On Orbital Labeling with Circular Contours

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Abstract

Schematic depictions in text books and maps often need to label specific point features with a text label. We investigate one variant of such a labeling, where the image contour is a circle and the labels are placed as circular arcs along the circumference of this circle. To map the labels to the feature points, we use orbital-radial leaders, which consist of a circular arc concentric with the image contour circle and a radial line to the contour. In this paper, we provide a framework, which captures various dimensions of the problem space as well as several polynomial time algorithms and complexity results for some problem variants.

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1 Introduction

Map labeling is an extensively studied topic in computational geometry [1,7,13] that typically involves annotating feature points with names or additional descriptions, ensuring non-overlapping annotations. While traditional maps often use *internal* label positions next to the feature points [12], *external* labeling models [4] place labels remotely along the contour

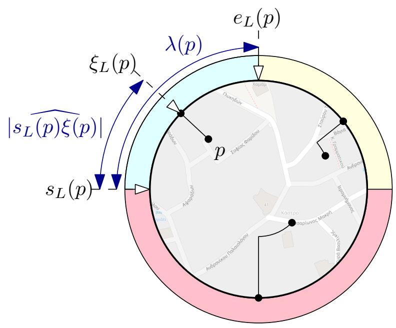

Figure 1 An orbital labeling on a map for illustrating our notation.

Table 1 A tabular overview of the problem space and our results. Empty cells remain open.

<table>
<tr><th colspan="3"></th><th>$A^{🔒}_{\equiv}$</th><th>$A^{📌}_{\equiv}$</th><th>$A^{🔒}_{≣}$</th><th>$A^{📌}_{≣}$</th></tr>
<tr><td rowspan="4">$C^{🔒}$</td><td rowspan="2">$O^{🔒}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n^2|C|)$ [Sec. 5.1]</td><td>$O(n^2|C|)$ [Sec. 5.1]</td><td></td></tr>
<tr><td>$S_{≣}$</td><td>$O(n^2|C|)$ [Sec. 5.1]</td><td></td><td>$O(n^2|C|)$ [Sec. 5.1]</td><td></td></tr>
<tr><td rowspan="2">$O^{📌}$</td><td>$S_{\equiv}$</td><td colspan="2">$O(n|C|^2)$ [Sec. 5.2]</td><td></td><td></td></tr>
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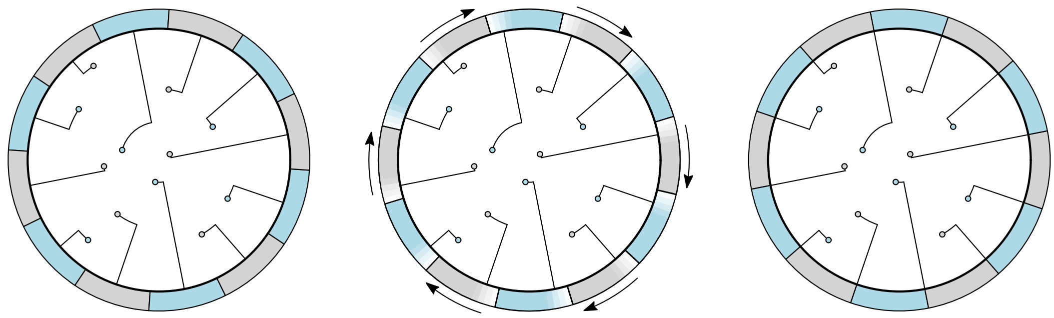
of a bounding shape and connect them to their feature points by crossing-free leaders. This model is frequently used in applications, where feature points are dense, the details of a map or an illustration should not be obscured by labels, or labels are relatively large, e.g., in anatomy atlases or assembly drawings. In this paper we study a novel variant of external labeling with a circular bounding shape, e.g., for displays of smartwatches; see Figure 1. The circular map is displayed in the center of the display and each label is bent and turned into a segment of the circular boundary of the map; we call these labels *orbital* labels. This is a special case of external and boundary labeling [3]. We assume that the lengths of the orbital labels are normalized and sum up to the perimeter of the boundary of the map. Previous research on circular map display considered either multirow circular labels where the sum of label lengths does not equal the map's boundary length [9], radial labels [2,6], or horizontal labels [6,10,11]. The latter two settings are relevant for circular maps on rectangular displays but not suitable for circular displays with a narrow annulus for labels.

Formally, we assume that we are given a disk D in the plane $\mathbb{R}^2$. The disk contains n points $P = \{p_1, \dots, p_n\}$. We call the set P of points *features* and we refer to the boundary of the disk as *the boundary* B. Every feature $p \in P$ has an associated *label* which represents additional information that is to be placed along a circular arc on the boundary starting at a point $b_1 \in B$ and ending at a point $b_2 \in B$. The circular arc along B is denoted as $\widehat{b_1 b_2}_B$. Usually, the start and endpoint of the label are not fixed in the input, however, the length of the arc is part of the input. We represent the associated label simply as a number $\lambda(p)$, which indicates the length of the associated label. We assume that $\sum_{i=1}^{n} \lambda(p_i)$ is equal to the circumference of D, i.e., if all labels are placed non-overlapping then there are no gaps between the arcs on B.

In a *labeling* L, every feature $p \in P$ is assigned a label with starting point $s_L(p) \in B$ and an endpoint $e_L(p) \in B$, s.t., $|s_L\widehat{(p)e_L}(p)| = \lambda(p)$. We assume that all labels are pairwise non-overlapping. Additionally, every feature p is connected to its label via a *leader*. In this paper, we consider *orbital-radial* leaders, which consist of two parts: (1) starting at the feature p with a (possibly empty) orbital circular arc that ends at a bend point q, and (2) a radial segment that connects q to the boundary B; see Figure 1. We call the leader endpoint, i.e., the point where the leader connects to B the *port* $\xi_L(p)$ of the leader starting at p. Note that q has the same distance to the circle center as p since the first part of the leader is an orbital-radial arc. We denote the length of the leader of feature p by $l(p)$.

Let the *port ratio* $\rho_L(p) = \frac{|s_L\widehat{(p)\xi(p)}|}{\lambda(p)}$ be the ratio of the arc from the starting point to the



(a) Uniform ratio of 0.5. **(b)** Clockwise rotation. **(c)** Uniform ratio of 0.

Figure 2 Any solution with uniform label sizes and a uniform ratio (e.g., 0.5) (a) can be rotated (b) to obtain a solution of any other ratio, e.g., 0 (c).

port and the arc from the start-point to the end-point. Now, we define the generic orbital labeling problem.

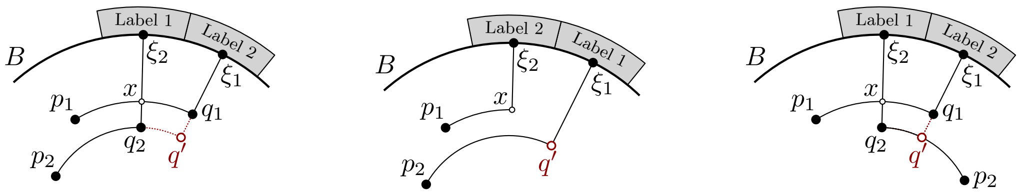
▶ **Problem 1** (Orbital Boundary Labeling). *Given a disk D, containing n feature points P compute a labeling L, in which all leaders are pairwise non-intersecting and the sum of leader lengths is minimal.*

In this paper we first provide a detailed overview of the different variants of Orbital Boundary Labeling in Section 2. For many of these variants, we obtain polynomial time algorithms and complexity results, see Table 1.After some general observations in Section 3, we present our main results about a set of variants, that allow the labels to be placed anywhere around the boundary, in Section 4. The more constrained variants, that consider only a given set of possible candidates for the ports, are discussed in Section 5.

2 Problem Space

In the following, we discuss the dimensions of our problem space. For the different dimensions, we use the notation based on the *COSA*-Orbital Boundary Labeling scheme and use each letter to describe the variants for the respective dimension.

- [C] **Candidate port positions on the boundary.** If we are given a set C of candidate positions on B and in any valid labeling L we require that for any port $\xi \in \Xi_L$ we have $\xi \in C$, we say *the port candidates are locked* (and use the symbol $C^{🔒}$) otherwise they are *free* ($C^{🔑}$).
- [O] **Order.** Next, we consider the cyclic order of labels around B. If a certain label order is pre-specified we say the label order is locked ($O^{🔒}$); otherwise, for the unconstrained setting, we say the label order is free ($O^{🔑}$).
- [S] **Size of labels.** Then, we distinguish the setting where $\forall p \in P : \lambda(p) = 1$, in which case we say that the label size is *uniform* ($S_{\equiv}$), otherwise the label size is *non-uniform* ($S_{\not\equiv}$).
- [A] **Port position on labels.** Lastly, we distinguish different positions of the ports on the labels. We differentiate between *uniform* port ratios, where $\forall i,j\ \rho_L(p_i) = \rho_L(p_j)$, and *non-uniform* port ratios. We also distinguish between the ratios being predefined as part of the input, in which case we call the ratios *locked*, or not, in which case we call them *free*. We obtain the following four settings:
  - Ratios are uniform and locked to a value $k \in [0,1]$ given in the input ($A^{🔒}_{\equiv}$).
  - Ratios are uniform and free, i.e., we have to find a value $k \in [0,1]$ for the ratios ($A^{🔑}_{\equiv}$).



(a) Case 1 **(b)** Rerouted leaders **(c)** Case 2

Figure 3 Given a free label order O^f we can reroute the leaders to arrive at a crossing-free solution with a shorter total leader length.

- Ratios are non-uniform and locked, meaning, we are given a set $K = \{k_1, \dots, k_n\}$ of ratios, s.t., in a valid labeling L, we have $\rho_L(p_i) = k_i$ ($A^{\text{lock}}_{\not\equiv}$).
- Ratios are non-uniform and free, i.e., ports can be chosen freely and independently ($A^{\text{free}}_{\not\equiv}$).

For our problem variants, we use the notation based on the $COSA$-Orbital Boundary Labeling scheme where we substitute C, O, S and A with $C^{\text{free}}/C^{\text{lock}}$, $O^{\text{free}}/O^{\text{lock}}$, $S_{\equiv}/S_{\not\equiv}$ and $A^{\text{free}}_{\not\equiv}/A^{\text{free}}_{\not\equiv}/A^{\text{lock}}_{\equiv}/A^{\text{lock}}_{\not\equiv}$, respectively. An overview of all variants and our results can be seen in Table 1. Whenever a statement applies to all variants along a certain dimension of the problem space, we drop the sub- or superscript of C, O, S, or A. For example, $C^{\text{free}}O^{\text{lock}}SA^{\text{lock}}_{\equiv}$ refers to the variants where the port candidates are free (C^{free}), the order is locked (O^{lock}), the label sizes could be fixed to be uniform or they could be non-uniform (S) and all port ratios are fixed to a given value ($A^{\text{lock}}_{\equiv}$). Therefore, $C^{\text{free}}O^{\text{lock}}SA^{\text{lock}}_{\equiv}$ covers a set of two problem variants.

3 Uniformly Spaced Ports

Using a simple shifting argument, illustrated in Figure 2, we show the following equivalence.

▶ **Observation 3.1.** *All problems in $COS_{\equiv}A^{\text{lock}}_{\equiv}$ are equivalent over all $k \in [0,1]$. Similarly all problems in $COS_{\equiv}A^{\text{free}}_{\equiv}$ are equivalent over all $k \in [0,1]$.*

This equivalence is based on the fact that the ports in these problems are necessarily equally spaced, which is only the case if both the label size and the port ratio are uniform. Based on the same property, we make the following statement, visualized in Figure 3.

▶ **Lemma 3.2.** *Given an instance of a problem variant in $CO^{\text{free}}SA_{\equiv}$ any leader-length minimal labeling L is crossing-free, assuming that all feature points in P lie on circles of different radii concentric with D.*

Proof. Assume a leader-length minimal labeling L contains two crossing leaders γ_1 and γ_2 connecting p_1 to its port ξ_1 and p_2 to its port ξ_2, respectively. Both leaders begin with an orbital segment $\widehat{p_1q_1}$ (or $\widehat{q_1p_1}$) and $\widehat{p_2q_2}$ (or $\widehat{q_2p_2}$), respectively, followed by their radial straight-line segment $q_1\xi_1$ and $q_2\xi_2$. Clearly the crossing x between the radial segment of the point closer to the center X of D and the orbital segment of the point closer to B. W.l.o.g. assume that $x = \widehat{p_1q_1} \cap q_2\xi_2$, and that p_1 is on the counter-clockwise end of $\widehat{p_1q_1}$. Let q' be the intersection of the supporting line of $q_1\xi_1$ and the circle containing $\widehat{q_2p_2}$. There are two cases shown in Figure 3.

In the first case (Figure 3a) we can replace γ_1 with a curve consisting of $\widehat{p_1x}$ and $x\xi_2$ and γ_2 with curve consisting of $\widehat{p_2q'}$ and $q'\xi_1$ (Figure 3b). Since $|q_2x| = |q'q_1|$ and $|\widehat{xq_1}| > |\widehat{q_2q'}|$, the total leader length has decreased. Note that the rerouting might have introduced new crossings, but since this method reduces the total leader length we can iteratively apply this

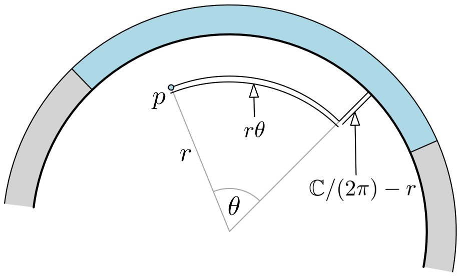


Figure 4 Determining the leader length by the length of the orbital and radial segment.

procedure and will never obtain an already seen labeling. Since there is a finite number of possible solutions, we have to arrive at a solution, which does not contain crossings anymore (otherwise we could apply the procedure infinitely many times contradicting the finite number of possible solutions). While the setup of the second case (Figure 3c) looks different, we can resolve the crossing identically to the first case and the sum of leader lengths again decreases concluding the proof. ◀

4 Free Candidates

In this section, we consider the problem set $C OSA$. Intuitively, these are problem variants, that allow solutions to be continuously rotated around B. Let $g : P \times [0, 2\pi] \to \mathbb{R}$ be a function which maps a feature $p \in P$ and an angle θ to the length of a leader that connects p and a port on B, s.t., the orbital segment of the leader spans the angle θ. Let r be the radius of the circle containing p concentric with D. If D has a circumference of $\mathcal{C}$ it has a radius of $\frac{\mathcal{C}}{2\pi}$. Then $g(p, \theta) = \frac{\mathcal{C}}{2\pi} - r + r\theta$; see Figure 4.

▶ **Observation 4.1.** *The function $g(p, \theta)$ is linear in θ.*

The total leader length of a labeling L can obtained as $h(L) = \sum_{i=1}^{n} g(p_i, \theta(p_i))$, where $\theta(p_i)$ is the angle spanned by the orbital segment of the leader connected to p_i.

Note that by fixing a port ξ on the boundary for a feature p, there are two orbital-radial leaders by which we could choose to connect them (with a clockwise or a counter-clockwise orbital segment). We call these *clockwise* and *counter-clockwise* leader, respectively.

▶ **Observation 4.2.** *The inner-most feature, i.e., the feature which lies on a circle concentric with D whose radius is smallest among all features can always be labeled with a clockwise or a counterclockwise leader.*

Observe that the leader of the inner-most feature p uses a radial line segment s starting on a circle concentric with D, which does not contain any other feature of P. Consider any other feature p' and any other point ξ' on B, then, p' and ξ' can be connected either with a clockwise or a counter-clockwise leader; see Figure 5. The orbital segments of the clockwise and the counter-clockwise leader of p' together form an entire circle concentric with D containing p and, hence, one of them has to intersect s.

▶ **Observation 4.3.** *A leader of the inner-most feature determines for every other leader γ connecting a feature and a point on B if γ is a clockwise or counter-clockwise leader.*

4.1 Locked Order.

Next, we consider problem variants in $C O SA$, in which the label order is locked.

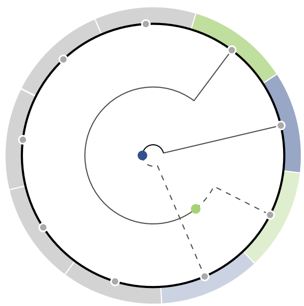


Figure 5 The port position of the inner-most feature (blue) determines labeling of other features: Depending on the labeling of the inner-most feature, the green feature point has access to different candidate ports (dark labels and solid leaders vs. light labels and dashed leaders).

▶ **Lemma 4.4.** *For $C^{\text{📍}}O^{\text{🔒}}SA^{\text{🔒}}$, the choice of a port point on B for the inner-most feature determines all other label placements including their port positions as well as their leaders. This also includes the length of their leaders and the angle that is spanned by the orbital segment of these leaders.*

Proof. This lemma directly follows from Observation 4.3 and the key point that the placement of one label not only determines the placement of others but also their port positions. ◀

With this, we state a method of solving the four problems $C^{\text{📍}}O^{\text{🔒}}SA^{\text{🔒}}$ and by extension $C^{\text{📍}}O^{\text{🔒}}S_{\equiv}A^{\text{📍}}_{\equiv}$ (recall Observation 3.1). By Lemma 4.4 the exact position of the port of the inner-most feature p_1 is the only degree of freedom when choosing a labeling. By Lemma 4.4, we immediately obtain the angle θ_1 spanned by its orbital segment. By Lemma 4.4, we likewise obtain all angles $\theta_2, \dots, \theta_n$ of all the other leaders. Therefore we can express the functions $g(p_2, \theta_2), \dots, g(p_n, \theta_n)$ all as piecewise linear functions of θ_1, which consist of exactly two linear pieces. The sum over all of these functions is therefore a piece-wise linear uni-variate function and we can find the minimum of it in $O(n)$ time.

To guarantee that a solution (if it exists) found in this way is crossing free we compute an *admissible range* $I_{i,j}$ for θ_1, s.t., if $\theta_1 \in I_{i,j}$ the leaders of p_i and p_j are crossing free; see Figure 6. $I_{i,j}$ is one continuous interval and therefore we can in $O(n^2)$ time determine all ranges as well as their (also continuous) intersection if it exists. Then we either restrict our search for a minimum to this intersection or – if the intersection is empty – know that no solution exists.

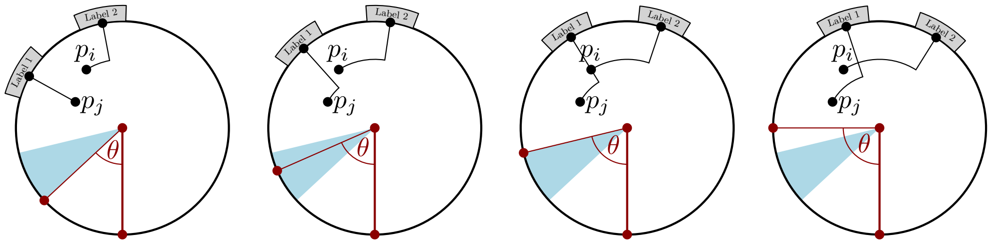
4.2 Free Order.

When the label order is not locked, we get problem variants in $C^{\text{📍}}O^{\text{📍}}SA$. We will present in Section 4.2.1 a polytime algorithm for uniform labels (or more specifically uniformly spaced ports, i.e., the problems in $C^{\text{📍}}O^{\text{📍}}S_{\equiv}A_{\equiv}$). However for most variants with non-uniform labels the problem turns out to be NP-hard (Section 4.2.2).

4.2.1 An algorithm for uniformly spaced ports

We will use a reduction of some of these problems to a non-circular variant called Boundary Labeling [5]. In particular we use the following lemma.

▶ **Lemma 4.5.** *In any (crossing-free) labeling of an instance of a problem in $C^{\text{📍}}O^{\text{📍}}SA$, there exists a point $b \in B$, s.t., db does not intersect any leader, where d is the center of D.*



(a) **(b)** **(c)** **(d)**

Figure 6 Two clockwise leaders whose ports are rotated, s.t., (a) the length of the orbital segment of p_j is 0, (b) the leaders are non intersecting, (c), the radial segment of p_j contains p_i and (d) the leaders intersect. The admissible range of θ is shown in blue.

Proof. Let x be the smallest angle between two points, two ports, or a point and a port in an optimal labeling L (measured with 0 as the center). Consider the radial segment of the leader of the inner-most feature p_1, and assume w.l.o.g. that the orbital segment is clockwise. Set b' to be $\xi(p_1)$ but rotate it clockwise by $x/2$. Since the leader of p_1 does not intersect any other leader and the next feature or port is at least at an angle x in clockwise position from $\xi(p_1)$, the segment db' must now be crossing free. ◀

The previous lemma argues the existence of this *splitting line* in any labeling. Next, we state that we only need to consider $O(n^2)$ possibilities for such a line.

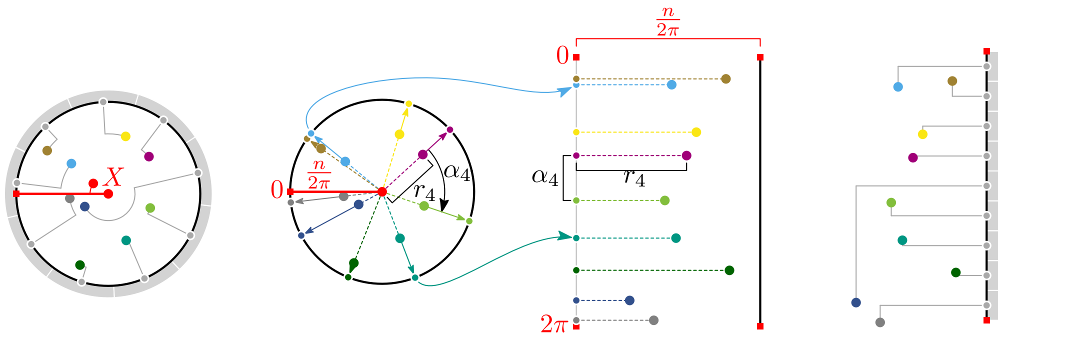
▶ **Lemma 4.6.** *For any problem in $C^{\nearrow}O^{\nearrow}S_{\equiv}A_{\equiv}$ there are only n^2 possibilities for the port of the inner-most feature.*

Proof. First note that if we can guarantee that a port in an optimal labeling is an intersection of B and a line through the center d of D and a feature, then we only need to consider all n such intersection points together with $n-1$ equally distributed points around B as ports.

Assume that L is an optimal labeling, where no port is such an intersection point. Now consider a small rotation of all ports clockwise, which does not change the order of labels and does not introduce any intersections. If such a rotation is not possible, the radial segment of a clockwise leader already contains another feature and therefore its port was an intersection point. If this rotation decreases the total leader length, then L was not optimal. If the leader length stays the same, we can continue the rotation until either the orbital segment of a counter-clockwise leader reaches length 0 or the radial segment of a leader hits another feature. In both cases its port is an intersection point. If the leader length increases, we rotate counter-clockwise. Again if now the total leader length decreases or stays the same, the arguments above apply. Assume therefore that the total leader length again increases. Since by Observation 4.1 the change in leader length is linear in θ, there must be a single leader that increases its length in both rotation directions, which implies that its orbital segment has length 0 in L.

Therefore in any optimal labeling, at least one port is an intersection point. By considering all n possible points, which each define a set of n ports we obtain at most n^2 possibilities for the port of the inner-most feature. ◀

The Boundary Labeling problem takes as input a set of features, which are entirely to the left of a straight vertical line V. The goal is to find a placement of uniformly sized labels along this line, s.t., every feature can be connected to one label with a *po*-leader, which is a piecewise linear line segment starting at the feature, continuing for some amount vertically



(a) Instance Orbital Boundary Labeling **(b)** Construction of new instance of Boundary Labeling **(c)** Solution Boundary Labeling

Figure 7 Based on an instance (a) of Orbital Boundary Labeling and a cutting line we use an appropriate mapping to construct an instance of Boundary Labeling, whose solution (c) corresponds to the solution of Orbital Boundary Labeling.

(parallel to V) and then turning 90° towards the line and continue horizontally until it meets V. For any feature p and any point $v \in V$, let $\text{bad}(p, v)$ be a function, which provides the 'badness' of a *po*-leader connecting p and v. A labeling, which optimizes an arbitrary badness function $\text{bad}(p, v)$ can be computed in $O(n^3)$ time [5]. In the next lemma we explain how we obtain an instance of Boundary Labeling based on an instance of Orbital Boundary Labeling.

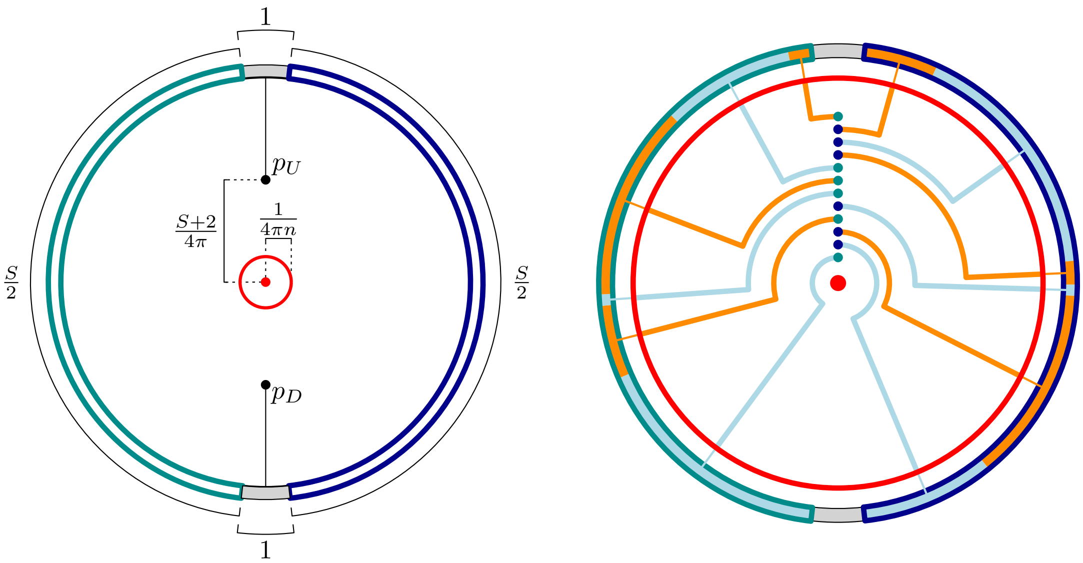
▶ **Lemma 4.7.** *Given a port ξ_1 for the inner-most feature p_1, we can reduce $C^{\bullet}O^{\bullet}S_{\equiv}A_{\equiv}$ to an instance of po-Boundary Labeling [5].*

Proof. The reduction is shown in Figure 7. Let ℓ_c be a line segment from X to ξ_1. Note that, if we remove the leader of p_1, ℓ_c does not intersect any leader of a leader-length optimal labeling. For any feature $p \in P$, let $\alpha_p = \measuredangle \xi_1 X p$, i.e., the size of the clockwise angle between ξ_1 and p centered at X. Let $r_p = |Xp|$, i.e., the distance of p to the center of D. For every feature $p \in P$ we create a feature $p' = (r_p, \alpha_p)$. Note that the visualization in Figure 7 is being drawn top to bottom and therefore technically uses the coordinates $(r_p, 2\pi - \alpha_p)$, which is of course equivalent up to symmetry. Finally we place the vertical line segment V of length $(n-1)2\pi$, with its lowest point at $(n/2\pi, 0)$.

Now the length of the radial segment of an orbital-radial leader connecting p to a point $b \in B$ in our problem is equal to the length of the orthogonal part of the *po*-leader connecting p' to a point $v \in V$. The relation between the length of the orthogonal part of the *po*-leader and the length of the orbital segment of the *or*-leader is more complicated. The length of the orthogonal part of the *po*-leader is simply the difference in y-coordinate between p' and v. Note that we mapped the clockwise angle of a point relative to ℓ_c to the y-coordinate of p'. However, the length of the circular segment of the *or*-leader is dependent on the distance of p to X, i.e., two *or*-leaders whose circular segments span the same angle can have different lengths. Specifically the length of a circular segment in an *or*-leader of a feature p is exactly $r_p\theta_p$. Therefore we define our badness function simply as $\text{bad}(p, v) = n/2\pi - r + r(|\theta_p - y(v)|)$, where $y(v)$ is the y-coordinate of v[1]. ◀

▶ **Theorem 4.8.** *Any problem in $C^{\bullet}O^{\bullet}S_{\equiv}A_{\equiv}$ can be solved exactly in $O(n^5)$.*

[1] It should be noted that the restriction of port placement on V to a specific range can be encoded in this badness function too, by setting the value of any leader that would exceed the permitted range to ∞.



(a) Placement of the two gray blocker labels p_U and p_D

(b) Detail of the placement of $p_1, \ldots, p_n$

Figure 8 Visualization of the reduction. The two points p_U and p_D are placed close two the boundary (a) and all points $p_1, \ldots, p_n$ are placed in the very small red circle. A zoomed in picture of the red circle is shown in (b).

Proof. This theorem follows immediately by applying the reduction of Lemma 4.7 to the instance of $C O S_{\equiv} A_{\equiv}$ and each one of the at most $O(n^2)$ possible ports of the innermost feature point (Lemma 4.6) and afterwards solving each of the resulting $O(n^2)$ instances using the existing $O(n^3)$ algorithm [5] to obtain a *po*-labeling minimizing the badness function, which has a one-to-one correspondence to a leader length minimal *or*-labeling of our original instance. ◀

4.2.2 Non-uniform label sizes are NP-hard.

Finally, we investigate problems without candidate ports, a free order on the labels, and non-uniform label sizes. We show NP-hardness for the problem variants $C O S_{\equiv} A_{\equiv}$. The hardness of $C O S_{\equiv} A_{\equiv}$ extends to $C O S_{\equiv} A_{\equiv}$, while $C O S_{\equiv} A_{\equiv}$ remains open.

▶ **Theorem 4.9.** *Given an instance of $C O S_{\equiv} A_{\equiv}$, $C O S_{\equiv} A_{\equiv}$ or $C O S_{\equiv} A_{\equiv}$ together with $k \in \mathbb{R}$ it is (weakly)* NP*-hard to decide if there exists a labeling L with a total leader length of less than k.*

Proof. The reduction is from Partition, where we are given a set X of n integers with $S = \sum_{x \in X} x$ and need to decide if X can be partitioned into two sets X_1 and X_2, s.t., $\sum_{x \in X_1} x = \sum_{x \in X_2} x = S/2$. For the reduction we place for every $x_i \in X$ a feature $p_i = (0, \frac{i}{4\pi n^2})$. Additionally we place two fetures $p_U = (0, r)$ and $p_D = (0, -r)$, where $r > \frac{S+2}{4\pi}$. We define $\lambda(p_i) = x_i$ for all $1 \leq i \leq n$ and $\lambda(p_U) = \lambda(p_D) = 1$. Note that $\sum_{i=1}^{n} \lambda(p_i) + \lambda(p_U) + \lambda(p_D) = S+2$ and the radius of the enclosing disk is therefore $(S+2)/2\pi$.

Any feature p_i, s.t., $1 \leq i \leq n$ is contained in a disk of radius 1 centered at the origin. Let $o(i)$ and $r(i)$ be the orbital and radial part of $\gamma_L^{p_i}$, respectively. Note that the sum over all $r(i)$ is equal in all labelings. Let this sum be equal to L_{radial}. Further note that for any p_i, $o(i) < 1/2n$. Therefore the sum of length over all $o(i)$ is smaller than $1/2$ in all labelings.

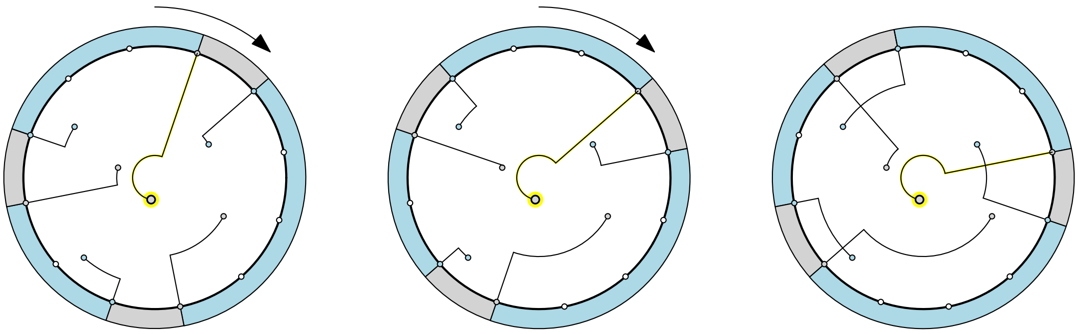


Figure 9 Three rotations for case C🔒O🔒SA🔒. The highlighted feature is the first that is placed and we iteratively test every port candidate. Due to the fixed order the other leaders are directly obtained.

For the problem variants C📌O📌S≡A📌≡ and C📌O📌S≡A🔒≡, in any labeling L the port ratios $\rho_L(p_U)$, and $\rho_L(p_D)$ are necessarily equal. For the variant C📌O📌S≡A🔒≡ port ratios are described as part of the input and we define them, s.t., $\rho_L(p_U) = \rho_L(p_D)$.

Assume there exists a partition of X into two sets X_1, X_2, s.t., $\sum_{x \in X_1} x = \sum_{x \in X_2} x$. We now make three observations. First, there exists a labeling L in which the length of the orbital part of $\gamma_L^{p_U}$ and $\gamma_L^{p_D}$ is equal to 0 and therefore $\gamma_L^{p_U}$ and $\gamma_L^{p_D}$ are straight lines. Second, in L both spaces between the labels of p_U and p_D are equally spaced, i.e., $|e_L(\widehat{p_U)s_L}(p_D)| = |e_L(\widehat{p_D)s_L}(p_U)|$, since $\rho_L(p_U) = \rho_L(p_D)$. Third, in a labeling L', in which the length of the orbital part of $\gamma_L^{p_U}$ or $\gamma_L^{p_D}$ is not equal to 0, the sum of the length of the orbital parts of $\gamma_{L'}^{p_U}$ and $\gamma_{L'}^{p_D}$ (and therefore the sum over the lengths of all orbital parts of leaders in L') is at least $\frac{2\pi}{S+2} \cdot \frac{S+2}{4\pi} = \frac{1}{2}$. This is because the difference between $e_{L'}(\widehat{p_U)s_{L'}}(p_D)$ and $e_{L'}(\widehat{p_D)L'}(p_U)$ is at least 1 (since the label sizes are integers).

Therefore the sum over all leader lengths in L is less than $1/2 + L_{\text{radial}}$, while in L' it is at least $1/2 + L_{\text{radial}}$ and L' can never be optimal. Finally we set $k = 1/2 + L_{\text{radial}}$.

Assume now that X can be partitioned into two subsets X_1, X_2, s.t., $\sum_{x \in X_1} x = \sum_{x \in X_2} x$. Then we know that the labels can be equally partitioned and the leaders of p_U and p_D can be straight lines. Therefore the total sum of leader lengths is less than k. Conversely assume that no such partition exists. Then the leaders of p_U and p_D must together always contain orbital segments of length at least 1/2 and the total sum of leader lengths is at least k, concluding the proof. ◀

5 Locked Port Candidates

Here we investigate the problem set C🔒OSA. Recall that we are given a set C of candidate positions for the ports. There are $2n|C|$ possible leaders: each of the n features can connect to each port candidate in C, via a clockwise or counter-clockwise or-leader.

5.1 Locked Order.

If we have a locked order and locked port ratios, i.e., C🔒O🔒SA🔒 the placement of a single label determines the position of all other labels. Therefore it is sufficient to place the first label with its port coinciding with one candidate ($O(|C|)$ possibilities) and then check in $O(n)$ time if the ports of the remaining labels placed in order of O also coincide with candidates; see Figure 9. To ensure that for each such choice of candidate port no leaders overlap, we can

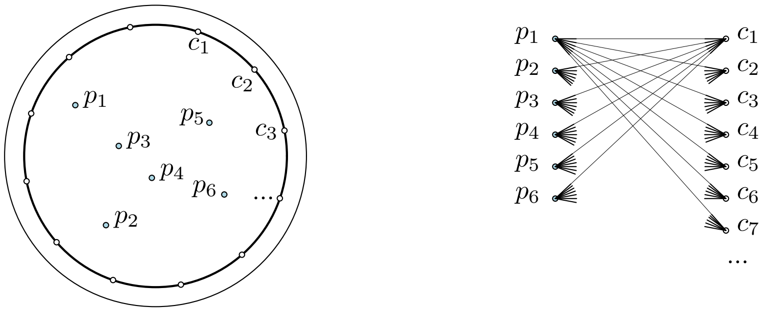


Figure 10 Case $C^{\text{🔒}}O^{\text{🔑}}S_{\equiv}A^{\text{🔒}}_{\equiv}$. Each feature $p_1, \cdots, p_6$ and port candidate $c_1, \cdots, c_{12}$(left) introduces a vertex in the weighted complete bipartite graph (right). An edge in the bipartite graph corresponds to a leader and is weighted with the leader's length.

naively check for each leader whether it overlaps with any of the other leaders in $O(n)$ time, or $O(n^2)$ time in total. Therefore we can in $O(n^2|C|)$ time compute a leader length optimal labeling or decide that no labeling exists given the port candidates. By Observation 3.1, the result of $C^{\text{🔒}}O^{\text{🔒}}S_{\equiv}A^{\text{🔒}}_{\equiv}$ extends to $C^{\text{🔒}}O^{\text{🔒}}S_{\equiv}A^{\text{🔑}}_{\equiv}$.

5.2 Free Order.

Now we consider the problem $C^{\text{🔒}}O^{\text{🔑}}S_{\equiv}A^{\text{🔒}}_{\equiv}$ (equivalent to $C^{\text{🔒}}O^{\text{🔑}}S_{\equiv}A^{\text{🔑}}_{\equiv}$ by Observation 3.1). We can create a weighted complete bipartite graph G between P and C (recall that $n \leq C$) using the length of the leader between a feature $p \in P$ and a potential port $c \in C$ as the weight of the edge (p, c); see Figure 10. Then a minimum weight bipartite matching in G corresponds to a leader-length minimal labeling. Such a matching in a bipartite graph with $|V|$ vertices and $|E|$ edges can be computed in $O(|V|^2 \log |V| + |V||E|)$ time [8]. In our case $|V| = n + |C|$ and $|E| = n|C|$ and $n \leq |C|$ (otherwise no solution exists). Therefore the runtime is $O(n|C|^2)$. We know by Lemma 3.2, that such a labeling is crossing free and therefore the optimal solution.

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