importance of Ulrich bundles is well-known (see for example [ES, Be, CMRPL] and references therein). Positivity properties of Ulrich bundles have been studied recently [L, LM, LS, LMS1, LMS2]. In particular, in [LS, Thm. 1], we showed that an Ulrich bundle $\mathcal{E}$ is ample (that is 0-ample) if and only if either X does not contain lines or $\mathcal{E}_{|L}$ is ample on any line $L \subset X$. We prove here a generalization of this result.

Theorem 1.
Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n \geq 1$. Let $\mathcal{E}$ be an Ulrich vector bundle and let $q \geq 0$ be an integer. Then the following are equivalent:

(i) *$\mathcal{E}$ is q-ample;*
(ii) *either X does not contain a linear space of dimension $q+1$, or $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$;*
(iii) *either X does not contain a linear space of dimension $q+1$, or $h^0(\mathcal{E}^*_{|M}) = 0$ for every linear space $M \subseteq X$ of dimension $q+1$.*

We also have the following consequence.

Corollary 1. *Let $\mathcal{E}$ be an Ulrich vector bundle on $X \subseteq \mathbb{P}^N$. Then:*

(i) *$\mathcal{E}$ is $(n-1)$-ample if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.*
(ii) *If $n \geq 2$, $(X, \mathcal{O}_X(1), \mathcal{E}) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ and $\rho(X) = 1$, then $\mathcal{E}$ is $(n-2)$-ample.*

In recent years, positivity of vector bundles have been measured by augmented and restricted base loci (see for example [BKKMSU, FM]). In the last section we will ask a question about augmented base loci of Ulrich bundles arising from the above theorem.

2. Notation

Throughout the paper we work over the field $\mathbb{C}$ of complex numbers. A *variety* is by definition an integral separated scheme of finite type over $\mathbb{C}$.

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3. Generalities on vector bundles

In this section we collect some general facts about vector bundles and some notation that will be used later.

Definition 3.1. Let $\mathcal{E}$ be a rank r vector bundle on X. We set $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ with projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. If $\mathcal{E}$ is globally generated we define the map determined by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ as

$$\varphi = \varphi_{\mathcal{E}} = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$$

and we set

$$\Pi_y = \pi(\varphi^{-1}(y)), y \in \varphi(\mathbb{P}(\mathcal{E})) \text{ and } P_x = \varphi(\mathbb{P}(\mathcal{E}_x)), x \in X.$$

We also define the map determined by $\mathcal{E}$ as

$$\Phi = \Phi_{\mathcal{E}} : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$$

given, for any $x \in X$, by $\Phi(x) = [P_x] \in \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$.

We record some simple but useful facts. The first one is essentially contained in [Tg, Proof of Lemma 2.4, page 426][1].

Lemma 3.2. *Let V be a vector space and let $P \in \mathbb{P}V$ be a point. Let $Y \subset \mathbb{G}(k, \mathbb{P}V)$ be a subvariety such that, for every $y \in Y$, the corresponding k-plane contains P. If $\mathcal{U}$ is the universal subbundle of $\mathbb{G}(k, \mathbb{P}V)$, then $\mathcal{U}_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, for some rank k vector bundle $\mathcal{G}$ on Y.*

Proof. The assertion being obvious if $\dim V = 1$, we assume that $\dim V \geq 2$. Let $P = \mathbb{P}V_1$, where $V_1 \subseteq V$ is 1-dimensional and choose a splitting $V = V_1 \oplus V'$. We have a closed embedding $j : G(k-1, V') \hookrightarrow G(k, V)$ defined by $j([W]) = [V_1 \oplus W]$, where $V_1 \oplus W \subset V$, or, equivalently, j is the morphism associated to the vector bundle $\mathcal{O}_{G(k-1,V')} \oplus (\mathcal{U}')^*$, where $\mathcal{U}'$ is the universal subbundle of $G(k-1, V')$. Let $G_P = \{[U] \in G(k, V) : P \in \mathbb{P}U\}$. Then j defines an isomorphism $G(k-1, V') \cong G_P$, hence

$$\mathcal{U}_{|G_P} \cong j^*\mathcal{U} \cong \mathcal{O}_{G(k-1,V')} \oplus \mathcal{U}' \cong \mathcal{O}_{G_P} \oplus \mathcal{G}'$$

for some rank k vector bundle $\mathcal{G}'$ on G_P. Since $Y \subseteq G_P$, we get that $\mathcal{U}_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, where $\mathcal{G} = \mathcal{G}'_{|Y}$. $\square$

Lemma 3.3. *Let $\mathcal{E}$ be a globally generated rank r vector bundle on X.*

(i) *For every $x \in X$ the restriction morphism $\varphi_{|\mathbb{P}(\mathcal{E}_x)} : \mathbb{P}(\mathcal{E}_x) \to P_x$ is an isomorphism onto a linear subspace of dimension $r-1$ in $\mathbb{P}H^0(\mathcal{E})$.*

Now let $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Then:

(ii) *$\pi_{|\varphi^{-1}(y)} : \varphi^{-1}(y) \to X$ is a closed embedding.*
(iii) *$\mathcal{E}_{|\Pi_y} \cong \mathcal{O}_{\Pi_y} \oplus \mathcal{G}$, for some rank $r-1$ vector bundle $\mathcal{G}$ on Π_y.*

Proof. To see (i), observe that we have $\mathbb{P}(\mathcal{E}_x) \cong \mathbb{P}^{r-1}$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{E}_x)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Let

$$W = \operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(1))\}.$$

Being $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ globally generated, we have that so is W, hence $\dim W \geq r$. It follows that $W = H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(1))$ and $\varphi_{|\mathbb{P}(\mathcal{E}_x)} = \varphi_{\mathcal{O}_{\mathbb{P}^{r-1}}(1)}$ is an isomorphism onto its image, which is then a linear subspace of dimension $r-1$ in $\mathbb{P}H^0(\mathcal{E})$. This proves (i) and then (i) implies that π and its differential are injective on the fibers of φ, proving (ii). As for (iii), set $M = \Pi_y$ and consider the globally generated rank r vector bundle $\mathcal{E}_{|M}$ on M. Let

$$U = \operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{|M})}(1)\}$$

[1] See also https://mathoverflow.net/questions/395472/trivial-subbundle-of-universal-bundle-on-the-grassmannian-mathbbgk-n.

so that $\varphi_{|\mathbb{P}(\mathcal{E}_{|M})} = \varphi_U : \mathbb{P}(\mathcal{E}_{|M}) \to \mathbb{P}U$. Set $\Phi_M = \Phi_{\mathcal{E}_{|M}}, \varphi_M = \varphi_{\mathcal{E}_{|M}}$ and, for any $x \in M$, $P_{M,x} = \varphi_M(\mathbb{P}((\mathcal{E}_{|M})_x)$. We have a commutative diagram

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}_{|M}) \arrow[r, "\varphi_M"] \arrow[dr, "\varphi_U"'] & \varphi_M(\mathbb{P}(\mathcal{E}_{|M})) \subset \mathbb{P}H^0(\mathcal{E}_{|M}) \arrow[d, "p"] \\ & \varphi_U(\mathbb{P}(\mathcal{E}_{|M})) \subset \mathbb{P}U \end{tikzcd}$$

where p is a finite map. For any $x \in M$, there is a $z \in \varphi^{-1}(y)$ such that $x = \pi(z)$. Hence $z \in \mathbb{P}(\mathcal{E}_x) = \mathbb{P}((\mathcal{E}_{|M})_x)$ and therefore $y = \varphi(z) = \varphi_U(z) = p(\varphi_M(z))$, so that $\varphi_M(z) \in p^{-1}(y) \cap P_{M,x}$. Therefore each $(r-1)$-plane $P_{M,x}$ passes through one of the points of $p^{-1}(y)$. On the other hand, the family of these $(r-1)$-planes is just $\Phi_M(M) \subset \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}_{|M}))$, thus it is irreducible. Since $p^{-1}(y)$ is finite and the condition of passing through a point is closed, we deduce that there is a point $y_M \in \mathbb{P}H^0(\mathcal{E}_{|M})$ such that $y_M \in P_{M,x}$ for every $x \in M$. Set $Y = \Phi_M(M) \subset \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}_{|M}))$. It follows by Lemma 3.2 that $\mathcal{U}^*_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, for some rank $r-1$ vector bundle $\mathcal{G}$ on Y. Since $\mathcal{E}_{|M} = \Phi_M^* \mathcal{U}^*$, this proves (iii). □

4. q-ample vector bundles

We discuss some generalities on q-ample vector bundles.

Definition 4.1. Let $\mathcal{E}$ be a vector bundle on X. We set $q_{\min}(\mathcal{E}) = \min\{q \geq 0 : \mathcal{E} \text{ is } q\text{-ample}\}$.

The definition of $q_{\min}(\mathcal{E})$ implies that $\mathcal{E}$ is q-ample if and only if $q \geq q_{\min}(\mathcal{E})$.

Remark 4.2. We have:

(i) If $\mathcal{E}$ is a globally generated vector bundle on X, then $\mathcal{E}$ is q-ample if and only if $\dim F \leq q$ for every fiber F of $\varphi = \varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$.

(ii) If $\mathcal{E}$ is globally generated, then it is n-ample. Also $n + r - 1 - \nu(\mathcal{E}) \leq q_{\min}(\mathcal{E}) \leq n$, where r is the rank of $\mathcal{E}$ and $\nu(\mathcal{E})$ is the numerical dimension of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Proof. (i) is just [S, Prop. 1.7]. The first part of (ii) follows either by [S, Prop. 1.7] or by (i), since $\dim \varphi^{-1}(y) = \dim \Pi_y \leq n$ for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Thus $q_{\min}(\mathcal{E}) \leq n$. Since $\mathcal{E}$ is $q_{\min}(\mathcal{E})$-ample, for any fiber F of φ, we have by (i) that $n + r - 1 - \nu(\mathcal{E}) \leq \dim F \leq q_{\min}(\mathcal{E})$. This proves (ii). □

We have the following characterization, which is a special case of [S, Prop. 1.7].

Proposition 4.3. *Let X be a smooth variety of dimension $n \geq 1$. Let $\mathcal{E}$ be a globally generated vector bundle on X and let $q \geq 0$ be an integer. Then the following are equivalent:*

(i) *$\mathcal{E}$ is q-ample;*

(ii) *$\mathcal{E}_{|Z}$ does not have a trivial direct summand for every subvariety $Z \subseteq X$ of dimension $q+1$;*

(iii) *$h^0(\mathcal{E}^*_{|Z}) = 0$ for every subvariety $Z \subseteq X$ of dimension $q+1$.*

Proof. The equivalence (ii)-(iii) follows by [O, Lemma 3.9]. As for the equivalence (i)-(ii), assume first that $\mathcal{E}_{|Z}$ does not have a trivial direct summand for every subvariety $Z \subseteq X$ of dimension $q+1$. If $\mathcal{E}$ is not q-ample, there exists by Remark 4.2(i) an $y \in \varphi(\mathbb{P}(\mathcal{E}))$ such that $\dim \varphi^{-1}(y) \geq q+1$. Set $M = \Pi_y$. By Lemma 3.3(ii) we have that $M \cong \varphi^{-1}(y)$, hence $\dim M \geq q+1$. Also, Lemma 3.3(iii) implies that $\mathcal{E}_{|M} \cong \mathcal{O}_M \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on M. But then, for any subvariety $Z \subseteq M$ with $\dim Z = q+1$, we have that $\mathcal{E}_{|Z} \cong \mathcal{O}_Z \oplus \mathcal{G}_{|Z}$, contradicting the hypothesis. Vice versa, assume that $\mathcal{E}$ is q-ample and let $Z \subseteq X$ be a subvariety of dimension $q+1$. If $\mathcal{E}_{|Z} \cong \mathcal{O}_Z \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on Z, then $Z \cong \mathbb{P}(\mathcal{O}_Z) \subseteq \mathbb{P}(\mathcal{E}_{|Z}) \subseteq \mathbb{P}(\mathcal{E})$ and

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}_{|Z})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} \cong \mathcal{O}_{\mathbb{P}(\mathcal{O}_Z)}(1) \cong \mathcal{O}_Z$$

hence $\varphi(\mathbb{P}(\mathcal{O}_Z))$ is a point. Therefore φ has a fiber of dimension at least $q+1$, contradicting Remark 4.2(i). □

5. Proofs of the main results

In the case of Ulrich vector bundles, we can do better than Proposition 4.3.

Proof of Theorem 1. Recall that $\mathcal{E}$ is globally generated since it is 0-regular. The equivalence (ii)-(iii) follows by [O, Lemma 3.9]. As for the equivalence (i)-(ii), assume first that $\mathcal{E}$ is q-ample. Then either X does not contain a linear space of dimension $q+1$ or it follows by Proposition 4.3 that $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$. To see the converse, let $y \in \varphi(\mathbb{P}(\mathcal{E}))$ and let $\Pi_y = \pi(\varphi^{-1}(y))$, so that $\Pi_y \cong \varphi^{-1}(y)$ by Lemma 3.3(ii). By [LS, Thm. 2] we have that Π_y is a linear space contained in X. Now, if X does not contain a linear space of dimension $q+1$, then $\dim \varphi^{-1}(y) = \dim \Pi_y \le q$ for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Hence $\mathcal{E}$ is q-ample by Remark 4.2(i). On the other hand, assume that $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$. If $\mathcal{E}$ is not q-ample, there exists by Remark 4.2(i) an $y \in \varphi(\mathbb{P}(\mathcal{E}))$ such that $\dim \varphi^{-1}(y) \ge q+1$. Hence $\dim \Pi_y \ge q+1$, and picking a linear subspace $M \subseteq \Pi_y$ with $\dim M = q+1$, we get a contradiction by Lemma 3.3(iii). $\square$

We also have.

Proof of Corollary 1. First we prove (i). If $\mathcal{E}$ is not $(n-1)$-ample, then it follows by Theorem 1 that $X = \mathbb{P}^n$ and $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on X. But then $\mathcal{O}_X$ is Ulrich and therefore $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ by [ACLR, Lemma 4.2](vi) and [ES, Prop. 2.1] (or [Be, Thm. 2.3]). On the other hand, if $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$, then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}^{r-1} \times \mathbb{P}^n$ and $\varphi = \pi_1 : \mathbb{P}^{r-1} \times \mathbb{P}^n \to \mathbb{P}^{r-1}$ has n-dimensional fibers, hence $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ is not $(n-1)$-ample by Remark 4.2(i). This proves (i). As for (ii), using Theorem 1, we just need to prove that $X \subset \mathbb{P}^N$ does not contain linear spaces of dimension $n-1$ unless $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$. To this end, let A be the ample generator of $N^1(X)$ and let $H \in |\mathcal{O}_X(1)|$, so that $H \equiv hA$. If X contains a linear space M of dimension $n-1$, then $M \equiv aA$ for some integer $a \ge 1$, and therefore

$$1 = MH^{n-1} = ah^{n-1}A^n$$

hence $a = h = A^n = 1$ and then $H^n = 1$, so that $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ by [ES, Prop. 2.1] (or [Be, Thm. 2.3]). This proves (ii). $\square$

6. Augmented base loci of Ulrich bundles

Given a vector bundle $\mathcal{E}$, it follows by [BKKMSU, Thm. 1.1] that $\mathbf{B}_+(\mathcal{E}) \ne \emptyset$ if and only if $\mathcal{E}$ is not ample if and only if $\mathcal{E}$ is not 0-ample. More generally, given $q \ge 0$, we have by Proposition 4.3 that $\mathcal{E}$ is not q-ample if and only if there exists a subvariety $Z \subseteq X$ of dimension $q+1$ such that $\mathcal{E}_{|Z}$ has a trivial direct summand. For any such subvariety, we have that $Z \cong \mathbb{P}(\mathcal{O}_Z) \subseteq \mathbb{P}(\mathcal{E})$ and, since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} = \mathcal{O}_{\mathbb{P}(\mathcal{O}_Z)}(1) \cong \mathcal{O}_Z$, it follows that $\mathbb{P}(\mathcal{O}_Z) \subseteq \mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. If $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the natural map, then [BKKMSU, Prop. 3.2] implies that

$$Z = \pi(\mathbb{P}(\mathcal{O}_Z)) \subseteq \pi(\mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))) = \mathbf{B}_+(\mathcal{E}).$$

It is well-known, using for example [BBP, Prop. 2.3], that one cannot expect, in general, that $\mathbf{B}_+(\mathcal{E})$ is the union of all such Z's, already in the case of line bundles.

Now assume that $\mathcal{E}$ is Ulrich and not ample. It follows by [LS, Thm. 1] that there is a line $L \subseteq X$ such that $\mathcal{E}_{|L}$ is not ample. It was recently proved by Buttinelli [Bu, Thm. 2] that

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_L L$$

where L runs among all lines contained in X such that $\mathcal{E}_{|L}$ is not ample. Equivalently L runs among all lines contained in X such that $\mathcal{E}_{|L}$ has a trivial direct summand. This is the case $q = 0$ of a more general question. In fact, when $\mathcal{E}$ is not q-ample, we have by Theorem 1 that there is a linear space $M \subseteq X$ of dimension $q+1$ such that $\mathcal{E}_{|M}$ has a trivial direct summand. As above, this implies that $M \subseteq \mathbf{B}_+(\mathcal{E})$. Question: is $\mathbf{B}_+(\mathcal{E})$ the union of all such M's?

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