On bipartite $(1, 1, k)$-mixed graphs *

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Abstract

Mixed graphs can be seen as digraphs with arcs and edges (or digons, that is, two opposite arcs). In this paper, we consider the case where such graphs are bipartite and in which the undirected and directed degrees are one. The best graphs, in terms of the number of vertices, are presented for small diameters. Moreover, two infinite families of such graphs with diameter k and number of vertices of the order of $2^{k/2}$ are proposed, one of them being totally regular $(1, 1)$-mixed graphs. In addition, we present two more infinite families

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called chordal ring and chordal double ring mixed graphs, which are bipartite and related to tessellations of the plane. Finally, we give an upper bound that improves the Moore bound for bipartite mixed graphs for $r = z = 1$.

Keywords: Mixed graph, degree/diameter problem, Moore bound, bipartite graph.

Mathematics Subject Classifications: 05C50, 05C20, 05C35.

1 Introduction

A mixed graph can be seen as a type of digraph containing some edges (two opposite arcs). Thus, a *mixed graph* G with vertex set V may contain (undirected) *edges* as well as directed edges (also known as *arcs*). From this point of view, a *graph* (respectively, *directed graph* or *digraph*) has all its edges undirected (respectively, directed). In fact, we can identify a mixed graph G with its associated digraph G^* obtained by replacing all the edges with digons (two opposite arcs or a directed 2-cycle). The *undirected degree* of a vertex v, denoted by $d(v)$, is the number of edges incident to v. The *out-degree* (respectively, *in-degree*) of vertex v, denoted by $d^+(v)$ (respectively, $d^-(v)$), is the number of arcs emanating from (respectively, to) v. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then G is said to be *totally regular* of degree (r, z), with $r + z = d$ (or simply (r, z)-*regular*).

The length of a shortest path from u to v is the *distance* from u to v, and it is denoted by $\mathrm{dist}(u, v)$. Note that $\mathrm{dist}(u, v)$ may differ from $\mathrm{dist}(v, u)$ when the shortest paths between u and v involve arcs. The *out-eccentricity* of a vertex u, denoted by $\mathrm{ecc}^+(u)$, is the maximum distance from u to any vertex in G. Analogously, the in-eccentricity of u is $\mathrm{ecc}^-(u) = \max\{\mathrm{dist}(v, u) : v \in V\}$. The maximum distance between any pair of vertices is the *diameter* $k = k(G)$ of G, that is, $k(G) = \max\{\mathrm{ecc}^+(u) : u \in V\}$. The *out-radius* and *in-radius* of G are $r^+(G) = \min\{\mathrm{ecc}^+(u) : u \in V\}$ and $r^-(G) = \min\{\mathrm{ecc}^-(u) : u \in V\}$, respectively. According to Knor [10], a *central vertex* is a vertex with minimum radius $r(G) = \max\{r^+(G), r^-(G)\}$. In the case of mixed graphs, we also use the concepts of *in-central* and *out-central* vertices, defined as the vertices having minimum in-radius and out-radius, respectively. The *converse* of a digraph G, denoted by $\overline{G}$, is the digraph obtained by reversing the orientation of all arcs in G. (Of course, if G is a mixed graph, the edges of $\overline{G}$ remain unchanged.) Notice that $k(G) = k(\overline{G})$, $r^+(G) = r^-(\overline{G})$, and $r^-(G) = r^+(\overline{G})$.

For results concerning diameter and order (Moore bounds), see, for instance, Nguyen and Miller [14], and Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [3].

In this paper, we consider bipartite mixed graphs with undirected and directed degrees both equal to 1. The structure of the paper is as follows. In Section 2, we present the best (in terms of the number of vertices) bipartite $(1,1,k)$-mixed graphs for small diameters. In Section 3, we propose two infinite families of such graphs with diameter k and number of vertices of the order of $2^{k/2}$, one of them being totally regular $(1,1)$-mixed graphs. In Section 4, we give two more infinite families, called chordal ring and chordal double ring mixed graphs, which are bipartite and related to tessellations of the plane. Finally, Section 5 gives an upper bound that improves the Moore bound for bipartite mixed graphs for $r = z = 1$.

1.1 The Moore bound for bipartite $(1,1,k)$-mixed graphs

The degree/diameter (optimization) problem for mixed graphs is the following.

Problem 1.1. *Given three natural numbers r, z, and k, find the largest possible number of vertices $N(r,z,k)$ in a mixed graph with maximum undirected degree r, maximum directed out-degree z, and diameter k.*

Here, we are interested in the case of bipartite mixed graphs. For this case, Dalfó, Fiol, and López [6] proved that the Moore bound for bipartite (r,z,k)-mixed graphs with diameter k is

$$M_B(r,z,k) = 2\left(A\,\frac{u_1^{k+1}-u_1}{u_1^2-1} + B\,\frac{u_2^{k+1}-u_2}{u_2^2-1}\right), \tag{1}$$

where, with $d = r+z$ and $v = (d-1)^2 + 4z$,

$$u_1 = \frac{d-1-\sqrt{v}}{2}, \qquad u_2 = \frac{d-1+\sqrt{v}}{2}, \tag{2}$$

$$A = \frac{\sqrt{v}-(d+1)}{2\sqrt{v}}, \qquad B = \frac{\sqrt{v}+(d+1)}{2\sqrt{v}}. \tag{3}$$

In the same paper [6], the following results were shown:

- Bipartite Moore (r,z,k)-mixed graphs do not exist for any $r \geq 1$, $z \geq 1$, and $k \geq 4$.
- Bipartite Moore mixed graphs with diameter $k = 3$ and $r = 1$ exist for any value of $z \geq 1$. In particular, some largest $(1,1,3)$- and $(1,1,4)$-mixed graphs were presented (see Section 2).

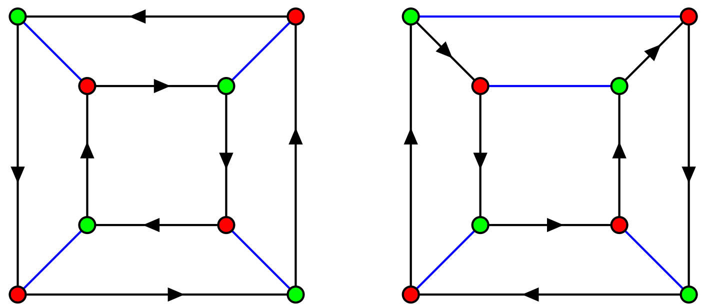

Figure 1: The only two bipartite Moore $(1,1,3)$-mixed graphs with 8 vertices.

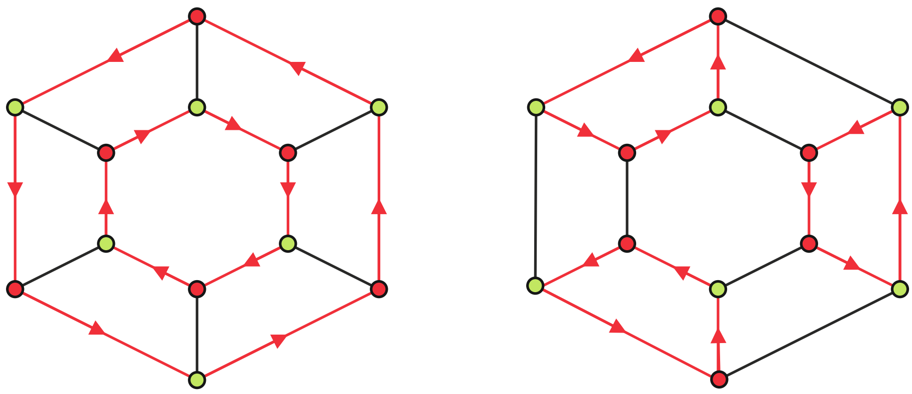

Figure 2: Two largest bipartite $(1,1,4)$-mixed graphs (with 12 vertices, 2 less than the corresponding bipartite Moore bound).

- There exist families of bipartite mixed graphs with diameter $k = 4, 5, 7$ and $r = 1$ that asymptotically attain the Moore bound, for large values of z being a power of a prime minus one.

Notice that, when $r = z = 1$, (2) and (3) become

$$u_1 = \frac{1-\sqrt{5}}{2}, \quad u_2 = \frac{1+\sqrt{5}}{2}, \quad A = \frac{\sqrt{5}-3}{2\sqrt{5}}, \quad B = \frac{\sqrt{5}+3}{2\sqrt{5}}, \tag{4}$$

and the numbers $M_B(k) = M_B(1,1,k)$ satisfy the Fibonacci-type recurrence

$$M_B(k) = M_B(k-1) + M_B(k-2) + 2$$

starting from $M_B(1) = 2$ and $M_B(2) = 4$. In Table 1, there are the values of $M_B(k)$ for $k = 3, \ldots, 16$.

	Upper bounds		Best graphs found	
k	Moore $M_B(k)$	Thm 5.1	$BDM(2,m)$	Computer search
3	8	8		8*
4	14	12		12*
5	24	18		18*
6	40	36	20	30
7	66	60		48
8	108	96	40	54
9	176	158		176
10	286	256	80	144
11	464	416		228
12	742	674	160	312
13	1208	1092		480
14	1952	1766	320	800
15	3162	2860		1024
16	5116	4628	640	1600

Table 1: Bounds for bipartite mixed graphs with $(r, z, k) = (1, 1, k)$.

From (1) and (4), note that the maximum possible number of vertices of a bipartite $(1, 1, k)$-mixed graph is of the order $M_B(k) \sim \varphi^k$, where φ is the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

2 Bipartite $(1, 1, k)$-mixed graphs with small diameter

In this section, we present some bipartite $(1, 1, k)$-mixed graphs that are best possible regarding the number of vertices. First, the mixed graphs of diameters $k = 3$ and $k = 4$, shown in Figures 1 and 2, respectively, were found by Dalfó, Fiol, and López in [6]. The ones with diameter $k = 3$ are Moore graphs, whereas the ones with $k = 4$ have order two less than the bipartite Moore bound and were proved to be the best possible. Notice that one of the mixed graphs with $k = 4$ (Figure 2) has 2 directed 6-cycles, whereas the other has 3 directed 4-cycles. Erskine found three other examples with the same underlying graph: one with two 6-cycles, one with 1 directed 12-cycle, and one with 1 directed 4-cycle plus 1 directed 8-cycle.

An exhaustive computational search by Exoo proved that the graphs with diameter $k = 5$ and order $N = 18$ (Figure 4) are also the best possible.

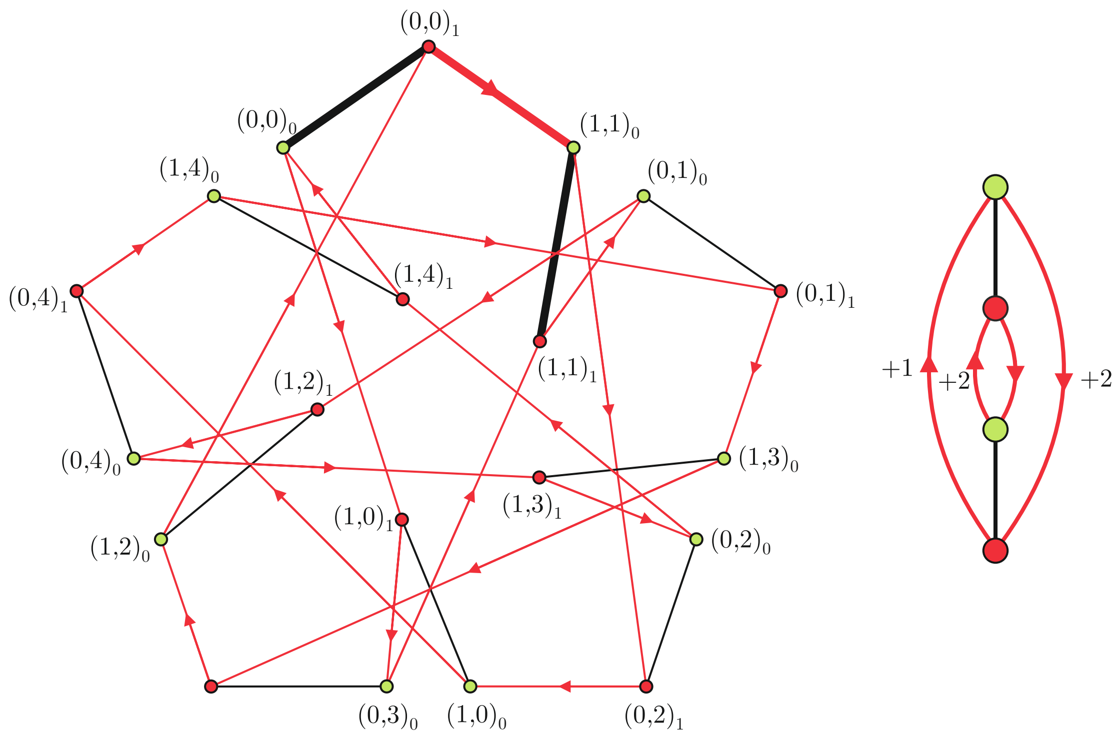

Figure 3: The bipartite mixed graph $BDM(2,5)$ and its base graph. The thick lines in $BDM(2,5)$ represent copy 0 of the lift.

In Figure 5, we show the best mixed graph with 30 vertices and diameter $k = 6$ that we found. In the figure, the 3 directed 10-cycles are shown in different colors. The question of whether or not this is the best possible solution is still open.

Table 1 lists the largest known mixed graphs with $r = z = 1$ for diameters from 3 to 16. The upper bounds are the Moore bound $M_B(k)$ from equations (1) and (4) and the tighter bound from Theorem 5.1 derived in Section 5 below. The values shown are the orders of the graphs $BDM(2,m)$ described in Section 3, and the largest graphs found by computer search.

The computer search was exhaustive for diameters 3, 4 and 5: these values are marked with an asterisk to show that the graphs are largest possible. For diameters 7 to 16, the graphs found are a combination of Cayley graphs on groups of the stated order, and voltage lifts of a 2-vertex base graph using groups of half the stated order. Full details of the groups and generators are available from the authors on request.

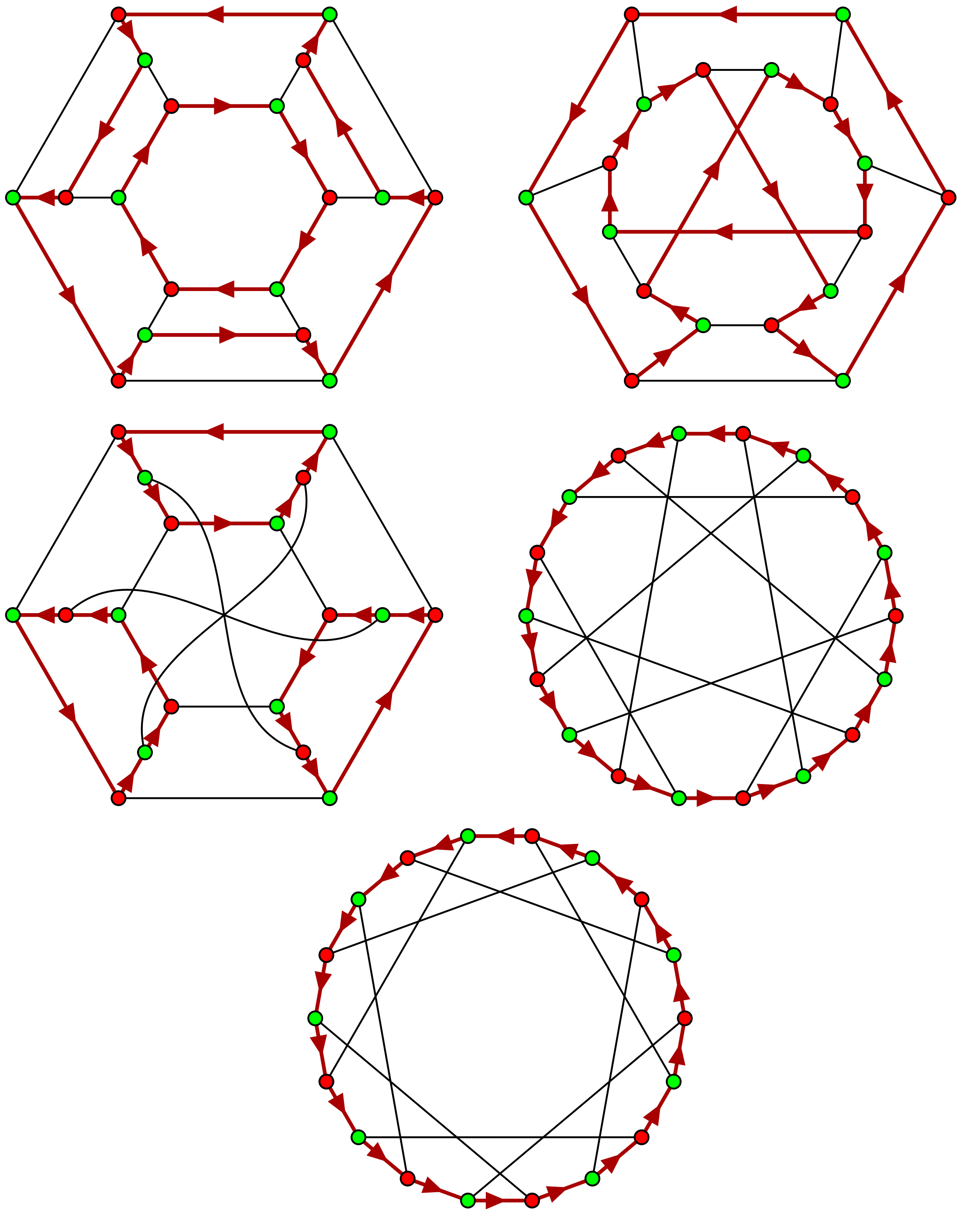

Figure 4: Five maximal bipartite $(1, 1, 5)$-mixed graph with 18 vertices (the corresponding bipartite Moore bound is 24).

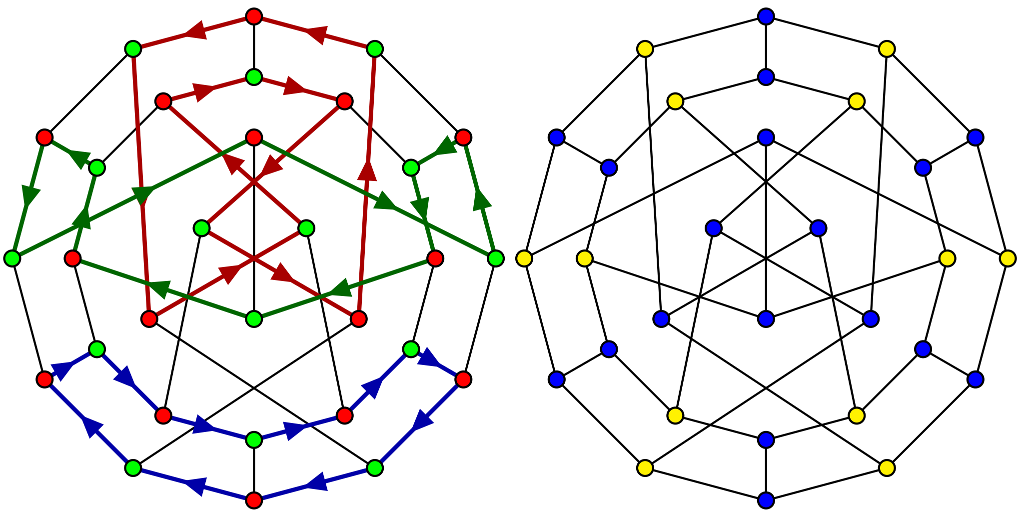

Figure 5: A bipartite $(1, 1, 6)$-mixed graph with 30 vertices (the corresponding bipartite Moore bound is 40) and its underlying cubic graph. Blue and yellow colors indicate vertex orbits.

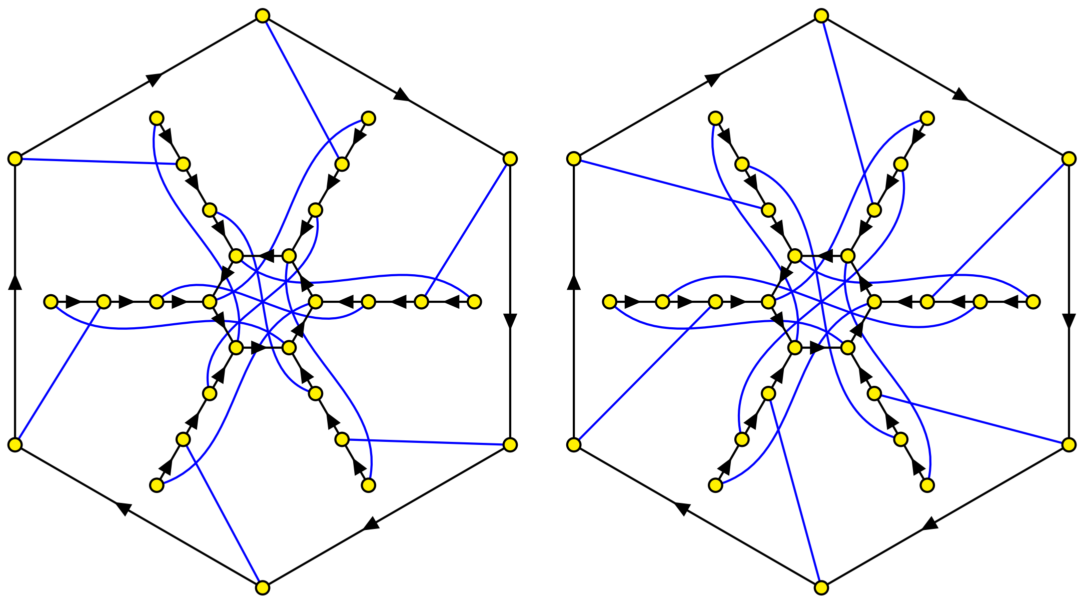

Figure 6: Two not in-regular bipartite $(1, 1, 6)$-mixed graphs with 30 vertices.

3 Some infinite families of bipartite $(1,1,k)$-mixed graphs

For some integer $n \geq 2$, let $m = 2^{n-1} + 2^{n-3}$. The mixed graph $BDM(2,m)$ has independent vertex sets

$$V_0 = \{(0,i)_0, (1,i)_0 : i \in \mathbb{Z}_m\} \quad \text{and} \quad V_1 = \{(0,i)_1, (1,i)_1 : i \in \mathbb{Z}_m\}.$$

The edges are

$$(0,i)_0 \sim (0,i)_1 \quad \text{and} \quad (1,i)_0 \sim (1,i)_1,$$

whereas the arcs are

$$\begin{aligned} (0,i)_0 &\to (1,2i)_1 \quad \text{and} \quad (0,i)_1 \to (1,2i+1)_0, \\ (1,i)_0 &\to (0,-2i-1)_1 \quad \text{and} \quad (1,i)_1 \to (0,-2i-2)_0, \end{aligned}$$

all with arithmetic modulo m. Thus, $BDM(2,m)$ is a bipartite $(1,1,k)$-mixed graph on $N = 4m = 2^{n+1} + 2^{n-1}$ vertices. For instance, in Figures 3 and 8, we show the mixed graph $BDM(2,5)$ and $BDM(2,10)$, respectively. Notice that $BDM(2,5)$ is totally $(1,1)$-regular, but this is not the case for $BDM(2,10)$, which has vertices with in-degree 0 and 2. In fact, this happens for any mixed graph $BDM(2, 2^{n-1} + 2^{n-3})$ with $n \geq 4$ (and thus $m \geq 10$). Indeed, for m even, notice that, from vertices $(0,i)_0$ and $(0, i + m/2)$, there is an arc to vertex $(1,2i)_1$. To overcome this drawback, in Subsection 3.1, we slightly modify the adjacency rules to obtain total $(1,1)$-regularity.

In the following results, we study some of the properties of $BDM(2,m)$.

Lemma 3.1. *Let* $\overline{0} = 1$ *and* $\overline{1} = 0$.

(*i*) *The mapping*

$$\Phi_1 : (\alpha, i)_\beta \to (\alpha, -i-1)_{\overline{\beta}},$$

where $\alpha, \beta \in \mathbb{Z}_2$, *is an involutive automorphism of* $BDM(2,m)$ *that interchanges its independent sets.*

(*ii*) *The mapping*

$$\Phi_2 : (\alpha, i)_\beta \to (\alpha, i + \alpha 2^{n-2} + \overline{\alpha} 2^{n-3})_\beta,$$

is an automorphism of $BDM(2,m)$ *of order* 5.

Proof. (i) Let Γ and Γ^+ denote adjacency through an edge and an arc, respectively. Since Φ_1 is trivially a bijection, we only need to verify that $\Phi_1\Gamma(\boldsymbol{v}) = \Gamma\Phi_1(\boldsymbol{v})$ and $\Phi_1\Gamma^+(\boldsymbol{v}) = \Gamma^+\Phi_1(\boldsymbol{v})$ for every vertex of $BDM(2,m)$:

$$\begin{aligned}\Phi_1\Gamma((\alpha,i)_\beta)) &= \Phi_1((\alpha,i)_{\overline{\beta}}) = (\alpha,-i-1)_\beta,\\ \Gamma\Phi_1((\alpha,i)_\beta)) &= \Gamma((\alpha,-i-1)_{\overline{\beta}}) = (\alpha,-i-1)_\beta.\end{aligned}$$

Now, assuming $\beta = 0$ (the case $\beta = 1$ is analogous), with $\alpha = 0$,

$$\begin{aligned}\Phi_1\Gamma^+((0,i)_0)) &= \Phi_1((1,2i)_1) = (1,-2i-1)_0,\\ \Gamma^+\Phi_1((0,i)_0)) &= \Gamma^+((0,-i-1)_1) = (1,-2i-1)_0,\end{aligned}$$

and, with $\alpha = 1$,

$$\begin{aligned}\Phi_1\Gamma^+((1,i)_0)) &= \Phi_1((0,-2i-1)_1) = (0,2i)_0,\\ \Gamma^+\Phi_1((1,i)_0)) &= \Gamma^+((1,-i-1)_1) = (0,2i)_0.\end{aligned}$$

(ii) We only check the directed adjacencies assuming that $\beta = 1$, with $\alpha = 0$,

$$\begin{aligned}\Phi_2\Gamma^+((0,i)_1)) &= \Phi_2((1,2i+1)_0) = (1,2i+1+2^{n+2})_0,\\ \Gamma^+\Phi_2((0,i)_1)) &= \Gamma_2^+((0,i+2^{n-3})_1) = (1,2i+2^{n-2}+1)_0,\end{aligned}$$

and, with $\alpha = 1$,

$$\begin{aligned}\Phi_2\Gamma^+((1,i)_1)) &= \Phi_2((0,-2i-2)_1)_0) = (0,-2i-2+2^{n-3})_0,\\ \Gamma^+\Phi_2((1,i)_1)) &= \Gamma^+((1,i+2^{n-2})_1) = (0,-2i-2^{n-1}-2)_0 = (0,-2i-2+2^{n-3})_0,\end{aligned}$$

where the last equality holds since $2^{n-3} = -2^{n-1} \pmod m$.

Moreover, since, with $\alpha = 0$, $\Phi_2^r((0,i)_\beta) = (0,i+r2^{n-3})_\beta$, we have that $\Phi_2^r = id$ if and only if $r2^{n-3} = 0 \pmod m$, and $r = 2^2+1 = 5$ is the smallest r satisfying this. The case for $\alpha = 1$ is similar. $\square$

In Figure 3 (left), we show the mixed graph $BDM(2,5)$ drawn according to the symmetries induced by the automorphisms Φ_1 and Φ_2. Notice that $BDM(2,5)$ is totally regular. Moreover, the action of Φ_2 allows us to construct it as a lift of the base graph with voltages on $\mathbb{Z}_5$ shown in Figure 3 (right). Then, its polynomial matrix is (see Dalfó, Fiol, Miller, Ryan, and Širáň [7])

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 & z^2\\ 1 & 0 & 1 & 0\\ 0 & z^2 & 0 & 1\\ z & 0 & 1 & 0\end{pmatrix},$$

which, with $z = e^{ri\frac{2\pi}{5}}$, has the eigenvalues shown in Table 2 and Figure 7.

$\zeta = e^{i\frac{2\pi}{5}}$, $z = \zeta^r$	$\lambda_{r,1}$	$\lambda_{r,2}$	$\lambda_{r,3}$	$\lambda_{r,4}$
$\mathrm{sp}(\boldsymbol{B}(\zeta^0))$	0	0	-2	2
$\mathrm{sp}(\boldsymbol{B}(\zeta^1)) = \mathrm{sp}(\boldsymbol{B}(\zeta^4))$	-0.8266-0.7015i	-0.8266+0.7015i	0.8266-0.7015i	0.8266+0.7015i
$\mathrm{sp}(\boldsymbol{B}(\zeta^2)) = \mathrm{sp}(\boldsymbol{B}(\zeta^4))$	-1.2671-0.5445i	-1.2671+0.5445i	1.2671-0.5445i	1.2671+0.5445i

Table 2: All the eigenvalues of the matrices $\boldsymbol{B}(\zeta^r)$, which yield the eigenvalues of the bipartite $(1,1,6)$-mixed graph $BDM(2,5)$.

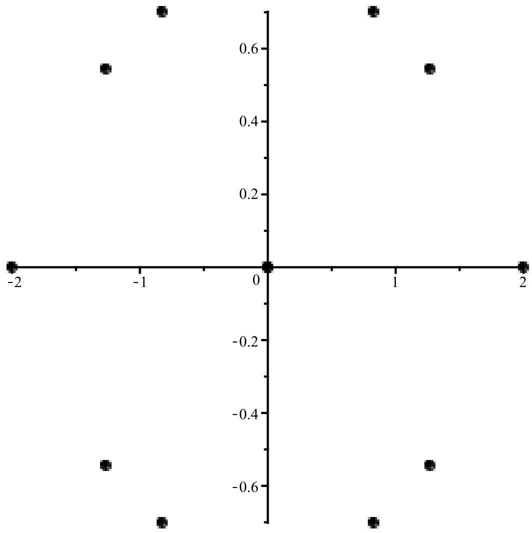

Figure 7: The eigenvalues of the bipartite $(1,1,6)$-mixed graph $BDM(2,5)$ in the complex plane (all of them, excepting ± 2, with multiplicity 2.)

Proposition 3.2. *The diameter of the bipartite* $(1,1,k)$*-mixed graph* $BDM(2,m)$*, with* $m = 2^{n-1} + 2^{n-3}$*, is* $k = 2n$*.*

Proof. First, notice that when we contract all the edges of $BDM(2,m)$, we get a bipartite digraph with a vertex set $V = \mathbb{Z}_2 \times \mathbb{Z}_m = \{(\alpha, i) : \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_m\}$, and adjacencies

$$\begin{aligned}(0,i) &\rightarrow (1,2i),\ (1,2i+1),\\ (1,i) &\rightarrow (0,-2i-1),\ (0,-2i-2).\end{aligned}$$

This is precisely the bipartite digraph $BD(2,m)$ proposed by Fiol and Yebra in [8]. For $m = 2^{n-1} + 2^{n-3}$, it was proved that $BD(2,m)$ has diameter $k = n$, see [8, Th. 4]. Thus, every path of length ℓ in $BD(2,m)$ induces a path of length $\ell' \leq 2\ell + 1$

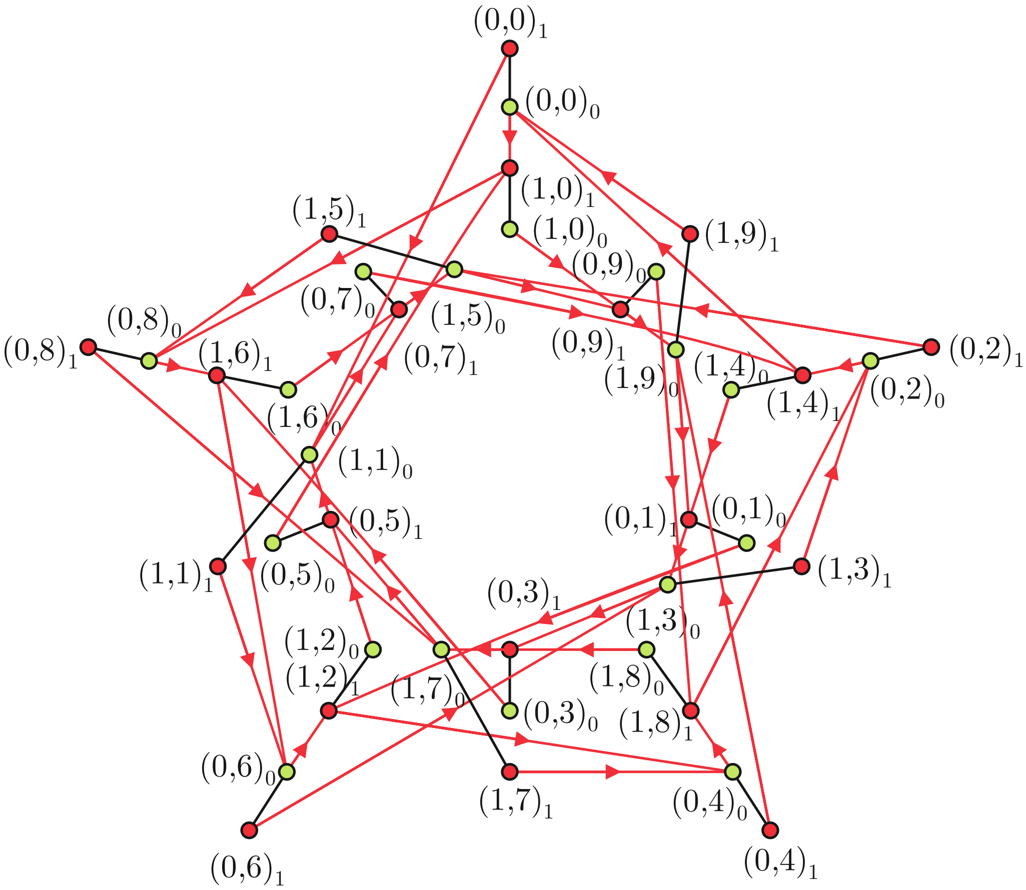

Figure 8: The automorphism Φ_2 acting on the bipartite $(1,1,8)$-mixed graph $BDM(2,10)$.

in $BDM(2,m)$ (because every vertex of the path in $BD(2,m)$ can turn into an edge in the corresponding path of $BDM(2,m)$). Then, starting from a given vertex of $BDM(2,m)$, the worst situation would be to reach a vertex through the path with adjacency pattern $EAEAE \overset{(2n+1)}{\cdots\cdots} AE$ (where E represents an edge, and A an arc). From Lemma 3.1, we only need to check two initial vertices of the same partite set, say $(0,i)_1$ and $(1,i)_1$. In the first column of Tables 3 and 4, we show the vertices $\boldsymbol{v}(n)$ and $\boldsymbol{u}(n)$ reached from such paths of length $2n+1$. To give a general formula, notice that the coefficient of 'i' is clearly a power of 2. Moreover, when we look at the adding terms of the vertices $\boldsymbol{v}(n)$ when $2n+1 = 5, 9, 13, 17, 21, \ldots$ (or the vertices $\boldsymbol{u}(n)$ when $2n+1 = 3, 7, 11, 15, 19, \ldots$) and omitting the signs, we have the sequence $1, 3, 13, 51, 205, 819, \ldots$ which corresponds to A015521 in [16]

$2n+1$	$\boldsymbol{v}(n)$	$\boldsymbol{v}'(n)$
3	$(1,2i)_0$	$(0,-4i-4)_0$
5	$(0,-4i-1)_0$	$(1,-8i-7)_0$
7	$(1,-8i-2)_0$	$(0,16i+13)_0$
9	$(0,16i+3)_0$	$(1,32i+26)_0$
11	$(1,32i+6)_0$	$(0,-64i-53)_0$
13	$(0,-64i-13)_0$	$(1,-128i,-106)_0$
15	$(1,-128i-26)_0$	$(0,256i+211)_0$
17	$(0,256i+51)_0$	$(1,512i+422)_0$
19	$(1,512i+102)_0$	$\cdots$
$\vdots$	$\vdots$	$\vdots$
n even	$(0,\phi(n))_0$	$(1,2\phi(n))_0$
n odd	$(1,2\phi(n-1))_0$	$(0,\phi(n+1))_0$

Table 3: The vertices $\boldsymbol{v}(n)$ reached from $(0,i)_1$ through a path of the form $EAEAE \overset{(2n+1)}{\cdots\cdots} AE$. The vertices $\boldsymbol{v}'(n)$ reached from $(0,i)_1$ through a path of the form $AEAEAEAE \overset{(2n+1)}{\cdots\cdots} AE$.

with general term $a(s) = \frac{1}{5}[4^s - (-1)^s]$ for $s = 1,2,3,\ldots$ Moreover, the adding terms of the vertices $\boldsymbol{v}(n)$ when $2n+1 = 3,7,11,15,19,\ldots$ (or the vertices $\boldsymbol{u}(n)$ when $2n+1 = 1,5,9,13,17,\ldots$), omitting again the signs, are $0,2,6,26,102,\ldots$, that is, $2a(s)$ for $s = 0,1,2,\ldots$. All this leads to the formulas shown in the tables, where the function $\phi(n)$ is given in (5). In order to show that the diameter of $BDM(2,m)$ is $k = 2n$, in the second column of Tables 3 and 4, there are the vertices $\boldsymbol{v}'(n)$ and $\boldsymbol{u}'(n)$ reached from the path of the form $AEAEAAEAE \overset{(2n+1)}{\cdots\cdots} AE$ and $AAEAEA \overset{(2n+1)}{\cdots\cdots} AE$, respectively, for $n \geq 2$. In this case, the general expression is also in the tables, with the function $\psi(n)$ given in (6). From these data, we observe that the above vertices $\boldsymbol{v}(n)$ and $\boldsymbol{u}'(n)$ can be reached, in fact, with a path of length at most $2n+1$. More precisely, when we compute the term modulo $m = 2^{n-1}+2^{n-3}$, we get

$$\boldsymbol{v}(n) = \boldsymbol{v}'(n-1) \quad \text{and} \quad \boldsymbol{u}(n) = \boldsymbol{u}'(n-3).$$

Consequently, the diameter of $BDM(2,m)$ must be $k \leq 2n$. Finally, the equality follows from the fact that, as commented, the digraph $BD(2,m)$, with $m = 2^{n-1} + 2^{n-3}$ has diameter $k = n$.

$$\phi(n) = (-1)^{\frac{n}{2}} \left[2^n i + \frac{1}{5}\left(2^n - (-1)^{\frac{n}{2}}\right)\right] \pmod{m}, \tag{5}$$

$$\psi(n) = (-1)^{\frac{n+1}{2}} \left[2^n i + \frac{1}{5}\left(2^{n+1} - (-1)^{\frac{n+1}{2}}\right)\right] \pmod{m}. \tag{6}$$

$2n+1$	$\boldsymbol{u}(n)$	$\boldsymbol{u}'(n)$
1	$\sim (1,i)_0$	$\rightarrow (0,-2i-2)_0$
3	$(0,-2i-1)_0$	$(1,-4i-4)_0$
5	$(1,-4i-2)_0$	$(0,-8i+7)_0$
7	$(0,8i+3)_0$	$(1,16i+14)_0$
9	$(1,16i+6)_0$	$(0,-32i-29)_0$
11	$(0,-32i-13)_0$	$(1,-64i-58)_0$
13	$(1,-64i-26)_0$	$(0,128i+115)_0$
15	$(0,128i+51)_0$	$\cdots$
17	$(1,256i+102)_0$	$\cdots$
19	$(0,-512i-205)_0$	$\cdots$
$\vdots$	$\vdots$	$\vdots$
n odd	$(0,\psi(n))_0$	$(0,\psi(n+3))_0$
n even	$(1,2\psi(n-1))_0$	$(1,2\psi(n+2))_0$

Table 4: The vertices $\boldsymbol{u}(n)$ reached from $(1,i)_1$ through a path of the form $EAEAE \overset{(2n+1)}{\cdots\cdots} AE$. The vertices $\boldsymbol{u}'(n)$ reached from $(1,i)_1$ through a path of the form A (for $n=0$), and $AAEAEA \overset{(2n+1)}{\cdots\cdots} AE$ $(n \geq 1)$.

This completes the proof. $\square$

To illustrate the situation of this proof, let us consider the case of the mixed graph $BDM(2,5)$ of Figure 3, the path of length $2n+1=7$ starting from $(0,i)_1$ is as follows:

$$\begin{aligned}(0,i)_1 \;\sim\; (0,i)_0 \;\rightarrow\; (1,2i)_1 \;\sim\; (1,2i)_0 \;\rightarrow\; (0,-4i-1)_1 \\ \sim\; (0,-4i-1)_0 \;\rightarrow\; (1,-8i-2)_1 \;\sim\; (1,-8i-2)_0,\end{aligned}$$

whereas the vertex $(1,-8i-2)_0$ can be reached through the following path of length 5:

$$\begin{aligned}(0,i)_1 \;\rightarrow\; (1,2i+1)_0 \;\sim\; (1,2i+1)_1 \;\rightarrow\; (0,-4i-4)_0 \\ \sim\; (0,-4i-4)_1 \;\rightarrow\; (1,-8i-7)_0 = (1,-8i-2)_0.\end{aligned}$$

Similarly, the path of length $2n+1=7$ from $(1,i)_1$ is

$$\begin{aligned}(1,i)_1 \;\sim\; (1,i)_0 \;\rightarrow\; (0,-2i-1)_1 \;\sim\; (0,-2i-1)_0 \;\rightarrow\; (1,-4i-2)_1 \\ \sim\; (1,-4i-2)_0 \;\rightarrow\; (0,8i+3)_1 \;\sim\; (0,8i+3)_0,\end{aligned}$$

whereas the vertex $(0,8i+3)_0$ is, in fact, at distance 1 from $(1,i)_1$:

$$(1,i)_1 \;\rightarrow\; (0,-2i-2)_0 = (0,8i+3)_0.$$

As a consequence of Proposition 3.2, and in comparison with the order of the corresponding Moore bound $MD(k) \sim 1.61803^k$, we get the following result.

Corollary 3.3. *The bipartite* $(1,1,k)$*-mixed graph* $BDM(2,m)$ *has number of vertices of the order of* $2^{k/2} \approx 1.4142^k$*.*

3.1 Totally regular bipartite $(1,1,k)$-mixed graphs

As commented above, the mixed graph $BDM(2,m)$ is not totally regular when $m \geq 10$. This subsection slightly modifies the adjacency conditions to ensure total $(1,1)$-regularity. The bipartite mixed graph $BDM^*(2,m)$ has the same vertex set and undirected adjacencies as $BDM(2,m)$, whereas the arcs are now (all arithmetic modulo $m = 2^{n-1} + 2^{n-3}$, with $n > 3$).

- If $i \in [0, m/2-1]$ (also as in $BDM(2,m)$):

$$(0,i)_0 \rightarrow (1,2i)_1, \quad (1,i)_0 \rightarrow (0,2i+1)_1,$$
$$(1,i)_0 \rightarrow (0,-2i-1)_1, \quad (1,i)_1 \rightarrow (0,-2i-2)_1;$$

- If $i \in [m/2, m-1]$:

$$(0,i)_0 \rightarrow (1,2i+1)_1, \quad (1,i)_0 \rightarrow (0,2i)_1$$
$$(1,i)_0 \rightarrow (0,-2i-2)_1, \quad (1,i)_1 \rightarrow (0,-2i-1)_1.$$

Lemma 3.4. *The bipartite graph* $BDM^*(2,m)$ *is a totally regular* $(1,1,k)$*-mixed graph on* $N = 4m = 2^{n+1} + 2^{n-1}$ *vertices.*

Proof. We only need to prove that every vertex $(\alpha, i)_\beta$ has in-degree one.

- If $\alpha = \beta = 0$: When i is odd, vertex $(0,i)_0$ is adjacent from $(1, \frac{-i-1}{2})_1$; and when i is even, vertex $(0,i)_0$ is adjacent from $(1, \frac{-i-2-m}{2})_1$.
- If $\alpha = 0$, $\beta = 1$: When i is odd, vertex $(0,i)_1$ is adjacent from $(1, \frac{-i-1-m}{2})_0$; and when i is even, vertex $(0,i)_1$ is adjacent from $(1, \frac{-i-2}{2})_1$.
- If $\alpha = 1$, $\beta = 0$: When i is odd, vertex $(1,i)_0$ is adjacent from $(0, \frac{i-1}{2})_1$; and when i is even, vertex $(1,i)_0$ is adjacent from $(0, \frac{i+m}{2})_1$.
- If $\alpha = \beta = 1$: When i is odd, vertex $(1,i)_1$ is adjacent from $(0, \frac{i-1+m}{2})_0$; and when i is even, vertex $(1,i)_1$ is adjacent from $(0, \frac{i}{2})_0$.

□

As in the case of $BDM(2,m)$, when we contract all the edges of $BDM^*(2,m)$, we obtain the bipartite digraph $BD(2,m)$, for $m = 2^{n-1}+2^{n-3}$, with diameter $k = n$ (see again [8]). Thus, reasoning as in the proof of Proposition 3.2, the diameter of $BDM^*(2,m)$ is $k \leq 2n+1$. Although, in this case, we are not able to prove that $k = 2n+1$, Corollary 3.3 also applies, and the number of vertices of $BDM^*(2,m)$ is also of the order of $\sqrt{2}^k$.

4 Chordal ring mixed graphs

In this section, we present a family of bipartite $(1,1,k)$-mixed graphs that are related to tessellations of the plane (see Yebra, Fiol, Morillo, and Alegre [18]). Let $n \geq 2$ and $c < n$ be, respectively, even and odd numbers. The chordal ring mixed graph $CRM(n,c)$ is a mixed graph with vertex set $V = \mathbb{Z}_n$ (all arithmetic will be modulo n), with arcs $i \to i+1$ (forming a directed cycle) and edges $i \sim i+c$ if i is odd (these are the 'chords'). Given the diameter k, we want to find the value of c such that the graph $CRM(n,c)$ has the maximum number of vertices. Arden and Lee studied this problem [1], and Yebra, Fiol, Morillo, and Alegre [18] in the case of undirected graphs, which were called 'chordal ring networks'. In fact, such graphs were already studied in another context by Coxeter [4].

In our case of $(1,1,k)$-mixed graphs, the following result gives a Moore-like bound for their number of vertices.

Lemma 4.1. *The maximum number of vertices of a $CRM(n,c)$ of a bipartite chordal ring mixed graph with diameter k is*

$$\begin{cases} \frac{1}{2}(k+1)^2 & \text{if } k \text{ is odd,} \\ \frac{1}{2}k(k+2) & \text{if } k \text{ is even.} \end{cases} \tag{7}$$

Proof. From the adjacency conditions, we observe that there are at most $d+1$ vertices at distance $d \geq 0$ from vertex 0 (that is, $\{0\}$, $\{-c,1\}$, $\{-c+1,2,1+c\}$, $\{-2c+2,-c+2,3,2+c\}$, ...). Then, if the mixed graph is bipartite with odd diameter, the maximum number of vertices is bounded above by twice the number of even vertices of a $CRM(n,c)$, which is equal to $2(1+3+\cdots+k) = \frac{1}{2}(k+1)^2$. Similarly, if the diameter is even, we have $2(2+4+\cdots+k = \frac{1}{2}k(k+2)$. □

Next, we show that the upper bound in (7) can be attained if k is odd, but not when k is even.

Theorem 4.2. (a) *If k is odd, $k = 2\ell - 1$ with $\ell \geq 2$, there exists a chordal ring mixed graph $CRM(n, c)$ with diameter k, order $n = 2\ell^2 = \frac{1}{2}(k+1)^2$, and chordal length $c = 2\ell - 1 = k$.*

(b) *If k is even, with $k \equiv 0 \pmod 4$, $k = 2\ell = 4t$ with $t \geq 1$, there exists a chordal ring mixed graph $CRM(n, c)$ with diameter k, order $n = 8t^2 + 2 = \frac{1}{2}k^2 + 2$, and chordal length $c = (2t-1)^2 + 2t = (\frac{k}{2} - 1)^2 + \frac{k}{2}$.*

$(c1)$ *If k is even, $k \equiv 6 \pmod 8$, $k = 2\ell = 8t - 2$ with $t \geq 1$, there exists a chordal ring mixed graph $CRM(n, c)$ with diameter k, order $n = 2(4t-3)(4t-4)+4 = k(\frac{k}{2} - 1) + 4$, and chordal length $c = 8t^2 - 8t + 3 = \frac{1}{8}(k+2)^2 - k + 1$.*

$(c2)$ *If k is even, $k \equiv 2 \pmod 8$, $k = 2\ell = 8t - 6$ with $t \geq 1$, there exists a chordal ring mixed graph $CRM(n, c)$ with diameter k, order $n = 2(4t-3)(4t-4)+4 = k(\frac{k}{2} - 1) + 4$, and chordal length $c = 24t^2 - 44t + 23 = \frac{3}{8}(k+6)^2 - \frac{11}{2}k - 10$.*

Proof. Consider the plane divided into unit squares and divide each square by its anti-diagonal line, forming two right triangles. Associate to each odd vertex i and even vertex $i + c$ (forming an edge) an upper and its adjacent lower triangle. The vertices $i \pm 1$ adjacent to and from i are represented with the adjacent triangles, as shown in Figure 10. Following this procedure, the vertices can be arranged in a planar pattern, as shown in the same figure. Notice that, when every triangle of the regular tessellation of the plane receives a number modulo n according to the adjacency rules, the distribution of these numbers in the plane repeats itself periodically. This fact is illustrated again in Figure 10 for the graph $CRM(32, 7)$. Moreover, n triangles form every tile, and they **periodically tessellate the plane.** Stated in this context, our concern is to find and 'construct' a given tile (that is, find an integer c that generates it) that tessellates the plane and has a maximum area (or number of unit triangles) for a given diameter k.
A more precise approach considers that the automorphism group of $CRM(n, c)$ has two orbits constituted by the even and odd vertices. Consequently, we must compute the number of vertices from, say, 0 and $-c$. This is shown in Figure 9 and Table 5, where we indicate the vertices whose maximum distance to 0 or $-c$ is $1, 2, \ldots, 8$. If we sum up odd diameters, we obtain the bounds in (7) again.

(a) To show that for odd diameter k the bound on the order is attained, the appropriate tile is a square of length ℓ, as it is shown for $k = 7$ in Figures 9 (left shaded area) and Figure 10, with $n = 2\ell^2$ triangles (vertices) and diameter $k = 2\ell - 1$, together with its tessellation.

It remains to show that a suitable choice of c can generate such a tile. For this, note that these values produce the given periodic pattern, which is characterized by the position of the 'zeros'. To obtain this distribution, we have to express the null

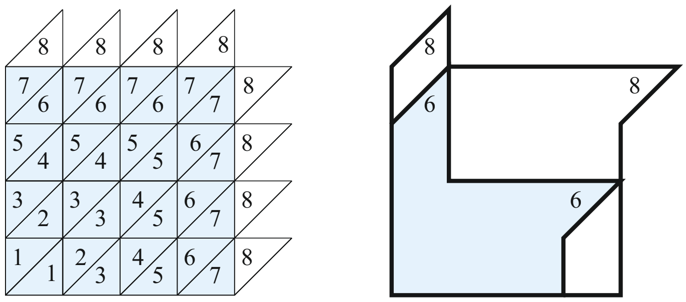

Figure 9: Vertices at a given distance $1, 2, \ldots, 8$ from 0 or $-c$, and optimal tiles.

$\max - \text{dist}$	vertices								
1				$-c+1$	1				
2				$-c+2$	2				
3			$-2c+2$	$-c+3$	3	$2+c$			
4			$-2c+3$	$-c+4$	4	$3+c$			
5		$-3c+3$	$-2c+4$	$-c+5$	5	$4+c$	$3+2c$		
6		$-3c+4$	$-2c+5$	$-c+6$	6	$5+c$	$4+2c$		
7	$-4c+4$	$-3c+5$	$-2c+6$	$-c+7$	7	$6+c$	$5+2c$	$4+3c$	
8	$-4c+5$	$-3c+6$	$-2c+7$	$-c+8$	8	$7+c$	$6+2c$	$5+3c$	

Table 5: The vertices i of $CRM(n,c)$ such that $\max\{\text{dist}(0,i), \text{dist}(-c,i)\}$ is $\max - \text{dist} = 1, 2, \ldots, 8$.

effect of translations along two linearly independent vectors (each of them associated with a path as shown in Figure 10) that generate the pattern. Choosing them as in the figure, c must satisfy

$$\begin{aligned} \ell + \ell c &\equiv 0 \pmod{2\ell^2} \quad \text{(horizontal path)}, \\ 2\ell - 1 - c &\equiv 0 \pmod{2\ell^2} \quad \text{(diagonal path)}, \end{aligned}$$

with trivial solution $c = 2\ell - 1$, as claimed.

(b) When the diameter is even, of the form $k = 2\ell = 4t$, see, for instance, Figure 9 (left) for $k = 8$, the optimal tiles clearly do not tessellate because of the bordering triangles. The best we can do is to remove all such triangles except two. This results in the tile of Figure 9 (on the right), with order $n = 2\ell^2 + 2$, or its equivalent (because of periodicity) L-shaped tile of Figure 11.

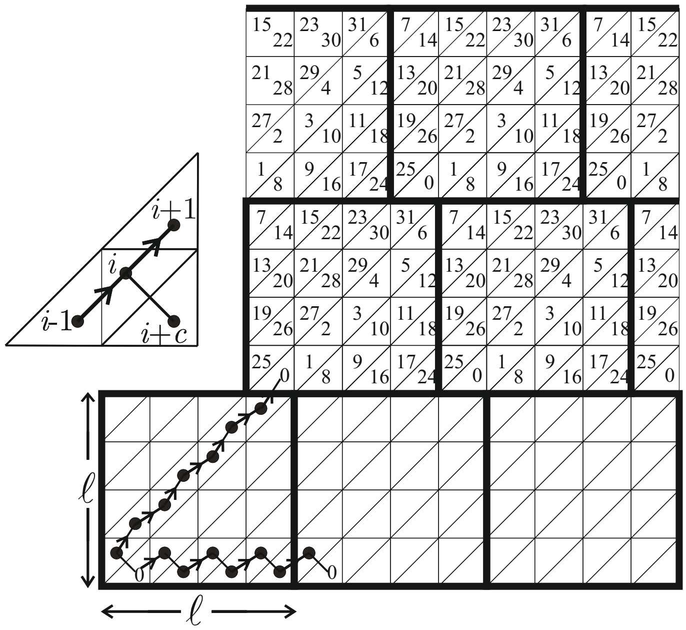

Figure 10: The plane tessellation corresponding to the chordal ring mixed graph $CRM(32,7)$, with diameter $k=7$.

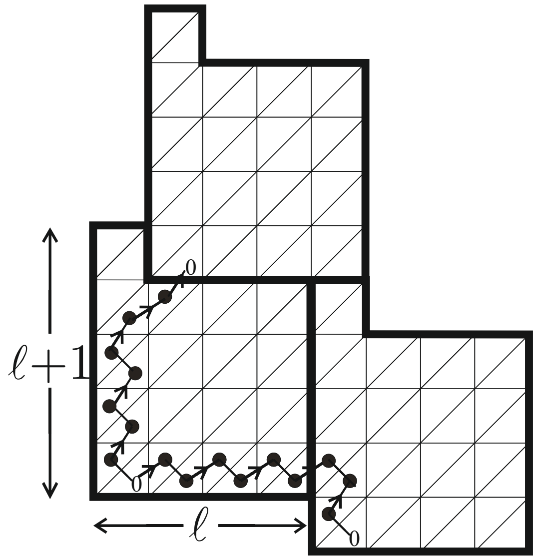

Figure 11: The plane tessellation corresponding to the chordal ring mixed graph $CRM(34,13)$, with diameter $k=8$.

Then, the equations to obtain the right value of c are

$$(\ell-1)+(\ell+1)c \equiv 0 \pmod{n},$$
$$(\ell+1)-(\ell-1)c \equiv 0 \pmod{n},$$

or, in matrix form,

$$\begin{pmatrix} \ell-1 & \ell+1 \\ \ell+1 & -\ell+1 \end{pmatrix} \begin{pmatrix} 1 \\ c \end{pmatrix} = n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Solving the system (notice that the determinant of the 2×2 matrix is $0 \pmod{n}$), we have

$$1 = (\ell-1)\alpha + (\ell+1)\beta = (2t-1)\alpha + (2t+1)c,$$
$$c = (\ell+1)\alpha - (\ell-1)\beta = (2t+1)\alpha - (2t-1)\beta.$$

Since c must be an (odd) integer, a solution is obtained by taking $\alpha = t$ and $\beta = -t+1$, so that the first equation holds and $c = (2t-1)^2 + 2t$, as claimed.

(c) When the diameter is even, of the form $k = 2\ell = 8t-2$ (case ($c1$)) or $k = 2\ell = 8t-6$ (case ($c2$)), we could use, in principle, the same tile as in (b). However, the corresponding tessellation yields no solution with 'step' 1, and, hence, we do not obtain a proper chordal ring mixed graph (see the remark after this proof). Then, the best solution is the right shaded tile shown in Figure 9 for $k=6$ (or the corresponding tile for $k=10$ in Figure 12) with order $n = 2\ell(\ell-1)+4$.

Then, in both cases, the equations to obtain the right value of c are

$$\begin{pmatrix} \ell-2 & \ell+2 \\ \ell & -\ell+2 \end{pmatrix} \begin{pmatrix} 1 \\ c \end{pmatrix} = n \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 1 = (\ell-2)\alpha + (\ell+2)\beta, \\ c = \ell\alpha + (-\ell+2)\beta. \end{cases}$$

In case ($c1$), we get $\alpha = t$ and $\beta = -t+1$; whereas in the case ($c2$), we have $\alpha = 3t-1$ and $\beta = -3t+4$. Then, depending on the values $\ell = \frac{1}{2}(8t-2)$ or $\ell = \frac{1}{2}(8t-6)$, we obtain the solutions $c = 8t^2 - 87 + 3$ and $c = 24t^2 - 44 + 23$. $\square$

Remark 4.3. As commented above, when k is even but not a multiple of 4, the tessellation with the optimal tile of Figure 11 does not give a directed ring with n vertices. Instead, we obtain two directed rings on $m = n/2$ vertices each, which we call a *chordal double ring mixed graph* $CDRM(n,c)$. Then, we work on the group $\mathbb{Z}_2 \times \mathbb{Z}_m$, and vertex (α, i) is adjacent to $(\alpha+1, i)$ (though an edge) and vertex $(\alpha, i+1)$ (through an arc). Thus, when k is an even number of the form $k = 4t+2$, the tile of Figure 11 with $\ell = 2t$ and order $n = 2\ell^2+2$ corresponds to a $CRM(n,c)$ with $c = (\frac{k}{2}-1)^2 + \frac{k}{2}$ (as in Theorem 4.2(b)). For instance, in the case $k=6$, Figure 13 shows the chordal ring mixed graph $CDRM(20,3)$ and its corresponding tessellation.

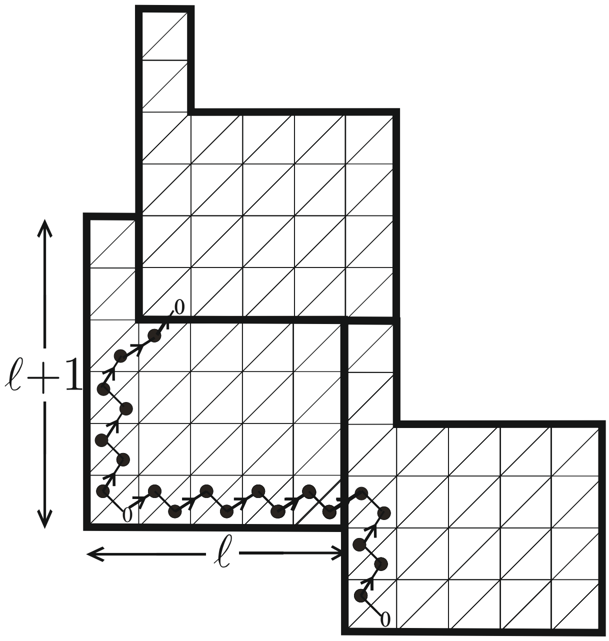

Figure 12: The plane tessellation corresponding to the chordal ring mixed graph $CRM(44, 31)$, with diameter $k = 10$.

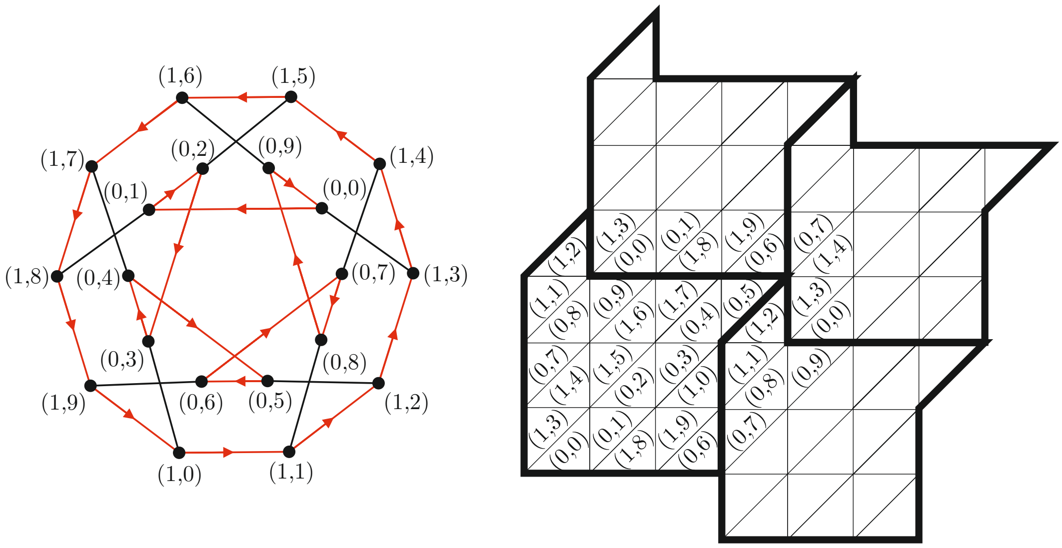

Figure 13: The chordal double ring mixed graph $CDRM(20, 7)$, with diameter $k = 6$, and its plane tessellation.

In Table 6, we show the values of the obtained chordal ring mixed graphs, given by Theorem 4.2, together with the upper bounds on the order of $CRM(n, c)$ and those from Theorem 5.1. For example, notice that the last maximal $(1, 1, 5)$-mixed graph of Figure 4 corresponds to the chordal ring mixed graph $CRM(18, 5)$.

k	n	c	Th. 4.2	max order $CRM(n, c)$	Th. 5.1
3	8	3	(a)	8	8
4	10	3	(b)	12	12
5	18	5	(a)	18	22
6	16	3	$(c1)$	24	36
7	32	7	(a)	32	60
8	34	13	(b)	40	96
9	50	9	(a)	50	158
10	44	31	$(c2)$	60	256
11	72	11	(a)	72	416
12	74	31	(b)	84	674
13	98	13	(a)	98	1092
14	88	19	$(c1)$	112	1766
15	128	15	(a)	128	2860
16	139	57	(b)	144	4628
17	162	17	(a)	162	7490
18	148	107	$(c2)$	180	12120
19	200	19	(a)	200	19612
20	202	91	(b)	220	31732
21	242	21	(a)	242	51346
22	224	51	$(c1)$	264	83080

Table 6: Bounds for the chordal rings mixed graph $CRM(n, c)$.

5 Upper bounds

This section gives an upper bound that improves the Moore bound for bipartite mixed graphs. We concentrate on our case of $r = z = 1$. The first step is to draw the Moore tree of depth k for $r = z = 1$, starting at level 0. When a vertex at level i has both an undirected edge and a directed arc to child vertices at level $i + 1$, we draw the undirected neighbour to the left of the directed out-neighbour. See the Moore tree for diameter $k = 5$ in Figure 14.

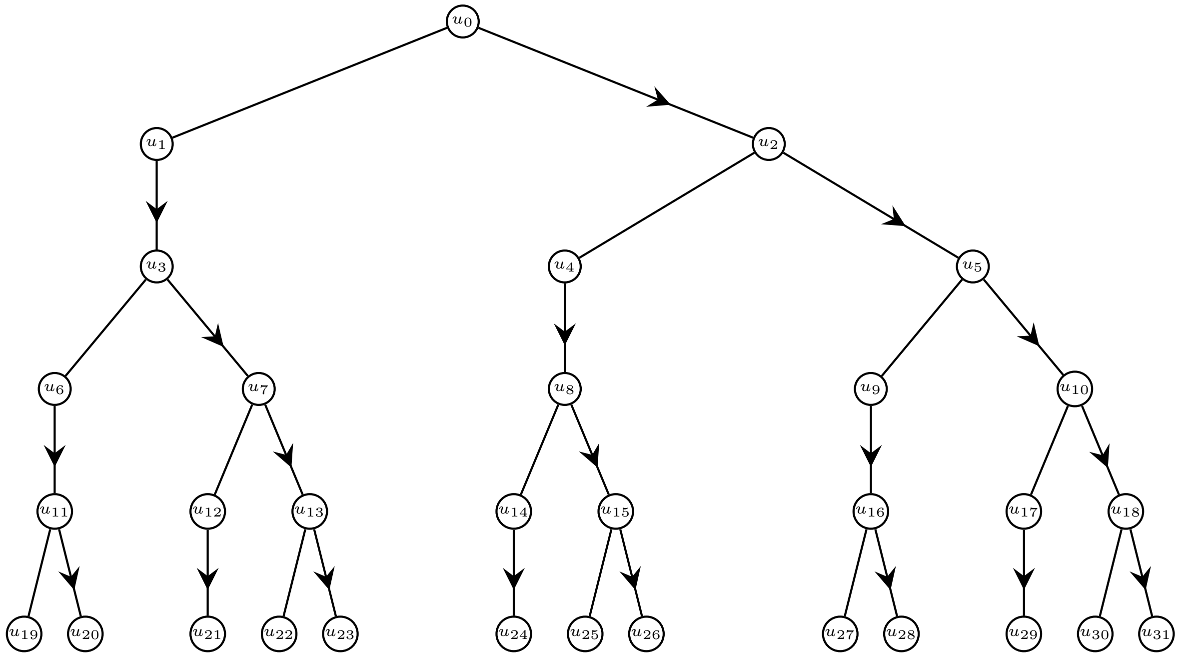

Figure 14: The Moore tree for $r = z = 1$ and $k = 5$.

We call a position in the Moore tree an *arrow vertex* if it lies in level i in the undirected branch of the tree for some $2 \leq i \leq k-1$ and arises as the endpoint of a directed arc from level $i-1$. Observe that if u is an arrow vertex, then both elements of $N^-(u)$ occur in the undirected branch. As the directed out-neighbour of the root of the tree must reach u by a mixed path of length at most k, at least one of these vertices of $N^-(u)$ must also occur in the directed branch of the tree, therefore contributing towards the defect of the graph. A complication is the fact that the in-neighbourhoods of arrow vertices can overlap. Hence, we require the smallest *transversal* of the in-neighbourhoods of the vertices in the undirected branch. We, therefore, partition the set of the union of these in-neighbourhoods in a convenient way. If v is a vertex at level i for $1 \leq i \leq k-2$, then we denote by $v^{\rightarrow}$ the undirected neighbour of the directed out-neighbour of v, which appears at level $i+2$ of the undirected branch.

By iterating this '$\rightarrow$' procedure, we obtain a chain of vertices that terminates at level $k-1$ or k. This chain is maximal if and only if it initiates at an arrow vertex or the undirected neighbour of the tree's root. These maximal chains are disjoint, and if there are t vertices contained in the chain, then the smallest number of vertices from the chain that must also appear in the directed branch is given by $\lceil \frac{t}{3} \rceil$ (this is the domination number of the path P_t).

Now, we show how this argument can be applied to improve the bound for bipartite mixed graphs. We first focus on the case of even diameter $k = 2\kappa$. Consider maximal chains that occupy odd levels of the tree (for example, the chain beginning at the undirected neighbour of the root of the tree has vertices at levels $1, 3, 5, \dots, k-1$); we call this an *odd level chain.* Recall that the Moore bound for bipartite mixed graphs is twice the number of vertices on odd levels of the tree. By the preceding argument, the directed branch must contain the vertices from a transversal of the undirected branch, but by bipartiteness, each of the vertices of a transversal of an odd level chain must also lie on odd levels.

Tuite and Erskine [17] showed that there are $\eta_t = \frac{1}{2^{t-1}\sqrt{5}}((1+\sqrt{5})^{t-1}-(1-\sqrt{5})^{t-1})$ chains that start at level t if $2 \leq t \leq k-2$, and $\eta_1 = 1$. Counting over the odd level chains (and replacing the summation index t by $2t-1$), we conclude that at least

$$\sum_{t=1}^{\kappa-1} \eta_{2t-1} \left\lceil \frac{\kappa - t + 1}{3} \right\rceil$$
$$= \left\lceil \frac{\kappa}{3} \right\rceil + \sum_{t=2}^{\kappa-1} \frac{1}{2^{2t-2}\sqrt{5}} \left((1+\sqrt{5})^{2t-2} - (1-\sqrt{5})^{2t-2}\right) \left\lceil \frac{\kappa - t + 1}{3} \right\rceil$$

vertices are repeated on the odd levels. As this applies to both partite sets, the defect is at least twice this figure.

By considering even level chains, we obtain the analogous conclusion for odd-diameter graphs.

Theorem 5.1. *The order of a totally regular bipartite graph with undirected degree $r = 1$, directed degree $z = 1$, and diameter $k = 2\kappa$ is at most*

$$M_B(1,1,2\kappa) - 2\left\lceil \frac{\kappa}{3} \right\rceil - 2\sum_{t=2}^{\kappa-1} \frac{1}{2^{2t-2}\sqrt{5}} \left((1+\sqrt{5})^{2t-2} - (1-\sqrt{5})^{2t-2}\right) \left\lceil \frac{\kappa - t + 1}{3} \right\rceil,$$

for $\kappa \geq 3$. For $\kappa = 2$, the order is at most $M_B(1,1,2\kappa) - 2\left\lceil \frac{\kappa}{3} \right\rceil$. If $k = 2\kappa + 1$, then

$$M_B(1,1,2\kappa+1) - 2\sum_{t=1}^{\kappa-1} \frac{1}{2^{2t-1}\sqrt{5}} \left((1+\sqrt{5})^{2t-1} - (1-\sqrt{5})^{2t-1}\right) \left\lceil \frac{\kappa - t + 1}{3} \right\rceil,$$

for $\kappa \geq 2$.

For diameter $k = 4$, this gives an upper bound of 12, and for $k = 6$, an upper bound of 36. Moreover, for diameter $k = 5$, our result gives the upper bound 22. Except for this case, for which we know that the tight bound is, in fact, 18, this theorem gives all the other upper bounds in Table 6.

Acknowledgments

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