# STABILIZATION OF LINEAR PORT-HAMILTONIAN DESCRIPTOR SYSTEMS VIA OUTPUT FEEDBACK

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**Abstract.** The structure preserving stabilization of (possibly non-regular) linear port-Hamiltonian descriptor (pHDAE) systems by output feedback is discussed. While for general descriptor systems the characterization when there exist output feedbacks that lead to an asymptotically stable closed loop system is a very hard and partially open problem, for systems in pHDAE representation this problem can be completely solved. Necessary and sufficient conditions are presented that guarantee that there exist a proportional output feedback such that the resulting closed-loop port-Hamiltonian descriptor system is (robustly) asymptotically stable. For this it is also necessary that the output feedback also makes the problem regular and of index at most one. A complete characterization when this is possible is presented as well.

**Keywords:** Port-Hamiltonian descriptor system, proportional output feedback, regularization, index reduction, asymptotic stability, derivative output feedback.

**AMS subject classification:** 93B05, 93B40, 93B52, 65F35

**1. Introduction.** In this paper we study proportional output feedback controls to make a descriptor system, often called differential-algebraic system (DAE) asymptotically stable. Consider a *general descriptor systems* of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu, \ x(t_0) = x_0 \\ y &= Cx + Du, \end{aligned} \tag{1.1}$$

with $E, A \in \mathbb{C}^{\ell,n}$, $B \in \mathbb{C}^{\ell,m}$, $C \in \mathbb{C}^{p,n}$ $D \in \mathbb{C}^{m,m}$. Here $\mathbb{C}^{p,n}$ denotes the complex $p \times n$ matrices, $u$ is the input, $y$ is the output and $x$ is the generalized state (descriptor) vector, and $\dot{x}$ denotes the time derivative.

We formulate our results for complex systems but the results hold analogously for systems with real coefficients. In the following, the real part of a complex number $z$ is denote by $\Re(z)$ and we denote that a Hermitian matrix $M$ is positive semidefinite (positive definite) by $M \geq 0$ ($M > 0$).

In our analysis and in the construction of feedbacks, we need to perform system transformations of the system. For general descriptor systems, these are changes of bases $x = T\tilde{x}$, $u = V\tilde{u}$, $y = Y\tilde{y}$ and multiplications of the state equation by $S$, where the matrices $S, T, V, Y$ are invertible.

For the matrix pencil $\lambda E - A$ associated with general descriptor systems of the form (1.1) one has the following classical result [16].

THEOREM 1.1. *Let $E, A \in \mathbb{C}^{\ell,n}$. Then there exist nonsingular matrices $S \in \mathbb{C}^{\ell,\ell}$ and $T \in \mathbb{C}^{n,n}$ such that*

$$S(\lambda E - A)T = \operatorname{diag}(\mathcal{L}_{\epsilon_1}, \ldots, \mathcal{L}_{\epsilon_p}, \mathcal{L}_{\eta_1}^\top, \ldots, \mathcal{L}_{\eta_q}^\top, \mathcal{J}_{\rho_1}^{\lambda_1}, \ldots, \mathcal{J}_{\rho_r}^{\lambda_r}, \mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_s}), \tag{1.2}$$

*where the block entries have the following properties:*

(i) *Every entry $\mathcal{L}_{\epsilon_j}$ is a bidiagonal block of size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$, of the form*

$$\lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.$$

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(ii) *Every entry $\mathcal{L}_{\eta_j}^{\top}$ is a bidiagonal block of size $(\eta_j+1)\times\eta_j$, $\eta_j\in\mathbb{N}_0$, of the form*

$$\lambda\begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}.$$

(iii) *Every entry $\mathcal{J}_{\rho_j}^{\lambda_j}$ is a Jordan block of size $\rho_j\times\rho_j$, $\rho_j\in\mathbb{N}$, $\lambda_j\in\mathbb{C}$, of the form*

$$\lambda\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

(iv) *Every entry $\mathcal{N}_{\sigma_j}$ is a nilpotent block of size $\sigma_j\times\sigma_j$, $\sigma_j\in\mathbb{N}$, of the form*

$$\lambda\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

*The Kronecker canonical form is unique up to permutation of the blocks.*

For real matrices there exists a real Kronecker canonical form which is obtained under real transformation matrices $S,T$. Here, the blocks $\mathcal{J}_{\rho_j}^{\lambda_j}$ with nonreal $\lambda_j$ are in real Jordan canonical form instead, but the other blocks are as in the complex case.

A value $\lambda_0\in\mathbb{C}$ is called a (finite) eigenvalue of $\lambda E-A$ if $\operatorname{rank}(\lambda_0 E-A)<\max_{\alpha\in\mathbb{C}}\operatorname{rank}(\alpha E-A)$. Furthermore, $\lambda_0=\infty$ is said to be an eigenvalue of $\lambda E-A$ if zero is an eigenvalue of $\lambda A-E$. The size of the largest block $\mathcal{N}_{\sigma_j}$ is called the *index* $\nu$ of the pencil $\lambda E-A$, where, by convention, $\nu=0$ if $E$ is invertible. The matrix pencil $\lambda E-A$ is called *regular* if $\ell=n$ and $\det(\lambda_0 E-A)\neq 0$ for some $\lambda_0\in\mathbb{C}$, otherwise it is called *singular*. For a given input $u$, an initial condition $x_0$ is called *consistent* if the initail value problem has at least one classical solution.

When descriptor systems are generated in an automated modularized modeling framework such as e.g. [15], then the resulting system typically is an over- or underdetermined (singular) system. For such singular systems, existence and uniqueness of the solutions for a given control input and given consistent initial values $x(t_0)=x_0$ only be guaranteed except if $E,A$ are square and the pencil $\lambda E-A$ is *regular*. If this is not the case then a regularization or reformulation is necessary, see [10, 23]. In control design this is often done via state or output feedback, see e.g. [9, 13]. Feedback design is also used classically to make the system (robustly) asymptotically stable [22, 37]. However, to do this with output feedback is a difficult and partially open problem even if $E=I$, the identity matrix, see e.g. [8, 34].

Note that the definition of stability and asymptotic stability is not defined in a uniform way in the literature. Some authors just require that the finite eigenvalues of $\lambda E-A$ are in the (open) left half plane, some require that the pencil $\lambda E-A$ is furthermore regular and of index at most one, since otherwise arbitrary small perturbations make the system unstable, see [14, 24, 29] for detailed discussions, which also include the robustness question when the pencil $\lambda E - A$ is close to singular or high index.

In this paper we address the problem of determining output feedback controls $u = Fy$ with $F \in \mathbb{C}^{m,p}$ that make the closed loop system regular and of index at most one, i.e. uniquely solvable for consistent initial conditions, and also (robustly) asymptotically stable. We study this problem for the important class of port-Hamiltonian descriptor system representations that are introduced in the next subsection.

**1.1. Port-Hamiltonian descriptor systems.** In this subsection we introduce the framework of port-Hamiltonian descriptor systems.

DEFINITION 1.2. *A linear time-invariant descriptor system of the form*

$$\begin{aligned} E\dot{x} &= (J-R)Qx + (B-P)u, \\ y &= (B+P)^H Qx + (S-N)u, \end{aligned} \tag{1.3}$$

*with* $E, Q \in \mathbb{C}^{\ell,n}$*,* $J, R \in \mathbb{C}^{\ell,\ell}$*,* $B, P \in \mathbb{C}^{\ell,m}$*,* $S = S^H, N = -N^H \in \mathbb{C}^{m,m}$ *is called* port-Hamiltonian differential-algebraic (pHDAE) system with quadratic nonnegative Hamiltonian

$$\mathcal{H}(x) := \frac{1}{2}\Re(x^H Q^H E x) \geq 0 \tag{1.4}$$

*if the following properties are satisfied:*

*(i)* $0 \leq Q^H E = E^H Q \in \mathbb{C}^{n,n}$ *and* $0 = \Re(Q^H(J - J^H)Q)$*;*

*(ii) the* dissipation matrix

$$W = \begin{bmatrix} Q^H R Q & Q^H P \\ P^H Q & S \end{bmatrix} \in \mathbb{C}^{n+m,n+m} \tag{1.5}$$

*is positive semidefinite, i.e.,* $W = W^H \geq 0$.

The class of pHDAE systems provides a unified and natural modeling framework for the simulation and control of almost all classes of real world physical systems, see [5, 21, 28, 30, 29, 35, 36] for detailed discussions and a multitude of applications. The great success of modeling with pHDAE systems is mainly due to its many important properties.

Key properties of pHDAEs, see e.g. [29], are the invariance of the class under power-conserving interconnection, which allows modularized automated modeling, the invariance under Galerkin projection which makes them ideal for discretization and model reduction, and in particular the encoding of properties like energy dissipation, stability and passivity in the algebraic structure of the coefficients of the equations. The class of pHDAE systems also provides an ideal framework for robust and physically interpretable control design. This follows, in particular, from the power balance equation and the resulting dissipation inequality, see e.g. [28].

THEOREM 1.3. *Consider a pHDAE system of the form* (1.3)*. Then for any input* $u(t)$ *the power balance equation*

$$\frac{d}{dt}\mathcal{H}(x(t)) = -\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^H W \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \Re(y^H(t)u(t)) \tag{1.6}$$

*holds along any solution* $x(t)$*. In particular, the* dissipation inequality

$$\mathcal{H}(x(t_2)) - \mathcal{H}(x(t_1)) \leq \int_{t_1}^{t_2} \Re(y(\tau)^H u(\tau))\, d\tau \tag{1.7}$$

*holds.*

In physical space, one can view pHDAE systems as modeling the interaction of three types of energies by encoding these in the structure of the coefficients. The *stored energy* is presented by the nonnegative Hamiltonian $\mathcal{H}(x)$, the *dissipated energy* by the nonnegative quadratic form $\mathcal{D}(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^H W \begin{bmatrix} x \\ u \end{bmatrix}$ and the *supplied energy* by $\mathcal{S}(y,u) = \Re(y^H u)$.

While for general descriptor systems it is computationally difficult to analyze whether a system is (robustly) asymptotically stable, see e.g. [7, 37], in the pHDAE modeling framework, using Theorem 1.3 easily allows to analyze when a pHDAE system is stable (asymptotically stable). It is well known [25] that if $Q$ has full column rank, then the pHDAE systems of the form (1.3) are stable (but not necessarily asymptotically stable) in the sense that all finite eigenvalues are in the closed left half plane and those on the imaginary axis are semisimple. Furthermore, it is shown in [25] that the index of a pHDAE system can be at most $\nu = 2$ and in [26] the singularity is characterized by a common nullspace property. Furthermore, if the system is in the pHDAE representation, it is only needed to check the semidefiniteness of $E^H Q$ and $W$, which can be done accurately and with perturbation bounds via the calculation of Cholesky decompositions, see e.g. [19].

There also exist structure preserving versions of the Kronecker canonical form, see [1, 6], where in order to preserve the structure and in particular the different types of energy $\mathcal{H}, \mathcal{D}, \mathcal{S}$, we require the transformations to satisfy $S = T^H$ and $Y = V^{-H}$, see [4, 30]. We will discuss such condensed forms in Section 2.

**1.2. Simplified pHDAE reformulation.** It has been addressed in [29] how one can reformulate a general linear pHDAE system to one with $\ell = n$ and $Q = I$ and how to remove the feedthrough term, so that $Du = (S - N)u = 0$. In the following we will briefly recall this reformulation which would always be the first step of a regularization procedure.

If in (1.3) $Q$ has full column rank then the state equation can be multiplied with $Q^H$ from the left, yielding the system

$$\begin{aligned} Q^H E\dot{x} &= Q^H (J - R)Qx + Q^H (B - P)u, \\ y &= (B + P)^H Qx + (S - N)u. \end{aligned}$$

Then setting $\widetilde{E} = Q^H E$, $\widetilde{J} = Q^H JQ$, $\widetilde{R} = Q^H RQ$, $\widetilde{B} = Q^H B$, and $\widetilde{P} = Q^H P$, the transformed system

$$\begin{aligned} \widetilde{E}\dot{x} &= (\widetilde{J} - \widetilde{R})x + (\widetilde{G} - \widetilde{P})u, \\ y &= (\widetilde{G} + \widetilde{P})^H x + (S - N)u \end{aligned}$$

is again a pHDAE system, but now has $\widetilde{Q} = I_n$ and hence $\widetilde{E} = \widetilde{E}^H \geq 0$. If $Q$ is not of full column rank then a subsystem of this form can be obtained by performing a singular value decomposition of $Q$ and then considering only the subsystem associated with the invertible part, see [29] for details.

One can also always remove the feedthrough term by extending the state space. Since the Hermitian part of $D$ is semidefinite, one can construct, see [1], a unitary matrix $U_D$, such that

$$D = U_D \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U_D^H,$$

with $D_1$ nonsingular and the Hermitian part of $D_1$ is positive semi-definite. Setting, with analogous partitioning,

$$(B-P)U_D = \begin{bmatrix} B_1 - P_1 & B_2 - P_2 \end{bmatrix}, \qquad U_D^H u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad U_D^H y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

the system can be written as

$$E\dot{x} = (J-R)x + (B_1 - P_1)u_1 + (B_2 - P_2)u_2, \tag{1.8a}$$

$$y_1 = (B_1 + P_1)^H x + D_1 u_1, \tag{1.8b}$$

$$y_2 = (B_2 + P_2)^H x. \tag{1.8c}$$

Since $R \geq 0$ we have $P_2 = 0$. Introducing $x_2 = D_1 u_1 + P_1^H x$ we obtain the extended system

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} J-R & 0 \\ D_1^{-1}P_1^H & -D_1^{-1} \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 - P_1 \\ I \end{bmatrix} u_1 + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u_2, \\ y_1 &= \begin{bmatrix} B_1^H & I \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix}, \\ y_2 &= B_2^H x. \end{aligned}$$

By this extension the Hamiltonian $\mathcal{H}$ and the dissipated energy $\mathcal{D}$ have not changed, they are just formulated in different variables and the added variables do not change the values of $\mathcal{H}$ and $\mathcal{D}$. Clearly also the supplied energy $\mathcal{S}$ stays the same. By multiplying the state equation with the nonsingular matrix $\begin{bmatrix} I & P_1 \\ 0 & I \end{bmatrix}$ from the left we obtain the extended pHDAE system

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= (\tilde{J} - \tilde{R})\tilde{x} + \tilde{B}u, \\ y &= \tilde{B}^H \tilde{x}, \end{aligned}$$

with extended state $\tilde{x} = [x^H, x_2^H]^H$ and coefficients

$$\begin{aligned} \tilde{E} &= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{J} = \begin{bmatrix} J + \frac{1}{2}\left(P_1 D_1^{-1} P_1^H - \left(P_1 D_1^{-1} P_1^H\right)^H\right) & -P_1 D_1^{-1} \\ D_1^{-1} P_1^H & -\frac{1}{2}\left(D_1^{-1} - D_1^{-H}\right) \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} B_1 & B_2 \\ I & 0 \end{bmatrix}, \ \tilde{P} = 0, \ \tilde{D} = 0, \\ \tilde{R} &= \begin{bmatrix} R - \frac{1}{2}\left(P_1 D_1^{-1} P_1^H + \left(P_1 D_1^{-1} P_1^H\right)^H\right) & 0 \\ 0 & \frac{1}{2}\left(D_1^{-1} + D_1^{-H}\right) \end{bmatrix}. \end{aligned}$$

In the following we therefore assume that we have a pHDAE system of the form

$$\begin{aligned} E\dot{x} &= (J-R)x + Bu, \\ y &= B^H x, \end{aligned} \tag{1.9}$$

with $E, J, R \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $E = E^H \geq 0$, $R = R^H \geq 0$, $J = -J^H$, with the quadratic Hamiltonian $\mathcal{H}(x) = \frac{1}{2}x^H E x \geq 0$ and the dissipation matrix $W = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. We also assume, without loss of generality, that $B$ has full column rank by restricting, if necessary, $u, y$ to an appropriate subspace.

**1.3. Problem statements.** For general unstructured descriptors systems it has been studied extensively, see e.g. [9, 10, 11, 12, 31, 32, 33], how to modify system properties like regularity or the eigenstructure of the system via different types of feedback. But such general feedback approaches do not necessarily preserve the pH-DAE structure. For pHDAE systems the natural feedback classes are proportional output feedbacks, since then the symmetry structure of the coefficients is preserved and it is sufficient if the feedback preserves the nonnegativity of the energy functions $\mathcal{H}$ and $\mathcal{D}$. We therefore discuss proportional output feedback of the form

$$u(t) = (F_S - F_H)y(t) + v(t)$$

where $F_S = -F_S^H$ and$F_H = F_H^H$ are such that the resulting closed loop system

$$\begin{array}{rcl} \dot{x}(t) & = & (J + BF_S B^H - (R + BF_H B^H))x(t) + Bv(t), \\ y(t) & = & B^H x(t), \end{array}$$

has desired properties. In particular, we study the following three problems:

**Problem 1** (Regularization of pHDAE system (1.9) by proportional output feedback): Determine matrices $F_S = -F_S^H, F_H = F_H^H$ such that the pair $(E, J+BF_S B^H-(R+BF_H B^H))$ is regular, and $R+BF_H B^H \geq 0$, i.e., the resulting closed-loop system is a regular pHDAE system.

**Problem 2** (Regularization and index reduction of pHDAE system (1.9) by proportional output feedback): Determine matrices $F_S = -F_S^H, F_H = F_H^H$ such that the pair $(E, J + BF_S B^H - (R + BF_H B^H))$ is regular, of index at most one, and $R + BF_H B^H \geq 0$, i.e., the resulting closed-loop system is a regular pHDAE system of index at most one.

**Problem 3.** (Stabilization of pHDAE system (1.9) by proportional output feedback): Determine matrices $F_S = -F_S^H, F_H = F_H^H$ such that the pair $(E, J+BF_S B^H-(R + BF_H B^H))$ is regular, of index at most one, has all its finite eigenvalues in the open left half complex plane, and

$$R + BF_H B^H \geq 0,$$

i.e., the resulting closed-loop system is a regular pHDAE system of index at most one and has all its finite eigenvalues in the open left half complex plane.

In some applications it is also possible to use derivative output feedback $u = K\dot{y}$ to perform regularization, index reduction and stabilization. The results that we present also extend to this case, see Appendix B.

All the constructions and conditions that we present are derived via structured condensed forms that we present in the next section. For completeness we also present coordinate free versions of the results for which we denote a full column rank matrix with its columns spanning the right nullspace of a matrix $M$ by $\mathcal{S}_\infty(M)$ and with its columns spanning the left nullspace of $M$ by $\mathcal{T}_\infty(M)$, respectively.

**2. Condensed forms.** The basis for the construction of regularizing feedbacks is the computation of condensed forms. In order to be able to construct the regularizing feedbacks in a numerically stable way we use unitary transformations. The following form is a modification of the condensed form presented in [6].

LEMMA 2.1. *Consider a pHDAE system of the form (1.9). Then there exist unitary matrices $U$ and $V$ such that*

$$U^H BV = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} m-n_3 & n_3 \end{array} \\ \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \\ 0 & B_{32} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad U^H EU = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & 0 \\ E_{12}^H & E_{22} & E_{23} & E_{24} & 0 & 0 \\ E_{13}^H & E_{23}^H & E_{33} & E_{34} & 0 & 0 \\ E_{14}^H & E_{24}^H & E_{34}^H & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{2.1}$$

$$U^H(J-R)U = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \begin{bmatrix} J_{11}-R_{11} & J_{12}-R_{12} & J_{13}-R_{13} & J_{14}-R_{14} & J_{15}-R_{15} & J_{16} \\ -J_{12}^H-R_{12}^H & J_{22}-R_{22} & J_{23}-R_{23} & J_{24}-R_{24} & J_{25}-R_{25} & J_{26} \\ -J_{13}^H-R_{13}^H & -J_{23}^H-R_{23}^H & J_{33}-R_{33} & J_{34}-R_{34} & J_{35}-R_{35} & 0 \\ -J_{14}^H-R_{14}^H & -J_{24}^H-R_{24}^H & -J_{34}^H-R_{34}^H & J_{44}-R_{44} & J_{45}-R_{45} & 0 \\ -J_{15}^H-R_{15}^H & -J_{25}^H-R_{25}^H & -J_{35}^H-R_{35}^H & -J_{45}^H-R_{45}^H & J_{55}-R_{55} & 0 \\ -J_{16}^H & -J_{26}^H & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

*where*

$$\operatorname{rank}\begin{bmatrix} J_{16} \\ J_{26} \end{bmatrix} = n_1+n_2,\ \operatorname{rank}(B_{21}) = n_2,\ \operatorname{rank}(B_{32}) = n_3,\ \operatorname{rank}(J_{55}-R_{55}) = n_5, \tag{2.2}$$

*and, furthermore,*

$$\operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & B_{12} \\ E_{12}^H & E_{22} & E_{23} & E_{24} & B_{21} & B_{22} \\ E_{13}^H & E_{23}^H & E_{33} & E_{34} & 0 & B_{32} \\ E_{14}^H & E_{24}^H & E_{34}^H & E_{44} & 0 & 0 \end{bmatrix} = n_1+n_2+n_3+n_4, \quad E_{44} > 0. \tag{2.3}$$

*Proof.* A constructive proof that can be directly implemented as a numerical method is presented in Appendix A. □

If one allows nonunitary transformations in Lemma 2.1, then one can reduce the condensed form further.

COROLLARY 2.2. *Consider a pHDAE system of the form (1.9). Then there exist nonsingular matrices $S$, $T$, and a unitary matrix $V$ such that*

$$SBV = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} m-n_3 & n_3 \end{array} \\ \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & B_{32} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad SET = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \begin{bmatrix} E_{11} & 0 & E_{13} & 0 & 0 & 0 \\ 0 & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{13}^H & E_{23}^H & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$S(J-R)T = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ -A_{16}^H & -A_{26}^H & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{2.4}$$

*where* $SB = T^H B$,

$$\operatorname{rank}\begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix} = n_1 + n_2, \quad \operatorname{rank}(B_{21}) = n_2, \quad \operatorname{rank}(B_{32}) = n_3, \quad \operatorname{rank}(A_{55}) = n_5, \tag{2.5}$$

*and*

$$E_{11} > 0,\ E_{44} > 0, \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{13}^H & E_{23}^H & E_{33} \end{bmatrix} \geq 0. \tag{2.6}$$

*Proof.* The proof follows by block Gaussian elimination in (2.1). ☐

Using Corollary 2.2 we immediately obtain the following coordinate-free descriptions of the dimensions in the condensed form (2.1).

COROLLARY 2.3. *Consider a pHDAE system of the form* (1.9) *in the condensed form (2.4). Then the following statements hold.*

*(a)*

$$\begin{aligned} n_1 + n_4 &= \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} - \operatorname{rank}(B), \\ n_3 + n_4 &= \operatorname{rank}(\mathcal{T}_\infty^H((J-R)\mathcal{S}_\infty(\begin{bmatrix} E \\ B^H \end{bmatrix}))\begin{bmatrix} E & B \end{bmatrix}), \\ \operatorname{rank}(E_{13}) &= \operatorname{rank}(\mathcal{T}_\infty^H(B)E\mathcal{S}_\infty(\mathcal{T}_\infty^H(\begin{bmatrix} E & B \end{bmatrix})(J-R)) - n_4. \end{aligned}$$

*(b)*

$$\operatorname{rank}\begin{bmatrix} E & J-R & B \end{bmatrix} = n \tag{2.7}$$

*if and only if*

$$n_6 = n_1 + n_2. \tag{2.8}$$

*(c)* $\operatorname{rank}(E_{13}) = n_1$ *if and only if*

$$\operatorname{rank}(\mathcal{T}_\infty^H(B)E\mathcal{S}_\infty(\mathcal{T}_\infty^H(\begin{bmatrix} E & B \end{bmatrix})(J-R))) = \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} - \operatorname{rank}(B). \tag{2.9}$$

*Proof.* The proof follows by direct calculation. ☐

The condensed forms in this section form the basis for the solution of problems 1-3 in the following section.

**3. Regularization and stabilization via proportional output feedback.** In this section we characterize the solutions of Problems 1.–3. The characterizations of the solution to the first two problems have similar conditions as in the unstructured case.

THEOREM 3.1. *Consider a pHDAE system (1.9). Then Problem 1. is solvable if and only if (2.7) holds.*

*Proof.* Suppose that there exist matrices $F_S = -F_S^H$ and $F_H = F_H^H$ such that $(E, J + BF_SB^H - (R + BF_HB^H)$ is regular. Then we have

$$\det(sE - (J + BF_SB^H - (R + BF_HB^H))) \neq 0, \quad \text{for some } s \in \mathbb{C},$$

which together with the condensed form (2.4) gives the condition (2.8), which, together with Corollary 2.3 yields condition (2.7). Hence, necessity follows.

To show the sufficiency, let condition (2.7) and thus equivalently (2.8) holds. Let $F_{22} \in \mathbb{C}^{n_3 \times n_3}$ be such that $F_{22} > 0$ and that $A_{33} - B_{32}F_{22}B_{32}^H$ is nonsingular. Then with

$$F_S = 0, \quad F_H = V \begin{bmatrix} 0 & 0 \\ 0 & F_{22} \end{bmatrix} V^H,$$

we have that $(E, J + BF_SB^H - (R + BF_HB^H)$ is regular and $R + BF_HB^H \geq 0$. ◻

If we further require that the index of the closed loop system pencil is reduced to one then we have the following result.

THEOREM 3.2. *Consider a pHDAE system of the form (1.9). Then Problem 2. is solvable if and only if*

$$\operatorname{rank} \begin{bmatrix} E & (J-R)\mathcal{S}_\infty(E) & B \end{bmatrix} = n. \tag{3.1}$$

*Proof.* Let $F = F_S - F_H$, with $F_S = -F_S^H$ and $F_H = F_H^H$ be such that $(E, J - R + BFB^H)$ is regular and of index at most one. Then condition (2.7) holds and

$$\operatorname{rank} \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{13}^H & E_{23}^H & E_{33} \end{bmatrix} = \operatorname{rank}(E_{33}). \tag{3.2}$$

Note that condition (3.2) is equivalent to

$$E_{22} = 0,\ E_{23} = 0,\ E_{11} = E_{13}E_{33}^+E_{13}^H, \tag{3.3}$$

where $E_{33}^+$ is the Moore-Penrose inverse of $E_{33}$.

A direct calculation yields that the conditions (2.7) and (3.3) imply condition (3.1). Hence, the necessity follows.

To show the sufficiency, take

$$F_S = 0, \quad F_H = V \begin{bmatrix} 0 & 0 \\ 0 & F_{22} \end{bmatrix} V^H,$$

with $F_{22} > 0$, and $\mathcal{T}_\infty^H(E_{33})(A_{33} - B_{32}F_{22}B_{32}^H)\mathcal{S}_\infty(E_{33})$ nonsingular. Then the pair $(E_{33}, A_{33} - B_{32}F_{22}B_{32}^H)$ is regular and of index at most one. Because condition (3.1) implies conditions (2.7) and (3.3), we have that $R + BF_HB^H \geq 0$ and the pair $(E, J + BF_SB^H - (R + BF_HB^H))$ is regular and of index at most one. ◻

After a pHDAE system of the form (1.9) has been regularized and made of index at most one, the next task is to design a propotional output feedback so that the resulting closed-loop system is *asymptotically stable*, i.e. all its finite eigenvalues have negative real part. While this a very hard and partially unsolved problem for general descriptor systems, for pHDAE systems the solution is surprisingly simple.

We need the following two lemmas.

LEMMA 3.3. *Consider $E, J, R \in \mathbb{C}^{n,n}$ with $E \geq 0$, $J = -J^H$, $R \geq 0$, and*

$$E = \begin{array}{c|c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{bmatrix} \end{array}, \quad R = \begin{array}{c|c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad J = \begin{array}{c|c} & \begin{array}{cc} n_1 & n_1 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} J_{11} & J_{12} \\ -J_{12}^H & J_{22} \end{bmatrix} \end{array},$$

*where $R_{11} > 0$. Then the following statements hold.*

*i)* $J-R$ *is nonsingular if and only if*

$$\operatorname{rank}\begin{bmatrix} J_{12} \\ J_{22} \end{bmatrix} = n_2.$$

*ii) The pair $(E, J-R)$ has all its finite eigenvalues in the open left half complex plane if and only if for all purely imaginary $s \in \mathbb{C}$*

$$\operatorname{rank}\begin{bmatrix} J_{12} - sE_{12} \\ J_{22} - sE_{22} \end{bmatrix} = n_2. \tag{3.4}$$

*Proof.* The proof of i) is trivial.

ii) If the pair $(E, J-R)$ has all its finite eigenvalues in the open left half plane, then obviously (3.4) holds for all purely imaginary $s \in \mathbb{C}$.

Conversely, let (3.4) hold for all purely imaginary $s \in \mathbb{C}$. It follows from i) that $(E, J-R)$ is regular. Next, let $s_0 \in \mathbb{C}$ be any finite eigenvalue of $(E, J-R)$, and let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^n$ (partitioned analogously) be a corresponding eigenvector normalized such that $x^H E x = 1$. Then we have

$$\begin{bmatrix} J_{11} - R_{11} & J_{12} \\ -J_{12}^H & J_{22} \end{bmatrix} x = s_0 \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{bmatrix} x, \tag{3.5}$$

and hence,

$$\begin{aligned} s_0 &= x^H \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{bmatrix} x \\ &= x^H \begin{bmatrix} J_{11} - R_{11} & J_{12} \\ -J_{12}^H & J_{22} \end{bmatrix} x = -x_1^H R_{11} x_1 + x^H \begin{bmatrix} J_{11} & J_{12} \\ -J_{12}^H & J_{22} \end{bmatrix} x, \end{aligned}$$

which gives

$$\Re(s_0) = -x_1^H R_{11} x_1 \leq 0.$$

We show that $x_1 \neq 0$. If we had $x_1 = 0$, then $s_0$ is purely imaginary, $x_2 \neq 0$ and

$$\begin{bmatrix} J_{12} - s_0 E_{12} \\ J_{22} - s_0 E_{22} \end{bmatrix} x_2 = 0.$$

This and the condition that $\operatorname{rank}\begin{bmatrix} J_{12} - s_0 E_{12} \\ J_{22} - s_0 E_{22} \end{bmatrix} = n_2$ yields that $x_2 = 0$ which is a contradiction. Hence, $x_1 \neq 0$ and $\Re(s_0) = -x_1^H R_{11} x_1 < 0$. Therefore, $(E, J-R)$ has all its finite eigenvalues in the open left half complex plane. ∎

LEMMA 3.4. *Given $J, R, \tilde{R} \in \mathbb{C}^{n,n}$ with $J = -J^H$, $R = R^H \geq 0$ and $\tilde{R} = \tilde{R}^H \geq 0$. If $J-R$ is nonsingular, then $J - (R + \tilde{R})$ is nonsingular.*

*Proof.* Suppose this were not the case, then for some vector $x \neq 0$ we have $(J - (R + \tilde{R}))x = 0$, which implies $Jx = 0$ and $(R + \tilde{R})x = 0$ and then $Rx = 0$ which is a contradiction. ∎

We now present necessary and sufficient solvability conditions for Problem 3.

THEOREM 3.5. *Consider a pHDAE system of the form (1.9). Then Problem 3. is solvable if and only if the condition (3.1) holds and for all purely imaginary $s \in \mathbb{C}$*

$$\operatorname{rank}\begin{bmatrix} J - R - sE & B \end{bmatrix} = n. \tag{3.6}$$

*Proof.* The necessity is obvious since the condition (3.1) follows by Theorem 3.2 and that condition (3.6) holds for all purely imaginary $s \in \mathbb{C}$ is well-known from the unstructured case, see e.g. [9].

To prove the sufficiency, let $U$ be a unitary matrix such that

$$U^H B = \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}, \quad U^H R U = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{12}^H & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

where

$$\operatorname{rank}(B_2) = n_2, \ R_{11} > 0.$$

Set

$$U^H J U = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ -J_{12}^H & J_{22} & J_{23} \\ -J_{13}^H & -J_{23}^H & J_{33} \end{bmatrix} \end{array}, \quad U^H E U = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12}^H & E_{22} & E_{23} \\ E_{13}^H & E_{23}^H & E_{33} \end{bmatrix} \end{array}.$$

Let $F_S = 0$ and $F_H = F_H^H$ be such that

$$\operatorname{rank}(R + BF_H B^H) = \operatorname{rank} \begin{bmatrix} R & B \end{bmatrix}, \quad R + BFB^H \geq 0,$$

i.e.,

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^H & R_{22} + B_2 F_H B_2^H \end{bmatrix} > 0.$$

Then it follows from Lemma 3.3 and the fact that (3.6) holds for all purely imaginary $s$ that the pair $(E, J - (R + BF_H B^H))$ has all its finite eigenvalues in the open left half complex plane.

Next, let $\tilde{U} \in \mathbb{C}^{n,n}$ be unitary such that

$$\tilde{U}^H E \tilde{U} = \begin{array}{c} \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & \tau_2 & \tau_3 \end{array} \\ \begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \tilde{U}^H (R + BF_H B^H) \tilde{U} = \begin{array}{c} \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & \tau_2 & \tau_3 \end{array} \\ \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & 0 \\ \tilde{R}_{12}^H & \tilde{R}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

where

$$\tilde{E}_{11} > 0, \quad \tilde{R}_{22} > 0.$$

Set

$$\tilde{U}^H J \tilde{U} = \begin{array}{c} \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & \tau_2 & \tau_3 \end{array} \\ \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13} \\ -\tilde{J}_{12}^H & \tilde{J}_{22} & \tilde{J}_{23} \\ -\tilde{J}_{13}^H & -\tilde{J}_{23}^H & \tilde{J}_{33} \end{bmatrix} \end{array}, \quad \tilde{U}^H B = \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix}.$$

Note that

$$\operatorname{rank}(R + BF_H B^H) = \operatorname{rank} \begin{bmatrix} R & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} R + BF_H B^H & B \end{bmatrix},$$

and thus

$$\tilde{B}_3 = 0.$$

Additionally, condition (3.1) implies that

$$\operatorname{rank} \begin{bmatrix} \tilde{J}_{23} \\ \tilde{J}_{33} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -\tilde{J}_{23}^H & \tilde{J}_{33} \end{bmatrix} = \tau_3,$$

and hence by Lemma 3.3 we have that $\begin{bmatrix} \tilde{J}_{22} - \tilde{R}_{22} & \tilde{J}_{23} \\ -\tilde{J}_{23}^H & \tilde{J}_{33} \end{bmatrix}$ is nonsingular. Therefore, the pair $(E, J - (R + BF_H B^H))$ is regular and of index at most one. $\square$

After the characterization of the existence of output feedbacks that make the pHDAE system regular and of index at most one as well as asymptotically stable an important question is to use the feedbacks in such a way that the resulting closed loo system is robustly regular, of index at most one and asymptotically stable. In order to do this one needs efficiently computable characterizations what the distance to the nearest non-regular pHDAE, higher index pHDAE are [17, 20, 26], respectively the distance to instability [2, 18, 17] are. Furthermore, it is necessary to analyze how the pHDAE structure can be exploited, and how to compute robust pHDAE representations, see [3, 27]. This topic is currently under investigation.

**4. Concluding Remarks.** In this paper, new characterizations have been derived for the regularization, index reduction and stabilization of port-Hamiltonian descriptor systems (1.9) by proportional output feedback while preserving the port-Hamiltonian structure. Future work will include the development and implementation of numerical methods for optimal robust output feedback stabilization.

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**Appendix A. Constructive proof of Lemma 2.1.** In this proof, we use QR decompositions and singular value decompositions, see [19] to determine the mentioned unitary matrices.

Step 1. Determine a unitary matrix $U_1$ such that

$$U_1^H B = \begin{array}{c} \mu_1 \\ n-\mu_1 \end{array} \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $\mathrm{rank}(B_1) = \mu_1$ and set

$$U_1^H E U_1 = \begin{array}{c} \\ \mu_1 \\ n-\mu_1 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & n-\mu_1 \end{array} \\ \begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} \\ (E_{12}^{(1)})^H & E_{22}^{(1)} \end{bmatrix} \end{array},$$

and

$$U_1^H (J-R) U_1 = \begin{array}{c} \\ \mu_1 \\ n-\mu_1 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & n-\mu_1 \end{array} \\ \begin{bmatrix} J_{11}^{(1)} - R_{11}^{(1)} & J_{12}^{(1)} - R_{12}^{(1)} \\ -(J_{12}^{(1)})^H - (R_{12}^{(1)})^H & J_{22}^{(1)} - R_{22}^{(1)} \end{bmatrix} \end{array},$$

where $E_{22}^{(1)} \geq 0$ since $E \geq 0$.

Step 2. Determine a unitary matrix $U_2$ such that

$$U_2^H E_{22}^{(1)} U_2 = \begin{array}{c} \\ \mu_2 \\ n-\mu_1-\mu_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_2 & n-\mu_1-\mu_2 \end{array} \\ \begin{bmatrix} \hat{E}_{22} & 0 \\ 0 & 0 \end{bmatrix} \end{array},$$

where $E_{22} > 0$ and set

$$U_2^H (J_{22}^{(1)} - R_{22}^{(1)}) U_2 = \begin{array}{c} \\ \mu_2 \\ n-\mu_1-\mu_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_2 & n-\mu_1-\mu_2 \end{array} \\ \begin{bmatrix} J_{22}^{(2)} - R_{22}^{(2)} & J_{23}^{(2)} - R_{23}^{(2)} \\ -(J_{23}^{(2)})^H - (R_{23}^{(2)})^H & J_{33}^{(2)} - R_{33}^{(2)} \end{bmatrix} \end{array}.$$

Step 3. Determine a unitary matrix $U_3$ such that

$$U_3^H (J_{33}^{(2)} - R_{33}^{(2)}) = \begin{array}{c} n_5 \\ n_6 \end{array} \begin{bmatrix} \tilde{J}_3 - \tilde{R}_3 \\ 0 \end{bmatrix},$$

where $\mathrm{rank}(\tilde{J}_3 - \tilde{R}_3) = n_5$. Set

$$U_3^H J_{33}^{(2)} U_3 = \begin{array}{c} \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} n_5 & n_6 \end{array} \\ \begin{bmatrix} J_{55} & J_{56}^{(3)} \\ -(J_{56}^{(3)})^H & J_{66}^{(3)} \end{bmatrix} \end{array}, \quad U_3 R_{33}^{(2)} U_3^H = \begin{array}{c} \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} n_5 & n_6 \end{array} \\ \begin{bmatrix} R_{55} & R_{56}^{(3)} \\ (R_{56}^{(3)})^H & R_{66}^{(3)} \end{bmatrix} \end{array}.$$

Then

$$\begin{aligned} U_3^H (J_{33}^{(2)} - R_{33}^{(2)}) U_3 &= \begin{array}{c} \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} n_5 & n_6 \end{array} \\ \begin{bmatrix} J_{55} - R_{55} & J_{56}^{(3)} - R_{56}^{(3)} \\ -(J_{56}^{(3)})^H - (R_{56}^{(3)})^H & J_{66}^{(3)} - R_{66}^{(3)} \end{bmatrix} \end{array} \\ &= \begin{array}{c} \\ n_5 \\ n_6 \end{array} \begin{array}{c} \begin{array}{cc} n_5 & n_6 \end{array} \\ \begin{bmatrix} J_{55} - R_{55} & J_{56}^{(3)} - R_{56}^{(3)} \\ 0 & 0 \end{bmatrix} \end{array}. \end{aligned}$$

Note that $R \geq 0$ and $J = -J^H$, so, $R_{33}^{(2)} \geq 0$, $J_{33}^{(2)} = -(J_{33}^{(2)})^H$, and thus,

$$R_{66}^{(3)} = J_{66}^{(3)} = 0, \quad R_{56}^{(3)} = 0, \quad J_{56}^{(3)} = 0.$$

Define

$$\tilde{U}_1 = \begin{bmatrix} I & \\ & U_3 \end{bmatrix} \begin{bmatrix} I & \\ & U_2 \end{bmatrix} U_1,$$

then

$$\tilde{U}_1^H B = \begin{array}{c} \mu_1 \\ \mu_2 \\ n_5 \\ n_6 \end{array} \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{U}_1^H E \tilde{U}_1 = \begin{array}{c|cccc} & \mu_1 & \mu_2 & n_5 & n_6 \\ \mu_1 & \hat{E}_{11} & \hat{E}_{12} & 0 & 0 \\ \mu_2 & \hat{E}_{12}^H & \hat{E}_{22} & 0 & 0 \\ n_5 & 0 & 0 & 0 & 0 \\ n_6 & 0 & 0 & 0 & 0 \end{array},$$

$$\tilde{U}_1 (J - R) \tilde{U}_1 = \begin{array}{c|cccc} & \mu_1 & \mu_2 & n_5 & n_6 \\ \mu_1 & \hat{J}_{11} - R_{11} & \hat{J}_{12} - \hat{R}_{12} & \hat{J}_{13} - \hat{R}_{13} & \hat{J}_{14} - \hat{R}_{14} \\ \mu_2 & -\hat{J}_{12}^H - \hat{R}_{12}^H & \hat{J}_{22} - \hat{R}_{22} & \hat{J}_{23} - \hat{R}_{23} & \hat{J}_{24} - \hat{R}_{24} \\ n_5 & -\hat{J}_{13}^H - \hat{R}_{13}^H & -\hat{J}_{23}^H - \hat{R}_{23}^H & J_{55} - R_{55} & 0 \\ n_6 & -\hat{J}_{14}^H - \hat{R}_{14}^H & -\hat{J}_{24}^H - \hat{R}_{24}^H & 0 & 0 \end{array},$$

where

$$\operatorname{rank}(B_1) = \mu_1, \ \operatorname{rank}(J_{55} - R_{55}) = n_5, \ \operatorname{rank}(\hat{E}_{22}) = \mu_2.$$

In addition, using $R \geq 0$, we also have that

$$\hat{R}_{14} = 0, \quad \hat{R}_{24} = 0,$$

Step 4. Construct unitary matrices $U_4$ and $V$ such that

$$U_4^H \begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} \\ \hat{E}_{12}^H & \hat{E}_{22} \end{bmatrix} U_4 = \begin{array}{c|cccc} & n_1 & n_2 & n_3 & n_4 \\ n_1 & E_{11} & E_{12} & E_{13} & E_{14} \\ n_2 & E_{12}^H & E_{22} & E_{23} & E_{24} \\ n_3 & E_{13}^H & E_{23}^H & E_{33} & E_{34} \\ n_4 & E_{14}^H & E_{24}^H & E_{34}^H & E_{44} \end{array},$$

$$U_4^H \begin{bmatrix} B_1 \\ 0 \end{bmatrix} V = \begin{array}{c|cc} & m - n_3 & n_3 \\ n_1 & 0 & B_{12} \\ n_2 & B_{21} & B_{22} \\ n_3 & 0 & B_{32} \\ n_4 & 0 & 0 \end{array}, \quad U_4^H \begin{bmatrix} \hat{J}_{14} \\ \hat{J}_{24} \end{bmatrix} = \begin{array}{c|c} & n_6 \\ n_1 & J_{16} \\ n_2 & J_{26} \\ n_3 & 0 \\ n_4 & 0 \end{array},$$

where

$$\operatorname{rank} \begin{bmatrix} J_{16} \\ J_{26} \end{bmatrix} = n_1 + n_2, \ \operatorname{rank}(B_{21}) = n_2, \ \operatorname{rank}(B_{32}) = n_3,$$

and

$$\operatorname{rank} \begin{bmatrix} E_{14}^H & E_{24}^H & E_{34}^H & E_{44} \end{bmatrix} = n_4,$$

which, together with $E \geq 0$, yields $E_{44} > 0$. Moreover,

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & B_{12} \\ E_{12}^H & E_{22} & E_{23} & E_{24} & B_{21} & B_{22} \\ E_{13}^H & E_{23}^H & E_{33} & E_{34} & 0 & B_{32} \\ E_{14}^H & E_{24}^H & E_{34}^H & E_{44} & 0 & 0 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} & B_1 \\ \hat{E}_{12}^H & \hat{E}_{22} & 0 \end{bmatrix} \\
&= \mu_1 + \mu_2 = n_1 + n_2 + n_3 + n_4.
\end{aligned}$$

Then

$$U = \begin{bmatrix} U_4 & \\ & I \end{bmatrix} \tilde{U}_1,$$

and $V$ are the transformation matrices to the form (2.1).

**Appendix B.** The results in this paper can be generalized to the case that one includes also derivative feedback. We present the extensions in this appendix.

Theorem 4.1. *Consider a pHDAE system of the form (1.9). There exists a derivative feedback matrix $K$ such that the pair $(E + BKB^H, J - R)$ is regular and $E + BKB^H \geq 0$ if and only if (2.7) holds.*

*Proof.* Suppose there exists matrix $K$ such that $(E + BKB^H, J - R)$ is regular. Then it follows that

$$\det(s(E + BKB^H) - (J - R)) \neq 0, \quad \text{for some } s \in \mathbb{C},$$

which together with the condensed form (2.4) gives condition (2.8), i.e. by equivalence also condition (2.7) holds. Hence, necessity is shown.

To show sufficiency, let $K_{22} \in \mathbb{C}^{n_3 \times n_3}$ be such that $K_{22} > 0$ and $E_{33}+B_{32}K_{22}B_{32}^H > 0$. Taking

$$K = V \begin{bmatrix} 0 & 0 \\ 0 & K_{22} \end{bmatrix} V^H,$$

it follows from (2.7) (or equivalently from (2.8)) that $(E + BKB^H, J - R)$ is regular and $E + BKB^H \geq 0$. Hence, the sufficiency is proved. ∎

We can also combine Theorems 3.1 and 4.1.

Theorem 4.2. *Consider a pHDAE system of the form (1.9). There exist feedback matrices $K$, $F_S = -F_S^H$ and $F_H = F_H^H$ such that the pair $(E+BKB^H, J+BF_SB^H - (R + BF_HB^H)$ is regular and $E + BKB^H \geq 0$, $R + BF_HB^H \geq 0$ if and only if (2.7) holds. Moreover, if condition (2.7) holds, then for any integer $r$ satisfying*

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} - \operatorname{rank}(B) \leq r \leq \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix}, \tag{4.1}$$

*there exist matrices $K$ and $F_H = F_H^H$ and $F_S = 0$ such that $(E+BKB^H, J+BF_SB^H - (R + BF_HB^H))$ is regular, and*

$$\operatorname{rank}(E + BKB^H) = r, \quad E + BKB^H \geq 0, \quad R + BF_HB^H \geq 0.$$

*Proof.* Suppose that there exist matrices $K$, $F_S = -F_S^H$, and $F_H = F_H^H$ such that $(E + BKB^H, J + BF_SB^H - (R + BF_HB^H)$ is regular. Then

$$\det(sE - (J + BF_SB^H - (R + BF_HB^H))) \neq 0, \quad \text{for some } s \in \mathbb{C},$$

from which we obtain condition (2.8), and equivalently (2.7). Hence, the necessity is shown.

By Corollary 2.3, conditions (2.7) and (4.1) are equivalent to the conditions (2.8) and

$$n_1 + n_4 \le r \le n_1 + n_2 + n_3 + n_4,$$

respectively. Since $E_{11} > 0$, $\operatorname{rank}(B_{21}) = n_2$, $\operatorname{rank}(B_{32}) = n_2$ and $E \ge 0$, there exists a matrix $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^H & K_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{13}^H & E_{23}^H & E_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & B_{32} \end{bmatrix} K \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & B_{32} \end{bmatrix}^H \ge 0,$$

$$\operatorname{rank}(\begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ E_{13}^H & E_{23}^H & E_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & B_{32} \end{bmatrix} K \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & B_{32} \end{bmatrix}^H) = r - n_4.$$

Let $F_{22} > 0$ be such that

$$\mathcal{T}_\infty^H (E_{33} + B_{32} K_{22} B_{32}^H)(A_{33} - B_{32} F_{22} B_{32}^H) \mathcal{S}_\infty (E_{33} + B_{32} K_{22} B_{32}^H)$$

is nonsingular and set

$$\mathcal{K} = V K V^H, \quad F_H = V \begin{bmatrix} 0 & 0 \\ 0 & F_{22} \end{bmatrix} V^H, \quad F_S = 0.$$

We then have

$$\operatorname{rank}(E + B\mathcal{K}B^H) = r, \quad E + B\mathcal{K}B^H \ge 0, \quad R + BF_H B^H \ge 0,$$

and $(E + B\mathcal{K}B^H, J + BF_S B^H - (R + BF_H B^H))$ is regular. $\square$

The corresponding results to achieve an index at most one are as follows.

Theorem 4.3. *Consider a pHDAE system of the form (1.9). There exists a matrix $K$ such that the pair $(E + BKB^H, J - R)$ is regular and of index at most one, and $E + BKB^H \ge 0$ if and only if conditions (2.7) and (2.9) hold.*

*Proof.* By Lemma 2.3, conditions (2.7) and (2.9) are equivalent to

$$n_6 = n_1 + n_2, \quad \operatorname{rank}(E_{13}) = n_1.$$

If the pair $(E + BKB^H, J - R)$ is regular and of index at most one for some $K$, then with

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = V^H K V,$$

we have condition (2.7) and

$$\operatorname{rank} \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} + B_{21} K_{11} B_{21}^H & E_{23} + B_{21} K_{12} B_{32}^H \\ E_{13}^H & E_{23}^H + B_{32} K_{21} B_{21}^H & E_{33} + B_{32} K_{22} B_{32}^H \end{bmatrix} = \operatorname{rank}(E_{33} + B_{32} K_{22} B_{32}^H). \tag{4.2}$$

It is obvious that (4.2) implies

$$E_{11} = E_{13}(E_{33} + B_{32}K_{22}B_{32}^H)^+ E_{13}^H,$$

which together with $E_{11} > 0$ gives $\operatorname{rank}(E_{13}) = n_1$, i.e., condition (2.9) holds. Hence, the necessity is shown.

To show the sufficiency, note that $E_{11} > 0$, $\operatorname{rank}(E_{13}) = n_1$, and $B_{32}$ is nonsingular, so there exists $K_{22} = K_{22}^H$ such that

$$E_{33} + B_{32}K_{22}B_{32}^H > 0,$$

and

$$\operatorname{rank} \begin{bmatrix} E_{11} & E_{13} \\ E_{13}^H & E_{33} + B_{32}K_{22}B_{32}^H \end{bmatrix} = \operatorname{rank}(E_{33} + B_{32}K_{22}B_{32}^H) = n_3.$$

Additionally, since $B_{21}$ is of full row rank, there exist $K_{11}$, $K_{12}$ such that

$$E_{22} + B_{21}K_{11}B_{21}^H = 0, \quad E_{23} + B_{21}K_{12}B_{32}^H = 0.$$

Taking

$$K = V \begin{bmatrix} K_{11} & K_{12} \\ K_{21}^H & K_{22} \end{bmatrix} V^H,$$

we have that

$$E + BKB^H \geq 0,$$

and $(E + BKB^H, J - R)$ is regular and of index at most one. $\square$

THEOREM 4.4. *Consider a pHDAE system of the form (1.9). There exist matrices $K$, $F_S = -F_S^H$, and $F_H = F_H^H$ such that the pair $(E + BKB^H, J + BF_SB^H - (R + BF_HB^H)$ is regular and of index at most one, and $E + BKB^H \geq 0$, $R + BF_HB^H \geq 0$ if and only if conditions (2.7) and (2.9) hold. Moreover, under conditions (2.7) and (2.9), for a given integer $r$, there exist matrices $K$, $F_S = -F_S^H$ and $F_H = F_H^H$ such that*

$$E + BKB^H \geq 0, \quad R + BF_HB^H \geq 0,$$

*$(E + BKB^H, (J + BF_SB^H) - (R + BF_HB^H))$ is regular, $(E + BKB^H, (J + BF_SB^H) - (R + BF_HB^H))$ has index at most one and* $\operatorname{rank}(E + BKB^H) = r$ *if and only if*

$$\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} - \operatorname{rank}(B) \leq r \leq \operatorname{rank}(\mathcal{T}_\infty^H((J - R)\mathcal{S}_\infty(\begin{bmatrix} E \\ B^H \end{bmatrix})) \begin{bmatrix} E & B \end{bmatrix}). \quad (4.3)$$

*Proof.* For any $K$ and $F$ with

$$K = V \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} V^H, \quad F = V \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} V^H, \tag{4.4}$$

it follows from direct calculation that $(E + BKB^H, J - R + BFB^H)$ is regular and of index at most one if and only if condition (2.7) holds,

$$\operatorname{rank} \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22} + B_{21}K_{11}B_{21}^H & E_{23} + B_{21}K_{12}B_{32}^H \\ E_{13}^H & E_{23}^H + B_{32}^H K_{21}B_{21}^H & E_{33} + B_{32}K_{22}B_{32}^H \end{bmatrix} = \operatorname{rank}(E_{33} + B_{32}K_{22}B_{32}^H), \tag{4.5}$$

and $(E_{33}+B_{32}K_{22}B_{32}^H, A_{33}+B_{32}F_{22}\mathcal{B}_{32}^H)$ is regular and of index at most one. Obviously, (4.5) implies $\mathrm{rank}(E_{13})=n_1$, i.e., condition (2.9) holds. Hence, necessity follows.

The sufficiency follows from the sufficiency of Theorem 4.3 with $F_S=0$ and $F_H=0$.

To study the possible rank of $E+BKB^H$, for any $K$ and $F$ of the form (4.4) with $(E+BKB^H, J-R+BFB^H)$ being regular and of index at most one, we obtain

$$\begin{aligned} n_1+n_4 &\le \mathrm{rank}\begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & E_{22}+B_{21}K_{11}B_{21}^H & E_{23}+B_{21}K_{12}B_{32}^H \\ E_{13}^H & E_{23}^H+B_{32}^HK_{21}B_{21}^H & E_{33}+B_{32}K_{22}B_{32}^H \end{bmatrix}+\mathrm{rank}(E_{44}) \\ &= \mathrm{rank}(E+BKB^H) \\ &= \mathrm{rank}(E_{33}+B_{32}K_{22}B_{32}^H)+\mathrm{rank}(E_{44}) \\ &\le n_3+n_4, \end{aligned}$$

which together with Lemma 2.3 gives condition (4.3).

Let $r$ be any integer satisfying the condition (4.3). We can assume without loss of generality that

$$E_{13} = \begin{array}{c} \begin{array}{cc} n_1 & n_3-n_1 \end{array} \\ \begin{bmatrix} E_{13}^{(1)} & 0 \end{bmatrix} \end{array}, \quad B_{32} = \begin{array}{c} \\ n_1 \\ n_3-n_1 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_3-n_1 \end{array} \\ \begin{bmatrix} B_{32}^{(1)} & 0 \\ 0 & B_{32}^{(4)} \end{bmatrix} \end{array},$$

where

$$\mathrm{rank}(E_{13}^{(1)})=n_1, \quad \mathrm{rank}(B_{32}^{(1)})=n_1, \quad \mathrm{rank}(B_{32}^{(4)})=n_3-n_1.$$

Set

$$E_{33} = \begin{array}{c} \\ n_1 \\ n_3-n_1 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_3-n_1 \end{array} \\ \begin{bmatrix} E_{33}^{(1)} & E_{33}^{(2)} \\ (E_{33}^{(2)})^H & E_{33}^{(4)} \end{bmatrix} \end{array}, \quad A_{33} = \begin{array}{c} \\ n_1 \\ n_3-n_1 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_3-n_1 \end{array} \\ \begin{bmatrix} A_{33}^{(1)} & A_{33}^{(2)} \\ A_{33}^{(3)} & A_{33}^{(4)} \end{bmatrix} \end{array}.$$

Let $K_{11}$, $K_{12}$, $K_{22}^{(1)}$, $K_{22}^{(2)}$ and $K_{22}^{(4)}$ be such that

$$E_{22}+B_{21}K_{11}B_{21}^H=0, \quad E_{23}+B_{21}K_{12}B_{32}^H=0,$$

$$E_{33}^{(1)}+B_{32}^{(1)}K_{22}^{(1)}(B_{32}^{(1)})^H=(E_{13}^{(1)})^HE_{11}^{-1}E_{13}^{(1)}, \quad E_{32}^{(2)}+B_{32}^{(1)}K_{22}^{(2)}(B_{32}^{(4)})^H=0,$$

$$E_{33}^{(4)}+B_{32}^{(4)}K_{22}^{(4)}(B_{32}^{(4)})^H=\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$K_{22}=\begin{bmatrix} K_{22}^{(1)} & K_{22}^{(2)} \\ (K_{22}^{(2)})^H & K_{22}^{(4)} \end{bmatrix},$$

where $\Lambda\in\mathbb{C}^{(r-n_1-n_4)\times(r-n_1-n_4)}$, $\Lambda>0$. Furthermore, let $F_{22}^{(4)}\in\mathbb{C}^{(n_1+n_3+n_4-r)\times(n_1+n_3+n_4-r)}$ satisfy that

$$F_{22}^{(4)}>0, \quad A_{33}^{(4)}-B_{32}^{(4)}F_{22}^{(4)}(B_{32}^{(4)})^H=\begin{bmatrix} \star & \star \\ \star & \Sigma \end{bmatrix},$$

where $\Sigma \in \mathbb{C}^{(n_1+n_3+n_4-r)\times(n_1+n_3+n_4-r)}$ is nonsingular. Take

$$K = V \begin{bmatrix} K_{11} & K_{12} \\ K_{21}^H & K_{22} \end{bmatrix} V^H, \quad F_H = V \begin{bmatrix} 0 & 0 \\ 0 & F_{22}^{(4)} \end{bmatrix} V^H, \quad F_S = 0.$$

We then have that

$$\operatorname{rank}(E + BKB^H) = r, \quad E + BKB^H \geq 0, \quad R + BF_H B^H \geq 0,$$

and the pair $(E + BKB^H, J - (R + BF_H B^H))$ is regular and of index at most one. □

The following corollary characterizes the case that the rank of $E + BKB^H$ is maximized.

COROLLARY 4.5. *Consider a pHDAE system of the form (1.9). There exists a matrix $K$ such that*

$$E + BKB^H \geq 0,$$

*the pair $(E + BKB^H, J - R)$ is regular and of index at most one and*

$$\operatorname{rank}(E + BKB^H) = \operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = \max_{\hat{K}\in\mathbb{C}^{m\times m}} \operatorname{rank}(E + B\hat{K}B^H),$$

*if and only if*

$$\operatorname{rank} \begin{bmatrix} E & (J-R)\mathcal{S}_\infty(\begin{bmatrix} E \\ B^H \end{bmatrix}) & B \end{bmatrix} = n \tag{4.6}$$

*Proof.* By the sufficiency proof of Theorem 4.3, there exists a matrix $K$ such that $E + BKB^H \geq 0$, the pair $(E + BKB^H, J - R)$ is regular and of index at most one and

$$\operatorname{rank}(E + BKB^H) = \operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = \max_{\hat{K}\in\mathbb{C}^{m\times m}} \operatorname{rank}(E + B\hat{K}B^H)$$

if and only if conditions (2.7) and (2.9) hold, and

$$n_3 + n_4 = \operatorname{rank} \begin{bmatrix} E & B \end{bmatrix},$$

and thus, if and only if

$$n_6 = n_1 + n_2 = 0,$$

or equivalently, condition (4.6) holds. □

We can also combine regularization, index reduction and stabilization via proportional and derivative output feedback.

THEOREM 4.6. *Consider a pHDAE system of the form (1.9). There exist feedback matrices $K$, $F_S = -F_S^H$, $F_H = F_H^H$ such that the pair $(E + BKB^H, J + BF_S B^H - (R + BF_H B^H))$ is regular, of index at most one, has all its finite eigenvalues in the open left half complex plane, and*

$$E + BKB^H \geq 0, \quad R + BF_H B^H \geq \tag{4.7}$$

*if and only if conditions (2.7), (2.9), and (3.6) for all purely imaginary $s$, hold. Moreover, under these conditions for a given integer $r$, there exist matrices $K$, $F_S =$*

*$-F_S^H$ and $F_H = F_H^H$ such that the pair $(E+BKB^H, (J+BF_SB^H)-(R+BF_HB^H))$ is regular, of index at most one, has all its finite eigenvalues in the open left half complex plane, (4.7) holds, and*

$$\operatorname{rank}(E+BKB^H) = r$$

*if and only if (4.3) holds.*

*Proof.* The necessity of conditions (2.7), (2.9) and (4.3) follow from Theorem 4.4 and the condition (3.6) is a standard condition in linear control [22].

For the sufficiency, for any integer $r$ satisfying (4.3), let $K = K^H$ and $F_H \geq 0$ be chosen as in the sufficiency proof of Theorem 4.4, i.e., such that the pair $(E + BKB^H, J - (R + BF_HB^H))$ is regular and of index at most one,

$$E + BKB^H \geq 0, \quad R + BF_HB^H \geq 0, \quad \operatorname{rank}(E + BKB^H) = r.$$

Let $\tilde{F}_H \geq 0$ be such that

$$\operatorname{rank}(R + B(F_H + \tilde{F}_H)B^H) = \operatorname{rank}\left[\begin{array}{cc} R + BF_HB^H & B \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} R & B \end{array}\right].$$

Note that for all purely imaginary $s$ we have that

$$\operatorname{rank}\left[\begin{array}{cc} J - (R + BF_HB^H) - s(E + BKB^H) & B \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} J - R - sE & B \end{array}\right] = n,$$

and it follows from the sufficiency proof of Theorem 3.5 that the pair

$$(E + BKB^H, J - B(F_H + \tilde{F}_H)B^H)$$

has all its finite eigenvalues in the open left half complex plane. Furthermore,

$$R + B(F_H + \tilde{F}_H)B^H = R + BF_HB^H + B\tilde{F}_HB^H \geq 0,$$

and

$$\mathcal{T}_\infty(E + BKB^H) = \mathcal{S}_\infty(E + BKB^H),$$

and by Lemma 3.3 it follows that

$$\begin{aligned} &\mathcal{T}_\infty^H(E + BKB^H)(J - (R + B(F_H + \tilde{F}_H)B^H))\mathcal{S}_\infty(E + BKB^H) \\ &\quad = \mathcal{T}_\infty^H(E + BKB^H)(J - (R + BF_HB^H))\mathcal{S}_\infty(E + BKB^H) \\ &\quad\quad -\mathcal{T}_\infty^H(E + BKB^H)(B\tilde{F}_HB^H)\mathcal{S}_\infty(E + BKB^H) \end{aligned}$$

is nonsingular. Therefore, the pair $(E + BKB^H, J - B(F_H + \tilde{F}_H)B^H)$ is of index at most one. ∎

REMARK 1. Consider the condensed form (2.4). Then for $K = V\left[\begin{array}{cc} K_{11} & K_{12} \\ K_{12}^H & K_{22} \end{array}\right]V^H$ the closed loop system $(E + BKB^H, J - R)$ has all its finite eigenvalues in the open left half complex plane if and only if the pair

$$\left[\begin{array}{cc} E_{33} + B_{32}K_{22}B_{32}^H & 0 \\ 0 & E_{44} \end{array}\right], \left[\begin{array}{cc} A_{33} & A_{34} \\ A_{43} & A_{44} \end{array}\right])$$

has all its finite eigenvalues in the open left half complex plane. So, the stabilization of the pHDAE system (1.9) by only derivative output feedback cannot be achieved in general.