

Local topology for periodic Hamiltonians and fuzzy tori

Nora Doll¹, Terry Loring², Hermann Schulz-Baldes³

¹Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, Germany

²Department of Mathematics and Statistics, University of New Mexico, Albuquerque, USA

³Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Abstract

A variety of local index formulas is constructed for quantum Hamiltonians with periodic boundary conditions. All dimensions of physical space as well as many symmetry constraints are covered, notably one-dimensional systems in Class DIII as well as two- and three-dimensional systems in Class AII. The constructions are based on several periodic variations of the spectral localizer and are rooted in the existence of underlying fuzzy tori. For these latter, a general invariant theory is developed.

MSC 2010: 81R60, 37B30, 46L80, 81V70, 47S40

1 Overview

In several works [14, 15, 5] it was shown that the index pairing between K -theory and K -homology elements can be computed as the half-signature of a suitably constructed finite dimensional matrix, called the spectral localizer. The main motivation for these works is the application to topological insulators for which the bulk topological invariants (Chern numbers and winding numbers) then become readily accessible in numerical computations. As will be described below, the technique is based on the principle of placing the physical system in a linearly growing Dirac trap and hence the spectral localizer is an intrinsically non-periodic object. On the other hand, it is well-known that periodic approximations often provide stable algorithms for bulk quantities in solid state systems (for invariants, this is described in [17]). This work constructs new periodic versions of the spectral localizer, for sake of conciseness referred to as *periodic spectral localizers*, which also allow to compute the topological invariants numerically, possibly in a more stable manner than with the non-periodic spectral localizer used in other works. Apart from this practical aspect, a further more theoretical insight is that the periodic spectral localizers can be understood as the K -theory representatives of associated fuzzy tori. Furthermore, the periodic spectral localizers may inspire extensions to interacting systems (with periodic boundary conditions) with a computable gapped ground state.

Let us directly describe the periodic spectral localizer for a bounded tight-binding Hamiltonian H on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ over an even dimensional lattice \mathbb{Z}^d with L local

degrees of freedom. This Hamiltonian is supposed to be of finite range R and periodic in space, namely the $L \times L$ matrices $\langle x|H|y \rangle$ vanish for $|x-y| > R$ and $\langle x+p_j e_j|H|y+p_j e_j \rangle = \langle x|H|y \rangle$ for some periods $p_j \in \mathbb{N}$ in the direction of the unit vectors e_j , $j = 1, \dots, d$. The final assumption on H is that it describes an insulator, *i.e.* it has a spectral gap at the Fermi level μ . After an energy shift, $\mu = 0$ and the Fermi projection is $P = \chi(H < 0)$. For any such insulator it is well-known (*e.g.* [18]) that there is an associated (even strong) Chern number $\text{Ch}_d(P) \in \mathbb{Z}$. This paper provides yet another way to compute this topological invariant. Moreover, the formula that is proven to work in the asymptotic regime of large volumes allows to associate numerical topological invariants to rather small systems.

For that purpose, let $\rho \in \mathbb{N}$ be a system size such that 2ρ is an integer multiple of each of the p_j . Then let H_ρ^{per} be the restriction of H to $\mathcal{H}_\rho = \ell^2((\mathbb{Z}/(2\rho\mathbb{Z}))^d, \mathbb{C}^L)$ with periodic boundary conditions. Then the associated (even) *periodic spectral localizer* is a finite dimensional matrix on $(\mathcal{H}_\rho \oplus \mathcal{H}_\rho) \otimes \mathbb{C}^d$ defined by

$$L_{\eta, \rho}^{\text{per}} = \begin{pmatrix} \sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) & \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \hat{\gamma}_j^* \\ \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \hat{\gamma}_j & -\sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) \end{pmatrix} + \frac{1}{\eta} \begin{pmatrix} -H_\rho^{\text{per}} & 0 \\ 0 & H_\rho^{\text{per}} \end{pmatrix}, \quad (1)$$

where X_j are the components $j = 1, \dots, d$ of the position operators on the lattice, furthermore $\hat{\gamma}_1, \dots, \hat{\gamma}_{d-1}$ is a selfadjoint irreducible representation on \mathbb{C}^d of the Clifford algebra with $d-1$ generators, namely $\hat{\gamma}_i \hat{\gamma}_j + \hat{\gamma}_j \hat{\gamma}_i = 2 \delta_{i,j}$ for $i, j = 1, \dots, d-1$, and $\hat{\gamma}_d = \iota \mathbf{1}$ with $\iota = \sqrt{-1}$. Finally, $\eta > 0$ is a parameter that should roughly be chosen as $\|H\|$, see Remark 3 below. Note that the first summand in (1) is the restriction of a diagonal operator onto $(\mathcal{H}_\rho \oplus \mathcal{H}_\rho) \otimes \mathbb{C}^d$ and it is periodic in all d directions of the discrete torus $(\mathbb{Z}/(2\rho\mathbb{Z}))^d \cong \mathbb{Z}^d \cap [-\rho + 1, \rho]^d$.

Theorem 1 *Let d be even and H be a finite-range periodic operator on $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^L)$. Also let $\rho \in \mathbb{N}$ be such that 2ρ is an integer multiple of the periods and $\rho \geq 2R$. Suppose*

$$\rho \geq \frac{C d^4 M \|H\|^3 \eta^2}{g^6}, \quad (2)$$

where $g = \|H^{-1}\|^{-1}$, $M = \max_{j=1, \dots, d} \| [X_j, H] \|$, and finally $C = 15 \cdot 10^6$. Moreover, $\eta \geq \frac{g}{4}$ is such that

$$\left(1 - \frac{g}{\|H\|}\right)^2 + 4\left(1 - \frac{\eta}{\|H\|}\right) \leq \frac{g^2}{4d\eta\|H\|}. \quad (3)$$

Then the periodic spectral localizer is gapped with lower bound

$$(L_{\eta, \rho}^{\text{per}})^2 \geq \frac{g^2}{600 d \eta^2} \mathbf{1}, \quad (4)$$

and the strong invariant given by the d -th Chern number of $P = \chi(H < 0)$ is equal to the half-signature of the periodic spectral localizer, namely

$$\text{Ch}_d(P) = \frac{1}{2} \text{Sig}(L_{\eta, \rho}^{\text{per}}). \quad (5)$$

Remark 2 Let us start out by comparing the periodic spectral localizer $L_{\eta,\rho}^{\text{per}}$ with the spectral localizer $L_{\kappa,\rho}$ used in prior works [14, 15, 5], and also explain the connection between the two of them. The latter matrix $L_{\kappa,\rho}$ is defined on the same finite-dimensional Hilbert space $(\mathcal{H}_\rho \oplus \mathcal{H}_\rho) \otimes \mathbb{C}^{d'}$ by

$$L_{\kappa,\rho} = \kappa \begin{pmatrix} 0 & \sum_{j=1}^d X_j \hat{\gamma}_j^* \\ \sum_{j=1}^d X_j \hat{\gamma}_j & 0 \end{pmatrix} + \begin{pmatrix} -H_\rho & 0 \\ 0 & H_\rho \end{pmatrix}. \quad (6)$$

Here the boundary condition on H_ρ merely needs to be local, notably it can be either periodic or simply be Dirichlet. The first matrix, without the factor κ , is called the Dirac operator D and is off-diagonal as the dimension d is even. The spectral localizer $L_{\kappa,\rho}$ is clearly not periodic in the above sense because the position operators take large positive and negative values at the boundaries of the discrete cube $\mathbb{Z}^d \cap [-\rho+1, \rho]^d$. The main result of [15] (see also [5]) is that the equality (5) holds with $L_{\eta,\rho}^{\text{per}}$ on the r.h.s. replaced by $L_{\kappa,\rho}$, provided conditions on κ and ρ hold that are quantitatively weaker than (2). In the latter regime, the spectral asymmetry of both operators $L_{\eta,\rho}^{\text{per}}$ and $L_{\kappa,\rho}$ is acquired near the center of the finite volume where both operators are roughly the same which gives an intuitive understanding why Theorem 1 should hold (based on the earlier results [14, 15]). Indeed, the proof of Theorem 1 consists of constructing a homotopy from $L_{\kappa,\rho}$ to $L_{\eta,\rho}^{\text{per}}$ inside the finite-dimensional invertible selfadjoint matrices. The essential step is a deformation of the first summand in (1), which is explicitly given in (25) below. It further results from the strategy of proof in Section 2 that the Hamiltonian can be tapered down, see (23). This means that in the regime of (2) the contribution to the signature results merely from the central part of the finite volume. Hence the half-signature in (5) is a *local topological marker* in this regime, just as the half-signature of the spectral localizer of [14, 15]. However, one can use the r.h.s. of (5) also for much smaller ρ for which periodic boundary conditions *are* relevant so that the signature invariant is a *global or bulk topological invariant*. All of this is numerically confirmed in Remark 6 on the example of a one-dimensional topological system, but there is definitely a need for further investigations. \diamond

Remark 3 Besides being the volume, the parameter ρ sets the length scale of the position operator close to the origin because $\sin(\frac{\pi}{\rho}x_j) \sim \frac{\pi}{\rho}x_j$ and $1 - \cos(\frac{\pi}{\rho}x_j) \sim \frac{1}{2}(\frac{\pi}{\rho}x_j)^2$. Comparing with the (non-periodic) spectral localizer (spelled out in (6)), $\frac{1}{\rho}$ hence plays the same role as the parameter κ in prior works [14, 15, 5]. Having this in mind, the condition in (2) is a more stringent version of the main hypothesis in these works. Note that, given a gapped Hamiltonian H , it can always be guaranteed by choosing ρ sufficiently large. The second bound (3) is a new supplementary condition. For a flat band Hamiltonian which by definition satisfies $g = \|H\|$, the condition becomes $\eta \geq \frac{g}{2}(1 + \sqrt{1 - \frac{1}{4d}})$. In particular, $\eta = 1$ is allowed for a flat band Hamiltonian with $g = \|H\| = 1$. As will be explained in second part of this introduction and Section 5, this is reminiscent of the fact that the flat band Hamiltonian together with suitable functions of the position operators forms a *graded fuzzy torus*. On the other hand, if one chooses $\eta = \|H\|$ (without imposing the flat band condition), then the bound (3) becomes $\frac{\|H\|}{g} - 1 \leq \frac{1}{2\sqrt{d}}$ which means that H has to be relatively close to a flat band Hamiltonian in the sense that $\|H\|$ is not allowed to be much bigger than $g = \|H^{-1}\|^{-1}$. Furthermore, (3) is always

satisfied if $\eta \geq \frac{5}{4}\|H\|$ (simply because then the l.h.s. becomes negative due to $g \leq \|H\|$). Note, however, that for large η , (2) enforces ρ to be larger, and furthermore the gap of $L_{\eta,\rho}^{\text{per}}$ closes, see the bound (4), and then its signature may not be numerically stable any more. Hence from a numerical perspective, it may be best to choose η of the order of $\|H\|$ and actually somewhat smaller than $\|H\|$ so that the periodic spectral localizer is associated to a fuzzy torus of small width (in the sense of Definition 19 below). In conclusion, the discussion shows that one should chose $\eta \approx \|H\|$ and in the sequel not consider it as a free parameter. \diamond

Remark 4 The bounds (2) and (3) have an intrinsic scale invariance. Actually, replacing H , M , g and η by λH , λM , λg and $\lambda \eta$ respectively where $\lambda > 0$ is a scaling parameter, leaves the conditions invariant. All four quantities are expressed in energy units, while ρ is a space unit. From a quantitative aspect, we believe that hypothesis (3) is relatively close to optimal, while the condition (2) is certainly off by several orders of magnitude and even the dependence on g is likely much worse than needed. Let us stress that once η is chosen as in Remark 3, there is no further parameter other than ρ . One can then analyze numerically the behavior for small ρ and safely use the half-signature as local topological marker, as long as it is stable. \diamond

Remark 5 As pointed out in Remark 3, the condition (3) is easiest to satisfy if H is already somewhat close to a flat band Hamiltonian. This can be attained by replacing a given initial gapped finite-range periodic Hamiltonian H' by a suitable polynomial $H = q(H')$ which is then also periodic and of finite range, even though the range is increased by a factor given by the degree of the polynomial q . The polynomial q should be chosen odd with $q(x) > 0$ for $x > 0$ so that H also has a spectral gap at 0, and, moreover, to have a degree as small as possible. Based on the spectral information of H' , it is straightforward to construct a suitable polynomial. \diamond

Remark 6 As already discussed in Remarks 2 and 4, we expect the signature of $L_{\eta,\rho}^{\text{per}}$ to be stable for much smaller system sizes ρ . Let us support this belief by some numerics in the numerically most simple situation of a chiral model in dimension $d = 1$, namely the so-called SSH model (see [18] for a detailed description of this much studied object). The chiral Hamiltonian is then an off-diagonal 2×2 matrix with off-diagonal entry A given by an invertible tight-binding operator on $\ell^2(\mathbb{Z})$ which in Dirac Bra-Ket notation is given by $A|x\rangle = (m + m_x)|x\rangle + (1 + t_x)|x + 1\rangle$, where $m > 0$, $(m_x)_{x \in \mathbb{Z}}$ and $(t_x)_{x \in \mathbb{Z}}$ are independent and identically distributed random variables in $[-\lambda, \lambda]$ with $\lambda < 1$. For λ and m sufficiently small, the operator A has a non-commutative winding number equal to -1 in the present situation. According to Theorem 15 (the odd-dimensional equivalent to Theorem 1), it can be computed using the following odd periodic spectral localizer discussed in Section 3:

$$L_{\eta,\rho}^{\text{per}} = \begin{pmatrix} \sin(\frac{\pi}{\rho}X) & \mathbf{1} - \cos(\frac{\pi}{\rho}X) \\ \mathbf{1} - \cos(\frac{\pi}{\rho}X) & -\sin(\frac{\pi}{\rho}X) \end{pmatrix} - \frac{1}{\eta} \begin{pmatrix} 0 & A_{\rho}^{\text{per}} \\ (A_{\rho}^{\text{per}})^* & 0 \end{pmatrix}. \quad (7)$$

Here $X = X_1$ is the position operator and the right summand is the periodized SSH Hamiltonian H_{ρ}^{per} in finite volume. Numerics readily show that its half-signature is -1 for a large span of ρ and λ , simply confirming Theorem 15. If the periodic boundary conditions are used, this reliably works for ρ as small as 4. Also interesting is that drastic modifications of the Hamiltonian do not alter the signature index if ρ is sufficiently large: instead of H_{ρ}^{per} , one can use $\tilde{H}_{\rho}^{\text{per}}$ obtained from H_{ρ}^{per} by setting all matrix elements $\langle x | H_{\rho}^{\text{per}} | y \rangle = 0$ for either $|x| > (1 - s)\rho$ or $|y| > (1 - s)\rho$

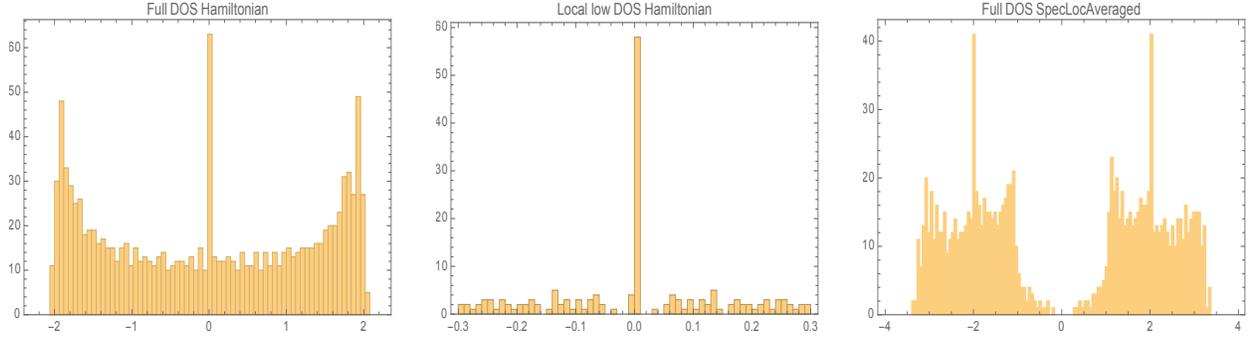


Figure 1: Plots of the full eigenvalue distribution for one realization of the Hamiltonian $\tilde{H}_\rho^{\text{per}}$ described in Remark 6, then its central part and in the third plot the eigenvalue distribution of $L_{1,\rho}^{\text{per}}$. The random variables are uniformly distributed, $\lambda = 0.5$, $m = 0.9\iota$, $\rho = 300$ and $s = 0.05$. Hence the Hilbert space is of dimension 600 states and due to $s = 0.05$ the kernel of $\tilde{H}_\rho^{\text{per}}$ has about 60 states, and $L_{1,\rho}^{\text{per}}$ has about 60 times the eigenvalues close to -2 and 2 each. The half-signature of $L_{1,\rho}^{\text{per}}$ is still -1 .

where $s \in [0, 1)$. Any $s > 0$ will eliminate the periodic boundary conditions and, moreover, leads to a large kernel of H , roughly of dimension $4\rho s$. Nevertheless, this kernel results from regions where the first summand in (7) has large off-diagonal parts and hence does not lead to a kernel of $L_{\eta,\rho}^{\text{per}}$. In the central region $[-(1-s)\rho, (1-s)\rho] \cap \mathbb{Z}$ where $L_{\eta,\rho}^{\text{per}}$ extracts the topology by means of its spectral asymmetry, the modified Hamiltonian $\tilde{H}_\rho^{\text{per}}$ coincides with the H_ρ^{per} . Indeed, numerics clearly show that the half-signature is still -1 as long as $(1-s)\rho \approx 30$. This clearly shows that the periodic spectral localizer in the regime of large ρ reads out the topology locally close to the origin (where \sin is linear and $1 - \cos$ vanishes). \diamond

Remark 7 In Section 4, it is shown how to deal with the strong \mathbb{Z}_2 -invariants for Hamiltonians lying in the suitable Cartan-Altdand-Zirnbauer symmetry classes. In principle, one can also access weak invariants by the techniques of the present work. Indeed, the experienced reader will easily locate the relevant formulas for fuzzy tori in Section 5, but this is here not explained in detail for solid state applications. \diamond

Remark 8 Theorem 1 is stated for periodic Hamiltonians. However, for space homogeneous random operators (in the sense of Bellissard [2], see also [18]), there is a natural construction of periodic approximants [17]. For sufficiently large sizes of these approximants, the Chern numbers coincide with those at infinite volume and can be computed using periodic boundary conditions [17]. In order to avoid introducing the notational machinery, these results are not spelled out in detail. \diamond

Remark 9 From a KK -theoretic perspective (explained in some detail elsewhere), both $L_{\eta,\rho}^{\text{per}}$ and $L_{\kappa,\rho}$ are representatives of a Kasparov product of two even KK -cycles $[P] \in KK^0(\mathbb{C}, \mathcal{A})$ and $[D] \in KK^0(\mathcal{A}, \mathbb{C})$ for a suitable algebra \mathcal{A} . The two gradings are apparent in (6), but once the Kasparov product is computed, one can deform it without respecting the grading of the separate Fredholm modules $[D]$ and $[P]$. This is reflected by the lack of grading of the first summand in expression (1) of the periodic spectral localizer. \diamond

Let us conclude this introduction with a brief discussion of the notion of a fuzzy d -torus in connection with the periodic spectral localizer. This is discussed in detail in Section 5 which we believe to be of considerable independent interest, possibly serving as a guideline to the construction of numerically computable local index formulas for other fuzzy versions of classical geometric objects. Abstract index formulas (not suitable for numerical implementation) have been known for a long time [8, 6, 7], and invariants for the special case of fuzzy spheres were already studied in other works, in particular [11, 15]. More specifically, a fuzzy d -torus consists by definition of d operators A_1, \dots, A_d which are almost unitary and almost commute, see Definition 19. Motivated by standard models of topological insulators (see Chapter 2 in [18]) and the work [12], let us associate a selfadjoint operator to the fuzzy torus:

$$G = \frac{i}{2} \sum_{j=1}^d (A_j^* - A_j) \otimes \gamma_j + \left((d-1)\mathbf{1} - \frac{1}{2} \sum_{j=1}^d (A_j^* + A_j) \right) \otimes \gamma_{d+1}, \quad (8)$$

where $\gamma_1, \dots, \gamma_{d+1}$ is an irreducible representation of the Clifford algebra with $d+1$ generators. In Section 5 it is shown that G is gapped and hence defines an even K -theory class which in the case of a matrix torus of even dimension d can be read out as half-signature. At the root of the construction of (8) is a classical map from the torus \mathbb{T}^d to the sphere \mathbb{S}^d of mapping degree 1. This map is analyzed in detail in Appendix B. Using variations of this map, one can construct a large set of invariants associated to the fuzzy torus, see Section 5. In the context of Theorem 1, there are two fuzzy tori of matrices, namely $e^{i\frac{\pi}{\rho}X_1}, \dots, e^{i\frac{\pi}{\rho}X_d}, H_\rho^{\text{per}}$ and $P_\rho e^{i\frac{\pi}{\rho}X_1} P_\rho, \dots, P_\rho e^{i\frac{\pi}{\rho}X_d} P_\rho$ where $P_\rho = \chi(H_\rho^{\text{per}} < 0)$. The first one is a $(d+1)$ -torus consisting of $d+1$ operators, but the last operator H_ρ^{per} in the list is selfadjoint; such a fuzzy $(d+1)$ -torus is called a graded d -torus (see again Definition 19). Essentially the G -operator associated to the graded fuzzy d -torus $e^{i\frac{\pi}{\rho}X_1}, \dots, e^{i\frac{\pi}{\rho}X_d}, H_\rho^{\text{per}}$ is the periodic spectral localizer. On the other hand, $P_\rho e^{i\frac{\pi}{\rho}X_1} P_\rho, \dots, P_\rho e^{i\frac{\pi}{\rho}X_d} P_\rho$ is an un-graded fuzzy d -torus. The latter is the reduced out version of the former and both have the same topological content (see Proposition 33). Indeed, for $d=2$, this second fuzzy torus already played a role in [6, 7] and the recent work by Toniolo on quantum Hall systems [23]. Combined with Theorem 1 one obtains:

Theorem 10 *Let $G = G(P_\rho e^{i\frac{\pi}{\rho}X_1} P_\rho, \dots, P_\rho e^{i\frac{\pi}{\rho}X_d} P_\rho)$ be constructed as in (8). For ρ sufficiently large, one has*

$$\text{Ch}_d(P) = \frac{1}{2} \text{Sig}(G). \quad (9)$$

The remainder of the paper is organized as follows. Section 2 is dedicated to the proof of Theorem 1. Section 3 describes the odd dimensional version of the periodic spectral localizer. Then Section 4 shows how to modify the periodic spectral localizer so that it can be used to compute \mathbb{Z}_2 -invariants in systems with real symmetries such as time-reversal and particle-hole symmetry. Finally Section 5 introduces the general notion of a fuzzy torus and shows how to extract K -theoretic topological invariants from it. Finally Appendix A recalls the tight connection between mapping degree and Chern number, which is then applied in Appendix B in order to analyze the classical maps behind the index construction of the periodic spectral localizer.

2 Periodic spectral localizer in even dimension

This section provides the proof of Theorem 1. Let $H = H^*$ be a bounded selfadjoint operator on $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ of finite range R which is periodic in all d directions with periods $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$. As above let $\rho \in \mathbb{N}$ be such that 2ρ is an integer multiple of all these p_j . Then H is 2ρ periodic in each of the d directions. It is well-known that such an operator can be partially diagonalized by a Bloch-Floquet transformation $\mathcal{F}_\rho : \ell^2(\mathbb{Z}^d, \mathbb{C}^L) \rightarrow L^2(\mathbb{T}_\rho^d, \mathbb{C}^{(2\rho)^d L})$ where $\mathbb{T}_\rho^d = (\mathbb{R}/4\pi\rho\mathbb{Z})^d$:

$$\mathcal{F}_\rho H \mathcal{F}_\rho^* = \int_{\mathbb{T}_\rho^d}^{\oplus} dk H(k). \quad (10)$$

Then the periodic Hamiltonian appearing in (1) is $H_\rho^{\text{per}} = H(0)$. Actually, any fiber $H(k)$ could be used as well and Theorem 1 remains valid. The first key observation, following directly from the direct integral representation is that

$$\text{spec}(H_\rho^{\text{per}}) \subset \text{spec}(H).$$

In particular, H_ρ^{per} also has a gap around the Fermi level $\mu = 0$ of size at least $g = \|H^{-1}\|^{-1}$.

Next let us introduce the periodic function $\xi : \mathbb{R} \rightarrow [-1, 1]$ by

$$\xi(x) = \sin\left(\frac{\pi}{2}x\right), \quad (11)$$

and then set $\xi_\rho(x) = \xi\left(\frac{x}{\rho}\right)$. Due to the addition theorems, one then has

$$e^{i\frac{\pi}{\rho}x} = 1 - 2\xi_\rho(x)^2 + i2\xi_\rho(x)\sqrt{1 - \xi_\rho(x)^2} \quad (12)$$

for $x \in [-\rho, \rho]$. The main estimates of the next lemma are folklore (*e.g.* [1]), but for the convenience of the reader a full proof is nevertheless provided.

Lemma 11 *For $j = 1, \dots, d$ and $\rho \in \mathbb{R}$, one has*

$$\|[\xi_\rho(X_j), H]\| \leq \frac{\pi}{2\rho} \|[X_j, H]\|. \quad (13)$$

Furthermore if $\rho \in \mathbb{N}$ is such that 2ρ is an integer multiple of the periods of H one has the following commutator bounds for H_ρ^{per} :

$$\|[\cos\left(\frac{\pi}{\rho}X_j\right), H_\rho^{\text{per}}]\| \leq \frac{\pi}{\rho} \|[X_j, H]\|, \quad (14)$$

$$\|[\sin\left(\frac{\pi}{\rho}X_j\right), H_\rho^{\text{per}}]\| \leq \frac{\pi}{\rho} \|[X_j, H]\|, \quad (15)$$

$$\|[\|\xi_\rho(X_j)\|, H_\rho^{\text{per}}]\|^2 \leq \frac{25\pi}{32\rho} \|H\| \|[X_j, H]\|. \quad (16)$$

Proof. Let us start out by noting that $\xi_\rho(X_j)$ is a linear combination of $e^{i\frac{\pi}{2\rho}X_j}$ and $e^{-i\frac{\pi}{2\rho}X_j}$, see (11). Therefore DuHamel's formula implies

$$\|[\xi(X_j), H]\| \leq \frac{1}{2} \left(\| [e^{i\frac{\pi}{2\rho}X_j}, H] \| + \| [e^{-i\frac{\pi}{2\rho}X_j}, H] \| \right) \leq \frac{\pi}{2\rho} \| [X_j, H] \| ,$$

which shows (13). As $\cos(\frac{\pi}{\rho}X_j)$ is a periodic multiplication operator one has

$$\| [\cos(\frac{\pi}{\rho}X_j), H_\rho^{\text{per}}] \| \leq \| [\cos(\frac{\pi}{\rho}X_j), H] \|$$

and then (14) follows from DuHamel's formula as above. Further, (15) holds by the same argument. To show (16) note that

$$|\xi_\rho(X_j)|^2 = \frac{\mathbf{1} - \cos(\frac{\pi}{\rho}X_j)}{2} .$$

Then using the main theorem in [16] stating that for any positive semidefinite bounded operator T on \mathcal{H}_ρ and any bounded operator S on \mathcal{H}_ρ

$$\| [T^{\frac{1}{2}}, S] \| \leq \frac{5}{4} \| S \|^{\frac{1}{2}} \| [T, S] \|^{\frac{1}{2}} ,$$

one obtains

$$\| [|\xi_\rho(X_j)|, H_\rho^{\text{per}}] \| \leq \frac{5}{4} \| H_\rho^{\text{per}} \|^{\frac{1}{2}} \left\| \left[\frac{1}{2} (\mathbf{1} - \cos(\frac{\pi}{\rho}X_j)), H_\rho^{\text{per}} \right] \right\|^{\frac{1}{2}} \leq \frac{5\sqrt{\pi}}{4\sqrt{2\rho}} \| H \|^{\frac{1}{2}} \| [X_j, H] \|^{\frac{1}{2}} ,$$

where in the last step (14) was used. The square of this bound is precisely (16). \square

The next result shows that the signature in (5) is well-defined. Some elements of the proof below are inspired by [12], others follow [15, 5].

Proposition 12 *If all conditions of Theorem 1 hold, then the periodic spectral localizer satisfies the bound (4). Moreover, if two parameter sets (η, ρ) and (η', ρ') both satisfy all conditions, then*

$$\text{Sig}(L_{\eta, \rho}^{\text{per}}) = \text{Sig}(L_{\eta', \rho'}^{\text{per}}) .$$

Proof. Let us start out with several preliminaries. To shorten notations let us denote H_ρ^{per} simply by H_ρ , $\sin(\frac{\pi}{\rho}X_j)$ by $s_{\rho, j}$ and $\cos(\frac{\pi}{\rho}X_j)$ by $c_{\rho, j}$. Further let us introduce the Clifford representation $\gamma_1, \dots, \gamma_{d+1}$ by

$$\gamma_1 = \hat{\gamma}_1 \otimes \sigma_1 , \dots , \gamma_{d-1} = \hat{\gamma}_{d-1} \otimes \sigma_1 , \gamma_d = \mathbf{1} \otimes \sigma_2 , \gamma_{d+1} = \mathbf{1} \otimes \sigma_3 ,$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. Finally let us set $\tilde{H}_\rho = \frac{1}{\eta} H_\rho$. Then the periodic spectral localizer as given in (1) becomes

$$L_{\eta, \rho}^{\text{per}} = \sum_{j=1}^d s_{\rho, j} \gamma_j + \left(\sum_{j=1}^d (1 - c_{\rho, j}) - \tilde{H}_\rho \right) \gamma_{d+1} . \quad (17)$$

Next let us introduce a tapering function as in [15, 5] by setting $G(x) = \frac{1}{2}(\chi(4x+3) - \chi(4x-3))$ with $\chi : \mathbb{R} \rightarrow [-1, 1]$ being the odd non-decreasing switch function with $\chi(\pm x) = \pm 1$ for $x \geq 1$ given by $\chi(x) = x(2 - |x|)$ for $x \in [-1, 1]$. Then set $G_\rho(x) = G(\frac{x}{\rho})$. One finds that G_ρ satisfies by construction $G_\rho(x) = 1$ for $|x| \leq \frac{\rho}{2}$ and $G_\rho(x) = 0$ for $|x| \geq \rho$, and, moreover, it is an even function. Furthermore, by Lemma 4 in [14] one has $\|[G_\rho(D), H \oplus H]\| \leq \frac{8}{\rho} \|[D, H \oplus H]\|$ where $D = \sum_{j=1}^d X_j \gamma_j$ (see also [5]). As G_ρ is even and $D_0 = \sum_{j=1}^d X_j \hat{\gamma}_j$ is normal, the operator $G_\rho(D)$ is diagonal with diagonal entry $G_\rho(|D_0|) = G_\rho(|D_0^*|)$. Therefore the commutator bound can also be stated as

$$\|[G_\rho(|D_0|), H]\| \leq \frac{8}{\rho} \|[D_0, H]\|. \quad (18)$$

Then introduce an interpolating function $G_{\rho,t} : \mathbb{R} \rightarrow [0, 1]$ by $G_{\rho,t}(x) = tG_{\frac{\rho}{2}}(x) + (1-t)$. Finally let us also set

$$G_t = G_{\rho,t}(|D_0|), \quad \tilde{H}_{\rho,\rho',t} = G_t \tilde{H}_{\rho'} G_t,$$

where $\rho' \in [\rho, 2\rho]$ satisfies all conditions of Theorem 1. The path

$$t \in [0, 1] \mapsto L_{\eta,\rho,\rho'}^{\text{per}}(t) = \sum_{j=1}^d s_{\rho',j} \gamma_j + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right) \gamma_{d+1} \quad (19)$$

connects $L_{\eta,\rho,\rho'}^{\text{per}}(0) = L_{\eta,\rho}^{\text{per}}$ to an operator on $(\mathcal{H}_{\rho'} \oplus \mathcal{H}_{\rho'}) \otimes \mathbb{C}^d$ that can be restricted to $(\mathcal{H}_\rho \oplus \mathcal{H}_\rho) \otimes \mathbb{C}^d$ easily. After these preparation, let us now start by computing the square

$$\begin{aligned} L_{\eta,\rho,\rho'}^{\text{per}}(t)^2 &= \left(\sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 \right) \otimes \mathbf{1} + \sum_{1 \leq j < l \leq d} [s_{\rho',j}, s_{\rho',l}] \otimes \gamma_j \gamma_l \\ &\quad + \sum_{j=1}^d [s_{\rho',j}, \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t}] \otimes \gamma_j \gamma_{d+1} \\ &\geq \left(\sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 \right) \otimes \mathbf{1} - \sum_{j=1}^d \|[s_{\rho',j}, \tilde{H}_{\rho,\rho',t}]\| \mathbf{1}. \end{aligned} \quad (20)$$

because $[s_{\rho',j}, s_{\rho',l}] = [s_{\rho',j}, c_{\rho',l}] = 0$ for all $j, l = 1, \dots, d$. Hence

$$\begin{aligned} &\sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 \\ &= \sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) \right)^2 + (\tilde{H}_{\rho,\rho',t})^2 - G_{\rho,t} \sum_{j=1}^d ((\mathbf{1} - c_{\rho',j}) \tilde{H}_{\rho'} + \tilde{H}_{\rho'} (\mathbf{1} - c_{\rho',j})) G_{\rho,t} \\ &= \sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) \right)^2 + (\tilde{H}_{\rho,\rho',t})^2 - \lambda G_{\rho,t} \sum_{j=1}^d ((\mathbf{1} - c_{\rho',j}) \hat{H}_{\rho'} + \hat{H}_{\rho'} (\mathbf{1} - c_{\rho',j})) G_{\rho,t}, \end{aligned}$$

where $\lambda > 0$ is a parameter to be chosen later and $\hat{H}_{\rho'} = \frac{\tilde{H}_{\rho'}}{\lambda}$. Using $s_{\rho',j}^2 + c_{\rho',j}^2 = \mathbf{1}$ for all $j = 1, \dots, d$ one directly checks

$$\sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (1 - c_{\rho',j}) \right)^2 \geq \sum_{j=1}^d s_{\rho',j}^2 + \sum_{j=1}^d (1 - c_{\rho',j})^2 = 2 \sum_{j=1}^d (1 - c_{\rho',j}).$$

Replacing in the above gives

$$\begin{aligned} \sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 &= 2(1 - \lambda G_{\rho,t}^2) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + (\tilde{H}_{\rho,\rho',t})^2 \\ &+ \lambda G_{\rho,t} \sum_{j=1}^d ((1 - c_{\rho',j})(\mathbf{1} - \hat{H}_{\rho'}) + (\mathbf{1} - \hat{H}_{\rho'}) (1 - c_{\rho',j})) G_{\rho,t}. \end{aligned}$$

Now let us use the elementary identity

$$\mathbf{1} - \hat{H}_{\rho'} = \frac{1}{2}(\hat{H}_{\rho'} - \mathbf{1})^2 + \frac{1}{2}(\mathbf{1} - \hat{H}_{\rho'}^2), \quad (21)$$

which implies that

$$\begin{aligned} \sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 \\ \geq 2(1 - \lambda G_{\rho,t}^2) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + (\tilde{H}_{\rho,\rho',t})^2 \\ + \frac{\lambda}{2} G_{\rho,t} \left(\sum_{j=1}^d ((1 - c_{\rho',j})(\hat{H}_{\rho'} - \mathbf{1})^2 + (\hat{H}_{\rho'} - \mathbf{1})^2 (1 - c_{\rho',j})) \right) G_{\rho,t} \\ + \frac{\lambda}{2} G_{\rho,t} \left(\sum_{j=1}^d ((1 - c_{\rho',j})(\mathbf{1} - \hat{H}_{\rho'}^2) + (\mathbf{1} - \hat{H}_{\rho'}^2)(1 - c_{\rho',j})) \right) G_{\rho,t}. \end{aligned}$$

Both of the last two summands require a detailed analysis. In order to deal with the first of them, let us use

$$(\mathbf{1} - c_{\rho',j})(\hat{H}_{\rho'} - \mathbf{1})^2 = ((\hat{H}_{\rho'} - \mathbf{1})(1 - c_{\rho',j}) + [\mathbf{1} - c_{\rho',j}, \hat{H}_{\rho'} - \mathbf{1}])(\hat{H}_{\rho'} - \mathbf{1})$$

and

$$(\hat{H}_{\rho'} - \mathbf{1})^2 (1 - c_{\rho',j}) = (\hat{H}_{\rho'} - \mathbf{1})((1 - c_{\rho',j})(\hat{H}_{\rho'} - \mathbf{1}) - [\mathbf{1} - c_{\rho',j}, \hat{H}_{\rho'} - \mathbf{1}])$$

as well as $(\hat{H}_{\rho'} - \mathbf{1})(1 - c_{\rho',j})(\hat{H}_{\rho'} - \mathbf{1}) \geq 0$. One gets

$$\begin{aligned} (\mathbf{1} - c_{\rho',j})(\hat{H}_{\rho'} - \mathbf{1})^2 + (\hat{H}_{\rho'} - \mathbf{1})^2 (1 - c_{\rho',j}) &\geq -2\|[\mathbf{1} - c_{\rho',j}, \hat{H}_{\rho'} - \mathbf{1}]\| \|\hat{H}_{\rho'} - \mathbf{1}\| \mathbf{1} \\ &\geq -2\|[\mathbf{1} - c_{\rho',j}, \hat{H}_{\rho'}]\| (\|\hat{H}_{\rho'}\| + 1) \mathbf{1}. \\ &\geq -4 \max\{\|\hat{H}_{\rho'}\|, 1\} \|[\mathbf{1} - c_{\rho',j}, \hat{H}_{\rho'}]\| \mathbf{1} \\ &\geq -\frac{4\pi}{\rho'\lambda^2\eta^2} \max\{\|H\|, \lambda\eta\} \|[\mathbf{1} - c_{\rho',j}, H]\| \mathbf{1}, \end{aligned}$$

where the last step follows from Lemma 11. This combined with $G_t^2 \leq 1$ leads to

$$\begin{aligned}
& \sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 \\
& \geq 2(1 - \lambda G_{\rho,t}^2) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + (\tilde{H}_{\rho,\rho',t})^2 - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} \\
& \quad + \frac{\lambda}{2} G_{\rho,t} \left(\sum_{j=1}^d ((\mathbf{1} - c_{\rho',j})(\mathbf{1} - \hat{H}_{\rho'}^2) + (\mathbf{1} - \hat{H}_{\rho'}^2)(\mathbf{1} - c_{\rho',j})) \right) G_{\rho,t}.
\end{aligned}$$

For the last summand, let us use that (12) implies $\frac{1-c_{\rho',j}}{2} = |\xi_{\rho',j}|^2$ where $\xi_{\rho',j} = \sin(\frac{\pi}{2\rho'} X_j)$. Thus

$$\begin{aligned}
& (\mathbf{1} - c_{\rho',j})(\mathbf{1} - \hat{H}_{\rho'}^2) + (\mathbf{1} - \hat{H}_{\rho'}^2)(\mathbf{1} - c_{\rho',j}) \\
& = 4|\xi_{\rho',j}|(\mathbf{1} - \hat{H}_{\rho'}^2)|\xi_{\rho',j}| - 2|\xi_{\rho',j}|[(\mathbf{1} - \hat{H}_{\rho'}^2), |\xi_{\rho',j}|] - 2[|\xi_{\rho',j}|, (\mathbf{1} - \hat{H}_{\rho'}^2)]|\xi_{\rho',j}| \\
& \geq -2\|\mathbf{1} - \hat{H}_{\rho'}^2\|(\mathbf{1} - c_{\rho',j}) - 8\frac{5\sqrt{\pi}}{\lambda^2\eta^2 4\sqrt{2\rho'}} \| [X_j, H] \|^{1/2} \|H\|^{3/2} \mathbf{1}
\end{aligned}$$

Replacing in the above, one then gets

$$\begin{aligned}
\sum_{j=1}^d s_{\rho',j}^2 + \left(\sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) - \tilde{H}_{\rho,\rho',t} \right)^2 & \geq \left(2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) G_{\rho,t}^2 \right) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) \\
& \quad + (\tilde{H}_{\rho,\rho',t})^2 - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} \\
& \quad - \frac{5\sqrt{\pi}}{\lambda\eta^2\sqrt{2\rho'}} \|H\|^{3/2} d\sqrt{M} \mathbf{1}.
\end{aligned}$$

Now let us focus on the summand $(\tilde{H}_{\rho,\rho',t})^2$. Due to $(\tilde{H}_{\rho,\rho',t})^2 \geq g^2 \mathbf{1}$ and again $G_{\rho,t}^2 \leq 1$

$$\begin{aligned}
(\tilde{H}_{\rho,\rho',t})^2 & = \frac{1}{\eta^2} (G_{\rho,t}^2 H_{\rho'}^2 G_{\rho,t}^2 + G_{\rho,t}^2 H_{\rho'} [G_{\rho,t}, H_{\rho'}] G_{\rho,t} + G_{\rho,t} [H_{\rho'}, G_{\rho,t}] G_{\rho,t} H_{\rho'} G_{\rho,t}) \\
& \geq \frac{g^2}{\eta^2} G_{\rho,t}^4 - \frac{2}{\eta^2} \|H\| \| [G_{\rho,t}, H_{\rho'}] \| \mathbf{1}.
\end{aligned}$$

The commutator can be bounded using (18):

$$\| [G_{\rho,t}, H_{\rho'}] \| = t \| [G_{\frac{\rho}{2}}(|D_0|), H_{\rho'}] \| = t \| [G_{\frac{\rho}{2}}(|D_0|), H] \| \leq \frac{16}{\rho} \| [D_0, H] \|,$$

where $[G_{\frac{\rho}{2}}(|D_0|), H_{\rho'} \oplus H_{\rho'}] = [G_{\frac{\rho}{2}}(|D_0|), H]$ as $R + \frac{\rho}{2} \leq \rho \leq \rho'$ so that periodic boundary

conditions do not interfere in the commutator. Replacing this, one concludes from (20)

$$\begin{aligned}
L_{\eta,\rho,\rho'}^{\text{per}}(t)^2 &\geq \left(2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) G_{\rho,t}^2\right) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + \frac{g^2}{\eta^2} G_{\rho,t}^4 \\
&\quad - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} - \frac{5\sqrt{\pi}}{\lambda\eta^2\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d\sqrt{M} \mathbf{1} \\
&\quad - \frac{32\|H\|}{\rho\eta^2} \|[D_0, H]\| \mathbf{1} - \frac{1}{\eta} \sum_{j=1}^d \|[s_{\rho',j}, H]\| \mathbf{1}.
\end{aligned}$$

Using $\sum_{j=1}^d (1 - c_{\rho',j}) \leq 2d$ and also bounding $\|[s_{\rho',j}, H_{\rho'}]\| \leq \frac{\pi}{\rho'} M$ by Lemma 11 and $\|[D_0, H]\| \leq dM$, this implies

$$\begin{aligned}
L_{\eta,\rho,\rho'}^{\text{per}}(t)^2 &\geq \left(2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) G_{\rho,t}^2 + \frac{g^2}{4d\eta^2} G_{\rho,t}^4\right) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + \frac{g^2}{2\eta^2} G_{\rho,t}^4 \\
&\quad - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} - \frac{5\sqrt{\pi}}{\lambda\eta^2\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d\sqrt{M} \mathbf{1} \\
&\quad - \frac{32\|H\|}{\rho\eta^2} d M - \frac{\pi}{\eta\rho'} d M \mathbf{1}.
\end{aligned}$$

Now the parenthesis in the first summand seen as a function of $G_{\rho,t}^2$ has a negative derivative for all $G_{\rho,t}^2 \in [0, 1]$ as long as

$$2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\| \geq \frac{g^2}{2d\eta^2},$$

which, after discarding the summand $\|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|$, actually always holds for $\lambda \geq \frac{g}{\eta}$ and $\eta \geq \frac{g}{4d}$ (which is required in Theorem 1). Then the minimum of the parenthesis is taken at $G_t^2 = 1$. Hence

$$\begin{aligned}
L_{\eta,\rho,\rho'}^{\text{per}}(t)^2 &\geq \left(2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) + \frac{g^2}{4d\eta^2}\right) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + \frac{g^2}{2\eta^2} G_{\rho,t}^4 \\
&\quad - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} - \frac{5\sqrt{\pi}}{\lambda\eta^2\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d\sqrt{M} \mathbf{1} \\
&\quad - \frac{32\|H\|}{\rho\eta^2} d M \mathbf{1} - \frac{\pi}{\eta\rho'} d M \mathbf{1} \\
&\geq \left(2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) + \frac{g^2}{8d\eta^2}\right) \sum_{j=1}^d (\mathbf{1} - c_{\rho',j}) + \frac{g^2\pi^2}{2560d\eta^2} \mathbf{1} \\
&\quad - \frac{2\pi}{\lambda\eta^2\rho'} \max\{\|H\|, \lambda\eta\} d M \mathbf{1} - \frac{5\sqrt{\pi}}{\lambda\eta^2\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d\sqrt{M} \mathbf{1} \\
&\quad - \frac{32\|H\|}{\rho\eta^2} d M \mathbf{1} - \frac{\pi}{\eta\rho'} d M \mathbf{1},
\end{aligned}$$

because $\sum_{j=1}^d (1 - c_{\rho', j}) \geq \frac{1}{5} \sum_{j=1}^d \frac{\pi^2}{(\rho')^2} X_j^2 \geq \frac{\pi^2 \rho^2}{80(\rho')^2} (\mathbf{1} - G_t^4) \geq \frac{\pi^2}{320} (\mathbf{1} - G_t^4)$ where the second step holds as $(\mathbf{1} - G_t^4) \chi(\sum_{j=1}^d X_j^2 \leq \frac{\rho^2}{16}) = 0$ and the final step used the bound $\rho' \leq 2\rho$. Because $\text{spec}(H_{\rho'}) \subset \text{spec}(H)$

$$\|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\| \leq \|\lambda \mathbf{1} - \lambda^{-1} \frac{H^2}{\eta^2}\|.$$

Thus let us minimize $f(\lambda) = 2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \frac{H^2}{\eta^2}\|$ over $\lambda \in [\frac{g}{\eta}, \frac{\|H\|}{\eta}]$. By spectral calculus and elementary analysis one finds

$$\min_{\eta\lambda \in [g, \|H\|]} f(\lambda) = \min_{\eta\lambda \in [g, \|H\|]} \lambda \left(2 + \max \left\{ 1 - \frac{g^2}{\lambda^2 \eta^2}, \frac{\|H\|^2}{\lambda^2 \eta^2} - 1 \right\} \right) = f(\lambda_c)$$

where $(\lambda_c)^2 = \frac{\|H\|^2 + g^2}{2\eta^2}$. Thus

$$\begin{aligned} f(\lambda_c) &= \frac{1}{\eta} \frac{g^2 + 3\|H\|^2}{\sqrt{2} \sqrt{g^2 + \|H\|^2}} = \frac{\|H\|}{\eta} \left(\sqrt{1 - \frac{1}{2} \left(1 - \frac{g^2}{\|H\|^2}\right)} + \frac{1}{\sqrt{1 - \frac{1}{2} \left(1 - \frac{g^2}{\|H\|^2}\right)}} \right) \\ &\leq \frac{\|H\|}{\eta} \left(2 + \frac{1}{8} \left(1 - \frac{g^2}{\|H\|^2}\right)^2 \right) \leq \frac{\|H\|}{\eta} \left(2 + \frac{1}{2} \left(1 - \frac{g}{\|H\|}\right)^2 \right), \end{aligned}$$

where the first inequality follows from $\sqrt{1 + \epsilon} + \frac{1}{\sqrt{1 + \epsilon}} \leq 2 + \frac{1}{2}\epsilon^2$ holding for $\epsilon \in [-\frac{1}{2}, 1]$. Then the term in the parenthesis satisfies

$$2 - (2\lambda + \|\lambda \mathbf{1} - \lambda^{-1} \tilde{H}_{\rho'}^2\|) + \frac{g^2}{4d\eta^2} \geq 2 - f(\lambda_c) + \frac{g^2}{8d\eta^2} \geq 2 - \frac{\|H\|}{\eta} \left(2 + \frac{1}{2} \left(1 - \frac{g}{\|H\|}\right)^2 \right) + \frac{g^2}{8d\eta^2} \geq 0,$$

where the last inequality is precisely the bound (3) divided by $\frac{\|H\|}{2\eta}$. Due to $\|H\| \geq \eta\lambda_c \geq g$ and the equality $\max\{\|H\|, \lambda_c \eta\} = \|H\|$ one hence deduces

$$\begin{aligned} L_{\eta, \rho, \rho'}^{\text{per}}(t)^2 &\geq \frac{g^2 \pi^2}{2560d\eta^2} \mathbf{1} - \frac{2\pi}{g\eta\rho'} \|H\| d M \mathbf{1} - \frac{5\sqrt{\pi}}{g\eta\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d \sqrt{M} \mathbf{1} \\ &\quad - \frac{128\|H\|}{\rho\eta g} d M \mathbf{1} - \frac{\pi}{\eta\rho' g} d \|H\| M \mathbf{1} \\ &\geq \frac{g^2}{300d\eta^2} \mathbf{1} - \frac{128 + 3\pi}{g\eta\rho} \|H\| d M \mathbf{1} - \frac{5\sqrt{\pi}}{g\eta\sqrt{2\rho'}} \|H\|^{\frac{3}{2}} d \sqrt{M} \mathbf{1}, \end{aligned}$$

where $1 \leq \frac{\|H\|}{g}$ and $\eta \geq \frac{g}{4}$ was used. Now ρ is bounded below by (2) and ρ' is bounded below by ρ . Then elementary numerical estimates show that

$$L_{\eta, \rho, \rho'}^{\text{per}}(t)^2 \geq \frac{g^2}{600 d \eta^2}, \tag{22}$$

uniformly in $t \in [0, 1]$. For $t = 0$ this implies (4), namely the first claim of the proposition.

Now let (η, ρ) and (η', ρ') both satisfy all the conditions of Theorem 1 and suppose, without restriction, that $\rho' \in [\rho, 2\rho]$. Continuity of $L_{\eta, \rho}^{\text{per}}$ in η together with the bound (22) shows that

$\text{Sig}(L_{\eta, \rho'}^{\text{per}}) = \text{Sig}(L_{\eta', \rho'}^{\text{per}})$. Hence one can assume $\eta' = \eta$. Then the above argument shows that the signature does not change along the paths $t \in [0, 1] \mapsto L_{\eta, \rho, \rho}^{\text{per}}(t)$ and $t \in [0, 1] \mapsto L_{\eta, \rho, \rho'}^{\text{per}}(t)$. But

$$L_{\eta, \rho, \rho'}^{\text{per}}(1) = L_{\eta, \rho, \rho'}^{\text{per}}(1) \mathcal{H}_{\frac{\rho}{2}} \oplus (D_{\rho'}^{\text{per}})_{\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}},$$

where $D_{\rho'}^{\text{per}}$ is the first summand in (1) with ρ replaced by ρ' and the lower index $\mathcal{H}_{\frac{\rho}{2}}$ and $(\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}})$ indicates its restriction to $(\mathcal{H}_{\frac{\rho}{2}} \oplus \mathcal{H}_{\frac{\rho}{2}}) \otimes \mathbb{C}^{d'}$ and $((\mathcal{H}_{\rho'} \oplus \mathcal{H}_{\rho'}) \ominus (\mathcal{H}_{\frac{\rho}{2}} \oplus \mathcal{H}_{\frac{\rho}{2}})) \otimes \mathbb{C}^{d'}$ respectively. As $\text{Sig}((D_{\rho'}^{\text{per}})_{\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}}) = 0$ it is sufficient to show

$$\text{Sig}(L_{\eta, \rho, \rho'}^{\text{per}}(1) \mathcal{H}_{\frac{\rho}{2}}) = \text{Sig}(L_{\eta, \rho, \rho}^{\text{per}}(1) \mathcal{H}_{\frac{\rho}{2}}).$$

This follows as the path

$$t \in [0, 1] \mapsto \sum_{j=1}^d \sin\left(\frac{\pi}{t\rho + (1-t)\rho'} X_j\right) \gamma_j + \left(\sum_{j=1}^d (\mathbf{1} - \cos\left(\frac{\pi}{t\rho + (1-t)\rho'} X_j\right)) - \frac{1}{\eta} G_{\frac{\rho}{2}}(|D_0|) H G_{\frac{\rho}{2}}(|D_0|) \right) \gamma_{d+1},$$

of operators on $(\mathcal{H}_{\frac{\rho}{2}} \oplus \mathcal{H}_{\frac{\rho}{2}}) \otimes \mathbb{C}^{d'}$ entirely lies in the invertibles, which can be checked directly by an argument very similar to the one showing that the path in (19) lies in the invertibles. \square

In the proof of Proposition 12 it was shown that $L_{\eta, \rho}^{\text{per}}$ can be homotopically deformed into

$$L_{\eta, \rho, \rho}^{\text{per}}(1) = \sum_{j=1}^d \sin\left(\frac{\pi}{\rho} X_j\right) \gamma_j + \sum_{j=1}^d (\mathbf{1} - \cos\left(\frac{\pi}{\rho} X_j\right)) \gamma_{d+1} - \frac{1}{\eta} G_{\rho} H_{\rho}^{\text{per}} G_{\rho} \gamma_{d+1} \quad (23)$$

without closing the gap, provided the conditions of Theorem 1 hold. In particular, one has $\text{Sig}(L_{\eta, \rho}^{\text{per}}) = \text{Sig}(L_{\eta, \rho, \rho}^{\text{per}}(1))$. Here $G_{\rho} = G_{\frac{\rho}{2}}(|D_0|)$ is a tapering function so that $G_{\rho} H_{\rho}^{\text{per}} G_{\rho}$ is a tempered Hamiltonian which is localized strictly inside the volume $[-\frac{\rho}{2}, \frac{\rho}{2}]^d$. In particular, the boundary conditions on the Hamiltonian are irrelevant, namely $G_{\rho} H_{\rho}^{\text{per}} G_{\rho} = G_{\rho} H G_{\rho}$. As already stressed in Section 1, this reflects that the signature is a local topological invariant associated to the Hamiltonian.

The next step in the proof of Theorem 1 will be to deform the first two summands in (23). To spell out that homotopy, it will be useful to express $L_{\eta, \rho, \rho}^{\text{per}}(1)$ through the function ξ by means of the formula (12):

$$L_{\eta, \rho, \rho}^{\text{per}}(1) = 2 \left(\sum_{j=1}^d \xi_{\rho, j} \sqrt{\mathbf{1} - \xi_{\rho, j}^2} \gamma_j + \sum_{j=1}^d \xi_{\rho, j}^2 \gamma_{d+1} \right) - \frac{1}{\eta} G_{\rho} H G_{\rho} \gamma_{d+1}, \quad (24)$$

where $\xi_{\rho, j} = \sin(\frac{\pi}{2\rho} X_j)$, see (11). Then the homotopy in the parameter $s \in [0, 1]$ will be given by

$$L_{\eta, \rho}^{\text{per}}(1, s) = 2 \left(\sum_{j=1}^d \xi_{\rho, j} \sqrt{\mathbf{1} - s^2 \xi_{\rho, j}^2} \gamma_j + \sum_{j=1}^d s \xi_{\rho, j}^2 \gamma_{d+1} \right) - \frac{1}{\eta} G_{\rho} H G_{\rho} \gamma_{d+1}. \quad (25)$$

Clearly $L_{\eta, \rho}^{\text{per}}(1, 1) = L_{\eta, \rho, \rho}^{\text{per}}(1)$, but moreover $L_{\eta, \rho}^{\text{per}}(1, 0)$ is essentially the spectral localizer with the damped Hamiltonian $G_{\rho} H G_{\rho}$. It was already proved in earlier works [14, 15, 5] that the

half-signature of the spectral localizer with this damped Hamiltonian is equal to the index pairing $\text{Ind}(PFP + \mathbf{1} - P)$ with F as below, which by an index theorem [18] is in turn equal to the Chern number. Hence a central element of the proof of Theorem 1 consists in checking that the homotopy $s \in [0, 1] \mapsto L_{\eta, \rho}^{\text{per}}(1, s)$ lies in the invertible matrices.

Proposition 13 *For (η, ρ) satisfying the conditions of Theorem 1,*

$$\frac{1}{2} \text{Sig}(L_{\eta, \rho}^{\text{per}}) = \text{Ind}(PFP + \mathbf{1} - P),$$

where $P = \chi(H < 0)$ and $F = D_0|D_0|^{-1}$ is the phase of $D_0 = \sum_{j=1}^d X_j \hat{\gamma}_j$, suitably regularized at the origin.

Proof. The main result of [15, 5] states that $PFP + \mathbf{1} - P$ is a Fredholm operator with index that can be computed as the half-signature of the finite-volume restrictions $L_{\kappa, \rho}$ of the spectral localizer defined in (6), provided that the parameters $\kappa > 0$ and $\rho < \infty$ are sufficiently small and large respectively. As also the signature of the periodic spectral localizer $L_{\eta, \rho}^{\text{per}}$ and its damped version $L_{\eta, \rho, \rho}^{\text{per}}(1)$ is stable for such parameters by Proposition 12, it hence merely has to be shown that for such parameters $L_{\eta, \rho, \rho}^{\text{per}}(1)$ is homotopic to $L_{\kappa, \rho}$ inside of the invertible matrices so that the signature does not change. Being able to choose ρ sufficiently large considerably simplifies the proof because one can simply neglect all commutators of the type $[G_{\frac{\rho}{2}}(|D_0|), H]$ and $[\xi_{\rho, j}^2, H]$ as they are of order $\mathcal{O}(\frac{1}{\rho})$. Here $G_{\frac{\rho}{2}}$ is the same function as used in the proof of Proposition 12. Furthermore, it is possible to choose $\eta = \frac{5}{4}\|H\|$ because then the bound (3) is automatically satisfied.

Let us start out by proving that the path $s \in [0, 1] \mapsto L_{\eta, \rho}^{\text{per}}(1, s)$ defined in (25) lies in the invertible matrices for ρ sufficiently large. The proof will essentially follow the first part of the proof of Proposition 12, namely one simply checks that $L_{\eta, \rho}^{\text{per}}(1, s)^2 > 0$ for all $s \in [0, 1]$. Setting $G_\rho = G_{\frac{\rho}{2}}(|D_0|)$ and $\hat{H} = \frac{1}{\eta}H = \frac{4}{5}\frac{H}{\|H\|}$ and, moreover, discarding commutators as described above, the square of (25) can be computed using

$$\left(\sum_{j=1}^d \xi_{\rho, j} \sqrt{\mathbf{1} - s^2 \xi_{\rho, j}^2} \gamma_j + \sum_{j=1}^d s \xi_{\rho, j}^2 \gamma_{d+1} \right)^2 \geq \sum_{j=1}^d \xi_{\rho, j}^2$$

and thus satisfies

$$\begin{aligned} L_{\eta, \rho}^{\text{per}}(1, s)^2 &\geq 4 \sum_{j=1}^d \xi_{\rho, j}^2 + G_\rho^2 \hat{H}^2 G_\rho^2 - 2s \left(\sum_{j=1}^d \xi_{\rho, j}^2 G_\rho \hat{H} G_\rho + G_\rho \hat{H} G_\rho \sum_{j=1}^d \xi_{\rho, j}^2 \right) + \mathcal{O}\left(\frac{1}{\sqrt{\rho}}\right) \\ &= 4 \sum_{j=1}^d \xi_{\rho, j}^2 (\mathbf{1} - s G_\rho^2) + G_\rho^2 \hat{H}^2 G_\rho^2 + 2s G_\rho \left(\sum_{j=1}^d \xi_{\rho, j}^2 (\mathbf{1} - \hat{H}) + (\mathbf{1} - \hat{H}) \sum_{j=1}^d \xi_{\rho, j}^2 \right) G_\rho + \mathcal{O}\left(\frac{1}{\sqrt{\rho}}\right), \end{aligned}$$

with a remainder that is uniformly bounded in $s \in [0, 1]$. In the first summand one can use the lower bound $\mathbf{1} - s G_\rho^2 \geq \mathbf{1} - G_\rho^2$. In the second summand, let us simply use $G_\rho^2 \hat{H}^2 G_\rho^2 \geq$

$\frac{16}{25} \frac{g^2}{\|H\|^2} G_\rho^4$. Finally, the last summand is non-negative up to errors $\mathcal{O}(\frac{1}{\rho})$ because $\mathbf{1} - \hat{H} \geq \frac{1}{5} \mathbf{1}$ and $\xi_{\rho,j}^2(\mathbf{1} - \hat{H}) = \xi_{\rho,j}(\mathbf{1} - \hat{H})\xi_{\rho,j} + \mathcal{O}(\frac{1}{\rho})$ by Lemma 11. Hence

$$L_{\eta,\rho}^{\text{per}}(1, s)^2 \geq 4 \sum_{j=1}^d \xi_{\rho,j}^2(\mathbf{1} - G_\rho^2) + \frac{16}{25} \frac{g^2}{\|H\|^2} G_\rho^4 + \mathcal{O}(\frac{1}{\sqrt{\rho}}).$$

Now one uses the geometric fact that $\sum_{j=1}^d \xi_{\rho,j}^2(\mathbf{1} - G_\rho^2) \geq \sin(\frac{\pi}{8\sqrt{d}})^2(\mathbf{1} - G_\rho^2)$ and the bound $1 - G^2 + G^4 \geq \frac{3}{4}$ holding for any number $G \in [0, 1]$ to conclude

$$\begin{aligned} L_{\eta,\rho}^{\text{per}}(1, s)^2 &\geq \min \left\{ \frac{16}{25} \frac{g^2}{\|H\|^2}, 4 \sin(\frac{\pi}{8\sqrt{d}})^2 \right\} (\mathbf{1} - G_\rho^2 + G_\rho^4) + \mathcal{O}(\frac{1}{\sqrt{\rho}}) \\ &\geq \frac{3}{4} \min \left\{ \frac{16}{25} \frac{g^2}{\|H\|^2}, 4 \sin(\frac{\pi}{8\sqrt{d}})^2 \right\} + \mathcal{O}(\frac{1}{\sqrt{\rho}}). \end{aligned}$$

The next (and essentially final) step is to homotopically deform

$$L_{\eta,\rho}^{\text{per}}(1, 0) = \begin{pmatrix} -G_\rho \hat{H} G_\rho & 2 \sum_{j=1}^d \xi_{\rho,j} \hat{\gamma}_j^* \\ 2 \sum_{j=1}^d \xi_{\rho,j} \hat{\gamma}_j & G_\rho \hat{H} G_\rho \end{pmatrix}$$

into

$$\hat{L}_{\kappa,\rho} = \begin{pmatrix} -G_\rho \hat{H} G_\rho & \kappa \sum_{j=1}^d X_j \hat{\gamma}_j^* \\ \kappa \sum_{j=1}^d X_j \hat{\gamma}_j & G_\rho \hat{H} G_\rho \end{pmatrix},$$

inside of the invertible matrices. This can readily be checked for the straight-line path from $L_{\eta,\rho}^{\text{per}}(1, 0)$ to $\hat{L}_{\kappa,\rho}$. But due to the stability of the signature, one has $\text{Sig}(\hat{L}_{\kappa,\rho}) = \text{Sig}(L_{\kappa,\rho})$ for κ sufficiently small and ρ sufficiently large. But by [15, Theorem 3] or [5, Theorem 10.3.1] the index of $PFP + \mathbf{1} - P$ equals the half-signature of $L_{\kappa,\rho}$. \square

Proof of Theorem 1: By a well-known index theorem [18] the Chern number $\text{Ch}_d(P)$ is equal to the index pairing appearing in Proposition 13, which hence directly implies the claim. \square

Remark 14 As already stressed in the introduction, the quantitative aspects of the proofs in this section are far from optimal. Considerably better (but still not optimal) estimates can be obtained by working with

$$L_{\kappa,\eta,\rho}^{\text{per}} = \begin{pmatrix} \sum_{j=1}^d (\mathbf{1} - \cos(\pi \chi_\kappa(X_j))) & \sum_{j=1}^d \sin(\pi \chi_\kappa(X_j)) \hat{\gamma}_j^* \\ \sum_{j=1}^d \sin(\pi \chi_\kappa(X_j)) \hat{\gamma}_j & - \sum_{j=1}^d (\mathbf{1} - \cos(\pi \chi_\kappa(X_j))) \end{pmatrix} + \frac{1}{\eta} \begin{pmatrix} -H_\rho^{\text{per}} & 0 \\ 0 & H_\rho^{\text{per}} \end{pmatrix},$$

where $\chi_\kappa(x) = \chi(\kappa x)$ is constructed from a suitable switch function χ and $\kappa > 0$ is a supplementary parameter. Modifying the above proofs one can show that $\text{Ch}_d(P) = \frac{1}{2} \text{Sig}(L_{\kappa,\eta,\rho}^{\text{per}})$ provided that (3) holds as well as the bounds

$$\kappa \leq \frac{g^3}{890 d^2 M \|H\| \eta}, \quad \rho \geq \frac{4\sqrt{d}}{\kappa}.$$

Note that this merely requires $\rho \geq C' M \|H\| \eta / g^3$ for some constant C' , which is considerably weaker than (2). However, the advantage of $L_{\eta,\rho}^{\text{per}}$ given in (1) over $L_{\kappa,\eta,\rho}^{\text{per}}$ is the simplicity of the formula as well as the fact that $L_{\eta,\rho}^{\text{per}}$ essentially contains no other free parameter than the volume (as discussed in Remark 3, one can safely choose $\eta \approx \|H\|$). \diamond

3 Odd periodic spectral localizer

In this brief section, the odd-dimensional counterpart to Theorem 1 is described. Hence let H be a finite-range tight-binding Hamiltonian on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ with d odd and L even and with a spectral gap at 0. On the fiber \mathbb{C}^L let J be a selfadjoint unitary with eigenvalues 1 and -1 of equal multiplicity $\frac{L}{2}$. The Hamiltonian is supposed to be chiral in the sense that $JHJ = -H$. This implies that it is off-diagonal in the grading of J :

$$H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad (26)$$

where hence A is an invertible short-range periodic operator on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{\frac{L}{2}})$. As such it has a strong invariant $\text{Ch}_d(A) \in \mathbb{Z}$ called either a (higher) winding number or also an odd Chern number [18]. Previous results [14, 5] allow to compute it as the signature of the spectral localizer. Here a similar connection is established to the (odd) *periodic spectral localizer* which is defined to be the finite-dimensional matrix on $\mathcal{H}_\rho \otimes \mathbb{C}^{d'}$ where $\mathcal{H}_\rho = \ell^2((\mathbb{Z}/(2\rho\mathbb{Z}))^d, \mathbb{C}^L)$ given by

$$L_{\eta,\rho}^{\text{per}} = \begin{pmatrix} \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \gamma_j & \sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) \\ \sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) & -\sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \gamma_j \end{pmatrix} - \frac{1}{\eta} H_\rho^{\text{per}}. \quad (27)$$

Just as in (1), the size ρ is a multiple of the periodicities of H , H_ρ^{per} is the Hamiltonian with periodic boundary conditions on \mathcal{H}_ρ and $\gamma_1, \dots, \gamma_d$ is an irreducible Clifford representation acting on $\mathbb{C}^{d'}$. Note that H_ρ^{per} is again off-diagonal and its upper right entry is denoted by A_ρ^{per} .

Theorem 15 *Let d be odd. Suppose that H is a finite-range periodic operator on $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ of the form (26). Let η and ρ satisfy the same conditions as in Theorem 1, in particular the bounds (2) and (3). Then $L_{\eta,\rho}^{\text{per}}$ defined in (27) is gapped with the bound (4) and the odd Chern number is*

$$\text{Ch}_d(A) = \frac{1}{2} \text{Sig}(L_{\eta,\rho}^{\text{per}}). \quad (28)$$

All the comments of Section 1 transpose to the odd-dimensional case. Example 26 in Section 5 explains that, in the case of a flat band Hamiltonian, the signature invariant in Theorem 15 is in fact associated to a fuzzy torus associated to H .

Sketch of proof of Theorem 15. A detailed proof will not be provided as it merely a modification of the proof of Theorem 1. However, let us briefly sketch the strategy of the argument. Unless differences are stressed, the same notations as in Section 2 will be used. Here $G_{\rho,t} = G_{\rho,t}(D)$ for $t \in [0, 1]$ and $D = \sum_{j=1}^d X_j \gamma_j$. By an argument similar to the one leading to Proposition 12, one shows that the path $t \in [0, 1] \mapsto L_{\eta,\rho,\rho'}^{\text{per}}(t)$ for

$$L_{\eta,\rho,\rho'}^{\text{per}}(t) = \begin{pmatrix} \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \gamma_j & \sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) \\ \sum_{j=1}^d (\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)) & -\sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j) \gamma_j \end{pmatrix} - \frac{1}{\eta} \begin{pmatrix} 0 & G_{\rho,t} A_{\rho'}^{\text{per}} G_{\rho,t} \\ G_{\rho,t} (A_{\rho'}^{\text{per}})^* G_{\rho,t} & 0 \end{pmatrix}_{\rho'}$$

lies in the invertibles and fulfills the bound (4) if η and ρ satisfy the conditions of Theorem 15 and ρ' fulfills $\rho \leq \rho' \leq 2\rho$. Then, as in the proof of Proposition 12 one can conclude that $\text{Sig}(L_{\eta,\rho}^{\text{per}})$ is independent of η and ρ in the permitted range of parameters.

To show that the half-signature of the periodic spectral localizer equals the Chern number of A let us use the path $s \in [0, 1] \mapsto L_{\kappa, \eta, \rho}^{\text{per}}(1, s)$ given by

$$L_{\eta, \rho}^{\text{per}}(1, s) = \begin{pmatrix} 2 \sum_{j=1}^d \xi_{\rho, j} \sqrt{\mathbf{1} - s^2 \xi_{\rho, j}^2} \gamma_j & 2s \sum_{j=1}^d \xi_{\rho, j}^2 - \frac{1}{\eta} G_{\rho} A_{\rho}^{\text{per}} G_{\rho} \\ 2s \sum_{j=1}^d \xi_{\rho, j}^2 - \frac{1}{\eta} G_{\rho} (A_{\rho}^{\text{per}})^* G_{\rho} & -2 \sum_{j=1}^d \xi_{\rho, j} \sqrt{\mathbf{1} - s^2 \xi_{\rho, j}^2} \gamma_j \end{pmatrix}_{\rho} \quad (29)$$

with $\xi_{\rho, j} = \sin(\frac{\pi}{2\rho} X_j)$ as in (11). As in the proof of Proposition 13, one checks that $L_{\eta, \rho}^{\text{per}}(1, s)$ is invertible for ρ sufficiently large. Thus $\text{Sig}(L_{\eta, \rho}^{\text{per}}) = \text{Sig}(L_{\eta, \rho}^{\text{per}}(1, 0))$. Finally, by transposing the techniques of the proof of Proposition 13, $L_{\eta, \rho}^{\text{per}}(1, 0)$ can be deformed inside the set of invertibles into the odd spectral localizer introduced in Section 1.4 of [14] but with A replaced by $-A$. Then as $\text{Ch}_d(A) = \text{Ch}_d(-A)$ Theorem 1 in [14] allows to conclude. \square

4 \mathbb{Z}_2 -invariants via periodic spectral localizer

This section addresses the real cases of the CAZ (Cartan-Altland-Zirnbauer) classification. They all impose a symmetry property on the Hamiltonian that involves a complex conjugation (real structure, denoted by an overline) on the complex Hilbert space. There are 64 such cases, stemming from an 8-periodicity in both dimension d and the CAZ classes (both routed in Bott periodicity). Only 16 of these cases are known to lead to \mathbb{Z}_2 -valued strong invariants [19, 10]. In previous works [13, 4] it was shown that a real skew-adjoint version of the spectral localizer, the so-called skew localizer, can be used to compute these \mathbb{Z}_2 -indices. In this section the associated skew periodic localizer is introduced for the physically most relevant low-dimensional cases, and it is shown that the sign of its Pfaffian is connected to the \mathbb{Z}_2 -invariants. An exhaustive treatment of all cases as in [4] is not provided here.

Let us now sketch the general common scheme. Like in [4], the skew periodic localizer is constructed from the periodic localizer by a basis change and multiplication by ι . More explicitly, in each of the relevant CAZ classes in even dimension d , there is a unitary $R : (\mathcal{H}_{\rho} \oplus \mathcal{H}_{\rho}) \otimes \mathbb{C}^{d'} \rightarrow (\mathcal{H}_{\rho} \oplus \mathcal{H}_{\rho}) \otimes \mathbb{C}^{d'}$ such that the skew periodic localizer given by

$$L_{\eta, \rho}^{\text{skew, per}} = \iota R^* L_{\eta, \rho}^{\text{per}} R$$

is a real and skew-adjoint operator on $(\mathcal{H}_{\rho} \oplus \mathcal{H}_{\rho}) \otimes \mathbb{C}^{d'}$. For odd d , the only modification is that $R : \mathcal{H}_{\rho} \otimes \mathbb{C}^{d'} \rightarrow \mathcal{H}_{\rho} \otimes \mathbb{C}^{d'}$ and $L_{\eta, \rho}^{\text{skew, per}}$ then acts on $\mathcal{H}_{\rho} \otimes \mathbb{C}^{d'}$. Clearly R depends on ρ , but as R is local, this dependence is suppressed in the notation. For even d and (η, ρ) as in Theorem 1 and for odd d and (η, ρ) as in Theorem 15 this operator is invertible and therefore has a non-vanishing Pfaffian. Moreover, let \tilde{D} be a suitable perturbation of the first summand of the periodic spectral localizer (1) or (27) by a term localized at the origin such that \tilde{D} is invertible and such that $D^{\text{skew}} = \iota R^* \tilde{D} R$ is real and skew-adjoint. Then its Pfaffian is well-defined, does not vanish and the \mathbb{Z}_2 -index associated to the Hamiltonian H is in each case proven to be given by

$$\text{Ind}_2(T) = \text{sgn}(\text{Pf}(L_{\eta, \rho}^{\text{skew, per}})) \text{sgn}(\text{Pf}(D_{\rho}^{\text{skew}})) \in \mathbb{Z}_2, \quad (30)$$

where $\text{Ind}_2(T) = \dim(\text{Ker}(T)) \bmod 2$ in the 16 relevant cases is defined as in [10] using the Fredholm operators $T = PFP + (\mathbf{1} - P)$ or $T = EAE + (\mathbf{1} - E)$ with F being the Dirac phase and E the Hardy projection. Let us stress that the sign of the Pfaffian depends on the choice of basis and that for a suitable choice one can always arrange that $\text{sgn}(\text{Pf}(D_\rho^{\text{skew}})) = 1$ so that the equality (30) takes a more simple form. Let us also note that the index pairing and therefore also the \mathbb{Z}_2 -index does depend on the perturbation of the first summand of the periodic spectral localizer at the origin. In the following this scheme is materialized in some of the important cases by constructing the unitary R . Also explicit formulas for $L_{\eta,\rho}^{\text{skew,per}}$ will be provided in these cases.

Case $d = 1$ for CAZ class DIII: For $d = 1$ the Dirac operator is just the position operator $D = X$. Let us add the projection onto its kernel to make it invertible, namely $\tilde{D} = X + p_0$ where p_0 is the orthogonal projection onto $\text{Ker}(D) = \text{span}(|0\rangle)$. Let H be a finite-range periodic Hamiltonian on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ with L even that is in CAZ class DIII namely that has a odd time-reversal symmetry and an even particle-hole symmetry. In a suitably chosen basis, H is of the form (26) with an $A \in \mathbb{B}(\ell^2(\mathbb{Z}, \mathbb{C}^{\frac{L}{2}}))$ fulfilling the additional symmetry

$$(\iota\sigma_2)^* A^* \iota\sigma_2 = \bar{A} \quad (31)$$

where σ_2 denotes the second Pauli matrix acting only on the fiber. The index pairing is $T = EAE + \mathbf{1} - E$ where $E = \chi(\tilde{D} \geq 0)$. Then set

$$Q = \begin{pmatrix} 0 & \iota\sigma_2 \\ -\iota\sigma_2 & 0 \end{pmatrix}, \quad R = \frac{1+\iota}{2} \begin{pmatrix} \mathbf{1} & \sigma_2 \\ -\sigma_2 & \mathbf{1} \end{pmatrix}. \quad (32)$$

Then Q is a self-adjoint real unitary and R is a particular choice for the root, namely $R^2 = Q$ (in principle one may choose other roots, but this choice leads to nice formulas below). The one-dimensional periodic spectral localizer is

$$L_{\eta,\rho}^{\text{per}} = \begin{pmatrix} \sin(\frac{\pi}{\rho}X) & \mathbf{1} - \cos(\frac{\pi}{\rho}X) \\ \mathbf{1} - \cos(\frac{\pi}{\rho}X) & -\sin(\frac{\pi}{\rho}X) \end{pmatrix} - \frac{1}{\eta} \begin{pmatrix} 0 & A_\rho^{\text{per}} \\ (A_\rho^{\text{per}})^* & 0 \end{pmatrix}$$

where A_ρ^{per} is the off-diagonal entry of H_ρ^{per} . As A_ρ^{per} fulfills the same symmetry relation as A , namely (31) holds with A replaced by A_ρ^{per} , one then has $Q\overline{L_{\kappa,\eta,\rho}^{\text{per}}}Q = -L_{\kappa,\eta,\rho}^{\text{per}}$. Then set

$$L_{\eta,\rho}^{\text{skew,per}} = \iota R^* L_{\eta,\rho}^{\text{per}} R, \quad D_\rho^{\text{skew}} = \iota R^* \tilde{D}_\rho R,$$

where

$$\tilde{D}_\rho = \begin{pmatrix} \sin(\frac{\pi}{\rho}X) + p_0 & \mathbf{1} - \cos(\frac{\pi}{\rho}X) \\ \mathbf{1} - \cos(\frac{\pi}{\rho}X) & -\sin(\frac{\pi}{\rho}X) - p_0 \end{pmatrix}_\rho.$$

Both $L_{\eta,\rho}^{\text{skew,per}}$ and D_ρ^{skew} are bounded real and skew-adjoint operators. Explicitly one finds:

$$L_{\eta,\rho}^{\text{skew,per}} = \begin{pmatrix} -(\mathbf{1} - \cos(\frac{\pi}{\rho}X) - \frac{1}{\eta} \Re(A_\rho^{\text{per}}))\iota\sigma_2 & \sin(\frac{\pi}{\rho}X)\iota\sigma_2 + \frac{1}{\eta} \Im(A_\rho^{\text{per}}) \\ \sin(\frac{\pi}{\rho}X)\iota\sigma_2 - \frac{1}{\eta} \Im(A_\rho^{\text{per}})^* & \iota\sigma_2(\mathbf{1} - \cos(\frac{\pi}{\rho}X) - \frac{1}{\eta} \Re(A_\rho^{\text{per}})) \end{pmatrix},$$

where $\Re(B) = \frac{1}{2}(B + \bar{B})$ and $\Im(B) = \frac{1}{2i}(B - \bar{B})$ are real (note that they are different from $\Re e(B) = \frac{1}{2}(B + B^*)$ and $\Im m(B) = \frac{1}{2i}(B - B^*)$).

Proposition 16 For (η, ρ) as in Theorem 1, $L_{\eta, \rho}^{\text{skew, per}}$ and D^{skew} are invertible and (30) holds.

Proof. First of all $L_{\eta, \rho}^{\text{per}}$ and therefore $L_{\eta, \rho}^{\text{skew, per}}$ is invertible by Theorem 15. Therefore and as D_{ρ}^{skew} is a real skew-adjoint invertible by construction, the r.h.s. of (30) is well-defined. One has to show that it is independent of η and ρ . For $L_{\eta, \rho, \rho'}^{\text{per}}(t)$, as in the proof of Theorem 15,

$$Q \overline{L_{\eta, \rho, \rho'}^{\text{per}}(t)} Q = -L_{\eta, \rho, \rho'}^{\text{per}}(t)$$

for all $t \in [0, 1]$. Therefore $\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(t) R$ is a real skew-adjoint invertible and thus its Pfaffian is well-defined, does not vanish and the sign of this Pfaffian is independent of t . Now let (η, ρ) and (η', ρ') both satisfy all the conditions of Theorem 1 and suppose, without restriction, that $\rho \leq \rho' \leq 2\rho$. Continuity of $L_{\eta, \rho}^{\text{skew, per}}$ in η allows to assume $\eta = \eta'$. By the above argument it is sufficient to show

$$\text{sgn}(\text{Pf}(\iota R^* L_{\eta, \rho, \rho}^{\text{per}}(1) R)) \text{sgn}(\text{Pf}(D_{\rho}^{\text{skew}})) = \text{sgn}(\text{Pf}(\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(1) R)) \text{sgn}(\text{Pf}(D_{\rho'}^{\text{skew}})).$$

But

$$L_{\eta, \rho, \rho'}^{\text{per}}(1) = L_{\eta, \rho, \rho'}^{\text{per}}(1) \mathcal{H}_{\frac{\rho}{2}} \oplus (\tilde{D}_{\rho'})_{\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}},$$

where the lower index $\mathcal{H}_{\frac{\rho}{2}}$ indicates the restriction to $\mathcal{H}_{\frac{\rho}{2}} \otimes \mathbb{C}^2$ and the lower index $(\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}})$ indicates the restriction to $(\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}) \otimes \mathbb{C}^2$. Thus

$$\begin{aligned} \text{sgn}(\text{Pf}(\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(1) R)) \text{sgn}(\text{Pf}(D_{\rho'}^{\text{skew}})) &= \text{sgn}(\text{Pf}((\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}})) \text{sgn}(\text{Pf}((D_{\rho'}^{\text{skew}})_{\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}})) \\ &\quad \text{sgn}(\text{Pf}((D_{\rho'}^{\text{skew}})_{\mathcal{H}_{\frac{\rho}{2}}})) \text{sgn}(\text{Pf}((D_{\rho'}^{\text{skew}})_{\mathcal{H}_{\rho'} \ominus \mathcal{H}_{\frac{\rho}{2}}})) \\ &= \text{sgn}(\text{Pf}((\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}})) \text{sgn}(\text{Pf}((D_{\rho'}^{\text{skew}})_{\mathcal{H}_{\frac{\rho}{2}}})) . \end{aligned}$$

By the same argument

$$\text{sgn}(\text{Pf}(\iota R^* L_{\eta, \rho, \rho}^{\text{per}}(1) R)) \text{sgn}(\text{Pf}(D_{\rho}^{\text{skew}})) = \text{sgn}(\text{Pf}((\iota R^* L_{\eta, \rho, \rho}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}})) \text{sgn}(\text{Pf}((D_{\rho}^{\text{skew}})_{\mathcal{H}_{\frac{\rho}{2}}})) .$$

Because the paths $t \in [0, 1] \mapsto (\iota R^* L_{\eta, \rho, t\rho+(1-t)\rho'}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}}$ and $t \in [0, 1] \mapsto (D_{t\rho+(1-t)\rho'}^{\text{skew}})_{\mathcal{H}_{\frac{\rho}{2}}}$ both lay in the real skew-adjoint invertibles

$$\text{sgn}(\text{Pf}((\iota R^* L_{\eta, \rho, \rho'}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}})) = \text{sgn}(\text{Pf}((\iota R^* L_{\eta, \rho, \rho}^{\text{per}}(1) R)_{\mathcal{H}_{\frac{\rho}{2}}}))$$

and

$$\text{sgn}(\text{Pf}((D_{\rho'}^{\text{skew}})_{\mathcal{H}_{\frac{\rho}{2}}})) = \text{sgn}(\text{Pf}(D_{\mathcal{H}_{\frac{\rho}{2}}}^{\text{skew}})) .$$

for any fixed basis of $\mathcal{H}_{\frac{\rho}{2}} \otimes \mathbb{C}^2$. This shows that the r.h.s. of (30) is independent of (η, ρ) in the permitted range of parameters.

Thus it remains to show (30) where ρ can be chosen as large as needed. For $L_{\eta, \rho}^{\text{per}}(1, s)$ as in the proof of Theorem 15

$$Q \overline{L_{\eta, \rho}^{\text{per}}(1, s)} Q = -L_{\eta, \rho, \rho'}^{\text{per}}(1, s)$$

for all $s \in [0, 1]$. Thus $\iota R^* L_{\eta, \rho}^{\text{per}}(1, s)R$ is a path of real skew-adjoint invertibles. Thus for fixed and sufficiently large ρ , the Pfaffian of the skew periodic localizer has the same sign as the Pfaffian of the skew localizer $\widehat{L}_{\kappa, \rho}$ for the considered index pairing introduced in Section 5.2 of [4] but for $-A$ instead of A . In the same way, one checks that the sign of the Pfaffian of D_ρ^{skew} equals the sign of the Pfaffian of $(\iota R^*((D + p_0) \oplus (D + p_0))R)_\rho$. Then the claim follows from Theorem 26 in [4] as $\text{Ind}_2(EAE + (\mathbf{1} - E)) = \text{Ind}_2(-EAE + (\mathbf{1} - E))$. \square

Case $d = 2$ for CAZ class AII: For $d = 2$ the Dirac operator is $D = X_1\gamma_1 + X_2\gamma_2$. Thus its off-diagonal entry is $D_0 = X_1 + \iota X_2$. Again in order to eliminate the kernel, D_0 is replaced by $D_0 + p_0$ with a projection p_0 on the origin. For sake of simplicity let us suppress this in the notations. In the present case, the index pairing is $T = PFP + \mathbf{1} - P$ where $F = D_0|D_0|^{-1}$ is the Dirac phase and $P = \chi(H < 0)$. Then the symmetry of the Hamiltonian is $\sigma_2 \overline{H} \sigma_2 = H$ where σ_2 is the second Pauli matrix which commutes with X_1 and X_2 . Then for Q as in (32) the periodic spectral localizer defined by (1) satisfies $Q \overline{L_{\eta, \rho}^{\text{per}}} Q = -L_{\eta, \rho}^{\text{per}}$. Finally the skew periodic localizer and D^{skew} are again defined by

$$L_{\eta, \rho}^{\text{skew, per}} = \iota R^* L_{\eta, \rho}^{\text{per}} R, \quad D^{\text{skew}} = \iota R^* D_\rho R,$$

for R as in (32). Both are real and skew-adjoint and the explicit form of the skew periodic localizer is

$$L_{\eta, \rho}^{\text{skew, per}} = \begin{pmatrix} -\iota \sigma_2 \sin(\frac{\pi}{\rho} X_1) & \iota \sigma_2 \sum_{j=1}^2 (1 - \cos(\frac{\pi}{\rho} X_j)) + \sin(\frac{\pi}{\rho} X_2) \\ \iota \sigma_2 \sum_{j=1}^2 (1 - \cos(\frac{\pi}{\rho} X_j)) - \sin(\frac{\pi}{\rho} X_2) & \iota \sigma_2 \sin(\frac{\pi}{\rho} X_1) \end{pmatrix} + \frac{1}{\eta} \begin{pmatrix} \Im(H) & -\iota \sigma_2 \Re(H) \\ -\iota \sigma_2 \Re(H) & -\Im(H) \end{pmatrix}.$$

By essentially the same proof as in Proposition 16 one obtains:

Proposition 17 *For (η, ρ) as in Theorem 1, $L_{\eta, \rho}^{\text{skew, per}}$ and D_ρ^{skew} are invertible and (30) holds.*

Case $d = 3$ for CAZ class AII: For $d = 3$ the Dirac operator is $D = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3$. Then let us set $\tilde{D} = D + \gamma_1 p_0$. Let H be a finite-range periodic Hamiltonian on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ that is in CAZ class AII namely that has an odd time-reversal symmetry $\sigma_2 \overline{H} \sigma_2 = H$. The index pairing is $T = E(\mathbf{1} - 2P)E + \mathbf{1} - E$ where $E = \chi(\tilde{D} \geq 0)$ and $P = \chi(H \leq 0)$. The skew periodic localizer can be obtained from the even or odd periodic spectral localizer.

The even periodic spectral localizer, given by (1), fulfills $Q \overline{L_{\eta, \rho}^{\text{per}}} Q = -L_{\eta, \rho}^{\text{per}}$ for

$$Q = \begin{pmatrix} 0 & \sigma_2 \gamma_2 \\ \sigma_2 \gamma_2 & 0 \end{pmatrix}. \quad (33)$$

One possible choice of R is

$$R = \frac{1}{2} \begin{pmatrix} (1 - \iota)\mathcal{R} & (1 + \iota)\mathcal{R} \\ (1 + \iota)\mathcal{R} & (1 - \iota)\mathcal{R} \end{pmatrix}, \quad (34)$$

where $\mathcal{R}^2 = \sigma_2 \gamma_2$ and $\overline{\mathcal{R}} = \mathcal{R}^* = \mathcal{R}^{-1}$. The skew periodic localizer is off-diagonal

$$L_{\eta, \rho}^{\text{skew, per}} = \begin{pmatrix} 0 & B_\rho \\ -B_\rho^* & 0 \end{pmatrix}, \quad (35)$$

for $B_\rho = \mathcal{R}^*(i \sum_{j=1}^3 \sin(\frac{\pi}{\rho} X_j) \gamma_j - \mathbf{31} + \sum_{j=1}^3 \cos(\frac{\pi}{\rho} X_j) + \frac{1}{\eta} H_\rho^{\text{per}}) \mathcal{R}$.

Proposition 18 *For (η, ρ) as in Theorem 1, $L_{\eta, \rho}^{\text{skew, per}}$ and D_ρ^{skew} are invertible and one has (30). Choosing the even periodic spectral localizer and R as in (34) one obtains*

$$\text{Ind}_2(T) = \text{sgn}(\det(B_\rho)) \text{sgn}(\det(C_\rho))$$

for B_ρ as above and $C_\rho = \mathcal{R}^*(i \sum_{j=1}^3 \sin(\frac{\pi}{\rho} X_j) \gamma_j + \gamma_1 p_0 - \mathbf{31} + \sum_{j=1}^3 \cos(\frac{\pi}{\rho} X_j)) \mathcal{R}$.

5 Fuzzy tori and their invariants

This section develops a general theory of fuzzy tori and their invariants. Here the terminology of fuzzy geometric object is meant in the following sense: consider a classical geometric object (compact or non-compact) as a subset of an Euclidean space \mathbb{R}^d defined by a set of equations in the components of $x = (x_1, \dots, x_d) \in \mathbb{R}^d$; replace these coordinates or functions of them by operators in some algebra and ask the defining equations to be satisfied only approximately, namely up to errors in operator norm; then these operators are called a fuzzy geometric object. As this is a particular case of algebras defined by relation, an abstract study of the K -theoretic invariants of such fuzzy objects has been known for a long time [8, 6, 7]. The construction of these invariants was essentially based on the replacement of the fuzzy object into classical maps from the geometric object to a sphere [11]. This leads to correct, but not very practical maps. Here we rather use relatively easy polynomial maps into the dotted Euclidean space and show that they do allow to construct the invariants, actually in a much easier manner that can be implemented numerically. The focus is only on fuzzy tori, because of their great relevance for solid state systems and hence connections to the first part of the paper. More precisely, it is shown how these abstract constructions applied to natural fuzzy tori associated to the situations analyzed in Sections 1 to 3 directly lead to the periodic spectral localizers.

Definition 19 *Let \mathcal{A} be a C^* -algebra of operators on a separable Hilbert space \mathcal{H} and let \mathcal{A}^\sim denote its unitization. Then d invertible operators $A_1, \dots, A_d \in \mathcal{A}^\sim$ form a d -dimensional fuzzy torus of width $\delta \in [0, 1)$, or simply a fuzzy d -torus, if for all $j, i = 1, \dots, d$*

$$\|A_j A_j^* - \mathbf{1}\| \leq \delta, \quad \|A_j^* A_j - \mathbf{1}\| \leq \delta, \quad \|[A_j, A_i]\| \leq \delta. \quad (36)$$

If, moreover, is given a selfadjoint,

$$(A_{d+1})^* = A_{d+1} \quad (37)$$

such that $A_1, \dots, A_{d+1} \in \mathcal{A}^\sim$ form a $(d+1)$ -dimensional fuzzy torus of width $\delta \in [0, 1)$, then $A_1, \dots, A_{d+1} \in \mathcal{A}^\sim$ are said to form a graded fuzzy d -torus of width $\delta \geq 0$.

Recall the definition of the real and imaginary part of an operator A :

$$\Re(A) = \frac{1}{2}(A + A^*), \quad \Im(A) = \frac{1}{2i}(A - A^*).$$

Definition 20 Associated to a fuzzy d -torus $A_1, \dots, A_d \in \mathcal{A}^\sim$ and a subset $I \subset \{1, \dots, d\}$, the operator $G_I = G_I(A_1, \dots, A_d)$ is defined by

$$G_I = \sum_{j=1}^{|I|} \Im(A_{i_j}) \otimes \gamma_j + \left((|I| - 1)\mathbf{1} - \sum_{j=1}^{|I|} \Re(A_{i_j}) \right) \otimes \gamma_{|I|+1}, \quad (38)$$

where $I = \{i_1, \dots, i_{|I|}\}$ and $\gamma_1, \dots, \gamma_{d+1}$ is an irreducible selfadjoint representation of the Clifford algebra with $d + 1$ generators. Furthermore, if $A_1, \dots, A_{d+1} \in \mathcal{A}^\sim$ form a graded fuzzy d -torus, then an operator $\widehat{G}_I = \widehat{G}_I(A_1, \dots, A_{d+1})$ is introduced by

$$\widehat{G}_I = \sum_{j=1}^{|I|} \Im(A_{i_j}) \otimes \gamma_j + \left(|I|\mathbf{1} - \sum_{j=1}^{|I|} \Re(A_{i_j}) - A_{d+1} \right) \otimes \gamma_{|I|+1}. \quad (39)$$

By construction, $G_I = G_I^*$ and $\widehat{G}_I^* = \widehat{G}_I$ are selfadjoint operators on $\mathcal{H} \otimes \mathbb{C}^{d'}$ for some d' . The results below show that G_I and \widehat{G}_I are invertible for sufficiently small δ , so that their positive spectral projections fix K -theory classes which for matrices can simply be read out via the signature. Underlying the construction in (38) are certain maps $g_{|I|, |I|-1} : \mathbb{T}^d \rightarrow \mathbb{R}^{d+1} \setminus \{0\}$ which for even $|I|$ are analyzed in detail in Appendix B. In particular, it is shown that the normalized maps $g_{|I|, |I|-1} / \|g_{|I|, |I|-1}\| : \mathbb{T}^d \rightarrow \mathbb{S}^d$ have a mapping degree equal to 1. One then gets $G_I : \mathbb{T}^d \rightarrow \mathbb{C}^{d' \times d'}$ by multiplying the coefficients with an irreducible representation of $|I| + 1$ Clifford generators:

$$G_I(e^{i\theta}) = \sum_{j=1}^{|I|} \sin(\theta_{i_j}) \otimes \gamma_j + \left((|I| - 1)\mathbf{1} - \sum_{j=1}^{|I|} \cos(\theta_{i_j}) \right) \otimes \gamma_{|I|+1}.$$

As $G_I(e^{i\theta})$ remains gapped, $\text{Sig}(G_I(e^{i\theta}))$ is independent of θ and $\text{Sig}(G_I(e^{i\theta})) = \text{Sig}(G_I(1)) = 0$. The same holds for fuzzy tori composed of commuting unitary matrices:

Proposition 21 *If a fuzzy d -torus consists of commuting unitary matrices A_1, \dots, A_d , one has $\text{Sig}(G_I) = 0$ and $\text{Sig}(\widehat{G}_I) = 0$ for all I .*

Proof. As the matrices can be simultaneously diagonalized, the above argument applies to all common eigenspaces. \square

For non-commuting (but almost commuting) matrices, the signatures of G_I and \widehat{G}_I can be different from zero though, see the examples below. Hence these signatures allow to distinguish different homotopy classes of fuzzy d -tori. As shown in Proposition 33 below, the definition (39) of \widehat{G}_I essentially reduces to the same map. Furthermore, one has the following elementary fact:

Lemma 22 For odd $|I|$, there exists a further Clifford generator $\Gamma = \gamma_{|I|+2}$ on the same representation space for which $\Gamma G_I \Gamma = -G_I$ and $\Gamma \widehat{G}_I \Gamma = -\widehat{G}_I$.

Remark 23 The formulas (38) and (39) look very much alike, but there is nevertheless a crucial difference that will be explained now. Given a graded fuzzy d -torus $A_1, \dots, A_{d+1} \in \mathcal{A}^\sim$, one can, of course view it by definition as an $(d+1)$ -dimensional fuzzy torus and hence associate the operators $G_I = G_I(A_1, \dots, A_{d+1})$. It requires the use of an irreducible selfadjoint representation of the Clifford algebra with $|I|+2$ generators even though, for $d+1 \in I$ and due to $\Im m(A_{d+1}) = 0$, the generator $\gamma_{|I|+1}$ does not appear in the formula. Therefore, the operator G_I satisfies the chirality relation $\gamma_{|I|+1} G_I \gamma_{|I|+1} = -G_I$, no matter whether $|I|$ is even or odd. For odd $|I|$, it is not possible to add a further Clifford generator on the representation space. On the other hand, if $|I|$ is even, there then *does* exist an extra generator $\gamma_{|I|+3}$. Choosing the representation such that

$$\gamma_1 = \hat{\gamma}_1 \otimes \sigma_1, \dots, \gamma_{|I|} = \hat{\gamma}_{|I|} \otimes \sigma_1, \quad \gamma_{|I|+2} = \hat{\gamma}_{|I|+1} \otimes \sigma_1, \quad \gamma_{|I|+1} = \mathbf{1} \otimes \sigma_2, \quad \gamma_{|I|+3} = \mathbf{1} \otimes \sigma_3,$$

where σ_1, σ_2 and σ_3 are the Pauli matrices and $\hat{\gamma}_1, \dots, \hat{\gamma}_{|I|+1}$ is an irreducible selfadjoint representation of the Clifford algebra with $|I|+1$ generators, one then has the two chirality relations

$$\sigma_2 G_I \sigma_2 = -G_I, \quad \sigma_3 G_I \sigma_3 = -G_I.$$

By an elementary argument with 2×2 matrices, the second relation implies that G_I is off-diagonal in the grading of the Pauli matrices, and the first relation that the off-diagonal entry is selfadjoint. Actually, setting $\hat{I} = I \setminus \{d+1\}$ and comparing with the definition of $\widehat{G}_{\hat{I}}$ written with the $\hat{\gamma}_j$ instead of the γ_j , one finds

$$G_I = \begin{pmatrix} 0 & \widehat{G}_{\hat{I}} \\ \widehat{G}_{\hat{I}} & 0 \end{pmatrix}, \quad |I| \text{ even}.$$

Hence $\widehat{G}_{\hat{I}}$ is the reduced-out form of G_I . Note that the spectra satisfy $\text{spec}(G_I) = \text{spec}(\widehat{G}_{\hat{I}}) \cup (-\text{spec}(\widehat{G}_{\hat{I}}))$. In particular, the spectrum of G_I is always symmetric, while that of $\widehat{G}_{\hat{I}}$ may have a spectral asymmetry. \diamond

Before starting with the analysis of the gap of G_I and \widehat{G}_I and their K -theoretic interpretations, let us provide several examples of fuzzy tori.

Example 24 Let $\theta \in \mathbb{R}^{d \times d}$ be an anti-symmetric matrix and $\mathcal{A}_\theta = C^*(U_1, \dots, U_d)$ be the d -dimensional rotation algebra generated by d unitaries U_1, \dots, U_d satisfying $U_i U_j = e^{2\theta_{i,j}} U_j U_i$. One has the bound $\|[U_i, U_j]\| \leq (2(1 - \cos(\theta_{i,j})))^{\frac{1}{2}}$. Hence if $|\theta_{i,j}| \leq \delta$, then $\|[U_i, U_j]\| \leq \delta$. If this holds for all $i \neq j$, then U_1, \dots, U_d form a fuzzy d -torus of width δ . \diamond

Example 25 Let $A \in C^1(\mathbb{T}^d, \mathbb{C}^{L \times L})$ be a continuously differentiable function from the classical d -torus to the invertible $L \times L$ -matrices. This is viewed as a fiberwise multiplication operator on $L^2(\mathbb{T}^d, \mathbb{C}^L)$. It is supposed to lead to small norms $\|A^* A - \mathbf{1}\|$ and $\|A A^* - \mathbf{1}\|$, namely is

almost unitary. It is well-known that A has odd Chern numbers (also called higher winding numbers) given by

$$\text{Ch}_d(A) = \frac{(\frac{1}{2}(d-1))!}{d!} \left(\frac{\iota}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathbb{T}^d} \text{Tr} \left((A^{-1} \mathbf{d}A)^d \right).$$

Let $\iota\partial_j$ for $j = 1, \dots, d$ be the (selfadjoint) coordinate vector fields on the torus. Then $e^{\iota\pi\chi_\kappa(\iota\partial_j)}$ is defined by functional calculus from a scaled smooth switch function $\chi_\kappa(x) = \chi(\kappa x)$. For κ sufficiently small, an argument similar to the one leading to Lemma 11 implies that

$$e^{\iota\pi\chi_\kappa(\iota\partial_1)}, \dots, e^{\iota\pi\chi_\kappa(\iota\partial_d)}, A$$

form a fuzzy $(d+1)$ -torus in the algebra of bounded operators on $L^2(\mathbb{T}^d, \mathbb{C}^L)$. Its width can be determined from κ and the above two norms. Note that, due to $\chi_\kappa(\iota\partial_j) = \chi(\kappa\iota\partial_j)$, κ plays the role of Planck's constant here. Also let us stress that associated to a classical d -torus \mathbb{T}^d is a fuzzy $(d+1)$ -torus. Hence there there is a natural dimensional shift here. There is an associated self-adjoint operator $G = G_{\{1, \dots, d+1\}}$. If one chooses the Clifford representation

$$\gamma_1 \otimes \sigma_3, \dots, \gamma_d \otimes \sigma_3, \mathbf{1} \otimes \sigma_2, \mathbf{1} \otimes \sigma_1,$$

it is given by

$$G = \begin{pmatrix} \sum_{j=1}^d \sin(\pi\chi_\kappa(\iota\partial_j)) \gamma_j & d - \sum_{j=1}^d \cos(\pi\chi_\kappa(\iota\partial_j)) \\ d - \sum_{j=1}^d \cos(\pi\chi_\kappa(\iota\partial_j)) & - \sum_{j=1}^d \sin(\pi\chi_\kappa(\iota\partial_j)) \gamma_j \end{pmatrix} - \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}. \quad (40)$$

Note that if d is even, then the Clifford representation admits another generator $\gamma_{d+1} \otimes \mathbf{1}$ with respect to which G is odd. \diamond

Example 26 Upon Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}^d, \mathbb{C}^L) \rightarrow L^2(\mathbb{T}^d, \mathbb{C}^L)$, Example 25 essentially becomes the situation described in Section 3 because $\mathcal{F}^* \iota\partial_j \mathcal{F} = X_j$ and the finite range condition in Section 3 corresponds to a finite frequency condition. Then $\mathcal{F}^* A \mathcal{F}$ is a 1-periodic operator on $\ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ which for sake of simplicity is simply denoted by A again. Then $e^{\iota\pi\chi_\kappa(X_1)}, \dots, e^{\iota\pi\chi_\kappa(X_d)}, A$ form a fuzzy $(d+1)$ -torus in the algebra of bounded operators on $\ell^2(\mathbb{Z}^d, \mathbb{C}^L)$, actually rather the much smaller algebra generated by the algebra \mathcal{A}_1 of 1-periodic short-range operators and the algebra \mathcal{K} compact operators on $\ell^2(\mathbb{Z}^d, \mathbb{C}^L)$ (as a vector space, this algebra is $\mathcal{A}_1 \oplus \mathcal{K}$, but the multiplication is not fiberwise). The associated operator $G = G_{\{1, \dots, d+1\}}$ is, as in (40),

$$G = \begin{pmatrix} \sum_{j=1}^d \sin(\pi\chi_\kappa(X_j)) \gamma_j & d - \sum_{j=1}^d \cos(\pi\chi_\kappa(X_j)) \\ d - \sum_{j=1}^d \cos(\pi\chi_\kappa(X_j)) & - \sum_{j=1}^d \sin(\pi\chi_\kappa(X_j)) \gamma_j \end{pmatrix} - \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.$$

In this formula, one can now let A be a 2ρ periodic operator. Furthermore, choosing $\chi(x) = x$ for $|x| \leq 1$ and $\chi(x) = \text{sgn}(x)$ for $|x| \geq 1$ as well as $\kappa = \frac{1}{\rho}$, one then obtains an operator G with a restriction G_ρ to $\mathcal{H}_\rho = \ell^2((\mathbb{Z}/(2\rho\mathbb{Z}))^d, \mathbb{C}^L)$ coinciding with the odd periodic spectral localizer

with $\eta = 1$, provided one replaces A by A_ρ with periodic boundary conditions. The spectral asymmetry of G entirely results from the spectral asymmetry of this finite-dimensional piece G_ρ (as can readily be shown for ρ sufficiently large and the arguments in Section 2 that modifying the boundary conditions does not alter the signature either) and can hence be measured by the half-signature of the odd periodic spectral localizer. The fuzzy torus (of square matrices of size $(2\rho)^d L$) leading to G_ρ is given by $e^{i\frac{\pi}{\rho}X_1}, \dots, e^{i\frac{\pi}{\rho}X_d}, A_\rho$. Actually, the framework can further be extended to operators A from the algebra \mathcal{A} of covariant operators as defined in [2, 18]. Then the fuzzy torus lies in the algebra $\mathcal{A} \oplus \mathcal{K}$. \diamond

Example 27 Suppose that $H = H^*$ is an invertible finite-range operator on $\ell^2(\mathbb{Z}^d, \mathbb{C}^L)$. Then the operators

$$e^{i\pi\chi_\kappa(X_1)}, \dots, e^{i\pi\chi_\kappa(X_d)}, H.$$

form a graded fuzzy d -torus. Associated is therefore an operator $\widehat{G} = \widehat{G}_{\{1, \dots, d\}}$. If the irreducible representation of the Clifford algebra with $d + 1$ generators is chosen to be

$$\gamma_1 \otimes \sigma_1, \dots, \gamma_{d-1} \otimes \sigma_1, \mathbf{1} \otimes \sigma_2, \mathbf{1} \otimes \sigma_3$$

where $\gamma_1, \dots, \gamma_{d-1}$ is an irreducible representation of the Clifford algebra with $d - 1$ generators (the tensor products will be dropped in the following), then one finds

$$\begin{aligned} \widehat{G} &= \sum_{j=1}^{d-1} \Im m(e^{i\pi\chi_\kappa(X_j)}) \gamma_j \sigma_1 + \Im m(e^{i\pi\chi_\kappa(X_d)}) \sigma_2 + \left(d\mathbf{1} - \sum_{j=1}^d \Re e(e^{i\pi\chi_\kappa(X_j)}) - H \right) \sigma_3 \\ &= \begin{pmatrix} -H + d\mathbf{1} - \sum_{j=1}^d \cos(\pi\chi_\kappa(X_j)) & \sum_{j=1}^{d-1} \sin(\pi\chi_\kappa(X_j)) \gamma_j - i \sin(\pi\chi_\kappa(X_d)) \\ \sum_{j=1}^{d-1} \sin(\pi\chi_\kappa(X_j)) \gamma_j + i \sin(\pi\chi_\kappa(X_d)) & H - d\mathbf{1} + \sum_{j=1}^d \cos(\pi\chi_\kappa(X_j)) \end{pmatrix}. \end{aligned}$$

Replacing the γ_j by $\hat{\gamma}_j$ and proceeding as in Example 26, one recovers the even periodic spectral localizer with $\eta = 1$ as the finite-volume restriction of \widehat{G} . \diamond

Example 28 Suppose given a C^* -dynamical system $(\mathcal{A}, \mathbb{R}^d, \alpha)$ consisting of a C^* -algebra (for simplicity given as subalgebra of the bounded operators on some Hilbert space \mathcal{H} and a continuous group action α of \mathbb{R}^d on \mathcal{A} . The action is implemented (in a unique manner, up to isomorphisms) by a strongly continuous unitary group action U on \mathcal{H} , namely $\alpha_x(A) = U_x^* A U_x$ for $A \in \mathcal{A}$ and $x \in \mathbb{R}^d$. If e_1, \dots, e_d is a basis of \mathbb{R}^d , then $t \in \mathbb{R} \mapsto U_{te_j}$ is a strongly continuous one-parameter group with generator X_j . Then define unitaries $e^{i\chi_\kappa(X_j)}$ by spectral calculus. Further be given an invertible operator $A \in \mathcal{A}$ fixing a class in $K_1(\mathcal{A})$. This operator is supposed to be sufficiently smooth w.r.t. the action α (see [20] for a detailed description of conditions that assure the existence of semi-finite index pairings in the presence of an α -invariant tracial state on \mathcal{A}). One then has an associated fuzzy $(d + 1)$ -torus $e^{i\chi_\kappa(X_1)}, \dots, e^{i\chi_\kappa(X_d)}, A$ of width converging to 0 as $\kappa \rightarrow 0$ (no detailed proof of this fact is provided here, as it readily follows from the techniques of Lemma 11 and [5]). Similarly, given a sufficiently smooth gapped selfadjoint operator $H \in \mathcal{A}$ specifying a class in $K_0(\mathcal{A})$, one has a graded fuzzy d -torus $e^{i\chi_\kappa(X_1)}, \dots, e^{i\chi_\kappa(X_d)}, H$. From both of these tori, one can build lower-dimensional tori associated to choices of a subset $I \subset \{1, \dots, d\}$. Note that these constructions are generalizations of the Examples 26 and 27. \diamond

The next two results give simple criteria on the width δ assuring that G -operators are invertible. We will focus on the case $G = G_{\{1, \dots, d\}}$ simply because the G_I with $|I| < d$ are associated to a fuzzy $|I|$ -torus so that the below results cover this case as well. As the estimate of the gap is more simple and transparent, let us first restrict to a fuzzy d -torus given by unitary instead of invertible operators. Note that this case is sufficient for Example 24.

Proposition 29 *For unitaries $U_1, \dots, U_d \in \mathbb{U}(\mathcal{H})$, the operator $G = G_{\{1, \dots, d\}}(U_1, \dots, U_d)$ satisfies*

$$G^2 \geq \left(1 - 7 \sum_{1 \leq j < i \leq d} \|[U_j, U_i]\| \right) \mathbf{1}. \quad (41)$$

In particular, G is invertible if the unitaries $U_1, \dots, U_d \in \mathbb{U}(\mathcal{H})$ form a fuzzy d -torus of width δ satisfying $1 - \binom{d}{2} 7\delta > 0$.

Proof. The argument is essentially identical to the one leading to Proposition 4.3 in [12]. Using the Clifford relations, one finds

$$\begin{aligned} G^2 &= \left(\sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \right) \otimes \mathbf{1} + \sum_{1 \leq j < i \leq d} [\Im m(U_j), \Im m(U_i)] \otimes \gamma_j \gamma_i \\ &\quad + \sum_{j=1}^d [\Im m(U_j), (d-1)\mathbf{1} - \sum_{i=1}^d \Re e(U_i)] \otimes \gamma_j \gamma_{d+1}. \end{aligned}$$

Using that

$$\|[\Im m(U_j), \Im m(U_i)]\| \leq \|[U_j, U_i]\|, \quad \|[\Im m(U_j), \Re e(U_i)]\| \leq \|[U_j, U_i]\|$$

and that γ_j is unitary for all j one obtains

$$G^2 \geq \left(\sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \right) \otimes \mathbf{1} - 3 \sum_{1 \leq j < i \leq d} \|[U_j, U_i]\| \mathbf{1}. \quad (42)$$

Moreover,

$$\begin{aligned} &\sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \\ &= \sum_{j=1}^d \Im m(U_j)^2 + (d-1)^2 \mathbf{1} - 2(d-1) \sum_{j=1}^d \Re e(U_j) + \left(\sum_{j=1}^d \Re e(U_j) \right)^2 \\ &= d\mathbf{1} + (d-1)^2 \mathbf{1} - 2(d-1) \sum_{j=1}^d \Re e(U_j) + \sum_{1 \leq j < i \leq d} (\Re e(U_j) \Re e(U_i) + \Re e(U_i) \Re e(U_j)), \end{aligned}$$

where the last step follows from $\Im m(U_j)^2 + \Re e(U_j)^2 = \mathbf{1}$. This simplifies to

$$\begin{aligned} & \sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \\ &= \mathbf{1} + \sum_{1 \leq j < i \leq d} \left((\mathbf{1} - \Re e(U_j))(\mathbf{1} - \Re e(U_i)) + (\mathbf{1} - \Re e(U_i))(\mathbf{1} - \Re e(U_j)) \right). \end{aligned} \quad (43)$$

One directly checks that $\mathbf{1} - \Re e(U_j) = \frac{1}{2}(U_j - \mathbf{1})(U_j - \mathbf{1})^*$ and therefore

$$\begin{aligned} & \sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \\ &= \mathbf{1} + \frac{1}{4} \sum_{1 \leq j < i \leq d} \left((U_j - \mathbf{1})(U_j - \mathbf{1})^*(U_i - \mathbf{1})(U_i - \mathbf{1})^* + (U_i - \mathbf{1})(U_i - \mathbf{1})^*(U_j - \mathbf{1})(U_j - \mathbf{1})^* \right) \\ &= \mathbf{1} + \frac{1}{4} \sum_{1 \leq j < i \leq d} \left((U_j - \mathbf{1})(U_i - \mathbf{1})(U_i - \mathbf{1})^*(U_j - \mathbf{1})^* + (U_j - \mathbf{1})[(U_j - \mathbf{1})^*, (U_i - \mathbf{1})](U_i - \mathbf{1})^* \right. \\ & \quad \left. + (U_j - \mathbf{1})(U_i - \mathbf{1})[(U_j - \mathbf{1})^*, (U_i - \mathbf{1})^*] + (U_i - \mathbf{1})(U_j - \mathbf{1})(U_j - \mathbf{1})^*(U_i - \mathbf{1})^* \right. \\ & \quad \left. + (U_i - \mathbf{1})[(U_i - \mathbf{1})^*, (U_j - \mathbf{1})](U_j - \mathbf{1})^* + (U_i - \mathbf{1})(U_j - \mathbf{1})[(U_i - \mathbf{1})^*, (U_j - \mathbf{1})^*] \right). \end{aligned}$$

Using that the first and fourth summand are non-negative and $\|(U_j - \mathbf{1})\| \leq 2$, $\|(U_i - \mathbf{1})\| \leq 2$, one obtains

$$\sum_{j=1}^d \Im m(U_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(U_j) \right)^2 \geq \left(1 - 4 \sum_{1 \leq j < i \leq d} \|[U_j, U_i]\| \right) \mathbf{1}.$$

Combining with (42) the claim (41) follows. \square

Example 30 Let us continue with Example 24 of the non-commutative torus. If $|\theta_{i,j}| \leq \delta$ for all $i \neq j$ and $\delta < \frac{2}{7}d(d-1)$, then $G_I = G_I(U_1, \dots, U_d)$ is gapped by Proposition 29. If $\frac{1}{2\pi}\theta$ consists only of rational numbers, it is well-known that the unitaries U_1, \dots, U_d can be chosen to be finite-dimensional matrices. In this case also G_I is a finite dimensional matrix and one can hence define the invariants $\nu_I = \frac{1}{2} \text{Sig}(G_I)$ which are integer-valued because the representation space of the Clifford algebra is even-dimensional so that also the selfadjoint matrix G_I acts on an even-dimensional vector space. Let us now focus on the case $d = 2$ and $I = \{1, 2\}$. Then θ is a scalar which is supposed to be $\theta = \frac{2\pi}{N}$. Choosing the Clifford representation to be the standard Pauli matrices, the associated operator $G = G_{\{1,2\}}(U_1, U_2)$ is then

$$G = \begin{pmatrix} \mathbf{1} - \Re e(U_1) - \Re e(U_2) & \Im m(U_1) - \imath \Im m(U_2) \\ \Im m(U_1) + \imath \Im m(U_2) & -\mathbf{1} + \Re e(U_1) + \Re e(U_2) \end{pmatrix}.$$

By Proposition 29, G is gapped provided that $N \geq 14\pi$. In reality, the gap is already open for N much smaller. Associated is then the invariant $\frac{1}{2} \text{Sig}(G(U_1, U_2)) \in \mathbb{Z}$. Due to [6, Proposition 5.1] and [7, Theorem 6.15] it is known that this integer is equal to the winding number of the path $t \in [0, 1] \mapsto \det(tU_1U_2 - (1-t)U_2U_1)$. \diamond

Example 31 This example is a continuation of Example 25, albeit with a function $A = U \in C^1(\mathbb{T}^d, \mathbb{U}(L))$ with values in the unitary matrices. For such a function, Proposition 29 applies. Therefore the G -operator given in (40) is gapped. It is still an operator on an infinite dimensional Hilbert space and has no finite-dimensional invariant subspaces (even if U only contains a finite number of frequencies). Nevertheless, the operator has a spectral asymmetry which can be extracted by projecting G down to frequencies of modulus less than ρ . If G_ρ denotes this restriction, then the proof of Theorem 1 shows that $\text{Ch}_d(U) = \frac{1}{2}\text{Sig}(G_\rho)$ for ρ sufficiently large. Let us note that the essential spectrum of the operator G in infinite volume consists of $\{-1, 1\}$. In particular, G does not have a compact resolvent so that it is not possible to define an η -invariant, other than for the spectral localizer [14, 5]. Furthermore, let us note that for d even, the symmetry $\gamma_{d+1}G_\rho\gamma_{d+1} = -G_\rho$ implies that $\text{Sig}(G_\rho) = 0$. However, for d even, one can choose I of odd cardinality $|I| < d$ and then the spectral asymmetry of the associated operators G_I determines the odd Chern numbers of lower degree. \diamond

Proposition 29 does not allow to show the G -operators associated to Examples 26 and 27 are gapped because A and H are not necessarily unitary, even though all other operators of the fuzzy tori are unitary which, moreover, commute with each other. Of course, the situation of Example 27 is dealt with in detail in the proof of Theorem 1 given in Section 2. The next result generalizes Proposition 29 to invertible operators $A_1, \dots, A_d \in \mathbb{B}(\mathcal{H})$ that form a fuzzy d -torus. This also provides the gap estimate of Theorem 1, albeit with considerably worse constants.

Proposition 32 *If $A_1, \dots, A_d \in \mathbb{B}(\mathcal{H})$ form a fuzzy d -torus of width $\delta \in (0, \frac{1}{2}]$ the selfadjoint operator $G = G(A_1, \dots, A_d)$ satisfies*

$$G^2 \geq (1 - 33d^2\delta)\mathbf{1}. \quad (44)$$

In particular, G is invertible if $\delta < \frac{1}{33d^2}$.

Proof. Using the Clifford relations one has

$$\begin{aligned} G^2 &= \left(\sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 \right) + \sum_{1 \leq j < i \leq d} [\Im m(A_j), \Im m(A_i)] \otimes \gamma_j \gamma_i \\ &\quad + \sum_{j=1}^d [\Im m(A_j), (d-1)\mathbf{1} - \sum_{i=1}^d \Re e(A_i)] \otimes \gamma_j \gamma_{d+1} \\ &\geq \left(\sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 \right) \\ &\quad - \left(\sum_{1 \leq j < i \leq d} \|[\Im m(A_j), \Im m(A_i)]\| - \sum_{j,i=1}^d \|[\Im m(A_j), \Re e(A_i)]\| \right). \end{aligned} \quad (45)$$

Using that $\text{spec}(|A_j|) \subset [(1-\delta)^{\frac{1}{2}}, (1+\delta)^{\frac{1}{2}}]$ and $\delta \leq \frac{1}{2}$, one checks that

$$\|A_j\| = \| |A_j| \| \leq (1+\delta)^{\frac{1}{2}} \leq 1 + \frac{\delta}{2}, \quad \|A_j^{-1}\|^2 \leq (1-\delta)^{-1} \leq 1 + 2\delta. \quad (46)$$

Therefore

$$\begin{aligned}
\| [A_i, A_j^*] \| &= \| A_j^{-1} (A_j A_i A_j^* A_j - A_j A_j^* A_i A_j) A_j^{-1} \| \\
&\leq \| A_j^{-1} \|^2 \| A_j A_i (A_j^* A_j - \mathbf{1}) + [A_j, A_i] + (\mathbf{1} - A_j A_j^*) A_i A_j \| \\
&\leq (1 + 2\delta)(2(1 + \delta)\delta + \delta) \leq 8\delta .
\end{aligned} \tag{47}$$

This leads to

$$\| [\Im m(A_j), \Im m(A_i)] \| \leq \frac{1}{4} (\| [A_j, A_i] \| + \| [A_i, A_j^*] \| + \| [A_i^*, A_j] \| + \| [A_j^*, A_i^*] \|) \leq \frac{1}{2}(\delta + 8\delta) = 5\delta .$$

In the same way one shows the bound

$$\| [\Im m(A_j), \Re e(A_i)] \| \leq \frac{1}{4} (\| [A_j, A_i] \| + \| [A_i, A_j^*] \| + \| [A_j, A_i^*] \| + \| [A_i^*, A_j^*] \|) \leq 5\delta ,$$

for $j \neq i$. For $j = i$ a slightly better estimate holds

$$\| [\Im m(A_j), \Re e(A_j)] \| = \frac{1}{2} \| A_j A_j^* - A_j^* A_j \| \leq \delta .$$

Inserting this into (45) leads to

$$G^2 - \left(\sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 \right) \geq - \left(d\delta + 15 \binom{d}{2} \delta \right) \mathbf{1} \geq -9d^2 \delta \mathbf{1} . \tag{48}$$

It thus remains to prove a lower bound on the term in the parenthesis. Using

$$\Im m(A_j)^2 + \Re e(A_j)^2 = \frac{1}{2} (A_j A_j^* + A_j^* A_j) \geq \mathbf{1} - \frac{1}{2} \| (A_j A_j^* - \mathbf{1}) + (A_j^* A_j - \mathbf{1}) \| \mathbf{1} \geq (1 - \delta) \mathbf{1} ,$$

one finds

$$\begin{aligned}
&\sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 \\
&\geq d(1 - \delta)\mathbf{1} + (d-1)^2 \mathbf{1} - 2(d-1) \sum_{j=1}^d \Re e(A_j) + \sum_{1 \leq j < i \leq d} (\Re e(A_j) \Re e(A_i) + \Re e(A_i) \Re e(A_j)) \\
&= \mathbf{1} + \sum_{1 \leq j < i \leq d} ((\mathbf{1} - \Re e(A_j))(\mathbf{1} - \Re e(A_i)) + (\mathbf{1} - \Re e(A_i))(\mathbf{1} - \Re e(A_j))) - d\delta \mathbf{1} .
\end{aligned} \tag{49}$$

One directly checks that

$$\frac{1}{2} (A_j - \mathbf{1})(A_j - \mathbf{1})^* - (\mathbf{1} - \Re e(A_j)) = \frac{1}{2} (A_j A_j^* - \mathbf{1})$$

and therefore, by the first part of (36),

$$\left\| \frac{1}{2}(A_j - \mathbf{1})(A_j - \mathbf{1})^* - (\mathbf{1} - \Re e(A_j)) \right\| \leq \frac{1}{2} \delta.$$

Moreover, using (46)

$$\|\mathbf{1} - \Re e(A_j)\| \leq 1 + (1 + \delta)^{\frac{1}{2}} \leq \frac{5}{2}, \quad \|A_j - \mathbf{1}\|^2 \leq (1 + (1 + \delta)^{\frac{1}{2}})^2 \leq 5.$$

Using the last two bounds one gets

$$\begin{aligned} & \left\| \frac{1}{4}(A_j - \mathbf{1})(A_j - \mathbf{1})^*(A_i - \mathbf{1})(A_i - \mathbf{1})^* - (\mathbf{1} - \Re e(A_j))(\mathbf{1} - \Re e(A_i)) \right\| \\ & \leq \left\| \frac{1}{2}(A_j - \mathbf{1})(A_j - \mathbf{1})^* \left(\frac{1}{2}(A_i - \mathbf{1})(A_i - \mathbf{1})^* - (\mathbf{1} - \Re e(A_i)) \right) \right\| \\ & \quad + \left\| \left(\frac{1}{2}(A_j - \mathbf{1})(A_j - \mathbf{1})^* - (\mathbf{1} - \Re e(A_j)) \right) (\mathbf{1} - \Re e(A_i)) \right\| \\ & \leq \frac{1}{2} 5 \frac{1}{2} \delta + \frac{1}{2} \delta \frac{5}{2} = \frac{5}{2} \delta. \end{aligned}$$

Inserting this into (49) leads to

$$\begin{aligned} & \sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 \\ & \geq \mathbf{1} + \frac{1}{4} \sum_{1 \leq j < i \leq d} \left((A_j - \mathbf{1})(A_j - \mathbf{1})^*(A_i - \mathbf{1})(A_i - \mathbf{1})^* + (A_i - \mathbf{1})(A_i - \mathbf{1})^*(A_j - \mathbf{1})(A_j - \mathbf{1})^* \right) \\ & \quad - d\delta \mathbf{1} - 2 \binom{d}{2} \frac{5}{2} \delta \\ & \geq \mathbf{1} + \frac{1}{4} \sum_{1 \leq j < i \leq d} \left((A_j - \mathbf{1})(A_i - \mathbf{1})(A_i - \mathbf{1})^*(A_j - \mathbf{1})^* \right. \\ & \quad + (A_j - \mathbf{1})[(A_j - \mathbf{1})^*, (A_i - \mathbf{1})](A_i - \mathbf{1})^* + (A_j - \mathbf{1})(A_i - \mathbf{1})[(A_j - \mathbf{1})^*, (A_i - \mathbf{1})^*] \\ & \quad + (A_i - \mathbf{1})(A_j - \mathbf{1})(A_j - \mathbf{1})^*(A_i - \mathbf{1})^* + (A_i - \mathbf{1})[(A_i - \mathbf{1})^*, (A_j - \mathbf{1})](A_j - \mathbf{1})^* \\ & \quad \left. + (A_i - \mathbf{1})(A_j - \mathbf{1})[(A_i - \mathbf{1})^*, (A_j - \mathbf{1})^*] \right) - \frac{7}{2} d^2 \delta \mathbf{1}. \end{aligned}$$

The first and fourth summand are non-negative and can thus be left out for a lower bound. In the other four summands, the commutators reduce to $[A_j^*, A_i]$ or $[A_j^*, A_i^*]$ which can be bound directly by (36) or by (47). Using, moreover, again $\|A_j - \mathbf{1}\| \leq 5$, one thus obtains

$$\begin{aligned} \sum_{j=1}^d \Im m(A_j)^2 + \left((d-1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right)^2 & \geq \left(1 - \frac{9}{4} \binom{d}{2} (\delta + 8\delta + 8\delta + \delta) \right) \mathbf{1} - \frac{7}{2} d^2 \delta \mathbf{1} \\ & \geq (1 - 24d^2 \delta) \mathbf{1}. \end{aligned}$$

Combining with (48) one obtains (44). \square

The next result shows that a graded fuzzy d -torus can always be reduced to a suitably associated ungraded fuzzy d -torus.

Proposition 33 *Let $A_1, \dots, A_d, A_{d+1} = A_{d+1}^* \in \mathcal{A}^\sim$ be graded fuzzy d -torus of a sufficiently small width $\delta \leq \frac{1}{2}$. Let P denote the Riesz projection on the positive spectrum of A_{d+1} . Then PA_1P, \dots, PA_dP form a fuzzy d -torus of width 6δ on the Hilbert space $P\mathcal{H}$. Setting $G^P = G_{\{1, \dots, d\}}(PA_1P, \dots, PA_dP)$, the operator $\widehat{G} = \widehat{G}_{\{1, \dots, d\}}(A_1, \dots, A_d, A_{d+1})$ is homotopic to $G^P \oplus (\mathbf{1} - P)\gamma_{d+1}$ inside the invertible operators.*

Proof. As $\|A_{d+1}^2 - \mathbf{1}\| \leq \delta$ by assumption, A_{d+1} is close to a symmetry and its spectrum is separated into two intervals $[-\sqrt{1+\delta}, -\sqrt{1-\delta}]$ and $[\sqrt{1-\delta}, \sqrt{1+\delta}]$, see e.g. (46). Let P denote the Riesz projection associated to $[\sqrt{1-\delta}, \sqrt{1+\delta}]$. Then $\|A_{d+1} - (2P - \mathbf{1})\| \leq 1 - \sqrt{1-\delta} \leq \delta$ by the spectral mapping theorem. Moreover, the other conditions in (36) imply

$$\|[A_j, P]\| \leq \max_{z \in \partial B_1(\pm 1)} \|(z\mathbf{1} - A_{d+1})^{-1}\|^2 \|[A_j, A_{d+1}]\| \leq 2\delta,$$

so that

$$\|A_j - PA_jP - (\mathbf{1} - P)A_j(\mathbf{1} - P)\| = \|(\mathbf{1} - P)[A_j, P]\| \leq 2\delta.$$

Moreover, since $\|A_j\| = \|A_j^*A_j\|^{\frac{1}{2}} \leq (1 + \delta)^{\frac{1}{2}} \leq 1 + \frac{1}{2}\delta$,

$$\|[PA_jP, PA_iP]\| \leq \|P[A_j, A_i]P\| + \|P[A_j, P]A_iP\| + \|[P[A_i, P]A_jP]\| \leq 6\delta,$$

the operators PA_1P, \dots, PA_dP forms an (ungraded) fuzzy d -torus on the Hilbert space $P\mathcal{H}$ of width 6δ . (Similarly, also $(\mathbf{1} - P)A_1(\mathbf{1} - P), \dots, (\mathbf{1} - P)A_d(\mathbf{1} - P)$ is an (ungraded) fuzzy d -torus on the Hilbert space $(\mathbf{1} - P)\mathcal{H}$ of width 6δ , but this torus will not be used.) The associated G -operator is denoted by G^P , see the statement of the proposition. One then has

$$\|\widehat{G} - G^P \oplus \widetilde{G}^{1-P}\| \leq 5d\delta,$$

where

$$\widetilde{G}^{1-P} = \sum_{j=1}^d (\mathbf{1} - P)\Im m(A_j)(\mathbf{1} - P) \otimes \gamma_j + \left((d+1)(\mathbf{1} - P) - \sum_{j=1}^d (\mathbf{1} - P)\Re e(A_j)(\mathbf{1} - P) \right) \otimes \gamma_{d+1}.$$

Let us stress that this is *not* the G -operator on $(\mathbf{1} - P)\mathcal{H}$ associated to the fuzzy d -torus $(\mathbf{1} - P)A_1(\mathbf{1} - P), \dots, (\mathbf{1} - P)A_d(\mathbf{1} - P)$ for $I = \{1, \dots, d\}$ by the definition (38), simply because one of the summands is $(d+1)(\mathbf{1} - P) \otimes \gamma_{d+1}$ rather than $(d-1)(\mathbf{1} - P) \otimes \gamma_{d+1}$. It hence remains to show that \widetilde{G}^{1-P} is homotopic to $(\mathbf{1} - P) \otimes \gamma_{d+1}$ inside the invertible operators. This follows directly from the next lemma. \square

Lemma 34 *Given a fuzzy d -torus A_1, \dots, A_d of a sufficiently small width δ , the operator*

$$\widetilde{G} = \sum_{j=1}^d \Im m(A_j) \otimes \gamma_j + \left((d+1)\mathbf{1} - \sum_{j=1}^d \Re e(A_j) \right) \otimes \gamma_{d+1}$$

is homotopic to γ_{d+1} inside the invertible operators.

Proof. The homotopy will be given by the straight-line path $t \in [0, 1] \mapsto \tilde{G}(t) = (1-t)\gamma_{d+1} + t\tilde{G}$. Explicitly

$$\tilde{G}(t) = \gamma_{d+1} + t \left(\sum_{j=1}^d \Im(A_j) \otimes \gamma_j + \left(\sum_{j=1}^d (\mathbf{1} - \Re(A_j)) \right) \otimes \gamma_{d+1} \right).$$

Now all commutators $[\Im(A_j), \Re(A_i)]$ are of order $\mathcal{O}(\delta)$, see the proof of Proposition 32. Hence

$$\tilde{G}(t)^2 = \mathbf{1} + 2t \sum_{j=1}^d (\mathbf{1} - \Re(A_j)) + t^2 \left(\sum_{j=1}^d \Im(A_j)^2 + \left(\sum_{j=1}^d (\mathbf{1} - \Re(A_j)) \right)^2 \right) + \mathcal{O}(\delta).$$

But up to errors of order $\mathcal{O}(\delta)$, one also has $\mathbf{1} - \Re(A_j) \geq 0$. Hence $\tilde{G}(t)^2 \geq \mathbf{1} + \mathcal{O}(\delta)$, which implies the claim. Note that the claim merely reflects that the maps $g_{d,d+1}$ defined in Appendix B have a vanishing mapping degree and are hence homotopic to the identity. \square

Now that the crucial property that the operators G_I are gapped is proved for fuzzy tori of sufficiently small width, it is possible to extract topological information from them. Recall that elements of the K -group $K_0(\mathcal{A})$ are homotopy equivalence classes of either projections or equivalently invertible selfadjoints in matrix algebras of \mathcal{A} , and that $K_1(\mathcal{A})$ are homotopy equivalence classes of invertibles in matrix algebras of \mathcal{A} or equivalent equivalence classes of selfadjoint invertible which anti-commute with some symmetry in the matrix degrees of freedom (see *e.g.* [14] for some further explanation of this). Based on Proposition 32 and Lemma 22 one therefore has the following.

Corollary 35 *Let A_1, \dots, A_d be a fuzzy d -torus in a C^* -algebra \mathcal{A} of sufficiently small width. For any index set $I \subset \{1, \dots, d\}$, one then obtains a class $[G_I]_0 \in K_0(\mathcal{A})$ if I is of even cardinality and a class $[G_I]_1 \in K_1(\mathcal{A})$ if I is of odd cardinality. For a graded fuzzy torus one obtains $[\hat{G}_I]_0 \in K_0(\mathcal{A})$ and $[\hat{G}_I]_1 \in K_1(\mathcal{A})$ respectively which can also be represented as the G -operators of the reduced out fuzzy torus given in Proposition 33, namely $[\hat{G}_I] = [G_I]$.*

If the algebra \mathcal{A} is given by matrices (or compact operators), then one can read out the K -theoretic content using the half-signatures $\frac{1}{2} \text{Sig}(G_I) \in \mathbb{Z}$ and $\frac{1}{2} \text{Sig}(\hat{G}_I) \in \mathbb{Z}$ if I is of even cardinality. Again, if G_I is obtained by reducing out a fuzzy $(d+1)$ -torus with G -operators \hat{G}_I , then $\text{Sig}(G_I) = \text{Sig}(\hat{G}_I)$.

Proof of Theorem 10. One only has to apply Corollary 35 to Example 27. \square

In the case of two almost commuting unitaries U_1, U_2 (see Example 30) satisfying that $\text{Sig}(G_{\{1,2\}}(U_1, U_2)) = 0$, it is known [7] that they can be deformed into two commuting unitaries. Hence one can expect the integers $\frac{1}{2} \text{Sig}(G_I(U_1, U_2))$ with $I = \{1\}, \{2\}, \{1, 2\}$ to fully classify all fuzzy 2-tori. We even suspect that fuzzy matrix tori are completely classified by the signature invariants constructed above:

Conjecture: *Two fuzzy d -tori of matrices having the same signature invariants $\text{Sig}(G_I)$ and $\text{Sig}(\hat{G}_I)$ can be homotopically deformed into each other without closing the gaps of G_I and \hat{G}_I .*

A Mapping degree versus Chern number

For the convenience of the reader, this appendix provides a detailed proof of the connection between mapping degree of a differentiable function $f : \mathbb{T}^d \rightarrow \mathbb{S}^d$ on an even-dimensional torus and the Chern number of an associated matrix-valued projection $P_f : \mathbb{T}^d \rightarrow \mathbb{C}^{d' \times d'}$ with $d' = 2^{\frac{d}{2}}$. This fact is used in Appendix B which is crucial for an understanding the motivation for the periodic spectral localizer and the G -operators in Section 5. While the main statement, Corollary 37 below, is certainly well-known in the community, we could not localize a detailed proof.

For d be even and let denote the restrictions of the euclidean coordinate functions to \mathbb{S}^d by $x_j : \mathbb{S}^d \rightarrow \mathbb{R}$ for $j \in \{1, \dots, d+1\}$. More precisely,

$$x_j(y) = y_j, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_{d+1} \end{pmatrix} \in \mathbb{S}^d.$$

Furthermore, let $f_j = x_j \circ f : \mathbb{T}^d \rightarrow \mathbb{R}$ be j th component of the function f for $j \in \{1, \dots, d+1\}$. Let $\gamma_1, \dots, \gamma_{d+1} \in \mathbb{C}^{d' \times d'}$ be an irreducible self-adjoint representation of the Clifford algebra with the convention that

$$\gamma_1 \cdots \gamma_{d+1} = i^{\frac{d}{2}} \mathbf{1}. \quad (50)$$

Then let us define the map $P_f : \mathbb{T}^d \rightarrow \mathbb{C}^{d' \times d'}$ by

$$P_f(k) = \frac{1}{2} \left(\sum_{j=1}^{d+1} f_j(k) \gamma_j + \mathbf{1} \right).$$

Then $P_f(k)$ is an (orthogonal) projection for all $k \in \mathbb{T}^d$. Its exterior derivative is $dP_f = \frac{1}{2} \sum_{j=1}^{d+1} \gamma_j df_j$, a matrix-valued 1-form on \mathbb{T}^d . Then let us set

$$\omega_f = \text{Tr}(P_f(dP_f \wedge dP_f)^{\wedge \frac{d}{2}}),$$

which is a d -form on \mathbb{T}^d . Using (50) as $\text{Tr}(\mathbf{1}) = 2^{\frac{d}{2}}$, it is explicitly given by

$$\omega_f = C_d \sum_{j=1}^{d+1} (-1)^{j+1} f_j df_1 \wedge \dots \wedge df_{j-1} \wedge df_{j+1} \wedge \dots \wedge df_{d+1}, \quad C_d = \frac{1}{2} \frac{1}{2^{\frac{d}{2}}} d! 2^{\frac{d}{2}} i^{\frac{d}{2}}$$

Similarly, let us define another projection-valued map $P^W : \mathbb{S}^d \rightarrow \mathbb{C}^{d' \times d'}$ (called the Weyl projection) by

$$P^W(p) = \frac{1}{2} \left(\sum_{j=1}^{d+1} x_j(p) \gamma_j + \mathbf{1} \right).$$

Similar as above, there is an associated d -form on \mathbb{S}^d given by

$$\omega^W = \text{Tr}(P^W(dP^W \wedge dP^W)^{\wedge \frac{d}{2}}) = C_d \sum_{j=1}^{d+1} (-1)^{j+1} x_j dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_{d+1}.$$

Proposition 36 For any differentiable map $f : \mathbb{T}^d \rightarrow \mathbb{S}^d$, the differential form ω_f on \mathbb{T}^d equals the pullback of the differential form ω^W by f :

$$\omega_f = f^* \omega^W .$$

Proof. For a point $k \in \mathbb{T}^d$ let $\eta_1, \dots, \eta_d : (-a, a) \rightarrow \mathbb{T}^d$ for $a > 0$ represent tangent vectors of \mathbb{T}^d at the point k , namely η_j is a differentiable curve fulfilling $\eta_j(0) = k$ for all $j \in \{1, \dots, d\}$. The tangent vector represented by η_j is denoted by $[\eta_j]$. Then one has to show

$$\omega_f([\eta_1], \dots, [\eta_d]) = \omega^W(df_k([\eta_1]), \dots, df_k([\eta_d])) , \quad (51)$$

where df_k denotes the differential of f at the point k . A direct computation shows

$$\begin{aligned} & \omega_f([\eta_1], \dots, [\eta_d]) \\ &= C_d \sum_{j=1}^{d+1} (-1)^{j+1} f_j df_1 \wedge \dots \wedge df_{j-1} \wedge df_{j+1} \wedge \dots \wedge df_{d+1}([\eta_1], \dots, [\eta_d]) \\ &= C_d \sum_{j=1}^{d+1} (-1)^{j+1} f_j(k) \frac{1}{d!} \sum_{\sigma \in S_d} df_1([\eta_{\sigma(1)}]) \cdots df_{j-1}([\eta_{\sigma(j-1)}]) df_{j+1}([\eta_{\sigma(j)}]) \cdots df_{d+1}([\eta_{\sigma(d)}]) \\ &= \frac{C_d}{d!} \sum_{j=1}^{d+1} (-1)^{j+1} f_j(k) \sum_{\sigma \in S_d} (f_1 \circ \eta_{\sigma(1)})'(0) \cdots (f_{j-1} \circ \eta_{\sigma(j-1)})'(0) \\ & \quad (f_{j+1} \circ \eta_{\sigma(j)})'(0) \cdots (f_{d+1} \circ \eta_{\sigma(d)})'(0) . \end{aligned}$$

In a similar manner one checks

$$\begin{aligned} & \omega^W(df_k([\eta_1]), \dots, df_k([\eta_d])) \\ &= C_d \sum_{j=1}^{d+1} (-1)^{j+1} x_j dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_{d+1}(df_k([\eta_1]), \dots, df_k([\eta_d])) \\ &= C_d \sum_{j=1}^{d+1} (-1)^{j+1} x_j(f(k)) \frac{1}{d!} \sum_{\sigma \in S_d} dx_1(df_k[\eta_{\sigma(1)}]) \cdots dx_{j-1}(df_k[\eta_{\sigma(j-1)}]) \\ & \quad dx_{j+1}(df_k[\eta_{\sigma(j)}]) \cdots dx_{d+1}(df_k[\eta_{\sigma(d)}]) \\ &= \frac{C_d}{d!} \sum_{j=1}^{d+1} (-1)^{j+1} f_j(k) \sum_{\sigma \in S_d} (x_1 \circ f \circ \eta_{\sigma(1)})'(0) \cdots (x_1 \circ f \circ \eta_{\sigma(j-1)})'(0) \\ & \quad (x_{j+1} \circ f \circ \eta_{\sigma(j)})'(0) \cdots (x_{d+1} \circ f \circ \eta_{\sigma(d)})'(0) \\ &= \frac{C_d}{d!} \sum_{j=1}^{d+1} (-1)^{j+1} f_j(k) \sum_{\sigma \in S_d} (f_1 \circ \eta_{\sigma(1)})'(0) \cdots (f_{j-1} \circ \eta_{\sigma(j-1)})'(0) \\ & \quad (f_{j+1} \circ \eta_{\sigma(j)})'(0) \cdots (f_{d+1} \circ \eta_{\sigma(d)})'(0) . \end{aligned}$$

Therefore (51) holds and the claim follows. \square

Now by a well-known pullback formula (*e.g.* tom Dieck's lecture notes [22] contain a detailed proof) one has

$$\int_{\mathbb{T}^d} \omega_f = \deg(f) \int_{\mathbb{S}^d} \omega^W ,$$

where $\deg(f)$ is the mapping degree of f , generically defined as the sum over all preimages (of a fixed point) of the signs of the determinants of the Jacobians. Next recall (*e.g.* [18]) the definition of the d th Chern number of a differentiable projection-valued map $P : \mathcal{M}^d \rightarrow \mathbb{C}^{L \times L}$ on a d -dimensional manifold \mathcal{M} :

$$\text{Ch}_d(P) = (-1)^{\frac{d}{2}} \left(\frac{1}{2\pi i}\right)^{\frac{d}{2}} \frac{1}{\frac{d}{2}!} \int_{\mathcal{M}^d} \text{Tr}(P(dP)^d) .$$

Then for $\omega_f = \text{Tr}(P_f(dP_f)^d)$ as above, one obtains

$$\text{Ch}_d(P_f) = \deg(f) (-1)^{\frac{d}{2}} \left(\frac{1}{2\pi i}\right)^{\frac{d}{2}} \frac{1}{\frac{d}{2}!} \int_{\mathbb{S}^d} \omega^W = \deg(f) \text{Ch}_d(P^W) .$$

It hence remains to compute $\text{Ch}_d(P^W)$ which is again well-known:

$$\text{Ch}_d(P^W) = (-1)^{\frac{d}{2}} .$$

(*E.g.* [3, 21] contains a detailed computation.) Summing up, one concludes:

Corollary 37 *For d even and a smooth map $f : \mathbb{T}^d \rightarrow \mathbb{S}^d$, the d th Chern number of P_f is*

$$\text{Ch}_d(P_f) = (-1)^{\frac{d}{2}} \deg(f) .$$

B Mapping degree of some maps from torus to sphere

This appendix is about the mapping degrees of the maps $g_{d,m} : \mathbb{T}^d \rightarrow \mathbb{R}^{d+1}$ with d even and $m \in \mathbb{R}$ given by

$$g_{d,m}(e^{i\theta_1}, \dots, e^{i\theta_d}) = \left(\sin(\theta_1), \dots, \sin(\theta_d), m - \sum_{n=1}^d \cos(\theta_n) \right) , \quad (52)$$

where $\theta_n \in [0, 2\pi)$ for $n \in \{1, \dots, d\}$ and $(e^{i\theta_1}, \dots, e^{i\theta_d}) \in \mathbb{T}^d$. It can readily be checked that the vector on the r.h.s. does not vanish if and only if $m \in \mathbb{R} \setminus \{-d, -d+2, \dots, d-2, d\}$. For such m , let us then set

$$f_{d,m}(e^{i\theta_1}, \dots, e^{i\theta_d}) = \|g_{d,m}(e^{i\theta_1}, \dots, e^{i\theta_d})\|^{-1} g_{d,m}(e^{i\theta_1}, \dots, e^{i\theta_d}) . \quad (53)$$

Then $f_{d,m} : \mathbb{T}^d \rightarrow \mathbb{S}^d$ is a map onto the unit sphere with same mapping degree $\deg(f_{d,m}) = \deg(g_{d,m})$. By Corollary 37, this mapping degree is equal to the d -th Chern number $\text{Ch}_d(P_{f_{d,m}})$. These Chern numbers were computed in Section 2.2.4 of [18] by analyzing the changes of the

Chern numbers at the transition points $\{-d, -d+2, \dots, d-2, d\}$. The argument involves rather delicate singular integrals, and this was revisited in detail in [21]. Here a direct alternative argument based on the computation of the mapping degree of $g_{d,m}$ is provided. Let us stress that the map $g_{d,d-1}$ is at the root of the construction of the periodic spectral localizer and the G -operators associated to fuzzy tori. It leads to a Chern number $\text{Ch}_d(P_{f_{d,d-1}}) = 1$.

To compactify notations, let us set $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_d}) \in \mathbb{T}^d$ and $\mathbf{x} = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ as well as $\hat{\mathbf{x}} = (x_1, \dots, x_d) \in \mathbb{R}^d$. Furthermore, let \mathbb{T}^d be equipped with the orientation inherited from the atlas $(U_j, \varphi_j)_{j \in \{1,2\}}$ given by

$$\begin{aligned} U_1 &= \{e^{i\theta} \in \mathbb{T}^d : \theta_n \in [0, 2\pi) \setminus \{\frac{\pi}{2}\} \text{ for } n \in \{1, \dots, d\}\}, \\ U_2 &= \{e^{i\theta} \in \mathbb{T}^d : \theta_n \in [0, 2\pi) \setminus \{\frac{3\pi}{2}\} \text{ for } n \in \{1, \dots, d\}\}, \end{aligned}$$

and the charts

$$\varphi_1(e^{i\theta}) = \left(\frac{\cos(\theta_1)}{1-\sin(\theta_1)}, \dots, \frac{\cos(\theta_d)}{1-\sin(\theta_d)} \right), \quad \varphi_2(e^{i\theta}) = \left(\frac{\cos(\theta_1)}{1+\sin(\theta_1)}, \dots, \frac{\cos(\theta_d)}{1+\sin(\theta_d)} \right).$$

Moreover, let \mathbb{S}^d be equipped with orientation inherited from the atlas $(V_j, \psi_j)_{j \in \{1,2\}}$ given by

$$V_1 = \{\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1} : x_{d+1} \neq 1\}, \quad V_2 = \{\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1} : x_{d+1} \neq -1\},$$

and

$$\psi_1(\mathbf{x}) = \left(\frac{x_1}{1-x_{d+1}}, \dots, \frac{x_d}{1-x_{d+1}} \right), \quad \psi_2(\mathbf{x}) = \left(\frac{x_2}{1+x_{d+1}}, \frac{x_1}{1+x_{d+1}}, \frac{x_3}{1+x_{d+1}}, \dots, \frac{x_d}{1+x_{d+1}} \right).$$

Proposition 38 *If \mathbb{T}^d is equipped with the orientation inherited from the atlas $(U_j, \varphi_j)_{j \in \{1,2\}}$ and \mathbb{S}^d is equipped with the orientation inherited from the atlas $(V_j, \psi_j)_{j \in \{1,2\}}$ the degree of the map $f_{d,m}$ defined by (53) is given by*

$$\deg(f_{d,m}) = \begin{cases} \sum_{k=0}^{\frac{d-n-1}{2}} (-1)^k \binom{d}{k}, & n \in (0, d+1) \cap 2\mathbb{N} + 1, m \in (n-1, n+1), \\ \sum_{k=0}^{\frac{d-n-1}{2}} (-1)^{k+1} \binom{d}{k}, & n \in (0, d+1) \cap 2\mathbb{N} + 1, m \in (-n-1, -n+1), \\ 0, & m \in (-\infty, -d) \cup (d, \infty). \end{cases}$$

Proof. By the homotopy invariance of the mapping degree, it is sufficient to consider the case $m \in \{-d-1, -d+1, \dots, d+1\}$. For the case $m = d+1$, one can consider the maps

$$g_{d,d+1,t}(e^{i\theta}) = \left(t \sin(\theta_1), \dots, t \sin(\theta_d), 1 - t \sum_{n=1}^d (\cos(\theta_n) - 1) \right),$$

where $t \in [0, 1]$. Then $g_{d,d+1,1} = g_{d,d+1}$ and $g_{d,d+1,0}(e^{i\theta}) = p_N$ where $p_N = (0, \dots, 0, 1) \in \mathbb{S}^d$ is the north pole. Now the norm satisfies

$$\|g_{d,d+1,t}(e^{i\theta})\|^2 = t^2 \sum_{j=1}^d \sin^2(\theta_j) + 1 + 2t \sum_{j=1}^d (1 - \cos(\theta_j)) + t^2 \left(\sum_{j=1}^d (1 - \cos(\theta_j)) \right)^2 \geq 1.$$

This implies that $t \in [0, 1] \mapsto f_{d,d+1,t} = \|g_{d,d+1,t}\|^{-1} g_{d,d+1,t}$ is a homotopy from $f_{d,d+1}$ to a constant map. Hence the mapping degree vanishes. (Note that this argument is essentially reproduced for fuzzy tori in Lemma 34.) Let us next focus on the case $m \in \{1, 3, \dots, d-1\}$. The other points are dealt with in a similar manner (with the north instead of the south pole used in the argument below). Consider the south pole given by $p_S = (0, \dots, 0, -1) \in \mathbb{S}^d$. Its inverse image is

$$f_{d,m}^{-1}(p_S) = \{e^{i\theta} \in \mathbb{T}^d : \theta_n \in \{0, \pi\} \text{ for } n \in \{1, \dots, d\}, \theta_j = \pi \text{ for at most } \frac{d-m-1}{2} \text{ many } j\}.$$

Then

$$\psi_1 \circ f_{d,m} \circ \varphi_2^{-1} : \mathbb{R}^d \setminus \{\hat{\mathbf{x}} \in \mathbb{R}^d : x_n \in \{\pm 1\}, \#\{j \in \{1, \dots, d\} : x_j = -1\} > \frac{d-m-1}{2}\} \rightarrow \mathbb{R}^d$$

is given by

$$(\psi_1 \circ f_{d,m} \circ \varphi_2^{-1})(\hat{\mathbf{x}}) = \left(\frac{1-x_1^2}{x_1^2+1} \left(1 - m + 2 \sum_{n=1}^d \frac{x_n}{x_n^2+1}\right)^{-1}, \dots, \frac{1-x_d^2}{x_d^2+1} \left(1 - m + 2 \sum_{n=1}^d \frac{x_n}{x_n^2+1}\right)^{-1} \right).$$

Its k th component is

$$(\psi_1 \circ f_{d,m} \circ \varphi_2^{-1})(\hat{\mathbf{x}})_k = \frac{(1-x_k^2) \prod_{n \neq k} (x_n^2 + 1)}{(1-m) \prod_{n=1}^d (x_n^2 + 1) + 2 \sum_{n=1}^d x_n \prod_{j \neq n} (x_j^2 + 1)}. \quad (54)$$

Let us denote the Jacobian matrix of this map by

$$J = J_{\psi_1 \circ f_{d,m} \circ \varphi_2^{-1}} : \mathbb{R}^d \setminus \{\hat{\mathbf{x}} \in \mathbb{R}^d : x_n \in \{\pm 1\}, \#\{j \in \{1, \dots, d\} : x_j = -1\} > \frac{d-m-1}{2}\} \rightarrow \mathbb{R}^d.$$

To determine the mapping degree of f (at the point p_S) it is sufficient to compute the restriction of J to $\mathbb{M} = \varphi_2(f^{-1}(p_S))$ explicitly given by

$$\mathbb{M} = \{\hat{\mathbf{x}} \in \mathbb{R}^d : x_n \in \{-1, 1\} \text{ for } n \in \{1, \dots, d\}, x_j = -1 \text{ for at most } \frac{d-m-1}{2} \text{ many } j\}.$$

One directly checks that the off-diagonal entries of $J|_{\mathbb{M}}$ vanish, namely $J(\hat{\mathbf{x}})_{k,l} = 0$ for all $\hat{\mathbf{x}} \in \mathbb{M}$ and $k, l \in \{1, \dots, d\}$, $k \neq l$. The diagonal entries of $J|_{\mathbb{M}}$ are

$$J(\hat{\mathbf{x}})_{k,k} = \frac{-x_k 2^d}{((1-m)2^d + 2^d \sum_{n=1}^d x_n)^2} \quad (55)$$

for $\hat{\mathbf{x}} \in \mathbb{M}$ and $k \in \{1, \dots, d\}$. Therefore $\text{sgn}(J(\hat{\mathbf{x}})_{k,k}) = -\text{sgn}(x_k)$ and

$$\text{sgn}(\det(J(\hat{\mathbf{x}}))) = (-1)^{\#\{j \in \{1, \dots, d\} : x_j = 1\}} = (-1)^{\#\{j \in \{1, \dots, d\} : x_j = -1\}},$$

where the last equality holds because d is even. Because

$$\#\{\hat{\mathbf{x}} \in \mathbb{R}^d : x_n \in \{-1, 1\} \text{ for } n \in \{1, \dots, d\}, x_j = -1 \text{ for } k \text{ many } j\} = \binom{d}{k},$$

one has

$$\deg(f_{d,m}, p_S) = \sum_{k=0}^{\frac{d-n-1}{2}} (-1)^k \binom{d}{k}.$$

As the mapping degree is independent of the point at which the preimage is taken (provided it has a finite preimage), this shows the claim for $m \in \{1, 3, \dots, d-1\}$. \square

Acknowledgements: T.L. acknowledges support from the National Science Foundation, grant DMS-2110398. This work was also supported by the DFG grant SCHU 1358/8-1. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study. The authors have no competing interests to declare that are relevant to the content of this article.

References

- [1] O. Bratteli, D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*, (Springer, Berlin, 1979).
- [2] J. Bellissard, *K-theory of C*-algebras in solid state physics*, in T. Dorlas, M. Hugenholtz, M. Winnink, editors, *Lecture Notes in Physics* **257**, 99-156, (Springer-Verlag, Berlin, 1986).
- [3] A. L. Carey, H. Schulz-Baldes, *Spectral flow of monopole insertion in topological insulators*, *Commun. Math. Phys.* **370**, 895-923 (2019).
- [4] N. Doll, H. Schulz-Baldes, *Skew localizer and \mathbb{Z}_2 -flows for real index pairings*, *Advances Math.* **392**, 108038 (2021).
- [5] N. Doll, H. Schulz-Baldes, N. Waterstraat, *Spectral Flow: A Functional Analytic and Index-Theoretic Approach*, (De Gruyter, Berlin, 2023).
- [6] S. Eilers, T. A. Loring, *Computing contingencies for stable relations*, *International J. Math.* **10**, 301-326 (1999).
- [7] S. Eilers, T. A. Loring, G. K. Pedersen, *Morphisms of extensions of C*-algebras: pushing forward the Busby invariant*, *Adv. Math.* **147**, 74-109 (1999).
- [8] R. Exel, T. A. Loring, *Invariants of almost commuting unitaries*, *J. Funct. Anal.* **95**, 364-376 (1991).
- [9] G. von Gersdorff, S. Panahiyan, W. Chen, *Unification of topological invariants in Dirac models*, *Phys. Rev.* **B 103**, 245146 (2021).
- [10] J. Grossmann, H. Schulz-Baldes, *Index pairings in presence of symmetries with applications to topological insulators*, *Commun. Math. Phys.* **343**, 477-513 (2016).
- [11] M. B. Hastings, T. A. Loring, *Topological insulators and C*-algebras: Theory and numerical practice*, *Annals of Physics* **326**, 1699-1759 (2011).

- [12] Y. Kubota, *The index theorem of lattice Wilson-Dirac operators via higher index theory*, *Annales H. Poincaré* **23**, 1297-1319 (2022).
- [13] T. A. Loring, *K-theory and pseudospectra for topological insulators*, *Annals of Physics* **356**, 383-416 (2015).
- [14] T. Loring, H. Schulz-Baldes, *Finite volume calculation of K-theory invariants*, *New York J. Math.* **22**, 1111-1140 (2017).
- [15] T. Loring, H. Schulz-Baldes, *The spectral localizer for even index pairings*, *J. Noncommutative Geometry* **14**, 1-23 (2020).
- [16] G. K. Pedersen, *A commutator estimate*, in R. Herman, B. Tanbay (Editors), *Operator Algebras, Mathematical Physics, and Low Dimensional Topology*, (CRC Press, New York, 1993).
- [17] E. Prodan, *A computational non-commutative geometry program for disordered topological insulators*, *Springer Briefs in Math. Phys.* Vol. 23, (Springer, Switzerland, 2017).
- [18] E. Prodan, H. Schulz-Baldes, *Bulk and boundary invariants for complex topological insulators: From K-theory to physics*, (Springer International, Switzerland, 2016).
- [19] S. Ryu, A. P. Schnyder, A. Furusaki, A. W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, *New J. Phys.* **12**, 065010 (2010).
- [20] H. Schulz-Baldes, T. Stoiber, *Harmonic analysis in operator algebras and its applications to index theory and topological solid state systems*, (Springer, Cham, 2022).
- [21] H. Schulz-Baldes, T. Stoiber, *Spectral localization for semimetals and Callias operators*, *J. Math. Phys.* **64**, 081901 (2023).
- [22] T. tom Dieck, *Differenzierbare Mannigfaltigkeiten*, available online, *Lecture Notes Göttingen*, 2009.
- [23] D. Toniolo, *On the Bott index of unitary matrices on a finite torus*, *Lett. Math. Phys.* **112**, 116 (2022).