Sister Celine's polynomials in the quantum theory of angular momentum

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Abstract

The polynomials introduced by Sister Celine cover different usual orthogonal polynomials as special cases. Among them, the Jacobi and discrete Hahn polynomials are of particular interest for the quantum theory of angular momentum. In this note, we show that characters of irreducible representations of the rotation group as well as Wigner rotation "d" matrices, can be expressed as Sister Celine's polynomials. Since many relations were proposed for the latter polynomials, such connections could lead to new identities for quantities important in quantum mechanics and atomic physics.

1 Introduction

Sister Celine introduced the polynomial [1, 2] (see also Refs. [3, 4]):

$$f_n\left[\begin{array}{c} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{array};x\right] = {}_{p+2}F_{q+2}\left[\begin{array}{c} -n,n+1,a_1,\cdots,a_p \\ 1,\frac{1}{2},b_1,\cdots,b_q \end{array};x\right] \tag{1}$$

defined ($|t|<1$) by the generating function

$$\sum_{n=0}^{\infty} f_n\left[\begin{array}{c} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{array};x\right] t^n = \frac{1}{(1-t)}\,{}_{p+2}F_{q+2}\left[\begin{array}{c} -n,n+1,a_1,\cdots,a_p \\ 1,\frac{1}{2},b_1,\cdots,b_q \end{array};-\frac{4xt}{(1-t)^2}\right], \tag{2}$$

where ${}_pF_q$ denotes the generalized hypergeometric function. The Sister Celine polynomials can be generalized in the following manner [5]:

$$\mathscr{I}_n^{(c,k)}(a_1,\cdots,a_p;b_1,\cdots,b_q;x) = \frac{(c)_n}{n!}\,{}_{p+k}F_{q+k}\left[\begin{array}{c} -n,\Delta(k-1,c+n),a_1,\cdots,a_p \\ \Delta(k,c),b_1,\cdots,b_q \end{array};(k-1)^{k-1}x\right], \tag{3}$$

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where $\Delta(k,c)$ is defined as the set of parameters $c/k,(c+1)/k,\cdots,(c+k-1)/k$. The original Sister Celine polynomials of Eq. (1) are recovered for $c=1$ and $k=2$. One has also

$$\mathscr{I}_n^{(1+\alpha+\beta,2)}\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta}{2}+1;1+\alpha;x\right)=\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}\,P_n^{(\alpha,\beta)}(1-2x),\tag{4}$$

as well as

$$\mathscr{I}_n^{(1+\alpha+\beta,2)}\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta}{2}+1,\xi;1+\alpha,p;v\right)=\frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}\,H_n^{(\alpha,\beta)}(\xi,p,v),\tag{5}$$

where $H_n^{(\alpha,\beta)}(\xi,p,v)$ is a generalized Rice polynomial [6], which can be related to the Jacobi polynomial by

$$\sqrt{\gamma}(1+t)^{-\gamma}\,P_n^{(\alpha,\beta)}\left(1-\frac{2x}{1+t}\right)=\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\sqrt{\lambda}\sqrt{\gamma-\lambda}\,H_n^{(\alpha,\beta)}(\lambda,\gamma,x)\,t^{-\lambda}d\lambda.\tag{6}$$

Jain also established integral relations, which were extended by Khan [7] using the Mellin inversion formula. In 1970, Shah defined [8]:

$$F_n(x)=x^{(m-1)n}\;{}_{p+m}F_q\left[\begin{array}{c}\Delta(m,-n),a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{array};\lambda x^{\mu}\right]\tag{7}$$

with $\Delta(m,-n)=-n/m,(-n+1)/m,\cdots,(-n+m+1)/m$. Setting $m=\lambda=\mu=1$, one gets

$$F_n(x)=\;{}_{p+1}F_q\left[\begin{array}{c}-n,n+\alpha+\beta+1,a_2,\cdots,a_p\\ 1+\alpha,\frac{1}{2},b_3,\cdots,b_q\end{array};x\right]\tag{8}$$

and for $p=q=3$, $a_2=1/2$, $a_3=\xi$, $b_3=p$,

$$F_n(x)=\;{}_3F_2\left[\begin{array}{c}-n,n+\alpha+\beta+1,\xi\\ 1+\alpha,p\end{array};x\right]=\frac{n!}{(1+\alpha)_n}\,H_n^{(\alpha,\beta)}(\xi,p,x)\tag{9}$$

and finally for $p=q=3$, $a_2=1/2$, $a_3=b_3=\xi=p$,

$$F_n(x)=\;{}_3F_2\left[\begin{array}{c}-n,n+\alpha+\beta+1\\ 1+\alpha\end{array};x\right]=\frac{n!}{(1+\alpha)_n}\,P_n^{(\alpha,\beta)}(1-2x).\tag{10}$$

Such generalized polynomials were also denoted by Khan in the slightly different form [9]:

$$f_n(k,\lambda,\mu;a_1,\cdots,a_p;b_1,\cdots,b_q;x)=\;{}_{p+k+1}F_{q+k+1}\left[\begin{array}{c}\Delta(k,-n),n+\lambda,a_1,\cdots,a_p\\ \Delta(k+1,\mu),b_1,\cdots,b_q\end{array};x\right],\tag{11}$$

which enables one to recover the Jacobi polynomials through

$$f_n(1,1+\alpha+\beta,1+2\alpha;\alpha+\frac{1}{2};-;\frac{1-x}{2})=\;{}_2F_1\left[\begin{array}{c}-n,n+\alpha+\beta+1\\ 1+\alpha\end{array};\frac{1-x}{2},\right],\tag{12}$$

i.e.,

$$f_n(1,1+\alpha+\beta,1+2\alpha;\alpha+\frac{1}{2};-;\frac{1-x}{2})=\frac{n!}{(1+\alpha)_n}\,P_n^{(\alpha,\beta)}(x).\tag{13}$$

Note that we have also

$$f_n(1,1+\alpha+\beta,1+2\alpha,\xi;\alpha+\frac{1}{2};p;v) = {}_3F_2\left[\begin{array}{c}-n,n+\alpha+\beta+1\\1+\alpha,p\end{array};v\right] = \frac{n!}{(1+\alpha)_n}\,H_n^{(\alpha,\beta)}(\xi,p,v). \tag{14}$$

More recently, Ahmad *et al.* [10] and Özmen [11] considered:

$$\mathscr{A}_n^{(\alpha,\beta)}\left[\begin{array}{c}a_1,\cdots,a_p\\b_1,\cdots,b_q\end{array};x\right] = \frac{(1+\alpha+\beta)_n}{n!}\,{}_{p+2}F_{q+2}\left[\begin{array}{c}-n,n+\alpha+\beta+1,a_1,\cdots,a_p\\1+\alpha,\frac{1}{2},b_1,\cdots,b_q\end{array};x\right], \tag{15}$$

giving

$$\mathscr{A}_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n\sqrt{\pi}}\int_0^1 t^{-1}(1-t)^{1/2-1}\,P_n^{(\alpha,\beta)}(1-2xt)dt. \tag{16}$$

In the following, we keep the notation of Eq. (11) [9].

2 Rotation matrices and characters of irreducible representations of the rotation group

The main idea of the present part is to take advantage of the relation

$$P_s^{(\mu,\nu)}(\cos\theta) = \frac{(1+\mu)_s}{s!}\,{}_2F_1\left[\begin{array}{c}-s,s+\mu+\nu+1\\1+\mu\end{array};\frac{1-\cos\theta}{2}\right] \tag{17}$$

yielding, with a Sister Celine polynomial

$$P_s^{(\mu,\nu)}(\cos\theta) = \frac{(1+\mu)_s}{s!}\,f_s\left(1,\mu+\nu+1,1+2\mu,\mu+\frac{1}{2};-;\frac{1-\cos\theta}{2}\right). \tag{18}$$

2.1 Characters of irreducible representations

The character of the irreducible representation of rank j of the rotation group can be put in the form [12]:

$$\chi^j(\omega) = \frac{(4j-2)!!}{2(4j+1)!!}\,P_{2j}^{(1/2,1/2)}\left[\cos\left(\frac{\omega}{2}\right)\right] \tag{19}$$

yielding

$$\chi^j(\omega) = \frac{(4j-2)!!}{2(4j+1)!!}\frac{\left(\frac{3}{2}\right)_{2j}}{(2j)!}\,{}_2F_1\left[\begin{array}{c}-2j,2j+2\\3/2\end{array};\sin^2\left(\frac{\omega}{4}\right)\right]. \tag{20}$$

or, in terms of Sister Celine's polynomials

$$\chi^j(\omega) = \frac{(4j-2)!!}{2(4j+1)!!}\frac{\left(\frac{3}{2}\right)_{2j}}{(2j)!}\,f_{2j}\left(1,2,2;1;-;\sin^2\left(\frac{\omega}{4}\right)\right). \tag{21}$$

Note that we have also

$$P_n^{(1/2,1/2)}(x) = \frac{2\Gamma(n+3/2)}{(n+1)!\sqrt{\pi}} U_n(x), \tag{22}$$

where $U_n(x)$ is a Chebyshev polynomial of the second kind, as well as

$$P_n^{(1/2,1/2)}(x) = \frac{\Gamma(n+3/2)}{(n+1)!\Gamma(3/2)} C_n^{(1)}(x), \tag{23}$$

where $C_n^{(\alpha)}$ represents an ultraspherical Gegenbauer polynomial. In the same way, the so-called generalized characters or order λ of the irreducible representation of rank j:

$$\chi_\lambda^j(\omega) = \frac{\sqrt{2j+1}}{(4j+1)!!}\sqrt{(2j-\lambda)!(2j+\lambda+1)!}2^{2j-\lambda}\left[\sin\left(\frac{\omega}{2}\right)\right]^\lambda P_{2j-\lambda}^{(\lambda+1/2,\lambda+1/2)}\left[\cos\left(\frac{\omega}{2}\right)\right] \tag{24}$$

which can be put in the form

$$\begin{aligned}\chi_\lambda^j(\omega) =& \frac{\sqrt{2j+1}}{(4j+1)!!}\sqrt{\frac{(2j+\lambda+1)!}{(2j-\lambda)!}}2^{2j-\lambda}\left[\sin\left(\frac{\omega}{2}\right)\right]^\lambda \left(\lambda+\frac{3}{2}\right)_{2j-\lambda} \\ & \times\ {}_2F_1\left[\begin{array}{c}\lambda-2j, 2j+\lambda+2 \\ \lambda+3/2\end{array};\sin^2\left(\frac{\omega}{4}\right)\right]\end{aligned} \tag{25}$$

or also

$$\begin{aligned}\chi_\lambda^j(\omega) =& \frac{\sqrt{2j+1}}{(4j+1)!!}\sqrt{\frac{(2j+\lambda+1)!}{(2j-\lambda)!}}2^{2j-\lambda}\left[\sin\left(\frac{\omega}{2}\right)\right]^\lambda \left(\lambda+\frac{3}{2}\right)_{2j-\lambda} \\ & \times\ f_{2j-\lambda}\left(1, 2\lambda+2, 2\lambda+2; \lambda+1; -; \sin^2\left(\frac{\omega}{4}\right)\right)\end{aligned} \tag{26}$$

which, for $\lambda = 0$, Eq. (25) reduces to Eq. (21).

2.2 Rotation matrix and Wigner d functions

The Wigner d function reads [12]:

$$d_{mk}^j(\theta) = \xi_{mk}\left[\frac{s!(s+\mu\nu)!}{(s+\mu)!(s+\nu)!}\right]^{1/2}\left[\sin\frac{\theta}{2}\right]^\mu\left[\cos\frac{\theta}{2}\right]^\nu P_s^{(\mu,\nu)}(\cos\theta) \tag{27}$$

with $\mu = |m-k|$, $\nu = |m+k|$, $s = j-(\mu+\nu)/2$, and $\xi_{mk} = 1$ if $k \geq m$ and $(-1)^{k-m}$ if $k < m$.

2.3 Other relations involving Jacobi polynomials

In 1999, Khan found [9]:

$$\frac{1}{\Gamma(\alpha+1/2)}\int_0^\infty t^{\alpha-\frac{1}{2}}\, e^{-t}\, f_n(1, 2\alpha+1; -; -; xt)dt = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\alpha)}(1-2x), \tag{28}$$

as well as

$$\frac{i}{2\sin\left[(\alpha+\frac{1}{2})\pi\right]\Gamma(\frac{1}{2}-\alpha)} \int_{\infty}^{(0+)} (-t)^{-\alpha-\frac{1}{2}}\, e^{-t}\, P_n^{(\alpha,\alpha)}\left(\frac{2x+t}{t}\right) dt = \frac{(1+\alpha)_n}{n!}\, f_n(1, 2\alpha+1; -; -; x). \tag{29}$$

Note that such investigations about the Sister Celine polynomials led to the introduction of generalized Rice polynomials [6], rediscovered by Khan in 1989.

3 Clebsch-Gordan coefficients and Wigner $3j$ symbols

In quantum mechanics, Clebsch-Gordan coefficients describe how individual angular-momentum states may be coupled to yield the total angular-momentum state of a system. In the literature, Clebsch-Gordan coefficients [13] are sometimes also known as Wigner coefficients or vector coupling coefficients. They are closely related to Wigner's $3j$ symbol [14] by

$$C_{a\alpha,\ b\beta}^{c\gamma} = (-1)^{a-b+\gamma}\sqrt{2c+1}\begin{pmatrix} a & b & c \\ \alpha & \beta & -\gamma \end{pmatrix}, \tag{30}$$

where $C_{a\alpha,\ b\beta}^{c\gamma}$ is the Clebsch-Gordan coefficient and

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \tag{31}$$

the $3j$ symbol for angular momenta a, b and c with respective projections α, β and γ. We have in particular [15]:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =& (-1)^{j_1-j_2+m_3}\frac{(2j_1)!(j_1+j_2+m_3)!}{(j_1-j_2+j_3)!(j_1+j_2-j_3)!} \\ & \times \frac{\Delta(j_1,j_2,j_3)}{\delta(j_1,m_1,j_2,m_2,j_3,m_3)} \\ & \times {}_3F_2\left[\begin{matrix} m_1-j_1, j_3-j_1-j_2, -j_1-j_2-j_3-1 \\ -2j_1, -j_1-j_2-m_3 \end{matrix}; 1\right] \end{aligned} \tag{32}$$

with

$$\Delta(j_1,j_2,j_3) = \sqrt{\frac{(-j_1+j_2+j)!(j_1-j_2+j)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}} \tag{33}$$

and

$$\delta(j_1,m_1,j_2,m_2,j_3,m_3) = \sqrt{(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j_3-m_3)!(j_3+m_3)!}. \tag{34}$$

The Hahn polynomials are a family of orthogonal polynomials in the Askey scheme [16] of hypergeometric orthogonal polynomials, introduced by Chebyshev in 1875 [17] and rediscovered by Wolfgang Hahn [18, 19]. They read [20]:

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix} -x, -n, n+\alpha+\beta+1 \\ -N, 1+\alpha \end{matrix}; 1\right] \tag{35}$$

and the Clebsch-Gordan coefficients can be put in the form [21]:

$$C^{\left(n+\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)}_{\frac{N}{2}\left(\frac{N}{2}-x\right),\ \left(\frac{N+\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta-N}{2}+x\right)} = \frac{(-1)^x N!}{\alpha!} \times \left[\frac{(2n+\alpha+\beta+1)(N-x+\beta)!(x+\alpha)!(n+\alpha)!(n+\alpha+\beta)!}{x!(N-x)!(n+\beta)!n!(N-n)!(N+n+\alpha+\beta+1)!}\right]^{1/2} \times Q_n(x;\alpha,\beta,N), \tag{36}$$

or with Sister Celine's polynomials

$$C^{\left(n+\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)}_{\frac{N}{2}\left(\frac{N}{2}-x\right),\ \left(\frac{N+\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta-N}{2}+x\right)} = \frac{(-1)^x N!}{\alpha!} \times \left[\frac{(2n+\alpha+\beta+1)(N-x+\beta)!(x+\alpha)!(n+\alpha)!(n+\alpha+\beta)!}{x!(N-x)!(n+\beta)!n!(N-n)!(N+n+\alpha+\beta+1)!}\right]^{1/2} \times f_n(1,1+\alpha+\beta,1+2\alpha;\alpha+\frac{1}{2};-x;-N;1). \tag{37}$$

One has also [21]:

$$C^{\left(N-n+\frac{\alpha+\beta}{2}\right)\left(\frac{\beta-\alpha}{2}\right)}_{\frac{N}{2}\left(\frac{N}{2}-x\right),\ \left(\frac{N+\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta-N}{2}+x\right)} = (\alpha+N)!N! \times \left[\frac{(2N-2n+\alpha+\beta+1)(n+\beta)!}{x!(N-x)!(\alpha+N-x)!(\beta+x)!}\right]^{1/2} \times \left[\frac{(n+\alpha+\beta)!}{n!(N-n)!(N-n+\beta)!(2N-n+\alpha+\beta+1)}\right]^{1/2} \times Q_n(x;-N-\alpha-1,-N-\beta-1,N). \tag{38}$$

which reads, in terms of Sister Celine's polynomials

$$C^{\left(N-n+\frac{\alpha+\beta}{2}\right)\left(\frac{\beta-\alpha}{2}\right)}_{\frac{N}{2}\left(\frac{N}{2}-x\right),\ \left(\frac{N+\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta-N}{2}+x\right)} = (\alpha+N)!N! \times \left[\frac{(2N-2n+\alpha+\beta+1)(n+\beta)!}{x!(N-x)!(\alpha+N-x)!(\beta+x)!}\right]^{1/2} \times \left[\frac{(n+\alpha+\beta)!}{n!(N-n)!(N-n+\beta)!(2N-n+\alpha+\beta+1)}\right]^{1/2} \times f_n(1,-2N-\alpha-\beta-1,-2N-2\alpha-1,-N-\alpha-\frac{1}{2};-x;-N;1). \tag{39}$$

Considering the $3j$ coefficient, using the Weber-Erdelyi identity [22]:

$$ {}_3F_2\left[\begin{matrix}-n,\alpha,\beta\\ \gamma,\delta\end{matrix};1\right] = \frac{\tilde{\Gamma}(\gamma,\gamma+n-\alpha)}{\tilde{\Gamma}(\gamma+n,\gamma-\alpha)}\ {}_3F_2\left[\begin{matrix}-n,\alpha,\delta-\beta\\ 1+\alpha-\gamma-n,\delta\end{matrix};1\right], \tag{40}$$

where we introduced the notation $\tilde{\Gamma}(x,y,z,\cdots) = \Gamma(x)\,\Gamma(y)\,\Gamma(z)\cdots$ (Γ being the usual Gamma function), we have also (see the notations of Eq. (32)) [23]:

$$
\begin{aligned}
f_n(1,1+\alpha+\beta,1+2\alpha;\alpha+\frac{1}{2};-x;-N;1) =& (-1)^{2j_2+m_1+n+x}\frac{(j_3-j_2-m_1)!}{(2j_2)!} \\
&\times\left[\frac{(j_2-n)!n!(2j_3+n+1)!}{(2j_1-2j_2+n)!(j_3-j_2+m_1+n)!}\right]^{1/2} \\
&\times\left[\frac{x!(2j_2-x)!(j_3-j_2-m_1+n)!}{(j_3-j_2+m_1+x)!(j_3+j_2-m_1-x)!}\right]^{1/2} \\
&\times\begin{pmatrix} j_3-j_2+n & j_2 & j_3 \\ m_1 & x-j_2 & j_2-m_1-x \end{pmatrix},
\end{aligned}
\tag{41}
$$

with

$$
\left\{\begin{array}{l}
n = j_1+j_2-j_3, \\
x = j_2+m_2, \\
N = 2j_2+1, \\
\alpha = j_3-j_2+m_1, \\
\beta = -j_2+j_3-m_1.
\end{array}\right.
\tag{42}
$$

Alternately, it is possible to write [23]:

$$
\begin{aligned}
f_n(1,1+\alpha+\beta,1+2\alpha;\alpha+\frac{1}{2};-x;-N;1) =& \frac{(-1)^{j_1-j_2-m_3}}{(2j_1)!(j_1+j_2+m_3)!} \\
&\times\left[\frac{(2j_1-n)!n!(2j_1+2j_2-n+1)!}{(2j_2-n)!(j_1+j_2-m_3-n)!}\right]^{1/2} \\
&\times\left[x!(2j_1-x)!(j_1+j_2+m_3-x)!\right]^{1/2} \\
&\times\left[(-j_1+j_2-m_3+x)!(j_1+j_2+m_3-n)!\right]^{1/2} \\
&\times\begin{pmatrix} j_1 & j_2 & j_1+j_2-n \\ j_1-x & -j_1+m_3+x & m_3 \end{pmatrix},
\end{aligned}
\tag{43}
$$

with

$$
\left\{\begin{array}{l}
n = j_1+j_2-j_3, \\
x = j_1-m_1, \\
N = 2j_1+1, \\
\alpha = -j_1-j_2-m_3-1, \\
\beta = -j_1-j_2+m_3-1.
\end{array}\right.
\tag{44}
$$

Smorodinskii and Suslov obtained the same result [24] using the Weber-Erdelyi transformation [22]:

$$
\begin{aligned}
{}_3F_2\left[\begin{array}{c} -n,\alpha,\beta \\ \gamma,\delta \end{array};1\right] =& \frac{\tilde{\Gamma}(\gamma,\delta,\delta+n-\alpha,\gamma+n-\alpha)}{\tilde{\Gamma}(\gamma+n,\delta+n,\delta-\alpha,\gamma-\alpha)} \\
&\times\ {}_3F_2\left[\begin{array}{c} -n,\alpha,\gamma-\delta-n \\ 1+\alpha-\delta-n,1+\alpha-\gamma-n \end{array};1\right].
\end{aligned}
\tag{45}
$$

Finally, the following asymptotic connection between Jacobi and Hahn polynomials is worth mentioning [25, 26]:

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \frac{n!}{(\alpha+1)_n}\, P_n^{(\alpha,\beta)}(1-2x). \tag{46}$$

4 Conclusion

We expressed (generalized) characters of irreducible representations of the rotation group as well as Wigner "d" matrices in terms of Sister Celine polynomials. This was made possible thanks to expressions of the formers involving Jacobi polynomials. In the second section we recast Clebsch-Gordan coefficients and Wigner $3j$ symbols as Sister Celine's polynomials, by harnessing their expressions in terms of Hahn polynomials. Using known relations, integral or discrete, as well as algorithms for Sister Celine's polynomials, the expressions obtained in the present work should yield new identities, or provide shorter derivations of known formulas related to the quantum theory of angular momentum.

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