ON THE DISCONTINUITIES OF HAUSDORFF DIMENSION IN GENERIC DYNAMICAL LAGRANGE SPECTRUM

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Abstract. Let φ_0 be a C^2-conservative diffeomorphism of a compact surface S and let Λ_0 be a mixing horseshoe of φ_0. Given a smooth real function f defined in S and some diffeomorphism φ, close to φ_0, let $\mathcal{L}_{\varphi,f}$ be the Lagrange spectrum associated to the hyperbolic continuation $\Lambda(\varphi)$ of the horseshoe Λ_0 and f. We show that, for generic choices of φ and f, if $L_{\varphi,f}$ is the map that gives the Hausdorff dimension of the set $\mathcal{L}_{\varphi,f} \cap (-\infty, t)$ for $t \in \mathbb{R}$, then there are at most two points that can be limit of a infinite sequence of discontinuities of $L_{\varphi,f}$.

Contents

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1. Introduction

1.1. **Classical spectra.** The classical Lagrange and Markov spectra are closed subsets of the real line related to Diophantine approximations. They arise naturally in the study of rational approximations of irrational numbers and of indefinite binary quadratic forms, respectively. More precisely, given an irrational number α, let

$$\ell(\alpha) := \limsup_{\substack{p,q\to\infty \\ p,q\in\mathbb{N}}} \frac{1}{|q(q\alpha-p)|}$$

be its best constant of Diophantine approximation. The set

$$\mathcal{L} := \{\ell(\alpha) : \alpha \in \mathbb{R} - \mathbb{Q} \text{ and } \ell(\alpha) < \infty\}$$

consisting of all finite best constants of Diophantine approximations is the so-called *Lagrange spectrum*.

Similarly, given a real quadratic form $q(x, y) = ax^2 + bxy + cy^2$, let $\Delta(q) = b^2 - 4ac$ its discriminant. We define the *Markov spectrum* as follows

$$\mathcal{M} := \left\{ \frac{\sqrt{\Delta(q)}}{\inf\limits_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} |q(x,y)|} < \infty : q \text{ is indefinite and } \Delta(q) > 0 \right\}.$$

The reader can find more information about the structure of these sets in the classical book [13] of Cusick and Flahive, but let us mention here that:

- Markov showed that $\mathcal{L} \cap (-\infty, 3) = \mathcal{M} \cap (-\infty, 3) = \{\sqrt{9 - 4/z_n^2} : n \in \mathbb{N}\}$ where z_n are the *Markov numbers*, that is, the largest coordinate of a triple $(x_n, y_n, z_n) \in \mathbb{N}^3$ verifying the Markov equation
$$x_n^2 + y_n^2 + z_n^2 = 3x_n y_n z_n.$$
- Hall showed that $\mathcal{L}$ (and then $\mathcal{M}$) contain a half-line and Freiman determined the biggest half-line contained in the spectra, namely $[c, +\infty)$ where
$$c = \frac{2221564096 + 283748\sqrt{462}}{491993569} \simeq 4.52782956\ldots$$
- Moreira proved in [11] several results on the geometry of the Markov and Lagrange spectra, for example, that the map $d : \mathbb{R} \to [0, 1]$, given by
$$d(t) = HD(\mathcal{L} \cap (-\infty, t)) = HD(\mathcal{M} \cap (-\infty, t)),$$
(where $HD(X)$ denotes the Hausdorff dimension of the set X) is continuous, surjective and such that $\max\{t \in \mathbb{R} : d(t) = 0\} = 3$.

For our purposes, it is worth to point out here that the Lagrange and Markov spectra have the following *dynamical* interpretation in terms of the continued fraction algorithm: Denote by $[a_0, a_1, \ldots]$ the continued fraction $a_0 + \frac{1}{a_1 + \frac{1}{\ddots}}$. Let $\Sigma = \mathbb{N}^{\mathbb{Z}}$ the space of bi-infinite sequences of positive integers, $\sigma : \Sigma \to \Sigma$ be the left-shift map $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$, and let $f : \Sigma \to \mathbb{R}$ be the function

$$f((a_n)_{n\in\mathbb{Z}}) = [a_0, a_1, \dots] + [0, a_{-1}, a_{-2}, \dots].$$

Then,

$$\mathcal{L} = \left\{ \limsup_{n\to\infty} f(\sigma^n(\underline{\theta})) < \infty : \underline{\theta} \in \Sigma \right\} \quad \text{and} \quad \mathcal{M} = \left\{ \sup_{n\to\infty} f(\sigma^n(\underline{\theta})) < \infty : \underline{\theta} \in \Sigma \right\}.$$

In the sequel, we consider the natural generalization of this dynamical version of the classical Lagrange and Markov spectra in the context of horseshoes[1] of smooth diffeomorphisms of compact surfaces.

1.2. **Dynamical spectra.** Let $\varphi : S \to S$ be a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and let $f : S \to \mathbb{R}$ be a differentiable function. Following the above characterization of the classical spectra, we define the maps $\ell_{\varphi,f} : \Lambda \to \mathbb{R}$ and $m_{\varphi,f} : \Lambda \to \mathbb{R}$ given by $\ell_{\varphi,f}(x) = \limsup_{n\to\infty} f(\varphi^n(x))$ and $m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x))$ for $x \in \Lambda$ and call $\ell_{\varphi,f}(x)$ the *Lagrange value* of x associated to f and φ and also $m_{\varphi,f}(x)$ the *Markov value* of x associated to f and φ. The sets[2]

$$\mathcal{L}_{\varphi,f} = \ell_{\varphi,f}(\Lambda) = \{\ell_{\varphi,f}(x) : x \in \Lambda\}$$

and

$$\mathcal{M}_{\varphi,f} = m_{\varphi,f}(\Lambda) = \{m_{\varphi,f}(x) : x \in \Lambda\}$$

are called *Lagrange Spectrum* of (φ, f) and *Markov Spectrum* of (φ, f).

In this paper, we are interested in the study of the real function

$$L_{\varphi,f}(t) = HD(\mathcal{L}_{\varphi,f} \cap (-\infty, t)). \tag{1.1}$$

The description of this function is closely related to the study of the behavior of the family of sets $\{\Lambda_t\}_{t\in\mathbb{R}}$, where for $t \in \mathbb{R}$

$$\Lambda_t = m_{\varphi,f}^{-1}((\infty, t]) = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(f|_\Lambda^{-1}((\infty, t])) = \{x \in \Lambda : \forall n \in \mathbb{Z},\ f(\varphi^n(x)) \le t\}.$$

In order to do that, we will explore the combinatorial nature of $\varphi|_\Lambda$ and its connection with the unstable and stable Cantor sets associated to Λ. More specifically, fix a Markov partition $\{R_a\}_{a\in\mathcal{A}}$ of Λ with sufficiently small diameter consisting of rectangles $R_a \sim I_a^u \times I_a^s$ delimited by compact pieces I_a^s, I_a^u, of stable and unstable manifolds of certain points of Λ, see [16] theorem 2, page 172. The set $\mathcal{B} \subset \mathcal{A}^2$ of

[1] i.e., a non-empty compact invariant hyperbolic set of saddle type which is transitive, locally maximal, and not reduced to a periodic orbit (cf. [16] for more details).

[2] we omit the reference to the horseshoe Λ because in our context it will be always determined by the diffeomorphism φ.

admissible transitions consist of pairs (a,b) such that $\varphi(R_a)\cap R_b\neq\emptyset$; so, we can define the transition matrix B by

$$b_{ab}=1 \text{ if } \varphi(R_a)\cap R_b\neq\emptyset \text{ and } b_{ab}=0 \text{ otherwise, for } (a,b)\in\mathcal{A}^2.$$

Let $\Sigma_{\mathcal{A}}=\{\underline{a}=(a_n)_{n\in\mathbb{Z}}:a_n\in\mathcal{A} \text{ for all } n\in\mathbb{Z}\}$ and consider the homeomorphism of $\Sigma_{\mathcal{A}}$, the shift, $\sigma:\Sigma_{\mathcal{A}}\to\Sigma_{\mathcal{A}}$ defined by $\sigma(\underline{a})_n=a_{n+1}$. Let $\Sigma_{\mathcal{B}}=\left\{\underline{a}\in\Sigma_{\mathcal{A}}:b_{a_na_{n+1}}=1\right\}$, this set is closed and σ-invariant subspace of $\Sigma_{\mathcal{A}}$. Still denote by σ the restriction of σ to $\Sigma_{\mathcal{B}}$, the pair $(\Sigma_{\mathcal{B}},\sigma)$ is a subshift of finite type, see [6] chapter 10. The dynamics of φ on Λ is topologically conjugate to the sub-shift $\Sigma_{\mathcal{B}}$, namely, there is a homeomorphism $\Pi:\Lambda\to\Sigma_{\mathcal{B}}$ such that $\varphi\circ\Pi=\Pi\circ\sigma$.

As we generally will deal with sequences, we transfer the function f from Λ to a function (still denoted f) on $\Sigma_{\mathcal{B}}$. In this way, we set

$$\Sigma_t=\Pi(\Lambda_t)=\{\theta\in\Sigma_{\mathcal{B}}:\sup_{n\in\mathbb{Z}}f(\sigma^n(\theta))\leq t\}.$$

Recall that the stable and unstable manifolds of Λ can be extended to locally invariant $C^{1+\alpha}$ foliations in a neighborhood of Λ for some $\alpha>0$. Using these foliations it is possible define projections $\pi_a^u:R_a\to I_a^s\times\{i_a^u\}$ and $\pi_a^s:R_a\to\{i_a^s\}\times I_a^u$ of the rectangles into the connected components $I_a^s\times\{i_a^u\}$ and $\{i_a^s\}\times I_a^u$ of the stable and unstable boundaries of R_a, where $i_a^u\in\partial I_a^u$ and $i_a^s\in\partial I_a^s$ are fixed arbitrarily. In this way, we have the unstable and stable Cantor sets

$$K^u:=\bigcup_{a\in\mathcal{A}}\pi_a^s(\Lambda\cap R_a) \text{ and } K^s:=\bigcup_{a\in\mathcal{A}}\pi_a^u(\Lambda\cap R_a).$$

In fact K^u and K^s are $C^{1+\alpha}$ dynamically defined, associated to some expanding maps ψ_s and ψ_u defined in the following way: If $y\in R_{a_1}\cap\varphi(R_{a_0})$ we put

$$\psi_s(\pi_{a_1}^u(y))=\pi_{a_0}^u(\varphi^{-1}(y))$$

and if $z\in R_{a_0}\cap\varphi^{-1}(R_{a_1})$ we put

$$\psi_u(\pi_{a_0}^s(z))=\pi_{a_1}^s(\varphi(z)).$$

Moreira's theorem of [11] was generalized first in [1] in the context of *conservative* diffeomorphism with some horseshoe with Hausdorff dimension smaller than 1 and later was removed the condition on the dimension of the horseshoe in [9]. More specifically, the authors proved that for typical choices of the dynamic and of the real function, the intersections of the corresponding dynamical Markov and Lagrange spectra with half-lines $(-\infty,t)$ have the same Hausdorff dimension, and this defines a continuous function of t whose image is $[0,\min\{1,D\}]$, where D is the Hausdorff dimension of the horseshoe.

Our main theorem (cf. Theorem 1.1 below) is quite related to the result of the previous paragraph but, in our case, we will work away from the two points that determine "the canonical interval" where $L_{\varphi,f}$ can have a discontinuity. Here, we drop the hypothesis of the neighborhood of the initial conservative diffeomorphism be in the space of conservative diffeomorphisms. However, we can only conclude finiteness of the number of discontinuities but not continuity else.

1.3. **Statement of the main theorem.** Let φ_0 be a smooth conservative diffeomorphism of a surface S possessing a mixing horseshoe Λ_0. Denote by $\mathcal{U}$ a C^2 neighborhood of φ_0 in the space $\mathrm{Diff}^2(S)$ of smooth diffeomorphisms of S such that Λ_0 admits a continuation Λ for every $\varphi \in \mathcal{U}$. Using the notations of the previous subsection, our objective is to study the discontinuities of the map $L_{\varphi,f}$ defined by

$$t \mapsto L_{\varphi,f}(t) = HD(\mathcal{L}_{\varphi,f} \cap (-\infty, t)).$$

In order to do this, we consider the interval $I_{\varphi,f} = [c_{\varphi,f}, \tilde{c}_{\varphi,f}]$, where

$$c_{\varphi,f} := \sup\{t \in \mathbb{R} : L_{\varphi,f}(t) = \min L_{\varphi,f} = 0\}$$

and

$$\tilde{c}_{\varphi,f} := \inf\{t \in \mathbb{R} : L_{\varphi,f}(t) = \max L_{\varphi,f} = HD(\mathcal{L}_{\varphi,f})\}$$

which is the interval where $L_{\varphi,f}$ can have discontinuities. With this notation, our main result is the following

Theorem 1.1. *If $\mathcal{U} \subset \mathrm{Diff}^2(S)$ is sufficiently small, then there exists a residual subset $\mathcal{U}^* \subset \mathcal{U}$ with the property that for every $\varphi \in \mathcal{U}^*$ and any $r \geq 2$, there exists a C^r-residual set $\mathcal{P}_{\varphi,\Lambda} \subset C^r(S, \mathbb{R})$ such that given $f \in \mathcal{P}_{\varphi,\Lambda}$ one has*

$$\max L_{\varphi,f} = HD(\mathcal{L}_{\varphi,f}) = \min\{1, HD(\Lambda)\}$$

and

$$c_{\varphi,f} = \min \mathcal{L}'_{\varphi,f} = \min\{x : x \text{ is an accumulation point of } \mathcal{L}_{\varphi,f}\}.$$

Even more,

- *If $HD(\Lambda) < 1$ then $L_{\varphi,f}$ has finitely many discontinuities in any closed sub interval $I \subset I_{\varphi,f}$ that doesn't contain $c_{\varphi,f}$.*
- *If $HD(\Lambda) \geq 1$ then $L_{\varphi,f}$ has finitely many discontinuities in any closed sub interval $I \subset I_{\varphi,f}$ that doesn't contain neither $c_{\varphi,f}$ nor $\tilde{c}_{\varphi,f}$.*

As a consequence, we immediately have the corollaries

Corollary 1.2. *If $HD(\Lambda_0) < 1$, then by choosing $\mathcal{U}$ small, given $\varphi \in \mathcal{U}^*$, $f \in \mathcal{P}_{\varphi,\Lambda}$ and $\epsilon > 0$ the function $L_{\varphi,f}$ has finitely many discontinuities in the interval $[c_{\varphi,f} + \epsilon, \infty)$. Therefore, $c_{\varphi,f}$ is the only possible limit of an infinite sequence of discontinuities of $L_{\varphi,f}$.*

Corollary 1.3. *If $HD(\Lambda_0) > 1$, then by choosing $\mathcal{U}$ small, given $\varphi \in \mathcal{U}^*$, $f \in \mathcal{P}_{\varphi,\Lambda}$ and $\epsilon > 0$ small, the function $L_{\varphi,f}$ has finitely many discontinuities in the interval $[c_{\varphi,f} + \epsilon, \tilde{c}_{\varphi,f} - \epsilon]$. Therefore, $c_{\varphi,f}$ and $\tilde{c}_{\varphi,f}$ are the only possible limits of an infinite sequence of discontinuities of $L_{\varphi,f}$.*

2. Preliminary results

2.1. **Stable and unstable dimensions.** Given a Markov partition $\mathcal{P} = \{R_a\}_{a\in\mathcal{A}}$, recall that the geometrical description of Λ in terms of the Markov partition $\mathcal{P}$ has a combinatorial counterpart in terms of the Markov shift $\Sigma_{\mathcal{B}} \subset \mathcal{A}^{\mathbb{Z}}$. Given an admissible finite sequence $\alpha = (a_1, ..., a_n) \in \mathcal{A}^n$ (i.e., $(a_i, a_{i+1}) \in \mathcal{B}$ for all $1 \le i < n$), we define

$$I^u(\alpha) = \{x \in K^u : \psi_u^i(x) \in I^u(a_i, a_{i+1}),\ i = 1, 2, ..., n-1\}$$

and if $\alpha^T = (a_n, a_{n-1}, ..., a_1)$, we define

$$I^s(\alpha^T) = \{y \in K^s : \psi_s^i(y) \in I^s(a_i, a_{i-1}),\ i = 2, ..., n\}.$$

In a similar way, let $\theta = (a_{s_1}, a_{s_1+1}, ..., a_{s_2}) \in \mathcal{A}^{s_2-s_1+1}$ an admissible word where $s_1, s_2 \in \mathbb{Z}$, $s_1 < s_2$ and fix $s_1 \le s \le s_2$. Define

$$R(\theta; s) = \bigcap_{m=s_1-s}^{s_2-s} \varphi^{-m}(R_{a_{m+s}}).$$

Note that if $x \in R(\theta; s) \cap \Lambda$ then the symbolic representation of x is in the way $\Pi(x) = (\dots, a_{s_1} \dots a_{s-1}; a_s, a_{s+1} \dots a_{s_2} \dots)$, where the letter following to ; is in the 0 position of the sequence.

In our context of dynamically defined Cantor sets, we can relate the length of the unstable and stable intervals determined by an admissible word to its length as a word in the alphabet $\mathcal{A}$ via the *bounded distortion property* that let us conclude that for some constant $c_1 > 0$

$$e^{-c_1} \le \frac{|I^u(\alpha\beta)|}{|I^u(\alpha)| \cdot |I^u(\beta)|} \le e^{c_1} \text{ and } e^{-c_1} \le \frac{|I^s((\alpha\beta)^T)|}{|I^s(\alpha^T)| \cdot |I^s(\beta^T)|} \le e^{c_1}, \tag{2.1}$$

and also, for some positive constants $\lambda_1, \lambda_2 < 1$, one has

$$e^{-c_1}\lambda_1^{|\alpha|} \le |I^u(\alpha)| \le e^{c_1}\lambda_2^{|\alpha|} \text{ and } e^{-c_1}\lambda_1^{|\alpha|} \le |I^s(\alpha^T)| \le e^{c_1}\lambda_2^{|\alpha|}. \tag{2.2}$$

We write $r^{(u)}(\alpha)$ for the unstable scale of α, that is, $r^{(u)}(\alpha) = \lfloor \log(1/|I^u(\alpha)|) \rfloor$ and similarly, $r^{(s)}(\alpha) = \lfloor \log(1/|I^s(\alpha^T)|) \rfloor$ for the stable scale of α. Write $\alpha^* = (a_1, a_2, ..., a_{n-1})$ if $\alpha = (a_1, a_2, ..., a_n)$ and for $r \in \mathbb{N}$ define the sets

$$P_r^{(u)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(u)}(\alpha) \ge r \text{ and } r^{(u)}(\alpha^*) < r\}$$

and

$$P_r^{(s)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(s)}(\alpha) \ge r \text{ and } r^{(s)}(\alpha^*) < r\}.$$

Now, given any $X \subset \Lambda$ compact and φ-invariant we define its projections

$$\pi^u(X) = \bigcup_{a\in\mathcal{A}} \pi_a^s(X \cap R_a) \text{ and } \pi^s(X) = \bigcup_{a\in\mathcal{A}} \pi_a^u(X \cap R_a).$$

We also set

$$\mathcal{C}_u(X, r) = \{\alpha \in P_r^{(u)} : I^u(\alpha) \cap \pi^u(X) \ne \emptyset\}$$

and

$$\mathcal{C}_s(X,r) = \{\alpha \in P_r^{(s)} : I^s(\alpha^T) \cap \pi^s(X) \neq \emptyset\}$$

whose cardinalities are denoted $N_u(X,r) = |\mathcal{C}_u(X,r)|$ and $N_s(X,r) = |\mathcal{C}_s(X,r)|$.

Note that by 2.2 for $\alpha \in \mathcal{C}_u(X,r)$ one has $e^{c_1}\lambda_2^{-1}\lambda_2^{|\alpha|} > |I^u(\alpha^*)| > e^{-r}$ and from this follows that $|\alpha| < r/\log(\lambda_2^{-1}) + \log(e^{c_1}\lambda_2^{-1})/\log(\lambda_2^{-1})$ and then

$$N_u(X,r) = |\mathcal{C}_u(X,r)| \leq e^{\alpha_1 r + \alpha_2} \tag{2.3}$$

where $\alpha_1 = \log|\mathcal{A}|/\log(\lambda_2^{-1}) > 0$ and $\alpha_2 = \log(e^{c_1}\lambda_2^{-1}) \cdot \log|\mathcal{A}|/\log(\lambda_2^{-1}) > 0$ depends only on φ and Λ. Note that the same inequality also holds for $N_s(X,r)$.

In the article [1] the authors proved the following lemma in the case of $X = \Lambda_t$, for completeness we give a proof here:

Lemma 2.1. *There exists a constant $c_2 = c_2(\varphi, \Lambda) \in \mathbb{N}$ such that if X is a compact, φ-invariant subset of Λ, then*

$$N_u(X, m+n) \leq |\mathcal{A}|^{c_2} \cdot N_u(X,m) \cdot N_u(X,n)$$

and

$$N_s(X, m+n) \leq |\mathcal{A}|^{c_2} \cdot N_s(X,m) \cdot N_s(X,n)$$

for all $n, m \in \mathbb{N}$.

Proof. By symmetry, it is suffices to show that the sequence $\{N_u(X,r)\}_{r\in\mathbb{N}}$ satisfies the conclusions of the lemma. By 2.1 and 2.2 we have for all α, β, γ finite words such that the concatenation $\alpha\beta\gamma$ is admissible

$$|I^u(\alpha\beta\gamma)| \leq e^{2c_1}|I^u(\alpha)| \cdot |I^u(\beta)| \cdot |I^u(\gamma)| \leq e^{3c_1}\lambda_2^{|\gamma|} \cdot |I^u(\alpha)| \cdot |I^u(\beta)|.$$

Now, we note that, for each $c \in \mathbb{N}$, one can cover $\pi^u(X)$ with no more than $|\mathcal{A}|^c \cdot N_u(X,n) \cdot N_u(X,m)$ intervals $I^u(\alpha\beta\gamma)$ with $\alpha \in \mathcal{C}_u(X,n)$, $\beta \in \mathcal{C}_u(X,m)$, $\gamma \in \mathcal{A}^c$ and $\alpha\beta\gamma$ admissible.

Therefore, by taking $c_2 = \left\lceil \frac{3c_1}{\log \lambda_2^{-1}} \right\rceil \in \mathbb{N}$ it follows that we can cover $\pi^u(X)$ with no more than $|\mathcal{A}|^{c_2} \cdot N_u(X,n) \cdot N_u(X,m)$ intervals $I^u(\alpha\beta\gamma)$ whose unstable scales satisfy

$$r^{(u)}(\alpha\beta\gamma) \geq r^{(u)}(\alpha) + r^{(u)}(\beta) \geq n + m.$$

Hence, by definition, we conclude that

$$N_{\mathbf{u}}(X, n+m) \leq |\mathcal{A}|^{c_2} \cdot N_u(X,n) \cdot N_u(X,m),$$

as we wanted to see. ∎

From this lemma we get that for each $X \subset \Lambda$ compact, φ-invariant there exist the limits

$$D_u(X) = \lim_{r\to\infty} \frac{\log N_u(X,r)}{r} = \inf_{r\in\mathbb{N}} \frac{\log(|\mathcal{A}|^{c_2} \cdot N_u(X,r))}{r}$$

and

$$D_s(X) = \lim_{r\to\infty} \frac{\log N_s(X,r)}{r} = \inf_{r\in\mathbb{N}} \frac{\log(|\mathcal{A}|^{c_2} \cdot N_s(X,r))}{r}$$

and that the numbers $D_u(X)$ and $D_s(X)$ are the limit capacities of $\pi^u(X)$ and $\pi^s(X)$ respectively.

By 2.2 we have for the constants $\tilde{C} = \log\lambda_1/\log\lambda_2 > 1$ and $C = e^{c_1\cdot(\tilde{C}+1)} > 1$ and any α admissible that

$$C^{-1}|I^u(\alpha)|^{\tilde{C}} \leq |I^s(\alpha^T)| \leq C|I^u(\alpha)|^{1/\tilde{C}} \tag{2.4}$$

and for this, we conclude that for every $X \subset \Lambda$, compact and φ-invariant, $D_s(X)$ and $D_u(X)$ are comparable:

$$\tilde{C}^{-1}D_u(X) \leq D_s(X) \leq \tilde{C}D_u(X) \tag{2.5}$$

and so,

$$HD(X) \leq D_s(X) + D_u(X) \leq (\tilde{C}+1)D_s(X) \tag{2.6}$$

and

$$HD(X) \leq D_s(X) + D_u(X) \leq (\tilde{C}+1)D_u(X). \tag{2.7}$$

2.2. **Sets of finite type and connection of subhorseshoes.** The following definitions and results can be found in [10]. Fix a horseshoe Λ of some diffeomorphism $\varphi : S \to S$ and $\mathcal{P} = \{R_a\}_{a\in\mathcal{A}}$ some Markov partition for Λ. Take a finite collection X of finite admissible words of the form $\theta = (a_{-n(\theta)}, \dots, a_{-1}, a_0, a_1, \dots, a_{n(\theta)})$, we said that the maximal invariant set

$$M(X) = \bigcap_{m\in\mathbb{Z}} \varphi^{-m}\Big(\bigcup_{\theta\in X} R(\theta;0)\Big)$$

is a *hyperbolic set of finite type*. Even more, it is said to be a *subhorseshoe* of Λ if it is nonempty and $\varphi|_{M(X)}$ is transitive. Observe that a subhorseshoe need not be a horseshoe; indeed, it could be a periodic orbit in which case it will be called trivial.

By definition, hyperbolic sets of finite type have local product structure. In fact, any hyperbolic set of finite type is a locally maximal invariant set of a neighborhood of a finite number of elements of some Markov partition of Λ.

Definition 2.2. Any $\tau \subset M(X)$ for which there are two different subhorseshoes $\Lambda(1)$ and $\Lambda(2)$ of Λ contained in $M(X)$ with

$$\tau = \{x \in M(X) : \ \omega(x) \subset \Lambda(1) \text{ and } \alpha(x) \subset \Lambda(2)\}$$

will be called a transient set or transient component of $M(X)$.

Note that by the local product structure, given a transient set τ as before,

$$HD(\tau) = HD(K^s(\Lambda(2))) + HD(K^u(\Lambda(1))). \tag{2.8}$$

Proposition 2.3. *Any hyperbolic set of finite type $M(X)$, associated with a finite collection of finite admissible words X as before, can be written as*

$$M(X) = \bigcup_{i\in\mathcal{I}} \tilde{\Lambda}_i$$

where $\mathcal{I}$ is a finite index set (that may be empty) and for $i \in \mathcal{I}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set.

Now, fix $r \geq 2$ and for $x \in \Lambda$, let e^s_x and e^u_x unit vectors in the stable and unstable directions of T_xS. Given some subhorseshoe $\tilde{\Lambda} \subset \Lambda$ we define

$$\mathcal{R}_{\varphi,\tilde{\Lambda}} := \{f \in C^r(S,\mathbb{R}) : \nabla f(x) \text{ is not perpendicular neither to } e^s_x \text{ nor } e^u_x \text{ for all } x \in \tilde{\Lambda}\}.$$

In other terms, $\mathcal{R}_{\varphi,\tilde{\Lambda}}$ is the class of C^r-functions $f : S \to \mathbb{R}$ that are locally monotone along stable and unstable directions for points in $\tilde{\Lambda}$. The next proposition follows from the results proved in [1] (see remark 1.4 in that paper):

Proposition 2.4. *Fix $r \geq 2$. If the subhorseshoe $\tilde{\Lambda} \subset \Lambda$ has Hausdorff dimension smaller than 1, then $\mathcal{R}_{\varphi,\tilde{\Lambda}}$ is C^r-open and dense and for $f \in \mathcal{R}_{\varphi,\tilde{\Lambda}}$ the functions $t \mapsto D_u(\tilde{\Lambda}_t)$ and $t \mapsto D_s(\tilde{\Lambda}_t)$ are continuous, where $\tilde{\Lambda}_t = \{x \in \tilde{\Lambda} : \forall n \in \mathbb{Z},\ f(\varphi^n(x)) \leq t\}$.*

Fix $f : S \to \mathbb{R}$ differentiable. A notion that plays an important role in our study of the discontinuities of the map $L_{\varphi,f}$ is the notion of *connection of subhorseshoes*

Definition 2.5. Given $\Lambda(1)$ and $\Lambda(2)$ subhorseshoes of Λ and $t \in \mathbb{R}$, we said that $\Lambda(1)$ *connects* with $\Lambda(2)$ or that $\Lambda(1)$ and $\Lambda(2)$ *connect* before t if there exist a subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and some $q < t$ with $\Lambda(1) \cup \Lambda(2) \subset \tilde{\Lambda} \subset \Lambda_q$.

For our present purposes, the next criterion of connection will be also important

Proposition 2.6. *Suppose $\Lambda(1)$ and $\Lambda(2)$ are subhorseshoes of Λ and for some $x, y \in \Lambda$ we have $x \in W^u(\Lambda(1)) \cap W^s(\Lambda(2))$ and $y \in W^u(\Lambda(2)) \cap W^s(\Lambda(1))$. If for some $t \in \mathbb{R}$, it is true that*

$$\Lambda(1) \cup \Lambda(2) \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_t,$$

then for every $\epsilon > 0$, $\Lambda(1)$ and $\Lambda(2)$ connect before $t + \epsilon$.

Corollary 2.7. *Let $\Lambda(1)$, $\Lambda(2)$ and $\Lambda(3)$ subhorseshoes of Λ and $t \in \mathbb{R}$. If $\Lambda(1)$ connects with $\Lambda(2)$ before t and $\Lambda(2)$ connects with $\Lambda(3)$ before t. Then also $\Lambda(1)$ connects with $\Lambda(3)$ before t.*

3. Proof of Theorem 1.1

The proof when the Hausdorff dimension of the horseshoe is less than 1 is by contradiction:[3] we suppose the existence of an infinite sequence of discontinuities of the map $L_{\varphi,f}$ in some closed sub interval of $I_{\varphi,f}$ that doesn't contain the first accumulation point of the Lagrange spectrum and associate to every term of such a sequence a pair of subhorseshoes that don't connect before the term but they connect little time after it. Then, from this sequence of pair of subhorseshoes, we extract an infinite sequence of subhorseshoes $\mathcal{S}$, with the property that it contains arbitrarily big finite subsequences of terms that don't connect two by two before the maximum

[3] the precise statements will be present in the sequel.

of the discontinuities that determine them. Choosing correct scales (at the level of sequences) we show that for every term of $\mathcal{S}$, we can associate a periodic orbit (with period bounded by a fixed constant) in such a way that it is possible to connect two subhorseshoes with the same associated periodic orbit before the maximum of the discontinuities that determine them, letting us obtain the desired contradiction. The proof when the Hausdorff dimension of the horseshoe is greater than or equal to 1 is reduced to the previous case.

3.1. **The residuals subsets.** In this short subsection we introduce the residuals sets with which we are going to work. First, using the spectral decomposition theorem, it follows the next result from [7]:

Proposition 3.1. *There exists a residual subset $\mathcal{U}^* \subset \mathcal{U}$ with the property that for every subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and any $f \in C^1(S, \mathbb{R})$ such that there exists some point in $\tilde{\Lambda}$ with its gradient not parallel neither the stable direction nor the unstable direction, one has*

$$HD(f(\widetilde{\Lambda})) = \min\{1, HD(\widetilde{\Lambda})\}.$$

that we use to prove the next proposition

Proposition 3.2. *If $\mathcal{U}^*$ is as in the proposition 3.1 and $r \geq 2$, then for any $\varphi \in \mathcal{U}^*$, there exists a C^r-residual subset $\mathcal{P}_{\varphi,\Lambda}$ such that for every subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and any $f \in \mathcal{P}_{\varphi,\Lambda}$ one has*

$$\min\{1, HD(\widetilde{\Lambda})\} = HD(\ell_{\varphi,f}(\widetilde{\Lambda})) = HD(m_{\varphi,f}(\widetilde{\Lambda})).$$

Even more, if $HD(\tilde{\Lambda}) < 1$ one has $\mathcal{P}_{\varphi,\Lambda} \subset \mathcal{R}_{\varphi,\tilde{\Lambda}}$.

Proof. Following the ideas of the proof of the theorem 1 of [12] we see that given a subhorseshoe $\widetilde{\Lambda} \subset \Lambda$, the set

$$H_{\widetilde{\Lambda}} = \{f \in C^r(S, \mathbb{R}) : |M_{\widetilde{\Lambda},f}| = 1 \text{ and if } z \in M_{\widetilde{\Lambda},f},\ Df_z(e^{s,u}_z) \neq 0\}$$

is C^r- open and dense, where $M_{\widetilde{\Lambda},f} = \{z \in \widetilde{\Lambda} : f(z) = \max f|_{\widetilde{\Lambda}}\}$.

If $HD(\tilde{\Lambda}) < 1$ set $\mathcal{H}_{\widetilde{\Lambda}} = H_{\widetilde{\Lambda}} \cap \mathcal{R}_{\varphi,\tilde{\Lambda}}$ (which is residual by proposition 2.4) and $\mathcal{H}_{\widetilde{\Lambda}} = H_{\widetilde{\Lambda}}$ in other case. Define then

$$\mathcal{P}_{\varphi,\Lambda} := \bigcap_{\substack{\widetilde{\Lambda} \subset \Lambda \\ subhorseshoe}} \mathcal{H}_{\widetilde{\Lambda}}.$$

In the mentioned paper is also proved that for any such subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and $f \in \mathcal{P}_{\varphi,\Lambda}$ if x_M is the unique element where $f|_{\widetilde{\Lambda}}$ take its maximum value, then for any $\epsilon > 0$ there exists some subhorseshoe $\widetilde{\Lambda}^\epsilon \subset \widetilde{\Lambda} \setminus \{x_M\}$ with

$$HD(\widetilde{\Lambda}^\epsilon) \geq HD(\widetilde{\Lambda})(1 - \epsilon)$$

and such that for some point $d \in \widetilde{\Lambda}^\epsilon$ there exists a local C^1-diffeomorphism $\tilde{A}$ defined in a neighborhood U_d of d such that

$$f(\varphi^{j_0}(\tilde{A}(\tilde{\Lambda}_{j_0}))) \subset \ell_{\varphi,f}(\widetilde{\Lambda}),$$

where j_0 is an integer and $\tilde{\Lambda}_{j_0} \subset \widetilde{\Lambda}^\epsilon$ has nonempty interior in $\widetilde{\Lambda}^\epsilon$ and then is such that $HD(\tilde{\Lambda}_{j_0}) = HD(\widetilde{\Lambda}^\epsilon)$. Moreover, it is proved also that $\dfrac{\partial \tilde{A}}{\partial e_x^{s,u}} \parallel e^{s,u}_{\tilde{A}(x)}$, for $x \in U_d \cap \widetilde{\Lambda}^\epsilon$ and then, $\nabla(f \circ \varphi^{j_0} \circ \tilde{A})(x) \nparallel e_x^{s,u}$ for every $x \in \tilde{\Lambda}_{j_0}$.

Extending properly $f \circ \varphi^{j_0} \circ \tilde{A}$, and letting ϵ tends to 0; it follows from this and proposition 3.1 that

$$\min\{1, HD(\tilde{\Lambda})\} \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})).$$

An elementary compactness argument shows that $\{\ell_{\varphi,f}(x) : x \in X\} \subset \{m_{\varphi,f}(x) : x \in X\} \subset f(X)$ whenever $X \subset M$ is a compact φ-invariant subset. It follows that

$$\min\{1, HD(\tilde{\Lambda})\} \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})) \leq HD(m_{\varphi,f}(\widetilde{\Lambda})) \leq HD(f(\widetilde{\Lambda})) \leq \min\{1, HD(\tilde{\Lambda})\},$$

as we wanted to see. □

Corollary 3.3. *Given $\varphi \in \mathcal{U}^*$ and $f \in \mathcal{P}_{\varphi,\Lambda}$, one has*

$$\max L_{\varphi,f} = HD(\mathcal{L}_{\varphi,f}) = \min\{1, HD(\Lambda)\}.$$

3.2. **A technical proposition.** Throughout this subsection we will suppose $HD(\Lambda) < 1$. Fix $f \in \mathcal{R}_{\varphi,\Lambda}$ and take $X \subset \Lambda$, compact and φ-invariant. Observe that the same proof of proposition 2.9 of [1] let us conclude that for every $0 < \eta < 1$ there exists $\delta > 0$ and a complete subshift $\Sigma(\mathcal{B}_u) \subset \Sigma_{\mathcal{B}} \subset \mathcal{A}^{\mathbb{Z}}$ associated to a finite set $\mathcal{B}_u$, of finite sequences such that

$$\Sigma(\mathcal{B}_u) \subset \Sigma_{\max f|_X - \delta} \quad \text{and} \quad D_u(\Lambda(\Sigma(\mathcal{B}_u))) > (1-\eta) D_u(X),$$

where $\Lambda(\Sigma(\mathcal{B}_u))$ denotes the subhorseshoe of Λ associated to $\mathcal{B}_u$. We point here that $\Lambda(\Sigma(\mathcal{B}_u))$ doesn't need to be contained in X.

For fixing ideas and for future use we will remember some facts about that proof: The construction of $\mathcal{B}_u$ depends on three combinatorial lemmas (2.13-2.15). In our case, to prove that lemmas, we take r_0 large so that

$$\left| \frac{\log N_u(X,r)}{r} - D_u(X) \right| < \frac{\tau}{2} D_u(X) \tag{3.1}$$

for all $r \in \mathbb{N}$, $r \geq r_0$ where $\tau = \eta/100$.

The alphabet $\mathcal{B}_u$ is obtained from the set

$$\widetilde{\mathcal{B}}_u = \{\beta = \beta_1 \dots \beta_k : \beta_j \in \mathcal{C}_u(X, r_0),\ \forall 1 \leq j \leq k \ \text{ and } \ \pi^u(X) \cap I^u(\beta) \neq \emptyset\}$$

where $k = 8N_u(X, r_0)^2 \lceil 2/\tau \rceil$.

Defining the notion of *good position* for positions $j \in \{1, ..., k\}$ (see definition 3.16 below for a generalization) is showed that most positions of most words of $\widetilde{\mathcal{B}}_u$ are good and for that set of words, say $\mathcal{E}$, we can find natural numbers $1 \leq s_1 \leq \cdots \leq s_{3N_0^2} \leq k$, ($N_0 = N_u(X, r_0)$) with

$$s_{m+1} - s_m \geq 2\lceil 2/\tau \rceil \quad \text{for} \quad 1 \leq m < 3N_0^2$$

and words $\widehat{\beta}_{s_1}, \widehat{\beta}_{s_1+1}, \dots, \widehat{\beta}_{s_{3N_0^2}}, \widehat{\beta}_{s_{3N_0^2}+1} \in \mathcal{C}_u(X, r_0)$ such that the set $\mathcal{P}$ of words in $\mathcal{E}$ with $s_m, s_m + 1$ good positions and $\beta_{s_m} = \widehat{\beta}_{s_m}, \beta_{s_m+1} = \widehat{\beta}_{s_m+1}$ for $1 \leq m < 3N_0^2$ has cardinality $|\mathcal{P}| > N_0^{(1-2\tau)k}$.

Then is proved that there are $1 \leq p_0 < q_0 \leq 3N_0^2$ such that $\widehat{\beta}_{s_{p_0}} = \widehat{\beta}_{s_{q_0}}$, $\widehat{\beta}_{s_{p_0+1}} = \widehat{\beta}_{s_{q_0+1}}$ and the cardinality of $\mathcal{B}_u = \pi_{p_0,q_0}(\mathcal{P})$ is

$$|\mathcal{B}_u| > N_0^{(1-10\tau)(s_{q_0} - s_{p_0})},$$

where

$$\pi_{p_0,q_0} : \mathcal{P} \to \mathcal{C}_u(X, r_0)^{s_{q_0} - s_{p_0}} \quad \text{is the projection} \quad \pi_{p_0,q_0}(\beta_1 \dots \beta_k) = (\beta_{s_{p_0+1}}, \dots, \beta_{s_{q_0}})$$

obtained by cutting a word $\beta_1 \dots \beta_k \in \mathcal{P}$ at the positions s_{p_0} and s_{q_0} and discarding the words β_j with $j \leq s_{p_0}$ and $j > s_{q_0}$.

Using the conclusion on the cardinality of $\mathcal{B}_u$ is showed that $D_u(\Lambda(\Sigma(\mathcal{B}_u))) > (1 - \eta)D_u(X)$ and using that s_{p_0}, $s_{p_0} + 1$, s_{q_0} and $s_{q_0} + 1$ are good positions for words in $\mathcal{P}$ that $\Sigma(\mathcal{B}_u) \subset \Sigma_{\max f|_X - \delta}$.

Even more, the proof of that proposition gives us the next formula: $\delta = \min\{\delta^1, \delta^2, \delta^3, \delta^4\}$ where if $\gamma_1 = \widehat{\beta}_{s_{p_0+1}} = a_1 \dots a_{\widehat{m}_1}$, $\beta_{s_{p_0}+2} \dots \beta_{s_{q_0}-1} = b_1 \dots b_{\widehat{m}}$ and $\gamma_2 = \widehat{\beta}_{s_{q_0}} = d_1 \dots d_{\widehat{m}_2}$ then

- $\delta^1 = c_3 \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \min\limits_{1 \leq j \leq \widehat{m}-1} |I^u(b_j \dots b_{\widehat{m}} \gamma_2)|$
- $\delta^2 = c_3 \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \min\limits_{1 \leq j \leq \widehat{m}-1} |I^s((\gamma_1 b_1 \dots b_{j-1})^T)|$
- $\delta^3 = c_3 \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \min\limits_{1 \leq \ell \leq \widehat{m}_1-1} |I^s((\gamma_2 a_1 \dots a_\ell)^T)|$
- $\delta^4 = c_3 \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \min\limits_{1 \leq \ell \leq \widehat{m}_1-1} |I^u(d_{\ell - \widehat{m_1} - \widehat{m}+1} \dots d_{\widehat{m}_2} \gamma_1)|$

and c_3 is a positive constant that only depends on the function f and φ.

We will give a more precise estimate of the value of $\delta = \delta(\eta, X)$ and show some uniformity property of it; we also want to describe better the horseshoe $\Lambda^u(X) = \Lambda(\Sigma(\mathcal{B}_u))$ obtained before. To do this, let us consider for $n \in \mathbb{N}$ the set $C(X, n)$ of admissible finite words θ of the form $\theta = (a_{-n}, \dots, a_0, \dots, a_n)$, such that the rectangle $R(a_{-n}, \dots, a_0, \dots, a_n; 0) = \bigcap\limits_{j=-n}^{n} \varphi^{-j}(R_{a_j})$ has nonempty intersection with X. Also, given $\epsilon > 0$ define $n(\epsilon) = \min\{n \in \mathbb{N} : \forall \theta \in C(\Lambda, n),\ \operatorname{diam}(R(\theta; 0)) \leq \epsilon/2\}$ where $\operatorname{diam}(R(\theta; 0))$ denotes the diameter of the set $R(\theta; 0)$.

Proposition 3.4. *Given $\epsilon > 0$ and $c_0 > 0$ there exists a constant $\delta = \delta(\epsilon, c_0) > 0$ such that if X is a compact φ-invariant subset of Λ that satisfies $D_u(X) \geq c_0$, then* *we can find some subhorseshoe* $\Lambda^u(X)$ *of* Λ *such that*

$$D_u(\Lambda^u(X)) > (1-\epsilon)D_u(X) \text{ and } \Lambda^u(X) \subset \Lambda_{\max f|_X - \delta}.$$

Furthermore, for every $x \in \Lambda^u(X)$ *the set*

$$X_\epsilon(x) = \{n \in \mathbb{Z} : \exists \theta \in C(X, n(\epsilon)) \text{ such that } \varphi^n(x) \in R(\theta; 0)\}$$

is neither bounded below nor bounded above.

Proof. Take $X \subset \Lambda$, compact and φ-invariant as in the statement of the proposition. It is clear from the construction given of $\mathcal{B}_u$ and from the fact that

$$s_{q_0} - s_{p_0} \geq 2\lceil 2/\tau \rceil (q_0 - p_0) \geq 2\lceil 2/\tau \rceil = 2\lceil 200/\eta \rceil$$

that for $\eta = \eta(\epsilon) < \epsilon$ small enough and $x \in \Lambda^u(X) = \Lambda(\Sigma(\mathcal{B}_u))$, the set $X_\epsilon(x)$ is neither bounded below nor bounded above. Also, because $\Lambda^u(X) \subset \Lambda_{\max f|_X - \delta}$, the proposition will be proved if we can choose δ depending only on η and c_0.

Without lose of generality, consider $0 < \eta < \min\{c_0, 5000/(c_2 \log|\mathcal{A}|), 3\lambda_1, \kappa\}$, where $\kappa > 0$ is such that the maps $x \to e^{e^x} - 8e^{2\alpha_1 x + 2\alpha_2} \cdot x^2$ and $x \to e^{e^x} - 8\log x \cdot e^{2\alpha_1 x + 2\alpha_2} \cdot x(\alpha_1 x + \alpha_2)$ are positive if $x > 1/\kappa^2$.

The crucial observation here is that in the proof sketched above (without the dimension estimate) we can replace the conditions on r_0 (and k) given by the equation 3.1 by the assumption that $r_0 > \lceil \frac{4(c_1+1)\log|\mathcal{A}|^{c_2}}{c_0\tau^2} \rceil$ and $k = 8N_u(X, r_0)^2 \lceil 2/\tau \rceil$ satisfy the inequality

$$\frac{\log N_u(X, r_0)}{r_0} < (1 + \frac{\tau}{2}) \frac{\log N_u(X, k(r_0 - c_1))}{k(r_0 - c_1)},$$

where c_1 comes from the bounded distortion property as in equation 2.1, because in that case, multiplying this inequality by $(1-\tau)r_0 k$ we have

$$\begin{aligned}
\log N_u(X, r_0)^{(1-\tau)k} &< (1-\tau)(1+\frac{\tau}{2})\frac{r_0}{r_0 - c_1} \log N_u(X, k(r_0 - c_1)) \\
&\leq (1-\frac{\tau}{2})(1+\frac{c_1}{r_0 - c_1}) \log N_u(X, k(r_0 - c_1)) \\
&< (1-\frac{\tau}{2})(1+\frac{\tau^2}{1-\tau^2}) \log N_u(X, k(r_0 - c_1)) \\
&< (1-\frac{\tau}{2})(1+\frac{\tau}{2}) \log N_u(X, k(r_0 - c_1)) \\
&= \log N_u(X, k(r_0 - c_1))^{1-\frac{\tau^2}{4}}
\end{aligned}$$

also, given any $r \geq r_0$ we have by definition of $D_u(X)$

$$(1-\frac{\tau}{2})D_u(X) \leq D_u(X) - \frac{\tau}{2}c_0 \leq D_u(X) - \frac{\log|\mathcal{A}|^{c_2}}{r} \leq \frac{\log N_u(X, r)}{r} \tag{3.2}$$

which implies that

$$\log 2 < \log|\mathcal{A}|^{c_2} < \frac{\tau^2}{4} r_0 c_0 \leq \frac{\tau^2}{4}(1-\frac{\tau}{2})k(r_0 - c_1)D_u(X) \leq \frac{\tau^2}{4} \log N_u(X, k(r_0 - c_1))$$

and then $2N_u(X,r_0)^{(1-\tau)k} < N_u(X,k(r_0-c_1))$ which is precisely the necessary condition to obtain the equation 2.4 in the proof of lemma 2.13 of [1] and the claims in other parts of the proof of the lemmas that use the assumptions that r_0 and k are large are satisfied provided $r_0 > \lceil \frac{4(c_1+1)\log|\mathcal{A}|^{c_2}}{c_0\tau^2} \rceil$.

On the other hand, given any regular Cantor set (K,ψ) with Markov partition $\mathcal{P} = \{I_1,\dots,I_k\}$ if we define inductively $\mathcal{R}_1 = \mathcal{P}$ and for $n \geq 2$, $\mathcal{R}_n$ as the set of connected components of $\psi^{-1}(J)$, $J \in \mathcal{R}_{n-1}$. And also, for each $R \in \mathcal{R}_n$ we denote by

$$\lambda_{n,R} = \inf|(\psi^n)'|_R| \quad \text{and} \quad \Lambda_{n,R} = \sup|(\psi^n)'|_R|,$$

the bounded distortion property shows the existence of some $a = a(K) \geq 1$, such that $\Lambda_{n,R} \leq a.\lambda_{n,R}$, for all $n \geq 1$. Even more, it is well known that for any such K, $D(K) = HD(K)$ where $D(K)$ denotes the limit capacity of K (cf. [16, chap 4]). Indeed, it follows from the proof of this result that for the sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ given by

$$\sum_{R\in\mathcal{R}_n}\left(\frac{1}{\Lambda_{n,R}}\right)^{\alpha_n} = 1 = \sum_{R\in\mathcal{R}_n}\left(\frac{1}{\lambda_{n,R}}\right)^{\beta_n}, \tag{3.3}$$

when ψ is a full Markov map i.e., $\psi(K\cap I_j) = K$ for $1 \leq j \leq k$, one has

$$\alpha_n \leq HD(K) = D(K) \leq \beta_n \tag{3.4}$$

and if $n \geq \log a/\log\lambda$, where $\lambda = \lambda(K) = \inf|\psi'| > 1$

$$\beta_n - \alpha_n \leq \frac{\log a \cdot HD(K)}{n\log\lambda - \log a}. \tag{3.5}$$

Now, if $z(\eta,\Lambda) \in \mathbb{N}$ is such that given $r_0 \geq z(\eta,\Lambda)$, for any complete subshift associated to a finite alphabet $\mathcal{B}_u = \mathcal{B}_u(r_0)$ of finite words as before, the Cantor set $K^u(\Sigma(\mathcal{B}_u))$ consisting of points of K^u whose trajectory under ψ_u follows an itinerary obtained from the concatenation of words in the alphabet $\mathcal{B}_u$[4], satisfies that $\lambda = \lambda(K^u(\Sigma(\mathcal{B}_u)))$ is big (we can take $a = a(K^u(\Sigma(\mathcal{B}_u))) = a(K^u(\Lambda))$), then by 3.4 and 3.5

$$\beta_1 - \alpha_1 \leq \frac{\tau}{2}HD(K^u(\Sigma(\mathcal{B}_u))) \leq \frac{\tau}{2}\beta_1.$$

Using this, 3.3 and 3.4 we obtain

$$HD(K^u(\Sigma(\mathcal{B}_u))) \geq \alpha_1 \geq \left(1-\frac{\tau}{2}\right)\beta_1 \geq \left(1-\frac{\tau}{2}\right)\frac{|\mathcal{B}_u|}{-\log(\min\limits_{\alpha\in\mathcal{B}_u}|I^u(\alpha)|)}$$

which is the equation used in [1] (together with 3.2) to obtain the dimension estimate.

[4] which is $C^{1+\alpha}$-dynamically defined associated to certain iterates of ψ_u on the intervals $I^u(\beta)$. with $\beta \in \mathcal{B}_u$.

Following the observations described above, we define the sequence $\{p_n\}$ as follows: $p_0 = \max\{\lceil \frac{4(c_1+1)\log|\mathcal{A}|^{c_2}}{c_0\tau^2}\rceil, z(\eta,\Lambda)\}$ and for $n \geq 0$ put

$$p_{n+1} = 8N_u(X,p_n)^2\lceil 2/\tau\rceil(p_n - c_1).$$

We claim that, for some integer $0 \leq s_0 < (1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$ one has

$$\frac{\log N_u(X,p_{s_0})}{p_{s_0}} < (1+\frac{\tau}{2})\frac{\log N_u(X,p_{s_0+1})}{p_{s_0+1}} = (1+\frac{\tau}{2})\frac{\log N_u(X,k(p_{s_0}-c_1))}{k(p_{s_0}-c_1)},$$

with $k = 8N_u(X,p_{s_0})^2\lceil 2/\tau\rceil$.

Indeed, if it is not the case, then for $0 \leq n < (1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$, we have

$$\frac{\log N_u(X,p_{n+1})}{p_{n+1}} < (1+\frac{\tau}{2})^{-1}\frac{N_u(X,p_n)}{p_n}$$

and then, for $M = \lceil(1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}\rceil$ we would have

$$\frac{\log N_u(X,p_M)}{p_M} \leq (1+\frac{\tau}{2})^{-M}\cdot\frac{\log N_u(X,p_0)}{p_0} < \frac{\eta}{4(\alpha_1+\alpha_2+1)}\frac{\log N_u(X,p_0)}{p_0}$$

because

$$(1+\frac{\tau}{2})^{-M} \leq ((1+\frac{\tau}{2})^{-(1+\frac{2}{\tau})})^{\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}} < e^{-\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}} = \frac{\eta}{4(\alpha_1+\alpha_2+1)}.$$

And so, by 2.3

$$\frac{\log N_u(X,p_M)}{p_M} \leq \frac{\eta}{4(\alpha_1+\alpha_2)}\frac{\log N_u(X,p_0)}{p_0} \leq \frac{\eta}{4(\alpha_1+\alpha_2)}\frac{\alpha_1.p_0+\alpha_2}{p_0} < \frac{\eta}{2}.$$

But this is a contradiction because by 3.2

$$\frac{\eta}{2} < (1-\frac{\tau}{2})c_0 \leq (1-\frac{\tau}{2})D_u(X) \leq \frac{\log N_u(X,p_M)}{p_M}.$$

Therefore, by taking $r_0 = p_{s_0}$ and $k = 8N_u(X,r_0)^2\lceil 2/\tau\rceil$, the argument for the construction of $\mathcal{B}_u$ works and then, because of the formula for δ, we have

$$\delta \geq c_3e^{-c_1}\lambda_1^{\widehat{m}_1+\widehat{m}_2+\widehat{m}} \geq c_3e^{-c_1}\lambda_1^{k\cdot\max\{|\alpha|:\alpha\in\mathcal{C}_u(X,r_0)\}} \geq c_3e^{-c_1}\lambda_1^{k\cdot(\alpha_1r_0+\alpha_2)}. \tag{3.6}$$

We will now give an explicit positive lower bound for δ in terms of η. In order to do that, we define recursively, for each integer $n \geq 0$ and $x \in \mathbb{R}$, the function $\mathcal{T}(n,x)$ by $\mathcal{T}(x,0) = x$, $\mathcal{T}(x,n+1) = e^{\mathcal{T}(x,n)}$. We have for $n \geq 0$

$$p_{n+1} = 8N_u(X,p_n)^2\lceil 2/\tau\rceil(p_n-c_1) < 8e^{2\alpha_1p_n+2\alpha_2}\cdot p_n^2 < e^{e^{p_n}},$$

since $p_n \geq p_0 > \lceil 2/\tau^2\rceil$. Therefore $r_0 = p_{s_0} < \mathcal{T}(p_0,2s_0)$ and

$$\begin{aligned}\log\lambda_1^{-1}\cdot k(\alpha_1r_0+\alpha_2) &= 8\log\lambda_1^{-1}\cdot N_u(X,r_0)^2\lceil 2/\tau\rceil(\alpha_1r_0+\alpha_2)\\ &< 8\log r_0\cdot e^{2\alpha_1r_0+2\alpha_2}\cdot r_0(\alpha_1r_0+\alpha_2) < e^{e^{r_0}}\end{aligned}$$

so, by 3.6

$$\delta \geq c_3 e^{-c_1} e^{\log \lambda_1 \cdot k(\alpha_1 r_0 + \alpha_2)} > c_3 e^{-c_1} e^{-e^{e^{r_0}}} > \frac{c_3 e^{-c_1}}{\mathcal{T}(p_0, 2s_0+3)}. \tag{3.7}$$

As $p_0 = \max\{\lceil \frac{40000(c_1+1)\log|\mathcal{A}|^{c_2}}{c_0\eta^2}\rceil, z(\eta,\Lambda)\}$ and $s_0 < (1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta} = (1+\frac{200}{\eta})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$, we have by 3.7

$$\delta > \frac{c_3 e^{-c_1}}{\mathcal{T}(p_0, 2s_0+3)} = \frac{c_3 e^{-c_1}}{\mathcal{T}(\max\{\lceil \frac{40000(c_1+1)\log|\mathcal{A}|^{c_2}}{c_0\eta^2}\rceil, z(\eta,\Lambda)\}, \lceil \frac{201}{\eta}\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}\rceil)},$$

that finishes the proof of the proposition. □

Now, if we suppose that $D_s(X) \geq c_0$, given $\epsilon > 0$ we can construct, as before, some complete subshift $\Sigma(\mathcal{B}_s)$ such that $\Lambda(\Sigma(\mathcal{B}_s))$ has similar properties as $\Lambda^u(X) = \Lambda(\Sigma(\mathcal{B}_u))$. Then, we immediately have

Corollary 3.5. *Given $\epsilon > 0$ and $c_0 > 0$ there exists a constant $\delta = \delta(\epsilon, c_0) > 0$ such that if X is a compact φ-invariant subset of Λ such that the limit capacities $D_u(X)$ and $D_s(X)$ satisfy both $D_u(X), D_s(X) \geq c_0$. Then there are subhorseshoes $\Lambda^s(X)$ and $\Lambda^u(X)$ of Λ such that*

$$D_u(\Lambda^u(X)) > (1-\epsilon)D_u(X), \quad D_s(\Lambda^s(X)) > (1-\epsilon)D_s(X)$$

and

$$\Lambda^u(X) \cup \Lambda^s(X) \subset \Lambda_{\max f|_X - \delta}.$$

Furthermore, for every $x \in \Lambda^u(X) \cup \Lambda^s(X)$ the set

$$X_\epsilon(x) = \{n \in \mathbb{Z} : \exists \theta \in C(X, n(\epsilon)) \text{ such that } \varphi^n(x) \in R(\theta; 0)\}$$

is neither bounded below nor bounded above.

3.3. **First accumulation point of the Lagrange spectrum.** In this subsection, we show the existence of the first accumulation point of the Lagrange spectrum and show that it is exactly at that point where the map $L_{\varphi,f}$ begins to be positive. In what follows, we will use the following result from [9]:

Lemma 3.6. *Given $\varphi \in \mathcal{U}$, any subhorseshoe $\tilde{\Lambda} \subset \Lambda$, $f \in C^1(S, \mathbb{R})$ and $t \in \mathbb{R}$, one has*

$$\ell_{\varphi,f}(\tilde{\Lambda}) \cap (-\infty, t) = \bigcup_{s<t} \ell_{\varphi,f}(\tilde{\Lambda}_s).$$

In particular

$$L_{\varphi,f}(t) = \sup_{s<t} HD(\ell_{\varphi,f}(\Lambda_s)) = \lim_{s\to\ t^-} HD(\ell_{\varphi,f}(\Lambda_s)).$$

From this we get

$$L_{\varphi,f}(t) = \sup_{s<t} HD(\ell_{\varphi,f}(\Lambda_s)) \leq HD(\ell_{\varphi,f}(\Lambda_t)) \leq HD(f(\Lambda_t)) \leq HD(\Lambda_t). \tag{3.8}$$

Proposition 3.7. *Take $\varphi \in \mathcal{U}^*$ and $f \in \mathcal{P}_{\varphi,f}$. Then*

$$\mathcal{L}'_{\varphi,f} = \{x : x \text{ is an accumulation point of } \mathcal{L}_{\varphi,f}\} \neq \emptyset$$

and $c_{\varphi,f} = \min L'_{\varphi,f}$.

Proof. First, by proposition 3.2

$$HD(\mathcal{L}_{\varphi,f}) = HD(\ell_{\varphi,f}(\Lambda)) = \min\{1, HD(\Lambda)\} > 0,$$

then, $\mathcal{L}_{\varphi,f}$ cannot be finite and as $\mathcal{L}_{\varphi,f} \subset f(\Lambda)$, it must be true that $\mathcal{L}'_{\varphi,f} \neq \emptyset$.

Let $c^*_{\varphi,f} = \min L'_{\varphi,f}$. Given $\epsilon > 0$, it is clearly that $L_{\varphi,f}(c^*_{\varphi,f} - \epsilon) = 0$ because $\mathcal{L}_{\varphi,f} \cap (-\infty, c^*_{\varphi,f} - \epsilon)$ is finite. On the other hand, take an injective sequence $(y_n)_{n\in\mathbb{N}} = (\ell_{\varphi,f}(x_n))_{n\in\mathbb{N}} \subset \mathcal{L}_{\varphi,f}$ such that $\lim\limits_{n\to\infty} y_n = c^*_{\varphi,f}$ and consider $N \in \mathbb{N}$ big enough such that for two elements $x, y \in \Lambda$ if their kneading sequences coincide in the central block (centered at the zero position) of size $2N+1$ then $|f(x) - f(y)| < \epsilon/6$.

Take first $n_0 \in \mathbb{N}$ large so that $|\ell_{\varphi,f}(x_n) - c^*_{\varphi,f}| < \epsilon/6$ for $n \geq n_0$ and there are infinitely many $j \in \mathbb{N}$ such that $|f(\varphi^j(x_n)) - c^*_{\varphi,f}| < \epsilon/6$. Given such a pair (j, n), consider the finite sequence with $2N+1$ terms $S(j,n) = (b^{(n)}_{j-N}, b^{(n)}_{j-N+1}, \cdots, b^{(n)}_{j}, \cdots, b^{(n)}_{j+N})$ where $\Pi^{-1}((b^{(n)}_j)_{j\in\mathbb{Z}}) = x_n$. There is a sequence S such that for infinitely many values of n, S appears infinitely many times as $S(j,n)$; i.e., there are $j_1(n) < j_2(n) < \cdots$ with $\lim\limits_{i\to\infty}(j_{i+1}(n) - j_i(n)) = \infty$ and $S(j_i(n), n) = S$ for all $i \geq 1$ and for all n in some infinite set $A \subset \mathbb{N}$.

Consider the sequences $\beta(i,n)$ for $i \geq 1$, $n \in A$ given by

$$\beta(i,n) = (b^{(n)}_{j_i(n)+N+1}, b^{(n)}_{j_i(n)+N+2}, \cdots, b^{(n)}_{j_{i+1}(n)+N}).$$

Taking $n_1, n_2 \in A$ distinct and $r = r(n_1, n_2)$ large enough such that for $j \geq r$, $f(\varphi^j(x_{n_1})) < \ell_{\varphi,f}(x_{n_1}) + \epsilon/6$ and $f(\varphi^j(x_{n_2})) < \ell_{\varphi,f}(x_{n_2}) + \epsilon/6$. There are $i_1 \geq r$ and $i_2 \geq r$ for which there is no a sequence γ such that $\beta(i_1, n_1)$ and $\beta(i_2, n_2)$ are concatenations of copies of γ, otherwise $y_{n_1} = y_{n_2}$ because for $n \in A$

$$\Pi(x_n) = (\cdots, b^{(n)}_1, \cdots b^{(n)}_{j_1(n)+N}, \beta(1,n), \beta(2,n), \cdots, \beta(m,n), \cdots).$$

This implies that, by taking

$$C = \{\beta(i_1,n_1)\beta(i_2,n_2), \beta(i_2,n_2)\beta(i_1,n_1)\},$$

we have $\Sigma(C)$ is a complete subshift and for $x \in \Lambda(\Sigma(C)) = \Lambda_C$ (the subhorseshoe associated to $\Sigma(C)$) we have $m_{\varphi,f}(x) < c^*_{\varphi,f} + \epsilon/2$. Indeed, for every $k \in \mathbb{Z}$ the kneading sequence of $\varphi^k(x)$ coincides in the central block of size $2N+1$ with the kneading sequence of $\varphi^l(x_\theta)$ where θ is either n_1 or n_2 and $l \geq r$. So

$$f(\varphi^k(x)) < f(\varphi^l(x_\theta)) + \frac{\epsilon}{6} < \ell_{\varphi,f}(x_\theta) + \frac{\epsilon}{3} < c^*_{\varphi,f} + \frac{\epsilon}{2}.$$

Therefore, using one more time proposition 3.2 and lemma 3.6 we conclude

$$0 < \min\{1, HD(\Lambda_C)\} = HD(\ell_{\varphi,f}(\Lambda_C)) \leq HD(\ell_{\varphi,f}(\Lambda_{c^*_{\varphi,f}+\epsilon/2})) \leq L_{\varphi,f}(c^*_{\varphi,f} + \epsilon).$$

Then, by definition $c^*_{\varphi,f} = c_{\varphi,f}$, which ends the proof of the proposition. $\square$

Corollary 3.8. *If $HD(\Lambda) < 1$ one has*

$$c_{\varphi,f} = \max\{t \in \mathbb{R} : HD(\Lambda_t) = 0\}.$$

Proof. It follows from the previous proposition and 3.8 that $0 < L_{\varphi,f}(c_{\varphi,f} + \epsilon) \leq HD(\Lambda_{c_{\varphi,f}+\epsilon})$. Now, if $HD(\Lambda_{c_{\varphi,f}}) > 0$ then by 2.7, $D_u(\Lambda_{c_{\varphi,f}}) > 0$ (also $D_s(\Lambda_{c_{\varphi,f}}) > 0$), and by proposition 3.4 we can find some horseshoe $\tilde{\Lambda} \subset \Lambda_{c_{\varphi,f}-\delta}$ for some $\delta > 0$ and arguing as before, we get the contradiction $L_{\varphi,f}(c_{\varphi,f} - \delta/2) > 0$. $\square$

Remark 3.9. This corollary remains true if $HD(\Lambda) \geq 1$ because proposition 1 of [9] let us also show the existence of $\tilde{\Lambda}$ and $\delta > 0$ as before.

Corollary 3.10. *If $HD(\Lambda) < 1$ then $L_{\varphi,f}$ is continuous in $c_{\varphi,f}$.*

Proof. Suppose $\lim\limits_{t \to c^+_{\varphi,f}} HD(\Lambda_t) = h > 0$, then by 2.7, for $t > c_{\varphi,f}$ one has $D_u(\Lambda_t) \geq h/(1+\tilde{C})$. On the other hand, proposition 3.4 let us find some $\delta = \delta(\frac{1}{2}, \frac{h}{1+\tilde{C}}) > 0$ such that for any $t > c_{\varphi,f}$ we can find some horseshoe $\Lambda^u(\Lambda_t) \subset \Lambda_{t-\delta}$ (the other conclusions of the proposition are not necessary here). By applying this to $t = c_{\varphi,f} + \delta/2$, we get the contradiction $0 < HD(\Lambda^u(\Lambda_{c_{\varphi,f}+\delta/2})) \leq HD(\Lambda_{c_{\varphi,f}-\delta/2})$. Then

$$0 = L_{\varphi,f}(c_{\varphi,f}) \leq \lim_{t \to c^+_{\varphi,f}} L_{\varphi,f}(t) \leq \lim_{t \to c^+_{\varphi,f}} HD(\Lambda_t) = 0,$$

as we wanted to see. $\square$

Remark 3.11. This corollary also holds when $HD(\Lambda) \geq 1$ because as we will see later, before $\tilde{c}_{\varphi,f}$, it is true some expression of the type $L_{\varphi,f} = \max\limits_i L_i$, where the functions L_i are defined like $L_{\varphi,f}$ but are associated to horseshoes with Hausdorff dimension less than 1.

3.4. **Geometric consequences of having a discontinuity.** In this subsection, we show how to associate to each discontinuity the pair of subhorseshoes described in the introduction of the section.

Take $\varphi \in \mathcal{U}^*$ with $HD(\Lambda) < 1$, $f \in \mathcal{P}_{\varphi,\Lambda}$ and suppose $t_0 \in \mathbb{R}$ is a discontinuity of the map $t \to L_{\varphi,f}(t) = HD(\mathcal{L}_{\varphi,f} \cap (-\infty, t))$. So, there exists an $a > 0$ such that

$$L_{\varphi,f}(q) + a < L_{\varphi,f}(s) \text{ for } q \leq t_0 < s. \tag{3.9}$$

By corollary 3.10 and 3.8 we have $0 < L_{\varphi,f}(t_0) \leq HD(\Lambda_{t_0})$, then $D_u(\Lambda_{t_0}) > 0$ and one more time, by proposition 3.4, we can find some horseshoe $\Lambda^0 \subset \Lambda_{t_0}$. For $0 < \epsilon < a/2$ and $c_0 = HD(\Lambda^0)/(\tilde{C} + 1) > 0$ take $\delta = \delta(\epsilon/2k, c_0) < \epsilon$ as in the corollary 3.5 where $k > 1$ is a Lipschitz's constant for f, and let us consider for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ the set $C(\Lambda_t, n)$. By compactness, one has

$$C(\Lambda_{t_0}, n) = \bigcap_{t > t_0} C(\Lambda_t, n).$$

In particular, for each n, there exists $t(n) > t_0$ such that for $t_0 < t \leq t(n)$

$$C(\Lambda_t, n) = C(\Lambda_{t_0}, n).$$

Take then, $n = n(\delta/2k)$ and consider the maximal invariant set

$$P = M(C(\Lambda_{t_0}, n)) = \bigcap_{m\in\mathbb{Z}} \varphi^{-m}(\bigcup_{\theta\in C(\Lambda_{t_0},n)} R(\theta;0)) = \bigcap_{m\in\mathbb{Z}} \varphi^{-m}(\bigcup_{\theta\in C(\Lambda_t,n)} R(\theta;0))$$

for $t_0 < t \leq t(n)$.

Observe that for $x \in P$ and $m \in \mathbb{Z}$ if $y \in \Lambda_{t_0}$ belongs to the same rectangle $R(\theta;0)$ as $\varphi^m(x)$ for some $\theta \in C(\Lambda_{t_0}, n)$ then

$$f(\varphi^m(x)) \leq f(\varphi^m(x)) - f(y) + t_0 \leq k \cdot d(\varphi^m(x), y) + t_0 \leq k \cdot \frac{\delta}{4k} + t_0 < \frac{\delta}{2} + t_0$$

and so $P \subset \Lambda_{t_0+\delta/2}$.

Remember that for any subhorseshoe $\tilde{\Lambda} \subset \Lambda$, being locally maximal, we have

$$\bigcup_{y\in\tilde{\Lambda}} W^s(y) = W^s(\tilde{\Lambda}) = \{y \in S : \lim_{n\to\infty} d(\varphi^n(y), \tilde{\Lambda}) = 0\}. \tag{3.10}$$

Now, by proposition 2.3, the set P admits a decomposition $P = \bigcup_{i\in\mathcal{I}} \tilde{\Lambda}_i$ where $\mathcal{I}$ is a finite index set and for any $i \in \mathcal{I}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set. In particular, given $i_1 \in \mathcal{I}$ we can find $i_2 \in \mathcal{I}$ such that $\tilde{\Lambda}_{i_2}$ is a subhorseshoe with $\omega(x) \subset \tilde{\Lambda}_{i_2}$ for every $x \in \tilde{\Lambda}_{i_1}$; and from this and 3.10, it follows that $\ell_{\varphi,f}(x) = \ell_{\varphi,f}(y)$ for some $y \in \tilde{\Lambda}_{i_2}$. We conclude then

$$\ell_{\varphi,f}(P) = \bigcup_{i\in\mathcal{I}} \ell_{\varphi,f}(\tilde{\Lambda}_i) = \bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i\ is\\ horseshoe}} \ell_{\varphi,f}(\tilde{\Lambda}_i) \cup \bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i\\ is\ orbit}} \ell_{\varphi,f}(\tilde{\Lambda}_i)$$

and by proposition 3.2

$$HD(\ell_{\varphi,f}(P)) = HD(\bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i\ is\\ horseshoe}} \ell_{\varphi,f}(\tilde{\Lambda}_i)) = \max_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i\ is\\ horseshoe}} HD(\ell_{\varphi,f}(\tilde{\Lambda}_i)) = \max_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i\ is\\ horseshoe}} HD(\tilde{\Lambda}_i).$$

Let $\tilde{\Lambda}_{i_0}$ with $HD(\ell_{\varphi,f}(P)) = HD(\tilde{\Lambda}_{i_0})$. As $\Lambda^0 \subset P$, by 2.6 and 2.7 one has

$$c_0 \leq HD(\tilde{\Lambda}_{i_0})/(\tilde{C}+1) \leq D_s(\tilde{\Lambda}_{i_0}) \text{ and also } c_0 \leq HD(\tilde{\Lambda}_{i_0})/(\tilde{C}+1) \leq D_u(\tilde{\Lambda}_{i_0})$$

then, corollary 3.5 applied to $\tilde{\Lambda}_{i_0}$ let us show the existence of two horseshoes $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ of Λ such that

$$D_u(\Lambda^u(t_0)) > D_u(\tilde{\Lambda}_{i_0}) - \epsilon/2k, \quad D_s(\Lambda^s(t_0)) > D_s(\tilde{\Lambda}_{i_0}) - \epsilon/2k,$$

$$\Lambda^u(t_0) \cup \Lambda^s(t_0) \subset \Lambda_{(t_0+\delta/2)-\delta} = \Lambda_{t_0-\delta/2},$$

and for every $x \in \Lambda^u(t_0) \cup \Lambda^s(t_0)$ the set $(\tilde{\Lambda}_{i_0})_{\epsilon/2k}(x)$ is neither bounded below nor bounded above.

Now, suppose there exists a subhorseshoe $\widetilde{\Lambda} \subset \Lambda_q$ for some $q < t_0$ with $\Lambda^u(t_0) \cup \Lambda^s(t_0) \subset \widetilde{\Lambda}$, then as $\Lambda_t \subset P$ for $t_0 < t \leq t(h)$, we have by 3.9 and lemma 3.6

$$\begin{aligned} L_{\varphi,f}(t_0) + a/2 &< L_{\varphi,f}(t_0) + a - \epsilon/k < HD(\ell_{\varphi,f}(P)) - \epsilon/k = HD(\tilde{\Lambda}_{i_0}) - \epsilon/k \\ &< D_u(\Lambda^u(t_0)) + D_s(\Lambda^s(t_0)) \leq HD(\widetilde{\Lambda}) = HD(\ell_{\varphi,f}(\widetilde{\Lambda})) \leq HD(\ell_{\varphi,f}(\Lambda_q)) \\ &\leq \sup_{s<t_0} HD(\ell_{\varphi,f}(\Lambda_s)) = L_{\varphi,f}(t_0) \end{aligned}$$

which is a contradiction. Then, by definition, $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ don't connect before t_0.

On the other hand, fix $x \in \Lambda^s(t_0)$, $y \in \Lambda^u(t_0)$ with kneading sequences $(x_n)_{n\in\mathbb{Z}}$, respectively $(y_n)_{n\in\mathbb{Z}}$. As the sets $(\tilde{\Lambda}_{i_0})_{\epsilon/2k}(x)$ and $(\tilde{\Lambda}_{i_0})_{\epsilon/2k}(y)$ are nonempty we can find two words θ and $\tilde{\theta}$ in $C(\tilde{\Lambda}_{i_0}, n(\epsilon/2k))$ that appear respectively in the sequences $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ as sub-words and also appear in the kneading sequence of two points $\tilde{x}_1, \tilde{y}_1 \in \tilde{\Lambda}_{i_0}$, i.e., $\tilde{x}_1 \in R(\theta; 0)$, and $\tilde{y}_1 \in R(\tilde{\theta}; 0)$, $(x_{N_1}, \ldots x_{N_1-|\theta|-1}) = \theta$ and $(y_{-N_2-|\tilde{\theta}|+1}, \ldots y_{-N_2}) = \tilde{\theta}$ for some $N_1, N_2 > 0$.

As $\tilde{\Lambda}_{i_0}$ is a horseshoe, we can find a point $z_1 \in \tilde{\Lambda}_{i_0}$ with kneading sequence of the form

$$\Pi(z_1) = (\ldots, z_{-2}, z_{-1}; \theta, z_{|\theta|}, \ldots, z_{|\theta|+r_1}, \tilde{\theta}, z_{|\theta|+r_1+|\tilde{\theta}|+1}, \ldots)$$

for some $r_1 > 0$. Then consider the point $z \in \Lambda$ with kneading sequence

$$\Pi(z) = (\ldots, x_{-2}, x_{-1}; x_0, \ldots, x_{N_1-1}, \theta, z_{|\theta|}, \ldots, z_{|\theta|+r_1}, \tilde{\theta}, y_{-N_2+1}, y_{-N_2+2}, y_{-N_2+3}, \ldots)$$

note that, by construction $z \in W^u(\Lambda^s(t_0)) \cap W^s(\Lambda^u(t_0)) \cap \tilde{P}$ where

$$\tilde{P} = M(C(\Lambda^u(t_0) \cup \Lambda^s(t_0) \cup \tilde{\Lambda}_{i_0}, n(\epsilon/2k))) = \bigcap_{m\in\mathbb{Z}} \varphi^{-m}\Big(\bigcup_{\theta \in C(\Lambda^u(t_0)\cup\Lambda^s(t_0)\cup\tilde{\Lambda}_{i_0}, n(\epsilon/2k))} R(\theta; 0)\Big).$$

Analogously we can find $\tilde{z} \in W^u(\Lambda^u(t_0)) \cap W^s(\Lambda^s(t_0)) \cap \tilde{P}$. Moreover, as $\Lambda^u(t_0) \cup \Lambda^s(t_0) \cup \tilde{\Lambda}_{i_0} \subset \Lambda_{t_0+\delta/2}$, reasoning as we did for P, we have $\tilde{P} \subset \Lambda_{k\cdot\epsilon/2k+t_0+\delta/2} = \Lambda_{\epsilon/2+t_0+\delta/2}$. That is,

$$\Lambda^s(t_0) \cup \Lambda^u(t_0) \cup \mathcal{O}(z) \cup \mathcal{O}(\tilde{z}) \subset \Lambda_{\epsilon/2+t_0+\delta/2}$$

and using proposition 2.6 we get that $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ connect before $t_0 + \epsilon$.

We summarize our conclusions in the following proposition

Proposition 3.12. *Take $\varphi \in \mathcal{U}^*$ with $HD(\Lambda) < 1$, $f \in \mathcal{P}_{\varphi,\Lambda}$ and some discontinuity t_0 of the map*

$$t \to L_{\varphi,f}(t) = HD(\mathcal{L}_{\varphi,f} \cap (-\infty, t)).$$

Then, given $\epsilon > 0$ there are two subhorseshoes $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ and some $0 < \eta < \epsilon$ such that

- $\Lambda^s(t_0) \cup \Lambda^u(t_0) \subset \Lambda_{t_0-\eta}$,
- $\Lambda^s(t_0)$ *and* $\Lambda^u(t_0)$ *don't connect before* t_0,

- $\Lambda^s(t_0)$ *and* $\Lambda^u(t_0)$ *connect before* $t_0+\epsilon$.

Remark 3.13. As in remark 3.11, this result also holds when $HD(\Lambda)\geq 1$ and $t_0<c_{\varphi,f}$. Note that, in our context, by corollary 3.3, $L_{\varphi,f}$ is discontinuous in $\tilde{c}_{\varphi,f}$ if and only if $L_{\varphi,f}(\tilde{c}_{\varphi,f})<1$.

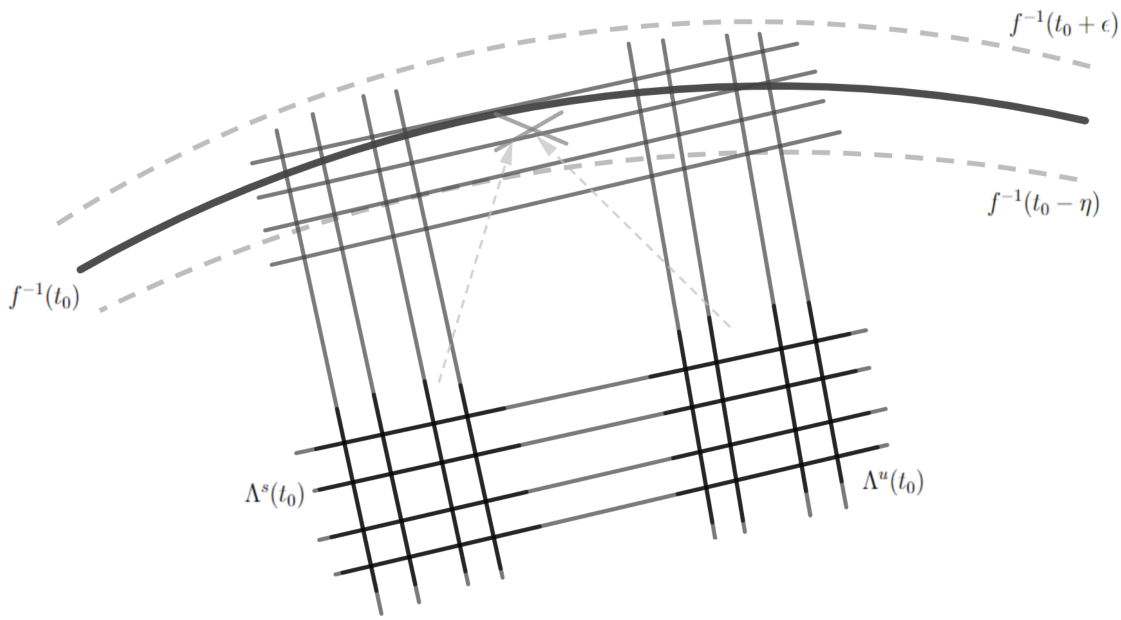

FIGURE 1. The subhorseshoes $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ in proposition 3.12.

3.5. **Sequences of subhorseshoes.** In this subsection, we suppose existence of an infinite sequence of discontinuities of the map $L_{\varphi,f}$ in some closed sub interval of $I_{\varphi,f}$ that doesn't contain the first accumulation point of the Lagrange spectrum and then construct arbitrary large finite sequences of subhorseshoes with some specific properties. Observe that here is the first time when we use the hypothesis of the diffeomorphism being close to a conservative one.

Remember that any subhorseshoe $\tilde{\Lambda}_0$ of Λ_0 has a continuation $\tilde{\Lambda}\subset\Lambda$ for any $\varphi\in\mathcal{U}$. In theorem A of [15], the authors showed that the maps $D_{\Lambda_0,u}:\mathcal{U}\to\mathbb{R}$ and $D_{\Lambda_0,s}:\mathcal{U}\to\mathbb{R}$ given by $D_{\Lambda_0,u}(\varphi)=D_u(\Lambda)$ and $D_{\Lambda_0,s}(\varphi)=D_u(\Lambda)$ are continuous and, in fact, the same proof also shows that the continuity of the maps $D_{\tilde{\Lambda}_0,u}(\varphi)=D_u(\tilde{\Lambda})$ and $D_{\tilde{\Lambda}_0,s}(\varphi)=D_u(\tilde{\Lambda})$ is uniform on the subhorseshoes. Moreover, as for φ_0 one can take $\tilde{C}=1$ in 2.4 (see remark 2.2 in [1]) then $D_u(\tilde{\Lambda}_0)=D_s(\tilde{\Lambda}_0)$ for any subhorseshoe $\tilde{\Lambda}_0$ of Λ_0 and, as a consequence, we can choose the neighborhood $\mathcal{U}$ of φ_0 small enough such that for some constants r_1,r_2 with $r_1/r_2>999/1000$ and for any subhorseshoe

$\tilde{\Lambda}$ of Λ one has

$$r_1 D_s(\tilde{\Lambda}) \leq D_u(\tilde{\Lambda}) \leq r_2 D_s(\tilde{\Lambda}). \tag{3.11}$$

Fix $\varphi \in \mathcal{U}^*$ with $HD(\Lambda) < 1$, $f \in \mathcal{P}_{\varphi,\Lambda}$, some closed sub interval $I \subset I_{\varphi,f}$ that doesn't contain $c_{\varphi,f}$ and suppose we have an infinite sequence of discontinuities of $L_{\varphi,f}$ with $s \in I$ for every s in the sequence. Then, as $L_{\varphi,f}(\min I) \leq L_{\varphi,f}(s) \leq HD(\Lambda_s)$, by 2.6 and 2.7

$$c \leq D_s(\Lambda_s) \text{ and } c \leq D_u(\Lambda_s), \tag{3.12}$$

where $c = L_{\varphi,f}(\min I)/(\tilde{C}+1)$.

Now, as the maps $t \mapsto D_u(\Lambda_t)$ and $t \mapsto D_s(\Lambda_t)$ are continuous (by proposition 2.4) and $D_u(\Lambda_t) = D_s(\Lambda_t) = 0$ for $t < \min(f)$ and $D_u(\Lambda_t) = D_u(\Lambda)$, $D_s(\Lambda_t) = D_s(\Lambda)$ for $t > \max(f)$. Then, they are uniformly continuous and so we can find some $\delta > 0$ such that

$$|t - \bar{t}| < \delta \text{ implies } |D_u(\Lambda_t) - D_u(\Lambda_{\bar{t}})| < 0.001c \text{ and } |D_s(\Lambda_t) - D_s(\Lambda_{\bar{t}})| < 0.001c.$$

Also, for the sequence of discontinuities we have some accumulation point and unless pass to a sub-sequence, change the index set and discard some terms, we can suppose that $\{t_n\}$ is of one of the next two types:

- The sequence is strictly increasing $\{t_n\}_{n\geq 1}$ with $\lim\limits_{n\to\infty} t_n := t_0$ and $t_0 - t_1 < \delta$,
- The sequence is strictly increasing $\{t_n\}_{n\leq 0}$ with $\lim\limits_{n\to-\infty} t_n := t^*$ and $t_0 - t^* < \delta$.

In particular, for each n

$$0.999 D_u(\Lambda_{t_0}) = D_u(\Lambda_{t_0}) - 0.001 D_u(\Lambda_{t_0}) \leq D_u(\Lambda_{t_0}) - 0.001c < D_u(\Lambda_{t_n}) \tag{3.13}$$

and

$$0.999 D_s(\Lambda_{t_0}) = D_s(\Lambda_{t_0}) - 0.001 D_s(\Lambda_{t_0}) \leq D_s(\Lambda_{t_0}) - 0.001c < D_s(\Lambda_{t_n}). \tag{3.14}$$

Now, in order to get the sequences of subhorseshoes, we will associate to every n a pair of subhorseshoes of Λ. In fact, the two subhorseshoes $\Lambda^s(t_n)$ and $\Lambda^u(t_n)$ are given by proposition 3.12 considering some $0 < \epsilon_n < \min\{0.001, (t_{n+1} - t_n)/2\}$ and they satisfy

- $\Lambda^s(t_n) \cup \Lambda^u(t_n) \subset \Lambda_{t_n - \eta_n}$ for some $0 < \eta_n < \epsilon_n$,
- $\Lambda^s(t_n)$ doesn't connect with $\Lambda^u(t_n)$ before t_n,
- $\Lambda^s(t_n)$ connects with $\Lambda^u(t_n)$ before t_{n+1}

We are ready to prove the next proposition

Proposition 3.14. *We can take $\theta \in \{s, u\}$ such that given $N \in \mathbb{N}$ arbitrary, there exists a sequence $n_1 < n_2 < \ldots < n_N$ of elements of $\mathcal{I}$ (where $\mathcal{I}$ is the index set of the sequence $\{t_n\}$) such that for $i, j \in \{1, \ldots, N\}$ with $i \neq j$, $\Lambda^\theta(t_{n_i})$ and $\Lambda^\theta(t_{n_j})$ doesn't connect before* $\max\{t_{n_i}, t_{n_j}\}$.

Proof. We said that a sequence $n_1 < n_2 < ... < n_r$ of elements of $\mathcal{I}$ is a *r-chain* if $\Lambda^s(t_{n_i})$ connects with $\Lambda^s(t_{n_{i+1}})$ before $t_{n_{i+1}}$ for $i = 1, \dots r-1$. Then we have two cases:

- There exists some $R \in \mathbb{N}$ such that there is no r-chain for $r > R$.
- There are r-chains with r arbitrarily big.

We do the proof when the index set of the sequence is $\mathcal{I} = \{n \in \mathbb{Z} : n \geq 1\}$, and the other case follows similarly.

In the first case take a maximal r_1-chain beginning with 1; that is, a r_1-chain $1 = n_1 < n_2 < ... < n_{r_1}$ such that for any $n > n_{r_1}$, $1 = n_1 < n_2 < ... < n_{r_1} < n$ is not a (r_1+1)-chain and then $\Lambda^s(t_{n_{r_1}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n. Next take a maximal r_2-chain beginning with $n_{r_1}+1$: $n_{r_1}+1 = n_1^{(r_1)} < n_2^{(r_1)} < \cdots < n_{r_2}^{(r_1)}$ then, as before, for $n_{r_2}^{(r_1)} < n$, $\Lambda^s(t_{n_{r_2}^{(r_1)}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n. Now consider a maximal r_3-chain beginning with $n_{r_2}^{(r_1)}+1$: $n_{r_2}^{(r_1)}+1 = n_1^{(r_1,r_2)} < n_2^{(r_1,r_2)} < \cdots < n_{r_3}^{(r_1,r_2)}$ then for $n_{r_3}^{(r_1,r_2)} < n$, $\Lambda^s(t_{n_{r_3}^{(r_1,r_2)}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n.

Continuing in this way we can construct inductively an increasing sequence

$$\{\tilde{n}_k\}_{k\geq 2} = \{n_{r_k}^{(r_1,r_2,\dots,r_{k-1})}\}_{k\geq 2}$$

such that for $k_1, k_2 \geq 2$ with $k_1 \neq k_2$, $\Lambda^s(t_{\tilde{n}_{k_1}})$ and $\Lambda^s(t_{\tilde{n}_{k_2}})$ doesn't connect before $\max\{t_{\tilde{n}_{k_1}}, t_{\tilde{n}_{k_2}}\}$.

On the other hand, in the second case, take $r \in \mathbb{N}$ arbitrarily big and $n_1 < n_2 < ... < n_r$ some r-chain, then we affirm that for $i, j \in \{1, ..., r\}$ with $i \neq j$, $\Lambda^u(t_{n_i})$ and $\Lambda^u(t_{n_j})$ doesn't connect before $\max\{t_{n_i}, t_{n_j}\}$. In other case if for some $i_0, j_0 \in \{1, ..., r\}$ with $i_0 < j_0$, $\Lambda^u(t_{n_{i_0}})$ and $\Lambda^u(t_{n_{j_0}})$ connect before $t_{n_{j_0}}$ then as by corollary 2.7, $\Lambda^s(t_{n_{j_0}})$ connect with $\Lambda^s(t_{n_{i_0}})$ before $t_{n_{j_0}}$ and as also $\Lambda^s(t_{n_{i_0}})$ connects with $\Lambda^u(t_{n_{i_0}})$ before $t_{n_{i_0}+1}$ (and then before $t_{n_{j_0}}$). Applying two times more that corollary we have that $\Lambda^s(t_{n_{j_0}})$ connect with $\Lambda^u(t_{n_{j_0}})$ before $t_{n_{j_0}}$ that is a contradiction. From this follows the result. □

Without loss of generality, we will suppose that in the previous proposition $\theta = u$ (for $\theta = s$ the argument is similar) and call $\Lambda^u(t_n) = \Lambda^n$. Observe that $\mathcal{S} = \{\Lambda^n\}_{n\in\mathcal{I}}$ is the sequence commented in the introduction of the section.

3.6. **Subhorseshoes and connection by periodic orbits.** In this subsection, we associate to every term of the sequence $\mathcal{S}$ a periodic orbit with the property that if Λ^n and Λ^m are associated with the same periodic orbit then they connect before $\max\{t_n, t_m\}$.

In order to do that, given some n, remember the construction of Λ^n given by proposition 3.12. A close inspection of the proof of that proposition shows that for some maximal invariant set, said M^n, that contains Λ_{t_n} we took the subhorseshoe with maximal Hausdorff dimension $\Lambda_0^n \subset M^n$ and then applied proposition 3.5 in order to obtain the subhorseshoe Λ^n with

$$D_u(\Lambda^n) > (1-\epsilon_n/2k)D_u(\Lambda_0^n) > (1-\epsilon_n)D_u(\Lambda_0^n) > 0.999D_u(\Lambda_0^n). \tag{3.15}$$

Next, if $D_u(M^n) = D_u(\Lambda_2^n)$ where $\Lambda_2^n \subset M^n$ is a subhorseshoe of Λ, then as Λ_0^n has maximal dimension, it follows that either $D_u(\Lambda_2^n) \le D_u(\Lambda_0^n)$ or $D_s(\Lambda_2^n) \le D_s(\Lambda_0^n)$. In the first case

$$D_u(\Lambda_{t_n}) \le D_u(M^n) = D_u(\Lambda_2^n) \le D_u(\Lambda_0^n) \le \frac{r_2}{r_1} D_u(\Lambda_0^n)$$

and in the second, 3.11 let us conclude that

$$D_u(\Lambda_{t_n}) \le D_u(M^n) = D_u(\Lambda_2^n) \le r_2 D_s(\Lambda_2^n) \le r_2 D_s(\Lambda_0^n) \le \frac{r_2}{r_1} D_u(\Lambda_0^n)$$

that is,

$$D_u(\Lambda_{t_n}) \le \frac{r_2}{r_1} D_u(\Lambda_0^n). \tag{3.16}$$

Now, take r_0 big enough such that $2^{2023} < N_u(\Lambda_{t_0}, r_0)$ and

$$\frac{\log N_u(\Lambda_{t_0}, r_0)}{r_0 - c_1} < 1.001 D_u(\Lambda_{t_0}). \tag{3.17}$$

We set $\mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$, $N_0 = N_u(\Lambda_{t_0}, r_0)$ and for $n \in \mathcal{I}$, $M \in \mathbb{N}$ define the set

$$\mathcal{B}_M(\Lambda^n) := \{\beta = \beta_1 \dots \beta_M : \forall\, 1 \le j \le M,\ \beta_j \in \mathcal{B}_0 \ \text{ and } \ \Pi^u(\Lambda^n) \cap I^u(\beta) \neq \emptyset\}.$$

Before continuing, we introduce some notation. Consider $\beta = \beta_{k_1}\beta_{k_2}...\beta_{k_\ell} = a_1...a_p \in \mathcal{A}^p$, $\beta_{k_i} \in \mathcal{B}_0$, $1 \le i \le \ell$. We say that $n \in \{1, ..., p\}$ is the n-th position of β. If $\beta_{k_i} \in \mathcal{A}^{n_{k_i}}$ we write $|\beta_{k_i}| = n_{k_i}$ for its length and $P(\beta_{k_i}) = \{1, 2, ..., n_{k_i}\}$ for its set of positions as a word in the alphabet $\mathcal{A}$ and given $s \in P(\beta_{k_i})$ we call $P(\beta, k_i; s) = n_{k_1} + ... + n_{k_{i-1}} + s$ the position in β of the position s of β_{k_i}.

Recall that the sizes of the intervals $I^u(\alpha)$ behave essentially submultiplicatively due the bounded distortion property of ψ_u (equation 2.1) so that, one has

$$|I^u(\beta)| \le \exp(-M(r_0 - c_1))$$

for any $\beta \in \mathcal{B}_M(\Lambda^n)$, and thus, $\{I^u(\beta) : \beta \in \mathcal{B}_M(\Lambda^n)\}$ is a covering of $\Pi^u(\Lambda^n)$ by intervals of sizes $\le \exp(-M(r_0 - c_1))$. In particular for $M(\Lambda^n) = M_n$ big enough

$$\begin{aligned}
\frac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{\log N_0^{M_n}} &= \frac{\dfrac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{-\log\exp(-M_n(r_0 - c_1))}}{\dfrac{M_n \cdot \log N_0}{M_n(r_0 - c_1)}} \\
&\ge \frac{\dfrac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{-\log\exp(-M_n(r_0 - c_1))}}{1.001 D_u(\Lambda_{t_0})} \quad \text{(by equation 3.17)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{0.999 D_u(\Lambda^n)}{1.001 D_u(\Lambda_{t_0})} \quad (M_n \text{ is big})\\
&\geq \frac{0.999\cdot 0.999 D_u(\Lambda_0^n)}{1.001 D_u(\Lambda_{t_0})} \quad (\text{by equation 3.15})\\
&\geq \frac{r_1}{r_2}\frac{0.999\cdot 0.999 D_u(\Lambda_{t_n})}{1.001 D_u(\Lambda_{t_0})} \quad (\text{by equation 3.16})\\
&\geq \frac{r_1}{r_2}\frac{0.999\cdot 0.999\cdot 0.999}{1.001} \quad (\text{by equation 3.13})\\
&> 0.999^4/1.001\\
&> 991/1000.
\end{aligned}
$$

Then we have proved the next result

Lemma 3.15. *Given $n \in \mathbb{N}$ and M_n large*

$$|\mathcal{B}_{M_n}(\Lambda^n)| \geq N_0^{\frac{991}{1000}\cdot M_n}.$$

Remember that $f \in \mathcal{R}_{\varphi,\Lambda}$ where $\mathcal{R}_{\varphi,\Lambda}$ was defined in Section 2 above. Then, we can suppose, unless refining the initial Markov partition $\{R_a\}_{a\in\mathcal{A}}$, that the restriction of f to each of the intervals $\{i_a^s\}\times I_a^u$, $a \in \mathcal{A}$, is strictly monotone and, furthermore, for some constant $c_4 > 0$, the following estimates hold

$$
\begin{aligned}
(3.18)\, &|f(\underline{\theta}^{(1)}; a_1\dots a_n a_{n+1}\underline{\theta}^{(3)}) - f(\underline{\theta}^{(1)}; a_1\dots a_n a'_{n+1}\underline{\theta}^{(4)})| > c_4\cdot |I^u(a_1\dots a_n)|\\
&|f(\underline{\theta}^{(1)} a_{m+1} a_m\dots; a_1\underline{\theta}^{(3)}) - f(\underline{\theta}^{(2)} a'_{m+1} a_m\dots; a_1\underline{\theta}^{(3)})| > c_4\cdot |I^s(a_1\dots a_m)|
\end{aligned}
$$

whenever $a_{n+1} \neq a'_{n+1}$, $a_{m+1} \neq a'_{m+1}$ and $\underline{\theta}^{(1)}, \underline{\theta}^{(2)} \in \mathcal{A}^{\mathbb{Z}^-}$, $\underline{\theta}^{(3)}, \underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$ are admissible.

Moreover, we observe that, since $f \in C^2$, there exists $c_5 > 0$ such that we also have the following estimates:

$$
\begin{aligned}
(3.19)\, &|f(\underline{\theta}^{(1)}; a_1\dots a_n a_{n+1}\underline{\theta}^{(3)}) - f(\underline{\theta}^{(1)}; a_1\dots a_n a'_{n+1}\underline{\theta}^{(4)})| < c_5\cdot |I^u(a_1\dots a_n)|\\
&|f(\underline{\theta}^{(1)} a_{m+1} a_m\dots; a_1\underline{\theta}^{(3)}) - f(\underline{\theta}^{(2)} a'_{m+1} a_m\dots; a_1\underline{\theta}^{(3)})| < c_5\cdot |I^s(a_1\dots a_m)|
\end{aligned}
$$

whenever $a_{n+1} \neq a'_{n+1}$, $a_{m+1} \neq a'_{m+1}$ and $\underline{\theta}^{(1)}, \underline{\theta}^{(2)} \in \mathcal{A}^{\mathbb{Z}^-}$, $\underline{\theta}^{(3)}, \underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$ are admissible.

Next, we give a definition

Definition 3.16. Given $n \in \mathcal{I}$, $M \in \mathbb{N}$ and $\beta = \beta_1\dots\beta_M \in \mathcal{B}_M(\Lambda^n)$ with $\beta_i \in \mathcal{B}_0$ for all $1 \leq i \leq M$, we say that $j \in \{1,\dots,M\}$ is a *M-right-good position* of β if there are two elements of $\mathcal{B}_M(\Lambda^n)$

$$\beta^{(p)} = \beta_1\dots\beta_{j-1}\beta_j^{(p)}\dots\beta_M^{(p)}, \quad p = 1,2$$

with $\beta_i^{(p)} \in \mathcal{B}_0$ for all $j \le i \le M,\ p = 1, 2$ and such that $\sup I^u(\beta_j^{(1)}) < \inf I^u(\beta_j) < \sup I^u(\beta_j) < \inf I^u(\beta_j^{(2)})$, i.e., the interval $I^u(\beta_j)$ is located between $I^u(\beta_j^{(1)})$ and $I^u(\beta_j^{(2)})$.

Similarly, we say that $j \in \{1, \ldots, M\}$ is a *M-left-good position* of β if there are two elements of $\mathcal{B}_M(\Lambda^n)$

$$\beta^{(p)} = \beta_1^{(p)} \ldots \beta_j^{(p)} \beta_{j+1} \ldots \beta_M, \quad p = 3, 4$$

with $\beta_i^{(p)} \in \mathcal{B}_0$ for all $1 \le i \le j,\ p = 3, 4$ such that $\sup I^s((\beta_j^{(3)})^T) < \inf I^s(\beta_j^T) < \sup I^s(\beta_j^T) < \inf I^s((\beta_j^{(4)})^T)$, i.e., the interval $I^s(\beta_j^T)$ is located between $I^s((\beta_j^{(3)})^T)$ and $I^s((\beta_j^{(4)})^T)$.

Finally, we say that $j \in \{1, \ldots, M\}$ is a *M-good position* of β if it is both a M-right-good and a M-left-good position of β.

The bounded distortion property (equation 2.2) let us fix $J \in \mathbb{N}$ big enough such that for $\beta_1\beta_2 \ldots \beta_J$ and $\beta_{J+1}\beta_{J+2}$ admissible with $\beta_1, \beta_2, \ldots, \beta_J, \beta_{J+1}, \beta_{J+2} \in \mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$

$$|I^u(\beta_1\beta_2 \ldots \beta_J)| \le |I^s((\beta_{J+1}\beta_{J+2})^T)|$$

and

$$|I^s((\beta_1\beta_2 \ldots \beta_J)^T)| \le |I^u(\beta_{J+1}\beta_{J+2})|.$$

Set $k := 8JN_0^2$ (observe that k does not depend on n). The next lemma says that most positions of some word of $\mathcal{B}_{5N_nk}(\Lambda^n)$ are $5N_nk$-good.

Lemma 3.17. *For N_n big enough, there exists $\beta_n \in \mathcal{B}_{5N_nk}(\Lambda^n)$ such that the number of $5N_nk$-good positions of β_n is greater or equal than $49N_nk/10$.*

Proof. Let us first estimate the cardinality of the subset of $\mathcal{B}_{5N_nk}(\Lambda^n)$ consisting of words β such that at least $N_nk/20$ positions are not $5N_nk$-right-good: Once we fix a set of $m \ge N_nk/20$, $5N_nk$-right-bad (i.e., not $5N_nk$-right-good) positions, if j is a $5N_nk$-right-bad position and $\beta_1, \ldots, \beta_{j-1} \in \mathcal{B}_0$ were already chosen, then by definition, it follows that there are at most two options for $\beta_j \in \mathcal{B}_0$ which correspond to the leftmost and rightmost subintervals of $I^u(\beta_1 \ldots \beta_{j-1})$ of the form $I^u(\beta_1 \ldots \beta_{5N_nk})$ intersecting $\pi^u(\Lambda^n)$.

In particular, once a set of $m \ge N_nk/20$, $5N_nk$-right-bad positions is fixed, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda_n)$ with this set of m, $5N_nk$-right-bad positions is less than or equal to

$$2^m \cdot N_0^{5N_nk-m} \le 2^{N_nk/20} \cdot N_0^{99N_nk/20}.$$

Therefore, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda^n)$ with at least $N_nk/20$, $5N_nk$-right-bad positions is less than or equal to

$$2^{5N_nk} \cdot 2^{N_nk/20} \cdot N_0^{99N_nk/20} = 2^{101N_nk/20} \cdot N_0^{99N_nk/20}.$$

Analogously, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda^n)$ with at least $N_nk/20$, $5N_nk$-left-bad positions is bounded by $2^{101N_nk/20} \cdot N_0^{99N_nk/20}$.

It follows that the set of words $\beta \in \mathcal{B}_{5N_nk}(\Lambda_n)$ with at least $N_nk/10$, $5N_nk$-bad (i.e., not $5N_nk$-good) positions has cardinality less or equal than $2.2^{101N_nk/20} \cdot N_0^{99N_nk/20}$.

Since $|\mathcal{B}_{5N_nk}(\Lambda^n)| > N_0^{991N_nk/200}$ (by lemma 3.15) and $2^{1+101N_nk/20} \cdot N_0^{99N_nk/20} < N_0^{991N_nk/200}$ (from our choices of r_0, N_0 large), we have that there exists some $\beta_n \in \mathcal{B}_{5N_nk}(\Lambda^n)$ with less than $N_nk/10$, $5N_nk$-bad positions. That is, with at least $5N_nk - N_nk/10 = 49N_nk/10$ good positions. □

Given $n \in \mathcal{I}$ take N_n big enough as in the lemma 3.17 and such that for two elements $x, y \in \Lambda$ if their kneading sequences coincide in the central block (centered at the zero position) of size $2N_n + 1$ then $|f(x) - f(y)| < \eta_n/2$.

The next proposition shows that the notion of good positions allows us to have some control over the values that f takes in some rectangles.

Proposition 3.18. *If $\beta_n = \beta_1^n\beta_2^n \dots \beta_{5N_nk}^n$ with $\beta_r^n \in \mathcal{B}_0$ for $i = 1, \dots, 5N_nk$ is as in the previous lemma and for some $1 < i < j < 5N_nk$, the positions $i-1, i, j, j+1$ are $5N_nk$-good positions of β_n and $j - i \geq J$. Then for each $i \leq s \leq j$ and $\bar{n} \in P(\beta_s^n)$ if $\eta = \beta_{i-1}^n\beta_i^n \dots \beta_j^n\beta_{j+1}^n$ and $x \in R(\eta; P(\eta, s; \bar{n})) \cap \Lambda$ we have $f(x) < t_n$.*

Proof. The arguments are similar to those of proposition 2.9 of [1]. Consider $\underline{\theta}^{(2)} \in \mathcal{A}^{\mathbb{N}}$ and $\underline{\theta}^{(1)} \in \mathcal{A}^{\mathbb{Z}^-}$ such that $\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)} \in \Sigma_{\mathcal{B}}$. With this notation, our task is equivalent to show that

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})) < t_n \tag{3.20}$$

for all $0 \leq \ell \leq m_1 + m + m_2 - 1$ where $\beta_i^n = a_1 \dots a_{m_1}$, $\beta_{i+1}^n \dots \beta_{j-1}^n = b_1 \dots b_m$ and $\beta_j^n = d_1 \dots d_{m_2}$.

First we deal with positions of the word $\beta_{i+1}^n\beta_{i+2}^n \dots \beta_j^n\beta_{j-1}^n$, that is, we consider $m_1 \leq \ell \leq m_1 + m - 1$. Write $\ell = m_1 - 1 + r$ so that

$$\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}) = \underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)} \tag{3.21}$$

and also suppose that $|I^s((\beta_i^n b_1 \dots b_{r-1})^T)| \leq |I^u(b_r \dots b_m\beta_j^n)|$ (the conclusion when $|I^u(b_r \dots b_m\beta_j^n)| < |I^s((\beta_i^n b_1 \dots b_{r-1})^T)|$ follows similarly).

By definition of $5N_nk$-good position, we have

$$\sup I^s((\beta_i')^T) < \inf I^s((\beta_i^n)^T) < \sup I^s((\beta_i^n)^T) < \inf I^s((\beta_i'')^T)$$

and

$$\sup I^u(\beta_j') < \inf I^u(\beta_j^n) < \sup I^u(\beta_j^n) < \inf I^u(\beta_j''),$$

for some words $\beta_i', \beta_i'', \beta_j', \beta_j'' \in \mathcal{B}_0$ verifying

$$I^u(\beta_i'\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n) \cap \pi^u(\Lambda^n) \neq \emptyset, \quad I^u(\beta_i''\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n) \cap \pi^u(\Lambda^n) \neq \emptyset,$$

$$I^u(\beta_{i-1}^n\beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j') \cap \pi^u(\Lambda^n) \neq \emptyset, \quad I^u(\beta_{i-1}^n\beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j'') \cap \pi^u(\Lambda^n) \neq \emptyset.$$

Choose $\beta_j^* \in \{\beta_j', \beta_j''\}$ such that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) < f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)})$$

for any $\underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$. By 3.18, it follows that

$$\begin{aligned} &f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) + c_4|I^u(b_r \dots b_m \beta_j^n)| \\ &< f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}). \end{aligned}$$

On the other hand, by (3.19), we also know that, for any $\underline{\theta}^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$

$$\begin{aligned} &|f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}) - f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)})| \\ &< c_5|I^s((\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1})^T)| \end{aligned}$$

From these estimates, we obtain that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) + c_4|I^u(b_r \dots b_m \beta_j^n)| <$$

$$f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}) + c_5 e^{c_1}|I^s((\beta_{i-1}^n)^T)| \cdot |I^s((\beta_i^n b_1 \dots b_{r-1})^T)|$$

for any $\underline{\theta}^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$.

Since we are supposing that $|I^s((\beta_i^n b_1 \dots b_{r-1})^T)| \leq |I^u(b_r \dots b_m \beta_j^n)|$, we conclude

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) <$$

$$f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}) - (c_4 - c_5 e^{c_1}|I^s((\beta_{i-1}^n)^T)|) \cdot |I^u(b_r \dots b_m \beta_j^n)|.$$

Next, we note that if $r_0 \in \mathbb{N}$ is sufficiently large, $c_5 e^{c_1}.|I^s((\beta_{i-1}^n)^T)| < c_4/2$. In particular, we have that

$$\begin{aligned} &f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) < \\ &f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}) - (c_4/2) \cdot |I^u(b_r \dots b_m \beta_j^n)| \end{aligned} \tag{3.22}$$

for any $\underline{\theta}^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$.

Now, we recall that as $\beta_j^* \in \{\beta_j', \beta_j''\}$, one has $I^u(\beta_{i-1}^n\beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^*) \cap \pi^u(\Lambda^n) \neq \emptyset$. By definition, this implies that there are $\underline{\theta}_*^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}_*^{(4)} \in \mathcal{A}^{\mathbb{N}}$ with

$$\underline{\theta}_*^{(3)}; \beta_{i-1}^n\beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^*\underline{\theta}_*^{(4)} \in \Sigma_{t_n},$$

and, in particular, by (3.21)

$$f(\sigma^{m_2+\ell}(\underline{\theta}_*^{(3)}; \beta_{i-1}^n\beta_i^n b_1 \dots b_m \beta_j^* \underline{\theta}_*^{(4)})) = f(\underline{\theta}_*^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}_*^{(4)})) \leq t_n.$$

Combining this with (3.22), we see that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) < t_n - (c_4/2) \cdot |I^u(b_r \dots b_m \beta_j^n)|$$

and then

$$f(\sigma^{\ell}(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n \dots \beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})) < t_n. \tag{3.23}$$

Finally, the case when we deal with positions of the words β_i^n or β_j^n is similar with the previous one. We write

$$\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n;\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}) = \underline{\theta}^{(1)}\beta_{i-1}^n a_1\dots a_\ell; a_{\ell+1}\dots a_{m_1}\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}$$

for $0 \le \ell \le\ m_1 - 1$, and

$$\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n;\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}) = \underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n d_1\dots d_{\ell-m_1-m}; d_{\ell-m_1-m+1}\dots d_{m_2}\beta_{j+1}^n\underline{\theta}^{(2)}$$

for $m_1 + m \le \ell \le m_1 + m + m_2 - 1$.

Since $j - i \ge J$ and $\beta_{i-1}^n, \beta_i^n, \dots, \beta_{j-1}^n, \beta_j^n \in \mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$, it follows from our choice of J that

$$|I^u(a_{\ell+1}\dots a_{m_1}\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n)| \le |I^s((\beta_{i-1}^n a_1\dots a_\ell)^T)|$$

for $0 \le \ell \le m_1 - 1$, and

$$|I^s((\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n d_1\dots d_{\ell-m_1-m})^T)| \le |I^u(d_{\ell-m_1-m+1}\dots d_{m_2}\beta_{j+1}^n)|$$

for $m_1 + m \le \ell \le m_1 + m + m_2 - 1$. Arguing as before, one deduces that

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n;\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})) < t_n - (c_4/2)\cdot|I^s((\beta_{i-1}^n a_1\dots a_\ell)^T)| < t_n \tag{3.24}$$

for $0 \le \ell \le m_1 - 1$, and

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n;\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})) < t_n - (c_4/2)\cdot|I^u(d_{\ell-m_1-m+1}\dots d_{m_2}\beta_{j+1}^n)| < t_n \tag{3.25}$$

for $m_1 + m \le \ell \le m_1 + m + m_2 - 1$.

In summary, from 3.23, 3.24, and 3.25 we deduce that 3.20 holds, as we wanted to see. □

Consider $\beta_n = \beta_1^n\beta_2^n\dots\beta_{5N_nk}^n$ and divide its position set $I = \{1, 2, \dots, 5N_nk\}$ in positions packages of size N_nk. In the central package $I^* = \{2N_nk+1, 2N_nk+2, \dots, 3N_nk\}$, the number of $5N_nk$-bad positions is less than $5N_nk - 49N_nk/10 = N_nk/10$ and then subdividing that package now in N_n package of positions of size k we can find some package of size k with less than $k/10$, $5N_nk$-bad positions, said

$$I^{**} = \{2N_nk + sk + 1, 2N_nk + sk + 2, \dots, 2N_nk + (s+1)k\} \text{ for some } 0 \le s < N_n.$$

Then we can find $\lceil 2k/5 \rceil$ positions

$$2N_nk + sk + 1 \le i_1 \le \dots \le i_{\lceil 2k/5\rceil} < 2N_nk + (s+1)k$$

such that $i_{r+1} \ge i_r + 2$ for all $1 \le r < \lceil 2k/5 \rceil$ and the positions $i_1, i_1+1, \dots, i_{\lceil 2k/5\rceil}$, $i_{\lceil 2k/5\rceil} + 1$ are $5N_nk$-good.

Since we took $k = 8JN_0^2$, it makes sense to set

$$j_r = i_{rJ} \quad \text{ for } 1 \le r \le 3N_0^2$$

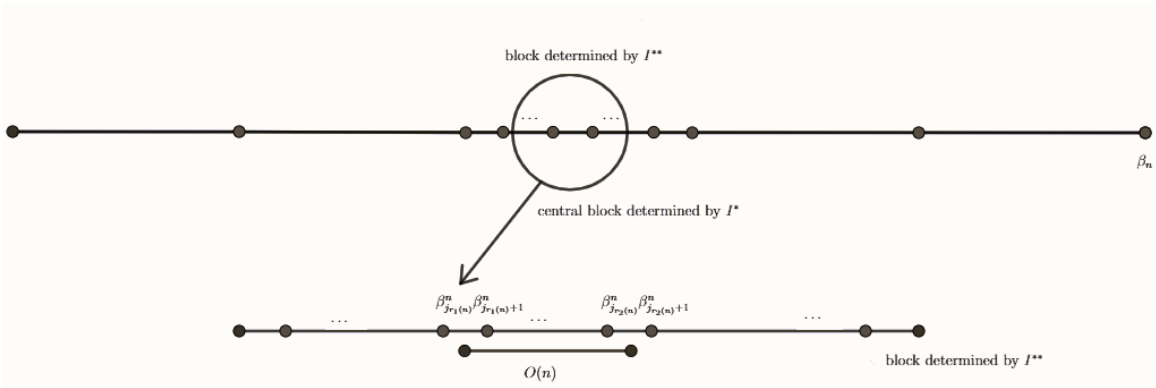

FIGURE 2. Construction of $O(n)$.

because $3JN_0^2 < (16/5)JN_0^2 = 2k/5$. In this way, we obtain positions such that

$$j_{r+1} - j_r \geq 2J \quad \text{for } 1 \leq r \leq 3N_0^2$$

and $j_1, j_1+1, \ldots, j_{3N_0^2}, j_{3N_0^2}+1$ are $5N_nk$-good positions.

Since for $1 \leq r \leq 3N_0^2$ the number of possibilities for $(\beta^n_{j_r}, \beta^n_{j_r+1})$ is at most N_0^2, we conclude that for some different $1 \leq r_1(n), r_2(n) \leq 3N_0^2$ one has

$$(\beta^n_{j_{r_1(n)}}, \beta^n_{j_{r_1(n)}+1}) = (\beta^n_{j_{r_2(n)}}, \beta^n_{j_{r_2(n)}+1})$$

then, we can define the following map:

$$\begin{aligned} O : \mathcal{I} &\rightarrow \bigcup_{j=2}^{k-1} \mathcal{B}_0^j \\ n &\rightarrow \beta^n_{j_{r_1(n)}+1}\beta^n_{j_{r_1(n)}+2}\ldots\beta^n_{j_{r_2(n)}} \end{aligned}$$

Next, we see that if for some $m, n \in \mathcal{I}$ we have $O(m) = O(n)$ then it is possible to go from Λ^m to Λ^n without leaving $\Lambda_{\max\{t_n,t_m\}}$ and staying arbitrarily close of the orbit of the periodic point $p = \Pi^{-1}(\overline{O(m)})$ for times arbitrarily big.

Proposition 3.19. *Take $m, n \in \mathcal{I}$ such that $O(m) = O(n)$. Then given $N \in \mathbb{N}$ and $\epsilon > 0$ there exist some $x = x(N,\epsilon) \in W^u(\Lambda^m) \cap W^s(\Lambda^n)$ and $\overline{m} = \overline{m}(N,\epsilon) \in \mathbb{N}$ such that for $\overline{m} \leq i \leq \overline{m} + N$, $d(\mathcal{O}(p), \phi^i(x)) < \epsilon$. Even more, we have $m_{\phi,f}(x) < \max\{t_n, t_m\}$.*

Remark 3.20. By symmetry, we also have the existence of some $y \in W^u(\Lambda^n) \cap W^s(\Lambda^m)$ and $\overline{n} \in \mathbb{N}$ with similar properties as x and $\overline{m}$.

Proof. As $\beta_m \in \mathcal{B}_{5N_mk}(\Lambda^m)$ and $\beta_n \in \mathcal{B}_{5N_nk}(\Lambda^n)$ we can find $\theta^1_m, \theta^1_n \in \mathcal{A}^{\mathbb{Z}^-}$ and $\theta^2_m, \theta^2_n \in \mathcal{A}^{\mathbb{N}}$ such that

$$\theta^1_m; \beta_m\theta^2_m \in \Pi(\Lambda^m) \quad \text{and} \quad \theta^1_n; \beta_n\theta^2_n \in \Pi(\Lambda^n).$$

By lemma 3.17, arguing as before; we can find positions $1 \le j_{r_0(m)} < N_m k$ and $1 \le j_{r_0(n)} < N_n k$ such that $j_{r_0(m)}$, $j_{r_0(m)}+1$ are $5N_m k$-good positions for β_m and $j_{r_0(n)}$, $j_{r_0(n)}+1$ are $5N_n k$-good positions for β_n; and also positions $4N_m k+1 \le j_{r_3(m)} < 5N_m k$ and $4N_n k + 1 \le j_{r_3(n)} < 5N_n k$ such that $j_{r_3(m)}$, $j_{r_3(m)} + 1$ are $5N_m k$-good positions for β_m and $j_{r_3(n)}$, $j_{r_3(n)} + 1$ are $5N_n k$-good positions for β_n.

Define then for $R \in \mathbb{N}$

$$x_R = \theta^1_m; \beta^m_1 \beta^m_2 \dots \beta^m_{j_{r_1(m)}} O(n)^R \beta^n_{j_{r_2(n)}+1} \beta^n_{j_{r_2(n)}+2} \dots \beta^n_{5N_n k} \theta^2_n.$$

Clearly, the proposition will be proved if we show that for some $t < \max\{t_n, t_m\}$, $x_R \in \Sigma_t$:

Let $l \in \mathbb{Z}$. In any of the next three cases:

- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^n_{j_{r_1(n)}} \beta^n_{j_{r_1(n)}+1} \dots \beta^n_{j_{r_2(n)}} \beta^n_{j_{r_2(n)}+1} (= \beta^m_{j_{r_1(m)}} \beta^m_{j_{r_1(m)}+1} \dots \beta^m_{j_{r_2(m)}} \beta^n_{j_{r_2(m)+1}})$, some $j_{r_1(n)} < s \le j_{r_2(n)}$ and $\bar{n} \in P(\beta^n_s)$.
- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^m_{j_{r_0(m)}} \beta^m_{j_{r_0(m)}+1} \dots \beta^m_{j_{r_1(m)}} \beta^m_{j_{r_1(m)+1}}$, some $j_{r_0(m)} < s \le j_{r_1(m)}$ and $\bar{n} \in P(\beta^m_s)$.
- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^2_{j_{r_2(n)}} \beta^2_{j_{r_2(2)}+1} \dots \beta^2_{j_{r_3(n)}} \beta^n_{j_{r_3(n)+1}}$, some $j_{r_2(n)} < s \le j_{r_3(n)}$ and $\bar{n} \in P(\beta^n_s)$

proposition 3.18 let us conclude that $f(\Pi^{-1}(\sigma^l(x_R))) < \max\{t_n, t_m\}$.

Let $r_1 = |\beta^m_1 \beta^m_2 \dots \beta^n_{j_{r_0(m)}}|$ then, for $l \le r_1 - 1$

$$f(\Pi^{-1}(\sigma^l(x_R))) < f(\Pi^{-1}(\sigma^l(\theta^1_m; \beta_m \theta^2_m))) + \eta_m/2 < t_m - \eta_m/2$$

because $\Lambda^m \subset \Lambda_{t_m - \eta_m}$ and as $j_{r_1(m)} - j_{r_0(m)} > 2N_m k - N_m k = N_m k$ we have that $\sigma^l(x_R)$ coincides with $\sigma^l(\theta^1_m; \beta_m \theta^2_m)$ in the central block of size $2N_m + 1$ centered at the zero position.

Analogously, for $r_2 = |\beta^m_1 \beta^m_2 \dots \beta^m_{j_{r_1(m)}} O(n)^R \beta^n_{j_{r_2(n)}+1} \beta^n_{j_{r_2(n)}+2} \dots \beta^n_{j_{r_3(n)}}|$, $j = r_2 - |\beta^n_1 \beta^n_2 \dots \beta^n_{j_{r_3(n)}}|$ and $l \ge r_2$

$$f(\Pi^{-1}(\sigma^l(x_R))) < f(\Pi^{-1}(\sigma^{l-j}(\theta^1_n; \beta_n \theta^2_n))) + \eta_n/2 < t_n - \eta_n/2$$

because $\Lambda^n \subset \Lambda_{t_n - \eta_n}$ and as $j_{r_3(n)} - j_{r_2(n)} > 4N_n k - 3N_n k = N_n k$ we have that $\sigma^l(x_R)$ coincides with $\sigma^{l-j}(\theta^1_n; \beta_n \theta^2_n)$ in the central block of size $2N_n + 1$ centered at the zero position.

As the previous cases describe all the possibilities for $l \in \mathbb{Z}$ and for $l \le r_1 - 1$ and $l \ge r_2$ we have uniform limitation for the values of $f(\Pi^{-1}(\sigma^l(x_R))) < \max\{t_n, t_m\}$ then we have proved the result. □

Using proposition 3.19 we can prove that if for some $m, n \in \mathbb{N}$, $O(m) = O(n)$ then we can connect Λ^m with Λ^n without leaving $\Lambda_{max\{t_n, t_m\}}$ as is expressed in definition 2.5

Corollary 3.21. *Let $m, n \in \mathcal{I}$ such that $O(m) = O(n)$. Then Λ^m connects with Λ^n before* $\max\{t_n, t_m\}$.

Proof. Proposition 3.19 let us find some $x, y \in \Lambda$ with $x \in W^u(\Lambda^m) \cap W^s(\Lambda^n)$, $y \in W^u(\Lambda^n) \cap W^s(\Lambda^m)$ and some $t < \max\{t_n, t_m\}$ such that

$$\Lambda^n \cup \Lambda^m \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_t.$$

Then proposition 2.6 let us conclude that Λ^n and Λ^m connects before $\max\{t_n, t_m\}$. □

3.7. **End of the proof of theorem 1.1 when the dimension is less than 1.** We are ready to obtain the desired contradiction. As the map O takes only a finite number of different values, said M. Then by corollary 3.21 it would be impossible to have a sequence $n_1 < n_2 < ... < n_{M+1}$ of elements of $\mathcal{I}$ such that for $i, j \in \{1, ..., M+1\}$ with $i \neq j$, Λ^{n_i} and Λ^{n_j} doesn't connect before $\max\{t_{n_i}, t_{n_j}\}$ in contradiction with proposition 3.14.

3.8. **Proof of theorem 1.1 when the dimension is greater than or equal to 1.** Consider $\varphi \in \mathcal{U}^*$ such that $HD(\Lambda) \geq 1$, $f \in \mathcal{P}_{\varphi,f}$ and some closed sub interval $I \subset I_{\varphi,f}$ that doesn't contain neither $c_{\varphi,f}$ nor $\tilde{c}_{\varphi,f}$. Observe that, in this case, by corollary 3.3, $\max L_{\varphi,f} = 1$ and then for $t < \tilde{c}_{\varphi,f}$ one has $L_{\varphi,f}(t) < 1$.

Take a hyperbolic set of finite type P such that

$$\Lambda_{\max I} \subset P \subset \Lambda_{\frac{\tilde{c}_{\varphi,f}+\max I}{2}}.$$

As before, the set P admits a decomposition $P = \bigcup\limits_{i \in \mathcal{I}} \tilde{\Lambda}_i$ where $\mathcal{I}$ is a finite index set and for any $i \in \mathcal{I}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set. Note that if $i_0, i_1 \in \mathcal{I}$ are different and $\tilde{\Lambda}_{i_0}$ and $\tilde{\Lambda}_{i_1}$ are subhorseshoes, then $\tilde{\Lambda}_{i_0}$ and $\tilde{\Lambda}_{i_1}$ don't connect before $\max I$.

Consider $s < \max I$, then we have

$$\ell_{\varphi,f}(\Lambda_s) = \bigcup_{i \in \mathcal{I}} \ell_{\varphi,f}(\tilde{\Lambda}_i \cap \Lambda_s) = \bigcup_{\substack{i \in \mathcal{I}:\ \tilde{\Lambda}_i \ is \\ subhorseshoe}} \ell_{\varphi,f}(\tilde{\Lambda}_i \cap \Lambda_s) = \bigcup_{\substack{i \in \mathcal{I}:\ \tilde{\Lambda}_i \ is \\ subhorseshoe}} \ell_{\varphi,f}((\tilde{\Lambda}_i)_s).$$

by taking union over $s < t$ where $t \leq \max I$, we conclude from this and lemma 3.6 that

$$\mathcal{L}_{\varphi,f} \cap (-\infty, t) = \ell_{\varphi,f}(\Lambda) \cap (-\infty, t) = \bigcup_{\substack{i \in \mathcal{I}:\ \tilde{\Lambda}_i \ is \\ subhorseshoe}} \ell_{\varphi,f}(\tilde{\Lambda}_i) \cap (-\infty, t)$$

and then, for $t \leq \max I$

$$L_{\varphi,f}(t) = \max_{\substack{i \in \mathcal{I}:\ \tilde{\Lambda}_i \ is \\ horseshoe}} HD(\ell_{\varphi,f}(\tilde{\Lambda}_i) \cap (-\infty, t)) = \max_{\substack{i \in \mathcal{I}:\ \tilde{\Lambda}_i \ is \\ horseshoe}} L_i(t),$$

where $L_i(t) = HD(\ell_{\varphi,f}(\tilde{\Lambda}_i) \cap (-\infty, t))$ is associated to the horseshoe $\tilde{\Lambda}_i$ with

$$HD(\tilde{\Lambda}_i) = HD(\ell_{\varphi,f}(\tilde{\Lambda}_i)) \leq HD(\ell_{\varphi,f}(\Lambda_{\frac{\tilde{c}_{\varphi,f}+\max I}{2}})) \leq L_{\varphi,f}\left(\frac{2\tilde{c}_{\varphi,f} + \max I}{3}\right) < 1.$$

Observe that, as in the proposition 2.4, the first part of the theorem also holds for subhorseshoes of Λ with Hausdorff dimension less than 1. Therefore, if we set $c_i = \min\{x : x \text{ is an accumulation point of } \ell_{\varphi,f}(\tilde{\Lambda}_i)\}$, by proposition 3.7 there is some $i_0 \in \mathcal{I}$ such that $c_{\varphi,f} = c_{i_0}$ and also by corollary 3.10 for any i such that $c_{\varphi,f} < c_i$ the function L_i doesn't contribute with any discontinuity close c_i to the discontinuity set of L (note that it is possible to have $c_i \geq \max I$ for some i). Then, we conclude that $L_{\varphi,f}$ has finitely many discontinuities in the interval I as we wanted to see.

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