SUFFICIENT CONDITIONS FOR SOLVABILITY OF OPERATORS OF SUBPRINCIPAL TYPE

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Abstract. In this paper we show that condition $\mathrm{Sub}_r(\Psi)$ on the subprincipal symbol is sufficient for local solvability of pseudodifferential operators of real subprincipal type. These are the operators having real principal symbol which vanish on an involutive manifold where the subprincipal symbol is of principal type. This condition has been shown in [5] and [6] to be necessary for local solvability of pseudodifferential operators of real subprincipal type.

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1. Introduction

In this paper we shall study the local solvability of classical pseudodifferential operators $P \in \Psi^m_{cl}$, which are given by an asymptotic expansion $p_m(x,\xi) + p_{m-1}(x,\xi) + \dots$ of terms $p_{m-j}(x,\xi)$ homogeneous of degree $m-j$ in ξ for $j \in \mathbf{N}$, where $p_m = p$ is the principal symbol. We are going to study operators which are not of principal type, i.e., when the principal symbol p vanishes of at least order 2, in particular the sufficiency in the case when the principal symbol is real and has involutive double characteristics. But we will also assume that the operator is of subprincipal type, so that the subprincipal symbol of the operator is of principal type, see Definition 1.9.

The definition that P is locally solvable at a compact subset of a manifold $K \subseteq X$ is that the equation

$$Pu = v \tag{1.1}$$

has a local solution $u \in \mathcal{D}'(X)$ in a neighborhood of K for any $v \in C^\infty(X)$ in a set of finite codimension. We can also define microlocal solvability of P at any compactly based cone $K \subset T^*X$, see Definition 1.10.

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For classical pseudodifferential operators $P \in \Psi^m_{cl}$ which are of principal type, local solvability is equivalent to condition (Ψ) on the principal symbol p, i.e.,

$$\text{Im}\, ap \text{ does not change sign from } - \text{ to } + \quad \text{along the oriented bicharacteristics of } \text{Re}\, ap \text{ when } \text{Re}\, ap = 0 \tag{1.2}$$

for any $0 \neq a \in C^\infty(T^*X)$, see [3] and [9]. The oriented bicharacteristics are the positive flow of the Hamilton vector field $H_{\text{Re}\, ap} \neq 0$ on which $\text{Re}\, ap = 0$, these are also called *semibicharacteristics* of p. Observe that if condition (Ψ) is satisfied on a set, then it is trivially satisfied on any subset.

For operators which are not of principal type, the invariant *subprincipal symbol*

$$p_s = p_{m-1} + \frac{i}{2} \sum_j \partial_{x_j} \partial_{\xi_j} p \tag{1.3}$$

becomes important. There are several conditions corresponding to condition (Ψ) on the subprincipal symbol, several necessary conditions for solvability are known, but not many sufficient conditions.

One of the earliest results are by Mendoza and Uhlman [16], who studied the case when principal symbol is equal to a product $p = p_1 p_2$ with p_j of real principal type with linearly independent differentials dp_1 and dp_2. Thus the double characteristic set $\Sigma_2 = \{p_1 = p_2 = 0\}$ is a intersection of two transversal hypersurfaces. In this case, they proved that P is *not solvable* if the imaginary part of the subprincipal symbol p_s changes sign on the bicharacteristics of p_1 or p_2 on Σ_2. These are the limits of the characteristics of the principal symbol at the double characteristic set Σ_2. They proved in [17] that P is solvable if the imaginary part of the subprincipal symbol p_s does not vanish on the double characteristics, thus there are no sign changes.

Mendoza [18] generalized the necessary condition to the case when the principal symbol is real, vanishes of second order on an involutive submanifold where it has an indefinite Hessian with rank equal to the codimension of the manifold. Then Hessian gives well-defined limit bicharacteristis on the submanifold, and P is not solvable if the imaginary part of the subprincipal symbol changes sign on any of these limit bicharacteristics.

There are several other necessary condition for solvability of operators that are not of principal type corresponding to condition (Ψ) on operators of principal type. The following one generalizes Mendoza's and Uhlmann's necessary conditions for solvability.

Example 1.1. A necessary condition on the subprincipal symbol for solvability of operators with real principal symbol p vanishing of at least second order on an involutive submanifold Σ_2 is condition $\mathrm{Lim}(\Psi)$:

$$\text{(1.4)} \quad \operatorname{Im} p_s \text{ does not change sign from } - \text{ to } + \text{ on the limit bicharacteristics of } p \text{ on } \Sigma_2$$

which follows from the necessary condition (2.9) of [5].

This condition is invariant under symplectic changes of variables and multiplication with nonvanishing real factors, since a negative factor changes both the direction of the limit bicharacteristic and the sign of the imaginary part. Thus, it is invariant under conjugations of the operator with Fourier integral operators with real principal symbols.

Observe that this is a condition on the sign changes of $\operatorname{Im} p_s$ at a (possibly empty) subset of directions on the leaves of Σ_2. The sufficient conditions that we are going to use will exclude any sign changes of $\operatorname{Im} p_s$ on the leaves of Σ_2. But even this stronger condition is not sufficient, one also needs conditions on the imaginary part of the subprincipal symbol in the direction of the bicharacteristics if the real part of the subprincipal symbol. For that, the operator has to be of *subprincipal type*, which means that the subprincipal symbol is of principal type, with Hamilton vector field that is tangent to Σ_2 at the characteristics, see Definition 2.2 in [6] or Definition 1.9 in this paper.

Example 1.2. A necessary conditions for solvability for operators of subprincipal type for involutive Σ_2 is given by Definition 2.4 in [6]. It is condition $\mathrm{Sub}(\Psi)$ which is $\mathrm{Sub}(\Psi)$ on subprincipal symbol p_s on Σ_2:

$$\text{(1.5)} \quad \operatorname{Im} ap_s \text{ does not change sign from } - \text{ to } + \\ \text{on the oriented bicharacteristics of } \operatorname{Re} ap_s \text{ when } \operatorname{Re} ap_s = 0 \text{ on } \Sigma_2.$$

for any $0 \neq a \in C^\infty$. It is known that this condition is invariant under symplectic changes of variables and multiplication with nonvanishing factors, since the subprincipal symbol then only get multiplied with these factors.

Observe that condition (1.5) is empty if p_s vanishes of second order, so for this condition we need that the operator is of subprincipal type, see Definition 1.9.

A stronger necessary condition for solvability involves the sign changes on the imaginary part of the subprincipal symbol on a larger set of curves on the double characteristic set, actually on the limits of the bicharacteristics of the real part of the *refined principal symbol*

$$p_r = p + p_s \tag{1.6}$$

where p_s is the subprincipal symbol given by (1.3), see Theorem 18.1.33 in [10]. This symbol is invariant under conjugation with elliptic Fourier integral operators, and multiplication with $a(x,D) \in \Psi^0$ gives the refined principal symbol ap_r modulo terms in S^{m-1} vanishing of order 1 on Σ_2. In fact, the refined principal symbol of $a(x,D)P(x,D)$ is $ap_r + \frac{i}{2}H_p a$ modulo S^{m-2}.

Example 1.3. *A necessary condition for solvability for operators of subprincipal type with principal symbol vanishing of second order on Σ_2 is* $\mathrm{Sub}_2(\Psi)$. *This condition is given by Definition 2.6 in* [7] *and is condition* (Ψ) on the symbol

$$p_{s,2} = J^2(p) + J^0(p_{m-1}) = J^2(p) + p_s \tag{1.7}$$

where $J^2(p)$ equal to the 2:nd jet of p at Σ_2.

This condition is invariant under symplectic changes of variables and multiplication with nonvanishing factors by Remark 2.3 in [7], since then $p_{s,2}$ gets multiplied with a nonvanishing factor.

Observe that this definition gives conditions on the sign changes of $\operatorname{Im} p_{s,2}$ on the limits of the bicharacteristics of $\operatorname{Re} p_{s,2}$ at Σ_2, which are the limits of of the bicharacteristics of $\operatorname{Re} p_r$, see (1.19). Condition $\mathrm{Sub}_2(\Psi)$ gives (1.4) and (1.5), but the directions of the limit characteristics depend on the sign of $\operatorname{Re} p_s$, see the following example.

Example 1.4. *If $p = |\xi'|^2 - |\xi''|^2$ with $(\xi', \xi'') \in \mathbf{R}^n \times \mathbf{R}^m$, then*

$$H_p = 2(\xi' \cdot \partial_{x'} - \xi'' \cdot \partial_{x''}) = 2|\xi|(\theta' \cdot \partial_{x'} - \theta'' \cdot \partial_{x''}) \qquad |\theta| = 1$$

which gives all directions in x when $\xi \to 0$. If we take the limit only when $p_r = 0$, i.e., when $p = -\operatorname{Re} p_s$, then we get the limit bicharacteristics $\theta' \cdot \partial_{x'} - \theta'' \cdot \partial_{x''}$ with $|\theta'| > |\theta''|$ when $\operatorname{Re} p_s < 0$, the ones with $|\theta'| < |\theta''|$ when $\operatorname{Re} p_s > 0$ and all directions in $\mathbf{R}^n \times \mathbf{R}^m$ when $\operatorname{Re} p_r = 0$.

Thus, when $\operatorname{Re} p_s = 0$ we may obtain that the sign of $\operatorname{Im} p_{s,2} = \operatorname{Im} p_s$ is constant on the leaves of Σ_2 when $\operatorname{Re} p_s = 0$, but by Example 1.5 that is not enough to get solvability. Observe that we shall assume that $\operatorname{Re} p_r$ is constant on the leaves by (1.13).

Also observe that the necessity of the conditions in Examples 1.1–1.3 only hold under some additional conditions on the symbol, for example finite order of the sign change. For the sufficiency, it is not enough that $\mathrm{Sub}(\Psi)$ holds when $p_s = 0$, by the following example.

Example 1.5. *Consider the PDO*

$$P_\pm = (1 + t^2)\Delta_x + D_t \pm itD_t \tag{1.8}$$

with symbol $(1+t^2)|\xi|^2+\tau \pm it\tau = (1+t^2)|\xi|^2+(1\pm it)\tau$. *This operator satisfies the condition* $\mathrm{Lim}(\Psi)$ *by* (1.4) *and* $\mathrm{Sub}(\Psi)$ *by* (1.5) *since* $\operatorname{Im} p_s = \pm t\tau = 0$ *when* $\operatorname{Re} p_s = \tau = 0$. *But multiplication by* $(1\pm it)^{-1} = (1\mp it)/(1+t^2)$ *gives the operator* $D_t + (1\mp it)\Delta_x$, *and conjugation with the Schrödinger kernel* $\exp(\pm it\Delta_x)$ *gives the operator* $Q_\mp = D_1 \mp it\Delta_x$. *Here* Q_+ *is the Mizohata operator which is a standard example of an unsolvable operator, and* Q_- *is solvable, since* $u(t,x) = i\int_0^t \exp((s^2-t^2)\Delta_x/2)f(s,x)\,ds$ *solves* $Q_- u = f \in C_0^\infty$.

Observe that the condition $\mathrm{Lim}(\Psi)$ in Example 1.1 does in general not imply that $\operatorname{Im} p_s$ has constant sign on the leaves of Σ_2.

Example 1.6. *If the principal symbol of* P *is* $D_{x_1}D_{x_2}$, *then the leaves of* Σ_2 *have dimension* 2. *Divide the leaves into a checkerboard and index the squares with* $(j,\,k) \in \mathbf{Z}^2$. *Denote the squares with index* $(2j, 2k)$ *with* S_+ *and the ones with index* $(2j+1, 2k+1)$ *with* S_- *and the rest with* S_0. *If* $\operatorname{Im} p_s > 0$ *in the interior of the squares in* S_+, $\operatorname{Im} p_s < 0$ *in the interior of the squares in* S_- *and* $\operatorname{Im} p_s = 0$ *on the squares in* S_0, *then* $\operatorname{Im} p_s$ *has constant sign along any* x_1 *and* x_2 *lines, but not on the whole plane.*

The conditions on $P \in \Psi^m_{cl}$ in the present paper will be the following. Let p be the real principal symbol, $\Sigma = p^{-1}(0)$ be the characteristics and $\Sigma_2 = \{\, p = |dp| = 0 \,\}$ be the double characteristics. We assume that Σ_2 is a nonradial involutive submanifold and that p is real and vanishes of exactly second order at Σ_2 so that

$$\operatorname{Hess} p \text{ is } \textit{nondegenerate} \text{ on } \Sigma_2 \tag{1.9}$$

This implies that p is of real principal type on $\Sigma_1 = \Sigma \setminus \Sigma_2$ in a sufficiently small conical neighborhood of Σ_2, since it cannot vanish of second order on Σ_1. Also, $\operatorname{Hess} p\big|_{\Sigma_2}$ has locally constant rank and index. That Σ_2 is nonradial means that if a function vanishes on Σ_2 then its Hamilton vector field does not have the radial direction on T^*X, which is a generic condition.

Remark 1.7. *The invariant condition is that* p *is proportional to a real function. This means that the quotient* $q = \operatorname{Im} p/\operatorname{Re} p$, *which is defined where* $\operatorname{Re} p \neq 0$, *can be extended to a* C^∞ *function with values on the extended real line* $\overline{\mathbf{R}}$, *i.e., either* q *or* q^{-1} *is smooth. In fact, if* $q \in C^\infty$ *then* $p = (1+iq)\operatorname{Re} p$ *and if* $q^{-1} \in C^\infty$ *then* $p = (q^{-1}+i)\operatorname{Im} p$.

In the following, we will assume that P is on the form so that the principal symbol p is real valued. In the case when p is not proportional to a real function, condition (Ψ) on the principal symbol has to be satisfied on Σ_1 since it is then necessary for solvability.

Recall that the subprincipal symbol

$$p_s = p_{m-1} + \frac{i}{2}\sum_j \partial_{x_j}\partial_{\xi_j} p \tag{1.10}$$

is invariantly defined on Σ_2 under conjugation with elliptic Fourier integral operators. In fact, p_s is the value of the Weyl symbol of $p + p_{m-1}$ at Σ_2 modulo S^{m-2}, see [8].

Remark 1.8. *When Σ_2 is involutive we may choose symplectic coordinates so that $\Sigma_2 = \{\, \xi_1 = \dots = \xi_k = 0 \,\}$ and then the subprincipal symbol $p_s = p_{m-1}$ at Σ_2.*

In fact, since $\partial_x \in T\Sigma_2$ we find that $\partial_x p$ vanishes of second order on Σ_2. If C is a pseudodifferential operator with principal symbol $c = \sigma(C)$, then the value of the subprincipal symbol of the composition CP is equal to $cp_s + \frac{i}{2}H_p c = cp_s$ on Σ_2. Observe that the subprincipal symbol is complexly conjugated when taking the adjoint of the operator, see [10, Theorem 18.1.34].

Since we shall assume that p is real, the real and imaginary parts of p_s are invariant under multiplication with elliptic pseudodifferential operators and conjugation with elliptic Fourier integral operators if these operators have real principal symbols. In order to study the invariants, we need some symplectic concepts.

The symplectic annihilator to a linear space consists of the vectors that are symplectically orthogonal to the space. Let $T\Sigma_2^\sigma$ be the symplectic annihilator to $T\Sigma_2$, which spans the symplectic leaves of Σ_2. If $\Sigma_2 = \{\, \xi = 0 \,\}$, $(x, y) \in \mathbf{R}^d \times \mathbf{R}^{n-d}$, then the leaves are spanned by ∂_x. Let

$$T^\sigma\Sigma_2 = T\Sigma_2 / T\Sigma_2^\sigma \tag{1.11}$$

which is a symplectic space over Σ_2. In these coordinates it is parametrized by

$$T^\sigma\Sigma_2 = \left\{\, ((x_0, y_0; 0, \eta_0); (0, y; 0, \eta)) \in T\Sigma_2 :\ (y, \eta) \in T^*\mathbf{R}^{n-d} \,\right\} \tag{1.12}$$

Thus the fiber is isomorphic to the symplectic manifold $T^*\mathbf{R}^{n-d}$ with $(x_0, y_0; 0, \eta_0) = w_0 \in \Sigma_2$ as a parameter.

Definition 1.9. *If the principal symbol is real valued, then we say that the operator P is of* real subprincipal type *if the following conditions hold:*

$$H_{\operatorname{Re} p_s} \subset T\Sigma_2 \tag{1.13}$$

which means that $d \operatorname{Re} p_s\big|_{TL} = 0$, and

$$d \operatorname{Re} p_s\big|_{T^\sigma\Sigma_2} \neq 0 \tag{1.14}$$

so that $H_{\mathrm{Re}\, p_s}$ is transversal to the leaves and we shall assume that it does not have the radial direction. The bicharacteristics of $\mathrm{Re}\, p_s$ *with respect to the symplectic structure of* Σ_2 *are called the* subprincipal bicharacteristics *for any value of* $\mathrm{Re}\, p_s$*.*

This definition is invariant under symplectic changes of variables and but *not* by multiplication with nonvanishing real factors when $\mathrm{Re}\, p_s \neq 0$. But it is invariant by multiplication with nonvanishing real factors that are constant on the leaves of Σ_2. When the coordinates are given as in (1.12), we find from (1.13) that $\partial_x \mathrm{Re}\, p_s = 0$ on Σ_2 and from (1.14) that $\partial_\eta \mathrm{Re}\, p_s \neq 0$ or $\partial_y \mathrm{Re}\, p_s \not\parallel \eta$ on Σ_2. Thus it follows that

$$\mathrm{Re}\, p_s \text{ is of real principal type} \tag{1.15}$$

and thus has simple zeroes. In [6, Definition 2.1] the definition was that P is of subprincipal type if Definition 1.9 hold with $\mathrm{Re}\, p_s$ replaced with p_s, but only when $p_s = 0$, which is invariant under multiplication with nonvanishing factors and symplectic changes of variables. But in that case the principal symbol may not be proportional to a real symbol ant then $\mathrm{Re}\, p_s$ is not well defined.

We shall study the microlocal solvability of the operator P, which is given by the following definition from [10]. Recall that $H^{loc}_{(s)}(X)$ is the set of distributions that are locally in the L^2 Sobolev space $H_{(s)}(X)$.

Definition 1.10. *If $P \in \Psi^m_{cl}$ and $K \subset T^*X$ is a compactly based cone, then we say that P is microlocally solvable at K if there exists an integer N so that for every $f \in H^{loc}_{(N)}(X)$ there exists $u \in \mathcal{D}'(X)$ such that $K \bigcap \mathrm{WF}(Pu - f) = \emptyset$.*

Observe that solvability at a compact set $M \subset X$ is equivalent to solvability at $T^*X\big|_M$ by [10, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by Proposition 26.4.4 in [10] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators.

To prove solvability we shall use *a priori* estimates. Let $\|u\|_{(k)}$ be the L^2 Sobolev norm of order k, $u \in C_0^\infty$. In the following, P^* will be the L^2 adjoint of P.

Remark 1.11. *Let $P \in \Psi^m_{cl}(X)$ and $K \subset T^*X$ be a compactly based cone, and assume that there exists $\nu \in \mathbf{R}$ and a pseudodifferential operator A so that $K \bigcap \mathrm{WF}(A) = \emptyset$ and*

$$\|u\|_{(-N)} \le C(\|P^*u\|_{(\nu)} + \|u\|_{(-N-n)} + \|Au\|_{(0)}) \qquad u \in C_0^\infty(Y) \tag{1.16}$$

Then P is microlocally solvable at K and one can take this N in Definition 1.10.

Observe that if $P \in \Psi^m_{cl}$ then there is a loss of $\nu+m+N$ derivatives in the estimate (1.16) compared with the elliptic case. One can have several operators A_j in (1.16) by taking $A = (A_1, \dots)$ vector valued.

Definition 1.12. *We say that $P \in \Psi^m$ satisfies condition* $\mathrm{Sub_r}(\Psi)$ *if there exists a homogeneous $0 \neq a \in S^0$ such that ap_r has real principal symbol that vanishes of order 2 at an involutive manifold Σ_2 with nondegenerate Hessian, ap_r is of real subprincipal type and satisfies condition (Ψ) at the limit Σ_2. This means that* $\operatorname{Im} ap_r$ *does not change sign from $-$ to $+$ on the limits of the bicharacteristic of* $\operatorname{Re} ap_r$ *at Σ_2.*

The refined principal symbol is equal to $p_r = p + p_s$ by (1.6). Observe that this condition gives conditions on the sign changes for *any* value of $\operatorname{Re} p_s$, and that is *not* the case in Example 1.5. This condition is stronger than the conditions in Examples (1.1)–(1.3). Observe that the factor a makes this condition invariant under multiplication with with nonvanishing factors. It is also invariant under symplectic changes of variables, thus the conditions is invariant under conjugation with Fourier integral operators and multiplication with elliptic pseudodifferential operators having real principal symbols.

Proposition 1.13. *If $P \in \Psi^m$ has real principal symbol that vanishes of order 2 at an involutive manifold Σ_2 with nondegenerate Hessian and is of real subprincipal type, then P satisfies condition* $\mathrm{Sub_r}(\Psi)$ *if and only if* $\operatorname{Im} p_r = \operatorname{Im} p_s$ *does not change sign on the leaves of Σ_2 and the sign of* $\operatorname{Im} p_s$ *on the leaves do not change from $-$ to $+$ on the subprincipal bicharacteristics, i.e., the bicharacteristics of* $\operatorname{Re} p_s$ *with respect to the symplectic structure of Σ_2 for* any value *of* $\operatorname{Re} p_s$*, see Definition 1.9.*

Here, the sign on the leaves is ± 1 if $\pm \operatorname{Im} p_s \geq 0$ and $\operatorname{Im} p_s \not\equiv 0$ on leaf L of Σ_2 and equal to 0 if $\operatorname{Im} p_s \equiv 0$ on L, see Definition 4.1. Thus, condition $\mathrm{Sub_r}(\Psi)$ implies the necessary conditions in Examples 1.1 and 1.2, and it is not hard to show that it implies the condition $\mathrm{Sub}_2(\Psi)$ in Example 1.3. In fact, the limits at Σ_2 of the Hamilton vector field of the refined principal symbol only depend on the values at Σ_2 of the Hessian of the principal symbol and the gradient of the subprincipal symbol, see (1.19).

Proof. By multiplying with $\langle D \rangle^{2-m}$ we assume that $P \in \Psi^2$, and we may choose symplectic coordinates so that $(x, y) \in \mathbf{R}^d \times \mathbf{R}^{nd}$ and $\Sigma_2 = \{\, \xi = 0 \,\}$. By Taylor's formula we can write the real principal symbol as

$$p_2(x, y; \xi, \eta) = \sum_{jk} a_{jk}(x, y; \xi, \eta) \xi_j \xi_k \tag{1.17}$$

where the Hessian $\{a_{jk}(x, y; 0, \eta)\}_{jk}$ is nondegenerate near w_0 by assumption. We have that $p_1 = p_s$ the subprincipal symbol on Σ_2 and by extending it we may assume it is constant in ξ.

Now the Poisson parentheses $\{\, \xi, \operatorname{Re} p_s \,\} \equiv \partial_x \operatorname{Re} p_s \equiv 0$ on Σ_2 and $\{\, x, \operatorname{Re} p_s \,\} \equiv -\partial_\xi \operatorname{Re} p_s \equiv 0$. Thus $\partial_x \partial_\xi \operatorname{Re} p_s \equiv 0$ which gives $\partial_x \operatorname{Re} p_s \equiv 0$. Since $\operatorname{Re} p_s$ is homogeneous and of principal type, we can thus complete x, ξ and $\tau = \operatorname{Re} p_s\big|_{\Sigma_2}$ to a symplectic homogeneous coordinate system $(x, t, y; \xi, \tau, \eta)$ microlocally near $w_0 \in \Sigma_2$. We then obtain that the refined principal symbol of P is equal to

$$\sum_{jk} a_{jk}(x, t, y; \xi, \tau, \eta)\xi_j\xi_k + \tau + i f(x, t, y; \tau, \eta) + C(x, t, y; \xi, \tau, \eta) \cdot \xi + p_0(x, t, y; \tau, \xi, \eta) \tag{1.18}$$

modulo terms in S^{-1}, where C homogeneous of degree 0 with values in $\mathbf{R}^m$, f is real and homogeneous of degree 1 and $p_0 \in S^0$.

The Hamilton vector field of the real part of the refined principal symbol is given by

$$H_{\operatorname{Re} p_r} \cong 2 \sum_{jk} a_{jk}(x, t, y; 0, \tau, \eta)\xi_j \partial_{x_k} + \partial_t + \operatorname{Re} C(x, t, y; 0, \tau, \eta) \cdot \partial_x \tag{1.19}$$

modulo terms with coefficients that are $\mathcal{O}((|\xi|^2 + |\xi|)/\Lambda)$ with $\Lambda = \sqrt{\tau^2 + |\eta|^2}$ in the base of homogeneous vector fields $V = (\partial_t, \partial_x, \partial_y, \Lambda\partial_\tau, \Lambda\partial_\xi, \Lambda\partial_\eta)$. We get the limit at Σ_2 when $|\xi|/\Lambda \to 0$.

One limit is when $\xi \to 0$ for fixed Λ and then the limit of (1.19) is equal to $\partial_t + \operatorname{Re} C \cdot \partial_x$ which gives the subprincipal bicharacteristics. We can also take $\tau = \Lambda\tau_0$, $\eta = \Lambda\eta_0$ and $\xi = \Lambda^{1/3}\theta$ with $|\tau_0|^2 + |\eta_0|^2 = |\theta| = 1$. Since the coefficients a_{jk} and C are homogeneous, one can write (1.19) as

$$2\Lambda^{1/3} \sum_{jk} a_{jk}(x, t, y_0; 0, \tau_0, \eta_0)\theta_j \partial_{x_k} + \partial_t + \operatorname{Re} C(x, t, y; 0, \tau_0, \eta_0) \cdot \partial_x \tag{1.20}$$

modulo vector fields in V with coefficients that are $\mathcal{O}(\Lambda^{-1/3})$. By dividing by $\lambda^{1/3}$ and taking the limit $\lambda \to \infty$ we obtain the limit Hamilton vector field

$$H_{p_2} = 2 \sum_{jk} a_{jk}(x, t, y_0; 0, \tau_0, \eta_0)\theta_j \partial_{x_k} \tag{1.21}$$

This vector field gives for fixed $\theta_0 = (\theta_1, \dots, \theta_d)$ a foliation of the leaves of Σ_2. Let $\Gamma(\xi)$ with $\xi = \varrho\theta$ be the flow-out of the Hamilton vector field ϱH_{p_2} with H_{p_2} given by (1.21). Then the Hessian $\operatorname{Hess}\Gamma(0)$ is non-degenerate so the mapping $\xi \mapsto \Gamma(\xi)$ is a diffeomorphism from $|\xi| < c$ to a neighborhood of the w_0 in the leaf through $w_0 = (x, t, y; 0, \tau_0, \eta_0)$. Since we can take $\pm\xi$ we obtain from condition $\operatorname{Sub_r}(\Psi)$ that there can be no sign changes of f on the leaves of Σ_2 near w_0.

We can also take the limit $(\tau, \eta) = \Lambda(\tau_0, \eta_0)$ and $\xi = \varrho \cdot \theta$ with $|\tau_0|^2 + |\eta_0|^2 = |\theta| = 1$ and $0 \le \varrho \ll \sqrt{\Lambda} \to \infty$. As before, we obtain the limit Hamilton vector field

$$2\sum_{jk} a_{jk}(x, t, y_0; 0, \tau_0, \eta_0)\xi_j \partial_{x_k} + \partial_t + \operatorname{Re} C(x, t, y_0; 0, \tau_0, \eta_0) \cdot \partial_x \tag{1.22}$$

As before, the orbits of (1.19) gives a foliation of Σ_2 for fixed θ. When $\xi = 0$ the orbit γ_0 of (1.19) through $w_0 = (x_0, t_0, y_0; 0, \tau_0, \eta_0)$ is a subprincipal bicharacteristic. When $\xi = \varrho\theta \ne 0$ then the orbits $\Gamma(t, \xi)$ of (1.19) through w_0 with $t > 0$ form a proper cone in Σ_2 with the subprincipal bicharacteristic in its interior. Now scaling gives that the Hessian of $\xi \mapsto \Gamma(\varrho t, \xi/\varrho)$ at $t = 0$ is constant in $\varrho > 0$, so by letting $\varrho \to 0$ we find for some $c > 0$ that $\bigcup_{0<\varrho\le c} \Gamma(\varrho t, \xi/\varrho)$ for $|\xi| < c$ forms a cylindrical neighborhood of the forward subprincipal bicharacteristic. Thus, we find from condition $\operatorname{Sub_r}(\Psi)$ that if $f \not\ge 0$ on a leaf in a neighborhood of $(t_0, y_0; 0, \tau_0, \eta_0)$ then $f \le 0$ on the leafs in a neighborhood of $(t, y_0; \tau_0, \eta_0)$ for $0 \le t - t_0 \le c$ which proves the proposition. $\square$

Remark 1.14. *The requirement that condition* (Ψ) *shall hold on the refined principal symbol for all values of the real part is only needed when* $f = \operatorname{Im} p_r = \operatorname{Im} p_s$ *depends on* $\tau = \operatorname{Re} p_s$*. In fact, if* f *does not depend on* τ*, then by choosing suitable* τ *so that* $\operatorname{Re} p_r = 0$ *we can get the same limits at* Σ_2 *of the Hamilton vector field of* $\operatorname{Re} p_r$ *as in the proof of Proposition 1.13. One may of course eliminate the* τ *dependence of* f *by the Malgrange preparation theorem, but that would change the imaginary part of the principal symbol* p*, see for example Example 1.5.*

The following is the main result of the paper.

Theorem 1.15. *Assume that* $P \in \Psi^m_{cl}(X)$ *satisfies* $\operatorname{Sub_r}(\Psi)$ *microlocally near* $w_0 \in \Sigma_2$*, then* P *is microolocally solvable near* w_0 *with a loss of* $5/2$ *derivatives by the* a priori *estimate* (1.16)*.*

Thus, by Example 1.3 condition $\operatorname{Sub_r}(\Psi)$ is both necessary (under additional conditions) and sufficient for local solvability for operators of subprincipal type with principal symbol that is real and vanishes of exactly second order at a nonradial involutive manifold Σ_2.

The solvability with a loss of 5/2 derivatives can be compared with the loss of 2 derivatives when the antisymmetric part of P is bounded. In fact, by using the normal form (2.1) with $f \equiv 0$ gives a Schrödinger type operator that is symmetric modulo bounded operators. By using a multiplier as in Lemma 6.9 one can obtain L^2 estimates with arbitrarily small constants. For small enough constant, these estimates may be perturbed by any bounded term. This is similar to the case of operators of principal part, where the loss of derivatives is 1/2 more when the antisymmetric part if the operator is unbounded and condition (Ψ) is satisfied.

This paper treats subprincipal type operators with involutive characteristics having nondegenerate second order vanishing of the principal symbol. For the noninvolutive or degenerate cases, see [19], [20] and the references there.

To prove Theorem 1.15 we shall use suitable *a priori* estimate and Remark 1.11. The proof will occupy most of the remaining paper.

2. The Preparation

As in the proof of Proposition 1.13 we may assume that the operator $P \in \Psi^2_{cl}(X)$ is of second order with real principal symbol, $X = \mathbf{R}^n$ and the coordinates are chosen so that $\Sigma_2 = \{\, \xi = 0 \,\}$, $(x, y; \xi, \eta) \in T^*(\mathbf{R}^d \times \mathbf{R}^{d-n})$ microlocally near $w_0 \in \Sigma_2$, where $x \mapsto (x, y_0; 0, \eta_0)$ spans the leaves of the symplectic foliation of Σ_2 and d is the codimension of Σ_2. We may multiply P with an elliptic operator of order zero so that the refined principal symbol satifies condition $\mathrm{Sub_r}(\Psi)$, see Definition 1.12. Thus, we have the operator on the normal form given by (1.18)

$$\sum_{jk} a_{jk}(x, t, y; \xi, \tau, \eta)\xi_j\xi_k + p_1(x, t, y; \xi, \tau, \eta) + p_0(x, t, y; \xi, \tau, \eta) \tag{2.1}$$

modulo terms in S^{-1}, where $\{\, a_{jk} \,\}_{jk}$ is nondegenerate on Σ_2,

$$p_1(x, t, y; \xi, \tau, \eta) = \tau + if(x, t, y; \tau, \eta) + C(x, t, y; \xi, \tau, \eta) \cdot \xi \tag{2.2}$$

with f homogeneous of degree 1, and C and p_0 homogeneous of degree 0. By condition $\mathrm{Sub_r}(\Psi)$ and Proposition 1.13 we find that f does not change sign on the leaves of Σ_2 and the sign on the leaves do not change from $-$ to $+$ as t increases by Definition 4.1.

We shall compute the symbol modulo the error terms

$$R = \{\langle C_2\xi, \xi\rangle + C_1 \cdot \xi + C_0 : \ C_j \in S^{-1}\} \tag{2.3}$$

These are sums of terms that are either in S^1 vanishing of second order on Σ_2, in S^0 vanishing on Σ_2 or in S^{-1}. Observe that homogeneous vector fields that are tangent to Σ_2 maps R into itself.

We shall use the Weyl quantization, which has the property that symmetric operators have real symbols. The Weyl quantization of symbols $a \in \mathcal{S}'(T^*\mathbf{R}^n)$ is defined by:

$$(a^w u, v) = (2\pi)^{-n} \iint \exp\left(i\langle x - y, \xi\rangle\right) a\big(\tfrac{x+y}{2}, \xi\big)\, u(x)\overline{v(y)}\, dx dy d\xi \qquad u, v \in C_0^\infty \tag{2.4}$$

Observe that $\operatorname{Re} a^w = (\operatorname{Re} a)^w$ is the symmetric part and $i \operatorname{Im} a^w = (i \operatorname{Im} a)^w$ the antisymmetric part of the operator a^w. Also, if $a \in S^m_{1,0}$ then $a(x, D_x) = b^w(x, D_x)$ modulo $\Psi^{m-2}_{1,0}$

where

$$b(x,\xi) = a(x,\xi) + \frac{i}{2}\sum_j \partial_{x_j}\partial_{\xi_j} a(x,\xi) \tag{2.5}$$

which gives the subprincipal symbol by [10, Theorem 18.5.10]. The equality (2.5) shows that $a \in R$ if and only if $b \in R$. I

Now by conjugating with $e^{\phi} \in S^0$ having phase $\phi(x,t,y;\tau,\eta)$ that is real and homogeneous of degree 0, we may obtain that $\operatorname{Im} p_0 = 0$ at Σ_2, i.e., $\operatorname{Im} p_0 \in R$. In fact, we obtain this by solving the equation

$$\partial_t \phi(x,t,y;\tau,\eta) + \operatorname{Re} C(x,t,y;\tau,0,\eta)\cdot\partial_x\phi(x,t,y;\tau,\eta) = \operatorname{Im} p_0(x,t,y;\tau,0,\eta) \tag{2.6}$$

but this may of course change the values of $\operatorname{Im} C$ and $\operatorname{Re} p_0$.

Next, we want to reduce to the case $\operatorname{Im} C = 0$ on Σ_2. i.e, $\operatorname{Im} C \in R$, but that can in general not be done by conjugation. Instead we shall use symplectic changes of variables given microlocally by Fourier integral operators. In the following, we shall for simplicity include the variable t in the y variables, and the variable τ in the η variables. The variables (y,η) will then parametrize the leaves of Σ_2.

Let $\mathbf{R}^d \ni x \mapsto \chi(x,y,\eta) \in \mathbf{R}^d$, where $\chi \in C^\infty$, homogeneous in η and $|\partial_x \chi| \neq 0$ and let

$$Fu(x,y) = (2\pi)^{-n}\iint e^{i(\langle \chi(x,y,\eta)-z,\xi\rangle + \langle y-w,\eta\rangle)} u(z,w)\, dz dw d\xi d\eta \qquad u \in C_0^\infty \tag{2.7}$$

which is an elliptic Fourier integral operator. This correspond to the homogeneous symplectic transformation

$$(x,y;\partial_x\chi(x,y,\eta)\cdot\xi, \eta + \partial_y\chi(x,y,\eta)\cdot\xi) \mapsto (\chi(x,y,\eta), y+\partial_\eta\chi\cdot\xi;\xi,\eta)$$

which preserves Σ_2, thus $|\xi|$ and $|\eta|$ are preserved modulo multiplicative constants. In fact, when $\xi = 0$ we get the mapping $(x,y;0,\eta) \mapsto (\chi(x,y,\eta),y;0,\eta)$ which gives a homogeneous change of x variables. We put the amplitude of F equal to 1 to simplify the notation, actually the amplitude only has to equal to 1 near the wave front set of the kernel of F.

By applying the operator P we find

$$PFu(x,y) = (2\pi)^{-n}\iint e^{i(\langle \chi(x,y,\eta)-z,\xi\rangle + \langle y-w,\eta\rangle)} Q(x,y;\xi,\eta)u(z,w)\, dz dw d\xi d\eta \tag{2.8}$$

so $PFu = FQu$, where

$$Q(x,y;\xi,\eta) = \sum_{\alpha,\,\beta\in\mathbf{N}} \partial_\xi^\alpha\partial_\eta^\beta P(x,y;\partial_x\chi\cdot\xi,\eta+\partial_y\chi\cdot\xi)\mathcal{M}^\chi_{\alpha,\beta}(x,y;\xi,\eta)/\alpha!\beta! \tag{2.9}$$

with

$$\mathcal{M}^\chi_{\alpha,\beta}(x,y;\xi,\eta) = D_z^\alpha D_w^\beta e^{i\chi_2(x,y,z,w,\eta)\cdot\xi}\Big|_{\substack{z=x\\ w=y}} \tag{2.10}$$

where the phase function

$$\chi_2(x,y,z,w,\eta) = \chi(z,w,\eta) - \chi(x,y,\eta) + \langle x-z, \partial_x\chi(x,y,\eta)\rangle + \langle y-w, \partial_y\chi(x,y,\eta)\rangle \tag{2.11}$$

vanishes of second order at $z = x$ and $w = y$, see [10, Th. 18.1.17] or [21, Chapter 7, Theorem 3.1]. Thus there are no terms with $|\alpha|+|\beta| = 1$ in the expansion of (2.9). Since we only need the symbols modulo terms in R it suffices to compute the first two terms of the expansion (2.9).

We obtain from (2.1) by a straighforward computation that

$$Q(x,y;\xi,\eta) \cong P(x,y;\partial_x\chi\cdot\xi, \eta + \partial_y\chi\cdot\xi) + \frac{1}{2i}\sum_{jk} a_{jk}(x,y;0,\eta)\partial_{x_j}\partial_{x_k}\chi(x,y,\eta)\cdot\xi \tag{2.12}$$

modulo terms that are in R. In fact, $\partial_\xi a_{jk} \in S^{-1}$, $\partial_\xi^\alpha\partial_\eta^\beta p_2 = \mathcal{O}(|\eta|^{-1}|\xi|)$ if $\beta \neq 0$ and $|\alpha+\beta| \geq 2$, and $\partial_\xi^\alpha\partial_\eta^\beta p_1 \in S^{-1}$ if $|\alpha+\beta| \geq 2$.

We shall first simplify by making a change of variables to diagonalize $A = \{a_{jk}\}_{jk} = \operatorname{Hess} p_2$. Since A is nondegenerate, we can use the spectral projections to obtain either that $A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$ or $A = A_\pm$, where $\pm A_\pm$ is positive definite. Then we can use $(\pm A_\pm)^{-1/2}$ to construct real valued $\chi(x,y,\eta)$ so that $(\partial_x\chi\cdot\xi)^t A\partial_x\chi\cdot\xi = |\xi'|^2 - |\xi''|^2 = L(\xi)$ has constant coefficients near $w_0 \in \Sigma_2$. Here $(\xi',\xi'') = \xi$, and $L(\xi)$ is the real quadratic form with the polarized bilinear form $L(\xi_0,\xi)$. Of course, this may also change the values of p_j for $j < 2$. But observe that the term homogeneous of order 0 in the expansion (2.12) of Q is equal to

$$q_0(x,y;\xi,\eta) = p_0(x,y;\partial_x\chi_k(x,y,\eta)\cdot\xi, \eta + \partial_y\chi(x,y,\eta)\cdot\xi)$$

modulo terms vanishing at Σ_2, so we find that $\operatorname{Im} q_0(x,y;0,\eta) = \operatorname{Im} p_0(x,y;0,\eta) = 0$.

Thus, we may assume that $A = L$ is constant in the following. Next we shall do another change of symplectic variables to make $\operatorname{Im} C = 0$ at Σ_2. Assume that $\chi = (\chi_1,\dots,\chi_d)$ where χ_j is parallel to e_j for the standard base $e_1,\dots,e_d$ of $\mathbf{R}^d$, so that we can write $\chi = (\chi_1 e_1,\dots,\chi_d e_d)$ with scalar χ_j. As before, we get the expansion (2.12) with $\{\, a_k \,\}_{jk} = A = L$.

In order to get the symmetric part of the operator we shall compute the Weyl symbol of Q which is given by $\widetilde{Q} \cong Q + \frac{i}{2}\sum_j \partial_{x_j}\partial_{\xi_j} q_2$ modulo S^0 where

$$q_2(\xi) = L(\partial_x\chi\cdot\xi) = |\partial_{x'}\chi\cdot\xi|^2 - |\partial_{x''}\chi\cdot\xi|^2 = \sum_{k=1}^{\ell}(\partial_{x_k}\chi\cdot\xi)^2 - \sum_{k=\ell+1}^{m}(\partial_{x_k}\chi\cdot\xi)^2$$

is the principal symbol of $\widetilde{Q}$, so (2.12) gives

$$\widetilde{Q} \cong q_2 + i\sum_{k=1}^{m}\left(2L(\partial_x\chi_k, \partial_x\partial_{x_k}\chi_k) - \frac{1}{2}L(\partial_x)\chi_k\right)\xi_k + p_1(x,y;\partial_x\chi\cdot\xi, \eta + \partial_y\chi\cdot\xi) \tag{2.13}$$

modulo S^0.

Let q_1 be the terms homogeneous of order 1 in the expansion (2.13) of $\widetilde{Q}$, then we have $q_1(x,y;0,\eta) = p_1(x,y;0,\eta)$ on Σ_2. We find from (2.13) that the ξ_k derivative of q_1 at Σ_2 is equal to

$$\begin{aligned}2iL(\partial_x\chi_k(x,y,\eta), \partial_x\partial_k\chi_k(x,y,\eta)) - \frac{i}{2}L(\partial_x)\chi_k(x,y,\eta)\\ + \partial_\xi p_1(x,y;0,\eta)\cdot\partial_x\chi_k(x,y,\eta) + \partial_\eta p_1(x,y;0,\eta)\cdot\partial_y\chi_k(x,y,\eta)\end{aligned} \tag{2.14}$$

Here $\operatorname{Im}\partial_\eta p_1 = \partial_\eta f$ and $\operatorname{Im}\partial_\xi p_1 = \operatorname{Im} C$ on Σ_2. Observe that the terms in q_1 vanishing of second order at Σ_2 are in R.

By taking the imaginary part and ignoring the term $\partial_\eta f\cdot\partial_y\chi_k$ for now, we find $\partial_\xi \operatorname{Im} q_1(x,y;0,\eta) = 0$ if for any k we have

$$\begin{aligned}L(\partial_x)\chi_k(x,y,\eta) - 4L(\partial_x\chi_k(x,y,\eta), \partial_x\partial_k\chi_k(x,y,\eta))\\ - 2\operatorname{Im} C(x,y;\eta)\cdot\partial_x\chi_k(x,y,\eta) = 0\end{aligned} \tag{2.15}$$

which is a quasilinear second order system of PDE on the leaves of Σ_2 having real coefficients. Observe that this system is completely decoupled with one equation for each χ_k. As before, we we find that $\operatorname{Im} q_0(x,y;0,\eta) = \operatorname{Im} p_0(x,y;0,\eta) = 0$. In order to get a suitable change of coordinates we shall solve system (2.15) in a neighborhood of $w_0 = (x_0, y_0, \eta_0)$ with initial data $\chi = 0$ and $|\partial_x\chi| \neq 0$ at w_0.

Proposition 2.1. *For any $v_k \in \mathbf{R}^d$, $1 \le k \le d$, the equation* (2.15) *with data $\chi_k = 0$ and $\partial_x\chi_k = v_k$ at $w_0 = (x_0, y_0, \eta_0)$ has a solution $\chi_k(x,y,\eta) \in C^\infty$ in a neighborhood of w_0.*

Proposition 2.1 follows from Theorem A.1 in Appendix A, and shows that this initial value problem has a C^∞ solution $\chi(x,y,\eta) = (\chi_1(x,y,\eta), \chi_2(x,y,\eta)\cdots)$ near w_0 such that $\chi(x_0,y_0,\eta_0) = 0$ and $\partial_x\chi(x_0,y_0,\eta_0) = \mathrm{Id}$. By restricting $\chi(x,y,\eta)$ to the set $|\eta| = 1$ and extending it by homogeneity in η, we obtain a homogeneous change of coordinates so that $\partial_\xi \operatorname{Im} p_1 = \operatorname{Im} C = 0$ and $\operatorname{Im} p_0 = 0$ at Σ_2 near w_0. Thus we have proved the following result.

Proposition 2.2. *Assume that $P \in \Psi^m_{cl}(X)$ satisfies the conditions in Theorem 1.15 microlocally near $w \in \Sigma_2$. By conjugation with elliptic Fourier integral operators and multiplication with symmetric elliptic pseudodifferential operators, we may assume that*

$X = T^*\mathbf{R}^n$, the coordinates are $(x, t, y; \xi, \tau.\eta)$ so that $\Sigma_2 = \{\, \xi = 0 \,\}$ and P is on the form

$$P \cong D_t + A^w + i f_1^w \tag{2.16}$$

microlocally near $w \in \Sigma_2$ modulo terms with symbols in R given by (2.3). Here $A = A_2 + A_1 + A_0 \in S^2_{cl}$, $A_j \in S^j$, is real valued, with principal symbol A_2 vanishing of second order at Σ_2 with $\operatorname{Hess} A_2$ nondegenerate on the normal bundle $N\Sigma_2$, and $A_1 = 0$ at Σ_2. Also, $f_1 = f + f_0$ is real and homogeneous of degree 1 where f does not depend on ξ and $f_0 = \partial_\eta f \cdot r \cdot \xi$ with $r \in S^0$. By condition $\operatorname{Sub_r}(\Psi)$ we have that $f = f_1\big|_{\Sigma_2}$ does not change sign on the leaves of Σ_2 and the sign of f on the leaves do not change from $-$ to $+$ as t increases. Here the sign of f is given by Definition 4.1.

Observe that P in (2.16) is an evolution operator and it is of subprincipal type.

Remark 2.3. *The normal form of the L^2 adjoint P^* is (2.16) with f_1 replaced by $-f_1$. Then P^* satisfies condition $\operatorname{Sub_r}(\overline{\Psi})$ which gives the opposite conditions on sign changes as t increases given by (3.3).*

3. The microlocal estimate

Next, we shall microlocalize and reduce the proof of Theorem 1.15 to the semiclassical multiplier estimate of Proposition 3.6 for a microlocal normal form of the adjoint operator. We shall consider operators given by Proposition 2.2

$$P^* \cong D_t + A^w + i f^w(x, t, y; D_t, D_y) \tag{3.1}$$

modulo terms with symbols in R given by (2.3). Here $f_1 = f + f_0$ is real and homogeneous of degree 1 where $f_0 = \partial_\eta f \cdot r \cdot \xi$ with $r \in S^0$, f does not depend on ξ and

$$A = \sum_{jk} a_{jk}\xi_j\xi_k + \sum_j a_j\xi_j + a_0 \tag{3.2}$$

where a_{jk} and $a_j \in S^0_{1,0}$ are real and homogeneous of degree 0, and $\{a_{jk}\}_{jk}$ is symmetric and nondegenerate.

In the following, we shall assume that P^* satisfies condition $\operatorname{Sub_r}(\overline{\Psi})$, so that the sign of $f(t, x, y; \tau, \eta)$ is constant in x and

$$f(t, x_0, y_0; \tau_0, \eta_0) > 0 \quad \text{and } s > t \implies f(s, x, y_0; \tau_0, \eta_0) \ge 0 \ \ \forall\, x \tag{3.3}$$

so that the sign on the leaves cannot change from $+$ to $-$ as t increases, see Definition 4.1 for the definition of the sign. Observe that if $\chi \ge 0$ then χf also satisfies condition $\operatorname{Sub_r}(\overline{\Psi})$, so this condition can be microlocalized.

In order to prove Theorem 1.15 we shall make a second microlocalization using the specialized symbol classes of the Weyl calculus. We shall therefore recall the definitions of the Weyl calculus: let g_w be a Riemannean metric on $T^*\mathbf{R}^n$, $w = (x, \xi)$, then we say that g is slowly varying if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $g_w \cong g_{w_0}$, i.e., $1/C \le g_w/g_{w_0} \le C$. Let σ be the standard symplectic form on $T^*\mathbf{R}^n$, and assume $g^\sigma(w) \ge g(w)$ where g^σ is the dual metric of $w \mapsto g(\sigma(w))$. We say that g is σ temperate if it is slowly varying and there exists $C > 0$ and $N \in \mathbf{N}$ so that

$$g_w \le C g_{w_0}(1 + g^\sigma_w(w - w_0))^N \qquad \forall\, w,\ w_0 \in T^*\mathbf{R}^n$$

Actually, σ temperate metrics with $g \le g^\sigma$ are called Hörmander metrics. A positive real valued function $m(w)$ on $T^*\mathbf{R}^n$ is g continuous if there exists $c > 0$ so that $g_{w_0}(w - w_0) < c$ implies $m(w) \cong m(w_0)$. We say that m is σ, g temperate if it is g continuous and there exists $C > 0$ and $N \in \mathbf{N}$ so that

$$m(w) \le C m(w_0)(1 + g^\sigma_w(w - w_0))^N \qquad \forall\, w,\ w_0 \in T^*\mathbf{R}^n.$$

If m is σ, g temperate, then m is a weight for g and we can define the symbol classes: $a \in S(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and

$$|a|^g_j(w) = \sup_{T_i \ne 0} \frac{|a^{(j)}(w, T_1, \dots, T_j)|}{\prod_1^j g_w(T_i)^{1/2}} \le C_j m(w) \qquad \forall\, w \in T^*\mathbf{R}^n \qquad \text{for } j \ge 0, \tag{3.4}$$

which gives the seminorms of $S(m, g)$. If $a \in S(m, g)$ then we say that the corresponding Weyl operator $a^w \in \operatorname{Op} S(m, g)$. For more on the Weyl calculus, see [10, Section 18.5].

Definition 3.1. *Let m be a weight for the metric g. We say that $a \in S^+(m, g)$ if $a \in C^\infty(T^*\mathbf{R}^n)$ and $|a|^g_j \le C_j m$ for $j \ge 1$.*

Observe that by the mean value theorem we find that

$$|a(w) - a(w_0)| \le C_1 \sup_{\theta \in [0,1]} g_{w_\theta}(w - w_0)^{1/2} m(w_\theta) \le C' m(w_0)(1 + g^\sigma_{w_0}(w - w_0))^{(3N+1)/2} \tag{3.5}$$

where $w_\theta = \theta w + (1 - \theta) w_0$, since $w_\theta - w_0 = \theta(w - w_0)$ for some $0 < \theta < 1$ and

$$g_{w_\theta}(w - w_0) \lesssim g^\sigma_{w_\theta}(w - w_0) \lesssim g^\sigma_{w_0}(w - w_0)(1 + g^\sigma_{w_0}(w - w_0))^N$$

Thus $m + |a|$ is a weight for g and $a \in S(m + |a|, g)$, so the operator a^w is well-defined.

Lemma 3.2. *Assume that m_j is a weight for $g_j = h_j g^\sharp \le g^\sharp = (g^\sharp)^\sigma \le g_j^\sigma \le h_j^{-1} g^\sharp$ and $a_j \in S^+(m_j, g_j)$, $j = 1, 2$. Let $g = g_1 + g_2$ and $h^2 = \sup g_1/g_2^\sigma = \sup g_2/g_1^\sigma = h_1 h_2$, then*

$$a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h, g) \tag{3.6}$$

with the usual expansion of (3.6) in terms in $S(m_1 m_2 h^k, g)$, $k \ge 1$. We also have that

$$\operatorname{Re} a_1^w a_2^w - (a_1 a_2)^w \in \operatorname{Op} S(m_1 m_2 h^2, g) \tag{3.7}$$

if $a_j \in C^\infty$ is real and $|a_j|^{g_j}_k \le C_k m_j$, $k \ge 2$, for $j = 1, 2$. In that case we have $a_j \in S(m_j + |a_j| + |a_j|^{g_j}_1, g_j)$.

Proof. As shown after Definition 3.1 we have that $m_j + |a_j|$ is a weight for g_j and $a_j \in S(m_j + |a_j|, g_j)$, $j = 1, 2$. Thus $a_1^w a_2^w \in \operatorname{Op} S((m_1 + |a_1|)(m_2 + |a_2|), g)$ is given by Proposition 18.5.5 in [10]. We find that $a_1^w a_2^w - (a_1 a_2)^w = a^w$ with

$$a(w) = E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2)\big|_{w_1=w_2=w} \tag{3.8}$$

where $E(z) = (e^z - 1)/z = \int_0^1 e^{\theta z}\, d\theta$. Here $\sigma(D_{w_1}, D_{w_2})a_1(w_1)a_2(w_2) \in S(MH, G)$ where $M(w_1, w_2) = m_1(w_1)m_2(w_2)$, $G_{w_1,w_2}(z_1, z_2) = g_{1,w_1}(z_1) + g_{2,w_2}(z_2)$ and $H^2(w_1, w_2) = h_1(w_1)h_2(w_2) = \sup G_{w_1,w_2}/G^\sigma_{w_1,w_2}$ so that $H(w, w) = h(w)$. The proof of Theorem 18.5.5 in [10] works when $\sigma(D_{w_1}, D_{w_2})$ is replaced by $\theta\sigma(D_{w_1}, D_{w_2})$, uniformly in $0 \le \theta \le 1$ (when $\theta = 0$ we just get the Poisson parenthesis $\frac{i}{2}\{a_1, a_2\}$). By integrating over $\theta \in [0, 1]$ we obtain that $a(w)$ has an asymptotic expansion in $S(m_1 m_2 h^k, g)$, which proves (3.6).

If $|a_j|^{g_j}_k \le C_k m_j$, $k \ge 2$, then we have by Taylor's formula as in (3.5) that

$$\begin{aligned}|a_j(w) - a_j(w_0)| &\le g_{w_0}(w - w_0)^{1/2}|a_j|^g_1(w_0) + C_1 \sup_{\theta\in[0,1]} g_{w_\theta}(w - w_0)m(w_\theta)\\ &\le C'(|a_j|^g_1(w_0) + m(w_0))(1 + g^\sigma_{w_0}(w - w_0))^{2N+1}\end{aligned}$$

$$\begin{aligned}|\langle T, \partial_w a_j(w)\rangle - \langle T, \partial_w a_j(w_0)\rangle| &\le C_2 \sup_{\theta\in[0,1]} g_{w_\theta}(T)^{1/2} g_{w_\theta}(w - w_0)^{1/2} m(w_\theta)\\ &\le C_3 g_{w_0}(T)^{1/2} m(w_0)(1 + g^\sigma_{w_0}(w - w_0))^{(4N+1)/2}\end{aligned}$$

thus $m_j + |a_j| + |a_j|^{g_j}_1$ is a weight for g_j and clearly $a_j \in S(m_j + |a_j| + |a_j|^{g_j}_1, g_j)$.

Now if a_1 and a_2 are real, then $\operatorname{Re} a_1^w a_2^w - (a_1 a_2)^w = a^w$ with

$$a(w) = \operatorname{Re} E(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))(\tfrac{i}{2}\sigma(D_{w_1}, D_{w_2}))^2 a_1(w_1)a_2(w_2)/2\big|_{w_1=w_2=w}$$

where $\sigma(D_{w_1}, D_{w_2})^2 a_1(w_1)a_2(w_2) \in S(MH^2, G)$, with the same E, M, G and H as before. The proof of (3.7) then follows in the same way as the proof of (3.6). □

Remark 3.3. *The conclusions of Lemma 3.2 also hold if a_1 has values in $\mathcal{L}(B_1, B_2)$ and a_2 in B_1 where B_1 and B_2 are Banach spaces, then $a_1^w a_2^w$ has values in B_2.*

For example, if $\{ a_j \}_j \in S(m_1, g_1)$ with values in ℓ^2, and $b_j \in S(m_2, g_2)$ uniformly in j, then $\{ a_j^w b_j^w \}_j \in \operatorname{Op}(m_1 m_2, g)$ with values in ℓ^2.

Remark 3.4. *For pseudodifferential operators with the Kohn-Nirenberg quantization, we have by Theorem 4.5 and (4.13) in [8] that $a_1(x, D)a_2(x, D) = a(x, D)$ with*

$$a(x, \xi) = e^{i\langle D_\xi, D_y\rangle} a_1(x, \xi)a_1(y, \eta)\big|_{\substack{y=x\\ \eta=\xi}} \tag{3.9}$$

As in the proof of Lemma 3.2 we find that $a_1(x, D)a_2(x, D) - a(x, D) = r(x, D)$ with

$$r(x, \xi) = E(i\langle D_\xi, D_y\rangle)\partial_\xi a_1(x, \xi)D_y a_1(y, \eta)\big|_{\substack{y=x\\ \eta=\xi}} \tag{3.10}$$

where $E(z) = (e^z - 1)/z = \int_0^1 e^{\theta z}\, d\theta$.

To prove Theorem 1.15 we shall prove an estimate for the microlocal normal form of the adjoint operator. Since the proof is rather long, we will take it in two steps, and the first is microlocal estimate. For that, we shall use the symbol classes $S^m_{1,0}$ and $S^m_{1/2,1/2}$.

Proposition 3.5. *Assume that P is as in Proposition 2.2 microlocally near $w_0 \in \Sigma_2$ and that $t = t_0$ and $x = x_0$ at w_0. Then there exist $T_0 > 0$ and a real valued symbol $b_T \in S^1_{1/2,1/2}$ with homogeneous gradient $\nabla b_T = (\partial_z b_T, |\zeta|\partial_\zeta b_T) \in S^1_{1/2,1/2}$ uniformly for $0 < T \le T_0$, $(z,\zeta) \in T^*\mathbf{R}^n$, such that for every $N > 0$ there exists $C_N > 0$ so that*

$$\|b^w_T u\|^2_{(-1/2)} + \|D_x u\|^2 + \|u\|^2 \le C_N \left(T \operatorname{Im}(P^*u, b^w_T u) + \|u\|^2_{(-N)}\right) + \|\psi^w u\|^2 \tag{3.11}$$

for $u \in C^\infty_0$ having support where $|t-t_0| \le T$ and $|x-x_0| \le T$. Here $\psi \in S^2$, $w_0 \notin \operatorname{WF}\psi^w$ and the constants T_0, C_N and the seminorms of b_T only depend on the seminorms of' the symbols in P.

Proof that Proposition 3.5 gives Theorem 1.15. We shall prove that there exists $\phi \in S^0_{1,0}$ such that $\phi \ge 0$ and $\phi = 1$ in a conical neighborhood of $w_0 \in \Sigma_2$, and $R \in S^{3/2}_{1,0}$ with $w_0 \notin \operatorname{WF} R^w$ so that for any $N > 0$ there exists $C_N > 0$ such that

$$\|\phi^w u\| \le C_N \left(\|\phi^w P^* u\|_{(1/2)} + \|R^w u\| + \|u\|_{(-N)}\right) \qquad u \in C^\infty_0 \tag{3.12}$$

Here $\|u\|_{(s)}$ is the usual L^2 Sobolev norm, so by Remark 1.11 we obtain that P is solvable with a loss of 5/2 derivatives in a conical neighborhood of w_0 since $w_0 \notin \operatorname{WF}(1-\phi)^w$ and $m = 2$.

We may assume that $m = 2$ and $P \in \Psi^2_{1,0}$ is on the form in Proposition 2.2 in a conical neighborhood of w_0. Let $\phi \ge 0$ have support in a smaller conical neighborhood such that $\max(|t|, |x|) \le T \le T_0$ in $\operatorname{supp}\phi$, ψ in (3.11) vanishes on $\operatorname{supp}\phi$ and $\phi = 1$ in a conical neighborhood of w_0. Then by applying the estimate (3.11) on $\phi^w u$ we obtain for any $N > 0$

$$\|b^w_T\phi^w u\|^2_{(-1/2)} + \|\phi^w u\|^2 + \|D_x\phi^w u\|^2 \le C_N \left(T \operatorname{Im}(P^*\phi^w u, b^w_T\phi^w u) + \|u\|^2_{(-N)}\right) \tag{3.13}$$

where $C_N > 0$ and $b^w_T \in \Psi^1_{1/2,1/2}$ is symmetric with homogeneous gradient $\nabla b_T \in S^1_{1/2,1/2}$.

By Cauchy-Schwarz,

$$|\,(P^*\phi^w u, b^w_T\phi^w u)\,| \lesssim \|P^*\phi^w u\|^2_{(1/2)} + \|b^w_T\phi^w u\|^2_{(-1/2)} \tag{3.14}$$

and $\|P^*\phi^w u\|^2_{(1/2)} \le \|\phi^w P^* u\|^2_{(1/2)} + \|[P^*, \phi^w]u\|^2_{(1/2)}$ where the commutator $[P^*, \phi^w] \in \Psi^1$ with $w_0 \notin \operatorname{WF}[P^*, \phi^w]$.

Thus, for T small enough, we obtain for any $N > 1$ the estimate

$$\|\phi^w u\|^2 \le \|b_T^w \phi^w u\|^2_{(-1/2)} + \|\phi^w u\|^2 + \|D_x \phi^w u\|^2 \le C_N \left(T\|\phi^w P^* u\|^2_{(1/2)} + T\|[P^*, \phi^w]u\|^2_{(1/2)} + \|u\|^2_{(-N)} \right) \tag{3.15}$$

This gives the estimate (3.12) with $R = T\langle D\rangle^{1/2}[P^*, \phi^w] \in \Psi^{3/2}$ which completes the proof of Theorem 1.15. □

Next we shall derive a semiclassical estimate for the proof of Proposition 3.5. We shall assume that the coordinates are chosen as in Proposition 2.2 so that $\Sigma_2 = \{\, \xi = 0 \,\}$. The proof involves a second microlocalization near $(t_0, x_0, y_0; \tau_0, 0, \eta_0) = (z_0; \zeta_0) \in \Sigma_2$ using the homogeneous metric $g = g_{1,0}$. Then $\sup g/g^\sigma = h^2 \le 1$ are constant and $|\xi| \lesssim h^{-1} \cong \langle(\tau, \eta)\rangle$. We have $g/h = g^\sharp \cong g_{1/2,1/2}$ where $g^\sharp = (g^\sharp)^\sigma$ is constant, $S^k_{1,0} = S(h^{-k}, g)$ and $S^k_{1/2,1/2} = S(h^{-k}, g^\sharp)$ for $k \in \mathbf{R}$. Observe that now the the symbols of the error terms R can be written $\langle R_2\xi, \xi\rangle + R_1 \cdot \xi + R_0$ where $R_j \in S(h, g)$.

Proposition 3.6. *Assume that* $P^* \cong D_t + A^w + i f_1^w$ *with real* $f_1 = f + f_0$ *where* $f \in S(h^{-1}, g)$ *is independent of* ξ *and satisfies condition* $\mathrm{Sub_r}(\overline{\Psi})$ *in* (3.3), $f_0 = \partial_\eta f \cdot r \cdot \xi \in S(h^{-1}, g)$ *with* $r \in S(1, g)$ *and*

$$A = \sum_{jk} a_{jk}\xi_j\xi_k + \sum_j a_j\xi_j + a_0 \tag{3.16}$$

where a_{jk} *and* $a_j \in S(1, g)$ *are real and* $\{a_{jk}\}_{jk}$ *is symmetric and nondegenerate, here* $0 < h \le 1$ *and* $g^\sharp = (g^\sharp)^\sigma$ *are constant. Then there exist* $T_0 > 0$ *and real valued symbols* $b_T(t, x, \xi) \in S(h^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp) + S(h^{-1/2}, g)$ *uniformly for any* $0 < T \le T_0$ *and* $|x| \le T$, *so that*

$$h^{1/2}\left(\|b_T^w u\|^2 + \|D_x u\|^2 + \|u\|^2\right) \le C_0 T \,\mathrm{Im}\,(P^* u, b_T^w u) + \|\Psi^w u\|^2 \tag{3.17}$$

when $u \in C_0^\infty$ *has support where* $|t| \le T$ *and* $|x| \le T$. *Here* $\Psi \in S^2$, $\Sigma_2 \bigcap \operatorname{supp}\Psi = \emptyset$ *and* C_0, T_0 *and the seminorms of* b_T *only depend on the seminorms of* f *in* $S(h^{-1}, g)$.

Proposition 3.6 will be proved at the end of Section 8.

Proof of that Proposition 3.6 gives Proposition 3.5. First note that in the estimate (3.11), P^* can be perturbed by operators with symbols in R. In fact, if $\widetilde{R} = \langle R_2 D_x, D_x\rangle + \langle R_1, D_x\rangle + R_0$ with $R_j \in \Psi^{-1}_{1,0}$, then $\mathrm{Re}\, b_T^w \widetilde{R} = \langle S_2 D_x, D_x\rangle + \langle S_1, D_x\rangle + S_0$ where $S_j \in \Psi^0_{1/2,1/2}$ is continuous on L^2. Thus, this term can be estimated by the last two terms in the left hand side of (3.11) for small enough T.

As before, we shall include τ in the variables η and use the coordinates $(z, \zeta) \in T^*\mathbf{R}^n$. For the localization, we shall take $\phi \in S^0_{1,0}$ such that $0 \le \phi \le 1$, $\phi = 1$ in a conical neighborhood of $w_0 \in \Sigma_2$ such that ϕ is supported where $|\xi| \lesssim h^{-1}$, $\Psi = 0$ and P is on the normal form (2.16). By taking $\phi(z,\zeta) = \phi_0(z)\phi_1(\zeta)$ as a product, we obtain that $\partial_{t,x}\varphi$ has support where $\max(|t|,|x|)| \ge T_0$ for some $T_0 > 0$.

Next, we shall microlocalize in $\zeta = (\tau, \xi, \eta)$ with respect to the homogeneous metric $g = g_{1,0}$ with a partition of unity $\{\, \varphi_j(\zeta) \,\}_j \in S^0_{1,0} = S(1,g)$ independent of z with values in ℓ^2 such that $\sum_j \varphi_j^2 = 1$, $0 \le \varphi_\le 1$ and φ_j is supported where $\langle \zeta \rangle \cong h_j^{-1}$. Then we can get a partition of unity in a conical neighborhood of w_0 by putting $\phi_j = \phi\varphi_j$, so that ϕ_j is supported where $|\xi| \lesssim h_j^{-1}$, $\sum_j \phi_j^2 = \phi^2$ and $\partial_{t,x}\phi_j$ has support where $\max(|t|,|x|)| \ge T_0$.

Since the functions ϕ_j are real, we find from the calculus and symmetry that $\sum_j \phi_j^w\phi_j^w = \phi^w\phi^w + r^w$ where $r \in S^{-2}$ is real valued, which gives $\|\phi^w v\|^2 \le \sum_j \|\phi_j^w v\|^2 + C\|v\|^2_{(-2)}$ for $v \in C_0^\infty$ and by continuity we have $\sum_j \|\phi_j^w v\|^2 \lesssim \|v\|^2$. By cutting off, we find that

$$\|v\|^2_{(-2)} \lesssim \sum_j \|h_j^2\phi_j^w v\|^2 + \|\langle D\rangle^{-2}(1-\phi)^w v\|^2 \tag{3.18}$$

Since the cut-off functions have values in ℓ^2 the calculus gives that the operators that we obtain from these will have values in ℓ^2 (or scalar values after summation) by Remark 3.3.

By possibly shrinking T_0 we can also choose real symbols $\{\, \psi_j \,\}_j \in S^0_{1,0}$ with values in ℓ^2, such that $0 \le \psi_j \le 1$ has support in a g neighborhood of w_j of radius $2T_0$ so that $\psi_j\phi_j = \phi_j$. If T_0 is small enough, we may assume that P is on the normal form (2.16) and $g_{1,0} \cong g = hg^\sharp$ is constant in $\operatorname{supp}\psi_j$, and that there is a fixed bound on number of overlapping supports of ψ_j, see [10, Section 18.5]. Then we obtain that $S^m_{1,0} = S(h_j^{-m}, g_j)$ and $S^m_{1/2,1/2} = S(h_j^{-m}, g^\sharp)$ in $\operatorname{supp}\psi_j$ for $m \in \mathbf{R}$, where $h_j \le 1$, $g_j = h_j g^\sharp$.

The microlocalization of P is $P_j = D_t + A_j^w + if_j^w$ where $A_j = \psi_j A + (1-\psi_j)A_{0,j} \in S(\langle\xi\rangle^2, g_j)$ with $A_{0,j}(t,x,y;\eta,\xi) = \sum_{k\ell} a_{k\ell}(w_j)\xi_k\xi_\ell$, $f_{1j} = \psi_j f_1 \in S(h_j^{-1}, g_j)$ uniformly in j satisfying condition $\mathrm{Sub_r}(\Psi)$. If the support of ψ_j is small enough, then the Hessian $\partial_\xi^2 A_j$ is nondegenerate at Σ_2.

Then, by using Proposition 3.6 with P_j and substituting $\phi_j^w u$ in (3.17), we obtain real $b_{j,T} \in S(h_j^{-1/2}, g_j^\sharp) \bigcap S^+(1, g_j^\sharp) + S(h_j^{1/2}\langle\xi\rangle, g_j)$ uniformly so that

$$\begin{aligned}\|b_{j,T}^w\phi_j^w u\|^2 + \|\phi_j^w u\|^2 + \|D_x\phi_j^w u\|^2 \\ \le C_0 T h_j^{-1/2}\operatorname{Im}\left(P_j^*\phi_j^w u, b_{j,T}^w\phi_j^w u\right) + C_N\|\phi_j^w u\|^2_{(-N)}\end{aligned} \tag{3.19}$$

for $u \in C_0^\infty$ having support where $\max(|t|,|x|) \le T \le T_0$. We have $P_j^*\phi_j^w = \phi_j^w P_j^* + Q_j^w$, where

$$Q_j^w = [D_t, \phi_j^w] + [A_j^w, \phi_j^w] - i[f_{1j}^w, \phi_j^w] \in \operatorname{Op} S(h_j^{-1}, g_j) \tag{3.20}$$

Since the commutator of symmetric operators is antisymmetric, the calculus gives that $\operatorname{Im} Q_j \in S(h_j\langle\xi\rangle^2, g_j)$ when $\max(|t|,|x|)| \le T_0$ since then $\partial_{t,x}\phi_j = 0$ which gives $[A_j^w, \phi_j^w] =$

$\sum_{k\ell}[a^w_{k\ell}, \phi^w_j]D_{x_k}D_{x_\ell}$ and $[D_t, \phi^w_j] = 0$. This also gives $D_x\phi^w_j u = \phi^w_j D_x u$ when u is supported where $\max(|t|, |x|)| \le T_0$. We also have $\operatorname{Re} Q_j \in S(1, g_j)$ since $\langle\xi\rangle \lesssim h_j^{-1}$ in $\operatorname{supp}\phi_j$.

By using the calculus we obtain that $\phi^w_j P = \phi^w_j P_j$ modulo $\operatorname{Op} S(h^N_j, g_j)$, $\forall N$, since $\psi_j\phi_j = \phi_j$. We obtain for any N that

$$
\begin{aligned}
(3.21)\quad & \|b^w_{j,T}\phi^w_j u\|^2 + \|\phi^w_j u\|^2 + \|\phi^w_j D_x u\|^2 \\
&\le C_0 T\left(\operatorname{Im}\left(P^*u, B^w_{j,T}u\right) + h_j^{-1/2}\operatorname{Im}\left(Q^w_j u, b^w_{j,T}\phi^w_j u\right)\right) + C_N\|h^N_j\phi^w_j u\|^2 \qquad \forall j
\end{aligned}
$$

if $u \in C^\infty_0$ supported where $\max(|t|, |x|) \le T \le T_0$. Here

$$
B^w_{j,T} = h_j^{-1/2}\phi^w_j b^w_{j,T}\phi^w_j \in \operatorname{Op} S(h_j^{-1}, g^\sharp_j)\bigcap \operatorname{Op} S^+(h_j^{-1/2}, g^\sharp_j) + \operatorname{Op} S(\langle\xi\rangle, g_j)
$$

uniformly and by symmetry $B_{j,T}$ is real. Since $\phi^w_j b^w_{j,T} \cong (b_{j,T}\phi_j)^w$ modulo $\operatorname{Op} S(h_j^{1/2}, g^\sharp_j)$ and $\operatorname{Re} Q_j \in S(1, g_j)$ we find that $\{h_j^{-1/2}\operatorname{Im}\phi^w_j b^w_{j,T}Q^w_j\}_j \in \operatorname{Op} S(\langle\xi\rangle^2, g^\sharp)$ with values in ℓ^2 when $\max(|t|, |x|)| \le T_0$. Thus we may find $\psi \in S^1$ with support outside a conical neighborhood of w_0 so that

$$
\sum_j h_j^{-1/2}\operatorname{Im}\left(Q^w_j u, b^w_{j,T}\phi^w_j u\right) \lesssim \|u\|^2 + \|D_x u\|^2 + \|\psi^w u\|^2
$$

if $u \in C^\infty_0$ supported where $\max(|t|, |x|) \le T_0$.

Let $b^w_T = \sum_j B^w_{j,T}$, then by the finite bound on the overlap of the supports we find that

$$
(3.22)\quad \|b^w_T u\|^2_{(-1/2)} \lesssim \left\|\sum_j h_j^{1/2}B^w_{j,T}u\right\|^2 = \left\|\sum_j \phi^w_j b^w_{j,T}\phi^w_j u\right\|^2 \lesssim \sum_j \|b^w_{j,T}\phi^w_j u\|^2 + \|u\|^2_{(-N)}
$$

since $\langle B^w_{j,T}u, B^w_{k,T}u\rangle = 0$ if $|j-k| \gg 1$. Thus, by summing up we obtain

$$
\begin{aligned}
(3.23)\quad & \|b^w_T u\|^2_{(-1/2)} + \|u\|^2 + \|D_x u\|^2 \\
&\le C_1\left(T(\operatorname{Im}(P^*u, b^w_T u) + \|u\|^2 + \|D_x u\|^2 + \|\psi^w u\|^2) + \|u\|^2_{(-N)} + \|(1-\phi)^w u\|^2_{(1)}\right)
\end{aligned}
$$

for $u \in C^\infty_0$ having support where $\max(|t|, |x|) \le T \le T_0$. Here we find that $w_0 \notin \operatorname{WF}\psi^w \bigcup \operatorname{WF}(1-\phi)^w$ which gives (3.11) for small enough T. We also have that $b^w_T = \sum_j h_j^{-1/2}\phi^w_j b^w_{j,T}\phi^w_j \in \Psi^1_{1/2,1/2}$ since $\phi_j \in S(1, g_j)$ is supported where $\langle\zeta\rangle \simeq h_j^{-1}$ and $b_{j,T} \in S(h_j^{-1/2}, g^\sharp)$. The homogeneous gradient $\nabla b_T \in S^1_{1/2,1/2}$ since $b_{j,T} \in S^+(1, g^\sharp)$ and the homogeneous gradient is equal to $h^{-1/2}$ times the gradient in coordinates which are $g^\sharp$ orthonormal. This finishes the proof of Proposition 3.5. $\square$

It remains to prove Proposition 3.6, which will be done at the end of Section 8. The proof involves the construction of a multiplier b^w_T, and it will occupy most of the remaining part of the paper.

4. The symbol classes

In this section we shall define the symbol classes we shall use. We shall follow Section 3 in [4] with some changes due to the different conditions and normal forms. We shall study the subprincipal symbol $p_s = \tau + i f_1$ with $f_1 = f + f_0$ where $f(t, x, y, \tau, \eta) \in S(h^{-1}, g)$ and $f_0 = \partial_\eta f \cdot r \cdot \xi$ where $\partial_\eta f \cdot r \in S(1, g)$. The metric is localized and assumed to be constant, but the result holds in general for σ temperate metrics $g \le h^2 g^\sigma$.

Since we are going to study the adjoint, we shall also assume that $f(t, x, \tau, w) \in S(h^{-1}, g)$ is independent of ξ and satisfies condition $\mathrm{Sub_r}(\overline{\Psi})$ in (3.3). Here $g = g_{1,0}$ is the usual homogeneous metric, $x \in \mathbf{R}^m$, $(t, \tau) \in T^*\mathbf{R}$ and $w = (y, \eta) \in T^*\mathbf{R}^{n-m-1}$ as in Section 2. We have that $g = h g^\sharp$ where $g^\sharp \le (g^\sharp)^\sigma$, in the case of the homogeneous metric we have

$$g^\sharp = (dt^2 + |dx|^2 + |dy|^2)/h + h(d\tau^2 + |d\xi|^2 + |d\eta|^2) \tag{4.1}$$

We shall construct a metric, weight and multiplier adapted to f, so the symbols in this section will be independent of ξ except for f_0, which will be handled as an error term in the estimates, see Remark 4.8. We shall suppress the ξ variables and assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp$ is the euclidean metric so that $g^\sharp(t, x, \tau, w) = |(t, x, \tau, w)|^2$. Then we have $|f'| = |f|_1^{g^\sharp} \lesssim h^{-1/2}$, $|f''| = |f|_2^{g^\sharp} \lesssim 1$ and $|f^{(k)}| = |f|_k^{g^\sharp} \lesssim h^{-1+k/2} \lesssim h^{1/2}$ for $k > 2$. By decreasing h we may obtain that $|f'| \le h^{-1/2}$ which we assume in what follows. Observe that after the change of coordinates $|\partial_\eta f| = h^{1/2} |\partial_\eta f|^{g^\sharp} \le h^{1/2}|f'| \le 1$ and $|\partial_t f| = h^{-1/2}|\partial_t f|^{g^\sharp} \le h^{-1/2}|f'| \le h^{-1}$. The results in this section are uniform in the sense that they depend only on the seminorms of f in $S(h^{-1}, g)$.

Since we assume that $f = \operatorname{Im} p_r$ does not change sign on the leaves of Σ_2, we may have the following definition of the sign of f.

Definition 4.1. *If f does not change sign on the leaves L of Σ_2, then we define the sign function*

$$\operatorname{sgn}(f) = \begin{cases} \pm 1 & \text{if } \pm f \ge 0 \text{ and } f \not\equiv 0 \quad \text{on } L \\ 0 & \text{if } f \equiv 0 \quad \text{on } L \end{cases} \tag{4.2}$$

which is then constant on the leaves of Σ_2 such that $\operatorname{sgn}(f) f \ge 0$.

Let

$$X_+ = \left\{ (t, \tau, w) : \exists\, s \le t,\ \max_x f(s, x, \tau, w) > 0 \right\} \tag{4.3}$$

$$X_- = \left\{ (t, \tau, w) : \exists\, s \ge t,\ \min_x f(s, x, \tau, w) < 0 \right\}. \tag{4.4}$$

Observe that by the definition, $X_\pm$ is open in Σ_2 and is a union of leaves of Σ_2. By condition $\mathrm{Sub_r}(\Psi)$ we find that $\pm f(t,x,\tau,w) \ge 0$ when $(t,\tau,w) \in X_\pm$ and that $X_- \bigcap X_+ = \emptyset$. Let $X_0 = \Sigma_2 \setminus (X_+ \bigcup X_-)$ which is a union of leaves and is relatively closed in Σ_2.

Definition 4.2. *Let*

$$d(t,\tau,w) = \inf\{\, |(t,\tau,w) - (s,\sigma,z)| : (s,\sigma,z) \in X_0 \,\} \tag{4.5}$$

be the $g^\sharp$ distance to X_0, it is constant in x and equal to $+\infty$ in the case when $X_0 = \emptyset$. We define the signed distance function $\delta(t,w)$ by

$$\delta = \operatorname{sgn}(f)\min(d, h^{-1/2}), \tag{4.6}$$

where d is given by (4.5) *and* $\operatorname{sgn}(f)$ *by Definition 4.1.*

We say that $a(t,x,\tau,w)$ is Lipschitz continuous if it is Lipschitz with respect to the metric $g^\sharp$.

Proposition 4.3. *The signed distance function $(t,\tau,w) \mapsto \delta(t,\tau,w)$ given by Definition 4.2 is Lipschitz continuous with Lipschitz constant equal to 1. We also find that $t \mapsto \delta(t,\tau,w)$ is nondecreasing, δ is constant in x, $0 \le \delta f$, $|\delta| \le h^{-1/2}$ and $|\delta| = d$ when $|\delta| < h^{-1/2}$.*

Proof. Clearly $\delta f \ge 0$, and by the definition we have that $|\delta| = \min(d, h^{-1/2}) \le h^{-1/2}$ so $|\delta| = d$ when $|\delta| < h^{-1/2}$. Now, it suffices to show the Lipschitz continuity of $(t,\tau,w) \mapsto \delta(t,\tau,w)$ locally, and thus locally on $\complement X_0$ when $d < \infty$. Then $d(t,\tau,w)$ is the distance function to X_0 so it is Lipschitz continuous with constant 1.

Next we show that $t \mapsto \delta(t,\tau,w)$ is nondecreasing. In fact, when t increases we can only go from X_- to X_0 and from X_0 to X_+. If $(t,\tau,w) \in X_+$ then $c = \delta(t,\tau,w) > 0$ is the distance to $\complement X_+$. If there exists $\varepsilon > 0$ so that $\delta(t+\varepsilon,\tau,w) < c$ then there would exists $(s,\sigma,z) \notin X_+$ so that $|(t+\varepsilon,\tau,w) - (s,\sigma,z)| < c$. But then $|(t,\tau,w) - (s-\varepsilon,\sigma,z)| < c$ where $(s-\varepsilon,\sigma,z) \notin X_+$ which gives a contradiction. By switching t to $-t$, δ to $-\delta$ and X_+ to X_- we similarly find that $\delta < 0$ is nondecreasing on X_- and δ is of course equal to 0 on X_0. $\square$

Next, we are going to define the metric that we are going to use for f.

Definition 4.4. *Let*

$$H^{-1/2} = 1 + |\delta| + \frac{|f'|}{|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}} \tag{4.7}$$

and $G = Hg^\sharp$.

Remark 4.5. *We have that*

$$1 \le H^{-1/2} \le 1 + |\delta| + h^{-1/4}|f'|^{1/2} \le 3h^{-1/2} \tag{4.8}$$

since $|f'| \le h^{-1/2}$ *and* $|\delta| \le h^{-1/2}$. *Moreover,* $|f'| \le H^{-1/2}(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2})$ *so by the Cauchy-Schwarz inequality we obtain*

$$|f'| \le 2|f''|H^{-1/2} + 3h^{1/2}H^{-1/2} \le C_2H^{-1/2}. \tag{4.9}$$

which gives that $f \in S^+(H^{-1}, G)$, *see Definition 3.1.*

Since the metric G does not depend on the values of f, we shall need a weight to define the symbol class of f.

Definition 4.6. *Let*

$$M = |f| + |f'|H^{-1/2} + |f''|H^{-1} + h^{1/2}H^{-3/2} = |f| + |f'|_1^G + |f''|_2^G + h^{1/2}H^{-3/2} \tag{4.10}$$

then we have that $h^{1/2} \le M \le C_3h^{-1}$ *by* (4.8).

In the following, we shall simplify the notation and include the variables x, t and τ in the w variables.

Proposition 4.7. *We find that* $H^{-1/2}$ *is Lipschitz continuous,* G *is* σ *temperate such that* $G = H^2G^\sigma$ *and*

$$H(w) \le C_0H(w_0)(1 + G_w(w - w_0)). \tag{4.11}$$

We have that M *is a weight for* G *such that* $f \in S(M, G)$ *and*

$$M(w) \le C_1M(w_0)(1 + G_{w_0}(w - w_0))^{3/2}. \tag{4.12}$$

In the case when $1 + |\delta(w_0)| \le H^{-1/2}(w_0)/2$ *we have* $|f'(w_0)| \ge h^{1/2}$,

$$|f^{(k)}(w_0)| \le C_k|f'(w_0)|H^{\frac{k-1}{2}}(w_0) \qquad k \ge 1, \tag{4.13}$$

and $1/C \le |f'(w)|/|f'(w_0| \le C$ *when* $|w - w_0| \le cH^{-1/2}(w_0)$ *for some* $c > 0$.

Remark 4.8. *The term* $f_0 = \partial_\eta f \cdot r \cdot \xi \in S(h^{-1}, g)$ *in Proposition 3.6 can be written* $f_0 = r_0 \cdot \xi$ *with* $r_0 \in S(MH^{1/2}h^{1/2}, G) \subset S(1, G)$ *since* $|\partial_\eta f| \le h^{1/2}|f'|$. *Now,* $\langle\xi\rangle$ *is not a weight for* $g^\sharp$ *near* Σ_2 *but* $f_0 \in S(MH^{1/2}h^{1/2}\langle\xi\rangle, G_0)$ *where* G_0 *is given by* (7.11).

Since $G \le g^\sharp \le G^\sigma$ we find that the conditions (4.11) and (4.12) are stronger than the property of being σ temperate (in fact, it is strongly σ temperate in the sense of [1, Definition 7.1]), and imply that G is slowly varying and M is G continuous. When $1 + |\delta| < H^{-1/2}/2$ we find from (4.13) that $|f'| > 0$ is a weight for G, $f' \in S(|f'|, G)$ and $f^{-1}(0)$ is a C^∞ hypersurface. Since that surface does not depend on x we find that

$\min_x H^{1/2}$ gives an upper bound on the curvature of $f^{-1}(0)$ by (4.13). Proposition 4.9 shows that (4.13) also holds for $k = 0$ when $1 + |\delta| \ll H^{-1/2}$.

Proof. First we note that if $H^{-1/2}$ is Lipschitz continuous, then

$$H^{-1/2}(w_0) \lesssim H^{-1/2}(w) + C_1|w - w_0| \tag{4.14}$$

which gives (4.11) since $G = Hg^\sharp$. Next, we shall show that $H_1^{-1/2}$ is Lipschitz continuous. Since the first terms of (4.7) are Lipschitz continuous, we only have to prove that

$$|f'|/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) = E/D$$

is Lipschitz continuous. Since this is a local property, it suffices to prove this when $|\Delta w| = |w - w_0| \le 1$. Then we have that $D(w) \cong D(w_0)$, in fact $D^2 \cong h + h^{1/2}|f'| + |f''|^2$ so

$$D^2(w) \le C(D^2(w_0) + |f''(w_0)|h^{1/2} + h) \le C' D^2(w_0)$$

when $|\Delta w| \le 1$. We find that

$$\left|\Delta \frac{E}{D}\right| = \left|\frac{E(w)}{D(w)} - \frac{E(w_0)}{D(w_0)}\right| \le \frac{|\Delta E|}{D(w)} + \frac{E(w_0)|\Delta D|}{D(w)D(w_0)}.$$

Taylor's formula gives that

$$|\Delta E| \le (|f''(w)| + Ch^{1/2})|\Delta w| \le CD(w) \tag{4.15}$$

when $|\Delta w| \le 1$. It remains to show that $E(w_0)|\Delta D| \le CD(w)D(w_0)|\Delta w|$, which is trivial if $E(w_0) = 0$. Else, we have

$$|\Delta|f''|| \le Ch^{1/2}|\Delta w| \le CD^2(w_0)|\Delta w|/E(w_0) \le C' D(w_0)D(w)||\Delta w|/E(w_0)$$

when $|\Delta w| \le 1$ since $h^{1/2} \le D^2/E$ and $D(w_0) \le CD(w)$. Finally, we have

$$\begin{aligned} h^{1/4}|\Delta|f'|^{1/2}| &\le h^{1/4}|\Delta E|/(|f'(w_0)|^{1/2} + |f'(w)|^{1/2}) \\ &\le Ch^{1/4}|f'(w_0)|^{1/2} D(w)|\Delta w|/|f'(w_0)| \le CD(w_0)D(w)|\Delta w|/E(w_0) \end{aligned}$$

when $|\Delta w| \le 1$ by (4.15), which proves the Lipschitz continuity.

Next, we study the case when $1 + |\delta(w_0)| \le H^{-1/2}(w_0)/2$, then $H^{1/2}(w_0) \le 1/2$. Then we find from (4.7) that

$$|f''(w_0)| + h^{1/4}|f'(w_0)|^{1/2} + h^{1/2} \le 2H^{1/2}(w_0)|f'(w_0)| \le |f'(w_0)|. \tag{4.16}$$

which gives $|f'(w_0)| \ge h^{1/2}$, $|f'(w_0)| \ge |f''(w_0)|$ and that $h^{1/2} \le 4H(w_0)|f'(w_0)|$. When $|w - w_0| \le cH^{-1/2}(w_0)$ we find from (4.16) by using Taylors formula that

$$|f'(w) - f'(w_0)| \le |f''(w_0)|cH^{-1/2}(w_0) + C_3 h^{1/2} c^2 H^{-1}(w_0) \le (c + 4C_3c^2)|f'(w_0)|$$

which gives $1/C \le |f'(w)|/|f'(w_0| \le C$ for small enough $c > 0$. Now (4.13) follows from (4.16) for $k = 2$. When $k \ge 3$ we have

$$|f^{(k)}(w_0)| \le C_k h^{\frac{k-2}{2}} \le 4C_k 3^{k-3}|f'(w_0)|H^{\frac{k-1}{2}},$$

since $h^{1/2} \le 4H|f'(w_0)|$ by (4.16) and $h^{(k-3)/2} \le 3^{k-3}H^{(k-3)/2}$ by (4.8).

Finally, we shall prove that M is a weight for G. By Taylor's formula we have

$$|f^{(k)}(w)| \le C_4 \sum_{j=0}^{2-k} |f^{(k+j)}(w_0)|w-w_0|^j + C_4 h^{1/2}|w-w_0|^{(3-k)} \qquad 0 \le k \le 2, \tag{4.17}$$

thus we obtain that

$$M(w) \le C_5 \sum_{k=0}^{2} |f^{(k)}(w_0)|(|w-w_0| + H^{-1/2}(w))^k + C_5 h^{1/2}(|w-w_0| + H^{-1/2}(w))^3.$$

By switching w and w_0 in (4.14) we find $H^{-1/2}(w)+|w-w_0| \le C_0(H^{-1/2}(w_0)+|w-w_0|)$. Thus we obtain that

$$\begin{aligned} M(w) \le C_6 \sum_{k=0}^{2} |f^{(k)}(w_0)|H^{-k/2}(w_0)(1+H^{1/2}(w_0)|w-w_0|)^k \\ + C_6 h^{1/2}H^{-3/2}(w_0)(1+H^{1/2}(w_0)|w-w_0|)^3 \le C_6 M(w_0)(1+G_{w_0}(w-w_0))^{3/2} \end{aligned}$$

which gives (4.12). It is clear from the definition of M that $|f^{(k)}| \le MH^{k/2}$ when $k \le 2$, and when $k \ge 3$ we have $|f^{(k)}| \le C_k h^{\frac{k-2}{2}} \le C_k 3^{k-3} M H^{\frac{k}{2}}$ since $h^{1/2} \le MH^{3/2}$ and $h^{(k-3)/2} \le 3^{k-3}H^{(k-3)/2}$ when $k \ge 3$. $\square$

Proposition 4.9. *We have that*

$$1/C \le M/(|f''|H^{-1} + h^{1/2}H^{-3/2}) \le C \tag{4.18}$$

When $|\delta| \le \kappa_0 H^{-1/2}$ and $H^{1/2} \le \kappa_0$ for $0 < \kappa_0 \le 1/4$ we find that

$$1/C_1 \le M/|f'|H^{-1/2} \le C_1 \tag{4.19}$$

which implies that $f \in S(H^{-1}, G)$ by (4.9).

Proof. First note that when $|\delta| \cong h^{-1/2}$ we have $H^{-1/2} \cong h^{-1/2}$, which gives $M \cong h^{-1}$ and proves (4.18) in this case. If $|\delta(w_0)| < h^{-1/2}$, then as before there exists $w \in f^{-1}(0)$ such that $|w-w_0| = |\delta(w_0)| \le H^{-1/2}(w_0)$. Since $f(w) = 0$, Taylor's formula gives that

$$|f(w_0)| \le |f'(w_0||\delta(w_0)| + |f''(w_0)||\delta(w_0)|^2/2 + Ch^{1/2}|\delta(w_0)|^3. \tag{4.20}$$

We obtain from (4.20) and (4.9) that

$$M \le C\big(|f'|H^{-1/2} + |f''|H^{-1} + h^{1/2}H^{-3/2}\big) \le C'\left(|f''|H^{-1} + h^{1/2}H^{-3/2}\right) \qquad \text{at } w_0,$$

which gives (4.18) since the opposite estimate is trivial.

If $|\delta| \le \kappa_0 H^{-1/2} < h^{1/2}$ and $H^{1/2} \le \kappa_0$ with $\kappa_0 \le 1/4$ then $\langle \delta \rangle \le H^{-1/2}/2$ so we we obtain by (4.13), (4.16) and (4.20) that

$$M \le C\big(|f'|H^{-1/2} + |f''|H^{-1} + h^{1/2}H^{-3/2}\big) \le C'|f'|H^{-1/2} \qquad \text{at } w_0.$$

since $h^{1/2} \le 4H|f'|$ by (4.16). This gives (4.19) since the opposite estimate is trivial, which completes the proof of the proposition. □

Proposition 4.10. *Let $H^{-1/2}$ be given by Definition 4.4 for $f \in S(h^{-1}, g)$. There exists positive κ_1 and c so that if $\langle \delta \rangle = 1 + |\delta| \le \kappa_1 H^{-1/2}$ at w_0 then*

$$f = \alpha\delta \qquad \textit{when } |w - w_0| \le cH^{-1/2}(w_0) \tag{4.21}$$

where $0 \le \alpha \in S(MH^{1/2}, G)$, such that $\alpha \ge \kappa_1 MH^{1/2}$, which implies that $\delta = f/\alpha \in S(H^{-1/2}, G)$.

Proof. Let $\kappa_0 > 0$ be given by Proposition 4.9. If $\kappa_1 \le \kappa_0$ and $\langle \delta \rangle \le \kappa_1 H^{-1/2}$ at w_0 then we find that $|f'(w_0)| \cong M(w_0)H^{1/2}(w_0)$ by (4.19). We may change coordinates so that $w_0 = 0$. Let $H^{1/2} = H^{1/2}(0)$ and $M = M(0)$, $w = H^{-1/2}z$ and

$$F(z) = H^{1/2}f(H^{-1/2}z)/|f'(0)| \cong f(H^{-1/2}z)/M \in C^\infty$$

Now $\delta_1(z) = H^{1/2}\delta(H^{-1/2}z)$ is the signed distance to $F^{-1}(0)$ in the z coordinates which is constant in x.

We have $|F(0)| \le C_0$, $|F'(0)| = 1$, $|F''(0)| \le C_2$ and $|F^{(3)}(z)| \le C_3$, $\forall\, z$, by (4.13) in Proposition 4.7. It is no restriction to assume that the coordinates $z = (z_1, z')$ are chosen so that $\partial_{z'}F(0) = 0$, and then $|\partial_{z_1}F(z)| \ge c > 0$ in a fixed neighborhood of the origin. If $|\delta_1(0)| = |\delta(0)H^{1/2}| \le \kappa_1 \ll 1$ then $F^{-1}(0)$ is a C^∞ manifold in this neighborhood, $\delta_1(z)$ is uniformly C^∞ and $\partial_{z_1}\delta_1(z) \ge c_0 > 0$ in a fixed neighborhood of the origin.

By choosing $(\delta_1(z), z')$ as local coordinates and using Taylor's formula we find that $F(z) = \alpha_1(z)\delta_1(z)$ for any x since $F = 0$ when $\delta_1 = 0$. Here $0 \le \alpha_1 \in C^\infty$ and $\alpha_1 \ge C > 0$ in a fixed neighborhood of the origin. Thus

$$f(w) = |f'(0)|H^{-1/2}\alpha_1(H^{1/2}w)\delta_1(H^{1/2}w) = |f'(0)|\alpha_1(H^{1/2}w)\delta(w)$$

when $|w| \le cH^{-1/2}$. Now $\alpha(w) = |f'(0)|\alpha_1(H^{1/2}w) \in S(MH^{1/2}, G)$ with $\alpha \cong MH^{1/2}$ which gives the proposition. □

5. Properties of the symbol

In this section we shall study the properties of the symbol near the sign changes. We shall follow Section in [4] with some minor changes, and we shall start with a one-dimensional result.

Lemma 5.1. *Assume that $f(t) \in C^3(\mathbf{R})$ such that $\|f^{(3)}\|_\infty = \sup_t |f^{(3)}(t)|$ is bounded. If*

$$\operatorname{sgn}(t)f(t) \ge 0 \qquad \text{when } \varrho_0 \le |t| \le \varrho_1 \tag{5.1}$$

for $\varrho_1 \ge 3\varrho_0 > 0$, then we find

$$|f(0)| \le \frac{3}{2}\left(\varrho_0 f'(0) + \varrho_0^3\|f^{(3)}\|_\infty/2\right) \tag{5.2}$$

$$|f''(0)| \le f'(0)/\varrho_0 + 7\varrho_0\|f^{(3)}\|_\infty/6. \tag{5.3}$$

Proof. By Taylor's formula we have

$$0 \le \operatorname{sgn}(t)f(t) = |t|f'(0) + \operatorname{sgn}(t)(f(0) + f''(0)t^2/2) + R(t) \qquad \varrho_0 \le |t| \le \varrho_1$$

where $|R(t)| \le \|f^{(3)}\|_\infty |t|^3/6$. We obtain that

$$\left|f(0) + t^2 f''(0)/2\right| \le f'(0)|t| + \|f^{(3)}\|_\infty |t|^3/6 \tag{5.4}$$

for any $|t| \in [\varrho_0, \varrho_1]$. By choosing $|t| = \varrho_0$ and $|t| = 3\varrho_0$, we obtain that

$$4\varrho_0^2|f''(0)| \le 4f'(0)\varrho_0 + 28\|f^{(3)}\|_\infty \varrho_0^3/6$$

which gives (5.3). By letting $|t| = \varrho_0$ in (5.4) and substituting (5.3), we obtain (5.2). □

Proposition 5.2. *Let $f(w) \in C^\infty(\mathbf{R}^n)$ such that $\|f^{(3)}\|_\infty < \infty$. Assume that there exists $0 < \varepsilon \le r/5$ such that*

$$\operatorname{sgn}(w_1)f(w) \ge 0 \quad \text{when } |w_1| \ge \varepsilon + |w'|^2/r \text{ and } |w| \le r \tag{5.5}$$

where $w = (w_1, w')$. Then we obtain

$$|f''(0)| \le 33(|\partial_{w_1} f(0)|/\varrho + \varrho\|f^{(3)}\|_\infty) \tag{5.6}$$

for any $\varepsilon \le \varrho \le r/\sqrt{10}$.

Proof. We shall consider the function $t \mapsto f(t, w')$ which satisfies (5.1) for fixed w' with

$$\varepsilon + |w'|^2/r = \varrho_0(w') \le |t| \le \varrho_1 \equiv 3r/\sqrt{10}$$

and $|w'| \le r/\sqrt{10}$ which we assume in what follows. In fact, then $t^2 + |w'|^2 \le r^2$ and $3\varrho_0(w') \le 9r/10 \le 3r/\sqrt{10} = \varrho_1$. We obtain from (5.2) and (5.3) that

$$|f(0, w')| \le \frac{3}{2}\partial_{w_1} f(0, w')\varrho + 3\varrho^3\|f^{(3)}\|_\infty/4 \tag{5.7}$$

$$|\partial_{w_1}^2 f(0, w')| \le \partial_{w_1} f(0, w')/\varrho + 7\varrho\|f^{(3)}\|_\infty/6 \tag{5.8}$$

for $\varepsilon + |w'|^2/r \le \varrho \le r/\sqrt{10}$ and $|w'| \le r/\sqrt{10}$. By letting $w' = 0$ in (5.8) we find that

$$|\partial_{w_1}^2 f(0)| \le \partial_{w_1} f(0)/\varrho + 7\varrho\|f^{(3)}\|_\infty/6 \tag{5.9}$$

for $\varepsilon \le \varrho \le r/\sqrt{10}$. By letting $\varrho = \varrho_0(w')$ in (5.7) and dividing with $3\varrho_0(w')/2$, we obtain that

$$0 \le \partial_{w_1} f(0, w') + 2\|f^{(3)}\|_\infty |w'|^2 \tag{5.10}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$ since then $\varrho_0(w') \le \varepsilon + |w'| \le 2|w'|$. By using Taylor's formula for $w' \mapsto \partial_{w_1} f(0, w')$ in (5.10), we find that

$$0 \le \partial_{w_1} f(0) + \langle w', \partial_{w'}(\partial_{w_1} f)(0)\rangle + \frac{5}{2}\|f^{(3)}\|_\infty |w'|^2$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus, by optimizing over fixed $|w'|$, we obtain that

$$|w'||\partial_{w'}(\partial_{w_1} f)(0)| \le \partial_{w_1} f(0) + \frac{5}{2}\|f^{(3)}\|_\infty |w'|^2 \qquad \text{when } \varepsilon \le |w'| \le r/\sqrt{10}. \tag{5.11}$$

By again putting $\varrho = \varrho_0(w')$ in (5.7), using Taylor's formula for $w' \mapsto \partial_{w_1} f(0, w')$ but this time substituting (5.11), we obtain

$$|f(0, w')| \le 6\partial_{w_1} f(0)|w'| + 15\|f^{(3)}\|_\infty |w'|^3 \qquad \text{when } \varepsilon \le |w'| \le r/\sqrt{10}. \tag{5.12}$$

We may also estimate the even terms in Taylor's formula by (5.12):

$$\begin{aligned} |f(0) + \langle \partial^2_{w'} f(0) w', w'\rangle/2| &\le \frac{1}{2}|f(0, w') + f(0, -w')| + \|f^{(3)}\|_\infty |w'|^3/6 \\ &\le 6\partial_{w_1} f(0)|w'| + \frac{91}{6}\|f^{(3)}\|_\infty |w'|^3 \end{aligned}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus, by using (5.7) with $\varrho = \varepsilon$ and $w' = 0$ to estimate $|f(0)|$ and optimizing over fixed $|w'|$, we obtain that

$$|\partial^2_{w'} f(0)||w'|^2/2 \le \frac{15}{2}|\partial_{w_1} f(0)||w'| + 16\|f^{(3)}\|_\infty |w'|^3 \tag{5.13}$$

when $\varepsilon \le |w'| \le r/\sqrt{10}$. Thus we obtain (5.6) by taking $\varepsilon \le |w'| = \varrho \le r/\sqrt{10}$ in (5.9)–(5.13). □

As before, if $f \in C^\infty(\mathbf{R}^n)$ then we define the *signed distance function* of f as $\delta = \operatorname{sgn}(f) d$ where d is the Euclidean distance to $f^{-1}(0)$.

Proposition 5.3. *Let $f_j(w) \in C^\infty(\mathbf{R}^n)$ and $\delta_j(w)$ be the signed distance functions of $f_j(w)$, for $j = 1, 2$. Assume that $f_1(w) > 0 \implies f_2(w) \ge 0$. There exists positive c_0 and c_1, such that if $|\delta_j(w_0)| \le c_0$, $|f_j'(w_0)| \ge c_1$, for $j = 1, 2$, and*

$$|\delta_1(w_0) - \delta_2(w_0)| = \varepsilon \tag{5.14}$$

then there exist $g^\sharp$ orthonormal coordinates $w = (w_1, w')$ so that $w_0 = (x_1, 0)$ with $x_1 = \delta_1(w_0)$ and

$$\operatorname{sgn}(w_1) f_j(w) > 0 \quad \text{when } |w_1| \ge (\varepsilon + |w'|^2)/c_0 \text{ and } |w| \le c_0 \qquad j = 1,\ 2 \tag{5.15}$$

$$|\delta_2(w) - \delta_1(w)| \le (\varepsilon + |w - w_0|^2)/c_0 \qquad \text{when } |w| \le c_0. \tag{5.16}$$

The constant c_0 only depends on the seminorms of f_1 and f_2 in a fixed neighborhood of w_0.

Proof. Observe that the conditions get stronger and the conclusions weaker when c_0 decreases. Assume that f_1 and f_2 are uniformly bounded in C^∞ near w_0. For any positive c_0 and c_1 there exists $c_2 > 0$ so that if $|f_j'(w_0)| \ge c_1$ and $|\delta_j(w_0)| \le c_0$, $j = 1, 2$ then $|f_j'(w)| > 0$ for $|w - w_0| \le c_2$, thus $f_j^{-1}(0)$ is a C^∞ hypersurface in $|w - w_0| \le c_2$. By decreasing c_0 we obtain as in the proof of Proposition 4.10 that there exists $c_3 > 0$ so that $w \mapsto \delta_j(w) \in C^\infty(\mathbf{R}^n)$ uniformly in $|w - w_0| \le c_3$, $j = 1, 2$. We may also choose $z_0 \in f_1^{-1}(0)$ so that $|\delta_1(w_0)| = |w_0 - z_0|$, and then choose $g^\sharp$ orthonormal coordinates so that $z_0 = 0$, $w_0 = (\delta_1(w_0), 0)$ and $\partial_{w'}\delta_1(0) = \partial_{w'}\delta_1(w_0) = 0$, $w = (w_1, w')$. If $c_0 \le c_3/3$ we find that $\delta_j \in C^\infty$ in $|w| \le c_4 = 2c_3/3$. Since $\operatorname{sgn}(f_1(w_0)) = \operatorname{sgn}(\delta_1(w_0))$ we find that $\partial_{w_1} f_1(0) > 0$.

We have that $|\partial_w^2 \delta_j(w)| \le C_0$ for $|w| \le c_4$, $j = 1, 2$, and $\Delta(w) = \delta_2(w) - \delta_1(w) \ge 0$ by the sign condition. By [10, Lemma 7.7.2] we obtain that $|\partial_w \Delta(w)|^2 \le C_1 \Delta(w) \le C_1\varepsilon$ when $w = w_0$ by (5.14). This gives

$$
\begin{aligned}
(5.17)\quad |\Delta(w)| \le |\Delta(w_0)| + |\partial_w \Delta(w_0)||w - w_0| + C_2|w - w_0|^2 \\
\le C_3(\varepsilon + |w - w_0|^2) \quad \text{for } |w| \le c_4
\end{aligned}
$$

which proves (5.16). Since $|\partial_{w'}\delta_1(w_0)| = 0$ we find that $|\partial_{w'}\delta_2(w)| \le C_4(\sqrt{\varepsilon} + |w - w_0|) \ll 1$ when $|w - w_0| \ll 1$ and $\varepsilon \le 2c_0 \ll 1$. Now $f_2(\overline{w}) = 0$ for some $|\overline{w}| \le \varepsilon$. Thus for $c_0 \ll 1$ we obtain $|\partial_{w'}\delta_2(\overline{w})| \ll 1$, which gives that $|\partial_{w_1} f_2(\overline{w})| \ge c_5 |\partial_w f_2(\overline{w})| \ge c_5^2 > 0$ for some $c_5 > 0$. Since $\operatorname{sgn}(f_2(w_1, 0)) = 1$ when $w_1 > 0$, we obtain that $\partial_{w_1} f_2(w) \ge c_6 |\partial_w f_2(w)| \ge c_6^2$ when $|w| \le c_6$ for some $c_6 > 0$.

By using the implicit function theorem, we obtain $b_j(w') \in C^\infty$, so that that $\pm f_j(w) > 0$ if and only if $\pm(w_1 - b_j(w')) \ge 0$ when $|w| \le c_7 > 0$, $j = 1, 2$. Since $f_1(0) = |\partial_{w'} f_1(0)| = 0$ we obtain that $b_1(0) = |b_1'(0)| = 0$. This gives that $|b_1(w')| \le C_5|w'|^2$ and proves the positive part of (5.15) by the sign condition. Observe that the sign condition gives that $b_1(w') \ge b_2(w')$. Now $|\delta_2(w_0)| \le |\delta_1(w_0)| + \varepsilon$, thus we find $-\varepsilon \le b_2(\overline{w}') \le b_1(\overline{w}')$ for some $|\overline{w}'| \le C\sqrt{\varepsilon} \le C\sqrt{2c_0} \le c_7$ for $c_0 \ll 1$. This gives $b_2(\overline{w}') \le C_5 C^2 \varepsilon$ and $|b_1'(\overline{w}')| \le C_6\sqrt{\varepsilon}$, and we obtain as before that $|b_1'(\overline{w}') - b_2'(\overline{w}')| \le C_7\sqrt{\varepsilon}$. As in (5.17), we find

$$
|b_2(w')| \le C_8(\varepsilon + |w' - \overline{w}'|^2) \le C_9(\varepsilon + |w'|^2) \qquad \text{for } |w| \le c_7
$$

which proves the negative part of (5.15) and the proposition. □

6. The Weight function

In this section, we shall define the weight m we shall use. We shall follow Section 5 in [4] with some necessary changes because of the different conditions and normal forms. We shall use the same notation as in Section 4, and let $\delta(t, \tau, w)$, $H^{-1/2}(t, x, \tau, w)$ and $M(t, x, \tau, w)$ be given by Definitions 4.2, 4.4 and 4.6 for $f(t, x, \tau, w) \in S(h^{-1}, hg^\sharp)$

satisfying condition $\mathrm{Sub_r}(\overline{\Psi})$ given by (3.3) such that $|f'| \le h^{-1/2}$. As before, we shall include τ in w but t will be a parameter.

The weight m will essentially measure how much $t \mapsto \delta(t,w)$ changes between the minima of $t \mapsto H^{1/2}(t,x,w)\langle\delta(t,w)\rangle^2$, which will give restrictions on the sign changes of the symbol. As before, we assume that we have choosen $g^\sharp$ orthonormal coordinates so that $g^\sharp(w)$ is the euclidean metric, and the results will only depend on the seminorms of f. The following definition uses that $t \mapsto \delta(t,w)$ is nondecreasing and δ is constant in x, and assumes that H is only defined in $|x| \le C$.

Definition 6.1. *Let $H_1(t,w) = \min_x H(t,x,w)$ and*

$$(6.1) \quad m(t,w) = \inf_{t_1 \le t \le t_2} \big\{ \delta(t_2,w) - \delta(t_1,w) \\ + \max\big(H_1^{1/2}(t_1,w)\langle\delta(t_1,w)\rangle^2, H_1^{1/2}(t_2,w)\langle\delta(t_2,w)\rangle^2\big)/2 \,\big\}$$

where $\langle\delta\rangle = 1 + |\delta| \le H_1^{-1/2} = \max_x H^{-1/2}$. Thus m is constant on the leaves of Σ_2.

This is actually Definition 5.1 in [4] with H replaced by H_1. It will be important in the proof that this weight is constant on the leaves of Σ_2.

Remark 6.2. *When $t \mapsto \delta(t,w)$ is constant for fixed w, we find that $t \mapsto m(t,w)$ is equal to the largest quasi-convex minorant of $t \mapsto H_1^{1/2}(t,w)\langle\delta(t,w)\rangle^2/2$, i.e., $\sup_I m = \sup_{\partial I} m$ for compact intervals $I \subset \mathbf{R}$, see* [11, Definition 1.6.3].

Remark 6.3. *One can also make a local definition of m by taking the infimum over $-T \le t_1 \le t \le t_2 \le T$ in (6.1) for some $0 < T \ll 1$. Then the results of this section will hold when $|t| \le T$. By making a translation in t we can of course define m in a neighborhood of any point.*

Proposition 6.4. *We have that $m \in L^\infty$, such that $w \mapsto m(t,w)$ is uniformly Lipschitz continous, $\forall\, t$, and*

$$h^{1/2}\langle\delta\rangle^2/6 \le m \le H_1^{1/2}\langle\delta\rangle^2/2 \le H^{1/2}\langle\delta\rangle^2/2 \le \langle\delta\rangle/2. \tag{6.2}$$

We may choose $t_1 \le t_0 \le t_2$ so that

$$\max_{j=0,1,2}\langle\delta(t_j,w_0)\rangle \le 2\min_{j=0,1,2}\langle\delta(t_j,w_0)\rangle. \tag{6.3}$$

and

$$H_0^{1/2} = \max(H_1^{1/2}(t_1,w_0),\ H_1^{1/2}(t_2,w_0)) \tag{6.4}$$

satisfies

$$H_0^{1/2} < 16m(t_0,w_0)/\langle\delta(t_j,w_0)\rangle^2 \qquad \textit{for } j = 0,\ 1,\ 2. \tag{6.5}$$

If $m(t_0,w_0) \le \varrho\langle\delta(t_0,w_0)\rangle$ *for* $\varrho \ll 1$*, then we may choose* $g^\sharp$ *orthonormal coordinates so that* $w_0 = (x_1, 0)$*,* $|x_1| < 2\langle\delta(t_0,w_0)\rangle < 32\varrho H_0^{-1/2}$*, and*

$$\operatorname{sgn}(w_1) f(t_0,w) \ge 0 \quad \text{when } |w_1| \ge (m(t_0,w_0) + H_0^{1/2}|w'|^2)/c_0 \tag{6.6}$$

$$|\delta(t_1,w) - \delta(t_2,w)| \le (m(t_0,w_0) + H_0^{1/2}|w-w_0|^2)/c_0 \tag{6.7}$$

when $|w| \le c_0 H_0^{-1/2}$*. The constant* c_0 *only depends on the seminorms of* f*.*

Observe that condition (6.6) is not empty when $m(t_0,w_0) \le \varrho\langle\delta(t_0,w_0)\rangle$ for ϱ sufficiently small since then $H_0^{-1/2} \gtrsim \langle\delta\rangle^2/m \gg m$ by (6.5) .

Proof. If we let

$$F(s,t,w) = |\delta(s,w) - \delta(t,w)| + \max(H_1^{1/2}(s,w)\langle\delta(s,w)\rangle^2, H_1^{1/2}(t,w)\langle\delta(t,w)\rangle^2)/2$$

then we find that $w \mapsto F(s,t,w)$ is uniformly Lipschitz continuous. Now, it suffices to show this when $|\Delta w| = |w - w_0| \ll 1$, and we know that $\langle\delta\rangle$ and $H^{-1/2}$ are uniformly Lipschitz continuous by Proposition 4.7 which gives that $H_1^{-1/2} = \max_x H^{-1/2}$ is uniformly Lipschitz continuous. The first term $|\delta(s,w) - \delta(t,w)|$ is obviously uniformly Lipschitz continuous. We have for fixed t that

$$\left|\Delta(H_1^{1/2}\langle\delta\rangle^2)\right| \le C(\langle\delta\rangle^2|\Delta H_1^{1/2}| + H_1^{1/2}\langle\delta\rangle|\Delta\delta|)$$

where $H_1^{1/2}\langle\delta\rangle \le 1$, $|\Delta\delta| \le |\Delta w|$ and $|\Delta H_1^{1/2}| \le CH_1|\Delta H_1^{-1/2}| \le C'H_1|\Delta w|$, which gives the uniform Lipschitz continuity of $F(s,t,w)$. By taking the infimum, we obtain (6.2) and the uniform Lipschitz continuity of m. In fact, $h^{1/2}/3 \le H_1^{1/2}$ by (4.8) and since $t \mapsto \delta(t,w)$ is monotone, we find that $t \mapsto \langle\delta(t,w)\rangle$ is quasi-convex. Thus $h^{1/2}\langle\delta(t_0,w_0)\rangle/6 \le F(s,t,w_0)$ when $s \le t_0 \le t$.

By approximating the infimum, we may choose $t_1 \le t_0 \le t_2$ so that $F(t_1,t_2,w_0) < m(t_0,w_0) + h^{1/2}/6$. Since $h^{1/2}/6 \le m \le H_1^{1/2}\langle\delta\rangle^2/2$ by (6.2), we find that

$$|\delta(t_1,w_0) - \delta(t_2,w_0)| < m(t_0,w_0) \le \langle\delta(t_0,w_0)\rangle/2 \qquad \text{and} \tag{6.8}$$

$$H_1^{1/2}(t_j,w_0)\langle\delta(t_j,w_0)\rangle^2/2 < 2m(t_0,w_0) \qquad \text{for } j = 1 \text{ and } 2. \tag{6.9}$$

Since $t \mapsto \delta(t,w_0)$ is monotone, we obtain (6.3) from (6.8), and (6.5) from (6.9) and (6.3).

Next assume that $m(t_0,w_0) \le \varrho\langle\delta(t_0,w_0)\rangle$ for some $0 < \varrho \le 1$. Then we find from (6.5) that

$$1 + |\delta(t_j,w_0)| < 16\varrho H_0^{-1/2} \qquad \text{for } j = 0,\ 1,\ 2. \tag{6.10}$$

We may choose $g^\sharp$ orthonormal coordinates so that $w_0 = 0$. Since δ, H_1, H_0 and m are constant in x, the results will hold for any x. I If we choose x_j so that $H_1(t_j,0) =$

$H(t_j, x_j, 0)$ then $\langle \delta(t_j, 0)\rangle < 16\varrho H^{-1/2}(t_j, x_j, 0)$ for $j = 1, 2$ by (6.10) so we find from Proposition 4.7 that

$$h^{1/2} \le |f'(t_j, x_j, 0)| \cong |f'(t_j, x_j, w)| \quad \text{for } |w| \le cH_0^{-1/2} \le cH^{-1/2}(t_j, x_j, 0) \tag{6.11}$$

when $\varrho \ll 1$ and $j = 1$, 2. Since $H_0^{-1/2} \le 3h^{-1/2}$ we find from (6.10) that $f(t_j, x_j, \widetilde{w}_j) = 0$ for some $|\widetilde{w}_j| < 16\varrho H_0^{-1/2}$ by (6.10) when $\varrho < 1/48$ and $j = 1$, 2. Thus, when $16\varrho \le c$ we obtain from (6.11) for $j = 1$, 2 that

$$|f(t_j, x_j, w)| \le C|f'(t_j, x_j, 0)|H_0^{-1/2} \qquad \text{when } |w| < cH_0^{-1/2}$$

and then (4.13) gives $f(t_j, x_j, w) \in S(|f'(t_j, x_j, 0)|H_0^{-1/2}, H_0 g^\sharp)$ since $H^{1/2}(t_j, x_j, 0) \le H_0^{1/2}$, $j = 1$, 2. By Proposition 4.10 we have that $f_j = \alpha\delta$ where $\delta \in S(H_0^{-1/2}, H_0 g^\sharp)$ and $\alpha \in S(|f'(t_j, x_j, 0)|, H_0 g^\sharp)$ in a $H_0 g^\sharp$ neighborhood of $(t_j, 0)$ such that $|\alpha| = |f'(t_j, x_j, 0)|$ and $|\delta'| = 1$ at $(t_j, 0)$. Now $\partial_t \delta \ge 0$ so if $|\partial_t \delta|^{g^\sharp} \ge \varepsilon > 0$ at $(t_j, 0)$ for $j = 1$ or 2 then $\partial_t \delta \ge c\varepsilon h^{-1/2}$ in a small $H_1 g^\sharp$ neighborhood. The interval $\{(t, 0) : |t - t_j| \le c_0 h^{1/2} H_1^{-1/2}\}$ is contained in this neighborhood for small enough $c_0 > 0$. Then we find

$$|\delta(t_0, 0) - \delta(t_j, 0)| \ge cc_0\varepsilon H_1^{-1/2}(t_j, 0) \ge cc_0\varepsilon H_0^{-1/2} \ge cc_0\varepsilon\langle \delta(t_j, 0)\rangle/16\varrho$$

by (6.10), which by (6.3) contradicts (6.8) for small enough ϱ. Thus, we may assume that $|\partial_w \delta| \ge 1/2$ at $(t_j, 0)$ for $j = 1$, 2.

Choose coordinates $z = H_0^{1/2} w$, we shall use Proposition 5.3 with

$$f_j(z) = H_0^{1/2} f(t_j, x_j, H_0^{-1/2} z)/|f'(t_j, 0)| \in C^\infty \qquad \text{for } j = 1,\ 2.$$

Let $\delta_j(z) = H_0^{1/2}\delta(t_j, H_0^{-1/2} z) \in C^\infty$ be the signed distance function to $f_j^{-1}(0)$ in z coordinates, then (6.10) gives that $|\delta_j(0)| \le 16\varrho$ for $= 0$, 1, 2. Now $|\partial_z \delta_j(t_j, 0)| \ge 1/2$, which for small enough ϱ gives $|\partial_z f_j(0)| \ge c_0$ for some $c_0 > 0$. In fact, we have that $f_j = a_j \delta_j$ where $a_j \in C^\infty$ is uniformly bounded and $a_j(0) = 1$. Then we obtain that $\partial_z f_j(0) = a_j(0)\partial_z \delta_j(0) + \delta_j(0)\partial_z a_j(0) \ge 1/2 - c\varrho$. Because of condition $\mathrm{Sub_r}(\overline{\Psi})$ given by (3.3) we find that $f_1(z) > 0 \implies f_2(z) \ge 0$. Since $|\delta_j(0)| < 16\varrho$ we find that

$$|\delta_1(0) - \delta_2(0)| = \varepsilon < H_0^{1/2} m(t_0, 0) \le H_0^{1/2}\langle \delta(t_0, 0)\rangle/2 < 8\varrho \tag{6.12}$$

by (6.8). Thus, for sufficiently small ϱ we may use Proposition 5.3 with this choice of f_j to obtain $g^\sharp$ orthogonal coordinates (z_1, z') so that $w_0 = z_0 = (y_1, 0)$, $|y_1| = |\delta_1(0)|$ and

$$\begin{cases} \operatorname{sgn}(z_1) f_j(z) > 0 \quad \text{when } |z_1| \ge (\varepsilon + |z'|^2)/c_0 \\ |\delta_1(z) - \delta_2(z)| \le (\varepsilon + |z - z_0|^2)/c_0 \end{cases}$$

when $|z| \le c_0$. Let $x_1 = H_0^{-1/2} y_1$ then $|x_1| < 2\langle \delta(t_0, 0)\rangle < 32\varrho H_0^{-1/2}$ by (6.3) and (6.10). We obtain (6.6)–(6.7) by the condition $\mathrm{Sub_r}(\overline{\Psi})$, since $H_0^{-1/2}\varepsilon < m(t_0, 0)$ by (6.12). $\square$

Proposition 6.5. *There exists $C > 0$ such that*

$$m(t_0, w) \le Cm(t_0, w_0)(1 + |w - w_0|/\langle\delta(t_0, w_0)\rangle)^3 \tag{6.13}$$

thus m is a weight for $g^\sharp$.

Proof. Since $m \le \langle\delta\rangle/2$ we only have to consider the case when

$$m(t_0, w_0) \le \varrho\langle\delta(t_0, w_0)\rangle \tag{6.14}$$

for some $\varrho > 0$. In fact, otherwise we have by (6.2) that

$$m(t_0, w) \le \langle\delta(t_0, w)\rangle/2 < m(t_0, w_0)(1 + |w - w_0|/\langle\delta(t_0, w_0)\rangle)/2\varrho$$

since the Lipschitz continuity of $w \mapsto \delta(t_0, w)$ gives

$$\langle\delta(t, w)\rangle \le \langle\delta(t, w_0)\rangle(1 + |w - w_0|/\langle\delta(t, w_0)\rangle) \qquad \forall\, t. \tag{6.15}$$

If (6.14) holds for $\varrho \ll 1$, then Proposition 6.4 gives $t_1 \le t_0 \le t_2$ such that (6.3), (6.5) and (6.7) hold when $|w| \le c_0 H_0^{-1/2}$ with $H_0^{1/2} = \max(H_1^{1/2}(t_1, w_0),\ H_1^{1/2}(t_2, w_0))$.

Now, for fixed w_0 it suffices to prove (6.13) when

$$|w - w_0| \le \varrho H_0^{-1/2} \tag{6.16}$$

for some $\varrho > 0$. In fact, when $|w - w_0| > \varrho H_0^{-1/2}$ we obtain from (6.5) that

$$\begin{aligned}|w - w_0|^2/\langle\delta(t_0, w_0)\rangle^2 > \varrho^2 H_0^{-1}/\langle\delta(t_0, w_0)\rangle^2 > \varrho^2\langle\delta(t_0, w_0)\rangle^2/256m^2(t_0, w_0)\\ \ge \varrho^2\langle\delta(t_0, w_0)\rangle m(t_0, w)/64\langle\delta(t_0, w)\rangle m(t_0, w_0)\end{aligned}$$

since $\langle\delta\rangle \ge 2m$. By (6.15) we obtain that (6.13) is satisfied with $C = 64/\varrho^2$. Thus in the following we shall only consider w such that (6.16) is satisfied for $\varrho \ll 1$. We find by (6.5) and (6.7) that

$$\begin{aligned}|\delta(t_1, w) - \delta(t_2, w)| &\le (m(t_0, w_0) + H_0^{1/2}|w - w_0|^2)/c_0\\ &< 16m(t_0, w_0)(1 + |w - w_0|^2/\langle\delta(t_0, w_0)\rangle^2)/c_0\end{aligned} \tag{6.17}$$

when $|w - w_0| \le c_0 H_0^{-1/2}$. Now G is slowly varying, thus we find for small enough $\varrho > 0$ that

$$H_1^{1/2}(t_j, w) \le CH_1^{1/2}(t_j, w_0) \qquad \text{when } |w - w_0| \le \varrho H_0^{-1/2} \le \varrho H_1^{-1/2}(t_j, w_0)$$

for $j = 1, 2$. By (6.15) and (6.3) we obtain that

$$H_1^{1/2}(t_j, w)\langle\delta(t_j, w)\rangle^2 \le 4CH_1^{1/2}(t_j, w_0)\langle\delta(t_j, w_0)\rangle^2(1 + |w - w_0|/\langle\delta(t_0, w_0)\rangle)^2 \tag{6.18}$$

when $j = 1,\ 2$, and $|w - w_0| \le c_0 H_0^{-1/2}$. Now $H_1^{1/2}(t_j, w_0)\langle\delta(t_j, w_0)\rangle^2 < 16m(t_0, w_0)$ by (6.5) for $j = 1, 2$. Thus, by using (6.17), (6.18) and taking the infimum we obtain

$$m(t_0, w) \le C_0 m(t_0, w_0)(1 + |w - w_0|/\langle\delta(t_0, w_0)\rangle)^2$$

when $|w - w_0| \le \varrho H_0^{-1/2}$ for $\varrho \ll 1$. □

The following result will be important for the proof of Proposition 3.6 in Section 8.

Proposition 6.6. *Let M be given by Definition 4.6 and m by Definition 6.1. Then there exists $C_0 > 0$ such that*

$$MH^{3/2} \le C_0 m/\langle\delta\rangle^2. \tag{6.19}$$

Proof. In the proof, we shall include the t variable in the w variables. Observe that since $h^{1/2}\langle\delta\rangle^2/6 \le m$ we find that (6.19) is equivalent to

$$|f''|H^{1/2} \le Cm/\langle\delta\rangle^2 \tag{6.20}$$

by Proposition 4.9. First we note that if $m \ge c\langle\delta\rangle > 0$, then $MH^{3/2}\langle\delta\rangle^2 \le C\langle\delta\rangle \le Cm/c$ since $\langle\delta\rangle \le H^{-1/2}$ and $M \le CH^{-1}$ by Proposition 4.9.

Thus, we only have to consider the case $m \le \varrho\langle\delta\rangle$ at w_0 for some $\varrho > 0$ to be chosen later. Then we may use Proposition 6.4 for $\varrho \ll 1$ to choose $g^\sharp$ orthonormal coordinates so that $|w_0| < 2\langle\delta(w_0)\rangle < 32\varrho H_0^{-1/2}$ and f satisfies (6.6) with

$$h^{1/2}/3 \le H_0^{1/2} < 16m(w_0)/\langle\delta(w_0)\rangle^2 \le 8H^{1/2}(w_0) \tag{6.21}$$

by (4.8), (6.2) and (6.5). Thus it suffices to prove the estimate

$$|f''|H^{1/2} \le CH_0^{1/2} \tag{6.22}$$

at w_0. Now it actually suffices to prove (6.22) at $w = 0$. In fact, (4.11) gives

$$H(w_0) \le C_0 H(0)(1 + H(w_0)|w_0|^2) \le 5C_0 H(0) \tag{6.23}$$

since $|w_0| < 2\langle\delta(w_0)\rangle \le 2H^{-1/2}(w_0)$. Thus Taylor's formula gives

$$|f''(w_0)|H^{1/2}(w_0) \le \left(|f''(0)| + C_3 h^{1/2}|w_0|\right) H^{1/2}(w_0) \le C_1(|f''(0)|H^{1/2}(w_0) + h^{1/2}) \tag{6.24}$$

since $|f^{(3)}| \le C_3 h^{1/2}$, which gives (6.22) at $w = 0$ by (6.21) and (6.23).

By Definition 4.4 we find that

$$\begin{aligned} H^{-1/2} &\ge 1 + |f'|/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) \\ &\ge (|f''| + |f'| + h^{1/2})/(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}), \end{aligned}$$

thus (6.22) follows if we prove

$$|f''|(|f''| + h^{1/4}|f'|^{1/2} + h^{1/2}) \le C\left(|f'| + |f''| + h^{1/2}\right) H_0^{1/2} \qquad \text{at } 0. \tag{6.25}$$

Since $h^{1/2}/3 \le H_0^{1/2}$ we obtain (6.25) by the Cauchy-Schwarz inequality if we prove that

$$|f''(0)| \le C(H_0^{1/4}|f'(0)|^{1/2} + h^{1/2}). \tag{6.26}$$

Let $F(z) = H_0 f(H_0^{-1/2} z)$, then (6.6) gives

$$\operatorname{sgn}(z_1)F(z) \ge 0 \quad \text{when } |z_1| \ge \varepsilon + |z'|^2/r \text{ and } |z| \le r$$

where $r = c_0$ and

$$\varepsilon = H_0^{1/2} m(w_0)/c_0 \le 16m^2(w_0)/c_0\langle\delta(w_0)\rangle^2 \le 16\varrho^2/c_0 \le c_0/5$$

by (6.21) when $\varrho \le c_0/4\sqrt{5}$ which we shall assume. Proposition 5.2 then gives that

$$|F''(0)| \le C_1\left(|F'(0)|/\varrho_0 + H_0^{-1/2}h^{1/2}\varrho_0\right) \qquad \varepsilon \le \varrho_0 \le c_0/\sqrt{10}$$

since $\|F^{(3)}\|_\infty \le C_3 H_0^{-1/2}h^{1/2}$. Observe that $|F'(0)| \le C_2$ since $H_0^{1/2} \le 8H^{1/2}(w_0) \le CH^{1/2}(0)$ by (6.21) and (6.23), and $|f'(0)| \le CH^{-1/2}(0)$ by (4.9). Choose

$$\varrho_0 = \varepsilon + \lambda|F'(0)|^{1/2} \le c_0/\sqrt{10}$$

with $\lambda = c_0(\sqrt{10}-2)/10\sqrt{C_2}$, then we obtain that

$$|F''(0)| \le C_4(|F'(0)|^{1/2} + h^{1/2}m(w_0))$$

since $H_0^{-1/2} \le 3h^{-1/2}$ and $\varepsilon = H_0^{1/2}m(w_0)/c_0$.

If $h^{1/2}m(w_0) \le |F'(0)|^{1/2}$ then we obtain (6.26) since $F' = H_0^{1/2}f'$ and $F'' = f''$. If $|F'(0)|^{1/2} \le h^{1/2}m(w_0)$, then we find

$$|f''(0)| \le 2C_4 h^{1/2}m(w_0) \le 4C_2 m(w_0)/\langle\delta(w_0)\rangle.$$

Then (6.20) follows from (6.13), (6.15) and (6.23) since $H^{1/2}(w_0) \le \langle\delta(w_0)\rangle^{-1}$, which completes the proof of the proposition. ☐

Next, we shall prove a convexity property of $t \mapsto m(t,w)$, which will be essential for the proof.

Proposition 6.7. *Let m be given by Definition 6.1. Then*

$$\sup_{t_1\le t\le t_2} m(t,w) \le \delta(t_2,w) - \delta(t_1,w) + m(t_1,w) + m(t_2,w) \qquad \forall\, w. \tag{6.27}$$

Proof. By definition we find that

$$\inf_{\pm(t-t_0)\ge 0}\left(|\delta(t,w)-\delta(t_0,w)| + H_1^{1/2}(t,w)\langle\delta(t,w)\rangle^2/2\right) \le m(t_0,w). \tag{6.28}$$

Let $t \in [t_1,t_2]$, then by taking the independent infima, we obtain that

$$\begin{aligned} m(t,w) \le \inf_{r\le t_1<t_2\le s} \delta(s,w) - \delta(r,w) + H_1^{1/2}(s,w)\langle\delta(s,w)\rangle^2/2 + H_1^{1/2}(r,w)\langle\delta(r,w)\rangle^2/2 \\ \le \delta(t_2,w) - \delta(t_1,w) + \inf_{t\ge t_2}\left(|\delta(t,w)-\delta(t_2,w)| + H_1^{1/2}(t,w)\langle\delta(t,w)\rangle^2/2\right) \\ + \inf_{t\le t_1}\left(|\delta(t,w)-\delta(t_1,w)| + H_1^{1/2}(t,w)\langle\delta(t,w)\rangle^2/2\right). \end{aligned}$$

By using (6.28) for $t_0 = t_1,\, t_2$, we obtain (6.27) after taking the supremum. ☐

Next, we shall construct the pseudo-sign $B = \delta + \varrho_0$, which we shall use in Section 8 to prove Proposition 3.6 with the multiplier $b^w = B^{Wick}$.

Proposition 6.8. *Assume that δ is given by Definition 4.2 and m is given by Definition 6.1. Then for $T > 0$ there exists real valued $\varrho_T(t,w) \in L^\infty$ with the property that $w \mapsto \varrho_T(t,w)$ is uniformly Lipschitz continuous, and*

$$|\varrho_T| \le m \tag{6.29}$$

$$T\partial_t(\delta + \varrho_T) \ge m/2 \qquad \text{in } \mathcal{D}'(\mathbf{R}) \tag{6.30}$$

when $|t| < T$.

Proof. (We owe this argument to Lars Hörmander [12].) Let

$$\varrho_T(t,w) = \sup_{-T\le s\le t}\left(\delta(s,w) - \delta(t,w) + \frac{1}{2T}\int_s^t m(r,w)\,dr - m(s,w)\right) \tag{6.31}$$

for $|t| \le T$, then

$$\delta(t,w) + \varrho_T(t,w) = \sup_{-T\le s\le t}\left(\delta(s,w) - \frac{1}{2T}\int_0^s m(r,w)\,dr - m(s,w)\right) + \frac{1}{2T}\int_0^t m(r,w)\,dr$$

which immediately gives (6.30) since the supremum is nondecreasing. Since $w \mapsto \delta(t,w)$ and $w \mapsto m(t,w)$ are uniformly Lipschitz continuous by Proposition 6.4, we find that $w \mapsto \varrho_T(t,w)$ is uniformly Lipschitz continuous by taking the supremum. Since $\delta(s,w) \le \delta(t,w)$ when $s \le t \le T$, we find from Proposition 6.7 that

$$\delta(s,w) - \delta(t,w) + \frac{1}{2T}\int_s^t m(r,w)\,dr - m(s,w) \le m(t,w) \qquad -T \le s \le t \le T.$$

By taking the supremum, we obtain that $-m(t,w) \le \varrho_T(t,w) \le m(t,w)$ when $|t| \le T$, which proves the result. $\square$

We shall also include a term in the multiplier to control the error terms involving $D_x u$.

Lemma 6.9. *Assume that A satisfies the conditions in Proposition 2.2 near $w_0 \in \Sigma_2$. Then there exists a matrix L and constant $c_1 > 0$ such that $\{A, \langle L(x-x_0),\xi\rangle\} \ge |\xi|^2 - c_1$ microlocally near $w_0 \in \Sigma_2$, where x_0 is the value of x at w_0. The constants only depend on the seminorms of A.*

Proof. Let $w = (x,\xi,z)$, then we find from the conditions that

$$A(x,\xi,z) = \langle C_2(x,\xi,z)\xi,\xi\rangle + \langle C_1(x,z),\xi\rangle + C_0(x,z)$$

where $C_j \in S^0$ is real valued, $\forall\, j$, and C_2 is a symmetric and nondegenerate matrix microlocally near w_0. By a translation we may assume that $x = 0$ at w_0. If we take $L = C_2^{-1}(0, 0, w_0)$ then we find that

$$\{A, \langle Lx, \xi\rangle\} = \langle L\xi, \partial_\xi A\rangle - \langle Lx, \partial_x A\rangle \ge c_2|\xi|^2 - c_0 \tag{6.32}$$

where $c_2 = 2$ at w_0, since $\partial_\xi A = 2C_2\xi + \langle\partial_\xi C_2\xi, \xi\rangle + C_1$. By continuity, we get the estimate in a neighborhood of w_0 where $|\partial_\xi C_2\xi| \ll 1$. □

Because of the cut-off in the estimate (3.17) we will only need the lower bound in a neighborhood of w_0.

Definition 6.10. *Let the multiplier symbol $B_T = \delta_0 + \varrho_T + \lambda_T$, where $\delta_0 = \delta$ is given by Definition 4.2, ϱ_T is given by Definition 6.8 for $T > 0$ so it is real valued and Lipschitz continuous, satisfying $|\varrho_T| \le m$ when $|t| \le T$, with $m \le \langle\delta_0\rangle/2$ given by Definition 6.1, and $\lambda_T = \epsilon h^{1/2}\langle L(x - x_0), \xi\rangle/T \in S(h^{-1/2}, g)$ uniformly when $|x - x_0| \le T$ and $|\xi| \lesssim h^{-1}$, where $0 < \epsilon \le 1$ and L is given by Lemma 6.9, so λ_T is Lipschitz continuous.*

7. The Wick quantization

In order to define the multiplier we shall use the Wick quantization. We shall start by recapitulating some results from Section 6 in [4] about the Wick operators. As before, we shall assume that $g^\sharp = (g^\sharp)^\sigma$ and the coordinates chosen so that $g^\sharp(w) = |w|^2$. For $a \in L^\infty(T^*\mathbf{R}^n)$ we define the Wick quantization:

$$a^{Wick}(x, D_x)u(x) = \int_{T^*\mathbf{R}^n} a(y, \eta)\Sigma^w_{y,\eta}(x, D_x)u(x)\, dyd\eta \qquad u \in C_0^\infty$$

using the orthonormal projections $\Sigma^w_{y,\eta}(x, D_x)$ with Weyl symbol

$$\Sigma_{y,\eta}(x, \xi) = \pi^{-n}\exp(-g^\sharp(x - y, \xi - \eta))$$

(see [2, Appendix B] or [13, Section 4]). We find that a^{Wick}: $\mathcal{S} \mapsto \mathcal{S}'$ so that

$$a \ge 0 \implies \left(a^{Wick}(x, D_x)u, u\right) \ge 0 \qquad u \in C_0^\infty \tag{7.1}$$

$(a^{Wick})^* = (\overline{a})^{Wick}$ and $\|a^{Wick}(x, D_x)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|a\|_{L^\infty(T^*\mathbf{R}^n)}$, which is the main advantage with the Wick quantization (see [13, Proposition 4.2]).

We obtain from the definition that $a^{Wick} = a_0^w$ where

$$a_0(w) = \pi^{-n}\int_{T^*\mathbf{R}^n} a(z)\exp(-|w - z|^2)\, dz \tag{7.2}$$

is the Gaussian regularization, thus Wick operators with real symbols have real Weyl symbols. This convolution also maps polynomials to polynomials.

Remark 7.1. *Observe that $a^{Wick} = a^w$ if $a(x, \xi)$ is affine in x for fixed ξ and affine in ξ for fixed x, for example if $a(x, \xi) = \langle Lx, \xi\rangle$ with a constant matrix L .*

In fact, $a^{Wick} = a^w$ if and only if

$$\pi^{-n} \int_{T^*\mathbf{R}^n} (a(z) - a(w)) \exp(-|w-z|^2)\, dz = 0$$

This vanishes if a is affine in x for fixed ξ and affine in ξ for fixed x since

$$a(x,\xi) - a(y,\eta) = a(x,\xi) - a(y,\xi) + a(y,\xi) - a(y,\eta)$$

which are two odd integrands giving vanishing integrals.

In the following, we shall assume that $G = Hg^\sharp \le g^\sharp$ is a slowly varying metric satisfying

$$H(w) \le C_0 H(w_0)(1 + |w - w_0|)^{N_0} \tag{7.3}$$

and m is a weight for G satisfying (7.3) with H replaced by m, by Propositions 4.7 and 6.5. This means that G and m are strongly σ temperate in the sense of [1, Definition 7.1]. Recall the symbol class $S^+(1, g^\sharp)$ given by Definition 3.1.

Proposition 7.2. *Assume that $a \in L^\infty$ such that $|a| \le Cm$, where m is a weight for $g^\sharp$, then $a^{Wick} = a_0^w$ where $a_0 \in S(m, g^\sharp)$ is given by (7.2). If $a \ge m$ we obtain that $a_0 \ge c_0 m$ for a fixed constant $c_0 > 0$. If $a \in S(M, G)$, where M is a weight for G, then $a_0 \cong a$ modulo $S(mH, G)$. If $|a| \le Cm$ and $a = 0$ in a fixed G ball with center w, then $a \in S(mH^N, G)$ at w for any N. If a is polynomial in the variable ξ then a_0 is polynomial in ξ with the same degree as a, and if a is Lipschitz continuous then we have $a_0 \in S^+(1, g^\sharp)$.*

By localization we find, for example, that if $|a| \le Cm$ and $a \in S(m, G)$ in a G neighborhood of w_0, then $a_0 \cong a$ modulo $S(mH, G)$ in a smaller G neighborhood of w_0. Observe that the results are uniform in the metrics and weights. The results are well known, but for completeness we give a proof.

Proof. Since a is measurable satisfying $|a| \le Cm$, where $m(z) \le C_0 m(w)(1 + |z-w|)^{N_0}$ by (7.3), we find that $a^{Wick} = a_0^w$ where $a_0 = \mathcal{O}(m)$ is given by (7.2). By differentiating on the exponential factor, we find $a_0 \in S(m, g^\sharp)$, and similarly we find that $a_0 \ge m/C$ if $a \ge m$ since $m(z) \gtrsim m(w)/(1 + |z-w|)^{N_0}$.

If $a = 0$ in a G ball of radius $\varepsilon > 0$ and center at w, then we can write

$$\pi^n a_0(w) = \int_{|z-w| \ge \varepsilon H^{-1/2}(w)} a(z) \exp(-|w-z|^2)\, dz = \mathcal{O}(m(w)H^N(w))$$

for any N even after repeated differentiation.

If $a \in S(m, G)$ then Taylor's formula gives

$$a_0(w) = a(w) + \pi^{-n} \int_0^1 \int_{T^*\mathbf{R}^n} (1-\theta)^2 \langle a''(w + \theta z)z, z\rangle e^{-|z|^2}\, dz d\theta/2$$

where $a'' \in S(mH, G)$ since $G = Hg^\sharp$. Since $m(w+\theta z) \le C_0 m(w)(1+|z|)^{N_0}$ and $H(w+\theta z) \le C_0 H(w)(1+|z|)^{N_0}$ when $|\theta| \le 1$, we find that $a_0(w) \cong a(w)$ modulo $S(mH, G)$.

If a is polynomial in the variable ξ of degree k then $\partial_\xi^\alpha a \equiv 0, \forall |\alpha| > k$, which gives $\partial_\xi^\alpha a_0 \equiv 0$. Thus a_0 is of degree $\le k$. The Lipschitz continuity of a means that $\partial a \in L^\infty(T^*\mathbf{R}^n)$. Since we have $\partial a_0(w) = \pi^{-n} \int_{T^*\mathbf{R}^n} \partial a(z) \exp(-|w-z|^2)\, dz$, we obtain the proposition. $\square$

We shall need the following result about the composition of Wick operators.

Proposition 7.3. *Assume that a and $b \in L^\infty$. If $|a| \le m_1$ and $|\partial b| \le m_2$, where m_j are weights for $g^\sharp$ satisfying* (7.3), *then*

$$a^{Wick} b^{Wick} = (ab)^{Wick} + r^w \tag{7.4}$$

with $r \in S(m_1 m_2, g^\sharp)$. If a and b are real such that $|a| \le m_1$ and $|\partial^2 b| \le m_2$, then

$$\operatorname{Re} a^{Wick} b^{Wick} = \left(ab - \frac{1}{2} \partial a \cdot \partial b \right)^{Wick} + R^w \tag{7.5}$$

with $R \in S(m_1 m_2, g^\sharp)$. By taking the adjoints, we get these results with a and b switched.

Observe that since $a \in L^\infty$ and ∂b is Lipschitz continuous in (7.5), we find that $\partial a \cdot \partial b$ is a well-defined distribution. In fact, we can define it as $\partial a \cdot \partial b(\varphi) = -\iint a \partial(\varphi \partial b) dw$. Proposition 7.3 essentially follows from Proposition 3.4 in [14] and Lemma A.1.5 in [15] but we shall for completeness give a proof.

Proof. By Proposition 7.2 we have $a^{Wick} b^{Wick} = a_0^w b_0^w$ in (7.4) where $a_0 \in S(m_1, g^\sharp)$ and $b_0 \in S^+(m_2, g^\sharp)$. By Lemma 3.2 we find $a^{Wick} b^{Wick} \cong (a_0 b_0)^w$ modulo $\operatorname{Op} S(m_1 m_2, g^\sharp)$, where

$$a_0(w) b_0(w) = \pi^{-2n} \iint a(w+z_1) b(w+z_2) e^{-|z_1|^2 - |z_2|^2}\, dz_1 dz_2. \tag{7.6}$$

By using the Taylor formula we find that $b(w+z_2) = b(w+z_1) + r_1(w, z_1, z_2)$ where $|r_1(w, z_1, z_2)| \le C m_2(w)(1+|z_1|+|z_2|)^N$ by (7.3). Integration in z_2 then gives (7.4).

For the proof of (7.5) we use that $\operatorname{Re} a_0^w b_0^w \cong (a_0 b_0)^w$ modulo $\operatorname{Op} S(m_1 m_2, g^\sharp)$ by Lemma 3.2, since a_0 and b_0 are real and $\partial^2 b_0 \in S(m_2, g^\sharp)$. We use the Taylor formula again:

$$b(w+z_2) = b(w+z_1) + \partial b(w+z_1) \cdot (z_2 - z_1) + r_2(w, z_1, z_2)$$

where $|r_2(w, z_1, z_2)| \le C m_2(w)(1+|z_1|+|z_2|)^N$. The term with z_2 is odd and gives a vanishing contribution in (7.6). Since $\partial_{z_1} e^{-|z_1|^2 - |z_2|^2} = -2z_1 e^{-|z_1|^2}$ we obtain (7.5) after an integration by parts, since $|a \partial^2 b| \le m_1 m_2$. $\square$

Example 7.4. *If* $a \in S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$ *and* $b \in S(M, G)$*, then* $\operatorname{Re} a^{Wick} b^{Wick} \cong (ab)^{Wick}$ *modulo* $\operatorname{Op} S(MH^{1/2}, g^\sharp)$*.*

We shall compute the Weyl symbol for the Wick operator $B_T^{Wick} = (\delta_0 + \varrho_T + \lambda_T)^{Wick}$ given by Definition 6.10. In the following, we shall suppress the T parameter.

Proposition 7.5. *Let* $B = \delta_0 + \varrho_0 + \lambda$ *be given by Definition 6.10, then we have* $B^{Wick} = b^w$ *where* $b = \delta_1 + \varrho_1 + \lambda$ *is real,* $\delta_1 \in S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$*, and* $\varrho_1 \in S(m, g^\sharp) \bigcap S^+(1, g^\sharp)$ *uniformly when* $|t| \le T$*. Also, there exists* $\kappa_2 > 0$ *so that* $\delta_1 \cong \delta_0$ *modulo* $S(H^{1/2}, G)$ *when* $\langle \delta_0 \rangle \le \kappa_2 H^{-1/2}$*. For any* $\varepsilon > 0$ *we find that* $|\delta_0| \ge \varepsilon H^{-1/2}$ *and* $H^{1/2} \le \varepsilon/3$ *imply that* $|\delta_0 + \varrho_0| \ge \varepsilon H^{-1/2}/3$*.*

Proof. Let $\delta_0^{Wick} = \delta_1^w$ and $\varrho_0^{Wick} = \varrho_1^w$. Since $|\delta_0| \le H_1^{-1/2}$, $|\varrho_0| \le m$ and the symbols are real valued, we obtain from Proposition 7.2 that $\delta_1 \in S(H^{-1/2}, g^\sharp)$ and $\varrho_1 \in S(m, g^\sharp)$ are real valued. Since δ_0 and ϱ_0 are uniformly Lipschitz continuous, we find that δ_1 and $\varrho_1 \in S^+(1, g^\sharp)$ by Proposition 7.2. By Remark 7.1 we have $\lambda^{Wick} = \lambda^w$.

If $\langle \delta_0 \rangle \le \kappa H^{-1/2}$ at w_0 for sufficiently small $\kappa > 0$, then we find by the Lipschitz continuity of δ_0 and the slow variation of G that $\langle \delta_0 \rangle \le C_0 \kappa H^{-1/2}$ in a fixed G neighborhood ω_κ of w_0 (depending on κ). For $\kappa \ll 1$ we find that $\delta_0 \in S(H^{-1/2}, G)$ in ω_κ by Proposition 4.9, which implies that $\delta_1 \cong \delta_0$ modulo $S(H^{1/2}, G)$ near w_0 by Proposition 7.2 after localization.

When $|\delta_0| \ge \varepsilon H^{-1/2} \ge \varepsilon > 0$ at w_0, then we find that

$$|\varrho_0| \le m \le \langle \delta_0 \rangle / 2 \le (1 + H^{1/2}/\varepsilon)|\delta_0|/2.$$

We obtain that $|\varrho_0| \le 2|\delta_0|/3$ and $|\delta_0 + \varrho_0| \ge |\delta_0|/3 \ge \varepsilon H^{-1/2}/3$ when $H^{1/2} \le \varepsilon/3$, which completes the proof. $\square$

Let m be given by Definition 6.1, then m is a weight for $g^\sharp$ according to Proposition 6.5. We are going to use the symbol classes $S(m^k, g^\sharp)$, $k \in \mathbf{R}$. The following proposition shows that the operator m^{Wick} dominates all operators in $\operatorname{Op} S(m, g^\sharp)$.

Proposition 7.6. *If* $c \in S(m, g^\sharp)$ *then there exists a positive constant* C_0 *such that*

$$|\langle c^w u, u \rangle| \le C_0 \left(m^{Wick} u, u \right) \qquad u \in C_0^\infty. \tag{7.7}$$

Here C_0 *only depends on the seminorms of* $c \in S(m, g^\sharp)$*.*

Proof. We shall use an argument by Hörmander [12]. Let $0 < \varrho \le 1$

$$M_\varrho(w_0) = \sup_w m(w)/(1 + \varrho|w - w_0|)^3 \tag{7.8}$$

then $m \le M_\varrho \le Cm/\varrho^3$ and

$$M_\varrho(w) \le CM_\varrho(w_0)(1+\varrho|w-w_0|)^3 \qquad \text{uniformly in } 0<\varrho\le 1 \tag{7.9}$$

by the triangle inequality. Thus, M_ϱ is a weight for $g_\varrho = \varrho^2 g^\sharp$, uniformly in ϱ. Take $0 \le \chi \in C_0^\infty$ such that $\int_{T^*\mathbf{R}^n} \chi(w)\,dw > 0$ and let

$$m_\varrho(w) = \varrho^{-2n}\int \chi(\varrho(w-z))M_\varrho(z)\,dz.$$

Then by (7.9) we find $1/C_0 \le m_\varrho/M_\varrho \le C_0$, and $|\partial^\alpha m_\varrho| \le C_\alpha \varrho^{|\alpha|} m_\varrho$ thus $m_\varrho \in S(m_\varrho, g_\varrho)$ uniformly in $0<\varrho\le 1$. Let $m_\varrho^{Wick} = \mu_\varrho^w$ then Proposition 7.2 and (7.9) give $m_\varrho/c \le \mu_\varrho \in S(m_\varrho, g_\varrho)$ uniformly in $0<\varrho\le 1$. Since $m \cong m_\varrho$ (depending on ϱ) we may replace m^{Wick} with $m_\varrho^{Wick} = \mu_\varrho^w$ in (7.7) for any fixed $\varrho > 0$.

Let $a_\varrho = \mu_\varrho^{-1/2} \in S(m_\varrho^{-1/2}, g_\varrho^\sharp)$ with $0<\varrho\le 1$ to be chosen later. Since g_ϱ is uniformly σ temperate, $g_\varrho/g_\varrho^\sigma = \varrho^4$, m_ϱ is uniformly σ, g_ϱ temperate, and $\mu_\varrho^{\pm 1/2} \in S(m_\varrho^{\pm 1/2}, g_\varrho)$ uniformly, the calculus gives that $a_\varrho^w(a_\varrho^{-1})^w = 1 + r_\varrho^w$ where $r_\varrho/\varrho^2 \in S(1,g^\sharp)$ uniformly for $0<\varrho\le 1$. Similarly, we find that $a_\varrho^w\mu_\varrho^w a_\varrho^w = 1 + s_\varrho^w$ where $s_\varrho/\varrho^2 \in S(1,g^\sharp)$ uniformly. We obtain that the L^2 operator norms

$$\|r_\varrho^w\|_{\mathcal{L}(L^2)} + \|s_\varrho^w\|_{\mathcal{L}(L^2)} \le C\varrho^2 \le 1/2$$

for sufficiently small ϱ. By fixing such a value of ϱ we find that $1/2 \le a_\varrho^w\mu_\varrho^w a_\varrho^w \le 2$ and

$$\frac{1}{2}\|u\| \le \|a_\varrho^w(a_\varrho^{-1})^w u\| \le 2\|u\| \tag{7.10}$$

thus $u \mapsto a_\varrho^w(a_\varrho^{-1})^w u$ is an homeomorphism on L^2. The estimate (7.7) then follows from

$$|\langle c^w a_\varrho^w(a_\varrho^{-1})^w u, a_\varrho^w(a_\varrho^{-1})^w u\rangle| \le C\|(a_\varrho^{-1})^w u\|^2 \le 2C\langle \mu_\varrho^w a_\varrho^w(a_\varrho^{-1})^w u, a_\varrho^w(a_\varrho^{-1})^w u\rangle$$

which holds since $a_\varrho^w c^w a_\varrho^w \in \operatorname{Op} S(1,g^\sharp)$ is bounded in L^2. Observe that the bounds only depend on the seminorms of c in $S(m,g^\sharp)$, since ϱ and a_ϱ are fixed. $\square$

We shall also need a weight to handle the calculus errors in the ξ variables. Let $\mu = h^{1/2}\langle\xi\rangle^2$ which is a weight for $g_0(dx,d\xi) = |dx|^2 + |d\xi|^2/\langle\xi\rangle^2$ so that $\lambda \in S(\mu, g_0)$ uniformly in h. In order to handle the compositions with λ and f_0 we shall need the following metric:

$$G_0 = H|dw|^2 + H(dt^2+|dx|^2)/h + Hh\,d\tau^2 + |d\xi|^2/\langle\xi\rangle^2 \tag{7.11}$$

which is strongly σ temperate in the sense that

$$G_{0,z} \lesssim G_{0,z_0}(1 + G_{0,z_0}(z-z_0)) \tag{7.12}$$

but since $G_0/G_0^\sigma \not\lesssim 1$ it is not σ temperate. But since λ and f_0 are linear in ξ we shall only use the calculus on Σ_2, i.e., in (t,x,w), and $G_0 = G$ on $T\Sigma_2$.

To handle the compositions with $b_1 = \delta_1 + \varrho_1 \in S(H^{-1/2}, g^\sharp)$ we shall use the metric:

$$g_0^\sharp = |dw|^2 + (dt^2 + |dx|^2)/h + hd\tau^2 + |d\xi|^2/\langle\xi\rangle^2 \tag{7.13}$$

which is strongly σ temperate on Σ_2, since $g_0^\sharp = g^\sharp$ on $T\Sigma_2$. Then $\lambda f \in S(Mh^{1/2}\langle\xi\rangle, G_0)$ and $b_1 f_0 \in S(MH^{1/2}h^{1/2}\langle\delta_0\rangle\langle\xi\rangle, g_0^\sharp)$.

Lemma 7.7. *If $|C| \lesssim \mu$ then we have*

$$|\langle C^{Wick}u, u\rangle| \lesssim \langle \mu^{Wick}u, u\rangle \lesssim h^{1/2}\|\langle D_x\rangle u\|^2 = h^{1/2}(\|D_x u\|^2 + \|u\|^2) \tag{7.14}$$

where $D_x u = (D_{x_1}u, D_{x_2}u, \dots)$.

Proof. By taking the real and imaginary part it suffices to prove the estimate for real valued C. We have $|C| \lesssim \mu$ so $\pm\langle C^{Wick}u, u\rangle \lesssim \langle\mu^{Wick}u, u\rangle$. Now $\mu^{Wick} = \nu^w$ where $\nu \in S(\mu, g_0^\sharp)$ by Proposition 7.2, so $\nu_0^w = \langle D_x\rangle^{-1}\nu^w\langle D_x\rangle^{-1} \in \operatorname{Op} S(h^{1/2}, g_0^\sharp)$ which gives

$$\langle\mu^{Wick}u, u\rangle = \langle\nu_0^w\langle D_x\rangle u, \langle D_x\rangle u\rangle \lesssim h^{1/2}\|\langle D_x\rangle u\|^2$$

which proves the result. $\square$

8. The multiplier estimate

In this section we shall obtain a proof of Proposition 3.6 which involves giving lower bounds on $\operatorname{Re} b_T^w f_1^w$, with the multipler $b_T^w = B_T^{Wick}$ having symbol $B_T = \delta_0 + \varrho_0 + \lambda$ given by Proposition 7.5. Also, $f_1 = f + f_0$, where $f \in S(M, G)$ and $f_0 = \partial_\eta f \cdot r \cdot \xi$ with $r \in S(1, g)$ so that $f_1 \in S(MH^{1/2}h^{1/2}\langle\xi\rangle, G_0)$ by Remark 4.8. Here $G = Hg/h = Hg^\sharp$, with constant $g \le h^2 g^\sigma$ and H given by Definition 4.4. The weight M is given by Definition 4.6, the metric G_0 by (7.11) and the weight m by Definition 6.1. The results will only depend on the seminorms of f in $S(h^{-1}, g)$, and we will assume the coordinates chosen so that $t = 0$ and $x = 0$ at $w_0 \in \Sigma_2$. We shall follow Section 7 in [4] with some necessary changes because of the different conditions, metrics and normal forms.

Proposition 8.1. *Let $B_T = \delta_0 + \varrho_0 + \lambda$ given by Definition 6.10, so $\delta_0 = \delta$ is given by Definition 4.2, $\varrho_0 = \varrho$ is real valued and Lipschitz continuous, satisfying $|\varrho_0| \le m$ when $|t| \le T$, with $m \le \langle\delta_0\rangle/2$ given by Definition 6.1 and $\lambda = \epsilon h^{1/2}\langle Lx, \xi\rangle/T \in S(h^{1/2}\langle\xi\rangle, G_0)$ uniformly when $|x| \le T$, where $0 < \epsilon \le 1$ and L is given by Lemma 6.9. Then for small enough T we can find $C \in S(m, g^\sharp) + S(\mu, g_0^\sharp)$ so that*

$$\operatorname{Re}\left(B_T^{Wick} f_1^w u, u\right) \ge (C^w u, u) \tag{8.1}$$

if $u \in C_0^\infty$ has support where $|t| \le T$ and $|x| \le T$.

Proof. In the proof we shall treat ξ as parameter and assume the coordinates $w = (t, x, y; \tau, \eta)$ chosen so that $g^\sharp(w) = |w|^2$, i.e., $g^\sharp$ ON coordinates on Σ_2. We shall localize in w with respect to the metric $G \ge g$ on Σ_2, i.e., when $\xi = 0$, and then estimate the localized operators. We shall include t and x in w and use the neighborhoods

$$\omega_{w_0}(\varepsilon) = \left\{ w : \ |w - w_0| < \varepsilon H^{-1/2}(w_0) \right\} \tag{8.2}$$

which gives that $\max(|t|, |x|) < \varepsilon H^{-1/2}(w_0) h^{1/2} \le 3\varepsilon$. Now, if ε_0 is small enough then $H(w)$ and $M(w)$ will only vary with a fixed factor in $\omega_{w_0}(2\varepsilon_0)$. By the uniform Lipschitz continuity of $w \mapsto \delta_0(w)$ we can find $\kappa_0 > 0$ with the following property: for $0 < \kappa \le \kappa_0$ there exist positive constants c_κ and $\varepsilon_\kappa \le \varepsilon_0$ so that

$$|\delta_0(w)| \le \kappa H^{-1/2}(w) \qquad w \in \omega_{w_0}(2\varepsilon_\kappa) \qquad \text{or} \tag{8.3}$$

$$|\delta_0(w)| \ge c_\kappa H^{-1/2}(w) \qquad w \in \omega_{w_0}(2\varepsilon_\kappa). \tag{8.4}$$

In fact, we have by the Lipschitz continuity that $|\delta_0(w) - \delta_0(w_0)| \le \varepsilon H^{-1/2}(w_0)$ when $w \in \omega_{w_0}(\varepsilon)$. Thus, if $\varepsilon_\kappa \ll \kappa$ we obtain that (8.3) holds when $|\delta_0(w_0)| \ll \kappa H^{-1/2}(w_0)$ and (8.4) holds when $|\delta_0(w_0)| \ge c\kappa H^{-1/2}(w_0)$.

By shrinking κ_0 we may assume that $M \cong |f'| H^{-1/2}$ when $|\delta_0| \le \kappa_0 H^{-1/2}$ and $H^{1/2} \le \kappa_0$ according to Proposition 4.9. Let κ_1 be given by Proposition 4.10, κ_2 by Proposition 7.5, and let ε_κ and c_κ be given by (8.3)–(8.4) for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. Using Proposition 7.5 with $\varepsilon = c_\kappa$ we find that

$$\operatorname{sgn}(f)(\delta_0 + \varrho_0) \ge c_\kappa H^{-1/2}/3 \qquad \text{in } \omega_{w_0}(2\varepsilon_\kappa) \tag{8.5}$$

if $H^{1/2} \le c_\kappa/3$ and (8.4) holds in $\omega_{w_0}(\varepsilon_\kappa)$.

Choose real symbols $\{\psi_j(w)\}_j$ and $\{\phi_j(w)\}_j \in S(1, G)$ with values in ℓ^2, such that $0 \le \psi_j \le 1$, $\sum_k \psi_j^2 \equiv 1$, $\psi_j \Psi_j = \psi_j$ with $0 \le \Psi_j = \phi_j^2 \le 1$ which gives $\{\Psi_j(w)\}_j \in S(1, G)$ with values in ℓ^2 so that

$$\operatorname{supp} \phi_j \subseteq \omega_j = \omega_{w_j}(\varepsilon_\kappa) \tag{8.6}$$

We shall suppress T, writing $B^{Wick} = b^w$ where $b = \delta_1 + \varrho_1 + \lambda$ is given by Proposition 7.5. In the following, we shall for $j \in \mathbf{N}$ denote $A_{jk} = \Psi_k f_j b = f_{jk} b$ for $k = 0,\ 1$, and $A_k = \Psi_k f b = f_k b$ where f_k should not be confused with f_0 and f_1.

Lemma 8.2. *When $|t| \le T$ and $|x| \le T$ we have $A_{1j} \in S(MH^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp) + S(Mh^{1/2}\langle\xi\rangle, g_0^\sharp)$ uniformly in j,*

$$\operatorname{Re} b^w f_1^w \cong \sum_j \psi_k^w A_{1k}^w \psi_k^w \qquad \textit{modulo } \operatorname{Op} S(m, g^\sharp) + \operatorname{Op} S(\mu, g_0^\sharp) \textit{ uniformly} \tag{8.7}$$

and $A_{1k}^w \cong \operatorname{Re} b^w f_{1k}^w$ modulo $\operatorname{Op} S(m, g^\sharp) + S(\mu, g_0^\sharp)$ uniformly in j.

Proof. Since $f_1 = f + f_0$ we have $A_{1k} = A_k + A_{0k}$ and we will start with f_0. First we note that $f_0 = r_0 \cdot \xi$ with $r_0 \in S(MH^{1/2}h^{1/2}, G) \bigcap S(1, g)$, which gives $\lambda^w f_0^w \in \operatorname{Op} S(\mu, G_0)$ so we can skip this term. Let $b_1 = \delta_1 + \varrho_1 \in S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp)$. Since b_1 is constant in x we obtain $\operatorname{Re} b_1^w f_0^w = (b_1 f_0)^w + R^w$, where $R = r\xi$ with $r \in S(h, g^\sharp)$ giving $R^w \in \operatorname{Op} S(\mu, g_0^\sharp)$. In fact, r has an asymptotic expansion in $S(h^{k/2}, g^\sharp)$ for $k \geq 0$.

We also find that $\psi_k^w A_{0k}^w \psi_k^w = (A_{0k}\psi_k^2)^w + C^w$ with $C = c_0 + c_1\xi$ with $c_0 \in S(MH^{3/2}, g^\sharp)$ and $c_1 \in S(MH^{3/2}h^{1/2}, g^\sharp)$ which gives $C^w \in \operatorname{Op} S(m, g^\sharp) + S(\mu, g_0^\sharp)$. In fact, c_0 has an asymptotic expansion in $S(MH^{k/2}, g^\sharp)$ for $k \geq -1$ and $c_1 \in S(MH^{k/2}h^{1/2}, g^\sharp)$ for $k \geq 0$ and any x derivative of the symbols gives the factor $H^{1/2}h^{-1/2}$.

Next, we will study $f \in S(M, G)$. In that case $\lambda^w f^w = (f\lambda - ih^{1/2}\langle Lx, \partial_x f\rangle/2T)^w$ since $\partial_\xi f \equiv 0$, which gives $\operatorname{Re} \lambda^w f^w = (f\lambda)^w$. We have $A_k \in S(MH^{-1/2}, g^\sharp) \bigcap S^+(M, g^\sharp) + \operatorname{Op} S(Mh^{1/2}\langle\xi\rangle, G_0)$ uniformly in j. Proposition 6.6 gives that

$$MH^{3/2}\langle\delta_0\rangle^2 \leq Cm \tag{8.8}$$

so we may ignore terms in $\operatorname{Op} S(MH^{3/2}\langle\delta_0\rangle^2, g^\sharp)$. Since $b \in S(H^{-1/2}, g^\sharp) + S(h^{1/2}\langle\xi\rangle, G_0)$, $\{\,\psi_k\,\}_k \in S(1, G)$, $A_k \in S(MH^{-1/2}, g^\sharp) + \operatorname{Op} S(Mh^{1/2}\langle\xi\rangle, G_0)$ uniformly with values in ℓ^2, $h^{1/2}\langle\xi\rangle \leq \mu$ and $H^{-1/2} \lesssim h^{-1/2}$, we find by Lemma 3.2 and Remark 3.3 that the symbols of $b^w f^w$, $b^w f_k^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ have expansions in $S(MH^{j/2}, g^\sharp) + S(MH^{k/2}\mu, G_0)$. Observe that in the domains ω_k where $H^{1/2} \geq c > 0$, we find that $M \lesssim H^{-1} \lesssim 1$ so the symbols of $\sum_k \psi_k^w A_k^w \psi_k^w$, $b^w f_k^w$ and $b^w f^w$ are in $S(MH^{3/2}, g^\sharp) + S(\mu, g_0^\sharp)$ giving the result in this case. Thus we may assume $H^{1/2} \leq \kappa_2/2$ in what follows. We shall consider the neighborhoods where (8.3) or (8.4) holds.

If (8.4) holds then we find $\langle\delta_0\rangle \cong H^{-1/2}$ so $S(MH^{1/2}, g^\sharp) \subseteq S(m, g^\sharp)$ in ω_k by (8.8) and $S(MH\mu, G_0) \subseteq S(\mu, G_0)$ since $M \lesssim H^{-1}$. Since $b_1 \in S^+(1, g^\sharp)$ we find that $A_k \in S^+(M, g^\sharp) + S(M\mu, G_0)$ and the symbols of both $b^w f^w$ and $\sum_k \psi_k^w A_k^w \psi_k^w$ are equal to $\sum_k \psi_k^2 A_k \cong fb$ modulo $S(MH^{1/2}, g^\sharp) + S(MH\mu, G_0)$ in ω_k. Similarly, we find that the symbol of $b^w f_k^w$ is equal to A_k modulo $S(MH^{1/2}, g^\sharp) + S(\mu, G_0)$, which proves the result in this case.

Next, we consider the case when (8.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H^{1/2} \leq \kappa_2/2$ in ω_k. Then $\langle\delta_0\rangle \leq \kappa_2 H^{-1/2}$ so $b = \delta_1 + \varrho_1 + \lambda \in S(H^{-1/2}, G) + S(m, g^\sharp) + S(\mu, G_0)$ in ω_k by Proposition 7.5. Since $\operatorname{Re} \lambda^w f^w = (\lambda f)^w$ we obtain from Lemma 3.2 that the symbol of $\operatorname{Re} b^w f^w - (fb)^w$ is in $S(MH^{3/2}, G) + S(MHm, g^\sharp) \subseteq S(m, g^\sharp)$ in ω_k. Similarly, we find that $A_k^w \cong \operatorname{Re} b^w f_k^w$ modulo $\operatorname{Op} S(m, g^\sharp)$. Since $A_k \in S(MH^{-1/2}, G) + S(Mm, g^\sharp) + S(M\mu, G_0)$ uniformly in this case, we find that the symbol of $\sum_k \psi_k^w A_k^w \psi_k^w$ is equal to bf modulo $S(m, g^\sharp) + S(\mu, G_0)$ in ω_k, which proves (8.7) and Lemma 8.2. $\square$

In order to estimate the localized operator we shall use the following

Lemma 8.3. *If $A_{1j} = \Psi_j f_1 b$ is given by Lemma 8.2 then there exists $C_j \in S(m, g^\sharp) + S(\mu, g_0^\sharp)$ uniformly when $|t| \le T$ and $|x| \le T$, such that*

$$\left(A_{1j}^w u, u\right) \ge \left(C_j^w u, u\right) \tag{8.9}$$

when $u \in C_0^\infty$.

We obtain from (8.7) and (8.9) that

$$\operatorname{Re}\left(b^w f^w u, u\right) \ge \sum_j \left(\psi_j^w C_j^w \psi_j^w u, u\right) + (R^w u, u) \qquad u \in C_0^\infty$$

where $\sum_j \psi_j^w C_j^w \psi_j^w$ and $R^w \in \operatorname{Op} S(m, g^\sharp) + \operatorname{Op} S(\mu, g_0^\sharp)$, which gives Proposition 8.1. □

Proof of Lemma 8.3. In the following, we shall assume that $\max(|t|, |x|) \le T$. As before we are going to consider the cases when $H^{1/2} \cong 1$ or $H^{1/2} \ll 1$, and when (8.3) or (8.4) holds in $\omega_{w_j}(2\varepsilon_\kappa)$ for $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$. When $H^{1/2} \ge c > 0$ we find that $A_{1j} \in S(MH^{3/2}, g^\sharp) + S(MHh^{1/2}\langle\xi\rangle, g_0^\sharp) \subseteq S(m, g^\sharp) + S(\mu, g_0^\sharp)$ uniformly by (8.8) which gives the lemma with $C_j = A_{1j}$ in this case. For handling this case where $H^{1/2} \ll 1$ we shall need the following result.

Lemma 8.4. *If $F \in S(M_0, G)$, where M_0 is a weight for G, and $\pm F \ge 0$ in $\omega_{w_j}(2\varepsilon_\kappa)$ then we have that $|\partial_\eta F| \le C_\kappa \sqrt{F} M_0^{1/2} H^{1/2} h^{1/2}$ in ω_j.*

Corollary 8.5. *We obtain from Lemma 8.4 that $\partial_\eta f = \partial_\eta \alpha_0 \delta_0 + \alpha_0 \partial_\eta \delta = \sqrt{\alpha_0}\, r_1$ where $r_1 \in S(M^{1/2} H^{1/4} h^{1/2}, G)$ in ω_j.*

In fact, we find from Lemma 8.4 that $\partial_\eta \alpha_0 \in S(\sqrt{\alpha_0} M^{1/2} H^{1/4} h^{1/2}, G)$ since $MH^{1/2} \lesssim \alpha_0 \in S(MH^{1/2}, G)$ and we have $\partial_\eta \delta_0 \in S(h^{1/2}, G)$ in ω_j.

Proof of Lemma 8.4. For any $w \in \omega_j$ we may choose $g^\sharp$ orthogonal coordinates so that $w = 0$ and $F \ge 0$ in $|\eta| < \varepsilon_\kappa H^{-1/2} h^{-1/2}$. Then by Taylor's formula and the slow variation there exists $C_\kappa > 0$ so that

$$|\eta \cdot \partial_\eta F(0)| \le F(0) + C_\kappa M_0 H h |\eta|^2 \tag{8.10}$$

Then by choosing $|\eta| = \varepsilon_\kappa \sqrt{F(0)/M_0 H h} \lesssim \varepsilon_\kappa H^{-1/2} h^{-1/2}$ we find that

$$|\partial_\eta F(0)| \le C'_\kappa \sqrt{F(0) M_0 H h} \tag{8.11}$$

which proves the result. □

In the following, we shall assume that

$$H^{1/2} \le \kappa_4 = \min(\kappa_0, \kappa_1, \kappa_2, \kappa_3)/2 \qquad \text{in } \omega_j \tag{8.12}$$

with $\kappa_3 = 2c_\kappa/3$ so that (8.5) follows from (8.4). First, we consider the case when $H^{1/2} \le \kappa_4$ and (8.4) holds in ω_j. Since $|\delta_0(w)| \ge c_\kappa H^{-1/2}(w)$, we find $\pm f \ge 0$ and $\langle\delta_0\rangle \cong H^{-1/2}$ in $\omega_{w_j}(2\varepsilon_\kappa)$. As before we may ignore terms in $S(MH\mu, g_0^\sharp) \subseteq S(\mu, g_0^\sharp)$ and $S(MH^{1/2}, g^\sharp) \subseteq S(m, g^\sharp)$ in ω_j by (8.8). We shall estimate $A_{1j} = A_j + A_{0j}$, starting with A_j. If $B_1 = \delta_0 + \varrho_0$ we find from (8.5) that $fB_1 \ge 0$ in $\omega_{w_j}(2\varepsilon_\kappa)$. We shall only consider the case $\operatorname{sgn}(f) = \operatorname{sgn}(B_1) = 1$, for the other case we may replace the symbols by their absolute values. Since $B_1 \gtrsim H^{-1/2}$ and $B_1 \in S(H^{-1/2}, g^\sharp)\bigcap S^+(1, g^\sharp)$ we find $B^{1/2} \in S(H^{-1/4}, g^\sharp)\bigcap S^+(H^{1/4}, g^\sharp)$ and $B^{-1/2} \in S(H^{1/4}, g^\sharp)\bigcap S^+(H^{3/4}, g^\sharp)$. Since $f_j \in S(M, G)$, we find $f_j^w \cong f_j^{Wick}$ modulo $\operatorname{Op} S(MH, G)$ by Proposition 7.2. As before, $\operatorname{Re} \lambda^w f_j^{Wick} = (\lambda f_j)^w$ so we find from Example 7.4 that

$$A_j^w \cong \operatorname{Re} b^w f_j^w \cong \operatorname{Re} B^{Wick} f_j^{Wick} \cong (f_j B)^{Wick}$$

modulo $\operatorname{Op} S(m, g^\sharp) + \operatorname{Op} S(\mu, G_0)$. In fact, we have $b^w = B^{Wick} = (B_1 + \lambda)^{Wick}$ and $\lambda f_j = s_1 \cdot \xi$ with $s \in S(Mh^{1/2}, G)$ so $(\lambda f_j)^w \cong (\lambda f_j)^{Wick}$ modulo $\operatorname{Op} S(\mu, G_0)$.

Similarly, since $f_{0j} \in S(MH^{1/2}h^{1/2}\langle\xi\rangle, G_0)$ is linear in ξ we find that $f_{0j}^w \cong f_{0j}^{Wick}$ modulo $S(MH^{3/2}h^{1/2}\langle\xi\rangle, G_0)$, thus

$$A_{0j}^w \cong \operatorname{Re} b^w f_{0j}^w \cong \operatorname{Re} B^{Wick} f_{0j}^{Wick} \cong (f_{0j} B)^{Wick}$$

modulo $\operatorname{Op} S(MHh^{1/2}\langle\xi\rangle, g_0^\sharp) + \operatorname{Op} S(MH^{3/2}h\langle\xi\rangle^2, G_0) \subset \operatorname{Op} S(\mu, g_0^\sharp)$. By Lemma 8.4 we have $|\partial_\eta f| \lesssim \sqrt{f} M^{1/2} H^{1/2} h^{1/2}$ in ω_j which gives

$$|f_{0j} B_1| \le \varepsilon f_j B_1 + C_\varepsilon \Psi_j M H^{1/2} h \langle\xi\rangle^2 \quad \forall\, \varepsilon > 0 \tag{8.13}$$

where $MH^{1/2}h\langle\xi\rangle^2 \lesssim \mu$. If $\varepsilon \le 1/2$ we find modulo $S(\mu, g_0^\sharp)$ that

$$f_{1j} B \gtrsim f_j(B_1/2 + \lambda) = f_j\left(\sqrt{B_1/2} + \lambda/\sqrt{2B_1}\right)^2 - f_j \lambda^2/2B_1$$

so $(f_{1j}B)^{Wick} \ge -(f_j\lambda^2/2B_1)^{Wick} \in \operatorname{Op} S(MH^{1/2}\lambda^2, g_0^\sharp) \subset \operatorname{Op} S(\mu, g_0^\sharp)$.

Finally, we consider the case when (8.3) holds with $\kappa = \min(\kappa_0, \kappa_1, \kappa_2)/2$ and $H^{1/2} \le \kappa_4 \le \kappa$ in ω_j. Then $\langle\delta_0\rangle \le 2\kappa H^{-1/2}$ so we obtain from Proposition 4.9 that $M \cong |f'|H^{-1/2}$ and $\delta_0 \in S(H^{-1/2}, G)$ in ω_j. We shall estimate $A_{1j} = A_j + A_{0j}$, starting with A_j using an argument of Lerner [15]. We have that $b^w = (\delta_0 + \varrho_0 + \lambda)^{Wick} = B^{Wick}$, where $|\varrho_0| \le m \le H^{1/2}\langle\delta_0\rangle^2/2$ by (6.2). Also, Lemma 8.2 gives $A_{1j}^w \cong \operatorname{Re} b^w f_{1j}^w = \operatorname{Re} B^{Wick} f_{1j}^w$ modulo $\operatorname{Op} S(m, g^\sharp) + \operatorname{Op} S(\mu, g_0^\sharp)$. As before, $f_{1j} = f_j + f_{0j}$ and we shall start with $\operatorname{Re} b^w f_j^w$. Take $\chi(t) \in C^\infty(\mathbf{R})$ such that $0 \le \chi(t) \le 1$, $|t| \ge 2$ in $\operatorname{supp} \chi(t)$ and $\chi(t) = 1$ for $|t| \ge 3$. Let $\chi_0 = \chi(\delta_0)$, then $\chi_0 \in S(1, g^\sharp)$, $2 \le |\delta_0|$ and $\langle\delta_0\rangle/|\delta_0| \le 3/2$ in $\operatorname{supp} \chi_0$, thus

$$1 + \chi_0\varrho_0/\delta_0 \ge 1 - \chi_0\langle\delta_0\rangle/2|\delta_0| \ge 1/4. \tag{8.14}$$

Since $|\delta_0| \le 3$ in $\operatorname{supp}(1-\chi_0)$ we find by Proposition 7.5 that

$$B^{Wick} \cong (\delta_0 + \chi_0\varrho_0 + \lambda)^{Wick}$$

modulo $\operatorname{Op} S(m/\langle\delta_0\rangle, g^\sharp) \subseteq \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ by (6.2). Since $|\chi_0\varrho_0/\delta_0| \le 3H^{1/2}\langle\delta_0\rangle/4$ and $\delta_0 \in S^+(1, g^\sharp)$ we find from (7.4) that

$$B_1^{Wick} \cong \delta_0^{Wick} B_0^{Wick} \qquad \text{modulo } \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp). \tag{8.15}$$

where $B_0 = 1 + \chi_0\varrho_0/\delta_0$. Proposition 7.2 gives $(\chi_0\varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ and $\delta_0^{Wick} = \delta_1^w$ where $\delta_1 = \delta_0 + \gamma$ with $\gamma \in S(H^{1/2}, G)$ in ω_j. Thus Lemma 3.2 gives

$$B^{Wick} \cong \delta_0^{Wick} B_0^{Wick} + \lambda^{Wick} \cong \delta_0^w B_0^{Wick} + \lambda^w + c^w \qquad \text{modulo } \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp) \tag{8.16}$$

where $c \in S(H^{-1/2}, g^\sharp)$ such that $\operatorname{supp} c \bigcap \omega_j = \emptyset$.

We find from Proposition 4.9 that $f = \alpha_0\delta_0$, where $MH^{1/2} \lesssim \alpha_0 \in S(MH^{1/2}, G)$, which gives $\alpha_0^{1/2} \in S(M^{1/2}H^{1/4}, G)$. Let

$$a_j = \alpha_0^{1/2}\delta_0\phi_j \in S(M^{1/2}H^{-1/4}, G) \tag{8.17}$$

then the calculus gives

$$\operatorname{Re} a_j^w (\alpha_0^{1/2}\phi_j)^w \cong f_j^w \qquad \text{modulo } \operatorname{Op} S(MH, G). \tag{8.18}$$

since $f_j = \Psi_j f = \phi_j^2 f$. Similarly, we find that $f_j^w c^w \in \operatorname{Op} S(MH^{3/2}, g^\sharp)$ by the expansion and

$$\operatorname{Re} f_j^w \delta_0^w \cong a_j^w a_j^w \qquad \text{modulo } \operatorname{Op} S(MH^{3/2}, G) \tag{8.19}$$

with imaginary part in $\operatorname{Op} S(MH^{1/2}, G)$. We obtain from (8.16) and (8.18) that

$$\begin{aligned} f_j^w B^{Wick} &\cong f_j^w(\delta_0^w B_0^{Wick} + \lambda^w + c^w + r^w) \\ &\cong f_j^w \delta_0^w B_0^{Wick} + f_j^w \lambda^w + a_j^w R_j^w \quad \text{modulo } \operatorname{Op} S(m, g^\sharp) \end{aligned} \tag{8.20}$$

where $r \in S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ which gives $R_j = (\alpha_0^{1/2}\phi_j)^w r^w \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. Since

$$\operatorname{Re} FB = \operatorname{Re}(\operatorname{Re} F)B + i[\operatorname{Im} F, B]$$

when $B^* = B$, we find from (8.19) by taking $F = f_j^w\delta_0^w$ and $B = B_0^{Wick}$ that

$$\operatorname{Re} f_j^w \delta_0^w B_0^{Wick} \cong \operatorname{Re} a_j^w a_j^w B_0^{Wick} \qquad \text{modulo } \operatorname{Op} S(m, g^\sharp). \tag{8.21}$$

In fact, since $B_0 = 1 + \chi_0\varrho_0/\delta_0$ and $(\chi_0\varrho_0/\delta_0)^{Wick} \in \operatorname{Op} S(H^{1/2}\langle\delta_0\rangle, g^\sharp)$ we find

$$[a^w, B_0^{Wick}] = [a^w, (\chi_0\varrho_0/\delta_0)^{Wick}] \in \operatorname{Op} S(MH^{3/2}\langle\delta_0\rangle, g^\sharp) \tag{8.22}$$

when $a \in S(MH^{1/2}, G)$. Similarly, since $a_j \in S(M^{1/2}H^{-1/4}, G)$ we obtain that

$$a_j^w a_j^w B_0^{Wick} \cong a_j^w (B_0^{Wick} a_j^w + s_j^w) \qquad \text{modulo } \operatorname{Op} S(m, g^\sharp) \tag{8.23}$$

where $s_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp)$. We also have

$$\operatorname{Re}\lambda^{Wick}f_j^w = \operatorname{Re}f_j^w\lambda^w \cong (f_j\lambda)^w \cong \operatorname{Re}a_j^w(\alpha_0^{1/2}\lambda\phi_j)^w \tag{8.24}$$

modulo $\operatorname{Op}S(\mu, G_0)$ where $\alpha_0^{1/2}\lambda\phi_j \in S(M^{1/2}H^{1/4}h^{1/2}\langle\xi\rangle, G_0)$. Since $B_0 \geq 1/4$ we find from (8.20)–(8.24) that

$$\operatorname{Re}B^{Wick}f_j^w \gtrsim \frac{1}{4}a_j^w a_j^w + \operatorname{Re}a_j^w S_j^w \qquad \text{modulo } \operatorname{Op}S(m, g^\sharp) + \operatorname{Op}S(\mu, G_0) \tag{8.25}$$

where $S_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp) + S(M^{1/2}H^{1/4}h^{1/2}\langle\xi\rangle, G_0)$.

We are going to complete the square in (8.25), but before that we must handle the term $\operatorname{Re}B^{Wick}f_{0j}^w = \operatorname{Re}b^w f_{0j}^w$. As before, we find that $\lambda^{Wick}f_{0j}^w \in \operatorname{Op}S(\mu, G_0)$ uniformly since we have $MH^{1/2}h^{1/2}\langle\xi\rangle\lambda \lesssim \mu$. For the term $\operatorname{Re}B_1^{Wick}f_{0j}^w$ with $B_1 = \delta_0 + \varrho_0$ we need the following result.

Lemma 8.6. *For any $\varepsilon > 0$ there exists $R_\varepsilon \in S(m, g^\sharp) + S(\mu, g_0^\sharp)$ so that*

$$|\operatorname{Re}\langle B_1^{Wick}f_{0j}^w u, u\rangle| \leq \varepsilon\langle a_j^w a_j^w u, u\rangle + \langle R_\varepsilon^w u, u\rangle \qquad u \in C_0^\infty \tag{8.26}$$

By using (8.25) and (8.26) we obtain for $\varepsilon \leq 1/12$ that

$$\operatorname{Re}B^{Wick}f_j^w \geq \frac{1}{6}a_j^w a_j^w + \operatorname{Re}a_j^w S_j^w \qquad \text{modulo } \operatorname{Op}S(m, g^\sharp) + \operatorname{Op}S(\mu, g_0^\sharp) \tag{8.27}$$

where $S_j \in S(M^{1/2}H^{3/4}\langle\delta_0\rangle, g^\sharp) + S(M^{1/2}H^{1/4}h^{1/2}\langle\xi\rangle, G_0)$. By completing the square, we find

$$A_j^w \cong \operatorname{Re}f_j^w B^{Wick} \gtrsim \frac{1}{6}\left(a_j^w + 3S_j^w\right)^*\left(a_j^w + 3S_j^w\right) \geq 0$$

modulo $\operatorname{Op}S(m, g^\sharp) + \operatorname{Op}S(\mu, G_0)$. In fact, $(S_j^w)^* S_j^w \in \operatorname{Op}S(m, g^\sharp) + \operatorname{Op}S(\mu, G_0)$ since we have $MH^{3/2}\langle\delta_0\rangle^2 \lesssim m$ and $MH^{1/2}h\langle\xi\rangle^2 \lesssim \mu$. This gives (8.9) and the lemma in this case. This completes the proof of Lemma 8.3. $\square$

Proof of Lemma 8.6. First, we note that $B_1^{Wick} = (\delta_0 + \varrho_0)^{Wick} \cong (\delta_0 B_0)^{Wick}$ modulo $S(H^{1/2}, g^\sharp)$, where $(\delta_0 B_0)^{Wick} \in \operatorname{Op}S^+(1, g^\sharp)$. Thus by Propositions 7.2 and 7.3 we find that $\operatorname{Re}B_1^{Wick}f_{0j}^w \cong \operatorname{Re}B_1^{Wick}f_{0j}^{Wick} \cong (f_{0j}\delta_0 B_0)^{Wick}$ modulo $\operatorname{Op}S(\mu, g_0^\sharp)$.

Now since $\Psi_j = \phi_j^2$ and $f_{0j} = \Psi_j \partial_\eta f \cdot r \cdot \xi$ we can factor the symbol $f_{0j}\delta_0 B_0 = AB$, where $A = \partial_\eta f \delta_0 B_0 M^{-1/2}H^{-1/4}h^{-1/2}\phi_j$ and $B = M^{1/2}H^{1/4}h^{1/2}r \cdot \xi\phi_j$. Then Corollary 8.5 gives $|A| \lesssim \sqrt{\alpha_0}|\delta_0|\phi_j$ and Proposition 7.2 gives

$$A^{Wick} \in \operatorname{Op}S(M^{1/2}H^{1/4}\langle\delta_0\rangle, g^\sharp) \bigcap \operatorname{Op}S^+(M^{1/2}H^{1/4}, g^\sharp)$$

and $B^{Wick} = \widetilde{B}^{Wick}D_x$ with $\widetilde{B} \in S(M^{1/2}H^{1/4}h^{1/2}, G) \subset S(h^{1/4}, G)$.

By Proposition 7.3 we have $(AB)^{Wick} \cong \operatorname{Re}A^{Wick}B^{Wick}$ modulo $\operatorname{Op}S(\mu, g_0^\sharp)$. Thus, it suffices to estimate

$$\operatorname{Re}\langle A^{Wick}B^{Wick}u, u\rangle \leq \varepsilon\|A^{Wick}u\|^2 + C_\varepsilon\|B^{Wick}u\|^2 \qquad u \in C_0^\infty \tag{8.28}$$

where we have $\|B^{Wick}u\|^2 \lesssim \|h^{1/4}D_xu\|^2 \le \langle \mu^w u, u\rangle$. We shall prove (8.26) by estimating $A^2 \lesssim \alpha_0\delta_0^2\phi_j^2 = f\delta_0\Psi_j = a_j^2$. Here $a_j = \sqrt{\alpha_0}\delta_0\phi_j \in S(M^{1/2}H^{-1/4}, G)$ is given by (8.17). We may then estimate $A^{Wick} \lesssim a_j^{Wick}$ where $a_j^{Wick} = a_j^w + r_2^w$ with $r_2^w \in S(M^{1/2}H^{3/4}, G)$. Thus we obtain

$$(A^{Wick})^2 \le \left(a_j^w + r_2^w\right)\left(a_j^w + r_2^w\right) \lesssim a_j^w a_j^w + r_2^w a_j^w + a_j^w r_2^w \lesssim 2a_j^w a_j^w \tag{8.29}$$

modulo $(r_2^w)^2 \in \operatorname{Op} S(m, g^\sharp)$, which gives (8.26) and Lemma 8.6. ☐

We shall finish the paper by giving a proof of Proposition 3.6.

Proof of Proposition 3.6. By the assumptions in Proposition 3.6 we have

$$P^* \cong D_t + A^w + if_1^w \qquad \text{modulo } R \tag{8.30}$$

microlocally near $w_0 \in \Sigma_2$, here

$$A = \sum_{jk} a_{jk}\xi_j\xi_k + \sum_j a_j\xi_j + a_0 \tag{8.31}$$

where a_{jk} and $a_j \in S(1, g)$ are real and $\{a_{jk}\}_{jk}$ is symmetric and nondegenerate, $f_1 = f + f_0$ where $f \in S(h^{-1}, g)$ is real valued satisfying condition $\mathrm{Sub_r}(\overline{\Psi})$ in (3.3) and $f_0 = \partial_\eta f \cdot r \cdot \xi$. Observe that P^* in (3.17) can be perturbed by r^w for $r \in R$ since $b_T \in S(h^{-1/2}, g^\sharp)$, so $|\langle b_T^w r^w u, u\rangle| \lesssim h^{1/2}\|\langle D_x\rangle u\|^2$ since $b_T^w s^w \in \operatorname{Op} S(h^{1/2}, g^\sharp)$ when $s \in S(h, g)$.

Let $B_T = \delta_0 + \varrho_T + \lambda_T$ be given by Definition 6.10, where $\lambda_T = \epsilon h^{1/2}\langle L(x - x_0), \xi\rangle/T$ given by Lemma 6.9 with x_0 being the value of x at w_0 and $0 < \epsilon \le 1$, $\delta_0 + \varrho_T$ is the Lipschitz continuous pseudo-sign for f given by Proposition 6.8 for $0 < T \le 1$, so that $|\varrho_T| \le m \le \langle\delta_0\rangle/2$ when $|t| \le T$. Proposition 6.8 also gives that

$$\partial_t(\delta_0 + \varrho_T) \ge m/2T \qquad \text{when } |t| \le T \tag{8.32}$$

We have $B_T^{Wick} = b_T^w$ where $b_T(t, w) \in S(H^{-1/2}, g^\sharp) \bigcap S^+(1, g^\sharp) + S(\mu, g_0^\sharp)$ uniformly by Proposition 7.5 when $\max(|t|, |x|) \le T$. In the following we shall assume that $u \in C_0^\infty$ has support where $\max(|t|, |x|) \le T$.

We are going to consider

$$\operatorname{Im}\left(P^*u, B_T^{Wick}u\right) = i\left([D_t + A^w, B_T^{Wick}]u, u\right)/2 + \operatorname{Re}\left(f_1^w u, B_T^{Wick}u\right) \tag{8.33}$$

We find by (7.1) and (8.32) that

$$i\left([D_t, B_T^{Wick}]u, u\right)/2 = \left(\partial_t B_T^{Wick}u, u\right)/2 \ge \left(m^{Wick}u, u\right)/4T \tag{8.34}$$

when $u \in C_0^\infty$. By Proposition 8.1, we find that

$$\operatorname{Re}\left(B_T^{Wick}f^w u, u\right) \ge (C^w u, u) \quad u \in C_0^\infty \tag{8.35}$$

with $C \in S(m, g^\sharp) + S(\mu, g_0^\sharp)$. Propositions 7.6 and 7.7 gives $C_0 > 0$ so that

$$|\,(C^w u, u)\,| \le C_0(\left(m^{Wick}u, u\right) + \left(\mu^{Wick}u, u\right)) \qquad u \in C_0^\infty \tag{8.36}$$

By Lemma 6.9 and (6.2), there exist $c_j > 0$ so that when $|\xi| \ll h^{-1}$ we have

$$i\left([A^w, B_T^{Wick}]u, u\right)/2 \ge (\epsilon - 2c_1T)\left(\mu^w u, u\right)/2T - 3c_0\epsilon\left(m^{Wick}u, u\right)/2T \tag{8.37}$$

for $u \in C_0^\infty$. In fact, $h^{1/2} \le 6m$ by (6.2) and since b_1 is constant in x we have $[A^w, b_1^w] = \langle s_2^w D_x, D_x\rangle + s_1^w D_x + s_0^w$ with $s_j \in S(h^{1/2}, g_0^\sharp)$.

We find from (8.34)–(8.36) that we can find $\Psi \in S^2$ such that $\Sigma_2 \bigcap \operatorname{supp}\Psi = \emptyset$ and

$$\begin{aligned}\operatorname{Im}\left(P^*u, B_T^{Wick}u\right) \ge \left(\frac{1}{4} - 6c_0\epsilon - C_0T\right)&\left(m^{Wick}u, u\right)/T \\ &+ (\epsilon - 2C_0T - 2c_1T))\left(\mu^w u, u\right))/2T - \|\Psi^w u\|^2/T\end{aligned} \tag{8.38}$$

for $u \in C_0^\infty$. By taking first ϵ and then T small enough we find

$$\operatorname{Im}\left(P^*u, B_T^{Wick}u\right) + \|\Psi^w u\|^2/T \gtrsim \left(m^{Wick}u, u\right)/T + (\mu^w u, u))/T \tag{8.39}$$

for $u \in C_0^\infty$.

Since $|\delta_0 + \varrho_T| \le |\delta_0| + m \le 3\langle\delta_0\rangle/2$, $h^{1/2}\langle\delta_0\rangle^2 \lesssim m$ by (6.2) and $|\lambda| \lesssim h^{1/4}\mu^{1/2}$, we find $|B_T| \lesssim h^{-1/4}m^{1/2} + h^{1/4}\mu^{1/2}$. Thus $h^{1/2}((B_T^{Wick})^2 + 1) \in \operatorname{Op} S(m, g^\sharp) + \operatorname{Op} S(h^{1/2}\mu, g_0^\sharp)$ so Propositions 7.6 and 7.7 give

$$h^{1/2}(\|B_T^{Wick}u\|^2 + \|u\|^2 + \|D_x u\|^2) \lesssim \left(m^{Wick}u, u\right) + (\mu^w u, u) \qquad u \in C_0^\infty \tag{8.40}$$

Summing up, we obtain that

$$\begin{aligned}h^{1/2}(\|B_T^{Wick}u\|^2 + \|u\|^2 + \|D_x u\|^2) &\le C_1\left(\left(m^{Wick}u, u\right) + (\mu^w u, u)\right) \\ &\le C_2\left(T\operatorname{Im}\left(Pu, B_T^{Wick}u\right) + \|\Psi^w u\|^2\right)\end{aligned} \tag{8.41}$$

if $u \in C_0^\infty$ has support where $\max(|t|, |x|) \le T$ which completes the proof of Proposition 3.6. $\square$

Appendix A. Proof of Proposition 2.1

In this appendix, we are going give a proof of Proposition 2.2 in Section 2. Let $f(x, w) \in C^\infty(\mathbf{R}^{n+m})$ be real valued and consider the equation

$$P(\partial_x u, x, w, \partial_x)u = f, \quad u(x_0, w_0) = u_0 \in \mathbf{R}, \quad \partial_x u(x_0, w_0) = u_1 \in \mathbf{R}^n \tag{A.1}$$

where P is a quasilinear second order PDO in the x variables with real C^∞ coefficients having $w \in \mathbf{R}^m$ as parameter such that

$$P(v, x, w, \partial_x) = p_2(v, x, w, \partial_x) + p_1(x, w)\partial_x + p_0(x, w) \tag{A.2}$$

with $v(x,w) \in C^\infty(\mathbf{R}^{n+m},\mathbf{R}^n)$, $p_1(x,w) \in C^\infty(\mathbf{R}^{n+m},\mathbf{R}^n)$ and $p_0(x,w) \in C^\infty(\mathbf{R}^{n+m},\mathbf{R})$. We assume that the principal symbol vanishes of second order, so that

$$p_2(v,x,w,\partial_x) = \sum_{j,k=1}^{n} L_{jk}(v,x,w)\partial_{x_j}\partial_{x_k} \tag{A.3}$$

where the real quadratic form

$$L(u_1,x_0,w_0) = \{\, L_{jk}(u_1,x_0,w_0)\,\}_{jk} \quad \text{has maximal rank } n \tag{A.4}$$

which then holds in a neighborhood of (u_1,x_0,w_0) so that P is of real principal type.

Theorem A.1. *Let P be given by* (A.2) *so that conditions* (A.3) *and* (A.4) *hold, then for any real valued $f \in C^\infty(\mathbf{R}^{m+n})$, $u_0 \in \mathbf{R}$ and $u_1 \in \mathbf{R}^n$ there exists a neighborhood U of (x_0,w_0) so that* (A.1) *has a real valued solution $u \in C^\infty(\mathbf{R}^{m+n})$ in U. The neighborhood U will only depend on the bounds on u_0, u_1, f and the coefficients of P.*

Observe that the solution is not unique, for uniqueness one needs hyperbolicity of P and initial values at a noncharacteristic surface. Since the system (2.17) is on the form (A.2)–(A.4) we obtain Proposition 2.1 from Theorem A.1.

We shall first reduce to the case with vanishing data by changing the dependent variable

$$u(x,w) = v(x,w) + u_0 + u_1 \cdot x \tag{A.5}$$

in (A.1), then we obtain the following equation for v:

$$P_0 v = P(\partial v + u_1, x, w, \partial)v = f(x,w) - p_1(x,w)u_1 - p_0(x,w)(u_0 + u_1\cdot x) = f_0(x,w) \tag{A.6}$$

with $v(x_0,w_0) = 0$ and $\partial_x v(x_0,w_0) = 0$. Now the right hand side of (A.6) depends linearly on both f, u_0 and u_1 and we have that (A.4) holds when $u_1 = 0$.

Renaming the operator, for the proof of Theorem A.1 we shall solve the linear equation

$$P(v(x,w),x,w,\partial_x)u(x,w) = f_0(x,w) \qquad u(x_0,w_0) = 0 \quad \partial_x u(x_0,w_0) = 0 \tag{A.7}$$

which is a second order real linear PDE with $w \in \mathbf{R}^m$ and $v(x,w) \in C^\infty(\mathbf{R}^{n+m},\mathbf{R}^n)$ as parameters such that $v(x_0,w_0) = 0$. By using iteration and compactness we shall obtain a solution to (A.1), the proof of Theorem A.1 will be at the end of the appendix.

To solve the linear equation (A.7) we shall microlocalize using pseudodifferential equations. In the following we will say that an pseudodifferential operator (or Fourier integral operator) $a(v,x,D)$ depends C^∞ on a parameter $v(x) \in C^\infty$ if any seminorm of the symbol (and phase function) is bounded by a finite number of seminorms of v. For operators with symbols in $S^{-\infty}$ this means that the C^∞ kernel is a C^∞ function of v. Observe that compositions and adjoints of such operators also depend C^∞ on v, see Lemma A.8 and Remark A.9.

Next, we shall microlocalize in cones in the ξ variables. In the following, we will use the classical Kohn-Nirenberg quantization having classical symbol expansions.

Definition A.2. *For any $\varepsilon > 0$ and $\xi_0 \in \mathbf{R}^n$ such $|\xi_0| = 1$ we let*

$$\Gamma_{\xi_0,\varepsilon} = \{\, \xi : |\xi/|\xi| - \xi_0| < \varepsilon \,\} \tag{A.8}$$

which is a conical neighborhood of ξ_0.

Recall that a partition of unity is a set $\{\, \phi_j \,\}_j$ such that $0 \le \phi_j \in C^\infty$ and $\sum_j \phi_j \equiv 1$.

Remark A.3. *For any $\varepsilon > 0$ small enough we can find a partition of unity on $S^*\mathbf{R}^n$ and extend it by homogeneity in ξ to get a partition of unity $\{\, \varphi_j(\xi) \,\}_j$ on $T^*\mathbf{R}^n \setminus 0$ such that $0 \le \varphi_j \in S^0$ is homogeneous and supported in $\Gamma_{\xi_j,\varepsilon}$ for some $|\xi_j| = 1$. We can also find $\{\, \psi_j \,\}_j$ such that $0 \le \psi_j \in S^0$ is homogeneous and supported in $\Gamma_{\xi_j,\varepsilon}$ so that $\psi_j = 1$ on* $\operatorname{supp} \phi_j$*.*

We shall also localize when $|\xi| \ge \varrho \ge 1$ by $\chi_\varrho(\xi) = \chi(|\xi|/\varrho) \in C^\infty$, where $\chi \in C^\infty(\mathbf{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 0$ when $t \le 1$ and equal to 1 when $t \ge 2$. Let $\varphi_{j,\varrho} = \chi_\varrho \varphi_j$ and $\psi_{j,\varrho} = \chi_\varrho \psi_j$ then $\varphi_j - \varphi_{j,\varrho}$ and $\psi_j - \psi_{j,\varrho}$ are in $S^{-\infty}$, $\forall\, j$ and $\forall\, \varrho \ge 1$. We also have that $\varrho\chi_\varrho \in S^1$ uniformly in $\varrho \ge 1$, $\forall\, j$.

In fact, since $0 \le \chi_\varrho \le 1$ and $\varrho \le |\xi|$ in the support of this symbol, we find that $|\varrho\chi_\varrho(\xi)| \le |\xi|$. Taking ξ derivatives of the symbol gives a factor ϱ^{-1} together with a symbol supported where $|\xi| \le \varrho \le 2|\xi|$.

Next, we have to prepare the linear operator P microlocally with respect to this partition of unity. We will then use microlocal pseudodifferential operators which may give complex solutions. But since P is a real PDO, we may take the real part of the solution to the linear equation. In the following we shall use the notation $\langle D \rangle = (1 + |D|^2)^{1/2}$.

Proposition A.4. *Let P be given by* (A.2)–(A.4) *with real $v(x,w) \in C^\infty$ such that $v(x_0,w_0) = 0$, and let $\Gamma = \Gamma_{\xi_0,\varepsilon}$ be defined by* (A.8) *for $|\xi_0| = 1$, $0 < \varepsilon \le \varepsilon_0$. Then for ε_0 small enough there exists real valued $0 \ne a(v,x,w,\xi) \in S^0$, $0 < c\langle \xi \rangle^{-1} \le b(\xi) \in S^{-1}$ and orthonormal variables $(t,x) \in \mathbf{R} \times \mathbf{R}^{n-1}$ so that*

$$P(v,t,x,w,D)b(D) = a(v,t,x,w,D)Q(v,t,x,w,D) + R(v,t,x,w,D) \tag{A.9}$$

where

$$Q(v,t,x,w,D) = D_t + \sum_{j=1}^{n-1} A_j(v,t,x,w,D_x)D_{x_j} + A_0(v,t,x,w,D_x) \tag{A.10}$$

Here a, A_j and R are operators that depend C^∞ on $v(x,w)$, $A_j \in C^\infty(\mathbf{R}, S^0)$ is real valued when $j > 0$ and $R = R_0 + R_1 \in \Psi^1$ where $R_0 \in \Psi^{-1}$ and $\mathrm{WF}\, R_1 \bigcap \Gamma_0 = \emptyset$*,*

$\Gamma_0 = (0, x_0, w_0) \times \Gamma_{\xi_0,\varepsilon}$. *The seminorms of* A_j*,* R *and the constant* ε_0 *only depend on the seminorms of* v *and the coefficients of* P*.*

Observe that since $a \neq 0$ we have by the calculus that $a^{-1}a \cong aa^{-1} \cong \mathrm{Id}$ modulo Ψ^{-1}.

Proof. For the proof, it is important that compositions of operators that depend C^∞ on v also depend C^∞ on v, see Lemma A.8. By taking an ON base of eigenvectors of $L(0, x_0, w_0)$ and choosing ON variables x, we may assume that $p_2(0, x_0, w_0, \xi) = \sum_j c_j \xi_j^2$ for $0 \neq c_j \in \mathbf{R}$. Choose j so that $\xi_j \neq 0$ at ξ_0, thus in a conical neighborhood of Γ_0 if ε is small enough. By an ON change of variables we may take $j = 1$, then $c_1 = -L_{11}(0, x_0, w_0)$. Letting $b(\xi) = \xi_1^{-1}$ near $\Gamma_0 \cap \{\, |\xi| \geq 1 \,\}$ we may extend $b(\xi)$ to a symbol in S^{-1} so that $b(\xi) \gtrsim \langle \xi \rangle^{-1}$. Then $Pb(D)$ has a symbol expansion with $p_j b \in S^{j-1}$ and

$$p_2(v, x, w, \xi) b(\xi) = -\sum_{jk} L_{jk}(v, x, w) \xi_j B_k(\xi)$$

where $B_k(\xi) = \xi_k b(\xi) \in S^0$. Since $B_j(D)D_k = B_k(D)D_j$, we find from (A.2)–(A.3) that

$$Pb(D) = \sum_{j=1}^{m} A_j(v, x, w, D) D_j + A_0(v, x, w, D)$$

where $A_0 = (p_1 \partial + p_0) b(D)$, $A_1(v, x, w, \xi) = -L_{11}(v, x, w) B_1(\xi) \neq 0$ near Γ_0 and $A_j \in S^0$ is real valued when $j > 0$. Observe that $B_1(\xi) = 1$ near $\Gamma_0 \bigcap \{\, |\xi| \geq 1 \,\}$ so it may be extended to be equal to 1 everywhere modulo terms having wave front set outside $\overline{\Gamma}_0$. This gives that $A_1 = -L_{11} \neq 0$ near $(0, x_0, w_0)$ thus we can extend $a = A_1$ so that $0 \neq a(v, x, w) \in C^\infty$. Replacing A_j with $a^{-1}A_j$ we find that $Pb(D) = a(v, x, w)Q \in \Psi^1$ where the symbol of Q is equal to $\xi_1 + \sum_{j>1} A_j(v, x, w, \xi)\xi_j + A_0(v, x, w, \xi)$ modulo Ψ^{-1} and terms having wave front set outside $\overline{\Gamma}_0$.

To obtain that A_j is independent of ξ_1 for $j > 0$, we shall use the Malgrange preparation theorem. If $\xi = (\xi_1, \xi')$ we find by homogeneity for small enough $\varepsilon_0 > 0$ that

$$\xi_1 + \sum_{j>1} A_j(v, x, w, \xi)\xi_j = q(v, x, w, \xi)\, (\xi_1 + r(v, x, w, \xi')) \tag{A.11}$$

in a conical neighborhood of Γ_0, where $q > 0$ is homogeneous and r is real, homogeneous of degree 1 and vanishes when $\xi' = 0$. Then we can extend $q > 0$ to a homogeneous symbol by a cut-off, observe that the symbols depend C^∞ on v. This replaces a by $0 \neq aq \in S^0$, and by using Taylor's formula we find that $r(v, x, w, \xi') = \sum_{j>1} r_j(v, x, w, \xi')\xi_j$ with r_j homogeneous in ξ'. This gives that $Pb(D) = aq(v, x, w, D)Q$ where Q is equal to (A.10) modulo Ψ^0 and terms having wave front set outside $\overline{\Gamma}_0$. The composition $aq(v, x, w, D)$ with Q also gives lower order terms in Ψ^0 which can be included in A_0.

Now the term $A_0 \in \Psi^0$ of Q can be replaced by $aq(v,x,w,D)R_0$ modulo Ψ^{-1} where the symbol $R_0 = A_0/aq \in S^0$. To make the term R_0 independent of ξ_1 we may use Malgrange division theorem and homogeneity for small enough $\varepsilon_0 > 0$ to obtain that

$$R_0(v,x,w,\xi) = q_0(v,x,w,\xi)\,(\xi_1 + r(v,x,w,\xi')) + r_0(v,x,w,\xi') \tag{A.12}$$

in a conical neighborhood of Γ_0, where $q_0 \in S^{-1}$ and $r_0 \in S^0$ by homogeneity. Cutting of q_0 we may replace $R_0(v,x,w,D)$ with $q_0(v,x,w,D)Q + r_0(v,x,w,D_{x'})$ modulo Ψ^{-1} near Γ_0. Since $R_0 \cong (1+q_0)R_0$ modulo Ψ^{-1}, we obtain (A.9)–(A.10) with a replaced by $aq(1+q_0)$. Cutting off q_0 where $|\xi| \gg 1$ only changes the operator with terms in $\Psi^{-\infty}$, but gives that $1+q_0 > 0$ making $aq(1+q_0) \neq 0$. The composition of $aq(v,x,w,D)$ with $q_0(v,x,w,D)Q$ will also give lower order terms in Ψ^{-1} which can be included in R together with any cut-off terms. This gives the proposition after putting $t = x_1$ and $x = x'$. ∎

Proposition A.4 shows that the linerarized equation $P(v,x,w,D)u = f$ may after ON changes of variables be microlocally be reduced to the system $Q_j(v,x,w,D)u_j \cong a_j^{-1}(v,x,w,D)f_j$ where $f_j = \varphi_j(D)f$ with φ_j given by Remark A.3. Observe that $u \cong \sum_j b_j(D)u_j$ where u_j also has to be microlocalized.

Now, the reduction and the calculus will give terms $S \in \Psi^{-\infty}$ which have smooth kernels. The errors $Sf(x) = \iint S(x,y)f(y)\,dy$ can be made small if f has support in a sufficiently small neighborhood of x_0 by cutting off the kernel S. Let $\phi_\delta(x) = \phi((x-x_0)/\delta)$ where $0 < \delta \le 1$ and $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le \phi \le 1$, has support where $|x| < 2$ and is equal to 1 when $|x| \le 1$, so that $\phi(x/\delta) \in C_0^\infty(B_{x_0,2\delta})$ if $B_{x_0,\delta} = \{\, x : |x - x_0| \le \delta \,\}$.

Lemma A.5. *Let $S(x,y) \in C^\infty$ and $S_\delta(x,y) = \phi_\delta(x)S(x,y)\phi_\delta(y) \in C_0^\infty(B_{x_0,2\delta} \times B_{x_0,2\delta})$. The mapping $S_\delta : C^\infty \mapsto C_0^\infty(B_{x_0,2\delta})$ is given by $S_\delta f(x) = \iint S_\delta(x,y)f(y)\,dy$, and for $f \in C_0^\infty(B_{x_0,\delta})$ we have $S_\delta f(x) = Sf(x)$ when $|x| \le \delta$. For δ small enough, $\operatorname{Id} + S_\delta$ has the inverse $(\operatorname{Id} + S_\delta)^{-1} = \sum_{j=0}^\infty (-S_\delta)^j \cong \operatorname{Id}$ modulo operators with kernels in $C_0^\infty(B_{x_0,2\delta} \times B_{x_0,2\delta})$.*

Proof. We may assume that $x_0 = 0$, clearly $S_\delta f(x) = Sf(x)$ if $f\phi_\delta = f$ and $\phi_\delta(x) = 1$. If $f \in C^\infty$ then L^∞ norm is $\|S_\delta f\|_\infty \le c_n 2^n \delta^n \|S\|_\infty \|f\|_\infty$. By induction we get

$$\|S_\delta^j f\|_\infty \le c_n 2^n \delta^n \|S_\delta\|_\infty \|S_\delta^{j-1} f\|_\infty \le c_n^j 2^{jn} \delta^{jn} \|S\|_\infty^j \|f\|_\infty \qquad j > 1 \tag{A.13}$$

where the kernels of S_δ^j are in $C_0^\infty(B_{0,2\delta} \times B_{0,2\delta})$. Thus the series $\sum_{j=0}^\infty (-S_\delta)^j$ converges on L^∞ if $c_n 2^n \delta^n \|S\|_\infty < 1$. Derivation of the terms in the series will only give factors $O(\delta^{-1})$ so the convergence is in $C_0^\infty(B_{0,2\delta} \times B_{0,2\delta})$. Then the inverse $(\operatorname{Id} + S_\delta)^{-1} = \operatorname{Id} + \sum_{j=1}^\infty (-S_\delta)^j \cong \operatorname{Id}$ modulo operators with kernels in $C_0^\infty(B_{x_0,2\delta} \times B_{x_0,2\delta})$. ∎

Next, we shall solve the microlocalized equations $Q_j u_j = a_j^{-1} f_j = a_j^{-1}\varphi_j f$ with $u_j = 0$ when $t = 0$. Here φ_j is given by Remark A.3, Q_j is given by (A.10) near $\Gamma_{\xi_j,\varepsilon}$ with $0 \ne a_j \in S^0$ given by Proposition A.4, but we shall treat terms $R \in \Psi^{-1}$ as perturbations. In the case when $A_j \equiv 0$, we would then find that $Q_j u_j \cong D_t u_j = a_j^{-1} f_j$, which has the approximate solution $u_j \cong \int_0^t a_j^{-1} f_j\, dt$. By using Fourier integral operators one can reduce to this case.

We shall use denote by I^k classical Fourier integral operators of order k with homogenous phase functions and classical symbol expansions depending C^∞ on v. But we shall also use operators $F \in C^\infty(\mathbf{R}, I^k)$ which are FIOs $F(t) \in I^k$ in x depending C^∞ on t and v, $(t, x) \in \mathbf{R} \times \mathbf{R}^{n-1}$. Observe that ΨDOs of order k in x depending C^∞ on t and v are also in $C^\infty(\mathbf{R}, I^k)$ and that $C^\infty(\mathbf{R}, I^k) \subset I^k$. By multiplying I^k by I^m we obtain operators in I^{k+m} by Remark A.9.

As before, we shall use ON coordinates $(t, x) \in \mathbf{R} \times \mathbf{R}^{n-1}$ and suppress the dependence on v and w. But the operators will depend C^∞ on v and w having symbols and phase functions that are uniformly bounded if $v \in C^\infty$ and $w \in \mathbf{R}^m$ are bounded.

Proposition A.6. *Assume that $Q = D_t + a_1(t, x, w, D_x) + a_0(t, x, w, D_x)$ depend C^∞ on $v \in C^\infty$, where $a_1 \in C^\infty(\mathbf{R}, \Psi^1)$ is real and homogeneous of degree 1 and $a_0 \in C^\infty(\mathbf{R}, \Psi^0)$. Then there exists elliptic Fourier integral operator $F_0(t)$ and $F_1(t) \in C^\infty(\mathbf{R}, I^0)$ such that $F_0(t)F_1(t) \cong \mathrm{Id}$ and $QF_0(t) \cong F_0(t)D_t$ modulo $C^\infty(\mathbf{R}, I^{-1})$. If $f \in C_0^\infty$ then we have that*

$$u(t, x) = iF_0(t) \int_0^t F_1(s) f(s, x)\, ds = \mathbb{F} f(t, x) \tag{A.14}$$

solves the initial value problem

$$Qu = (\mathrm{Id} + S) f \in C^\infty \qquad u(0, x) \equiv 0 \tag{A.15}$$

where $S \in C^\infty(\mathbf{R}, I^{-1})$ and $\mathbb{F} \in I^0$. Here $F_0(t)$, $F_1(t)$ and $\mathbb{F}$ have wave front sets close to the diagonal when $|t| \ll 1$. In fact, the canonical transformations given by $F_0(t)$ and $F_1(t)$ maps bicharacteristics of $D_t + a_1$ to t lines and vice versa.

Corollary A.7. *If $c_j \in \Psi^k$, $j = 1,\ 2$, and $\operatorname{supp} c_1 \bigcap \operatorname{supp} c_2 = \emptyset$, then $c_1 F_0(t) F_1(s) c_2 \in I^{-\infty}$ having a smooth kernel for small enough s and t.*

Proof. It is a classical result that there exists elliptic Fourier integral operators $F_0(t)$ and $F_1(t) \in C^\infty(\mathbf{R}, I^0)$ with the properties in the proposition. The construction of the homogeneous phase function of the FIO involves solving the Hamilton-Jacobi equations, which depend on the derivatives of the principal symbol $\tau + a_1$ of Q. Then the amplitude is given by the transport equations depending on the lower order term a_0 of Q modulo terms in $C^\infty(\mathbf{R}, S^{-1})$. For the approximate inverse, one takes the phase function for the inverse canonical relation and the inverse amplitude.

If $f \in C_0^\infty$ and $u = iF_0(t)v$ with $v = \int_0^t F_1(s)f(s,x)\,ds$ as in (A.14), then we find

$$\begin{aligned}(A.16)\quad Qu = iQF_0(t)v = (iF_0(t)D_t + S_0)v = F_0(t)F_1(t)f + S_0v \\ = f + S_1f + S_0v = f + S_1f + S_0\int_0^t F_1(s)f\,ds = f + Sf\end{aligned}$$

where S_0, S_1 and $S \in C^\infty(\mathbf{R}, I^{-1})$. □

The approximate solution u in (A.14) depends C^∞ on the data f and v, but we shall need stronger estimates. For that we shall use the L^2 Sobolev norms:

$$(A.17)\qquad \|\varphi\|_{(k)}^2 = \|\langle D\rangle^k\varphi\|^2 \qquad \varphi \in C_0^\infty$$

We shall use the following estimates, were we shall suppress the parameter w.

Lemma A.8. *If $a(u,x,D) \in \Psi^0$ depends C^∞ on $u(x) \in C^\infty$ then there exists $\ell \in \mathbf{N}$ so that for any $k \in \mathbf{N}$ there exists $C_k(t) \in C^\infty(\mathbf{R}_+)$ so that*

$$(A.18)\qquad \|a(u,x,D)\varphi\|_{(k)} \le C_k(\|u\|_{(\ell)})\|\varphi\|_{(k)} \qquad \forall\,\varphi \in C_0^\infty$$

If $a(u,x,D) \in \Psi^{m_1}$ and $b(u,x,D) \in \Psi^{m_2}$ depend C^∞ on $u(x)$ then $a(u,x,D)b(u,x,D) \in \Psi^{m_1+m_2}$ also depends C^∞ on $u(x)$. Then there exists $\ell \in \mathbf{N}$ so that for any k there exists $C_k(t) \in C^\infty(\mathbf{R}_+)$ so that

$$(A.19)\qquad \|\,[a(u,x,D), b(u,x,D)]\varphi\|_{(k)} \le C_k(\|u\|_{(\ell)})\|\varphi\|_{(k+m_1+m_2-1)} \qquad \forall\,\varphi \in C_0^\infty$$

If $a(u,x,D) \in \Psi^m$ depends C^∞ on $u(x)$ having real valued symbol modulo S^{m-1} then

$$(A.20)\qquad \|\operatorname{Im} a(u,x,D)\varphi\|_{(k)} \le C_k(\|u\|_{(\ell)})\|\varphi\|_{(k+m-1)} \qquad \forall\,\varphi \in C_0^\infty$$

where $2i\operatorname{Im} a(u,x,D) = a(u,x,D) - a^*(u,x,D)$ also depends C^∞ on u.

Proof. First we note that by definition any seminorm of $a(u,x,\xi) \in S^m$ is bounded by $\|u\|_{C^k}$ when $|\xi| = 1$ for some $k \in \mathbf{N}$. By the Sobolev embedding theorem, the C^k norm of u can be bounded by by the norm $\|u\|_{(k+s)}$ with $s > n/2$.

If $a(u,x,\xi) \in S^{m_1}$ and $b(u,x,\xi) \in S^{m_2}$ then $a(u,x,D)b(u,x,D) = c(u,x,D)$ is given by

$$(A.21)\qquad c(u,x,\xi) = e^{i\langle D_\xi, D_y\rangle}a(u,x,\xi)b(u,y,\eta)\big|_{\substack{y=x\\ \eta=\xi}}$$

The mapping $a, b \mapsto c$ is weakly continuous on the symbol classes S^m so that any seminorm of c only depends on some seminorms of a and b, see [10, Th. 18.4.10']. (Here weak continuity means that the restriction to a bounded set is continuous.) Thus if $a(u,x,D)$ and $b(u,x,D)$ depend C^∞ on $u \in C^\infty$ then $c(u,x,D)$ also does. Observe that the number of seminorms that is needed does not depend on the symbol classes S^k, it only depends on the symbol metric and the dimension.

We may reduce the estimate (A.18) to the case $k = 0$ by replacing $a(u, x, D)$ with $A(x, D) = \langle D\rangle^k a(u, x, D)\langle D\rangle^{-k} \in \Psi^0$ and φ with $\langle D\rangle^k\varphi$. Then the L^2 norm of $A(x, D)$ depends on a fixed seminorm of $A(x, \xi)$, see [10, Th. 18.6.3]. This seminorm in turn depends on a fixed seminorm of $a(u, x, \xi)$ which gives (A.18).

Any seminorm of the symbol of the commutator $[a(u, x, D), b(u, x, D)] \in S^{m_1+m_2-1}$ depends on the same seminorm of the symbols of the compositions $a(u, x, D)b(u, x, D)$ and $b(u, x, D)a(u, x, D)$. These seminorms in turn depend on some seminorms of $a(u, x, \xi)$ and $b(u, x, \xi)$. Thus we obtain (A.19) from (A.18) for some ℓ and C_k.

If $a \in S^m$ then the adjoint $a^*(u, x, D)$ is given by

$$a^*(u, x, \xi) = e^{i\langle D_\xi, D_x\rangle}\overline{a}(u, x, \xi) \tag{A.22}$$

which is weakly continuous in the symbol class S^m by [10, Th. 18.1.7]. If a is real modulo S^{m-1} then $\operatorname{Im} a(u, x, D) \in \Psi^{m-1}$. Thus any seminorm of the symbol of $\operatorname{Im} a(u, x, D)$ is bounded by some seminorms of $a(u, x, \xi)$, which gives (A.20) for some C_k. $\square$

Remark A.9. *The results of Lemma A.8 also holds for the ΨDOs Ψ^m depending C^∞ on u composed by FIOs I^k depending C^∞ on u, e.g., the FIO given by Proposition A.6. Operators in $I^{-\infty}$ have smooth kernels which are C^∞ functions of u.*

In fact, Theorem 9.1 in [8] shows that the conjugation of ΨDOs with FIOs gives symbol expansions similar to (A.21) after change of variables, see for example (9.2)" in [8]. This result is about Weyl operators, but by Theorem 4.5 in [8] it can be extended to operators having the Kohn-Nirenberg quantization. This gives a calculus with symbol expansions of classical homogeneous FIOs with homogeneous phases and symbols, see pages 441–442 in [8]. For example, if $a \in \Psi^m$ and $F \in I^k$ then we have $\|aFu\|^2 = \langle F^*a^*aFu, u\rangle$ where $F^*a^*aF = b \in \Psi^{2(m+k)}$, and similar result holds for $\|Fau\|^2$.

For $S \in I^{-\infty}$ the C^∞ dependence means that for any k we have $S \in I^{-k}$ depending C^∞ on u. Since the kernel is obtained by taking the Fourier transform in ξ of the symbol, we find that the kernel of S is smooth and a is C^∞ function of u.

Next, we are going to prove estimates for the microlocalized operators. Then we will use ON coordinates $(t, x) \in \mathbf{R} \times \mathbf{R}^{n-1}$ and for $k \in \mathbf{N}$ and $T > 0$ define the local norms

$$\|\varphi\|^2_{k,T} = \int_{|t|\le T} \|\varphi\|^2_k(t)\, dt \qquad \varphi \in C_0^\infty \tag{A.23}$$

and $\|\varphi\|_{k,j,T} = \|\langle D_t\rangle^j\varphi\|_{k,T}$, $\forall\, j \in \mathbf{Z}_+$, with $\|\varphi\|^2_k(t) = \iint |\langle D_x\rangle^k\varphi(t, x)|^2\, dx$.

Proposition A.10. *Let v, $f \in C_0^\infty$ and $u \in C^\infty$ be a solution to*

$$Q(v,t,x,w,D)u = \partial_t u + \sum_{j=1}^{n} A_j(v,t,x,w,D_x)\partial_{x_j} u + A_0(v,t,x,w,D_x)u = f \qquad u(0,x,w) = 0 \tag{A.24}$$

where $A_j \in C^\infty(\mathbf{R}, \Psi^0)$ depends C^∞ on v, $\forall\, j$, and A_j is real valued modulo S^{-1} for $j > 0$. Then there exists $\ell \in \mathbf{N}$ so that for any $k \in \mathbf{N}$ there exists $C_k(r) \in C^\infty(\mathbf{R}_+)$ so that

$$\|\phi u\|_{(k)}^2 \le C_k(\|v\|_{(\ell)})\|f\|_{(k)}^2 \tag{A.25}$$

if $\phi \in C_0^\infty$ has support where $|t| \le 1$. The estimate only depends on the seminorms of the symbol of Q and ϕ.

Thus, for any $k \in \mathbf{N}$ we get uniform local bounds on $\|u\|_{(k)}$ when $\|v\|_{(\ell)}$ is uniformly bounded. Now Q is a differential operator in t but a ΨDO in x, so in the proof we shall use Lemma A.8 in the x variables.

Proof. Let $A\partial_x = \sum_{j=1}^n A_j \partial_{x_j}$ and $\langle u, u\rangle_k(t) = \|u\|_k^2(t)$ be the sesquilinear form, then

$$\partial_t \|u\|_k^2(t) = 2\operatorname{Re}\langle \partial_t u, u\rangle_k(t) = 2\operatorname{Re}\langle f, u\rangle_k(t) - 2\operatorname{Re}\langle A\partial_t u, u\rangle_k(t) - 2\operatorname{Re}\langle A_0 u, u\rangle_k(t) \tag{A.26}$$

Conjugating with e^{-Ct} gives

$$\partial_t(e^{-Ct}\|u\|_k^2(t)) = e^{-Ct}\big(2\operatorname{Re}\langle f, u\rangle_k(t) - 2\operatorname{Re}\langle A\partial_x u, u\rangle_k(t) - 2\operatorname{Re}\langle A_0 u, u\rangle_k(t) - C\|u\|_k^2(t)\big) \tag{A.27}$$

where

$$2\operatorname{Re}\langle A\partial_x u, u\rangle_k(t) = 2\operatorname{Re}\langle [\langle D_x\rangle^k, A]\partial_x \langle D_x\rangle^{-k} w, w\rangle_0(t) + \langle [\operatorname{Re} A, \partial_x,]w, w\rangle_0(t) + 2\operatorname{Re}\langle i \operatorname{Im} A\partial_x w, w\rangle_0(t) = \langle Rv, v\rangle_0(t) \tag{A.28}$$

where $[\operatorname{Re} A, \partial_x]$ and $\operatorname{Im} A\partial_x \in C^\infty(\mathbf{R}, \Psi^0)$ and $w = \langle D_x\rangle^k u$. The calculus gives that the operator $[\langle D_x\rangle^k, A]\partial_x \langle D_x\rangle^{-k} \in C^\infty(\mathbf{R}, \Psi^0)$, so that $R \in C^\infty(\mathbf{R}, \Psi^0)$ depends C^∞ on v.

Since $\|w\|_0 = \|u\|_k$ we find by using Lemma A.8 that

$$|\langle Rw, w\rangle_0(t)| \le C_k(\|v\|_\ell(t))\|w\|_0^2(t) = C_k(\|v\|_\ell(t))\|u\|_k^2(t) \tag{A.29}$$

for some $\ell \in \mathbf{N}$ and $C_k(t) \in C^\infty(\mathbf{R})$, and clearly

$$|\langle f, u\rangle_k(t)| \le \|f\|_k^2(t) + \|u\|_k^2(t) \tag{A.30}$$

We also obtain from Lemma A.8 that

$$|\langle A_0 u, u\rangle_k(t)| \le C_k(\|v\|_\ell(t))\|u\|_k^2(t) \tag{A.31}$$

where in the following we will take the maximum of ℓ and $C_k(t)$. Summing up, we have

$$\partial_t(e^{-Ct}\|u\|_k^2(t)) \le e^{-Ct}\left((C_k(\|v\|_\ell(t)) + 1 - C)\|u\|_k^2(t) + C_k(\|v\|_\ell(t))\|f\|_k^2(t)\right) \tag{A.32}$$

Now, we may replace $C_k(t)$ by a nondecreasing function. Then we put

$$C = \max_{|t|\le T} C_k(\|v\|_\ell(t)) + 1 \le C_k\left(\max_{|t|\le T}\|v\|_\ell(t)\right) + 1$$

where $\|v\|_\ell(t) \le C_0\|v\|_{(\ell+1)}$ $\forall\, t$ by Sobolev's inequality. Since $\|u\|_k(0) = 0$ we find by integrating that

$$e^{-Ct}\|u\|_k^2(t) \le e^{CT}C_k(\|v\|_{(\ell+1)})\|f\|_{k,T}^2 \qquad t \in [-T, T] \tag{A.33}$$

for some $C_k(t)$. Integrating again over $[-T, T]$ we obtain that

$$\|u\|_{k,T}^2 \le 2Te^{2CT}C_k(\|v\|_{(\ell+1)})\|f\|_{k,T}^2 \tag{A.34}$$

By replacing $2Te^{2CT}C_k(\|v\|_{(\ell+1)})$ by $C_k(\|v\|_{(\ell+1)})$ and changing ℓ we obtain

$$\|u\|_{k,T}^2 \le C_k(\|v\|_{(\ell)})\|f\|_{k,T}^2 \qquad \forall\, k \ge 0 \tag{A.35}$$

Next, we shall estimate $\|u\|_{k,j,T}$ when $j > 0$, for $j = 1$ it suffices to estimate $\|\partial_t u\|_{k,T}$. Now $Q\partial_t u = \partial_t f + Bu$ where $B = [Q, \partial_t] = -\partial_t A\partial_x - \partial_t A_0 \in C^\infty(\mathbf{R}, \Psi^1)$ is an ΨDO in x depending on C^∞ on v and t. By applying (A.35) on $\partial_t u$ we obtain that

$$\|\partial_t u\|_{k,T}^2 \le C_k(\|v\|_{(\ell)})\left(\|\partial_t f\|_{k,T}^2 + \|Bu\|_{k,T}^2\right) \qquad \forall\, k \ge 0 \tag{A.36}$$

where $\|Bu\|_{k,T}^2 \le C'_k(\|v\|_{(\ell)})\|u\|_{k+1,T}$ by Lemma A.8. By estimating $\|u\|_{k+1,T}$ by (A.35) and using that $\|f\|_{k,j,T} \le \|f\|_{(k+j)}$ we obtain $C_{k,1}(t) \in C^\infty(\mathbf{R})$ so that

$$\|\partial_t u\|_{k,T}^2 \le C_{k,1}(\|v\|_{(\ell)})\|f\|_{(k+1)}^2 \qquad \forall\, k \ge 0$$

Next, we proceed by induction. Thus,, we assume that we have proved that for a fixed $j > 0$ we have for $i \le j$ the estimate

$$\|\partial_t^i u\|_{k,T}^2 \le C_{k,i}(\|v\|_{(\ell)})\|f\|_{(k+i)}^2 \qquad \forall\, k \ge 0 \tag{A.37}$$

for some $C_{k,i}(t)$. Then $Q\partial_t^{j+1}u = \partial_t^{j+1}f + [Q, \partial_t^{j+1}]u$ where $[Q, \partial_t^{j+1}] = \sum_{0\le i\le j} B_i\partial_t^i$ with $B_i(t, x, D_x) \in C^\infty(\mathbf{R}, \Psi^1)$ being a ΨDO in x depending C^∞ on t and v. We obtain that

$$\|\partial_t^{j+1}u\|_{k,T}^2 \le C_k(\|v\|_{(\ell)})\left(\|\partial_t^{j+1}f\|_{k,T}^2 + \sum_{0\le i\le j}\|B_i\partial_t^i u\|_{k,T}^2\right) \qquad \forall\, k \ge 0 \tag{A.38}$$

by using (A.35). As before, $\|B_i\partial_t^i u\|_{k,T}^2 \le C'_k(\|v\|_{(\ell)})\|\partial_t^i u\|_{k+1,T}$ for $i \le j$ which we can use (A.37) to estimate. This gives (A.37) with i replaced by $j + 1$, so induction over j gives this estimate for any i.

Finally, we shall show that

$$\|\phi u\|_{(k)}^2 \le C_k(\|v\|_{(\ell)})\|f\|_{(k)}^2 \tag{A.39}$$

if $\phi \in C^\infty$ is supported where $|t| \le 1$. To estimate $\|\phi u\|^2_{(k)}$ it suffices to estimate $\|D_x^\alpha D_t^j \phi u\|$ for $|\alpha| + j \le k$. We have that

$$[D_x^\alpha D_t^j, \phi] = \sum_{\substack{|\beta| \le |\alpha| \\ i \le j}} B_{\beta.i} D_x^\beta D_t^i$$

where $B_{\beta.i} \in C^\infty$ has support where $|t| \le 1$. Thus, (A.37) gives that

$$\|D_x^\beta D_t^j \phi u\| \le C \sum_{0 \le i \le j} \|D_t^i u\|_{k-i,T} \le \sum_{0 \le i \le j} C_{k-i,i}(\|v\|_{(\ell)}) \|f\|^2_{(k)} \tag{A.40}$$

which completes the proof. □

Next, we shall solve the IVP for the linearized equation

$$P(v(x,w), x, w, \partial) u(x,w) = f(x,w) \tag{A.41}$$

where f and $v \in C^\infty(\mathbf{R}^{m+n}, \mathbf{R}^n)$ with $v(x_0, w_0) = 0$ and P is on the form (A.2) satisfying (A.3) and (A.4) with $u_1 = 0$. In the following, we shall suppress the parameters v and w, the preparation will only depend on the bounds on these parameters.

To solve equation (A.41), we shall assume that $x_0 = 0$ and use the microlocal normal forms given by Proposition A.4. In fact, for any small enough $\varepsilon > 0$ we can by Remark A.3 find a partition of unity $\{\varphi_j(\xi)\}_j$ with $\varphi_j \in S^0$ supported in cones $\Gamma_{\xi_j,\varepsilon}$ and ON variables (x_1, x') so that $Pb_j = a_j Q_j + Rj$ satisfies the conditions in Proposition A.4 with $\Gamma_0 = (0, 0, w_0) \times \Gamma_{\xi_j,\varepsilon}$ after the change of variables. Here $0 \ne a_j \in S^0$, $\langle \xi \rangle^{-1} \lesssim b_j(\xi) \in S^{-1}$ and $Q_j = D_{x_1} + A_j D_{x'} + A_{0,j}$ satisfies the conditions in Proposition A.6. The operator $R_j \in \Psi^1$ has symbol in S^{-1} in a conical neighborhood of Γ_0. I Ignoring the operator R_j, which will be handled as a perturbation, we obtain from Proposition A.6 that if $f \in C^\infty$ then $u_j = \mathbb{F}_j a_j^{-1} \varphi_j f$ solves

$$Q_j u_j = (\mathrm{Id} + s_j) a_j^{-1} \varphi_j f = (a_j^{-1} + r_j) \varphi_j f \qquad u_j\big|_{t=0} = 0 \tag{A.42}$$

where s_j and $r_j \in I^{-1}$.

But u_j may not be localized near Γ_0. To handle the localization and the error term R_j we shall microlocalize u_j depending on parameters. Let $\Phi_T(x) = \Phi(x/T)$ with $0 < T \le 1$ and $\Phi(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le \Phi \le 1$, Φ has support where $|x| \le 1$ and is equal to 1 when $|x| \le 1/2$. We shall also use the cut-off $\psi_{j,\varrho}(\xi) = \psi_j(\xi)\chi_\varrho(\xi)$ given by Remark A.3 with $\varrho \ge 1$ such that $0 \le \chi_\varrho \le 1$ has support where $|\xi| \ge \varrho$, $\psi_j \varphi_j = \varphi_j$ and $\operatorname{supp} \psi_j \in \Gamma_{\xi_j,\varepsilon}$. Since $u_j = \Phi_T u_j + (1 - \Phi_T) u_j$ we find that

$$u_j = \psi_{j,\varrho} \Phi_T u_j + (1 - \psi_{j,\varrho}) \Phi_T u_j + (1 - \Phi_T) u_j = u_{j,\varrho,T} + S_{j,\varrho,T} f \tag{A.43}$$

where $u_{j,\varrho,T} = \psi_{j,\varrho} \Phi_T u_j$ and $S_{j,\varrho,T} = (1 - \psi_{j,\varrho}) \Phi_T \mathbb{F}_j a_j^{-1} \varphi_j + (1 - \Phi_T) \mathbb{F}_j a_j^{-1} \varphi_j \in I^0$ since $\mathbb{F}_j a_j^{-1} \varphi_j \in I^0$. Thus we find that $S_{j,\varrho,T} f(x)$ depends on the values of $f(y)$ when y_1 is in the interval between 0 and x_1. Since $\psi_j\varphi_j = \varphi_j$ we find that $(1-\psi_{j,\varrho})\varphi_j = (1-\chi_\varrho)\varphi_j$ which gives

$$S_{j,\varrho,T} = \Phi_T\mathbb{F}_j a_j^{-1}(1-\chi_\varrho)\varphi_j - [\psi_{j,\varrho}, \Phi_T\mathbb{F}_j a_j^{-1}]\varphi_j + (1-\Phi_T)\mathbb{F}_j a_j^{-1}\varphi_j$$

where $(1-\chi_\varrho) \in \Psi^{-\infty}$ and $1-\Phi_T = 0$ when $|x| \le T/2$. Since $\psi_{j,\varrho}$ does not depend on x we find from (3.8) in the proof of Lemma 3.2 that $[\psi_{j,\varrho}, \Phi_T\mathbb{F}_j a_j^{-1}]$ has symbol

$$E(i\langle D_\xi, D_y\rangle)\partial\psi_{j,\varrho}(\xi)D_y\Phi_T(y)\mathbb{F}_j a_j^{-1}(y,\eta)\big|_{\substack{y=x\\ \eta=\xi}} \tag{A.44}$$

where $E(z) = (e^z-1)/z = \int_0^1 e^{\theta z}\,d\theta$. Since $\varphi_j\partial^\alpha\psi_{j,\varrho} = \varphi_j\partial^\alpha\chi_\varrho\ \forall\,\alpha$, we find that all the terms in the expansion of the commutator have support where $|\xi| \cong \varrho$ and $1/2 \le |x| \le 1$. Thus $S_{j,\varrho,T} \in I^{-\infty}$ has a C^∞ kernel depending on ϱ and T.

Since $a_j \ne 0$ we have $a_j a_j^{-1} = \mathrm{Id} + B_j$ with $B_j \in \Psi^{-1}$. We find from (A.9), (A.42) and (A.43) that

$$\begin{aligned} Pb_j u_{j,\varrho,T} &= a_j Q_j(u_j - S_{j,\varrho,T}f) + R_j u_{j,\varrho,T} \\ &= ((\mathrm{Id} + B_j + a_j r_j)\varphi_j - a_j Q_j S_{j,\varrho,T} + R_{j,\varrho,T})\,f \end{aligned} \tag{A.45}$$

where $a_jQ_jS_{j,\varrho,T} \in I^{-\infty}$ when $|x| < T/2$, $a_jr_j \in I^{-1}$ and $R_{j,\varrho,T} = R_j\psi_{j,\varrho}\Phi_T\mathbb{F}_j a_j^{-1}\varphi_j \in I^{-1}$ when $T \ll 1$ since the symbol $R_j\psi_{j,\varrho} \in S^{-1}$ for $|x| \ll 1$ by Proposition A.4. Here $Q \in I^k$ in the open set $\Omega \subset T^*\mathbf{R}^n$ means that $Q = Q_0 + Q_1$ where $Q_1 \in I^k$ and $\mathrm{WF}\,Q_0 \bigcap \Omega = \emptyset$. Since $\varrho\psi_{j,\varrho} \in S^1$ uniformly by Remark A.3, we find $\varrho R_{j,\varrho,T} \in I^0$ uniformly for $T \ll 1$.

Now we define

$$u_{\varrho,T}(x) = \sum_j b_j(D)\psi_{j,\varrho}(D)\Phi_T u_j(x) = \sum_j b_j(D)u_{j,\varrho,T}(x) \tag{A.46}$$

where $\varrho b_j\psi_{j,\varrho} \in S^0$ uniformly when $\varrho \ge 1$. Since $Pb_j = a_jQ_j + R_j$ we obtain from (A.45) and (A.46) that

$$Pu_{\varrho,T} = f + \sum_j((B_j + a_jr_j)\varphi_j + R_{j,\varrho,T} - a_jQ_jS_{j,\varrho,T})\,f = (\mathrm{Id} + R_{\varrho.T})f \tag{A.47}$$

where $R_{\varrho.T} = \sum_j(B_j + a_jr_j)\varphi_j + R_{j,\varrho,T} - a_jQ_jS_{j,\varrho,T} \in I^{-1}$ when $|x| < T/2 \ll 1$. We shall localize the first terms in ξ by writing $B_j + a_jr_j = (B_j + a_jr_j)(1-\chi_\varrho) + (B_j + a_jr_j)\chi_\varrho$ which gives

$$R_{\varrho.T} = R_{\varrho,T,0} + R_{\varrho,T,1} \tag{A.48}$$

with $R_{\varrho,T,0} = \sum_j(B_j + a_jr_j)\varphi_j(1-\chi_\varrho) - a_jQ_jS_{j,\varrho,T} \in I^{-\infty}$ and $R_{\varrho,T,1} = \sum_j(B_j + a_jr_j)\varphi_{j,\varrho} + R_{j,\varrho,T} \in I^{-1}$ when $|x| < T/2 \ll 1$. This gives that $\varrho R_{\varrho,T,1} \in I^0$ uniformly when $\varrho \ge 1$ and $|x| < T/2 \ll 1$, where we assume T fixed in the following.

Since we are only need local solutions, we may cut off near $x = 0$. To solve the equation near $x = 0$ it is enough that $\Phi_\delta Pu_{\varrho,T} = \Phi_\delta f$ for small enough $0 < \delta < T/2$.

Here $\Phi_\delta(x) = \Phi(x/\delta)$ with the same $\Phi \in C_0^\infty$ as before, then $\Phi_\delta\Phi_T = \Phi_\delta$. If f has support where $|x| \le \delta/2$ then we obtain that $\Phi_\delta P u_{\varrho,T} = \Phi_\delta(\mathrm{Id} + R_{\varrho,T})\Phi_\delta f = (\mathrm{Id} + R_{\delta,\varrho,T})f$ where $R_{\delta,\varrho,T} = \Phi_\delta R_{\varrho,T}\Phi_\delta$. By (A.48) we have

$$R_{\delta,\varrho,T} = R_{\delta,\varrho,T,0} + R_{\delta,\varrho,T,1} \tag{A.49}$$

with $R_{\delta,\varrho,T,j} = \Phi_\delta R_{\varrho,T,j}\Phi_\delta$. For fixed $0 < \delta < T/2$ we have $\varrho R_{\delta,\varrho,T,1} \in I^0$ uniformly when $\varrho \ge 1$ and $R_{\delta,\varrho,T,0} \in I^{-\infty}$ having C^∞ kernel depending on ϱ and δ.

It remains to invert the term $\mathrm{Id} + R_{\delta,\varrho,T}$ in order solve equation (A.41). This will be done in two steps, first making $R_{\delta,\varrho,T,1}$ small by taking large enough ϱ. This may increase the seminorms of $R_{\delta,\varrho,T,0}$, but this term can then be made small by localizing in a sufficiently small neighborhood of $x = 0$.

Since $\varrho R_{\delta,\varrho,T,1} \in I^0$ uniformly when $\varrho \ge 1$, we find by Remark A.9 that there exists $\varrho_{\delta,T} \ge 1$ so that if $\varrho \ge \varrho_{\delta,T}$ we have $\|R_{\delta,\varrho,T,1}f\|_{(0)} < \|f\|_{(0)}/2$ for $f \in \mathcal{S}$. Then we find that $(\mathrm{Id} + R_{\delta,\varrho,T,1})^{-1} = \mathrm{Id} + \sum_{k>0}(-R_{\delta,\varrho,T,1})^k \in I^0$ uniformly. Observe that $(-R_{\delta,\varrho,T,1})^k$ has kernel supported where $|x| \le \delta$ and $|y| \le \delta$. If we then solve

$$Q_j u_j = a_j^{-1}\varphi_j(\mathrm{Id} + R_{\delta,\varrho,T,1})^{-1} f \qquad u_j\big|_{x_1=0} = 0 \tag{A.50}$$

for f supported where $|x| \le \delta/2$, then the earlier reduction gives

$$\Phi_\delta P u_{\varrho,T} = (\mathrm{Id} + R_{\delta,\varrho,T})(\mathrm{Id} + R_{\delta,\varrho,T,1})^{-1} f = (\mathrm{Id} + R_{\delta,\varrho,T,2})f \tag{A.51}$$

where $R_{\delta,\varrho,T,2} = R_{\delta,\varrho,T,0}(\mathrm{Id} + R_{\delta,\varrho,T,1})^{-1} \in I^{-\infty}$ with C^∞ kernel supported where $|x| \le \delta$ and $|y| \le \delta$. Observe that we have uniform bounds for fixed δ and T when $\varrho \ge \varrho_{\delta,T}$ and these bounds depend on the bounds on the symbol of P and the parameters $v \in C^\infty$ and w. We shall later put more restraints on the lower bound of ϱ because of conditions on the estimates, see (A.64), and the values of $u_{\varrho,T}(x_0)$ and $\partial u_{\varrho,T}(x_0)$, see (A.68).

Now we have to shrink the support of $R_{\delta,\varrho,T,2}$ to lower the norm of the kernel without changing $R_{\delta,\varrho,T,1}$. With fixed $0 < \delta < T/2$ and $\varrho_{\delta,T} \ge 1$, we assume $\varrho \ge \varrho_{\delta,T}$ and multiply the equation (A.51) with Φ_{δ_0} with $0 < \delta_0 \le \delta/2 < T/4$ so that $\Phi_\delta = 1$ on $\operatorname{supp}\Phi_{\delta_0}$. If f is supported where $|x| \le \delta_0/2 < T/8$, then we obtain as before that

$$\Phi_{\delta_0} P u_{\varrho,T} = (\mathrm{Id} + R_{\delta_0,\varrho,T,2})\, f \tag{A.52}$$

where $R_{\delta_0,\varrho,T,2} = \Phi_{\delta_0} R_{\delta,\varrho,T,2}\Phi_{\delta_0}$. By Lemma A.5 there exists $0 < \delta_0 \le \delta/2$ so that $(\mathrm{Id} + R_{\delta_0,\varrho,T,2}(x,D))^{-1} = \sum_{j\ge0}(-R_{\delta_0,\varrho,T,2}(x,D))^j \cong \mathrm{Id}$ modulo an operator in $I^{-\infty}$ with C_0^∞ kernel supported where $|x| \le \delta_0$ and $|y| \le \delta_0$. By replacing f in (A.50) by $(\mathrm{Id} + R_{\delta_0,\varrho,T,2})^{-1} f$ we obtain the first part of the following result.

Proposition A.11. *Let φ_j be given by Remark A.3, $Pb_j = a_jQ_j + R_j$ by Proposition A.4 and $R_{\delta,\varrho,T,1}$ and $R_{\delta_0,\varrho,T,2}$ be given by* (A.49) *and* (A.52) *depending C^∞ on v and w. Then* *there exist* $0 < T \le 1$, $0 < \delta < T/2$ *and* $\varrho_{\delta,T} \ge 1$ *so that if* $\varrho \ge \varrho_{\delta,T}$, $0 < \delta_0 \le \delta/2$ *is small enough,* $f \in C_0^\infty$ *has support where* $\Phi_{\delta_0} = 1$ *and* $u_j \in C_0^\infty$ *solves*

$$Q_j u_j = a_j^{-1}\varphi_j(\mathrm{Id} + R_{\delta,\varrho,T,1})^{-1}(\mathrm{Id} + R_{\delta_0,\varrho,T,2})^{-1} f \qquad u_j\big|_{x_1=0} = 0 \quad \forall\, j \tag{A.53}$$

then $u_{\varrho,T}(x) = \sum_j b_j \varphi_{j,\varrho}\Phi_T u_j(x)$ *solves*

$$P u_{\varrho,T} = f \tag{A.54}$$

when $|x| \le \delta_0/2 \le T/8$ *and* $|w - w_0| \ll 1$. *We also have that*

$$u_{\varrho,T}(0) = c_\varrho(f) \qquad \partial u_{\varrho,T}(0) = d_\varrho(f) \tag{A.55}$$

where ϱc_ϱ *and* $\varrho d_\varrho \in \mathcal{D}'$ *uniformly when* $\varrho \ge 1$ *independently on* δ *and* δ_0.

Proof. In only remains to prove the statement about the values at x_0. Since $u_{\varrho,T} = \sum_j b_j u_{j,\varrho,T}$ it suffices to consider the terms $b_j u_{j,\varrho,T} = b_j\psi_{j,\varrho}\Phi_T u_j$, $\forall\, j$. By Proposition A.6 we have that $u_j = \mathbb{F}_j a_j^{-1}\varphi_j f$, which gives that

$$b_j\psi_{j,\varrho}(D)\Phi_T u_j(0) = b_j\psi_{j,\varrho}(D)\Phi_T\mathbb{F}_j a_j^{-1}\varphi_j f(0) \tag{A.56}$$

which does not depend on δ and δ_0. Now we shall use the following result.

Lemma A.12. *Let* $\phi_\varrho \in S^k$ *uniformly and supported where* $|\xi| \ge \varrho \ge 1$, *then for any* $u \in C_0^\infty$ *and* x *we find that*

$$|\phi_\varrho u(x)| \le C\varrho^{-1}\|u\|_{\left(\frac{n+2k+3}{2}\right)} \tag{A.57}$$

where the constant only depends on the seminorms of ϕ.

Proof of Lemma A.12. Since $\phi_\varrho \in S^k$ uniformly and is supported where $|\xi| \ge \varrho$, we find that $\varrho\phi_\varrho \in S^{k+1}$ uniformly when $\varrho \ge 1$. This gives by the Sobolev embedding theorem and continuity that

$$|\varrho\phi_\varrho u(x)| \le C\|\varrho\phi_\varrho u\|_{\left(\frac{n+1}{2}\right)} \le C_0\|u\|_{\left(\frac{n+2k+3}{2}\right)} \tag{A.58}$$

which gives the result. □

Since $\psi_{j,\varrho} \in S^0$ uniformly and supported where $|\xi| \ge \varrho \ge 1$ we find from Lemmas A.8 and A.12 and continuity that (A.56) is a distribution $c_{j,\varrho}$ such that $\varrho c_{j,\varrho} \in \mathcal{D}'$ uniformly when $\varrho \ge 1$. By replacing b_j by ∂b_j in (A.56) we find $\varrho d_{j,\varrho} \in \mathcal{D}'$ uniformly when $\varrho \ge 1$. □

Proof of Theorem A.1. To solve (A.1) we may first assume $x_0 = 0$ and make the reduction (A.5) to the case with vanishing data. Then we find that f is replaced by $f_0 = f - (p_1 + p_0\cdot x)u_1 - p_0 u_0$, $P = P_0$ is given by (A.6) depending on u_1 and (A.4) holds with $u_1 = 0$. Since we only need a local solutions, it is no restriction to assume that f, p_0 and p_1 have compact support. Starting with $v^0 \equiv 0$ we shall solve the linearized equation

$$P(\partial v^j(x,w), x, w, \partial)v^{j+1}(x,w) = f_0(x,w) \qquad \forall\, j \geq 0 \tag{A.59}$$

with real valued solution v^{j+1} such that $v^{j+1}(x_0,w_0)$ and $\partial_x v^{j+1}(x_0,w_0) = O(\varrho^{-1})$ depending linearly on f_0. Since we are going to construct local solutions near $x = 0$, we may cut off ∂v^j with $\Phi \in C_0^\infty$ such that $0 \leq \Phi \leq 1$, Φ is supported where $|x| \leq 1$ and equal to 1 when $|x| \leq 1/2$. This will give a solution to (A.59) when $|x| < 1/2$. As before, we shall use $\Phi_\delta(x) = \Phi(x/\delta)$, $\delta > 0$, to cut off.

To microlocalize, we use Propositions A.4 and A.11 to find $0 < T \leq 1$, $0 < \delta < T/2$ and $\varrho_{\delta,T} \geq 1$ so that if $\varrho \geq \varrho_{\delta,T}$ and $0 < \delta_0 \leq \delta/2 < 1/4$ is small enough and f_0 has support where $\Phi_{\delta_0} = 1$, then (A.59) reduces to the coupled system of equations given by (A.53):

$$Q_k(\Phi\partial v^j(x), x, D)v_k^{j+1} = a_k^{-1}\varphi_k(\operatorname{Id} + R_{\delta,\varrho,T,1})^{-1}(\operatorname{Id} + R_{\delta_0,\varrho,T,2})^{-1} f_0 \quad 1 \leq k \leq N \tag{A.60}$$

where $v^j(x) = \operatorname{Re}\sum_{k=1}^N b_k\psi_{k,\varrho}(D)\Phi v_k^j(x) \in C^\infty$ with $\varrho b_k \psi_{k,\varrho} \in S^0$ uniformly when $\varrho \geq 1$. If one cuts off f_0 with $\Phi_{\delta_0/2}$ this would give a solution to (A.59) when $|x| < \delta_0/4$. Observe that a_k^{-1}, $(1 + R^k_{\delta,\varrho,T,1})^{-1}$ and $(1 + R^k_{\delta_0,\varrho,T,2})^{-1} \in I^0$ uniformly depending C^∞ on $\Phi\partial v^j(x)$ and w.

By Proposition A.11 we find that solving (A.60) using Proposition A.6 will give a solution to (A.59) when $|x| < \delta_0/2 \leq \delta/4 \leq T/8$ such that $v_k^{j+1}(x_0)$ and $\partial_x v_k^{j+1}(x_0)$ are distributions of f_0 which are $O(\varrho^{-1})$ as $\varrho \to \infty$. We are going to prove that the solutions v^{j+1} to (A.59) are uniformly bounded in C^∞ near $x = 0$, so we can use the Arzela-Ascoli theorem to get convergence of a subsequence to a solution to the nonlinear equation (A.6).

First we obtain from Lemmas A.8 and Remark A.9 that there exists $\ell \in \mathbf{N}$ so that for any $m \in \mathbf{N}$ there exists $C_m(t) \in C^\infty(\mathbf{R}_+)$ so that

$$\|a_k^{-1}\varphi_k(\operatorname{Id} + R_{\delta,\varrho,T,1})^{-1}(\operatorname{Id} + R_{\delta_0,\varrho,T,2})^{-1} f_0\|_{(m)} \leq C_m(\|\Phi\partial v^j\|_{(\ell)})\|f_0\|_{(m)} \tag{A.61}$$

since the operators are in I^0 depending C^∞ on $\Phi\partial v^j(x)$.

By (A.60), (A.61) and Proposition A.10 we also find that there exists $\ell \in \mathbf{N}$ such that for any $m \in \mathbf{N}$ there exists $C_m(t) \in C^\infty(\mathbf{R}_+)$ so that

$$\|\Phi v_k^{j+1}\|^2_{(m)} \leq C_m(\|\Phi\partial v^j\|_{(\ell)})\|f_0\|^2_{(m)} \tag{A.62}$$

since $\Phi \in C_0^\infty$ has support when $|x| \leq 1$. Now $\varrho\Phi\partial b_k\psi_{k,\varrho} \in \Psi^0$ uniformly when $\varrho \geq 1$ and $\|\operatorname{Re} u\|_{(m)} \leq \|u\|_{(m)}$ for $u \in \mathcal{S}$, which gives

$$\|\Phi\partial v^j\|_{(m)} \leq \varrho^{-2}\widetilde{C}_m \sum_{1 \leq k \leq N} \|\Phi v_k^j\|_{(m)} \qquad \forall\, m \in \mathbf{N} \quad \forall\, \varrho \geq 1 \tag{A.63}$$

By using this for $m = \ell$ we find that

$$\|\Phi v_k^{j+1}\|_{(m)}^2 \le C_m\left(\varrho^{-2}\widetilde{C}_\ell\sum_{k=1}^N \|\Phi v_k^j\|_{(\ell)}\right)\|f_0\|_{(m)}^2 \qquad \forall\, m \in \mathbf{N} \quad \forall\, \varrho \ge 1 \tag{A.64}$$

Here we may replace $C_m(t)$ with a nondecreasing function for any m.

Thus, for any $m \in \mathbf{N}$ we will obtain uniform bounds on $\|\Phi v_k^j\|_{(m)}$ if we have uniform bounds when $m = \ell$. Since $v_k^{-1} = 0$, $\forall\, k$, we find by taking $m = \ell$ in (A.62) that

$$\|\Phi v_k^0\|_{(\ell)}^2 \le C_\ell(0)\|f_0\|_{(\ell)}^2 \tag{A.65}$$

where $C_\ell(0) \le C_\ell(1)$ since $C_\ell(t)$ is nondecreasing. If we assume for some $j \ge 0$ that

$$\|\Phi v_k^j\|_{(\ell)}^2 \le C_\ell(1)\|f_0\|_{(\ell)}^2 \tag{A.66}$$

then by choosing $\varrho^2 \ge N\widetilde{C}_\ell C_\ell(1)\|f_0\|_{(\ell)}^2$ we obtain that $\|\Phi v_k^{j+1}\|_{(\ell)} \le 1/\widetilde{C}_\ell N$. Then (A.64) with $m = \ell$ gives that (A.66) holds with j replaced by $j+1$. Since this is true for $j = 0$ we obtain by induction that (A.66) holds for any j. By (A.64) we obtain for any m uniform bounds on $\|\Phi v_k^j\|_{(m)}$ for any j, k, which by (A.63) gives that $\|\Phi v^j\|_{(m)}$ is uniformly bounded for any j.

By the Arzela-Ascoli theorem there exists a subsequence $\{v^{j_k}\}_{j_k}$ that converges in C^∞ to a real valued limit v on $\Phi^{-1}(1)$, i.e., $|x| \le 1/2$, as $j_k \to \infty$. By taking the limit of the equation (A.59) we find by continuity that

$$P(\partial v(x,w), x, w, \partial)v(x,w) = f_0(x,w) \tag{A.67}$$

when $|x| < \delta_0/4 < 1/16$. We also obtain by taking the limit that $v(x_0,w_0) = c_\varrho(f_0) \in \mathbf{R}$ and $\partial_x v(x_0,w_0) = d_\varrho(f_0) \in \mathbf{R}^n$ where ϱc_ϱ and $\varrho d_\varrho \in \mathcal{D}'$ uniformly when $\varrho \ge 1$.

This means that we have a solution the original equation (A.1) when $|x| < \delta_0/4$ with f_0 replaced by $f - (p_1 + p_0 x)\cdot u_1 - p_0 u_0$ by (A.6) and v replaced by $v + u_0 + u_1 \cdot x$ by (A.5), which gives by linearity that $v(0,w_0) = u_0 + c_\varrho(f) - c_\varrho(p_0)u_0 - c_\varrho(p_1 + p_0 x)\cdot u_1$ and $\partial_x v(0,w_0) = u_1 + d_\varrho(f) - d_\varrho(p_0)u_0 - d_\varrho(p_1 + p_0 x)\cdot u_1$. If we replace u_j with indeterminate w_j for $j = 1$ and 2, then we obtain the linear system

$$\begin{cases} v(0,w_0) = (1 + \varrho^{-1}a_\varrho)w_0 + \varrho^{-1}b_\varrho \cdot w_1 + \varrho^{-1}c_\varrho = u_0 \\ \partial_x v(0,w_0) = \varrho^{-1}d_\varrho w_0 + (\mathrm{Id}_n + \varrho^{-1}e_\varrho)w_1 + \varrho^{-1}f_\varrho = u_1 \end{cases} \tag{A.68}$$

where the coefficients $a_\varrho, \dots, f_\varrho$ are uniformly bounded when $\varrho \ge 1$. Observe that b_ϱ is a $1\times n$, d_ϱ and f_ϱ are $n \times 1$ and e_ϱ is an $n \times n$ matrix. This is a linear $(n+1)\times(n+1)$ system in (w_0, w_1) which converges to the identity when $\varrho \to \infty$. Thus, there exists $\varrho_{iv} \ge 1$ so that (A.68) has a unique solution w_0, w_1 that is uniformly bounded when $\varrho \ge \varrho_{iv}$. By solving (A.6) with u_j replaced by w_j, $j = 1$, 2, when $\varrho \ge \varrho_{iv}$ we get a solution to (A.1) which finishes the proof of Theorem A.1. $\square$

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