LONG-TIME DYNAMICS OF A COMPETITION MODEL WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES: VANISHING AND SPREADING OF THE INVADER

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ABSTRACT. In this work, we investigate the long-time dynamics of a two species competition model of Lotka-Volterra type with nonlocal diffusions. One of the species, with density $v(t,x)$, is assumed to be a native in the environment (represented by the real line $\mathbb{R}$), while the other species, with density $u(t,x)$, is an invading species which invades the territory of v with two fronts, $x=g(t)$ on the left and $x=h(t)$ on the right. So the population range of u is the evolving interval $[g(t),h(t)]$ and the reaction-diffusion equation for u has two free boundaries, with $g(t)$ decreasing in t and $h(t)$ increasing in t, and the limits $h_\infty:=h(\infty)\leq\infty$ and $g_\infty:=g(\infty)\geq-\infty$ thus always exist. We obtain detailed descriptions of the long-time dynamics of the model according to whether $h_\infty-g_\infty$ is ∞ or finite. In the latter case, we reveal in what sense the invader u vanishes in the long run and v survives the invasion, while in the former case, we obtain a rather satisfactory description of the long-time asymptotic limit for both $u(t,x)$ and $v(t,x)$ when a certain parameter k in the model is less than 1. This research is continued in a separate work, where sharp criteria are obtained to distinguish the case $h_\infty-g_\infty=\infty$ from the case $h_\infty-g_\infty$ is finite, and new phenomena are revealed for the case $k\geq 1$. The techniques developed in this paper should have applications to other models with nonlocal diffusion and free boundaries.

Keywords: Nonlocal diffusion; Competition; Free boundary.

AMS Subject Classification (2000): 35K57, 35R20

1. INTRODUCTION

We are interested in the long-time dynamics of the following Lotka-Volterra type competition model with nonlocal diffusion and free boundaries

$$
\begin{cases}
u_t = d_1\displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)\mathrm{d}y - d_1u + u(1-u-kv), & t>0,\ g(t)<x<h(t),\\
v_t = d_2\displaystyle\int_{\mathbb{R}} J_2(x-y)v(t,y)\mathrm{d}y - d_2v + \gamma v(1-v-hu), & t>0,\ x\in\mathbb{R},\\
u(t,x)=0, & t\geq 0,\ x\notin (g(t),h(t)),\\
h'(t)=\mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
g'(t)=-\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
h(0)=-g(0)=h_0>0,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x), & x\in\mathbb{R},
\end{cases}
\tag{1.1}
$$

where d_1,d_2,h,k,γ,μ are given positive constants, and the initial functions satisfy

$$
u_0\in C(\mathbb{R}),\ u_0(x)=0 \text{ for } |x|\geq h_0,\ u_0(x)>0 \text{ for } |x|<h_0,\ v_0\in C_b(\mathbb{R}),\ v_0(x)\geq 0 \text{ in } \mathbb{R},
\tag{1.2}
$$

where $C_b(\mathbb{R})$ is the space of continuous and bounded functions in $\mathbb{R}$. The kernel functions J_1,J_2 are assumed to satisfy

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$$(\mathbf{J}): \quad J_i \in C_b(\mathbb{R}),\ J_i(x) = J_i(-x) \geq 0,\ J_i(0) > 0,\ \int_{\mathbb{R}} J_i(x)dx = 1 \text{ for } i = 1, 2.$$

System (1.1) may be viewed as a model describing the invasion of a species with density u into an environment (represented by $\mathbb{R}$ here) where a native competitor, with density v, has already appeared. The population range of u is given by the time-dependent interval $[g(t), h(t)]$, with $x = g(t)$ and $x = h(t)$ known as the free boundaries in the model, which represent the range boundary of u, or its invading fronts.

The corresponding local diffusion version of (1.1) was first studied in [10]. As will be explained below, the nonlocal diffusion model (1.1) poses several technical difficulties in the mathematical treatment, and is capable of exhibiting strikingly different behaviour from its local diffusion correspondent.

We remark that (1.1) is a reduced version of the following equivalent but more general looking system:

$$(1.3)\quad \begin{cases} U_t = D_1 \displaystyle\int_{G(t)}^{H(t)} J_1(x-y)U(t,y)\mathrm{d}y - D_1 U + U(a_1 - b_1 U - c_1 V), & t > 0,\ G(t) < x < H(t), \\ V_t = D_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)V(t,y)\mathrm{d}y - D_2 V + V(a_2 - b_2 V - c_2 U), & t > 0,\ x \in \mathbb{R}, \\ U(t,x) = 0, & t \geq 0, x \notin (G(t), H(t)), \\ H'(t) = \hat{\mu} \displaystyle\int_{G(t)}^{H(t)} \int_{H(t)}^{\infty} J_1(x-y)U(t,x)\mathrm{d}y\mathrm{d}x, & t > 0, \\ G'(t) = -\hat{\mu} \displaystyle\int_{G(t)}^{H(t)} \int_{-\infty}^{G(t)} J_1(x-y)U(t,x)\mathrm{d}y\mathrm{d}x, & t > 0, \\ H(0) = -G(0) = H_0 > 0, U(0,x) = U_0(x), & -H_0 \leq x \leq H_0, \\ V(0,x) = V_0(x), & t \geq 0, x \in \mathbb{R}, \end{cases}$$

Indeed, let

$$u(t,x) = \frac{b_1}{a_1} U\left(\frac{t}{a_1}, x\right),\ v(t,x) = \frac{b_2}{a_2} V\left(\frac{t}{a_1}, x\right),\ h(t) = H\left(\frac{t}{a_1}\right),\ g(t) = G\left(\frac{t}{a_1}\right),$$

$$d_1 = \frac{D_1}{a_1},\ d_2 = \frac{D_2}{a_1},\ \gamma = \frac{a_2}{a_1},\ k = \frac{a_2 c_1}{a_1 b_2},\ h = \frac{a_1 c_2}{a_2 b_1},\ \mu = \frac{1}{b_1}\hat{\mu}.$$

Then a simple calculation shows that (1.3) is reduced to (1.1).

Several closely related models have been studied recently. In [16], the situation that the population range of v is the same evolving interval $[g(t), h(t)]$ was considered, and a technical difficulty in treating the nonlocal competition model has been revealed: Due to the lack of compactness of the solutions of the nonlocal diffusion problem, the fact $[g(t), h(t)]$ remains uniformly bounded for all time $t > 0$, does not lead to the conclusion of vanishing as in the corresponding local diffusion models. To recover the vanishing property, [16] relied on the following extra assumption

$$(1.4)\qquad J_i(x) > 0 \text{ for all } x \in \mathbb{R},\ i = 1, 2,$$

and a trick relating the left and right derivatives of $M(t) := \max_{x \in [g(t),h(t)]} u(t,x)$ to $u_t(t, \xi(t))$ for some suitable $\xi(t) \in [g(t), h(t)]$. It was conjectured in [16] that the assumption (1.4) is unnecessary.

In [7], the same system (1.1) was considered, while [27] considered the special case $d_1 = d_2$ and $J_1 = J_2$. In [20], a predator-prey system with both species evolving over the same interval $[0, h(t)]$ was investigated. In these papers, the existence and uniqueness of the solution was established, together with various results on the long-time dynamics of the model for certain selected ranges of the parameters. However, to prove vanishing of u, they used the same trick as in [16] and made the same additional assumption (1.4) for $J_1(x)$.

In this work, we will introduce new techniques to treat (1.1) without requiring (1.4), and obtain precise description of the long-time dynamics of the model for broader parameter ranges. Moreover, we will reveal several new behaviours of the model.

Let us start by recalling the following well-posedness result:

Theorem A. ([7]) *Assume* (**J**) *holds, and the initial functions satisfy* (1.2). *Then* (1.1) *admits a unique solution* (u, v, g, h) *defined for all* $t > 0$.

For the long-time dynamics, we will use some terminologies arising from the corresponding ODE version of (1.1), namely

$$\begin{cases} u' = u(1 - u - kv), \\ v' = \gamma v(1 - v - hu), \\ u(0) > 0, \ v(0) > 0, \end{cases} \tag{1.5}$$

which always has the trivial equilibrium $R_0 = (0, 0)$ and semi-trivial equilibria $R_1 = (1, 0)$ and $R_2 = (0, 1)$. Moreover, if $\min\{h, k\} > 1$ or $\max\{h, k\} < 1$, the problem has a unique positive equilibrium $R_* := (\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$. Regarding the long-time dynamics of (1.5), the following conclusions are well known:

(1) R_0 is always unstable;
(2) when $\max\{h, k\} < 1$, R_* is globally asymptotically stable;
(3) when $k < 1 < h$, R_1 is globally asymptotically stable;
(4) when $h < 1 < k$, R_2 is globally asymptotically stable;
(5) when $\min\{h, k\} > 1$, both R_1 and R_2 are locally asymptotically stable while R_* is unstable.

In case (2), the competitors co-exist in the long run, and it is often referred to as the *weak competition* case, where no competitor wins or loses in the competition. Cases (3) and (4) are known as the *weak-strong competition* cases. In case (3), the competitor u wipes v out in the long run and wins the competition; so we will call u the strong competitor and v the weak competitor. Analogously u is the weak competitor and v is the strong competitor in case (4). Case (5) is the *strong competition* case, and depending on the location of $(u(0), v(0))$ in the first quadrant of $\mathbb{R}^2$, one of R_1 and R_2 is the global attractor of (1.5) (except when $(u(0), v(0))$ lies on the one dimensional stable manifold of R_*). We will keep using these terminologies which are determined solely by h and k for (1.1).

To state our main results in this paper, we also need to recall some properties of the principle eigenvalue for some associated nonlocal diffusion operators. For $\Omega = (a, b)$ a finite interval, under our assumption (**J**), it is well known that the following eigenvalue problem

$$\lambda\varphi = \mathcal{L}_\Omega[\varphi](x) := d_1 \left[\int_\Omega J_1(x - y)\varphi(y)\mathrm{d}y - \varphi(x) \right], \ \varphi \in C(\overline{\Omega}),$$

has a principal eigenvalue $\lambda = \lambda_p(\mathcal{L}_\Omega)$ associated with a positive eigenfunction φ (e.g., [5,8,18]), and it has the following properties:

Proposition B. ([6, Proposition 3.4]) *Assume that* $l > 0$, *and* J_1 *satisfies* (**J**). *Then*

(i) $\lambda_p(\mathcal{L}_{(a,a+l)}) = \lambda_p(\mathcal{L}_{(0,l)})$ *for all* $a \in \mathbb{R}$,
(ii) $\lambda_p(\mathcal{L}_{(0,l)})$ *is strictly increasing and continuous in* l,
(iii) $\lim\limits_{l\to\infty} \lambda_p(\mathcal{L}_{(0,l)}) = 0$,
(iv) $\lim\limits_{l\to 0} \lambda_p(\mathcal{L}_{(0,l)}) = -d_1$.

We are now ready to state our main results in this paper. Let (u, v, g, h) be the solution of (1.1). Then $g_\infty := \lim_{t\to\infty} g(t) \in [-\infty, -h_0)$ and $h_\infty := \lim_{t\to\infty} h(t) \in (h_0, \infty]$ always exist. We will describe the long-time dynamics of (1.1) according to the following two cases:

$$(\mathbf{a})\colon h_\infty - g_\infty < \infty, \quad (\mathbf{b})\colon h_\infty - g_\infty = \infty.$$

In case **(a)** the limit of the population range of u is finite and one expects u to vanish in the long run, while in case **(b)**, the limit of the size of the population range of u is infinite, and successful invasion of u is expected.

For case **(a)**, by introducing new techniques for the analysis of the function

$$U(t, x) := \int_{x-L}^{x+L} u(t, y)\mathrm{d}y$$

for suitable values of L, we will prove the following result.

Theorem 1.1. *Assume that* $(\mathbf{J})$ *holds and* (u, v, g, h) *is the unique solution of* (1.1). *If* $h_\infty - g_\infty < \infty$, *then necessarily*

$$d_1 > 1 - k \text{ and } \lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}) \leq k - 1; \tag{1.6}$$

moreover

$$\begin{cases} \lim\limits_{t\to\infty} \displaystyle\int_{\mathbb{R}} u(t,x)dx = 0, \\ \lim\limits_{t\to\infty} \displaystyle\int_{L}^{L} |v(t,x) - 1)|dx = 0 \text{ for every } L > 0, \\ \lim\limits_{t\to\infty} v(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R} \setminus (g_\infty, h_\infty). \end{cases} \tag{1.7}$$

One naturally wonders whether in Theorem 1.1 the conclusions in (1.7) imply the following stronger statements, as in the corresponding local diffusion model [10]:

$$\lim_{t\to\infty} \max_{x\in[g(t),h(t)]} u(t,x) = 0 \quad \text{and} \quad \lim_{t\to\infty} v(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R}. \tag{1.8}$$

It turns out that this question is not easy to answer. Using (1.7) and viewing (1.1) as a perturbation of the corresponding ODE problem (1.5), we are able to obtain a partial answer to this question.

Let us now be more precise. Denote $\tilde{d}_1 := d_1 + k - 1$ and

$$F(s) := \gamma(1 - hk)s^2 + [\tilde{d}_1\gamma h - \gamma(1 - hk) - d_2]s - \tilde{d}_1\gamma h, \tag{1.9}$$

which arises from the analysis of (1.5). Define the sets Θ_1 and Θ_2 by

$$\begin{cases} \Theta_1 := \{(\gamma, h, k, d_1, d_2) \in \mathbb{R}_+^5 : \ F(s) \neq 0 \text{ for } s \in [0, 1]\}, \\ \Theta_2 := \{(\gamma, h, k, d_1, d_2) \in \mathbb{R}_+^5 : \ F(s) = 0 \text{ has at least one root in } [0, 1]\}. \end{cases} \tag{1.10}$$

Clearly $\Theta_2 = \mathbb{R}_+^5 \setminus \Theta_1$, so for any given parameters (γ, h, k, d_1, d_2) in (1.1), it belongs to either Θ_1 or Θ_2. It is easy to show (see Remark 3.2 below) that $(\gamma, h, k, d_1, d_2) \in \Theta_1$ if

$$d_1 \geq 1 \text{ or } kh \leq 1 + \frac{d_2}{\gamma}.$$

(So in the weak competition case where $k, h \in (0, 1)$, we always have $(\gamma, h, k, d_1, d_2) \in \Theta_1$.)

Regarding the above question on the validity of (1.8), we have the following answer.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the following conclusions hold:*

(i) *If* $d_1 \geq 1$ *or if* $d_1 < 1$ *and* $(\gamma, h, k, d_1, d_2) \in \Theta_1$, *then* (1.8) *holds.*

(ii) *If* $d_1 < 1$ *and* $(\gamma, h, k, d_1, d_2) \in \Theta_2$, *then either* (1.8) *still holds or there is an open set* $\Omega \subset (g_\infty, h_\infty)$ *with* $|\Omega| = h_\infty - g_\infty$, $\Omega \neq (g_\infty, h_\infty)$, *such that*

$$\lim_{t\to\infty} (u(t,x), v(t,x)) = \begin{cases} (0, 1) & \text{locally uniformly for } x \in \Omega, \\ (kx_* - \tilde{d}_1, 1 - x_*) & \text{for } x \in (g_\infty, h_\infty)\setminus\Omega, \end{cases} \tag{1.11}$$

where $x_* \in (0, 1)$ *is the smallest positive root of* $F(s) = 0$ *in* $[0, 1]$.

Remark 1.3. *We do not know whether* (1.11) *can actually happen, and conjecture that it never happens.*

For the case $h_\infty - g_\infty = \infty$, we will prove the following result.

Theorem 1.4. *Assume that* $(\mathbf{J})$ *holds and* (u, v, g, h) *is the unique solution of* (1.1). *If* $h_\infty - g_\infty = \infty$ *and* $k < 1$, *then* $h_\infty = \infty$, $g_\infty = -\infty$ *and*

$$\lim_{t\to\infty} (u(t,x), v(t,x)) = \begin{cases} (1, 0) & \text{if } h \geq 1, \\ (\frac{1-k}{1-hk}, \frac{1-h}{1-hk}) & \text{if } h < 1, \end{cases}$$

where the convergence is locally uniform for $x \in \mathbb{R}$.

The assumption $k<1$ in Theorem 1.4 cannot be removed. In a separate following up work [14], we will show that, when $k\geq 1$, it is possible to have $h_\infty=\infty$ while g_∞ is finite. Since (1.1) generates a monotone dynamical system, it is possible to obtain sharp criteria for $h_\infty-g_\infty=\infty$ or $h_\infty-g_\infty<\infty$ to happen; this will also be carried out in this following up paper.

For nonlocal diffusion models with free boundary, compared with the corresponding models with local diffusion, a new phenomena occurs on the asymptotic spreading speed, namely accelerated spreading may happen. For the Fisher-KPP single species model, and for certain cooperative models, such a phenomena was investigated recently in [9, 11–13, 15]. We leave the study of this behaviour of the competition system here to future work.

There are extensive recent works on competition systems with nonlocal diffusion over a fixed bounded domain or over the entire Euclidean space. Since no free boundaries appear in these situations, compared to our work here, usually significantly different techniques are used; indeed, the technical difficulties here do not appear in these problems. For works on a bounded domain, we mention as examples [1–3, 19] and the references therein. For works on nonlocal competition systems over the entire space, an incomplete sample includes [17, 21, 22, 24, 26] and the references therein. In [17], the authors obtained conditions for coexistence and extinction of the species, and also considered the system in higher dimensions. In [21], the authors constructed entire solutions behaving like two monotone traveling-wave solutions moving toward each other from the ancient time $-\infty$, giving rise to the phenomena of successful invasion of a species with speed c_1 from $x=-\infty$ and with speed c_2 from $x=+\infty$. In [24] and [22], the propagation behaviour under a shifting environment was investigated. In [26], a strong competition system was considered, where interesting results on the stability of bistable traveling waves and the long-time propagation behaviour of the system were obtained.

The rest of this paper is organised as follows. In Sections 2 and 3, we prove Theorems 1.1 and 1.2, respectively. The proof of Theorem 1.4 is given in Section 4.

2. Proof of Theorem 1.1

We prove Theorem 1.1 by a sequence of lemmas.

Lemma 2.1. *If $h_\infty-g_\infty<\infty$, then* (1.6) *holds.*

Proof. To prove the second inequality in (1.6), we argue indirectly: Assume, on the contrary, that $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)})>k-1$. Then $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}>k(1+\epsilon)-1$ for sufficiently small $\epsilon>0$, say $\epsilon\in(0,\epsilon_1)$. For such ϵ, we can find $T_\epsilon>0$ such that

$$\begin{aligned}&h(t)>h_\infty-\epsilon,\ g(t)<g_\infty+\epsilon\quad\text{for } t>T_\epsilon,\\&v(t,x)<1+\epsilon\quad\text{for } t>T_\epsilon,\ x\in\mathbb{R}.\end{aligned}$$

Let w_ϵ be the unique solution of the auxiliary problem

$$\begin{cases}w_t=d_1\displaystyle\int_{g_\infty+\epsilon}^{h_\infty-\epsilon}J_1(x-y)w(t,y)\mathrm{d}y-d_1w+w(1-w-k(1+\epsilon)), & t>T_\epsilon,\ x\in[g_\infty+\epsilon,h_\infty-\epsilon],\\ w(T_\epsilon,x)=u(T_\epsilon,x), & x\in[g_\infty+\epsilon,h_\infty-\epsilon].\end{cases}$$

Since $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)})>k(1+\epsilon)-1$, from [6, Proposition 3.5], it follows that w_ϵ converges to the unique positive steady-state $W_\epsilon(x)$ uniformly in $[g_\infty+\epsilon,h_\infty-\epsilon]$ as $t\to\infty$. Moreover, a simple comparison argument yields that $u(t,x)\geq w_\epsilon(t,x)$ for $t>T_\epsilon$, and $x\in[g_\infty+\epsilon,h_\infty-\epsilon]$. Therefore, there exists $T_{\epsilon_1}>T_\epsilon$ such that

$$u(t,x)\geq\frac12 W_\epsilon(x)>0,\ \forall\ t\geq T_{\epsilon_1},\ x\in[g_\infty+\epsilon,h_\infty-\epsilon].$$

It follows that, for $t\geq T_{\epsilon_1}$,

$$h'(t)=\mu\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty}J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&\geq \mu \int_{h_\infty-2\epsilon}^{h_\infty-\epsilon}\int_{h_\infty}^{h_\infty+\epsilon} J_1(x-y)\frac{1}{2}W_\epsilon(x)dydx\\
&\geq \mu \int_{h_\infty-2\epsilon}^{h_\infty-\epsilon}\int_{h_\infty}^{h_\infty+\epsilon} \min_{z\in[-3\epsilon,-\epsilon]} J_1(z)\frac{1}{2}\min_{z\in[0,h_\infty]} W_\epsilon(z)dydx\\
&= \mu\epsilon^2 \min_{z\in[-3\epsilon,-\epsilon]} J_1(z)\frac{1}{2}\min_{z\in[0,h_\infty]} W_\epsilon(z) =: \sigma_\epsilon > 0
\end{aligned}
$$

provided that $\epsilon > 0$ is sufficiently small (recall that $J_1(0) > 0$). But this implies $h_\infty = \infty$, a contradiction. Thus $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}) \leq k-1$ holds true. This proves the first inequality in (1.6).

The first inequality in (1.6) is a consequence of the second. Indeed, by Proposition B, we have $-d_1 < \lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}) \leq k-1$, i.e., $d_1 > 1-k$. The proof is finished. □

Our analysis below will make use of Barbalat's Lemma [4], which is recalled below and can be proved by elementary calculus directly.

Lemma 2.2 (Barbalat's Lemma)**.** *Suppose that* $\psi : [0,\infty) \to \mathbb{R}$ *is uniformly continuous and that* $\lim_{t\to\infty}\int_0^t \psi(s)\mathrm{d}s \in \mathbb{R}$ *exists. Then* $\lim_{t\to\infty}\psi(t) = 0$.

Lemma 2.3. *If* $h_\infty - g_\infty < \infty$*, then*

$$
\lim_{t\to\infty}\int_{\mathbb{R}} u(t,y)\mathrm{d}y = 0. \tag{2.1}
$$

Proof. By the argument in the proof of Lemma 3.1 in [16], we obtain that $\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0$. Due to $J_i(0) > 0$, there is small $\epsilon > 0$ such that $\inf_{x\in[-\epsilon,\epsilon]} J_1(x) > 0$. Then for large $t > 0$,

$$
\begin{aligned}
h'(t) &= \mu \int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x\\
&\geq \mu \int_{h(t)-\epsilon/2}^{h(t)}\int_{h(t)}^{h(t)+\epsilon/2} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x\\
&= \mu \int_{h(t)-\epsilon/2}^{h(t)}\int_{x-(h(t)+\epsilon/2)}^{x-h(t)} J_1(z)u(t,x)\mathrm{d}z\mathrm{d}x\\
&\geq \mu\frac{\epsilon}{2}\inf_{x\in[-\epsilon,\epsilon]} J_1(x)\int_{h(t)-\epsilon/2}^{h(t)} u(t,x)\mathrm{d}x \geq 0,
\end{aligned}
$$

which implies

$$
\lim_{t\to\infty}\int_{h(t)-\epsilon/2}^{h(t)} u(t,x)\mathrm{d}x = 0 \ \text{ and hence } \ \lim_{t\to\infty}\int_{h_\infty-\epsilon/4}^{\infty} u(t,x)\mathrm{d}x = 0. \tag{2.2}
$$

Similarly, using $\lim_{t\to\infty} g'(t) = 0$ we deduce

$$
\lim_{t\to\infty}\int_{g(t)}^{g(t)+\epsilon/2} u(t,x)\mathrm{d}x = 0 \ \text{ and hence } \ \lim_{t\to\infty}\int_{-\infty}^{g_\infty+\epsilon/4} u(t,x)\mathrm{d}x = 0. \tag{2.3}
$$

In the following, we show that

$$
\lim\nolimits_{t\to\infty} U(t,x) = 0 \text{ for every } x \in \mathbb{R}, \tag{2.4}
$$

where

$$
U(t,x) := \int_{x-L}^{x+L} u(t,y)\mathrm{d}y, \ \ L = \epsilon/8.
$$

Due to (2.2), (2.3) and the fact that $u(t,x) \equiv 0$ for $t \geq 0$ and $x \in \mathbb{R}\backslash[g(t),h(t)]$, clearly (2.4) holds for $x \in \mathbb{R}\backslash(g_\infty + 3\epsilon/16, h_\infty - 3\epsilon/16)$. It remains to show (2.4) for $x \in (g_\infty + 3\epsilon/16, h_\infty - 3\epsilon/16)$.

In view of the equation satisfied by u, we deduce for large t and $x \in [g_\infty + 3\epsilon/16, h_\infty - 3\epsilon/16]$,

$$U_t = d_1 \int_{x-L}^{x+L} \int_{g(t)}^{h(t)} J_1(y-z)u(t,z)\mathrm{d}z\mathrm{d}y + \int_{x-L}^{x+L} [-d_1 u + u(1-u-kv)]\mathrm{d}y. \tag{2.5}$$

From the boundedness of u and v, we easily see from the above equation that $|U_t(t,x)|$ is uniformly bounded for all large t and $x \in [g_\infty + 3\epsilon/16, h_\infty - 3\epsilon/16]$. Hence for any fixed $x \in [g_\infty + 3\epsilon/16, h_\infty - 3\epsilon/16]$, $t \to U(t,x)$ is uniformly continuous in t for all large t, and hence for all $t \in [0,\infty)$.

Now take $x_0 = h_\infty - 3\epsilon/16$ and recall that (2.4) holds with $x = x_0$; so we have

$$\lim_{t\to\infty} \int_0^t U_t(s,x_0)ds = \lim_{t\to\infty} U(t,x_0) - U(0,x_0) = -U(0,x_0).$$

We may now apply Lemma 2.2 with $\psi(t) = U_t(t,x_0)$ to conclude that $\lim_{t\to\infty} U_t(t,x_0) = 0$. Moreover, using (2.4) with $x = x_0$ we further obtain

$$\lim_{t\to\infty} \int_{x_0-L}^{x_0+L} [-d_1 u + u(1-u-kv)]\mathrm{d}y = 0.$$

Hence we can use (2.5) to deduce

$$\lim_{t\to\infty} d_1 \int_{x_0-L}^{x_0+L} \int_{g(t)}^{h(t)} J_1(y-z)u(t,z)\mathrm{d}z\mathrm{d}y = 0.$$

It follows, due to $\inf_{x\in[-\epsilon,\epsilon]} J_1(x) > 0$ and $L = \epsilon/8$, that

$$\begin{aligned} 0 = \lim_{t\to\infty} \int_{x_0-L}^{x_0+L} \int_{g(t)}^{h(t)} J_1(y-z)u(t,z)\mathrm{d}z\mathrm{d}y &\geq \limsup_{t\to\infty} \int_{x_0-L}^{x_0+L} \int_{x_0-2L}^{x_0+2L} J_1(y-z)u(t,z)\mathrm{d}z\mathrm{d}y \\ &\geq 2L \inf_{x\in[-\epsilon,\epsilon]} J_1(x) \limsup_{t\to\infty} \int_{x_0-2L}^{x_0+2L} u(t,z)\mathrm{d}z \geq 0, \end{aligned}$$

which implies

$$\lim_{t\to\infty} \int_{x_0-2L}^{x_0+2L} u(t,z)\mathrm{d}z = 0.$$

In particular, $\lim_{t\to\infty} U(t,x) = 0$ for all $x \in [x_0 - L, x_0]$.

Now we may repeat the above argument with x_0 replaced by $x_1 := x_0 - L$, and similarly show $\lim_{t\to\infty} U(t,x) = 0$ for $x \in [x_1 - L, x_1] = [x_0 - 2L, x_0 - L]$.

Analogously we may take $y_0 = g_\infty + 3\epsilon/16$ and show that (2.4) holds for $x \in [y_0, y_0 + L]$ and then continue to show that (2.4) holds for $x \in [y_0 + L, y_0 + 2L]$, etc.

Clearly after finitely many steps we reach the conclusion that (2.4) holds for all $x \in [y_0, x_0]$, as desired. We have thus proved that (2.4) holds for every $x \in \mathbb{R}$, namely

$$\lim_{t\to\infty} \int_{x-L}^{x+L} u(t,y)dy = 0 \text{ for every } x \in \mathbb{R}.$$

Since $u(t,y) = 0$ for all $t > 0$ and $y \notin [g_\infty, h_\infty]$, this implies (2.1). The proof of the lemma is now complete. □

Remark 2.4. *If $g_\infty = -\infty$ and $h_\infty < +\infty$, then we could take $x_1 = x_0 - L$, ... $x_{n+1} = x_n - L$ and repeat the argument in the proof of Lemma 2.3 finitely many times to obtain*

$$\lim_{t\to\infty} \int_{-M}^{\infty} u(t,x)dx = 0 \textit{ for every } M > 0.$$

When $g_\infty > -\infty$ and $h_\infty = +\infty$, we can show

$$\lim_{t\to\infty} \int_{-\infty}^{M} u(t,x)dx = 0 \textit{ for every } M > 0.$$

To prove the next lemma, we will need a trick introduced in the proof of Theorem 3.3 in [16], which is formulated in a more general form below.

Lemma 2.5. *Suppose that $s_1(t)$ and $s_2(t)$ are continuous bounded functions over $[0,\infty)$ satisfying $s_1(t) < s_2(t)$ for all $t \geq 0$. Let $U(t,x)$ be a continuous bounded function over $\Omega := \{(t,x) : t \geq 0,\ x \in [s_1(t), s_2(t)]\}$, with $U_t(t,x)$ also continuous in Ω. Then there exist sequences $(\underline{t}_n, \underline{x}_n)$ and $(\bar{t}_n, \bar{x}_n)$ with $\underline{x}_n \in [s_1(\underline{t}_n), s_2(\underline{t}_n)]$, $\bar{x}_n \in [s_1(\bar{t}_n), s_2(\bar{t}_n)]$ and $\lim_{n\to\infty} \underline{t}_n = \lim_{n\to\infty} \bar{t}_n = \infty$ such that*

$$\begin{cases} \lim_{n\to\infty} U(\underline{t}_n, \underline{x}_n) = \underline{U},\ \lim_{n\to\infty} U_t(\underline{t}_n, \underline{x}_n) = 0, \\ \lim_{n\to\infty} U(\bar{t}_n, \bar{x}_n) = \overline{U},\ \lim_{n\to\infty} U_t(\bar{t}_n, \bar{x}_n) = 0, \end{cases}$$

where

$$\underline{U} := \liminf_{t\to\infty} \min_{x\in[s_1(t),s_2(t)]} U(t,x),\ \overline{U} := \limsup_{t\to\infty} \max_{x\in[s_1(t),s_2(t)]} U(t,x).$$

Proof. We only prove the existence of $(\bar{t}_n, \bar{x}_n)$ since the existence of $(\underline{t}_n, \underline{x}_n)$ then follows by considering the function $V(t,x) = -U(t,x)$.

Denote

$$M(t) := \max_{x\in[s_1(t),s_2(t)]} U(t,x) \text{ and } X(t) := \{x \in [s_1(t), s_2(t)] : U(t,x) = M(t)\}.$$

Then $X(t)$ is a compact set for each $t > 0$. Therefore, there exist $\underline{\xi}(t),\ \overline{\xi}(t) \in X(t)$ such that

$$U_t(t, \underline{\xi}(t)) = \min_{x\in X(t)} U_t(t,x), \quad U_t(t, \overline{\xi}(t)) = \max_{x\in X(t)} U_t(t,x).$$

We claim that $M(t)$ satisfies, for each $t > 0$,

$$\begin{cases} M'(t+0) := \lim_{s>t, s\to t} \dfrac{M(s) - M(t)}{s-t} = U_t(t, \overline{\xi}(t)), \\ M'(t-0) := \lim_{s<t, s\to t} \dfrac{M(s) - M(t)}{s-t} = U_t(t, \underline{\xi}(t)). \end{cases} \tag{2.6}$$

Indeed, for any fixed $t > 0$ and $s > t$, we have

$$U(s, \overline{\xi}(t)) - U(t, \overline{\xi}(t)) \leq M(s) - M(t) \leq U(s, \overline{\xi}(s)) - U(t, \overline{\xi}(s)).$$

It follows that

$$\liminf_{s>t, s\to t} \frac{M(s) - M(t)}{s-t} \geq U_t(t, \overline{\xi}(t)), \tag{2.7}$$

and

$$\limsup_{s>t, s\to t} \frac{M(s) - M(t)}{s-t} \leq \limsup_{s>t, s\to t} \frac{U(s, \overline{\xi}(s)) - U(t, \overline{\xi}(s))}{s-t}.$$

Let $s_n \searrow t$ satisfy

$$\lim_{n\to\infty} \frac{U(s_n, \overline{\xi}(s_n)) - U(t, \overline{\xi}(s_n))}{s_n - t} = \limsup_{s>t, s\to t} \frac{U(s, \overline{\xi}(s)) - U(t, \overline{\xi}(s))}{s-t}.$$

By passing to a subsequence if necessary, we may assume that $\overline{\xi}(s_n) \to \xi$ as $n \to \infty$. Then $U(t,\xi) = \lim_{n\to\infty} M(s_n) = M(t)$ and hence $\xi \in X(t)$. Due to the continuity of $U_t(t,x)$, it follows immediately that

$$\lim_{n\to\infty} \frac{U(s_n, \overline{\xi}(s_n)) - U(t, \overline{\xi}(s_n))}{s_n - t} = U_t(t,\xi) \leq U_t(t, \overline{\xi}(t)).$$

We thus obtain

$$\limsup_{s>t, s\to t} \frac{M(s) - M(t)}{s-t} \leq U_t(t, \overline{\xi}_i(t)).$$

Combining this with (2.7) we obtain

$$M'(t+0) = U_t(t, \overline{\xi}(t)).$$

Analogously we can show

$$M'(t-0) = U_t(t, \underline{\xi}(t)).$$

Let us note from (2.6) that $M'(t-0) \le M'(t+0)$ for all $t > 0$. Therefore if $M(t)$ has a local maximum at $t = t_0$, then $M'(t_0)$ exists and $M'(t_0) = 0$.

Regarding the function $M(t)$ we have three possibilities:

(a) it has a sequence of local maxima $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} M(t_n) = \limsup_{t\to\infty} M(t),$$

(b) it is monotone nondecreasing for all large t and so $\lim_{t\to\infty} M(t)$ exists,

(c) it is monotone nonincreasing for all large t and so $\lim_{t\to\infty} M(t)$ exists.

In case (a) we take $\bar{t}_n = t_n,\ \bar{x}_n = \overline{\xi}(t_n)$ and so

$$U_t(\bar{t}_n, \bar{x}_n) = M'(t_n) = 0,\ U(\bar{t}_n, \bar{x}_n) = U(t_n, \overline{\xi}(t_n)) = M(t_n) \to \limsup_{t\to\infty} M(t) \text{ as } n \to \infty.$$

In case (b) necessarily $M'(t_n - 0) \to 0$ along some sequence $t_n \to \infty$ for otherwise $M'(t+0) \ge M'(t-0) \ge \delta > 0$ for some $\delta > 0$ and all large t, which leads to the contradiction $M(t) \to \infty$ as $t \to \infty$. We now take $\bar{t}_n = t_n$ and $\bar{x}_n = \underline{\xi}(t_n)$, and obtain

$$\begin{cases} U_t(\bar{t}_n, \bar{x}_n) = U_t(t_n, \underline{\xi}(t_n)) = M'(t_n - 0) \to 0, \\ U(\bar{t}_n, \bar{x}_n) = U(t_n, \underline{\xi}(t_n)) = M(t_n) \to \lim_{t\to\infty} M(t) \text{ as } n \to \infty. \end{cases}$$

In case (c), necessarily $M'(s_n + 0) \to 0$ along some sequence $s_n \to \infty$ for otherwise $M'(t-0) \le M'(t+0) \le -\delta < 0$ for some $\delta > 0$ and all large t, which leads to the contradiction $M(t) \to -\infty$ as $t \to \infty$. We now take $\bar{t}_n = s_n$ and $\bar{x}_n = \overline{\xi}(s_n)$, and obtain

$$\begin{cases} U_t(\bar{t}_n, \bar{x}_n) = U_t(s_n, \overline{\xi}(s_n)) = M'(s_n + 0) \to 0, \\ U(\bar{t}_n, \bar{x}_n) = U(s_n, \overline{\xi}(s_n)) = M(s_n) \to \lim_{t\to\infty} M(t) \text{ as } n \to \infty. \end{cases}$$

The proof is complete. □

Lemma 2.6. *If $h_\infty - g_\infty < \infty$, then for any given $L > 0$, we have*

$$\liminf_{t\to\infty} \min_{x\in[-L,L]} v(t,x) > 0, \tag{2.8}$$

and for any $\epsilon > 0$, there is $L_* = L_*(\epsilon, L) \gg 1$ such that $L_1 \ge L_*$ leads to

$$\liminf_{t\to\infty} \min_{|x|\in[L_1, L_1+L]} v(t,x) \ge 1 - \epsilon, \tag{2.9}$$

which implies

$$\liminf_{t\to\infty} v(t,x) \ge 1 - \epsilon \textit{ for every } x \ge L_1.$$

Proof. **Step 1.** We first show (2.9), and only consider the case of $x > 0$, as the case $x < 0$ can be treated similarly.

Recalling that $u(t,x) = 0$ for all $x \notin [g_\infty, h_\infty]$, we see that v satisfies

$$v_t \ge d_2 \int_{h_\infty}^{\infty} J_2(x-y) v(t,y) \mathrm{d}y - d_2 v + \gamma v(1-v),\ t > 0,\ x \in [h_\infty, \infty).$$

To prove (2.9), we will utilize the conclusions of [6, Propositions 3.5 and 3.6] about the solution of a nonlocal diffusion problem over a fixed spatial interval. To be precise, for constants $a < b$, let $w(t,x)$ be the solution of the following problem

$$w_t = d_2 \int_a^b J_2(x-y) w(t,y) \mathrm{d}y - d_2 w + \gamma w(1-w) \quad t > 0,\ x \in (a,b),$$

with continuous initial function $w(0,x) \ge, \not\equiv 0$ in $[a,b]$. Then by [6, Proposition 3.5], for sufficient large $b-a$, the solution $w(t,x)$ converges to $w_{ab}(x)$ uniformly for $x \in [a,b]$ as $t \to \infty$ with $\lambda_p(\mathcal{L}_{(a,b)} + \gamma) > 0$, where $w_{ab}(x)$ satisfies

$$d_2 \int_a^b J_2(x-y) w_{ab}(y) \mathrm{d}y - d_2 w_{ab} + \gamma w_{ab}(1 - w_{ab}) = 0, \quad x \in (a,b).$$

[6, Proposition 3.6] then asserts

$$\lim_{a\to-\infty,\ b\to\infty} w_{ab}(x)=1 \ \text{ locally uniformly in } \mathbb{R},$$

Hence, there is large $L_*>L$ such that for $-a,b\geq L_*$,

$$w_{ab}(x)>1-\epsilon/2 \text{ in } [-L,L].$$

Recalling from [6, Proposition 3.6] that w_{ab} has the shifting invariance property: for $\tilde{a}=a+\delta$ and $\tilde{b}=b+\delta$, there holds $w_{\tilde{a}\tilde{b}}(x)=w_{ab}(x-\delta)$ for $x\in[\tilde{a},\tilde{b}]$. Taking $-a=b=\tilde{L}_1\geq L_*$, $\tilde{a}=h_\infty$ and $\tilde{b}=2\tilde{L}_1+h_\infty$, we obtain

$$w_{\tilde{a}\tilde{b}}(x)\geq 1-\epsilon/2 \text{ for } \ x\in[L_1,L_1+L]$$

with $L_1:=\tilde{L}_1+h_\infty$, due to $w_{\tilde{a}\tilde{b}}(x)=w_{ab}(x-h_\infty-\tilde{L}_1)$ for $x\in[\tilde{a},\tilde{b}]$. Then, the convergence $\lim_{t\to\infty}w(t,x)=w_{\tilde{a}\tilde{b}}(x)$ as a solution defined on $(t,x)\in[0,\infty)\times[\tilde{a},\tilde{b}]$ gives

$$\liminf_{t\to\infty}\min_{|x|\in[L_1,L_1+L]} w(t,x)\geq 1-\epsilon. \tag{2.10}$$

Letting $w(0,x):=v(1,x)$ for $x\in[\tilde{a},\tilde{b}]=[h_\infty,h_\infty+2\tilde{L}_1]$, by the comparison principle, we obtain

$$v(t,x)\geq w(t,x) \ \text{ for } \ t\geq 0,\ x\in[h_\infty,h_\infty+2\tilde{L}_1], \tag{2.11}$$

and so (2.9) follows from (2.10).

Step 2. We now verify (2.8).

From (2.11) we see that for any $[A,B]\subset\mathbb{R}\backslash(g_\infty,h_\infty)$,

$$\liminf_{t\to\infty}\min_{x\in[A,B]} v(t,x)>0,$$

Hence, to show the validity of (2.8), we only need to check

$$\liminf_{t\to\infty}\min_{x\in[g_\infty,h_\infty]} v(t,x)>0. \tag{2.12}$$

Denote

$$\delta_0:=\liminf_{t\to\infty}\min_{x\in[h_\infty,h_\infty+1]} v(t,x)>0$$

and choose $\epsilon_0>0$ small so that $J_2(x)>0$ for $x\in[-2\epsilon_0,2\epsilon_0]$. We show that

$$\delta_1:=\liminf_{t\to\infty}\min_{x\in[h_\infty-\epsilon_0,h_\infty]} v(t,x)>0.$$

Suppose, on the contrary, $\delta_1=0$, we are going to derive a contradiction. Using $\delta_1=0$ and Lemma 2.5, we can find two sequences $t_n\to\infty$ and $x_n\in[h_\infty-\epsilon_0,h_\infty]$ satisfying $x_n\to x_0\in[h_\infty-\epsilon_0,h_\infty]$ such that

$$\lim_{n\to\infty} v(t_n,x_n)=0,\ \lim_{n\to\infty} v_t(t_n,x_n)=0.$$

Then, letting $(t,x)=(t_n,x_n)$ and $n\to\infty$ in the equation satisfied by v, we obtain

$$\begin{aligned}0&=\lim_{n\to\infty}\int_{\mathbb{R}} J_2(x_n-y)v(t_n,y)\mathrm{d}y\geq\liminf_{n\to\infty}\int_{h_\infty}^{h_\infty+1} J_2(x_n-y)v(t_n,y)\mathrm{d}y\\&\geq\delta_0\int_{h_\infty}^{h_\infty+1} J_2(x_0-y)\mathrm{d}y>0,\end{aligned}$$

where we have used Fatou's Lemma in the third inequality. This is a contradiction. Hence $\delta_1>0$.

We may now similarly show that

$$\delta_2=\liminf_{t\to\infty}\min_{x\in[h_\infty-2\epsilon_0,h_\infty-\epsilon_0]} v(t,x)>0.$$

Repeating the argument several times, one can obtain the desired inequality (2.12). The proof is completed. □

Lemma 2.7. *If $h_\infty - g_\infty < \infty$, then for every $L > 0$, we have*

$$\lim_{t\to\infty}\int_{-L}^{L}|v(t,y)-1|\mathrm{d}y = 0 \tag{2.13}$$

and

$$\lim_{t\to\infty} v(t,x) \to 1 \quad \text{uniformly for } x\in[-L,L]\backslash(g_\infty,h_\infty). \tag{2.14}$$

Proof. **Step 1.** We prove that

$$\lim_{t\to\infty} V(t,x) = 1 \text{ for every } x\in\mathbb{R}, \tag{2.15}$$

where

$$V(t,x) := \frac{1}{2L}\int_{x-L}^{x+L} v(t,y)\mathrm{d}y.$$

Since

$$\begin{aligned}
&\frac{1}{2L}\int_{x-L}^{x+L}\int_{\mathbb{R}} J_2(y-z)v(t,z)\mathrm{d}z\mathrm{d}y = \frac{1}{2L}\int_{x-L}^{x+L}\int_{\mathbb{R}} J_2(z)v(t,y-z)\mathrm{d}z\mathrm{d}y\\
=&\frac{1}{2L}\int_{\mathbb{R}} J_2(z)\left(\int_{x-L}^{x+L} v(t,y-z)\mathrm{d}y\right)\mathrm{d}z = \int_{\mathbb{R}} J_2(z)V(t,x-z)\mathrm{d}z\\
=&\int_{\mathbb{R}} J_2(x-y)V(t,y)\mathrm{d}y,
\end{aligned}$$

by the equation for v, the function $V(t,x)$ satisfies

$$\begin{aligned}
V_t &= d_2\frac{1}{2L}\int_{x-L}^{x+L}\int_{\mathbb{R}} J_2(y-z)v(t,z)\mathrm{d}z\mathrm{d}y - d_2V + \frac{1}{2L}\int_{x-L}^{x+L}\gamma v(1-v-hu)\mathrm{d}y\\
&= d_2\int_{\mathbb{R}} J_2(x-y)V(t,y)\mathrm{d}y - d_2V + \frac{1}{2L}\int_{x-L}^{x+L}\gamma v(1-v-hu)\mathrm{d}y \text{ for } t>0,\ x\in\mathbb{R}.
\end{aligned} \tag{2.16}$$

To show (2.15), it suffices to verify for any given $0<\delta<1$,

$$\liminf_{t\to\infty} V(t,x) = \liminf_{t\to\infty}\frac{1}{2L}\int_{x-L}^{x+L} v(t,y)\mathrm{d}y \ge \delta \text{ for every } x\in\mathbb{R}.$$

We first show that this is true when $|x|$ is large. In fact, by (2.9), for any $\epsilon\in(0,1-\delta)$, any $a>0$ and all large $L_1>0$, say $L_1\ge L^*(a)>0$,

$$\liminf_{t\to\infty}\min_{|x|\in[L_1,L_1+a]} v(t,x) \ge 1-\epsilon > \delta.$$

Hence, for $|x|\ge L^*(2L)+L$, by Fatou's Lemma,

$$\liminf_{t\to\infty} V(t,x) = \liminf_{t\to\infty}\frac{1}{2L}\int_{x-L}^{x+L} v(t,y)\mathrm{d}y \ge \frac{1}{2L}\int_{x-L}^{x+L}\liminf_{t\to\infty} v(t,y)\mathrm{d}y > \delta. \tag{2.17}$$

For $|x|\le L^*(2L)+L$, the desired conclusion is a consequence of the following stronger result:

$$\liminf_{t\to\infty}\min_{|x|\le L_1} V(t,x) \ge \delta \text{ for every } L_1>0. \tag{2.18}$$

Otherwise, there exists $L_1>0$ such that

$$\delta_1 := \liminf_{t\to\infty}\min_{|x|\le L_1} V(t,x) < \delta.$$

Without loss of generality, we may assume that $L_1 > L^*(2L)+L$. By Lemma 2.5 we can choose two sequences $t_n\to\infty$ and $x_n\in[-L_1,L_1]$ satisfying $x_n\to x_0\in[-L_1,L_1]$ such that

$$\lim_{n\to\infty} V(t_n,x_n) = \delta_1,\ \lim_{n\to\infty} V_t(t_n,x_n) = 0.$$

We may also require, by passing to a subsequence if necessary, that the following limits exist

$$\lim_{n\to\infty}\int_{\mathbb{R}} J_2(x_n-y)V(t_n,y)\mathrm{d}y, \quad \lim_{n\to\infty}\frac{1}{2L}\int_{x_n-L}^{x_n+L}\gamma v(t_n,y)[1-v(t_n,y)]\mathrm{d}y.$$

Then from (2.16) we deduce, upon using (2.1),

$$\begin{aligned}0&=\lim_{n\to\infty} d_2\int_{\mathbb{R}} J_2(x_n-y)V(t_n,y)\mathrm{d}y-d_2\delta_1+\lim_{n\to\infty}\frac{\gamma}{2L}\int_{x_n-L}^{x_n+L} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y\\ &\geq d_2\int_{\mathbb{R}}\liminf_{n\to\infty}\left[J_2(x_n-y)V(t_n,y)\right]\mathrm{d}y-d_2\delta_1\\ &\quad+\frac{\gamma}{2L}\int_{-L}^{L}\liminf_{n\to\infty}\big(v(t_n,y+x_n)[1-v(t_n,y+x_n)]\big)\mathrm{d}y.\end{aligned}$$

where Fatou's Lemma is used in the last inequality. In view of the definition of δ_1 and (2.17), we have

$$d_2\int_{\mathbb{R}}\liminf_{n\to\infty} J_2(x_n-y)V(t_n,y)\mathrm{d}y-d_2\delta_1\geq d_2\int_{\mathbb{R}} J_2(x_0-y)\delta_1\mathrm{d}y-d_2\delta_1=0.$$

From $v\geq 0$ and $\limsup\limits_{t\to\infty} v(t,y)\leq 1$, we obtain

$$\int_{-L}^{L}\liminf_{n\to\infty}\big(v(t_n,y+x_n)[1-v(t_n,y+x_n)]\big)\mathrm{d}y\geq 0.$$

Thus we have

$$0=\lim_{n\to\infty} d_2\int_{\mathbb{R}} J_2(x_n-y)V(t_n,y)\mathrm{d}y-d_2\delta_1+\lim_{n\to\infty}\frac{\gamma}{2L}\int_{x_n-L}^{x_n+L} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y\geq 0,$$

which implies

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}} J_2(x_n-y)V(t_n,y)\mathrm{d}y-d_2\delta_1=0\\ &\lim_{n\to\infty}\int_{x_n-L}^{x_n+L} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y=0.\end{aligned}\tag{2.19}$$

Moreover, comparing v with the solution w of the ODE

$$w'=\gamma w(1-w),\ w(0)=k_1:=\max_{x\in\mathbb{R}} v_0(x)+1,$$

we obtain

$$v(t,x)\leq w(t)\leq 1+(k_1-1)e^{-\gamma t}\text{ for } t\geq 0,\ x\in\mathbb{R}.\tag{2.20}$$

Claim. For small $\epsilon_1>0$,

$$\lim_{n\to\infty}\int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1}[1-v(t_n,y)]\mathrm{d}y=0.\tag{2.21}$$

In view of (2.19), $x_n\to x_0$, and $\liminf\limits_{n\to\infty} v(t_n,y+x_n)[1-v(t_n,y+x_n)]\geq 0$, we have

$$\lim_{n\to\infty}\int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y=0.\tag{2.22}$$

Denote

$$\begin{aligned}\Omega_1(t_n)&:=\{x\in[x_0-L+\epsilon_1,x_0+L-\epsilon_1]: v(t_n,x)\leq 1\},\\ \Omega_2(t_n)&:=\{x\in[x_0-L+\epsilon_1,x_0+L-\epsilon_1]: v(t_n,x)>1\}.\end{aligned}$$

Then clearly

$$\int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y$$

$$
\begin{aligned}
&= \int_{\Omega_1(t_n)} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y + \int_{\Omega_2(t_n)} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y \\
&\geq k_0 \int_{\Omega_1(t_n)} [1-v(t_n,y)]\mathrm{d}y + \int_{\Omega_2(t_n)} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y,
\end{aligned}
$$

where $k_0 := \min_{\{t\geq 1,\ y\in[x_0-L+\epsilon_1, x_0+L-\epsilon_1]\}} v(t,y) > 0$ by (2.8).

By (2.20),

$$\lim_{n\to\infty} \max_{y\in\Omega_2(t_n)} |1-v(t_n,y)| = 0,$$

which implies

$$\lim_{n\to\infty} \int_{\Omega_2(t_n)} [1-v(t_n,y)]\mathrm{d}y = \lim_{n\to\infty} \int_{\Omega_2(t_n)} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y = 0.$$

Therefore, as $n\to\infty$, by (2.22),

$$
\begin{aligned}
0 &\leq k_0 \int_{\Omega_1(t_n)} [1-v(t_n,y)]\mathrm{d}y \\
&\leq \int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y - \int_{\Omega_2(t_n)} v(t_n,y)[1-v(t_n,y)]\mathrm{d}y \to 0.
\end{aligned}
$$

It follows that

$$\lim_{n\to\infty} \int_{\Omega_1(t_n)} [1-v(t_n,y)]\mathrm{d}y = 0.$$

Hence

$$\lim_{n\to\infty} \int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} [1-v(t_n,y)]\mathrm{d}y = \lim_{n\to\infty} \left[\int_{\Omega_1(t_n)} [1-v(t_n,y)]\mathrm{d}y + \int_{\Omega_2(t_n)} [1-v(t_n,y)]\mathrm{d}y \right] = 0.$$

This proves (2.21).

Now we are ready to use the above information to get a contradiction. Recalling the definition of V and $x_n\to x_0$ as $n\to\infty$, one sees that for large n,

$$V(t_n,x_n) = \frac{1}{2L}\int_{x_n-L}^{x_n+L} v(t_n,y)\mathrm{d}y \geq \frac{1}{2L}\int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} v(t_n,y)\mathrm{d}y.$$

Then by (2.21) we obtain, for sufficiently small $\epsilon_1>0$,

$$\lim_{n\to\infty} V(t_n,x_n) \geq \lim_{n\to\infty} \frac{1}{2L}\int_{x_0-L+\epsilon_1}^{x_0+L-\epsilon_1} v(t_n,y)\mathrm{d}y = \frac{L-\epsilon_1}{L} > \delta_1,$$

which contradicts to the fact $\lim\limits_{n\to\infty} V(t_n,x_n) = \delta_1$. Therefore (2.15) holds.

Step 2. We show (2.13).

We first observe that (2.18) and (2.20) imply

$$\lim_{t\to\infty} V(t,x) = 1 \quad \text{locally uniformly for } x\in\mathbb{R}. \tag{2.23}$$

Next by a simple computation we have

$$\frac{1}{2L}\int_{-L}^{L} |v(t,y)-1|\mathrm{d}y = 1 - V(t,0) + \frac{1}{L}\int_{\Omega_2} [v(t,y)-1]\mathrm{d}y$$

where $\Omega_2 = \Omega_2(t) := \{y\in[-L,L] : v(t,y) > 1\}$. Now (2.13) follows directly from (2.23) and (2.20).

Step 3. We prove (2.14).

Since $u(t,x)\equiv 0$ for $x\notin(g_\infty,h_\infty)$, the function v satisfies, for any interval $[a,b]\subset\mathbb{R}\backslash(g_\infty,h_\infty)$,

$$v_t = d_2\int_{\mathbb{R}} J_2(x-y)v(t,y)\mathrm{d}y - d_2 v + \gamma v(1-v), \quad t>0,\ x\in[a,b]. \tag{2.24}$$

Denote

$$\delta_2 = \liminf_{t\to\infty} \min_{x\in[a,b]} v(t,x).$$

To show (2.14), we only need to prove $\delta_2 = 1$ due to the arbitrariness of $[a,b]$.

By Lemma 2.5 we can choose two sequences $t_n \to \infty$ and $x_n \in [a,b]$ satisfying $x_n \to x_0 \in [a,b]$ as $n \to \infty$ such that

$$\lim_{n\to\infty} v(t_n, x_n) = \delta_2, \ \lim_{n\to\infty} v_t(t_n, x_n) = 0.$$

We may also require that the following limit exists

$$B := \lim_{n\to\infty} \int_{\mathbb{R}} J_2(x_n - y) v(t_n, y) \mathrm{d}y.$$

We claim that $B = 1$. Otherwise, $B \neq 1$. Since $\limsup_{t\to\infty} v \le 1$, we must have $B < 1$. Due to $\int_{\mathbb{R}} J_2(x)\mathrm{d}x = 1$, there is large $\hat{L} > 0$ such that

$$\int_{-\hat{L}}^{\hat{L}} J_2(x_0 - y)\mathrm{d}y > B.$$

By (2.13),

$$\begin{aligned}
&\int_{\mathbb{R}} J_2(x_n - y) v(t_n, y)\mathrm{d}y \ge \int_{-\hat{L}}^{\hat{L}} J_2(x_n - y) v(t_n, y)\mathrm{d}y \\
&= \int_{-\hat{L}}^{\hat{L}} J_2(x_n - y)\mathrm{d}y + \int_{-\hat{L}}^{\hat{L}} J_2(x_n - y)[v(t_n, y) - 1]\mathrm{d}y \\
&\ge \int_{-\hat{L}}^{\hat{L}} J_2(x_n - y)\mathrm{d}y - \|J_2\|_\infty \int_{-\hat{L}}^{\hat{L}} |v(t_n, y) - 1|\mathrm{d}y \\
&\to \int_{-\hat{L}}^{\hat{L}} J_2(x_0 - y)\mathrm{d}y > B \ \text{ as } n \to \infty.
\end{aligned}$$

This is a contradiction. Hence $B = 1$.

Taking $(t,x) = (t_n, x_n)$ in (2.24) and letting $n \to \infty$, in view of $B = 1$, we obtain

$$0 = d_2 - d_2\delta_2 + \gamma\delta_2(1 - \delta_2) = (1 - \delta_2)(d_2 + \gamma\delta_2),$$

which implies $\delta_2 = 1$, as desired. The proof is finished. □

Theorem 1.1 clearly follows directly from Lemmas 2.1, 2.3 and 2.7.

3. Proof of Theorem 1.2

Th arguments in the corresponding local diffusion case considered in [10] indicate that $h_\infty - g_\infty < \infty$ implies (1.8). Next we investigate whether this is also true in our nonlocal situation here. To this end we will make use of the estimates in Theorem 1.1 and regard (1.1) as a perturbation of an ODE system.

Denote

$$m_1(t,x) := d_1 \int_{g(t)}^{h(t)} J_1(x-y) u(t,y)\mathrm{d}y \ge 0, \quad m_2(t,x) := d_2 \int_{\mathbb{R}} J_2(x-y) v(t,y)\mathrm{d}y - d_2.$$

Then by Lemma 2.3 and Lemma 2.7, it is easily seen that, as $t \to \infty$,

$$m_1(t,x) \to 0 \text{ uniformly for } x \in \mathbb{R}, \ m_2(t,x) \to 0 \text{ locally uniformly for } x \in \mathbb{R}. \tag{3.1}$$

The functions $u(t,x)$ and $v(t,x)$ satisfy, for $t \ge 0$, $x \in (g(t), h(t))$,

$$\begin{cases} u_t = m_1 + u(1 - d_1 - u - kv), \\ v_t = m_2 + d_2(1 - v) + \gamma v(1 - v - hu), \end{cases}$$

which can be viewed as an ODE system for each fixed x.

Recall that under the assumption $h_\infty - g_\infty < \infty$, necessarily

$$\tilde{d}_1 = d_1 + k - 1 > 0.$$

Let $F(s)$ and Θ_1, Θ_2 be given by (1.9) and (1.10). Note that $F(0) = -\tilde{d}_1\gamma h < 0$ and $F(1) = -d_2 < 0$. Therefore if $(\gamma, h, k, d_1, d_2) \in \Theta_1$, then

$$F_* := \max_{s\in[0,1]} F(s) < 0. \tag{3.2}$$

Lemma 3.1. *Assume* $\tilde{d}_1 = d_1 + k - 1 > 0$. *Denote*

$$a := \gamma(1 - hk),\ \ b := \tilde{d}_1\gamma h - \gamma(1 - hk) - d_2,\ \ c := -\tilde{d}_1\gamma h,$$

and so $F(s) = as^2 + bs + c$. *Then*

$$(\gamma, h, k, d_1, d_2) \in \Theta_2 \Longleftrightarrow a \le c \text{ and } \sqrt{\frac{c}{a}} \le \frac{b}{-2a} \le 1.$$

Proof. Clearly $a + b + c + d_2 = 0$ and $c < 0$. If $a = 0$ then $F(s) = 0$ implies $s = \frac{-c}{b} = \frac{-c}{-c-d_2}$ which can never be in [0,1]. If $a > 0$ then $F(s) = 0$ and $s \in [0, 1]$ imply

$$s = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \le 1.$$

It follows that

$$\sqrt{b^2 - 4ac} \le 2a + b \implies b^2 - 4ac \le b^2 + 4ab + 4a^2 \implies a(a + b + c) \ge 0 \implies a + b + c \ge 0.$$

But $a + b + c = -d_2 < 0$. We have thus proved that

$$(\gamma, h, k, d_1, d_2) \in \Theta_2 \implies a < 0.$$

With $a < 0$ and $b^2 - 4ac \ge 0$ we easily see that $\sqrt{b^2 - 4ac} < |b|$ and so

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ has the same sign as } \frac{-b}{2a}.$$

Thus

$$(\gamma, h, k, d_1, d_2) \in \Theta_2 \Longleftrightarrow a < 0 < b, b^2 - 4ac \ge 0 \text{ and } \frac{-b + \sqrt{b^2 - 4ac}}{2a} \le 1.$$

With $a < 0 < b$, clearly $b^2 - 4ac \ge 0$ is equivalent to $b \ge 2\sqrt{ac}$, and $\frac{-b+\sqrt{b^2-4ac}}{2a} \le 1$ is equivalent to $\sqrt{b^2 - 4ac} \ge 2a + b$, which holds trivially when $b \le -2a$. If $b > -2a$ then

$$\sqrt{b^2 - 4ac} \ge 2a + b \implies b^2 - 4ac \ge b^2 + 4ab + 4a^2 \implies a(a + b + c) \le 0 \implies a + b + c \ge 0$$

which is impossible since $a + b + c = -d_2 < 0$. We have thus proved that

$$(\gamma, h, k, d_1, d_2) \in \Theta_2 \Longleftrightarrow a < 0 \text{ and } 2\sqrt{ac} \le b \le -2a.$$

Finally, it is easily seen that

$$a < 0 \text{ and } 2\sqrt{ac} \le b \le -2a \Longleftrightarrow a \le c \text{ and } \sqrt{\frac{c}{a}} \le \frac{b}{-2a} \le 1.$$

This completes the proof. □

Remark 3.2. *Suppose* $\tilde{d}_1 > 0$. *It can be easily checked that* $(\gamma, h, k, d_1, d_2) \in \Theta_1$ *if*

$$d_1 \ge 1 \text{ or } kh \le 1 + \frac{d_2}{\gamma}.$$

Indeed,

$$d_1 \ge 1 \implies \tilde{d}_1 \ge k \implies c \le -\gamma kh = a - \gamma < a,$$

and by Lemma 3.1 we see $(\gamma, h, k, d_1, d_2) \notin \Theta_2$. *Thus* $d_1 \ge 1$ *implies* $(\gamma, h, k, d_1, d_2) \in \Theta_1$.

If $kh \le 1 + \frac{d_2}{\gamma}$, then for $s \in (0,1)$,

$$\begin{aligned} F(s) &= \gamma(1-hk)s^2 - [\gamma(1-hk)+d_2]s + \tilde{d}_1\gamma hs - \tilde{d}_1\gamma h \\ &< \gamma(1-hk)s^2 - [\gamma(1-hk)+d_2]s \\ &= \gamma(hk-1)(s-s^2) - d_2 s \le d_2(s-s^2) - d_2 s = -d_2 s^2 < 0. \end{aligned}$$

Recalling $F(0) = -\tilde{d}_1\gamma h < 0$, $F(1) = -d_1 < 0$, we see that $kh \le 1 + \frac{d_2}{\gamma}$ implies $(\gamma, h, k, d_1, d_2) \in \Theta_1$.

Lemma 3.3. *Suppose $h_\infty - g_\infty < \infty$ and denote $\tilde{v} := 1 - v$. Then there exists $M \in (0,1)$ such that*

$$\limsup_{t\to\infty} \max_{x\in[g_\infty,h_\infty]} u(t,x) \le \max\{0, kM - \tilde{d}_1\}, \quad \limsup_{t\to\infty} \max_{x\in[g_\infty,h_\infty]} |\tilde{v}(t,x)| \le M. \tag{3.3}$$

If further $d_1 \ge 1$, then (1.8) holds.

Proof. From (2.8), there is $m_* > 0$ such that

$$\liminf_{t\to\infty} \min_{x\in[g_\infty,h_\infty]} v(t,x) \ge m_* > 0,$$

which, combined with the fact that $\limsup_{t\to\infty} \max_{x\in\mathbb{R}} v(t,x) \le 1$, implies

$$\limsup_{t\to\infty} \max_{x\in[g_\infty,h_\infty]} |\tilde{v}(t,x)| \le 1 - m_*.$$

Then using the equation of u_t and the uniformly convergence of $m_1(t,x)$ to 0 in $\mathbb{R}$, we deduce by a simple comparison argument that

$$\limsup_{t\to\infty} \max_{x\in[g_\infty,h_\infty]} u(t,x) \le \max\{0, k(1-m_*) - \tilde{d}_1\}.$$

We have thus proved (3.3).

Suppose $d_1 \ge 1$, and so $\tilde{d}_1 \ge k$. In this case, from $\tilde{v} = 1 - v \le 1$, we deduce

$$u_t \le m_1 + u(k - \tilde{d}_1 - u) \le m_1 - u^2 \text{ for } t > 0,\ x \in [g(t), h(t)].$$

For any given small $\epsilon > 0$, since $m_1(t,x)$ converges to 0 as $t \to \infty$ uniformly for x in $\mathbb{R}$, we can find $T = T_\epsilon > 0$ so that

$$m_1(t,x) \le \epsilon \text{ for } t \ge T,\ x \in \mathbb{R}.$$

Let $U(t)$ be the solution of the ODE problem

$$U' = \epsilon - U^2,\ U(T) = \|u(T,\cdot)\|_\infty.$$

Then by the comparison principle we deduce $u(t,x) \le U(t)$ for $t \ge T$, $x \in [g(t), h(t)]$. It follows that

$$\limsup_{t\to\infty} \max_{x\in[g(t),h(t)]} u(t,x) \le \lim_{t\to\infty} U(t) = \sqrt{\epsilon}.$$

Letting $\epsilon \to 0$ we deduce

$$\lim_{t\to\infty} \max_{x\in[g(t),h(t)]} u(t,x) = 0.$$

Thus for any small $\delta > 0$ there exists $T_\delta > 0$ so that $u(t,x) \le \delta$ for $t \ge T_\delta$ and all $x \in \mathbb{R}$. It follows that

$$v_t \ge d_2 \int_{\mathbb{R}} J_2(x-y)v(t,y)\mathrm{d}y - d_2 v + \gamma v(1 - v - h\delta) \text{ for } t \ge T_\delta,\ x \in \mathbb{R}.$$

Let $V(t,x)$ be the unique solution of

$$\begin{cases} V_t = d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)V(t,y)\mathrm{d}y - d_2 V + \gamma V(1 - h\delta - V) \text{ for } t \ge T_\delta,\ x \in \mathbb{R}, \\ V(T_\delta) = v(T_\delta, x) \text{ for } x \in \mathbb{R}. \end{cases}$$

Since $1 - h\delta > 0$ for all small $\delta > 0$, it is well known that

$$V(t,x) \to 1 - h\delta \text{ as } t \to \infty \text{ locally uniformly for } x \in \mathbb{R}.$$

By the comparison principle we have $v(t,x) \ge V(t,x)$ for $t > T$ and $x \in \mathbb{R}$. It follows that

$$\liminf_{t\to\infty} v(t,x) \ge 1 - h\delta \text{ locally uniformly in } x \in \mathbb{R}.$$

Letting $\delta \to 0$ we obtain

$$\liminf_{t\to\infty} v(t,x) \ge 1 \text{ locally uniformly in } x \in \mathbb{R}.$$

This implies, in view of (2.20),

$$\lim_{t\to\infty} v(t,x) = 1 \text{ locally uniformly in } x \in \mathbb{R}. \tag{3.4}$$

Hence (1.8) holds. The proof is complete. □

Lemma 3.4. *Suppose* $h_\infty - g_\infty < \infty$. *If*

$$d_1 < 1 \text{ and } (\gamma, h, k, d_1, d_2) \in \Theta_1,$$

then (1.8) *holds.*

Proof. Recall that $\tilde{v} = 1 - v$. So the functions u and $\tilde{v}$ satisfy

$$\begin{cases} u_t = m_1 + u(-\tilde{d}_1 - u + k\tilde{v}), \\ \tilde{v}_t = -m_2 + \gamma h u - (d_2 + \gamma)\tilde{v} + \gamma\tilde{v}(\tilde{v} - hu). \end{cases} \tag{3.5}$$

Claim 1. $\lim_{t\to\infty} \max_{x\in[g_\infty,h_\infty]} u(t,x) = 0$.

Suppose by way of contradiction that the desired conclusion in Claim 1 is not true. Then there exists a sequence (t_n, x_n) with $t_n \to \infty$ and $x_n \in (g_\infty, h_\infty)$ such that

$$u(t_n, x_n) > \epsilon_0 > 0 \text{ for all } n \ge 1 \text{ and some small } \epsilon_0 > 0.$$

We may assume that $t_n > T_*$ for $n \ge 1$. For later arguments we also assume that $\epsilon_0 > 0$ is small enough such that $\epsilon_0^2 + \gamma h \epsilon_0 + F_* < F_*/2 < 0$, where F_* is given by (3.2).

We will derive a contradiction by constructing a family of invariant sets for the solution pair (u, v). For

$$\sigma \in (\frac{\tilde{d}_1}{k}, 1) \text{ and } \epsilon_0 \text{ chosen above},$$

define

$$\epsilon(\sigma) := \min\{k\sigma - \tilde{d}_1, \epsilon_0\}, \;\; M(\sigma) := k\sigma - \tilde{d}_1 + \epsilon(\sigma),$$

and

$$A_\sigma := \{(p,q) \in \mathbb{R}^2 : 0 \le p < M(\sigma), \; q < \sigma\}.$$

We will show that A_σ is the desired invariant family.

Clearly $M(\sigma)$ is a continuous and strictly increasing function of σ over the interval $[\frac{\tilde{d}_1}{k}, 1]$, with $M(\frac{\tilde{d}_1}{k}) = 0$. Let $\sigma_0 \in (\frac{\tilde{d}_1}{k}, 1)$ be uniquely determined by

$$M(\sigma_0) = \epsilon_0.$$

By (3.3), there exists $\sigma_* \in (\sigma_0, 1)$ and $T_* > 0$ such that

$$(u(t,x), \tilde{v}(t,x)) \in A_{\sigma_*} \text{ for all } x \in [g_\infty, h_\infty], \; t \ge T_*.$$

In view of (3.1), by enlarging T_* we may assume that

$$m_1(t,x) \le \epsilon^2(\sigma_0), \quad |m_2(t,x)| \le \epsilon^2(\sigma_0) \;\text{ for }\; t \ge T_*, \; x \in [g_\infty, h_\infty]. \tag{3.6}$$

For fixed $\sigma \in [\sigma_0, \sigma_*]$, $x \in [g_\infty, h_\infty]$, $T \ge T_*$, $s > 0$ and each $(p,q) \in A_\sigma$, we consider the solution map

$$S_x(T+s, T)(p,q) := (\bar{p}, \bar{q}),$$

defined by $(\bar{p},\bar{q}) = (u(T+s,x),\tilde{v}(T+s,x)$, where $(u(t,x),\tilde{v}(t,x))$ solves

$$\begin{cases} u_t = m_1 + u(-\tilde{d}_1 - u + k\tilde{v}), & t > T, \\ \tilde{v}_t = -m_2 + \gamma hu - (d_2+\gamma)\tilde{v} + \gamma\tilde{v}(\tilde{v} - hu), & t > T, \\ u(T,x) = p, \ v(T,x) = q. \end{cases}$$

Claim 2. For each $\sigma \in [\sigma_0,\sigma_*]$, $t \geq T_*$ and $x \in [g_\infty, h_\infty]$,

$$S_x(t+s,t)(\partial A_\sigma) \subset A_\sigma \text{ for all } s > 0. \tag{3.7}$$

To prove (3.7), it suffices to show that

$$u(t,x) = M(\sigma) \text{ and } \tilde{v}(t,x) \leq \sigma \implies u_t(t,x) < \epsilon(\sigma)(-k\sigma + \tilde{d}_1) < 0, \tag{3.8}$$

$$u(t,x) \leq M(\sigma) \text{ and } \tilde{v}(t,x) = \sigma \implies \tilde{v}_t(t,x) < F_*/2 < 0. \tag{3.9}$$

Indeed, by (3.5) and (3.6), $u(t,x) = M(\sigma) > 0$ and $\tilde{v}(t,x) \leq \sigma$ imply $x \in (g(t),h(t))$ and

$$u_t(t,x) \leq \epsilon^2(\sigma) - (k\sigma - \tilde{d}_1 + \epsilon(\sigma))\epsilon(\sigma) = \epsilon(\sigma)(-k\sigma + \tilde{d}_1) < 0,$$

which proves (3.8). To verify (3.9), suppose $\tilde{v}(t,x) = \sigma$ and $u(t,x) \leq M(\sigma)$. Then by (3.5) and (3.6) we obtain

$$\begin{aligned} \tilde{v}_t(t,x) &= -m_2 + \gamma hu(1-\tilde{v}) - (d_2+\gamma)\tilde{v} + \gamma\tilde{v}^2 \\ &\leq \epsilon^2(\sigma) + \gamma h(k\sigma - \tilde{d}_1 + \epsilon(\sigma))(1-\sigma) - (d_2+\gamma)\sigma + \gamma\sigma^2 \\ &= \epsilon^2(\sigma) + (1-\sigma)\gamma h\epsilon(\sigma) + \gamma(1-hk)\sigma^2 + [\gamma(hk-1) - d_2 + \tilde{d}_1\gamma h]\sigma - \tilde{d}_1\gamma h \\ &= \epsilon^2(\sigma) + (1-\sigma)\gamma h\epsilon(\sigma) + F(\sigma) \leq \epsilon_0^2 + \gamma h\epsilon_0 + F_* < F_*/2 < 0. \end{aligned}$$

Hence (3.9) holds. This proves (3.7) and so Claim 2 holds true.

We are now ready to reach a contradiction and hence complete the proof of Claim 1. Consider $P_n(t) := (u(t,x_n),\tilde{v}(t,x_n))$ for $t \in [T_*,t_n]$. Since $u(t_n,x_n) > \epsilon_0$ and $t_n > T_*$, by Claim 2 and the definition of σ_*, there exists $\sigma_n(t) \in [\sigma_0,\sigma_*]$ such that $P_n(t) \in \partial A_{\sigma_n(t)}$ and $\sigma_n(t)$ is nonincreasing in t for $t \in [T_*,t_n]$. In particular,

$$P_n([T_*,t_n]) \subset A := A_{\sigma_*} \setminus \overline{A}_{\sigma_0}.$$

If we define

$$\begin{cases} A^+ := \{(p,q) \in A : p > M(q)\}, \\ A^- := \{(p,q) \in A : p < M(q)\}, \\ \Gamma := \{(p,q) \in A : p = M(q)\}, \end{cases}$$

then from (3.8) and (3.9) we see that, for $t \in [T_*,t_n]$,

$$\begin{cases} P_n(t) \in A^+ \cup \Gamma \implies \tilde{v}_t \leq -c_0, \\ P_n(t) \in A^- \cup \Gamma \implies u_t \leq -c_0, \end{cases}$$

where $c_0 := \min\{F_*/2, \epsilon(\sigma_0)(k\sigma_0 - \tilde{d}_1)\} > 0$.

Define

$$\begin{cases} I_n^+ := \{t \in (T_*,t_n) : P_n(t) \in A^+\}, \\ I_n^- := \{t \in (T_*,t_n) : P_n(t) \in A^-\}, \\ I_n^0 := \{t \in (T_*,t_n) : P_n(t) \in \Gamma\}. \end{cases}$$

Then I_n^+ and I_n^- are open sets (possibly empty for one of them). Hence each of them is the union of some (at most countably many) non-overlapping intervals when it is not the empty set. In the following, we derive a contradiction in each of the possible cases.

If one of I_n^+ and I_n^- is empty, say $I_n^- = \emptyset$, then

$$-\sigma_* \leq \tilde{v}(t_n,x_n) - \tilde{v}(T_*,x_n) = \int_{T_*}^{t_n} \tilde{v}_t(t,x_n)dt \leq -c_0(t_n - T_*) \to -\infty \text{ as } n \to \infty,$$

which is a contradiction. Similarly $I_n^+ = \emptyset$ leads to a contradiction.

If both I_n^+ and I_n^- are the union of some non-overlapping intervals, say for some non-empty but at most countable index sets K_n^+ and K_n^-,

$$I_n^+ = \cup_{k\in K_n^+}(s_k,t_k),\ \ I_n^- = \cup_{k\in K_n^-}(\tilde{s}_k,\tilde{t}_k),$$

then

$$\begin{cases} k\in I_n^+ \text{ and } s_k\neq T_* \implies s_k\in\Gamma,\\ k\in I_n^- \text{ and } \tilde{s}_k\neq T_* \implies \tilde{s}_k\in\Gamma. \end{cases}$$

By the invariance property of A_σ we have

$$\begin{cases} s_k\in\Gamma \implies \tilde{v}(t_k,x_n)\leq \tilde{v}(s_k,x_n), i.e., \displaystyle\int_{s_k}^{t_k}\tilde{v}_t(t,x_n)dt\leq 0\\ \tilde{s}_k\in\Gamma \implies u(\tilde{t}_k,x_n)\leq u(\tilde{s}_k,x_n), i.e., \displaystyle\int_{\tilde{s}_k}^{\tilde{t}_k}u_t(t,x_n)dt\leq 0, \end{cases}$$

and

$$\begin{cases} s_k=T_* \implies \tilde{v}(t_k)\leq\sigma_* \implies \displaystyle\int_{s_k}^{t_k}\tilde{v}_t(t,x_n)dt\leq\sigma_*,\\ \tilde{s}_k=T_* \implies u(\tilde{t}_k)\leq M(\sigma_*) \implies \displaystyle\int_{\tilde{s}_k}^{\tilde{t}_k}u_t(t,x_n)dt\leq M(\sigma_*). \end{cases}$$

It follows that

$$\int_{I_n^+}u_t(t,x_n)dt\leq M(\sigma_*),\ \ \int_{I_n^-}\tilde{v}_t(t,x_n)dt\leq\sigma_*.$$

Hence

$$-\sigma_*\leq\tilde{v}(t_n,x_n)-\tilde{v}(T_*,x_n)\leq\sigma_*+\int_{I_n^+\cup I_n^0}\tilde{v}_t(t,x_n)dt\leq\sigma_*-c_0|I_n^+\cup I_n^0|,$$
$$-M(\sigma_*)\leq u(t_n,x_n)-u(T_*,x_n)\leq M(\sigma_*)+\int_{I_n^-\cup I_n^0}u_t(t,x_n)dt\leq M(\sigma_*)-c_0|I_n^-\cup I_n^0|.$$

Adding the above inequalities we obtain the following contradiction:

$$-2[\sigma_*+M(\sigma_*)]\leq -c_0(|I_n^-\cup I_n^0|+|I_n^+\cup I_n^0|)\leq -c_0(t_n-T_*)\to-\infty \text{ as } n\to\infty.$$

Claim 1 is thus proved.

Claim 3. $\lim_{t\to\infty}v(t,x)=1$ locally uniformly in $x\in\mathbb{R}$.

This follows from Claim 1 in the same way as argued in the proof of Lemma 3.3 for the case $d_1\geq 1$. □

Lemma 3.5. *Suppose $h_\infty-g_\infty<\infty$. If*

$$d_1<1 \text{ and } (\gamma,h,k,d_1,d_2)\in\Theta_2,$$

then either (i) (1.8) *holds or* (ii) *there is an open set $\Omega\subset(g_\infty,h_\infty)$ with $|\Omega|=h_\infty-g_\infty$, $\Omega\neq(g_\infty,h_\infty)$ such that*

$$\lim_{t\to\infty}(u(t,x),v(t,x))=(0,1) \ \ \text{uniformly for } x \text{ in any compact subset of } \Omega, \tag{3.10}$$

$$\lim_{t\to\infty}(u(t,x),v(t,x))=(kx_*-\tilde{d}_1,1-x_*) \ \ \text{for } x\in(g_\infty,h_\infty)\backslash\Omega, \tag{3.11}$$

where $x_*>0$ is the smallest positive root of $F(s)=0$ in $[0,1]$.

Proof. We must have $kx_*-\tilde{d}_1>0$, since $kx_*-\tilde{d}_1\leq 0$ implies

$$0=F(x_*)=\gamma h(kx_*-\tilde{d}_1)(1-x_*)-(d_2+\gamma)x_*+\gamma x_*^2\leq-(d_2+\gamma)x_*+\gamma x_*^2<0,$$

which is clearly impossible.

Step 1. We show that there exists an open set Ω as described in (3.10).

For

$$\sigma\in(\frac{\tilde{d}_1}{k},x_*) \text{ and } \epsilon_1>0 \text{ small to be determined as later argument desires,}$$

define

$$\tilde{\epsilon}(\sigma) := \min\{k\sigma - \tilde{d}_1, \epsilon_1\},\ \ \tilde{M}(\sigma) := k\sigma - \tilde{d}_1 + \tilde{\epsilon}(\sigma),$$

and

$$\tilde{A}_\sigma := \{(p,q) \in \mathbb{R}^2 : 0 \le p < \tilde{M}(\sigma),\ \ q < \sigma\}.$$

Fix $\tilde{\sigma}_* \in (\frac{\tilde{d}_1}{k}, x_*)$. Since $F(0) < 0$ necessarily

$$\tilde{F}_* := \max_{s\in[0,\tilde{\sigma}_*]} F(s) < 0.$$

From (2.1) and (2.13), we see

$$\lim_{t\to\infty} \int_{g_\infty-1}^{h_\infty+1} [u(t,y) + |\tilde{v}(t,y)|]\mathrm{d}y = 0.$$

Hence $u(t,x) + |\tilde{v}(t,x)| \to 0$ in measure for x over $[g_\infty, h_\infty]$. By Egorov's theorem, for each small $\epsilon > 0$, there exists a set $\Omega_\epsilon \subset (g_\infty, h_\infty)$ such that

$$\begin{cases} |\Omega_\epsilon| \ge h_\infty - g_\infty - \epsilon, \\ u(t,x) + |\tilde{v}(t,x)| \to 0 \text{ uniformly for } x \in \Omega_\epsilon \text{ as } t \to \infty. \end{cases}$$

Therefore, there exists $T_\epsilon > 0$ such that

$$(u(t,x), \tilde{v}(t,x)) \in \tilde{A}_{\tilde{\sigma}_*/2} \text{ for all } t \ge T_\epsilon \text{ and } x \in \Omega_\epsilon.$$

By the continuous dependence of $u(T_\epsilon, x)$ and $\tilde{v}(T_\epsilon, x)$ on x, we see that there exists an open set O_ϵ such that

$$\begin{cases} (g_\infty, h_\infty) \supset O_\epsilon \supset \Omega_\epsilon, \\ (u(T_\epsilon, x), \tilde{v}(T_\epsilon, x)) \in \tilde{A}_{\tilde{\sigma}_*} \text{ for all } x \in \overline{O}_\epsilon. \end{cases}$$

We are now in a position to show

$$\lim_{t\to\infty} \max_{x\in\overline{O}_\epsilon} u(t,x) = 0 \tag{3.12}$$

by repeating the argument that leads to the conclusion in Claim 1 of the proof of Lemma 3.4, except that we replace $(A_{\sigma_*}, [g_\infty, h_\infty], \sigma_*, F_*)$ there by the above defined $(\tilde{A}_{\tilde{\sigma}_*}, \overline{O}_\epsilon, \tilde{\sigma}_*, \tilde{F}_*)$.

Since $\epsilon > 0$ is arbitrary, choosing $\epsilon_n > 0$ converging to 0 monotonically as n increasing to ∞, we can obtain a sequence of open sets $\{O_{\epsilon_n}\}$ such that (3.12) holds for each O_{ϵ_n}. Let $\Omega := \cup_{n=1}^\infty O_{\epsilon_n}$. Then $\Omega \subset (g_\infty, h_\infty)$ is an open set with $|\Omega| = h_\infty - g_\infty$, and since (3.12) holds for every O_{ϵ_n} we see that

$$\lim_{t\to\infty} u(t,x) = 0 \text{ uniformly for } x \text{ in any compact subset of } \Omega. \tag{3.13}$$

This implies that

$$\tilde{v}_t = -\tilde{m}_2 - (d_2 + \gamma)\tilde{v} + \gamma\tilde{v}^2$$

with $\tilde{m}_2 = \tilde{m}_2(t,x) \to 0$ as $t \to \infty$ uniformly for x in any compact subset of Ω. This fact, together with $\tilde{v}(t,x) \le 1$ and $\liminf_{t\to\infty} \tilde{v}(t,x) \ge 0$ uniformly in x, leads to

$$\lim_{t\to\infty} \tilde{v}(t,x) = 0 \text{ uniformly for } x \text{ in any compact subset of } \Omega$$

by a simple comparison argument; we omit the details. Step 1 is now completed.

Step 2. Let Ω be the maximal open set contained in (g_∞, h_∞) such that (3.10) holds. If $\Omega = (g_\infty, h_\infty)$, then (1.8) holds.

Arguing indirectly we assume that $\Omega = (g_\infty, h_\infty)$, but (1.8) does not hold. Since the first identity in (1.8) implies the second, we see that there must exist sequences $t_n \to \infty$ and $x_n \in (g(t_n), h(t_n))$ such that

$$u(t_n, x_n) \ge \epsilon_0 \text{ for all } n \ge 1 \text{ and some } \epsilon_0 > 0. \tag{3.14}$$

By passing to a subsequence we may assume that $x_n \to x^* \in [g_\infty, h_\infty]$. By our assumption, (3.10) holds and so $(u(t,x^*), v(t,x^*)) \to (0,1)$ as $t \to \infty$ if $x^* \in (g_\infty, h_\infty)$. If $x^* = g_\infty$ or h_∞, then $u(t,x^*) \equiv 0$ and by (2.14), $v(t,x^*) \to 1$ as $t \to \infty$. Thus we always have

$$(u(t,x^*), v(t,x^*)) \to (0,1) \text{ as } t \to \infty.$$

Let $\tilde{A}_\sigma$, $\tilde{\sigma}_*$ and $\tilde{F}_*$ be defined as in Step 1 above. Then there exists $T_0 > 0$ such that

$$(u(T_0,x^*), \tilde{v}(T_0,x^*)) \in \tilde{A}_{\tilde{\sigma}_*/2}.$$

By continuous dependent of $u(T_0,x)$ and $\tilde{v}(T_0,x)$ on x, there exists a $\delta > 0$ small such that

$$(u(T_0,x), \tilde{v}(T_0,x)) \in \tilde{A}_{\tilde{\sigma}_*} \text{ for all } x \in O_\delta := [x^* - \delta, x^* + \delta] \cap [g_\infty, h_\infty].$$

This implies, as in Step 1,

$$\lim_{t\to\infty} \max_{x\in O_\delta} u(t,x) = 0 \tag{3.15}$$

by repeating the argument that leads to the conclusion in Claim 1 of the proof of Lemma 3.4. But this is a contradiction to (3.14). This completes Step 2.

Step 3. Let Ω be the maximal open set contained in (g_∞, h_∞) such that (3.10) holds. Suppose that $\Omega \neq (g_\infty, h_\infty)$ and let $x_1 \in (g_\infty, h_\infty) \setminus \Omega$. We are going to show that

$$\lim_{t\to\infty} u(t,x_1) = kx_* - \tilde{d}_1, \quad \lim_{t\to\infty} \tilde{v}(t,x_1) = x_*.$$

Let

$$v_1 := \liminf_{t\to\infty} \tilde{v}(t,x_1).$$

Then there is a sequence $t_n \to \infty$ such that $\tilde{v}(t_n,x_1) \to v_1$ and $\tilde{v}_t(t_n,x_1) \to 0$ as $n \to \infty$. By passing to a subsequence we may also assume that $u(t_n,x_1) \to u_1$. From the equation of $\tilde{v}_t$ and $m_2 \to 0$ as $t \to \infty$, we deduce

$$0 = \gamma h u_1(1 - v_1) - (d_2 + \gamma)v_1 + \gamma v_1^2. \tag{3.16}$$

We show next that

$$v_1 \geq x_*. \tag{3.17}$$

Arguing indirectly we assume $v_1 < x_*$. Using $F(x_*) = 0$ we see that the function

$$G(s) := \frac{(d_2+\gamma)s - \gamma s^2}{\gamma h(1-s)} = \frac{s}{\gamma h}\Big(\frac{d_2}{1-s} + \gamma\Big)$$

satisfies $G(x_*) = kx_* - \tilde{d}_1$. By (3.16) we obtain $G(v_1) = u_1$. Clearly $G(s)$ is strictly increasing for $s \in (0,1)$. Therefore

$$v_1 < x_* \text{ implies } u_1 = G(v_1) < G(x_*) = kx_* - \tilde{d}_1.$$

We show below that this leads to a contradiction and therefore (3.17) must hold. Indeed, $u_1 < kx_* - \tilde{d}_1$ and $v_1 < x_*$ imply that for any given $\hat{\sigma}_* < x_*$ close enough to x_*, with $\tilde{A}_\sigma$ as defined in Step 1,

$$(u(t_n,x_1), \tilde{v}(t_n,x_1)) \subset \tilde{A}_{\hat{\sigma}_*} \text{ for all large } n.$$

Clearly $\hat{F}_* := \max_{s\in[0,\hat{\sigma}_*]} F(s) < 0$. We may now repeat the argument that leads to the conclusion in Claim 1 of the proof of Lemma 3.4 but with $(A_{\sigma_*}, [g_\infty, h_\infty], \sigma_*, F_*)$ there replaced by $(\tilde{A}_{\hat{\sigma}_*}, x_1, \hat{\sigma}_*, \hat{F}_*)$, to conclude that

$$\lim_{t\to\infty} u(t,x_1) = 0.$$

This implies $\lim_{t\to\infty} \tilde{v}(t,x_1) = 0$ by making use of the equation satisfied by $\tilde{v}(t,x_1)$. But this is a contradiction to our assumption that $x_1 \notin \Omega$. We have thus proved (3.17).

Let

$$v_2 = \limsup_{t\to\infty} \tilde{v}(t,x_1).$$

Then there exists a sequence t_n such that

$$v_2 = \lim_{n\to\infty} \tilde{v}(t_n, x_1), \quad 0 = \lim_{n\to\infty} \tilde{v}_t(t_n, x_1).$$

By passing to a subsequence there exists $u_2 \geq 0$ such that

$$u_2 := \lim_{n\to\infty} u(t_n, x_1).$$

Then the equation of $\tilde{v}_t$ and $m_2 \to 0$ as $t \to \infty$ yield

$$0 = \gamma h u_2(1 - v_2) - (d_2 + \gamma)v_2 + \gamma v_2^2. \tag{3.18}$$

By (3.3), we know

$$u_2 \leq kM - \tilde{d}_1 \text{ and } v_2 \leq M \text{ for some } 0 < M < 1.$$

We now set to show that

$$v_2 \leq x_*. \tag{3.19}$$

Argue indirectly we assume that $v_2 > x_*$ and seek a contradiction.

Since $F(s)$ is a quadratic function satisfying $F(0) < 0$ and $F(1) < 0$, either x_* is a degenerate root, namely

$$F(s) < 0 \text{ for } s \in [0, x_*) \cup (x_*, 1], \tag{3.20}$$

or there is another root $x^* \in (x_*, 1)$ such that

$$F(s) < 0 \text{ for } s \in [0, x_*) \cup (x^*, 1], \;\; F(s) > 0 \text{ for } s \in (x_*, x^*). \tag{3.21}$$

Using the function $G(s)$ defined earlier we obtain from (3.18) and $F(x_*) = 0$ that

$$u_2 = G(v_2) > G(x_*) = kx_* - \tilde{d}_1.$$

We will prove (3.19) by deriving a contradiction under the assumption $v_2 > x_*$ for both cases (3.20) and (3.21).

Claim 1. Case (3.20) leads to a contradiction.

When (3.20) happens, we fix

$$\bar{\sigma}_* \in (M, 1) \text{ such that } k\bar{\sigma}_* - \tilde{d}_1 > u_2,$$

and then fix

$$\bar{\sigma}_0 \in (x_*, \bar{\sigma}_*) \text{ close to } x_* \text{ such that } k\bar{\sigma}_0 - \tilde{d}_1 < u_2, \;\; \bar{\sigma}_0 < v_2.$$

It is now clear that

$$\bar{F}_* := \max_{s\in[\bar{\sigma}_0, \bar{\sigma}_*]} F(s) < 0.$$

We next choose $\epsilon_1 > 0$ small enough such that

$$\epsilon_1^2 + \gamma h \epsilon_1 + \bar{F}_* < \bar{F}_*/2 < 0, \;\; k\bar{\sigma}_0 - \tilde{d}_1 + \epsilon_1 < u_2,$$

and define, for $\sigma \in (x_*, \bar{\sigma}_*]$,

$$\bar{\epsilon}(\sigma) := \min\{\sigma - x_*, \epsilon_1\}, \;\; \bar{M}(\sigma) := k\sigma - \tilde{d}_1 + \bar{\epsilon}(\sigma),$$

and

$$\bar{A}_\sigma := \{(p, q) \in \mathbb{R}^2 : 0 \leq p < \bar{M}(\sigma), \; q < \sigma\}.$$

Clearly $\bar{M}(\sigma)$ is continuous and strictly increasing in σ with

$$\bar{M}(x_*) = kx_* - \tilde{d}_1 < \bar{M}(\bar{\sigma}_0) < u_2 < \bar{M}(\bar{\sigma}_*).$$

We also have

$$\bar{\sigma}_0 < v_2 \leq M < \bar{\sigma}_*.$$

Therefore, for all large n,

$$(u(t_n, x_1), \tilde{v}(t_n, x_1)) \in \bar{A}_{\bar{\sigma}_*} \setminus \bar{A}_{\bar{\sigma}_0},$$

and by (3.1),

$$m_1(t,x) \le \bar\epsilon(\bar\sigma_0)^2, \quad |m_2(t,x)| \le \bar\epsilon(\bar\sigma_0)^2 \ \text{ for } \ t \ge t_n. \tag{3.22}$$

Fix such an n and let the solution map $S_x(t+s,t)$ be defined as in Claim 1 of the proof of Lemma 3.4; then the same calculations as in the proof there yield the following:

For each $\sigma \in [\bar\sigma_0, \bar\sigma_*]$, $t \ge t_n$,

$$\begin{cases} S_{x_1}(t+s,t)(\partial \bar A_\sigma) \subset \bar A_\sigma \text{ for all } s > 0, \\ u(t,x_1) = \bar M(\sigma) \text{ and } \tilde v(t,x_1) \le \sigma \implies u_t(t,x_1) < \bar\epsilon(\sigma)(-k\sigma + \tilde d_1) < 0, \\ u(t,x_1) \le \bar M(\sigma) \text{ and } \tilde v(t,x_1) = \sigma \implies \tilde v_t(t,x_1) < \bar F_*/2 < 0. \end{cases} \tag{3.23}$$

Consider $P(t) := (u(t,x_1), \tilde v(t,x_1))$ for $t \in [t_n, t_{n+m}]$, $m \ge 1$. Since for every $m \ge 1$,

$$(u(t_{n+m},x_1), \tilde v(t_{n+m},x_1)) \in \bar A_{\bar\sigma_*} \setminus \bar A_{\bar\sigma_0},$$

by (3.23) there exists $\sigma(t) \in [\bar\sigma_0, \bar\sigma_*]$ such that $P(t) \in \partial \bar A_{\sigma(t)}$ and $\sigma(t)$ is nonincreasing in t for $t \in [t_n, t_{n+m}]$.

We may now apply the same argument used to prove Claim 1 in the proof of Lemma 3.4 to obtain a contradiction, and Claim 1 is proved.

Claim 2. Case (3.21) also leads to a contradiction.

Since $u_2 > kx_* - \tilde d_1$ and $v_2 > x_*$, we can find $\hat\sigma \in (x_*, x^*)$ such that

$$u_2 > k\hat\sigma - \tilde d_1, \ v_2 > \hat\sigma.$$

Fix $\hat\epsilon > 0$ small so that

$$-\hat\epsilon^2 - \gamma h\hat\epsilon + F(\hat\sigma) > 0, \quad k\hat\sigma - \tilde d_1 - \hat\epsilon > 0.$$

Then choose $\hat T > 0$ so that

$$m_1(t,x) \le \hat\epsilon^2, \ |m_2(t,x)| \le \hat\epsilon^2 \text{ for } t \ge \hat T \text{ and } x \in [g_\infty, h_\infty].$$

We now define

$$\hat B := \{(p,q) : p > k\hat\sigma - \tilde d_1 - \hat\epsilon, \ q > \hat\sigma\}.$$

Clearly $(u_2, v_2) \in \hat B$ and therefore $(u(t_n,x_1), \tilde v(t_n,x_1)) \in \hat B$ for all large n, say $n \ge n_0$. By enlarging n_0 we may also assume that $t_{n_0} \ge \hat T$. By the continuous dependence of $u(t_{n_0},x)$ and $\tilde v(t_{n_0},x)$ on x, there exists $\epsilon > 0$ sufficiently small so that

$$(u(t_{n_0},x), \tilde v(t_{n_0},x)) \in \hat B \text{ for all } x \in [x_1 - \epsilon, x_1 + \epsilon].$$

We show below that for each $x \in [x_1-\epsilon, x_1+\epsilon]$, the trajectory $\{(u(t,x), \tilde v(t,x)) : t \ge t_{n_0}\}$ is trapped inside $\hat B$. It suffices to show that for any $t \ge t_{n_0}$,

$$u(t,x) = k\hat\sigma - \tilde d_1 - \hat\epsilon \text{ and } \tilde v(t,x) \ge \hat\sigma \text{ implies } u_t(t,x) > 0, \tag{3.24}$$

$$u(t,x) \ge k\hat\sigma - \tilde d_1 - \hat\epsilon \text{ and } \tilde v(t,x) = \hat\sigma \text{ implies } \tilde v_t(t,x) > 0. \tag{3.25}$$

Indeed, (3.24) follows from the simple calculation below

$$u_t = m_1 + u(-\tilde d_1 - u + k\tilde v) \ge (k\hat\sigma - \tilde d_1 - \hat\epsilon)\hat\epsilon > 0,$$

and to verify (3.25), we calculate

$$\begin{aligned} \tilde v_t(t,x) &= -m_2 + \gamma h u(1-\tilde v) - (d_2+\gamma)\tilde v + \gamma \tilde v^2 \\ &\ge -\hat\epsilon^2 + \gamma h(k\hat\sigma - \tilde d_1 - \hat\epsilon)(1-\hat\sigma) - (d_2+\gamma)\hat\sigma + \gamma\hat\sigma^2 \\ &= -\hat\epsilon^2 - (1-\hat\sigma)\gamma h\hat\epsilon + \gamma(1-hk)\hat\sigma^2 + [\gamma(hk-1) - d_2 + \tilde d_1\gamma h]\hat\sigma - \tilde d_1\gamma h \\ &= -\hat\epsilon^2 - (1-\hat\sigma)\gamma h\hat\epsilon + F(\hat\sigma) > -\hat\epsilon^2 - \gamma h\hat\epsilon + F(\hat\sigma) > 0. \end{aligned}$$

Thus (3.25) holds and we have proved that

$$(u(t,x), \tilde v(t,x)) \in \hat B \text{ for all } t \ge t_{n_0}, \ x \in [x_1-\epsilon, x_1+\epsilon].$$

It follows that $[x_1-\epsilon, x_1+\epsilon] \subset [g_\infty, h_\infty] \setminus \Omega$, which is a contradiction to $|\Omega| = h_\infty - g_\infty$. Claim 2 is now proved.

The above Claims 1 and 2 prove that (3.19) holds, namely $v_2 \le x_*$. As we have already proved in (3.17) that $v_1 \ge x_*$, by the definitions of v_1 and v_2 we must have $v_1 = v_2 = x_*$ and

$$\tilde{v}(t, x_1) \to x_* \text{ as } t \to \infty.$$

It follows that $u(t, x_1)$ satisfies

$$u_t = \tilde{m}_1 + u(kx_* - \tilde{d}_1 - u)$$

with $\tilde{m}_1 = \tilde{m}_1(t, x_1) \to 0$ as $t \to \infty$, which implies

$$u(t, x_1) \to kx_* - \tilde{d}_1 \text{ as } t \to \infty.$$

This concludes Step 2 and the proof of the lemma is now complete. □

Theorem 1.2 now follows directly from Lemmas 3.3, 3.4 and 3.5.

4. Proof of Theorem 1.4

We divide the proof into three steps.

Step 1. We show that $h_\infty = -g_\infty = \infty$.

We only show that $h_\infty = \infty$ implies $g_\infty = -\infty$, as it can be shown similarly that $g_\infty = -\infty$ implies $h_\infty = \infty$.

Assume on the contrary that $h_\infty = \infty > -g_\infty$. Then by Remark 2.4,

$$\lim_{t\to\infty} \int_{-\infty}^{L} u(t, y)\mathrm{d}y = 0 \text{ for every } L > 0. \tag{4.1}$$

Since we always have

$$\limsup_{t\to\infty} v(t, x) \le 1 \text{ uniformly for } x \in \mathbb{R},$$

for any given $\epsilon > 0$ small, there exists a large $T = T_\epsilon > 0$ such that

$$u_t \ge d_1 \int_{g(T)}^{h(T)} J_1(x-y)u(t, y)\mathrm{d}y - d_1 u + u[1-(k+\epsilon) - u] \text{ for } t > T,\ g(T) < x < h(T).$$

However, as $1-k-\epsilon > 0$ and we can make $h(T) - g(T)$ as large as we want by enlarging T, the above inequality for u implies, for such ϵ and T, by the comparison principle and [6, Propositions 3.5 and 3.6], that

$$\liminf_{t\to\infty} \inf_{x\in[g(T),h(T)]} u(t, x) > 0.$$

This contradicts (4.1). Step 1 is finished.

Step 2. We show that $h \ge 1$ implies

$$\lim_{t\to\infty} u(t, x) = 1, \lim_{t\to\infty} v(t, x) = 0 \text{ locally uniformly for } x \in \mathbb{R}. \tag{4.2}$$

We always have

$$\limsup_{t\to\infty}[\sup_{x\in\mathbb{R}} u(t, x)] \le 1, \quad \limsup_{t\to\infty}[\sup_{x\in\mathbb{R}} v(t, x)] \le 1. \tag{4.3}$$

In view $1-k > 0$, by making use of Lemma 3.14 in [16], we derive

$$\liminf_{t\to\infty} u(t, x) \ge 1 - k := \underline{u}_1 \ \text{ locally uniformly in } \mathbb{R}.$$

The following proof will be presented according to two cases.

Case 1. $1 - h(1-k) \le 0$.

In this case $1 - h\underline{u}_1 \le 0$. By Proposition 4 (i) in [27], we have

$$\limsup_{t\to\infty} v(t, x) \le 0 \ \text{ locally uniformly in } \mathbb{R}.$$

Since $v(t, x) \ge 0$, it follows that $\lim_{t\to\infty} v(t, x) = 0$ locally uniformly in $\mathbb{R}$.

Fix $L \gg 1$ and $0 < \varepsilon_1 \ll 1$. There exists $T = T_{\varepsilon_1, L} > 0$ such that $v(t,x) \le \varepsilon_1$ and $h(T) - g(T) \gg L$ for $t > T$ and $x \in [-L, L]$. Thus, u satisfies

$$u_t \ge d_1 \int_{-L}^{L} J_1(x-y)u(t,y)\mathrm{d}y - d_1 u + u(1 - k\epsilon_1 - u) \quad \text{for } t > T,\ -L < x < L.$$

Since both $0 < \epsilon_1 \ll 1$ and $L \gg 1$ can be chosen arbitrarily, a simple comparison argument can be used to show that $\liminf_{t\to\infty} u(t,x) \ge 1$ locally uniformly for $x \in \mathbb{R}$. This, combined with $\limsup_{t\to\infty} u(t,x) \le 1$, gives $\lim_{t\to\infty} u(t,x) = 1$ locally uniformly for $x \in \mathbb{R}$. Thus (4.2) holds in Case 1.

Case 2. $1 - h(1-k) > 0$.

Now

$$1 - h\underline{u}_1 = 1 - h(1-k) > 0.$$

By Lemma 3.14 (ii) in [16], we have

$$\limsup_{t\to\infty} v(t,x) \le 1 - h\underline{u}_1 := \overline{v}_2 \quad \text{locally uniformly in } \mathbb{R}.$$

Clearly $\underline{u}_2 := 1 - k\overline{v}_2 = 1 - k(1 - h(1-k)) = (1-k)(1+kh) > 0$. According to Lemma 3.14 (i) in [16], we have

$$\liminf_{t\to\infty} u(t,x) \ge \underline{u}_2 \quad \text{locally uniformly in } \mathbb{R}.$$

If $1 - h\underline{u}_2 \le 0$, then similar to Case 1, we deduce $\lim_{t\to\infty} v(t,x) = 0$ and then $\lim_{t\to\infty} u(t,x) = 1$ locally uniformly for $x \in \mathbb{R}$.

If $1 - h\underline{u}_2 > 0$, then

$$\limsup_{t\to\infty} v(t,x) \le 1 - h\underline{u}_2 := \overline{v}_3 \quad \text{locally uniformly in } \mathbb{R}.$$

This and $\underline{u}_3 := 1 - k\overline{v}_3 = (1-k)(1 + kh + kh^2) > 0$ imply, as above,

$$\liminf_{t\to\infty} u(t,x) \ge \underline{u}_3 \quad \text{locally uniformly in } \mathbb{R}.$$

Continue with this procedure, we will obtain a sequence $\{\underline{u}_j\}$ and $\{\overline{v}_j\}$ such that

$$\overline{v}_{j+1} = 1 - h\underline{u}_j,\ \underline{u}_{j+1} = 1 - k\overline{v}_{j+1} \text{ for } j = 1, 2, ...$$

and there are two possibilities:

(a) there is a first $j \ge 1$ such that $h\underline{u}_j \ge 1$, then as in Case 1 we deduce (4.2).

(b) $h\underline{u}_j < 1$ for all $j = 1, 2, \dots$. Then repeating the above analysis we obtain

$$\limsup_{t\to\infty} v(t,x) \le \overline{v}_j \quad \text{locally uniformly in } \mathbb{R},$$

$$\liminf_{t\to\infty} u(t,x) \ge \overline{u}_j \quad \text{locally uniformly in } \mathbb{R}.$$

Moreover, $1 \ge \underline{u}_j = (1-k)\sum_{i=0}^{j-1}(hk)^i$ for every $j \ge 1$. This implies that $hk < 1$ and

$$\lim_{j\to\infty} (\underline{u}_j, \overline{v}_j) = (\frac{1-k}{1-hk}, \frac{1-h}{1-hk}).$$

Since $\overline{v}_j > 0$ for all j we further deduce $h \le 1$.

Summarising, we see that case (a) must happen when $h > 1$. When $h = 1$, if case (a) happens then the above discussion indicates that (4.2) holds, and if case (b) happens, then

$$\lim_{j\to\infty} (\underline{u}_j, \overline{v}_j) = (\frac{1-k}{1-hk}, \frac{1-h}{1-hk}) = (1, 0),$$

which, in view of (4.3), again implies (4.2). Therefore $h \ge 1$ always leads to (4.2). This concludes Step 2.

Step 3. We show that $h \in (0,1)$ implies

$$\lim_{t\to\infty} u(t,x) = \frac{1-k}{1-hk},\ \lim_{t\to\infty} v(t,x) = \frac{1-h}{1-hk} \text{ locally uniformly in } \mathbb{R}.$$

In this situation, apart from the sequences $\{\underline{u}_j\}$ and $\{\overline{v}_j\}$ obtained in Step 2, we can use $0 < h < 1$ to define another two analogous sequences $\{\overline{u}_j\}$ and $\{\underline{v}_j\}$ with $\underline{v}_1 = 1 - h$ such that

$$\overline{u}_{j+1} = 1 - k\underline{v}_j,\ \underline{v}_{j+1} = 1 - h\overline{u}_{j+1} \text{ for } j = 1, 2, ...,$$
$$\limsup_{t\to\infty} u(t,x) \le \overline{u}_j \ \text{ locally uniformly in } \mathbb{R},$$
$$\liminf_{t\to\infty} v(t,x) \ge \underline{v}_j \ \text{ locally uniformly in } \mathbb{R}.$$

It should be noted that $0 < k < 1$ and $0 < h < 1$ guarantee that these sequences are defined for all $j \ge 1$.

It follows that

$$\underline{v}_j = (1-h)\sum_{i=0}^{j-1}(hk)^i \to \frac{1-h}{1-hk} \text{ as } j \to \infty,$$

and so $\overline{u}_j = 1 - k\underline{v}_j \to \frac{1-k}{1-hk}$ as $j \to \infty$. We thus obtain

$$\limsup_{t\to\infty} u(t,x) \le \frac{1-k}{1-hk},\ \liminf_{t\to\infty} v(t,x) \ge \frac{1-h}{1-hk} \text{ locally uniformly for } x \in \mathbb{R}.$$

From the sequences $\{\underline{u}_j\}$ and $\{\overline{v}_j\}$ obtained in Step 2, we also have

$$\liminf_{t\to\infty} u(t,x) \ge \frac{1-k}{1-hk},\ \limsup_{t\to\infty} v(t,x) \le \frac{1-h}{1-hk} \text{ locally uniformly for } x \in \mathbb{R}.$$

The proof of Theorem 1.4 is now complete. □

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