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LONG-TIME DYNAMICS OF A COMPETITION MODEL WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES: CHANCES OF SUCCESSFUL INVASION

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Abstract. This is a continuation of our work [10] to investigate the long-time dynamics of a two species competition model of Lotka-Volterra type with nonlocal diffusions, where the territory (represented by the real line $\mathbb{R}$) of a native species with density $v(t,x)$, is invaded by a competitor with density $u(t,x)$, via two fronts, $x=g(t)$ on the left and $x=h(t)$ on the right. So the population range of u is the evolving interval $[g(t),h(t)]$ and the reaction-diffusion equation for u has two free boundaries, with $g(t)$ decreasing in t and $h(t)$ increasing in t. Let $h_\infty:=h(\infty)\leq\infty$ and $g_\infty:=g(\infty)\geq-\infty$. In [10], we obtained detailed descriptions of the long-time dynamics of the model according to whether $h_\infty-g_\infty$ is ∞ or finite. In the latter case, we demonstrated in what sense the invader u vanishes in the long run and v survives the invasion, while in the former case, we obtained a rather satisfactory description of the long-time asymptotic limits of $u(t,x)$ and $v(t,x)$ when the parameter k in the model is less than 1. In the current paper, we obtain sharp criteria to distinguish the case $h_\infty-g_\infty=\infty$ from the case $h_\infty-g_\infty$ is finite. Moreover, for the case $k\geq 1$ and u is a weak competitor, we obtain biologically meaningful conditions that guarantee the vanishing of the invader u, and reveal chances for u to invade successfully. In particular, we demonstrate that both $h_\infty=\infty=-g_\infty$ and $h_\infty=\infty$ but g_∞ is finite are possible; the latter seems to be the first example for this kind of population models, with either local or nonlocal diffusion.

Keywords: Nonlocal diffusion; Free boundary; Competition.

AMS Subject Classification (2000): 35K57, 35R20

1. Introduction

We continue our work [10] on the following Lotka-Volterra type competition model with nonlocal diffusion and free boundaries

$$
\begin{cases}
u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)\mathrm{d}y - d_1 u + u(1-u-kv), & t>0,\ g(t)<x<h(t),\\
v_t = d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)v(t,y)\mathrm{d}y - d_2 v + \gamma v(1-v-hu), & t>0,\ x\in\mathbb{R},\\
u(t,x)=0, & t>0,\ x\notin(g(t),h(t)),\\
h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
h(0)=-g(0)=h_0>0,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x), & x\in\mathbb{R},
\end{cases}
\tag{1.1}
$$

where d_1,d_2,h,k,γ,μ are given positive constants, and the initial functions satisfy

$$
\begin{cases}
u_0\in C(\mathbb{R}),\ u_0(x)=0 \text{ for } |x|\geq h_0,\ u_0(x)>0 \text{ for } |x|<h_0,\\
v_0\in C_b(\mathbb{R}),\ v_0(x)\geq 0,\ v_0(x)\not\equiv 0 \text{ in } \mathbb{R},
\end{cases}
\tag{1.2}
$$

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where $C_b(\mathbb{R})$ is the space of continuous and bounded functions in $\mathbb{R}$.

We assume that the kernel functions J_1 and J_2 satisfy

$$(\mathbf{J}): \quad J_i \in C_b(\mathbb{R}),\ J_i(x) = J_i(-x) \geq 0,\ J_i(0) > 0,\ \int_{\mathbb{R}} J_i(x)dx = 1 \text{ for } i = 1, 2.$$

Under these assumptions, it is known that system (1.1) has a unique solution (u, v, g, h) defined for all $t > 0$ (see [3]). Moreover,

$$g_\infty := \lim_{t\to\infty} g(t) \in [-\infty, -h_0) \text{ and } h_\infty := \lim_{t\to\infty} h(t) \in (h_0, \infty]$$

always exist.

In [10], the long-time dynamics of (1.1) are described according to the following two cases:

$$(\mathbf{a})\colon\ h_\infty - g_\infty < \infty, \quad (\mathbf{b})\colon\ h_\infty - g_\infty = \infty.$$

For case **(a)**, we have proved the following result.

Theorem A. *Assume that* $(\mathbf{J})$ *holds and* (u, v, g, h) *is the unique solution of* (1.1). *If* $h_\infty - g_\infty < \infty$, *then necessarily*

$$d_1 > 1 - k \text{ and } \lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}) \leq k - 1. \tag{1.3}$$

moreover

$$\begin{cases} \lim\limits_{t\to\infty} \displaystyle\int_{\mathbb{R}} u(t,x)dx = 0, \\ \lim\limits_{t\to\infty} \displaystyle\int_{L}^{L} |v(t,x) - 1)|dx = 0 \text{ for every } L > 0, \\ \lim\limits_{t\to\infty} v(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R} \setminus (g_\infty, h_\infty). \end{cases} \tag{1.4}$$

Whether (1.4) in Theorem A can be strengthened to

$$\lim_{t\to\infty} \max_{x\in[g(t),h(t)]} u(t,x) = 0 \quad\text{and}\quad \lim_{t\to\infty} v(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R} \tag{1.5}$$

was partially answered in [10] (see Theorem 1.2 there).

For case **(b)**, we have obtained in [10] the following conclusion.

Theorem B. *Assume that* $(\mathbf{J})$ *holds and* (u, v, g, h) *is the unique solution of* (1.1). *If* $h_\infty - g_\infty = \infty$ *and* $k < 1$, *then* $h_\infty = \infty$, $g_\infty = -\infty$ *and*

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = \begin{cases} (1, 0) & \text{if } h \geq 1, \\ (\frac{1-k}{1-hk}, \frac{1-h}{1-hk}) & \text{if } h < 1, \end{cases}$$

where the convergence is locally uniform for $x \in \mathbb{R}$.

Let us recall that, in (1.3), $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)})$ denotes the principal eigenvalue of the following eigenvalue problem

$$\lambda\varphi = \mathcal{L}_\Omega[\varphi](x) := d_1 \left[\int_\Omega J_1(x-y)\varphi(y)\mathrm{d}y - \varphi(x)\right],\ \varphi \in C(\overline{\Omega}), \tag{1.6}$$

with $\Omega = (g_\infty, h_\infty)$. It is well known that, under our assumption $(\mathbf{J})$, for any finite interval Ω, (1.6) has a unique principal eigenvalue $\lambda = \lambda_p(\mathcal{L}_\Omega)$ associated with a positive eigenfunction φ (e.g. [1,5,12]), and it has the following properties:

Proposition C. ([2, Proposition 3.4]) *Assume that* $l > 0$, *and* J_1 *satisfies* $(\mathbf{J})$. *Then*

(i) $\lambda_p(\mathcal{L}_{(a,a+l)}) = \lambda_p(\mathcal{L}_{(0,l)})$ *for all* $a \in \mathbb{R}$,
(ii) $\lambda_p(\mathcal{L}_{(0,l)})$ *is strictly increasing and continuous in* l,
(iii) $\lim\limits_{l\to\infty} \lambda_p(\mathcal{L}_{(0,l)}) = 0$,
(iv) $\lim\limits_{l\to 0} \lambda_p(\mathcal{L}_{(0,l)}) = -d_1$.

Therefore, for every $\sigma \in (0, d_1)$, there exists a unique $l_\sigma > 0$ such that

$$\lambda_p\left(\mathcal{L}_{(0,l_\sigma)}\right) = -\sigma.$$

We are now ready to describe our results in this paper. Firstly, we examine exactly when $h_\infty - g_\infty < \infty$ and $h_\infty - g_\infty = \infty$, respectively, happens. Then we focus on the situation that u is a weak competitor ($k \geq 1 > h$) and reveal some interesting phenomena; in particular, we will find conditions for u to invade successfully, with $h_\infty = \infty$, $g_\infty = -\infty$, as well as with $h_\infty = \infty$ and g_∞ finite.

By Theorem A, the fact $h_\infty - g_\infty < \infty$ implies that, $\int_{g(t)}^{h(t)} u(t,x)dx$, the total population of u at time t, converges to 0 as $t \to \infty$, so the invading competitor u vanishes in the long run. We will call this the **vanishing** (of u) case.

The indentity $h_\infty - g_\infty = \infty$ means that the size of the population range of u at time t, given by $h(t) - g(t)$, converges to ∞ as $t \to \infty$, and we will call this the **spreading** (of u) case. Theorem B gives a precise description for the population densities $u(t,x)$ and $v(t,x)$ in this case when $k < 1$. We will demonstrate below that when $k \geq 1$, more complicated dynamics may arise (see Theorems 1.2, 1.3, 1.4 and 1.5).

To describe the criteria governing spreading and vanishing (of u), we will regard μ as a parameter in certain situations.

Theorem 1.1. *Suppose that* (**J**) *holds. Then the following conclusions are valid:*

(i) *If $k < 1$ and $d_1 \leq 1 - k$, then we always have $h_\infty - g_\infty = \infty$.*
(ii) *If $k < 1$, $d_1 > 1 - k$ and $2h_0 \geq l_{1-k}$, then again $h_\infty - g_\infty = \infty$ always holds.*
(iii) *If $k < 1$, $d_1 > 1 - k$ and $2h_0 < l_{1-k}$, then there exists $\mu_* \in [0, \infty)$, depending on (u_0, v_0), such that $h_\infty - g_\infty = \infty$ exactly when $\mu > \mu_*$; moreover, $\mu_* > 0$ if $1 < d_1$ and $2h_0 < l_1$.*
(iv) *If $k \geq 1$, then there exists $\mu_* \in [0, \infty]$, depending on (u_0, v_0, h_0), such that $h_\infty - g_\infty = \infty$ exactly when $\mu > \mu_*$; moreover, $\mu_* \in (0, \infty]$ when $k > 1 > h$.*

The above results indicate that $1 - k$ serves as a critical diffusion rate for u: If its diffusion rate $d_1 \leq 1-k$, then successful invasion is guaranteed regardless of the choice of the initial data (u_0, v_0, h_0) as long as they are admissible, namely satisfying (1.2); if $d_1 > 1 - k > 0$, then the size of the initial population range $2h_0$ becomes crucial and l_{1-k} is a critical value for this initial size, and successful invasion is guaranteed when $2h_0 \geq l_{1-k}$. When $k < 1$, if both the diffusion rate and the size of the initial population range of u are below their respective critical values, then the value of the parameter μ becomes important, and there exists a critical value μ_*, depending on (u_0, v_0), such that the invasion is successful if and only if $\mu > \mu_*$. Similarly, when $k \geq 1$, there exists a critical value μ_*, depending on (u_0, v_0, h_0), such that the invasion is successful if and only if $\mu > \mu_*$.

Next we regard $\mu > 0$ as a fixed given constant, and look for biologically meaningful sufficient conditions guaranteeing vanishing and spreading of u, respectively. We will focus on the weak-strong competition case with u the weak competitor, namely

$$k > 1 > h.$$

In some of our results, $k = 1$ is also allowed.

Similar to the corresponding local diffusion model considered in [8] (see Theorem 3.3 there), the invasion of the weak competitor u will definitely fail if the native species v is already well established at time $t = 0$, namely

$$\inf_{x\in\mathbb{R}} v_0(x) > 0. \tag{1.7}$$

Indeed, following the proof of Theorem 3.3 in [14], one can easily show the following result:

Proposition D. *If v_0 satisfies* (1.7) *and $k > 1 > h$, then $h_\infty - g_\infty < \infty$, and as $t \to \infty$,*

$$\begin{cases} u(t,x) \to 0 \text{ uniformly for } x \in [g(t), h(t)], \\ v(t,x) \to 1 \text{ uniformly for } x \in \mathbb{R}. \end{cases}$$

Another situation that the invasion of u always fails is when u has the same dispersal strategy and the same growth rate to the stronger native species v, as described in the following result.

Theorem 1.2. *If $k \geq 1 > h$, $d_1 = d_2 = d$, $\gamma = 1$, $J_1 = J_2 = J$ and*

$$\int_0^\infty e^{\lambda x} J(x)dx < \infty \quad \text{for some } \lambda > 0, \tag{1.8}$$

then for any (u_0, v_0) satisfying (1.2), $h_\infty - g_\infty < \infty$ and as $t \to \infty$,

$$\begin{cases} u(t,x) \to 0 \text{ uniformly for } x \in [g(t), h(t)], \\ v(t,x) \to 1 \text{ locally uniformly for } x \in \mathbb{R}. \end{cases}$$

The assumption (1.8) guarantees that, in the absence of u, the species v has a finite spreading speed, which is helpful for our proof, but we believe it is only a technical condition. A kernel function satisfying (1.8) is known as a "thin-tailed" kernel in the literature.

Let use note that Proposition D and Theorem 1.2 are examples where $\mu_* = \infty$ in Theorem 1.1 (iv).

The next result indicates that when $k > 1 > h$, the native species v never vanishes.

Theorem 1.3. *If $k > 1 > h$, then for any (u_0, v_0) satisfying (1.2), and any $L > 0$, we have*

$$\begin{cases} \lim_{t\to\infty} u(t,x) = 0 & \text{uniformly for } x \in [-L, L], \\ \lim_{t\to\infty} v(t,x) = 1 & \text{uniformly for } x \in [-L, L]. \end{cases}$$

If the native species v is not well established at time $t = 0$, for example, if v_0 is compactly supported (see Remark 1.5 (i) below for other natural choices of v_0), we will show that there are indeed chances for the weak competitor u to invade successfully and establish itself in an increasing band behind the invasion front which goes to infinity as $t \to \infty$. Moreover, the invasion can succeed in both fronts (i.e., $h_\infty = \infty$, $g_\infty = -\infty$), or just in one front ($h_\infty = \infty$, g_∞ is finite).

To achieve these, we assume that J_1 and J_2 both have compact support (for technical reasons), and the dispersal strategy of u makes it a faster spreader than v, in the sense explained in the next several paragraphs, based on the notion of asymptotic spreading speed described in Proposition E and in [7].

For a kernel function J satisfying (**J**) and (1.8), consider the Cauchy problem of the logistic equation

$$\begin{cases} w_t = d\big[\int_{\mathbb{R}} J(x-y)w(t,y)dy - w\big] + aw - bw^2, & t > 0,\ x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

where a, b and d are positive constants, $w_0(x) \geq 0$ is a continuous function with nonempty compact support.

It is well known (see, for example, [4,6,13,15]) that the following results hold for (1.9).

Proposition E. *The asymptotic spreading speed determined by (1.9) is given by*

$$c^* := \min_{\lambda>0} \frac{1}{\lambda}\left(d\int_{\mathbb{R}} J(x)e^{\lambda x}dx - d + a\right),$$

namely, for any small $\epsilon > 0$,

$$\begin{cases} \lim_{t\to\infty} \max_{|x|\leq(c^*-\epsilon)t} |w(t,x) - \frac{a}{b}| = 0, \\ \lim_{t\to\infty} \max_{|x|\geq(c^*+\epsilon)t} w(t,x) = 0. \end{cases}$$

Moreover, for any $c \geq c^*$, there exists a monotone function $\phi = \phi_c \in C^1(\mathbb{R})$, called a traveling wave of (1.9) *with speed c, satisfying*

$$\begin{cases} d\displaystyle\int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi + c\phi' + a\phi - b\phi^2 = 0, \quad -\infty < x < \infty, \\ \phi(-\infty) = 1, \quad \phi(\infty) = 0. \end{cases}$$

Such a traveling wave is unique up to a translation of x, and no such traveling wave exists for speed $c < c^*$.

Since

$$\int_{\mathbb{R}} J(x)e^{\lambda x}dx = \int_0^{\infty} J(x)(e^{\lambda x} + e^{-\lambda x})dx > \int_{\mathbb{R}} J(x)dx = 1,$$

we see that

$$c^* > d \min_{\lambda > 0} \frac{1}{\lambda}\left(\int_{\mathbb{R}} J(x)e^{\lambda x}dx - 1\right) \to \infty \text{ as } d \to \infty. \tag{1.10}$$

For our competition system (1.1), in the absence of u, clearly v satisfies (1.9) with $(d, J, a, b, w_0) = (d_2, J_2, \gamma, \gamma, v_0)$. So the asymptotic spreading speed of v (in the absence of u, and with v_0 compactly supported) is

$$C_2 := \min_{\lambda>0} \frac{1}{\lambda}\left(d_2 \int_{\mathbb{R}} J_2(x)e^{\lambda x}dx - d_2 + \gamma\right).$$

By [7], in the absence of v, the asymptotic spreading speed of u, denoted by $c_1 = c_1(\mu)$, satisfies

$$0 < c_1(\mu) < C_1, \ \lim_{\mu\to\infty} c_1(\mu) = C_1 := \min_{\lambda>0} \frac{1}{\lambda}\left(d_1 \int_{\mathbb{R}} J_1(x)e^{\lambda x}dx - d_1 + 1\right).$$

We will say that **u is a faster spreader than v** if $c_1 > C_2$, which is guaranteed, for instance, if $C_1 > C_2$ and μ is sufficiently large. By (1.10) we see that $C_1 > C_2$ is guaranteed if d_1 is sufficiently large when the other parameters are fixed.

Theorem 1.4. *Suppose that v_0 is compactly supported, $k \geq 1 > h$, J_1 and J_2 have compact support and $c_1 > C_2$. Then one can find initial functions u_0 for u such that $(g_\infty, h_\infty) = (-\infty, \infty)$.*

Theorem 1.5. *In Theorem 1.4, suppose additionally $k(1-h) > 1$, then there exist initial functions u_0 such that $h_\infty = \infty$ and g_∞ is finite.*

Remark 1.6. *In Theorems 1.4 and 1.5, the following additional results hold:*

(i) *The assumption that v_0 has compact support can be replaced by some more natural assumptions. For example, if $V(t,x)$ stands for the solution of* (1.9) *with $(d, J, a, b) = (d_2, J_2, \gamma, \gamma)$, where the initial function w_0 is compactly supported, then it is easy to check that the conclusions in Theorems 1.4 and 1.5 remain valid for any v_0 satisfying*

$$0 < v_0(x) \leq V(t_0, x) \text{ for some } t_0 > 0 \text{ and all } x \in \mathbb{R}.$$

(ii) *The behaviours of the density functions $u(t,x)$ and $v(t,x)$ in Theorems 1.4 and 1.5 are described in Remarks 3.1 and 3.2 later in the paper, immediately after the respective proofs of these theorems.*

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.2. Section 3 is devoted to the proof of Theorems 1.4 and 1.5. The proof of Theorems 1.1 and 1.3 are given in Section 4. The final Section 5 is an appendix, where we list several comparison principles used in this paper, whose proofs are not included, as they are simple variations of existing ones.

2. Proof of Theorem 1.2

It is clear that $v(t,x)>0$ for all $t>0$ and $x\in\mathbb{R}$. Therefore, for fixed $t_0>0$, there exists $\alpha_0>0$ such that

$$u(t_0,x)\le \alpha_0 v(t_0,x)\ \text{ for }\ x\in\mathbb{R},$$

where we have used the assumption $u(t_0,x)=0$ for all $x\in\mathbb{R}\backslash[g(t_0),h(t_0)]$.

Step 1. We show $u(t,x)\le\alpha_0 v(t,x)$ for all $t\ge t_0$ and $x\in\mathbb{R}$.

Denote $\tilde v(t,x):=\alpha_0 v(t,x)$. Then $(u,\tilde v)$ satisfies

$$\begin{cases} u_t=d\displaystyle\int_{g(t)}^{h(t)}J(x-y)u(t,y)\mathrm{d}y-du+u(1-u-k\tilde v/\alpha_0), & t>0,\ g(t)<x<h(t),\\ \tilde v_t=d\displaystyle\int_{\mathbb{R}}J(x-y)\tilde v(t,y)\mathrm{d}y-d\tilde v+\tilde v(1-\tilde v/\alpha_0-hu), & t>0,\ x\in\mathbb{R},\end{cases}$$

Let $w:=\tilde v-u$. Then due to $h<1\le k$, the function $w(t,x)$ satisfies

$$\begin{aligned} w_t&=d\int_{g(t)}^{h(t)}J(x-y)w(t,y)\mathrm{d}y-dw+w\left(1-\frac{\tilde v}{\alpha_0}-\frac{\alpha_0 h+1-k}{\alpha_0}u\right)+\left(1-h+\frac{k-1}{\alpha_0}\right)u^2\\ &\ge d\int_{g(t)}^{h(t)}J(x-y)w(t,y)\mathrm{d}y-dw+w\left(1-\frac{\tilde v}{\alpha_0}-\frac{\alpha_0 h+1-k}{\alpha_0}u\right)\ \text{ for }\ t\ge t_0,\ x\in[g(t),h(t)].\end{aligned}$$

Clearly $w(t,x)\ge 0$ for $x=g(t)$ and $h(t)$, and $w(t_0,x)\ge0$ for $x\in[g(t_0),h(t_0)]$. It then follows from the comparison principle Lemma 5.1 that

$$w(t,x)\ge 0,\ \text{i.e.,}\ u(t,x)\le\alpha_0 v(t,x)\ \text{for}\ \ t\ge t_0,\ x\in\mathbb{R}.$$

Step 2. Estimates of u and v leading to the desired conclusion.

Using the result in Step 1, we see that (u,v) satisfies

$$\begin{cases} u_t\le d\displaystyle\int_{g(t)}^{h(t)}J(x-y)u(t,y)\mathrm{d}y-du+u[1-(1+k/\alpha_0)u], & t\ge t_0,\ g(t)<x<h(t),\\ v_t\ge d\displaystyle\int_{\mathbb{R}}J(x-y)v(t,y)\mathrm{d}y-dv+v[1-(1+h\alpha_0)v], & t\ge t_0,\ x\in\mathbb{R},\end{cases}$$

This allows us to compare u and v with $\hat U$, U and V defined below:

- $(\hat U,\hat h,\hat g)$ is the solution of

$$\begin{cases} \hat U_t=d\displaystyle\int_{\hat g(t)}^{\hat h(t)}J(x-y)\hat U(t,y)\mathrm{d}y-d\hat U+\hat U[1-(1+k/\alpha_0)\hat U], & t>t_0,\ \hat g(t)<x<\hat h(t),\\ \hat h'(t)=\mu\displaystyle\int_{\hat g(t)}^{\hat h(t)}\int_{\hat h(t)}^{\infty}J(x-y)\hat U(t,x)\mathrm{d}y\mathrm{d}x, & t>t_0,\\ \hat g'(t)=-\mu\displaystyle\int_{\hat g(t)}^{\hat h(t)}\int_{-\infty}^{\hat g(t)}J(x-y)\hat U(t,x)\mathrm{d}y\mathrm{d}x, & t>t_0,\\ \hat U(t,x)=0, & t\ge t_0,\ x\notin(\hat g(t),\hat h(t)),\\ \hat h(t_0)=h(t_0),\ \hat g(t_0)=g(t_0),\ \hat U(t_0,x)=u(t_0,x), & g(t_0)\le x\le h(t_0).\end{cases}\tag{2.1}$$

- U is the solution of

$$\begin{cases} U_t=d\displaystyle\int_{\mathbb{R}}J(x-y)U(t,y)\mathrm{d}y-dU+U[1-(1+k/\alpha_0)U], & t>t_0,\ x\in\mathbb{R},\\ U(t_0,x)=u(t_0,x), & t\ge 0,x\in\mathbb{R}.\end{cases}\tag{2.2}$$

- V be the solution of

$$
\begin{cases} V_t = d\displaystyle\int_{\mathbb{R}} J(x-y)V(t,y)\mathrm{d}y - dV + V[1-(1+h\alpha_0)V], & t > t_0,\ x\in\mathbb{R}, \\ V(t_0,x) = v(t_0,x), & x\in\mathbb{R}. \end{cases} \tag{2.3}
$$

By [7, Theorem 1.1 and Proposition 1.3], there exist $C_1 > 0$ and $c_1 = c_1(\mu) \in (0, C_1)$ such that, for any small $\epsilon > 0$,

$$
\begin{cases} \lim\limits_{t\to\infty} \dfrac{\hat{g}(t)}{t} = \lim\limits_{t\to\infty} \dfrac{\hat{h}(t)}{t} = c_1, \\ \lim\limits_{t\to\infty} \max\limits_{|x|\le(C_1-\epsilon)t} |U(t,x) - \dfrac{1}{1+k/\alpha_0}| = 0, \\ \lim\limits_{t\to\infty} \max\limits_{|x|\le(C_1-\epsilon)t} |V(t,x) - \dfrac{1}{1+h\alpha_0}| = 0. \end{cases} \tag{2.4}
$$

Using suitable versions of the comparison principle we have

$$
u(t,x) \le U(t,x),\ v(t,x) \ge V(t,x) \text{ for } t\ge t_0,\ x\in\mathbb{R},
$$

and

$$
u(t,x) \le \hat{U}(t,x),\ [g(t),h(t)] \subset [\hat{g}(t),\hat{h}(t)] \text{ for } t\ge t_0,\ x\in[g(t),h(t)].
$$

Hence, by (2.4), for $\tilde{C} := (c_1 + C_1)/2$ and all large t, say $t \ge t_1$, we have

$$
\begin{cases} u(t,x) \le \dfrac{1}{1+k/\alpha_0} + o_t(1) & \text{for } \ x\in[g(t),h(t)] \subset [-\tilde{C}t, \tilde{C}t], \\ v(t,x) \ge \dfrac{1}{1+\alpha_0 h} - o_t(1) & \text{for } \ x\in[-\tilde{C}t,\tilde{C}t], \end{cases} \tag{2.5}
$$

where $o_t(1)$ denotes a generic constant satisfying $o_t(1) \to 0$ as $t\to\infty$.

Case 1: $\beta_0 := 1 - \frac{k}{1+\alpha_0 h} < 0$. In this case, by (2.5), for all large t, say $t \ge t_2 > t_1$,

$$
u_t \le d_1 \int_{g(t)}^{h(t)} J(x-y)u(t,y)\mathrm{d}y - d_1 u + \frac{\beta_0}{2}u \ \text{ for } \ g(t)\le x\le h(t),
$$

which implies, by comparison with the corresponding ODE, that

$$
u(t,x) \le e^{(t-t_2)\beta_0/2} \max_{x\in[g(t_2),h(t_2)]} u(t_2,x) \text{ for } t\ge t_2 \text{ and } x\in[g(t),h(t)].
$$

Thus, $u(t,x) \to 0$ as $t\to\infty$ uniformly for $x\in[g(t),h(t)]$, and by the free boundary equation for $h'(t)$ and the estimate

$$
\infty > J_{\max} := \int_0^\infty J(y)y\mathrm{d}y = \int_{-\infty}^0\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x \ge \int_{g(t)}^{h(t)}\int_{h(t)}^\infty J(x-y)\mathrm{d}y\mathrm{d}x,
$$

we obtain, for some constant $C > 0$,

$$
\begin{aligned} h(t) &= h(t_2) + \int_{t_2}^t h'(s)\mathrm{d}s \le h(t_2) + C\mu J_{\max}\int_{t_2}^t e^{\beta_0(s-t_2)/2}\mathrm{d}s \\ &\le h(t_2) + \frac{2C\mu J_{\max}}{|\beta_0|} < \infty \ \text{ for } \ \ t\ge t_2. \end{aligned}
$$

So $h_\infty < \infty$. Similarly we can show $g_\infty > -\infty$. Moreover, using the fact $u(t,x)\to 0$ as $t\to\infty$ uniformly for $x\in[g(t),h(t)]$, and the comparison principle, it can be easily shown that $v(t,x)\to 1$ locally uniformly for $x\in\mathbb{R}$ as $t\to\infty$.

So in the case $\beta_0 < 0$, the desired conclusions hold.

Case 2: $\beta_0 = 1 - \frac{k}{1/\alpha_0 + h} \ge 0$.

In this case we can repeat the method leading to (2.5) in the following way. Firstly since $u(t,x)\equiv 0$ for $x\notin[g(t),h(t)]$, we see from (2.5) that

$$
u(t,x) \le [\alpha_1 + o_t(1)]v(t,x) \text{ for all large } t \text{ and } \ x\in\mathbb{R},
$$

with

$$\alpha_1 := \frac{\alpha_0+\alpha_0^2 h}{\alpha_0+k} = \alpha_0 - \frac{\alpha_0(k-1)+\alpha_0^2(1-h)}{\alpha_0+k}.$$

Clearly, $\alpha_1 < \alpha_0$. We are now in a position to repeat the earlier argument to obtain an analogue of (2.5), namely

$$\begin{cases} u(t,x) \le \frac{1}{1+k/\alpha_1} + o_t(1) & \text{for all large } t \text{ and } \ x \in [g(t),h(t)] \subset [-\tilde{C}t, \tilde{C}t], \\ v(t,x) \ge \frac{1}{1+\alpha_1 h} - o_t(1) & \text{for all large } t \text{ and } \ x \in [-\tilde{C}t, \tilde{C}t]. \end{cases}$$

Then, we estimate u and v according to whether $\beta_1 := 1 - \frac{k}{1+\alpha_1 h} < 0$ or $\beta_1 \ge 0$.

If $\beta_1 < 0$, then we can deduce the desired conclusions by similar arguments as in Case 1 above.

If $\beta_1 \ge 0$, then we analogously define

$$\alpha_2 := \frac{\alpha_1+\alpha_1^2 h}{\alpha_1+k} = \alpha_1 - \frac{\alpha_1(k-1)+\alpha_1^2(1-h)}{\alpha_1+k},$$

and obtain

$$u(t,x) \le [\alpha_2 + o_t(1)]v(t,x) \text{ for all large } t \text{ and } \ x \in \mathbb{R}.$$

Following this procedure, we obtain two decreasing sequences $\{\alpha_m\}$ and $\{\beta_m\}$ given by

$$\alpha_m := \frac{\alpha_{m-1}+\alpha_{m-1}^2 h}{\alpha_{m-1}+k}, \ \beta_m := 1 - \frac{k}{(1+\alpha_{m-1}h)}, \ m = 1, 2, ...$$

We claim that there is an integer $m_0 \ge 2$ such that $\beta_{m_0-1} \ge 0$ and

$$\beta_{m_0} = 1 - \frac{k}{1+\alpha_{m_0} h} < 0 \ \text{ or equivalently } \ \alpha_{m_0} < \frac{k-1}{h}.$$

Note that the cases of $m_0 = 0$ and $m_0 = 1$ have already been considered above. If $\alpha_m \ge \alpha_* := (k-1)/h$ for all $m \ge 1$, then

$$\begin{aligned} \alpha_m - \alpha_{m-1} &= \frac{\alpha_{m-1}+\alpha_{m-1}^2 h}{\alpha_{m-1}+k} - \alpha_{m-1} \\ &= -\frac{\alpha_{m-1}(k-1)+\alpha_{m-1}^2(1-h)}{\alpha_{m-1}+k} \\ &\le -\frac{\alpha_*(k-1)+\alpha_*^2(1-h)}{\alpha_*+k} < 0, \end{aligned}$$

which implies $a_m \to -\infty$ as $m \to \infty$. This is a contradiction to the assumption $\alpha_m \ge (k-1)/h$. Hence, there is a finite integer $m_0 \ge 2$ such that $\alpha_{m_0} < (k-1)/h$ and so $\beta_{m_0} < 0$.

Using $\beta_{m_0} < 0$, we could show $h_\infty - g_\infty < \infty$, $u(t,x) \to 0$ as $t \to \infty$ uniformly for $x \in [g(t), h(t)]$, and $v(t,x) \to 1$ as $t \to \infty$ locally uniformly for $x \in \mathbb{R}$, as in Case 1 above. The proof of Theorem 1.2 is now complete. □

3. Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Let $V(t,x)$ denote the solution of (1.9) with $(d,J,a,b,w_0) = (d_2, J_2, \gamma, \gamma, v_0)$. The comparison principle yields $v(t,x) \le V(t,x)$ for all $t \ge 0$ and $x \in \mathbb{R}$.

By [7], with

$$f_\rho(u) := u(1-\rho-u),$$

for each small $\rho > 0$, there is a monotone function $\phi = \phi_\rho \in C^1((-\infty,0])$ and a constant $c_1^\rho > 0$ such that

$$\begin{cases} d_1 \displaystyle\int_{-\infty}^0 J_1(x-y)\phi(y)dy - d_1\phi(x) + c_1^\rho \phi'(x) + f_\rho(\phi(x)) = 0, & -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \\ c_1^\rho = \mu \displaystyle\int_{-\infty}^0 \int_0^{+\infty} J(x-y)\phi(x)dydx. \end{cases}$$

Moreover, $c_1^\rho \to c_1$ as $\rho \to 0$. Therefore we can fix $\rho > 0$ small so that $c_1^\rho > C_2$.

Fix small $\epsilon > 0$ so that $C_2 + \epsilon < c_1^\rho$. By Proposition E in Section 1 here and Lemma 2.2 in [?], there is $L_0 > 0$ such that

$$\sup_{|x|\geq (C_2+\epsilon)t+L_0} v(t,x) \leq \sup_{|x|\geq (C_2+\epsilon)t+L_0} V(t,x) \leq \frac{\rho}{k} \ \text{ for all } \ t \geq 0.$$

Fix $S > 0$ such that the support of J_1 is contained in the interval $[-S, S]$, and fix K satisfying

$$1 > K > \frac{(1-2\epsilon)c_1^\rho + (C_2+\epsilon)}{2(1-2\epsilon)c_1^\rho}. \tag{3.1}$$

Then define, for some large constant $L > L_0/(2K-1)$,

$$\underline{h}(t) := c_1^\rho(1-2\epsilon)t + L,$$

and

$$\underline{u}(t,x) := \begin{cases} (1-\epsilon)\phi(x - \underline{h}(t)), & t \geq 0,\ K\underline{h}(t) \leq |x| \leq \underline{h}(t), \\ (1-\epsilon)\phi\big((2K-1)\underline{h}(t) - x\big), & t \geq 0,\ (2K-1)\underline{h}(t) \leq |x| \leq K\underline{h}(t). \end{cases}$$

The choice of K implies

$$(2K-1)\underline{h}(t) > (C_2+\epsilon)t + L_0 \ \text{ for } \ t \geq 0,$$

and so $kv(t,x) \leq \rho$ for $t \geq 0$ and $|x| \geq (2K-1)\underline{h}(t)$. Therefore, if $T > 0$ and $h(t) > (2K-1)\underline{h}(t)$ for $t \in [0,T]$, then

$$\begin{cases} u_t \geq d_1 \displaystyle\int_{(2K-1)\underline{h}(t)}^{h(t)} J_1(x-y)u(t,y)\mathrm{d}y - d_1 u + f_\rho(u), & t \in (0,T],\ x \in ((2K-1)\underline{h}(t), h(t)), \\ h'(t) \geq \mu \displaystyle\int_{(2K-1)\underline{h}(t)}^{h(t)} \int_{h(t)}^{+\infty} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t \in (0,T], \\ u(t,(2K-1)\underline{h}(t)) > 0 = u(t,h(t)), & t \in (0,T]. \end{cases} \tag{3.2}$$

We will show that, with $\underline{h}$ denoting $\underline{h}(t)$, for sufficiently large L,

$$\begin{cases} \underline{u}_t \leq d_1 \displaystyle\int_{(2K-1)\underline{h}}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u} + f_\rho(\underline{u}), & t > 0,\ x \in ((2K-1)\underline{h}, \underline{h})\backslash\{K\underline{h}\}, \\ \underline{h}' \leq \mu \displaystyle\int_{(2K-1)\underline{h}}^{\underline{h}} \int_{\underline{h}}^{+\infty} J_1(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x, & t > 0, \\ \underline{u}(t,\underline{h}) = \underline{u}(t,(2K-1)\underline{h}) = 0, & t \geq 0. \end{cases} \tag{3.3}$$

Assuming the validity of (3.3) for the time being, let us see how (u_0, h_0) can be chosen to obtain the desired conclusion.

Clearly, if $h_0 \geq \underline{h}(0)$ and $u_0(x) \geq \underline{u}(0,x)$ for $x \in [(2K-1)\underline{h}(0), \underline{h}(0)]$, then due to (3.2), (3.3) and $g(t) \leq -h_0 < 0 < (2K-1)\underline{h}(t)$,

$$u(t,x) > 0 = \underline{u}(t,x) \text{ for } x = (2K-1)\underline{h}(t),\ t \geq 0,$$

we can use the comparison principle to conclude that

$$h(t) \geq \underline{h}(t),\ u(t,x) \geq \underline{u}(t,x) \text{ for } \ t \geq 0,\ x \in [(2k-1)\underline{h}(t), \underline{h}(t)].$$

Hence, $h(t) \to \infty$ as $t \to \infty$.

Similarly, we can show that if $h_0 \geq \underline{h}(0)$ and $u_0(x) \geq \underline{u}(0,-x)$ for $x \in [-\underline{h}(0), -(2K-1)\underline{h}(0)]$, then

$$g(t) \leq -\underline{h}(t),\ u(t,x) \geq \underline{u}(t,-x) \text{ for } \ t \geq 0,\ x \in [-\underline{h}(t), -(2k-1)\underline{h}(t)].$$

Therefore $g(t) \to -\infty$ as $t \to \infty$.

Hence the desired conclusion $(g_\infty, h_\infty) = (-\infty, \infty)$ holds if the initial data (u_0, h_0) satisfies

$$h_0 > \underline{h}(0),\ u_0(x) \geq \underline{u}(0,x) \text{ for } |x| \in [(2k-1)\underline{h}(0), \underline{h}(0)].$$

To complete the proof, it remains to prove (3.3). Clearly, the third equation in (3.3) follows directly from the definition of $\underline{u}$. Now, we verify the first two inequalities in (3.3). Recalling that J_1 has compact support contained in $[-S,S]$, by a direct computation we obtain, for large L and all $t>0$,

$$\begin{aligned}
&\mu\int_{(2K-1)\underline{h}}^{\underline{h}}\int_{\underline{h}}^{+\infty} J_1(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \geq \mu\int_{K\underline{h}}^{\underline{h}}\underline{u}(t,x)\int_{\underline{h}}^{+\infty} J_1(x-y)\mathrm{d}y\mathrm{d}x\\
=&\mu(1-\epsilon)\int_{(K-1)\underline{h}}^{0}\phi(x)\int_{0}^{+\infty} J_1(x-y)\mathrm{d}y\mathrm{d}x = \mu(1-\epsilon)\int_{-\infty}^{0}\phi(x)\int_{0}^{+\infty} J_1(x-y)\mathrm{d}y\mathrm{d}x\\
=&(1-\epsilon)c_1 > \underline{h}'(t).
\end{aligned}$$

This showed the validity of the second inequality in (3.3).

We next varify the first inequality in (3.3).

Case 1. $x\in(K\underline{h},\underline{h})$.

In view of the equation satisfied by ϕ, we deduce for $t>0$ and $x\in(K\underline{h},\underline{h}]$,

$$\begin{aligned}
\underline{u}_t(t,x) &= -(1-\epsilon)c_1^\rho(1-2\epsilon)\phi'(x-\underline{h}) \leq -(1-\epsilon)c_1^\rho\phi'(x-\underline{h})\\
&= (1-\epsilon)\left[d_1\int_{-\infty}^{\underline{h}} J_1(x-y)\phi(y-\underline{h})\mathrm{d}y - d_1\phi(x-\underline{h}) + f_\rho(\phi(x-\underline{h}))\right]\\
&= d_1(1-\epsilon)\int_{x-S}^{\underline{h}} J_1(x-y)\phi(y-\underline{h})\mathrm{d}y - d_1\underline{u}(t,x) + (1-\epsilon)f_\rho(\phi(x-\underline{h})) := A.
\end{aligned}$$

If $x\in[K\underline{h}+S,\underline{h}]$, then $x-y\geq S$ when $y\leq K\underline{h}$, and therefore

$$\begin{aligned}
\underline{u}_t(t,x) \leq A &= d_1\int_{K\underline{h}}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u}(t,x) + (1-\epsilon)f_\rho(\phi(x-\underline{h}))\\
&\leq d_1\int_{K\underline{h}}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u} + f_\rho(\underline{u}),
\end{aligned}$$

where we have used the fact that $(1-\epsilon)f_\rho(\phi(x-\underline{h})) \leq f_\rho((1-\epsilon)\phi(x-\underline{h}))$.

For $x\in(K\underline{h},K\underline{h}+S]$, we have

$$\begin{aligned}
\underline{u}_t(t,x) \leq A = d_1 &\int_{K\underline{h}-S}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u}(t,x) + f_\rho(\underline{u})\\
&+\int_{K\underline{h}-S}^{K\underline{h}} J_1(x-y)[(1-\epsilon)\phi(y-\underline{h}) - \underline{u}(t,y)]\mathrm{d}y + (1-\epsilon)f_\rho(\phi(x-\underline{h})) - f_\rho(\underline{u})\\
= d_1 &\int_{K\underline{h}-S}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u}(t,x) + f_\rho(\underline{u})\\
&+\int_{K\underline{h}-S}^{K\underline{h}} J_1(x-y)[(1-\epsilon)\phi(y-\underline{h}) - \underline{u}(t,y)]\mathrm{d}y - (\epsilon-\epsilon^2)\phi^2(x-\underline{h}).
\end{aligned}$$

From the fact that $\phi(s)\to 1-\rho$ as $s\to-\infty$, we deduce

$$\begin{aligned}
&\int_{K\underline{h}-S}^{K\underline{h}} J_1(x-y)[(1-\epsilon)\phi(y-\underline{h}) - \underline{u}(t,y)]\mathrm{d}y - (\epsilon-\epsilon^2)\phi^2(x-\underline{h})\\
&= o(1) - (\epsilon-\epsilon^2)(1-\rho) < 0
\end{aligned}$$

for large L and all $t>0$. Therefore we also have

$$\underline{u}_t(t,x) \leq d_1\int_{K\underline{h}-S}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u} + f_\rho(\underline{u}).$$

Case 2. $x\in[(2K-1)\underline{h},K\underline{h}]$.

Note that for each $t\geq 0$, the function $\underline{u}(t,x)$ is symmetric with respect to $x=Kh(t)$:

$$\underline{u}(t,x) = \underline{u}(t,2K\underline{h}-x).$$

Thus, for $x \in [(2K-1)\underline{h}, K\underline{h}]$,

$$\begin{aligned} A_1 &:= d_1 \int_{(2K-1)\underline{h}}^{\underline{h}} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u}(t,x) + f_\rho(\underline{u}(t,x)) \\ &= d_1 \int_{(2K-1)\underline{h}}^{\underline{h}} J_1(x-y)\underline{u}(t,2K\underline{h}-y)\mathrm{d}y - d_1\underline{u}(t,2K\underline{h}-x) + f_\rho(\underline{u}(t,2K\underline{h}-x)) \\ &= d_1 \int_{(2K-1)\underline{h}}^{\underline{h}} J_1(2K\underline{h}-x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u}(t,2K\underline{h}-x) + f_\rho(\underline{u}(t,2K\underline{h}-x)). \end{aligned}$$

Since $2K\underline{h}-x \in [K\underline{h}, \underline{h}]$, from the calculations in Case 1, one finds that $A_1 \ge 0$. On the other hand, a direct computation gives, for $x \in ((2K-1)\underline{h}, K\underline{h})$,

$$u_t(t,x) = 2K(1-\epsilon)c_1^\rho(1-2\epsilon)\phi'\big(2Kh(t)-x\big) \le 0.$$

This immediately gives $\underline{u}_t(t,x) \le 0 \le A_1$. Therefore, the first inequality in (3.3) always holds.

The proof of Theorem 1.4 is now complete. □

Remark 3.1. *From the proof of Theorem 1.4, we actually know that*

$$\begin{cases} \limsup_{t\to\infty} \frac{h(t)}{t} \le c_1, \\ \liminf_{t\to\infty} \frac{h(t)}{t} \ge c_1^\rho, \end{cases} \quad \begin{cases} \liminf_{t\to\infty} \frac{g(t)}{t} \ge -c_1, \\ \limsup_{t\to\infty} \frac{g(t)}{t} \le -c_1^\rho, \end{cases}$$

$$\begin{cases} \liminf_{t\to\infty} \min_{(C_2+\epsilon)t\le|x|\le(c_1^\rho-3\epsilon)t} u(t,x) \ge 1-\epsilon, \\ \lim_{t\to\infty} \max_{|x|\ge(C_2+\epsilon)t} v(t,x) = 0. \end{cases}$$

Proof of Theorem 1.5. In this proof, for convenience we will replace the initial population range $[-h_0, h_0]$ of u by some general interval $[g_0, h_0]$ not necessarily symmetric about $x = 0$. Note that such a non-symmetric case can always be reduced to a symmetric one by a simple shift of the variable x: if $(u(t,x), v(t,x), h(t), g(t))$ has the required property with initial range (g_0, h_0) for u, then, with $x_0 := \frac{g_0+h_0}{2}$,

$$(\tilde{u}(t,x), \tilde{v}(t,x), \tilde{h}(t), \tilde{g}(t)) := (u(t,x-x_0), v(t,x-x_0), h(t)-x_0, g(t)-x_0)$$

is a solution with symmetric initial range $(-\frac{h_0-g_0}{2}, \frac{h_0-g_0}{2})$ for u and the desired property. So no generality is lost by considering a general initial range $[g_0, h_0]$ of u.

Since v_0 is compactly supported, from the proof of Theorem 1.4 we see that a lower solution can be constructed on one side of the population range $[g(t), h(t)]$ of u, for example in a subset of $[0, h(t)]$ to guarantee that $h_\infty = \infty$, while there are no specific restrictions imposed on the other side of the population range of u, namely the side $[g(t), 0]$. This suggests that it is perhaps possible to choose an initial function $u_0(x)$ which is sufficiently small for $x \le 0$ such that $g(t)$ remains uniformly bounded for all t while $h(t) \to \infty$ as $t \to \infty$. We show below that this is indeed possible.

Step 1. Construction of an upper solution $(\bar{u}, \bar{g})$.

Let $\alpha > 1$ be a fixed number, and $T > 0$ be a large constant to be specified. Denote $\tilde{t} = t + T$ for $t \in \mathbb{R}$, and define

$$\bar{g} = \bar{g}(t) := -\tilde{L} + \delta(\ln \tilde{t})^{1-\alpha}, \quad t \ge 0$$

and

$$\bar{u}(t,x) := \begin{cases} \dfrac{M}{\tilde{t}[(\ln\tilde{t})^\alpha - 2\bar{g}]}(x-2\bar{g}), & t\ge 0,\ x \in [\bar{g}, (\ln\tilde{t})^\alpha], \\ \dfrac{1-M/\tilde{t}}{C_2\tilde{t}/2-(\ln\tilde{t})^\alpha}[x-(\ln\tilde{t})^\alpha] + \dfrac{M}{\tilde{t}}, & t\ge 0,\ x\in((\ln\tilde{t})^\alpha, +\infty), \end{cases}$$

for some $\delta > 0$, $M > 0$ and large $\tilde{L} > 0$. Let us recall that C_2 is the spreading speed of v in the absence of u. Clearly, $\bar{u}$ is continuous and piecewise linear in x.

We next show that, with $\bar g$ denoting $\bar g(t)$,

$$
\begin{cases}
\bar u_t \geq d_1 \displaystyle\int_{\bar g}^{+\infty} J_1(x-y)\bar u(t,y)\mathrm{d}y - d_1\bar u - \frac{k(1-h)-1}{2}\bar u, & t>0,\ |x| \in (\bar g, C_2\tilde t/2)\backslash\{(\ln\tilde t)^\alpha\},\\
\bar g' \leq -\mu \displaystyle\int_{\bar g}^{+\infty} \bar u(t,x)\int_{-\infty}^{\bar g} J_1(x-y)\mathrm{d}y\mathrm{d}x, & t>0,\\
\bar u(t,\bar g)\geq 0,\ \ \bar u(t,x)\geq 1, & t\geq 0,\ x\geq C_2\tilde t/2.
\end{cases}
\tag{3.4}
$$

It is clearly that the two inequalities in the third line of (3.4) follow directly from the definition of $\bar u$. In the following we check the other inequalities in (3.4).

(a) We verify the second inequality in (3.4).

Suppose that the support of J_1 is contained in the interval $[-S,S]$ for some $S>0$. Then for $T\gg S$,

$$
\begin{aligned}
&-\mu\int_{\bar g}^{+\infty}\bar u(t,x)\int_{-\infty}^{\bar g}J_1(x-y)\mathrm{d}y\mathrm{d}x = -\mu\int_{\bar g}^{\bar g+S}\bar u(t,x)\int_{-\infty}^{\bar g}J_1(x-y)\mathrm{d}y\mathrm{d}x\\
\geq& -\mu\int_{\bar g}^{\bar g+S}\bar u(t,x)\mathrm{d}x \geq -\mu S\bar u(t,\bar g+S) = \frac{-\mu M}{\tilde t[(\ln\tilde t)^\alpha-2\bar g]}(-\bar g+S)\\
\geq& \frac{-\mu M(\tilde L+S)}{\tilde t[(\ln\tilde t)^\alpha+2\tilde L]} \geq \frac{-2\mu M(\tilde L+S)}{\tilde t(\ln\tilde t)^\alpha}.
\end{aligned}
$$

Note that

$$
\bar g'(t) = -\frac{(\alpha-1)\delta}{\tilde t(\ln\tilde t)^\alpha}\ \ \text{for}\ \ t\geq 0.
$$

Hence, the desired inequality holds if

$$
\delta \geq \frac{2\mu M(\tilde L+S)}{\alpha-1}.
$$

(b) We prove the first inequality in (3.4).

A direct calculation gives, for $x\in[\bar g,(\ln\tilde t)^\alpha)$ and $t>0$,

$$
\begin{aligned}
\bar u_t =& \frac{M}{\tilde t[(\ln\tilde t)^\alpha-2\bar g]}(-2\bar g') - \frac{M}{\tilde t[(\ln\tilde t)^\alpha-2\bar g]}(x-2\bar g)\frac{(\ln\tilde t)^\alpha+\alpha(\ln\tilde t)^{\alpha-1}-2\bar g-2\tilde t\bar g'}{\tilde t[(\ln\tilde t)^\alpha-2\bar g]}\\
\geq& -\bar u\frac{(\ln\tilde t)^\alpha+\alpha(\ln\tilde t)^{\alpha-1}-2\bar g-2\tilde t\bar g'}{\tilde t[(\ln\tilde t)^\alpha-2\bar g]} \geq -\frac{k(1-h)-1}{4}\bar u \quad \text{for large } T>0,
\end{aligned}
$$

where we have used the fact that $0>\bar g' = -\dfrac{(\alpha-1)\delta}{\tilde t(\ln\tilde t)^\alpha}$.

For $x>(\ln\tilde t)^\alpha$ and $t>0$, we have

$$
\begin{aligned}
\bar u_t =& \frac{1-M/\tilde t}{C_2\tilde t/2-(\ln\tilde t)^\alpha}[-\alpha\tilde t^{-1}(\ln\tilde t)^{\alpha-1}] - \frac{M}{\tilde t^2}\\
&+\frac{x-(\ln\tilde t)^\alpha}{C_2\tilde t/2-(\ln\tilde t)^\alpha}\frac{M\tilde t^{-2}(C_2\tilde t/2-(\ln\tilde t)^\alpha)-(1-M/\tilde t)[C_2/2-\alpha\tilde t^{-1}(\ln\tilde t)^{\alpha-1}]}{C_2\tilde t/2-(\ln\tilde t)^\alpha}\\
\geq& \frac{4}{C_2\tilde t}[-\alpha\tilde t^{-1}(\ln\tilde t)^{\alpha-1}] - \frac{M}{\tilde t^2} - \frac{x-(\ln\tilde t)^\alpha}{C_2\tilde t/2-(\ln\tilde t)^\alpha}\frac{2}{C_2\tilde t}\\
\geq& -\frac{k(1-h)-1}{4}\bar u \quad \text{for large } T>0.
\end{aligned}
$$

We now estimate the nonlocal diffusion term for such x and t, and show that

$$
A := d_1\int_{\tilde g}^{+\infty} J_1(x-y)\bar u(t,y)\mathrm{d}y - d_1\bar u(t,x) \leq \frac{k(h-1)-1}{4}\bar u \text{ for all large } T>0.
\tag{3.5}
$$

Since $\bar{u}(t,x)$ is a linear function of x for $x \in [\bar{g}, (\ln \tilde{t})^\alpha)$ and for $x \in ((\ln \tilde{t})^\alpha, +\infty)$, respectively, we have, for $x \in [\bar{g}, (\ln \tilde{t})^\alpha - S] \cup [(\ln \tilde{t})^\alpha + S, C_2\tilde{t}/2)$ and $t > 0$,

$$
\begin{aligned}
A &= d_1 \int_{\bar{g}}^{+\infty} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u}(t,x) \le d_1 \int_{x-S}^{x+S} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u}(t,x) \\
&= d_1 \int_{x}^{x+S} J_1(x-y)[\bar{u}(t,y) + \bar{u}(t,2x-y)]\mathrm{d}y - d_1\bar{u}(t,x) \\
&= 2d_1\bar{u}(t,x) \int_{x}^{x+S} J_1(x-y)\mathrm{d}y - d_1\bar{u}(t,x) = 0.
\end{aligned}
$$

Hence (3.5) holds,

For $x \in [(\ln \tilde{t})^\alpha - S, (\ln \tilde{t})^\alpha)$ and $t > 0$, we introduce the linear function (in x)

$$
\tilde{u}_1 := \frac{M}{\tilde{t}[(\ln \tilde{t})^\alpha - 2\bar{g}]}(x - 2\bar{g}),
$$

and obtain, for large $T > 0$,

$$
\begin{aligned}
A &= d_1 \int_{\bar{g}}^{+\infty} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u}(t,x) = d_1 \int_{x-S}^{x+S} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u}(t,x) \\
&= d_1 \int_{x-S}^{x+S} J_1(x-y)\tilde{u}_1(t,y)\mathrm{d}y - d_1\bar{u}(t,x) + d_1 \int_{(\ln \tilde{t})^\alpha}^{x+S} J_1(x-y)[\bar{u}(t,y) - \tilde{u}_1(t,y)]\mathrm{d}y \\
&= d_1 \int_{(\ln \tilde{t})^\alpha}^{x+S} J_1(x-y)[\bar{u}(t,y) - \tilde{u}_1(t,y)]\mathrm{d}y \le d_1 \int_{(\ln \tilde{t})^\alpha}^{(\ln \tilde{t})^\alpha + S} J_1(x-y)[\bar{u}(t,y) - M/\tilde{t}]\mathrm{d}y \\
&= \frac{1 - M/\tilde{t}}{C_2\tilde{t}/2 - (\ln \tilde{t})^\alpha} d_1 \int_{(\ln \tilde{t})^\alpha}^{(\ln \tilde{t})^\alpha + S} J_1(x-y)[y - (\ln \tilde{t})^\alpha]\mathrm{d}y \\
&\le \frac{d_1 S(1 - M/\tilde{t})}{C_2\tilde{t}/2 - (\ln \tilde{t})^\alpha} \le \frac{4d_1 S}{C_2\tilde{t}}.
\end{aligned}
$$

On the other hand, by the definition of $\bar{u}$, for large $T > 0$, we have

$$
\begin{aligned}
\frac{k(1-h)-1}{4}\bar{u}(t,x) &= \frac{k(1-h)-1}{4} \frac{M}{\tilde{t}[(\ln \tilde{t})^\alpha - 2\bar{g}]}(x - 2\bar{g}) \\
&\ge \frac{k(1-h)-1}{4} \frac{M}{\tilde{t}[(\ln \tilde{t})^\alpha - 2\bar{g}]}((\ln \tilde{t})^\alpha - S - 2\bar{g}) \ge \frac{k(1-h)-1}{8\tilde{t}} M.
\end{aligned}
$$

Thus it is clear that (3.5) holds if

$$
M \ge \frac{32 d_1 S}{C_2[k(1-h)-1]}. \tag{3.6}
$$

For $x \in ((\ln \tilde{t})^\alpha, (\ln \tilde{t})^\alpha + S)$ and $t > 0$, we similarly derive, for large $T > 0$ and linear function (in x)

$$
\tilde{u}_2 := \frac{1 - M/\tilde{t}}{C_2\tilde{t}/2 - (\ln \tilde{t})^\alpha}[x - (\ln \tilde{t})^\alpha] + \frac{M}{\tilde{t}},
$$

$$
\begin{aligned}
A &= d_1 \int_{x-S}^{x+S} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u}(t,x) \\
&= d_1 \int_{x-S}^{x+S} J_1(x-y)\tilde{u}_1(t,y)\mathrm{d}y - d_1\bar{u}(t,x) + d_1 \int_{x-S}^{(\ln \tilde{t})^\alpha} J_1(x-y)[\bar{u}(t,y) - \tilde{u}_2(t,y)]\mathrm{d}y \\
&= d_1 \int_{x-S}^{(\ln \tilde{t})^\alpha} J_1(x-y)[\bar{u}(t,y) - \tilde{u}_2(t,y)]\mathrm{d}y \le d_1 \int_{(\ln \tilde{t})^\alpha - S}^{(\ln \tilde{t})^\alpha} J_1(x-y)[M/\tilde{t} - \tilde{u}_2(t,y)]\mathrm{d}y
\end{aligned}
$$

$$= \frac{1-M/\tilde t}{C_2\tilde t/2-(\ln\tilde t)^\alpha} d_1 \int_{(\ln\tilde t)^\alpha}^{(\ln\tilde t)^\alpha+S} J_1(x-y)[y-(\ln\tilde t)^\alpha]\mathrm{d}y \le \frac{d_1 S(1-M/\tilde t)}{C_2\tilde t/2-(\ln\tilde t)^\alpha} \le \frac{4d_1 S}{C_2\tilde t}.$$

Since $\bar u(t,x)\ge M/\tilde t$ for $x\ge(\ln\tilde t)^\alpha$, we see immediately that (3.5) is satisfied provided (3.6) holds.

By (3.5) and our estimates of u_t we see immediately that (3.4) holds for large $T>0$ and M satisfying (3.6).

Step 2. We choose (u_0,h_0,g_0) to have the desired long-time limit for $g(t)$ and $h(t)$.

Firstly we fix $(T,\tilde L,\delta,M)$ such that $\bar g(t)$ and $\bar u(t,x)$ satisfy (3.4). So in particular,

$$\bar u(T_1,x)\ge 1 \text{ for } x\ge \frac{C_2}{2}(T+t).$$

Without loss of generality we may assume that T has been chosen large enough such that

$$\frac{C_2}{2}T>S+1.$$

We aim to choose (u_0,g_0,h_0) such that $h_\infty=\infty$ and

$$\begin{cases} g(T_1)\ge\bar g(T_1),\ u(T_1,x)\le\bar u(T_1,x) \text{ for } x\in[g(T_1),\min\{h(T_1),\frac{C_2}{2}(T+T_1)],\\ u(1-u-kv)\le-\frac{k(1-h)-1}{2}u \text{ for } t>T_1,\ x\in[\bar g(t),\min\{h(t),\frac{C_2}{2}(T+t)\}]. \end{cases} \tag{3.7}$$

If these inequalities are proven, then we can use the comparison principle to conclude that

$$g(t)\ge\bar g(t),\ u(t,x)\le\bar u(t,x) \text{ for } t>T_1,\ x\in[g(t),\min\{h(t),\frac{C_2}{2}(T+t)\}],$$

which clearly implies $g_\infty>-\infty$.

To complete the proof, it remains to choose (u_0,g_0,h_0) such that $h_\infty=\infty$ and all the inequalities in (3.7) hold.

Since $\limsup_{t\to\infty}u(t,x)\le 1$ uniformly in x, for any given small $\bar\epsilon>0$, we can find $T_0>0$ large such that

$$v_t\ge d_2\int_{\mathbb R}J_2(x-y)v(t,y)dy-d_2v+\gamma v(1-v-h(1+\bar\epsilon)) \text{ for } t\ge T_0,\ x\in\mathbb R.$$

It follows that $v(T_0+t,x)\ge V(t,x)$ where V is the unique solution of (1.9) with $(d,J,a,b,w_0)=(d_2,J_2,\gamma[1-h(1+\epsilon)],\gamma,v(T_0,x))$. By Proposition E in Section 1, for any small $\tilde\epsilon>0$,

$$\lim_{t\to\infty}\max_{|x|\le(C_2-\tilde\epsilon)t}|V(t,x)-1-h(1+\epsilon)|=0.$$

Therefore we can find $T_1>T_0$ such that

$$v(t,x)>1-h-\bar\epsilon \ \text{ for } \ t\ge T_1,\ |x|\le\frac{3C_2}{4}(T+t).$$

This clearly implies the validity of the inequality in the second line of (3.7).

Since $\bar g(T_1)=-\tilde L+\frac{\delta}{(\ln T_1)^{\alpha-1}}$, we may assume that T_1 is large enough to also guarantee

$$\bar g(T_1)<0.$$

Since $\bar u(T,x)>0$ for $x\in[\bar g(T_1),\min\{h(T_1),\frac{C_2}{2}(T+T_1)\}]$, there exists $\epsilon_1>0$ such that

$$\bar u(T,x)\ge\epsilon_1 \text{ for } x\in[\bar g(T_1),\min\{h(T_1),\tfrac{C_2}{2}(T+T_1)\}].$$

By Proposition E, equation (1.9) with $(d,J,a,b)=(d_1,J_1,1,1)$ has a traveling wave $\phi_1(x)$ with speed C_1. Let $\tilde L_0>0$ be chosen such that

$$\phi_1(x)<\tilde\epsilon_1:=\min\{\epsilon_1,\tfrac{1}{\mu S T_1}\} \text{ for } x\ge\tilde L_0.$$

We then fix $\hat L_0>0$ such that

$$\bar g(T_1)-C_1T_1+\hat L_0>\tilde L_0.$$

Let $\underline{h}(t)$ and $\underline{u}(t,x)$ be defined as in the proof of Theorem 1.4 so that (3.3) holds, and note that (3.3) holds for every large L used in the definition of $\underline{h}(t)$. We recall that

$$\underline{h}(0)=L,\ \underline{u}(0,x)=\begin{cases}(1-\epsilon)\phi(x-L) & \text{for } x\in[KL,L],\\ (1-\epsilon)\phi((2K-1)L-x) & \text{for } x\in[(2K-1)L,KL].\end{cases}$$

Moreover, $(2K-1)L>\frac{C_2}{c_1^\rho}L$, where c_1^ρ is defined in the proof of Theorem 1.4.

Let $L_1>0$ be chosen such that $\phi_1(x)\geq 1-\epsilon$ for $x\leq -L_1$. We now assume that L in the definition of $\underline{h}(t)$ is chosen so large that apart from (3.3) we also have

$$\begin{cases}L>\frac{C_2}{2}(T+T_1),\\ (2K-1)L>\bar{g}(T_1)+\epsilon_1\mu ST_1+S,\\ (2K-1)L\geq L_1+L_0+C_1T_1+\frac{C_2}{2}(T+T_1).\end{cases}$$

Define

$$U(t,x):=\phi_1(-x-C_1t-\hat{L}_1),\ \hat{L}_1:=L_1-(2K-1)L.$$

Clearly for $x\in[(2K-1)L,L]$, we have

$$U(0,x)\geq\phi_1(-(2K-1)L-\hat{L}_1)=\phi_1(-L_1)\geq 1-\epsilon>\underline{u}(0,x).$$

We now fix $h_0\geq L$, and note that the above estimate for $U(0,x)$ and $(2K-1)L>\bar{g}(T_1)+\epsilon_1\mu ST_1+S$ allow us to choose (u_0,g_0) such that

$$\begin{cases}g_0:=\bar{g}(T_1)+\tilde{\epsilon}_1\mu ST_1,\\ u_0(x)\leq U(0,x) & \text{for } x\in[g_0,h_0],\\ u_0(x)\geq\underline{u}(0,x) & \text{for } x\in[(2K-1)L,L].\end{cases}\tag{3.8}$$

By the proof of Theorem 1.4, the third inequality in (3.8) guarantees that $h_\infty=\infty$. Moreover, we note that

$$h(t)\geq h_0\geq L>\frac{C_2}{2}(T+t)\ \text{for } t\in[0,T_1].$$

Since $U(t,x)$ satisfies

$$U_t=d_1\int_{\mathbb{R}}J_1(x-y)U(t,y)dy-d_1U+U(1-U)\ \text{for } t>0,\ x\in\mathbb{R},$$

while

$$u_t\leq d_1\int_{\mathbb{R}}J_1(x-y)u(t,y)dy-d_1u+u(1-u)\ \text{for } t>0,\ x\in(g(t),h(t)),$$

by the comparison principle and the first inequality in (3.8) we have

$$u(t,x)\leq U(t,x)\ \text{for } t>0,\ x\in(g(t),h(t)).$$

Hence for $t\in[0,T_1]$ and $x\in[g(t),\frac{C_2}{2}(T+t)]$, we have

$$\begin{aligned}u(t,x)\leq U(t,x)&=\phi_1(-x-C_1t-\hat{L})\\ &\leq\phi_1(-\frac{C_2}{2}(T+t)-C_1t-\hat{L})\\ &=\phi_1(-\frac{C_2}{2}(T+T_1)-C_1T_1-L_1+(2K-1)L)\\ &\leq\phi_1(L_0)\leq\tilde{\epsilon}_1.\end{aligned}$$

In particular, for $x\in[g(T_1),\frac{C_2}{2}(T+T_1)]$, we have

$$u(T_1,x)\leq\tilde{\epsilon}_1\leq\bar{u}(T_1,x).$$

We now estimate $g(T_1)$. For $t\in(0,T_1]$,

$$g(t)+S\leq g_0+S=\bar{g}(T_1)+\tilde{\epsilon}_1\mu ST_1+S<1+S<\frac{C_2}{2}(T+t),$$

and so $u(t,x) \le \tilde\epsilon_1$ for $x \in [g(t), g(t)+S]$. It follows that, for $t \in (0, T_1]$,

$$\begin{aligned} g'(t) &= -\mu \int_{-\infty}^{g(t)} \int_{g(t)}^{h(t)} J_1(x-y)u(t,y)dydx \\ &= -\mu \int_{-\infty}^{g(t)} \int_{g(t)}^{g(t)+S} J_1(x-y)u(t,y)dydx \\ &\ge -\mu S\tilde\epsilon_1. \end{aligned}$$

Therefore

$$g(T_1) \ge g_0 - \mu S \tilde\epsilon_1 T_1 = \bar g(T_1).$$

Now all the inequalities in (3.7) are satisfied and the proof of the theorem is complete. □

Remark 3.2. *From the above proof of Theorem 1.5 we can easily obtain the following estimates:*

$$\begin{cases} \limsup\limits_{t\to\infty} \dfrac{h(t)}{t} \le c_1, \\ \liminf\limits_{t\to\infty} \dfrac{h(t)}{t} \ge c_1^\rho, \end{cases} \qquad \begin{cases} \liminf\limits_{t\to\infty} \min\limits_{(C_2+\epsilon)t \le x \le (c_1^\rho - 3\epsilon)t} u(t,x) \ge 1-\epsilon, \\ \lim\limits_{t\to\infty} \max\limits_{|x|\ge (C_2+\epsilon)t} v(t,x) = 0, \end{cases}$$

and for any given positive function $\xi(t) = o(t)$ as $t \to \infty$,

$$\begin{cases} \lim\limits_{t\to\infty} \max\limits_{x\in[g(t),\xi(t)]} u(t,x) = 0, \\ \lim\limits_{t\to\infty} \min\limits_{-(C_2-\epsilon)t \le x \le \xi(t)} v(t,x) = 1. \end{cases}$$

4. General criteria for spreading and vanishing

In this section, we prove Theorem 1.1, which gives sharp conditions for spreading or vanishing to happen. Theorem 1.3 will follow from one of the lemmas used to prove Theorem 1.1.

The comparison principle infers that if vanishing happens for a particular value μ_0 of μ, then it happens also for any $\mu < \mu_0$. Similarly, if spreading happens for $\mu = \mu_1$, then it also happens for $\mu > \mu_1$.

To emphasize the dependence of the solution (u, v, g, h) on the parameter μ, we denote $(u, v, g, h) = (u^\mu, v^\mu, g^\mu, h^\mu)$, and similarly $g_\infty = g^\mu_\infty$ and $h_\infty = h^\mu_\infty$. Define

$$\Omega^\mu := \{\mu > 0 : h^\mu_\infty - g^\mu_\infty < \infty\},\ \mu_* := \begin{cases} \sup \Omega^\mu & \text{if } \Omega^\mu \ne \emptyset, \\ 0 & \text{if } \Omega^\mu = \emptyset. \end{cases} \tag{4.1}$$

Lemma 4.1. *The following conclusions hold true:*

(i) $\mu_* = 0$ *implies* $h^\mu_\infty - g^\mu_\infty = \infty$ *for every* $\mu > 0$*;*
(ii) $\mu_* \in (0, \infty)$ *implies* $h^\mu_\infty - g^\mu_\infty < \infty$ *for every* $\mu \in (0, \mu_*]$*, and* $h_\infty - g_\infty = \infty$ *for every* $\mu > \mu_*$*;*
(iii) $\mu_* = \infty$ *implies* $h^\mu_\infty - g^\mu_\infty < \infty$ *for every* $\mu > 0$.

Proof. These conclusions follow directly from the definition of μ_*, except that in part (ii), the conclusion $h_\infty - g_\infty < \infty$ for $\mu = \mu_*$ is derived by a similar discussion as in the proof of [2, Lemma 3.14]. □

Note that part (i) of Theorem 1.1 follows directly from Theorem A, while the first half of part (iv) is a consequence of Lemma 4.1. To complete the proof of Theorem 1.1, it remains to consider the following two cases:

$$d_1 > 1-k > 0 \quad \text{and} \quad k > 1 > h.$$

Lemma 4.2. *Suppose* $d_1 > 1-k > 0$.

(i) *If* $2h_0 \ge l_{1-k}$*, then* $\mu_* = 0$.
(ii) *If* $2h_0 < l_{1-k}$*, then* $\mu_* \in [0, \infty)$.
(iii) *If* $d_1 > 1$ *and* $2h_0 < l_1$*, then* $\mu_* \in (0, \infty)$.

Proof. (i) Arguing indirectly, we suppose that for some $\mu > 0$, vanishing happens, namely $h_\infty - g_\infty < \infty$. Then by (1.3) we obtain $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}) \le k-1$. Since $h_\infty - g_\infty > 2h_0 \ge l_{1-k}$, by Proposition C we deduce $\lambda_p(\mathcal{L}_{(g_\infty,h_\infty)}) > k-1$. This contradiction shows that $h_\infty - g_\infty = \infty$ always holds.

(ii) It suffices to prove that $h_\infty - g_\infty = \infty$ for all large μ. In view of the conclusion (i) above, it is sufficient to find some $t_0 > 0$ such that

$$h(t_0) - g(t_0) \ge l_{1-k}, \tag{4.2}$$

which is a consequence of [11, Lemma 3.9]. Indeed, from the equation for u we obtain, for some positive constant $C > 1 + \max_{x\in[-h_0,h_0]} u_0(x) + \max_{x\in\mathbb{R}} kv_0(x)$,

$$\begin{cases} u_t(t,x) \ge d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)\mathrm{d}y - d_1 u - Cu, & t>0,\ g(t)<x<h(t),\\ u(t,g(t)) = u(t,h(t)) = 0, & t \ge 0,\\ h'(t) \ge \mu \displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t\ge 0,\\ g'(t) \le -\mu \displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t \ge 0,\\ u(0,x) = u_0(x), & |x| \le h_0,\\ h(0) = -g(0) = h_0. \end{cases}$$

Hence we can use [11, Lemma 3.9] to conclude that for any given $t_0 > 0$, there exists $\mu_1 > 0$ such that (4.2) holds for all $\mu > \mu_1$.

(iii) Suppose $d_1 > 1$ and $2h_0 < l_1$. Since $l_1 < l_{1-k}$ by (ii) we have $\mu_* < \infty$. It remains to show that $h_\infty - g_\infty < \infty$ for sufficient small μ. We will demonstrate this by constructing an upper solution following the method of [2, Theorem 3.12] and [9, Lemma 4.7]. Since $2h_0 < l_1$, we have $\lambda_p^\epsilon := \lambda_p(\mathcal{L}_{(-h_0-\epsilon,h_0+\epsilon)}) < -1$ for small $\epsilon > 0$ satisfying $2h_0 + 2\epsilon < l_1$. Let $\phi = \phi_\epsilon$ be a positive eigenfunction corresponding to λ_p^ϵ. Define, for $t \ge 0$, $x \in [-h_0-\epsilon, h_1+\epsilon]$,

$$\bar{h}(t) := h_0 + \epsilon(1-e^{-\delta t}),\ \ \bar{g}(t) := -\bar{h}(t)\ \text{ and }\ \bar{u}(t,x) := Me^{-\delta t}\phi(x)$$

with

$$\delta := -\lambda_p^\epsilon/2 > 0,\ M := \delta\frac{\epsilon}{\mu}\Big(\int_{-h_0-\epsilon}^{h_0+\epsilon}\phi(x)\mathrm{d}x\Big)^{-1} > 0.$$

Clearly $\bar{h}(t) \in [h_0, h_0+\epsilon) \subset [h_0, l_*)$ for $t \ge 0$. We next show that for all small $\mu > 0$, $(\bar{u},\bar{g},\bar{h})$ is an upper solution to the problem satisfied by (u,g,h), when $v(t,x)$ is viewed as a known function. If this is proved, then it follows from the comparison principle, see Lemma 5.1, that

$$[g(t),h(t)] \subset [\bar{g}(t),\bar{h}(t)] \text{ and hence } h_\infty - g_\infty \le \bar{h}(\infty) - \bar{g}(\infty) = 2h_0 + 2\epsilon,$$

as desired.

It remains to prove that $(\bar{u},\bar{g},\bar{h})$ is an upper solution. A simple computation gives

$$\begin{aligned} &\bar{u}_t - d_1\int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}(t,y)\mathrm{d}y + d_1\bar{u}(t,x) - \bar{u}(1-\bar{u}-kv)\\ \ge\ &\bar{u}_t - d_1\int_{-h_0-\epsilon}^{h_0+\epsilon} J_1(x-y)\bar{u}(t,y)\mathrm{d}y + d_1\bar{u}(t,x) - \bar{u}\\ =\ &-\delta\bar{u} - \lambda_p^\epsilon\bar{u} = \delta\bar{u} \ge 0 \qquad \text{for } t>0,\ x\in(\bar{g}(t),\bar{h}(t)). \end{aligned}$$

Recalling that $[\bar{g}(t),\bar{h}(t)] \subset (-h_0-\epsilon, h_0+\epsilon)$, we further deduce

$$\mu\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{\infty} J_1(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x \le \mu\int_{\bar{g}(t)}^{\bar{h}(t)}\bar{u}(t,x)\mathrm{d}x = \mu Me^{-\delta t}\int_{\bar{g}(t)}^{\bar{h}(t)}\phi(x)\mathrm{d}x$$

$$\le \mu M e^{-\delta t}\int_{-h_0-\epsilon}^{h_0+\epsilon}\phi(x)\mathrm{d}x = \epsilon\delta e^{-\delta t} = \bar h'(t) \text{ for } t>0,$$

and by symmetry,

$$-\mu\int_{\bar g(t)}^{\bar h(t)}\int_{-\infty}^{\bar g(t)} J_1(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x \ge \bar g'(t) \quad \text{for} \quad t>0.$$

It is clear that $\bar u(t,\bar g(t))>0$ and $\bar u(t,\bar h(t))>0$ for all $t\ge 0$, and by appropriately selecting a small value for $\mu>0$, we can make M as large as we want and hence ensure that $u_0(x)\le \bar u(0,x)$ for $x\in[-h_0,h_0]$. The above calculations indicate that $(\bar u,\bar g,\bar h)$ is an upper solution for (u,g,h), as we wanted. □

Lemma 4.3. *If $k>1>h$, then for any (u_0,v_0) satisfying (1.2), and any $L>0$, we have*

$$\begin{cases}\lim_{t\to\infty} u(t,x)=0 & \text{uniformly for } x\in[-L,L],\\ \lim_{t\to\infty} v(t,x)=1 & \text{uniformly for } x\in[-L,L].\end{cases} \tag{4.3}$$

Moreover, $h_\infty-g_\infty<\infty$ for all small $\mu>0$, and hence $\mu_*>0$.

Proof. Let $\bar u, \underline{v}$ be the solution of

$$\begin{cases}\bar u_t = d_1\displaystyle\int_{\mathbb R} J_1(x-y)\bar u(t,y)\mathrm{d}y - d_1\bar u + \bar u(1-\bar u-k\underline{v}), & t>0,\ x\in\mathbb R,\\ \underline{v}_t = d_2\displaystyle\int_{\mathbb R} J_2(x-y)\underline{v}(t,y)\mathrm{d}y - d_2\underline{v} + \gamma\underline{v}(1-\underline{v}-h\bar u), & t>0,\ x\in\mathbb R\end{cases}$$

with $\bar u(0,x)=u_0(x)$ and $\underline{v}(0,x)=v_0(x)$ for $x\in\mathbb R$. Then from the comparison principle, see Lemma 5.2, we obtain

$$u(t,x)\le \bar u(t,x),\ v(t,x)\ge \underline{v}(t,x) \quad \text{for} \quad t\ge 0,\ x\in\mathbb R.$$

Following a well known iteration argument (see, for example, the proof of Theorem 3.2 in [14]) one can show that

$$\lim_{t\to\infty}\bar u(t,x)=0,\ \lim_{t\to\infty}\underline{v}(t,x)=1 \text{ locally uniformly in } \mathbb R.$$

This clearly implies (4.3). Moreover, there is $t_1>0$ such that

$$v(t,x)\ge \underline{v}(t,x)\ge \frac{1+k}{2k} \quad \text{for} \quad t\ge t_1,\ x\in[-2h_0,2h_0].$$

Note that $kv(t,x)\ge \frac{1+k}{2}>1$ for $t\ge t_1$ and $x\in[-2h_0,2h_0]$. Then, as in Lemma 4.2 (iii), we could construct an upper solution for (u,g,h) and show that $h_\infty-g_\infty<\infty$ for sufficient small μ. The desired conclusion then follows from the comparison principle. □

Clearly Theorem 1.3 follows directly from Lemma 4.3, and Theorem 1.1 follows from Lemmas 4.1, 4.2 and 4.3.

5. Appendix: some comparison principles

Let (u,v,g,h) be the solution of (1.1). For $T>0$, $g_1,h_1\in C([0,T])$ with $g_1(t)<h_1(t)$, we will use the notation

$$[0,T]\times[g_1,h_1] := \{(t,x): t\in[0,T],\ x\in[g_1(t),h_1(t)]\}.$$

Let $m\in C([0,T]\times\mathbb R)$ be a bounded function and

$$F(t,x,s) := s(m(t,x)-s) \text{ for } s\ge 0,\ t\ge 0,\ x\in\mathbb R.$$

We list below several comparison principles, whose proofs are easily obtained by following the proof in [2], and are omitted here.

Lemma 5.1 (Comparison principle 1)**.** *Assume that* $\bar g$, $\bar h$, $\underline{g}$, $\underline{h} \in C([0,T])$ *satisfy* $\underline{g}(t) < \underline{h}(t)$ *and* $\bar g(t) < \bar h(t)$*; the functions* $\bar u \in C([0,T]\times\mathbb{R}) \cap C^{1,0}([0,T]\times[\bar g,\bar h])$ *and* $\underline{u} \in C([0,T]\times\mathbb{R}) \cap C^{1,0}([0,T]\times[\underline{g},\underline{h}])$ *are nonnegative and bounded.*

(i) (Two sides free boundaries) *Suppose* $\bar g' \le 0 \le \bar h'$ *and* $\underline{g}' \le 0 \le \underline{h}'$*. If* $(\bar u, \bar g, \bar h)$ *satisfies*

$$\begin{cases} \bar u_t \ge d_1 \displaystyle\int_{\bar g(t)}^{\bar h(t)} J_1(x-y)\bar u(t,y)\mathrm{d}y - d_1\bar u + F(t,x,\bar u), & (t,x)\in(0,T]\times(\bar g,\bar h),\\ \bar u(t,x)\ge 0, & t\in[0,T],\ x\notin(\bar g(t),\bar h(t)),\\ \bar h'(t) \ge \mu \displaystyle\int_{\bar g(t)}^{\bar h(t)}\int_{\bar h(t)}^{\infty} J_1(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x, & t\in(0,T],\\ \bar g'(t) \le -\mu \displaystyle\int_{\bar g(t)}^{\bar h(t)}\int_{-\infty}^{\bar g(t)} J_1(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x, & t\in(0,T], \end{cases} \tag{5.1}$$

and $(\underline{u}, \underline{g}, \underline{h})$ *satisfies* (5.1) *with all the inequality signs reversed, and*

$$[\underline{g}(0),\underline{h}(0)] \subset [\bar g(0),\bar h(0)],\ \bar u(0,x) \ge \underline{u}(0,x) \text{ for } x\in[\underline{g}(0),\underline{h}(0)],$$

then

$$\begin{cases} [\underline{g}(t),\underline{h}(t)] \subset [\bar g(t),\bar h(t)] \text{ for } t\in[0,T],\\ \underline{u}(t,x) \le u(t,x) \text{ for } \ t\in[0,T],\ x\in[\underline{g}(t),\underline{h}(t)]. \end{cases}$$

(ii) (One side free boundary) *Suppose* $\bar g < \bar h$, $\bar g' \le 0 \le \bar h'$ *and* $(\bar u,\bar g,\bar h)$ *satisfies* (5.1)*. If* $\bar g(t) \le \underline{g}(t) < \underline{h}(t)$ *and* $\underline{h}'(t) \ge 0$ *for* $t\in(0,T]$*, and* $(\underline{u},\underline{g},\underline{h})$ *satisfies*

$$\begin{cases} \underline{u}_t \le d_1 \displaystyle\int_{\underline{g}(t)}^{\underline{h}(t)} J_1(x-y)\underline{u}(t,y)\mathrm{d}y - d_1\underline{u} + F(t,x,\underline{u}), & (t,x)\in(0,T]\times(\underline{g},\underline{h}),\\ \underline{u}(t,x) = 0, & t\in[0,T],\ x\in\{\underline{g}(t),\underline{h}(t)\},\\ \underline{h}'(t) \le \mu \displaystyle\int_{\underline{g}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty} J_1(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x, & t\in(0,T],\\ \underline{h}(0) \le \bar h(0),\ \underline{u}(0,x) \le \bar u(0,x), & x\in[\underline{g}(0),\underline{h}(0)], \end{cases}$$

then

$$\underline{h}(t) \le \bar h(t), \quad \underline{u}(t,x) \le \bar u(t,x) \text{ for } t\in[0,T] \text{ and } x\in[\underline{g}(t),\underline{h}(t)].$$

(iii) (One side free boundary) *Suppose* $\underline{g} < \underline{h}$, $\underline{g}' \le 0 \le \underline{h}'$ *and* $(\underline{u},\underline{g},\underline{h})$ *satisfies* (5.1) *with all the inequality signs reversed. If* $h_1 \in C([0,T])$ *satisfies* $h_1(t) \le \underline{h}(t)$ *for* $t\in[0,T]$*, and* $(\bar u,\bar g,\bar h)$ *satisfies*

$$\begin{cases} \bar u_t \ge d_1 \displaystyle\int_{\bar g(t)}^{\infty} J_1(x-y)\bar u(t,y)\mathrm{d}y - d_1\bar u + F(t,x,\bar u), & (t,x)\in(0,T]\times(\bar g,\infty),\\ \bar u(t,x) \ge \underline{u}(t,x) \ge 0, & t\in[0,T],\ x\in\{\bar g(t)\}\cup[h_1(t),\infty)\\ \bar g'(t) \le -\mu \displaystyle\int_{\bar g(t)}^{\infty}\int_{-\infty}^{\bar g(t)} J_1(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x, & t\in(0,T],\\ \bar g(0) \le \underline{g}(0),\ \bar u(0,x) \ge \underline{u}(0,x), & x\in[\bar g(0),\infty) \end{cases}$$

then

$$\bar g(t) \le \underline{g}(t) \text{ for } t\ge 0, \quad \underline{u}(t,x) \le \bar u(t,x) \text{ for } t\in[0,T] \text{ and } x\in[\bar g(t),\infty).$$

Lemma 5.2 (Comparison principle 2)**.** *Assume that* $\bar g$, $\bar h$, $\underline{g}$, $\underline{h} \in C([0,T])$ *satisfy* $\underline{g}(t) < \underline{h}(t)$ *and* $\bar g(t) < \bar h(t)$*, and have the monotone properties* $\bar g' \le 0$, $\underline{g}' \le 0$, $\bar h' \ge 0$ *and* $\underline{h}' \ge 0$*. Suppose that*

$$\bar u \in C([0,T]\times\mathbb{R}) \cap C^{1,0}([0,T]\times[\bar g,\bar h]),\ \underline{v} \in C([0,T]\times\mathbb{R}) \cap C^{1,0}([0,T]\times[\underline{g},\underline{h}]),$$

and $\bar u$, $\underline{v} \in C^{1,0}([0,T]\times\mathbb{R})$ *are nonnegative and bounded.*

(i) <u>(Two sides free boundaries)</u> *If* $(\bar{u}, \underline{v}, \bar{g}, \bar{h})$ *satisfies*

$$(5.2)\qquad \begin{cases} \bar{u}_t \geq d_1 \displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u} + \bar{u}(1-\bar{u}-k\underline{v}), & (t,x) \in (0,T] \times (\bar{g},\bar{h}),\\ \underline{v}_t \leq d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)\underline{v}(t,y)\mathrm{d}y - d_2\underline{v} + \gamma\underline{v}(1-\underline{v}-h\bar{u}), & (t,x) \in (0,T] \times \mathbb{R},\\ \bar{u}(t,x) \geq 0, & t \in [0,T],\ x \notin (\bar{g}(t),\bar{h}(t)),\\ \bar{h}'(t) \geq \mu \displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J_1(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x, & t \in (0,T],\\ \bar{g}'(t) \leq -\mu \displaystyle\int_{\bar{g}(t)}^{\bar{h}(t)} \int_{-\infty}^{\bar{g}(t)} J_1(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x, & t \in (0,T],\\ \bar{u}(0,x) \geq u_0(x),\ \underline{v}(0,x) \leq v_0(x), & x \in \mathbb{R}, \end{cases}$$

and $\bar{u}(0,x) \geq 0$ *for* $x \in \mathbb{R}$, *then the unique solution* (u,v,g,h) *of* (1.1) *satisfies*

$$\begin{aligned} &[g(t),h(t)] \subset [\bar{g}(t),\bar{h}(t)], \quad t \in [0,T],\\ &u(t,x) \leq \bar{u}(t,x),\ \underline{v}(t,x) \leq v(t,x), \quad t \in [0,T],\ x \in \mathbb{R}. \end{aligned}$$

If $(\underline{u}, \bar{v}, \underline{g}, \underline{h})$ *satisfies* (5.2) *with all the inequality signs reversed, then*

$$\begin{aligned} &[\underline{g}(t),\underline{h}(t)] \subset [g(t),h(t)], \quad t \in [0,T],\\ &\underline{u}(t,x) \leq u(t,x),\ v(t,x) \leq \bar{v}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}. \end{aligned}$$

(ii) <u>(Fixed boundaries)</u> *Suppose* $g(t) < h(t)$, $g'(t) \leq 0 \leq h'(t)$, *and* $A \in C([0,T] \times [g,h]$ *is a nonnegative function. Assume that* $\bar{u}, \underline{v}$ *are nonnegative and satisfy*

$$(5.3)\qquad \begin{cases} \bar{u}_t \geq d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u} + \bar{u}(1-\bar{u}-k\underline{v}), & (t,x) \in (0,T] \times (g,h),\\ \underline{v}_t \leq d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)\underline{v}(t,y)\mathrm{d}y - d_2\underline{v} + A + \gamma\underline{v}(1-\underline{v}-h\bar{u}), & (t,x) \in (0,T] \times (g,h), \end{cases}$$

and $\underline{u}, \bar{v}$ *are nonnegative and satisfy* (5.3) *with all the inequality signs reversed. If*

$$(5.4)\qquad \begin{cases} \bar{u}(t,x) \geq \bar{v}(t,x) \geq 0, & t \in [0,T],\ x \in \{g(t),h(t)\},\\ \bar{u}(0,x) \geq \underline{u}(0,x),\ \bar{v}(0,x) \geq \underline{v}(0,x), & x \in [g(0),h(0)], \end{cases}$$

then

$$\bar{u}(t,x) \geq \underline{u}(t,x),\ \bar{v}(t,x) \geq \underline{v}(t,x) \quad \text{for} \quad t \in [0,T],\ x \in [g(t),h(t)].$$

We remark that the condition in first inequality of (5.4) *can be removed if* g *and* h *are constants.*

(iii) <u>(Comparison with the Cauchy problem)</u> *Let* (u,v,g,h) *be the solution of* (1.1). *If* $\bar{u}, \underline{v}$ *are nonnegative and satisfy*

$$\begin{cases} \bar{u}_t \geq d_1 \displaystyle\int_{\mathbb{R}} J_1(x-y)\bar{u}(t,y)\mathrm{d}y - d_1\bar{u} + \bar{u}(1-\bar{u}-k\underline{v}), & (t,x) \in (0,T] \times \mathbb{R},\\ \underline{v}_t \leq d_2 \displaystyle\int_{\mathbb{R}} J_2(x-y)\underline{v}(t,y)\mathrm{d}y - d_2\underline{v} + \gamma\underline{v}(1-\underline{v}-h\bar{u}), & (t,x) \in (0,T] \times \mathbb{R},\\ \bar{u}(0,x) \geq u_0(x),\ 0 \leq \underline{v}(0,x) \leq v_0(x), & x \in \mathbb{R}, \end{cases}$$

then $u(t,x) \leq \bar{u}(t,x)$ *and* $v(t,x) \geq \underline{v}(t,x)$ *for* $t \in [0,T]$ *and* $x \in \mathbb{R}$.

Corollary 5.3. *The solution* u *is nondecreasing with respect to* μ, h *and* u_0, *while nonincreasing with respect to* k *and* v_0.

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