THE CARTAN-HELGASON THEOREM FOR SUPERSYMMETRIC SPACES: SPHERICAL WEIGHTS FOR KAC-MOODY SUPERALGEBRAS

ALEXANDER SHERMAN

Abstract. Let $(\mathfrak{g},\mathfrak{k})$ be a supersymmetric pair arising from a finite-dimensional, symmetrizable Kac-Moody superalgebra $\mathfrak{g}$. An important branching problem is to determine the finite-dimensional highest-weight $\mathfrak{g}$-modules which admit a $\mathfrak{k}$-coinvariant, and thus appear as functions in a corresponding supersymmetric space $\mathcal{G}/\mathcal{K}$. This is the super-analogue of the Cartan-Helgason theorem. We solve this problem via a rank one reduction and an understanding of reflections in singular roots, which generalize odd reflections in the theory of Kac-Moody superalgebras. An explicit presentation of spherical weights is provided for every pair when $\mathfrak{g}$ is indecomposable.

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, and let $\mathfrak{k}\subseteq\mathfrak{g}$ be a symmetric subalgebra, i.e. $\mathfrak{k}$ is the fixed points of an involution on $\mathfrak{g}$. The Cartan-Helgason theorem (see [3]) describes which irreducible $\mathfrak{g}$-modules admit a $\mathfrak{k}$-invariant vector, and thus define functions on a corresponding symmetric space G/K. As a generalization of the Peter-Weyl theorem, this allows one to describe the space of polynomial functions on G/K, and similarly to describe the L^2 functions on G_c/K_c, where G_c and K_c are compact real forms of G and K, respectively.

1.1. Questions in the super-setting.

We are interested in the analogous question in the super setting. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody superalgebra, e.g. $\mathfrak{g}=\mathfrak{gl}(m|n),\mathfrak{osp}(m|2n),\dots$ (We work with $\mathfrak{gl}(m|n)$ instead of $\mathfrak{sl}(m|n)$ for simplicity.) Let θ be an involution on $\mathfrak{g}$ preserving an invariant form, and giving fixed points $\mathfrak{k}$ and (-1)-eigenspace $\mathfrak{p}$. We call $(\mathfrak{g},\mathfrak{k})$ a supersymmetric pair. Broadly speaking, we are interested in the following very important branching problem:

Question A: For which finite-dimensional, indecomposable $\mathfrak{g}$-modules V do we have $(V^*)^{\mathfrak{k}}\neq 0$?

Note that by Frobenius reciprocity, this question is directly related to when we can realize V inside $\mathbb{C}[\mathcal{G}/\mathcal{K}]$ (the algebra of polynomial functions on $\mathcal{G}/\mathcal{K}$), where $\mathcal{G},\mathcal{K}$ are global forms of $\mathfrak{g}$ and $\mathfrak{k}$. We immediately see a difference between the super setting and the classical setting in that representations need not be semisimple, forcing us to consider indecomposable representations.

It is generally thought that a full answer to question A is extremely challenging, if not hopeless. However, in this text we make progress in answering question A in the case when V is simple, and more generally when it is a highest-weight module. We write these questions more explicitly for later reference.

Question B: For which finite-dimensional highest-weight $\mathfrak{g}$-modules V do we have $(V^*)^{\mathfrak{k}} \neq 0$?

Question C: For which finite-dimensional simple $\mathfrak{g}$-modules V do we have $(V^*)^{\mathfrak{k}} \neq 0$?

Questions B and C are of considerable importance in representation theory, and a full understanding of their answer would yield great insight into the structure of $\mathbb{C}[\mathcal{G}/\mathcal{K}]$.

As already stated, the answer to these questions in the classical situation is given by the Cartan-Helgason theorem, see [3]. In that case, representations are completely reducible, and thus one only needs to look at simple modules, making questions A, B, and C equivalent.

1.2. **Previously known results.** Let $\mathfrak{a} \subseteq \mathfrak{p}_{\overline{0}}$ be a maximal abelian subspace (a Cartan subspace). Then we obtain a restricted root system $\Delta \subseteq \mathfrak{a}^*$, and the choices of positive systems for Δ are equivalent to choices of simple roots $\Sigma \subseteq \Delta$, which we also refer to as a base.

Given a base $\Sigma \subseteq \Delta$, let $\mathfrak{n}_\Sigma$ be the corresponding nilpotent subalgebra, which is generated by $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma$. Then we assume the Iwasawa decomposition holds, i.e. that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_\Sigma$. Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{a} \oplus \mathfrak{n}$; we call $\mathfrak{b}$ an Iwasawa Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}_\Sigma = \mathfrak{c}(\mathfrak{a}) \oplus \mathfrak{n}_\Sigma$, a parabolic subalgebra which contains any Iwasawa Borel subalgebra that contains $\mathfrak{a} \oplus \mathfrak{n}_\Sigma$. (Here $\mathfrak{c}(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.)

In [1] it was shown that if V is a $\mathfrak{b}$-highest weight module of highest weight λ with respect to an Iwasawa Borel $\mathfrak{b}$ such that $(V^*)^{\mathfrak{k}} \neq 0$, then in fact $\lambda \in \mathfrak{a}^*$, and the $\mathfrak{b}$-highest weight vector is actually a $\mathfrak{p}_\Sigma$-eigenvector. Further, $\dim(V^*)^{\mathfrak{k}} \leq 1$. The proof of this result works just like in the classical setting of the Cartan-Helgason theorem. Since the parabolic $\mathfrak{p}_\Sigma$ is determined by our base Σ, we define $P_\Sigma^+ \subseteq \mathfrak{a}^*$ to be the Σ-*spherical weights*, i.e. those $\lambda \in \mathfrak{a}^*$ for which there exists a highest weight, finite-dimensional $\mathfrak{g}$-module V of highest weight λ with respect to $\mathfrak{p}_\Sigma$ such that $(V^*)^{\mathfrak{k}} \neq 0$.

In [1] they use a super-generalization of the Harish-Chandra c-function to prove a partial converse: if $\lambda \in \mathfrak{a}^*$ is integral and 'high enough' with respect to Σ, in a sense we do not discuss here, then in fact $\lambda \in P_\Sigma^+$.

1.3. **Main (new) results.** We improve on the work in [1] by computing entirely the Σ-spherical weights P_Σ^+ with respect to particular bases Σ for every supersymmetric pair. We state this more precisely below:

Theorem 1.1. *For the following supersymmetic pairs, we determine P_Σ^+ for every base $\Sigma \subseteq \Delta$, thus giving a full answer to question B:*

$$(\mathfrak{gl}(m|n), \mathfrak{gl}(r|s) \times \mathfrak{gl}(m-r|n-s)), \quad (\mathfrak{osp}(m|2n), \mathfrak{osp}(r|2n) \times \mathfrak{osp}(m-r|2n)),$$
$$(\mathfrak{osp}(m|2n), \mathfrak{osp}(m|2s) \times \mathfrak{osp}(m|2n-2s)), \quad (\mathfrak{d}(2,1:a), \mathfrak{osp}(2|2) \times \mathfrak{so}(2)),$$
$$(\mathfrak{ab}(1|3), \mathfrak{gosp}(2|4)) \quad (\mathfrak{ag}(1|2), \mathfrak{d}(2,1;a)), \quad (\mathfrak{ab}(1|3), \mathfrak{d}(2,1;2) \times \mathfrak{sl}(2)).$$

For the remaining supersymmetric pairs, which are

$$(\mathfrak{osp}(2m|2n), \mathfrak{gl}(m|n)), \quad (\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n)),$$
$$(\mathfrak{osp}(m|2n), \mathfrak{osp}(m-r|2n-2s) \times \mathfrak{osp}(r|2s)), \quad (\mathfrak{ab}(1|3), \mathfrak{sl}(1|4)),$$

we compute the spherical weights with respect to certain positive systems. Thus we obtain the answer to question B for certain positive systems.

See Section 4.1 for tables describing the sets P^+_Σ explicitly for each pair $(\mathfrak{g},\mathfrak{k})$.

1.4. **Method of computation.** Our method of computation is directly inspired by the ideas of Serganova used in [13] to compute the dominant weights of a Kac-Moody superalgebra $\mathfrak{g}$. Indeed, her result is a special case of the arguments we give in the case of the diagonal pair $(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$.

The idea is that for any base $\Sigma\subseteq\Delta$ and any $\lambda\in\mathfrak{a}^*$, we may define a $\mathfrak{g}$-module $V_\Sigma(\lambda)$ which is of highest weight λ with respect to Σ and admits a $\mathfrak{k}$-coinvariant (i.e. $(V_\Sigma(\lambda)^*)^{\mathfrak{k}}\neq 0$). Further, we have that $\lambda\in P^+_\Sigma$ if and only if $V_\Sigma(\lambda)$ is finite-dimensional, i.e. integrable. Checking integrability must be done on all so-called 'principal roots' $\Pi\subseteq\Delta$, which are those that generate the even part Δ_0 of the root system Δ. If $\Pi\subseteq\Sigma$, which happens in certain cases (e.g. for the 'standard' choice of Σ when $(\mathfrak{g},\mathfrak{k})=(\mathfrak{gl}(m|2n),\mathfrak{osp}(m|2n))$), then integrability becomes easy to check, giving a straightforward description of P^+_Σ.

1.4.1. *Singular reflections.* In most cases we have $\Pi\not\subseteq\Sigma$, and thus we must reflect Σ in so-called *singular roots* to deal with principal roots not lying in Σ. We say $\alpha\in\Delta$ is a singular root if both α and 2α are not restrictions of even roots of $\mathfrak{g}$. Singular roots come in 2 flavours: isotropic, meaning that $(\alpha,\alpha)=0$, and non-isotropic, so that $(\alpha,\alpha)\neq 0$. If $\alpha\in\Sigma$ is singular, we define $r_\alpha\Sigma$ to be the base associated to the positive system $(\Delta\setminus\{\alpha\})\sqcup\{-\alpha\}$. See Lemma 2.20 for an explicit description of $r_\alpha\Sigma$.

As we show, if α is singular isotropic, then the reflection in α, which we write as r_α, is well behaved on representations. Namely, we have $V_\Sigma(\lambda)\cong V_{r_\alpha\Sigma}(r_\alpha\lambda)$, where $r_\alpha\lambda=\lambda$ if $(\lambda,\alpha)=0$, and $r_\alpha\lambda=\lambda-2\alpha$ if $(\lambda,\alpha)\neq 0$.

However, trouble occurs if α is singular non-isotropic. In this case $\mathfrak{g}_\alpha$ will be of dimension $(0|2n)$ for some $n\in\mathbb{Z}_{\geq 0}$. Write $h_\alpha\in\mathfrak{a}^*$ for the coroot of α. Then if $\lambda(h_\alpha)/2\notin\{n+1,\dots,2n\}$, we have that $V_\Sigma(\lambda)\cong V_{r_\alpha\Sigma}(r_\alpha\lambda)$, where the formula for $r_\alpha\lambda$ is given in Lemma 3.17.

However if $\lambda(h_\alpha)/2=n+k\in\{n+1,\dots,2n\}$, then $V_\Sigma(\lambda)$ must contain $V_\Sigma(\lambda-2k\alpha)$, implying that it is never simple, and further it is not highest weight with respect to $r_\alpha\Sigma$. In this case we say that λ is an α-*critical weight*. Reflecting α-critical weights to other simple roots systems becomes a tricky business. Nevertheless, if one performs only one simple reflection in a singular root, we do have control over what happens: see Lemma 3.20. This allows us to compute P^+_Σ in cases where non-isotropic singular roots appear.

1.5. **Consequences for simple spherical modules.** As explained, our work has an ad-hoc element to it when reflecting α-critical weights. This prevents us from computing spherical weights with respect to arbitrary positive systems in every case. However, if we are interested in Question C, such issues don't arise, because if λ is α-critical then necessarily $V_\Sigma(\lambda)$ is not simple. Hence, if $V_\Sigma(\lambda)$ is finite-dimensional and simple, we must have $V_\Sigma(\lambda)\cong V_{r_\alpha\Sigma}(r_\alpha\lambda)$ for any $\alpha\in\Sigma$.

If Σ is a base, we call $\lambda\in P^+_\Sigma$ *fully reflectable* if for any base $\Sigma'\subseteq\Delta$, we have $V_\Sigma(\lambda)\cong V_{\Sigma'}(\lambda_{\Sigma'})$ for some $\lambda_{\Sigma'}\in P^+_{\Sigma'}$.

Conjecture 1.2. *Let $\Sigma\subseteq\Delta$ be a base, and let $\lambda\in P^+_\Sigma$ be fully reflectable. Then $V_\Sigma(\lambda)$ is simple if and only if for any base $\Sigma'\subseteq\Delta$ and any non-isotropic $\beta\in\Sigma'$ with $m(\beta):=-\operatorname{sdim}(\mathfrak{g}_\beta)/2-\operatorname{sdim}\mathfrak{g}_{2\beta}>0$, we have*

$$\lambda_{\Sigma'}(h_\beta)/2\notin\{m(\beta)+1,\dots,2m(\beta)\}.$$

The numerical conditions in the above conjecture arise from considering the rank one cases, and studying when two dominant weights will have the same eigenvalue for the Casimir. We do not prove the necessity of these conditions in the article. The proof for the case of $(\mathfrak{osp}(m|2n), \mathfrak{osp}(m-1|2n))$ is written in [17]. We note that Conjecture 1.2 was shown to hold under an extra genericity hypothesis on λ in Sec. 6.4 of [18].

1.6. **Supersymmetric spaces.** Our work is part of an ongoing project to improve understanding of supersymmetric spaces and their connections to representation theory. Supersymmetric spaces are homogeneous superspaces of the form $\mathcal{G}/\mathcal{K}$, where $\mathcal{K}$ is a symmetric subgroup of the supergroup $\mathcal{G}$. Such spaces are natural in the study of super harmonic analysis, see for instance [2] and [5]. They also have important connections to interpolation polynomials (see [8], [9], and [10]), integrable systems (see [12]), and physics (see [22] and [11]).

1.7. **Outlook for the queer Kac-Moody setting.** We expect our techniques to generalize to the queer Kac-Moody setting (see [20]), which includes the supersymmetric pairs $(\mathfrak{q}(n), \mathfrak{q}(r) \times \mathfrak{q}(n-r))$. This will be the subject of future work.

1.8. **Outline.** In Section 2 we develop the necessary facts about restricted root systems we will use, including about singular reflections. Section 3 studies the modules $V_\Sigma(\lambda)$ and the tools for checking integrability. Section 4 explicitly describes the sets P_Σ^+ for each supersymmetric pair and a choice of base Σ.

1.9. **Acknowledgements.** We would like to thank Vera Serganova for providing feedback on an initial version of this article, and for many stimulating discussions. We also thank Shifra Reif, Siddhartha Sahi, and Hadi Salmasian for many helpful discussions about supersymmetric pairs and restricted root systems. This project was partially made possible by a SQuaRE at the American Institute for Mathematics, and we thank AIM for providing a supportive and mathematically rich environment. The author was partially supported by ARC grant DP210100251.

2. Restricted root systems

In what follows, for a super vector space V we write $V = V_{\overline{0}} \oplus V_{\overline{1}}$ for its parity decomposition. We always work over an algebraically closed field $\Bbbk$ of characteristic 0.

2.1. **Supersymmetric pairs.** Let $\mathfrak{g}$ be a finite-dimensional, symmetrizable Kac-Moody Lie superalgebra (see [13]). Let θ be an involution of $\mathfrak{g}$ which preserves a nondegenerate, invariant bilinear form $(-,-)$ on $\mathfrak{g}$. Write the eigenspace decomposition for θ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{g}^\theta$. Then $\mathfrak{k}$ is naturally a subalgebra, and $\mathfrak{p}$ is a $\mathfrak{k}$-module. We call $(\mathfrak{g}, \mathfrak{k})$ a supersymmetric pair.

Set $\mathfrak{a} \subseteq \mathfrak{p}_{\overline{0}}$ to be a Cartan subspace, meaning a maximal abelian subspace. It is known that $\mathfrak{a}$ is unique to conjugacy by the action of $\exp(\mathfrak{k}_{\overline{0}}) \subseteq GL(\mathfrak{p}_{\overline{0}})$ (see Sec. 26 of [21]). Set $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{c}(\mathfrak{a})$ to be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Assumption $(\star)$: We assume that $\mathfrak{c}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{m}$. By the classical picture (see Sec. 26 of [21]), we have $\mathfrak{c}(\mathfrak{a})_{\overline{0}} = \mathfrak{a} \oplus \mathfrak{m}_{\overline{0}}$. Thus our assumption is equivalent to $\mathfrak{c}(\mathfrak{a})_{\overline{1}} = \mathfrak{m}_{\overline{1}}$, i.e. $\mathfrak{c}(\mathfrak{a})_{\overline{1}} \subseteq \mathfrak{k}$. By [16], if $\mathfrak{g}$ is indecomposable then either $\mathfrak{c}(\mathfrak{a})_{\overline{1}} \subseteq \mathfrak{k}$ or $\mathfrak{c}(\mathfrak{a})_{\overline{1}} \subseteq \mathfrak{p}$.

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, then by Cor. 26.13 of [21], $\mathfrak{h}$ is θ-stable. We fix a choice of such a Cartan subalgebra $\mathfrak{h}$ throughout.

2.2. **Restricted root system.** We may consider the action of $\mathfrak{a}$ on $\mathfrak{g}$ by the adjoint action. Writing $\Delta \subseteq \mathfrak{a}^* \setminus \{0\}$ for the non-zero weights of this action, we have

$$\mathfrak{g} = \mathfrak{c}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where we have used out assumption $(\star)$ for the last equality. We refer to elements of Δ as restricted roots, or just roots when the context is clear.

Remark 2.1. Let $\widetilde{\Delta} \subseteq \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$. Then another description of Δ may be given as

$$\Delta = \{\alpha|_{\mathfrak{a}} : \alpha \in \widetilde{\Delta}\} \setminus \{0\}.$$

The following definition is from Sec. 5 of [16]; we write $\mathbb{Z}\Delta$ for the abelian subgroup of $\mathfrak{a}^*$ generated by Δ.

Definition 2.2. Let $\phi : \mathbb{Z}\Delta \to \mathbb{R}$ be a group homomorphism such that $\phi(\alpha) \neq 0$ for all $\alpha \in \Delta$. Then we write $\Delta^+ := \{\alpha \in \Delta | \phi(\alpha) > 0\}$, and call $\Delta^+ \subseteq \Delta$ a choice of positive system of $\Delta \subseteq \mathfrak{a}^*$.

Definition 2.3. We call a base $\Sigma \subseteq \Delta$ a linearly independent set in $\mathfrak{a}^*$ such that

$$\Delta \subseteq \mathbb{N}\Sigma \sqcup (-\mathbb{N}\Sigma).$$

In other words, every $\alpha \in \Delta$ is either a non-negative or non-positive integral linear combination of elements of Σ. We call elements of a base *simple (restricted) roots*.

Definition 2.4. We call the rank of a supersymmetric pair $(\mathfrak{g}, \mathfrak{k})$ the size of any base $\Sigma \subseteq \Delta$ of the restricted root system.

Given a base $\Sigma \subseteq \Delta$, set $\Delta^+_\Sigma := \mathbb{N}\Sigma$.

Lemma 2.5. *The set Δ^+_Σ is a positive system. Further, the correspondence $\Sigma \mapsto \Delta^+_\Sigma$ is bijective.*

Proof. For $\Sigma \subseteq \Delta$ a base, define $\phi_\Sigma : \mathbb{Z}\Delta \to \mathbb{R}$ by $\phi(\alpha) = 1$ for $\alpha \in \Sigma$, and extend linearly. Then it is clear that for ϕ_Σ the corresponding positive system is Δ^+_Σ. The injectivity of the correspondence $\Sigma \mapsto \Delta^+_\Sigma$ follows from the definition of a base.

For surjectivity, Sec. 5 of [16] explains that we may extend any positive system $\Delta^+ \subseteq \Delta$ to a choice of positive system for all $\mathfrak{g}$ with simple roots $\widetilde{\Sigma} \subseteq \mathfrak{h}^*$. Let us call Σ the projection of $\widetilde{\Sigma}$ to $\mathfrak{a}^*$. Then Σ will be a base by Prop. 5.7 and Lem. 5.10 of [16], and it is clear that $\Delta^+ = \Delta^+_\Sigma$. □

2.3. **Nilpotent subalgebra $\mathfrak{n}_\Sigma$ and Iwasawa decomposition.** Given a base Σ, we obtain a nilpotent subalgebra $\mathfrak{n}_\Sigma$, which by definition is

$$\mathfrak{n}^+_\Sigma = \bigoplus_{\alpha\in\Delta^+_\Sigma} \mathfrak{g}_\alpha.$$

Lemma 2.6. *For any choice of Σ we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+_\Sigma$.*

Proof. This follows from our assumption that $\mathfrak{c}(\mathfrak{a})_{\overline{1}} \subseteq \mathfrak{k}$ and a standard argument (see Thm. 5.3 of [16]). □

Lemma 2.7. *The subalgebra $\mathfrak{n}^+_\Sigma$ is generated by the subspaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma$.*

Proof. Extending $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, by Sec. 5 of [16] we may lift Σ to a base $\widetilde{\Sigma} \subseteq \mathfrak{h}^*$ such that every element $\alpha \in \widetilde{\Sigma}$ is either fixed by θ, and thus $\mathfrak{g}_\alpha \subseteq \mathfrak{m}$, or $\theta\alpha \neq \alpha$ in which case $\alpha|_{\mathfrak{a}} \in \Sigma$.

It is clear that the subalgebra generated by $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma$ is $\mathfrak{c}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$-stable. From the decomposition

$$\mathfrak{g} = \mathfrak{n}_\Sigma^- \oplus \mathfrak{c}(\mathfrak{a}) \oplus \mathfrak{n}_\Sigma^+,$$

the result is now clear. □

Remark 2.8. We have shown that for a base $\Sigma = \{\alpha_1, \dots, \alpha_k\} \subseteq \Delta$, we obtain a presentation of $\mathfrak{g}$ such that it is generated by $\mathfrak{c}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{m}$, $\mathfrak{g}_{\alpha_1}, \dots, \mathfrak{g}_{\alpha_k}, \mathfrak{g}_{-\alpha_1}, \dots, \mathfrak{g}_{-\alpha_k}$, subject to the relations that each $\mathfrak{g}_{\alpha_i}$ is a $\mathfrak{c}(\mathfrak{a})$-module, and

$$[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_j}] = 0 \quad \text{for} \quad i \neq j.$$

Of course there are more relations, but we don't concern ourselves with this here. We only mention that $\mathfrak{g}$ does not contain any ideals which do not intersect $\mathfrak{h}$ because it is Kac-Moody; in particular it does not have any ideals contained in $\mathfrak{n}_\Sigma^+$.

Corollary 2.9. *Let $\alpha \in \Sigma$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$.*

Proof. This follows from Lemma 2.7. □

Corollary 2.10. *A set $\Sigma \subseteq \Delta$ is a base if and only if the following conditions hold:*

(1) *Σ is linearly independent;*
(2) *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}] = 0$ for distinct $\alpha, \beta \in \Sigma$;*
(3) *$\mathfrak{g}$ is generated by $\mathfrak{c}(\mathfrak{a})$ along with the root spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ where α runs over the elements of Σ.*

Proof. Indeed, under the conditions we have $\Delta \subseteq (\mathbb{N}\Sigma \sqcup (-\mathbb{N}\Sigma))$, as required. □

2.4. **Principal roots.** Observe that the involution θ defines a symmetric pair $(\mathfrak{g}_{\overline{0}}, \mathfrak{k}_{\overline{0}})$ with the same Cartan subspace $\mathfrak{a}$, and along with it a restricted root system $\Delta_0 \subseteq \Delta \subseteq \mathfrak{a}^*$. This will be a (potentially non-reduced) root system with possibly several irreducible components (Lem. 26.16, [21]). Suppose that we have a positive system Δ_Σ^+ for $(\mathfrak{g}, \mathfrak{k})$. Then this induces a positive system $\Delta_0 = \Delta_0^+ \sqcup \Delta_0^-$, and along with it a base $\Pi \subseteq \Delta_0$.

Definition 2.11. Given a base Σ of Δ, we call the base $\Pi \subseteq \Delta_0^+$ determined by Σ the *principal* roots (of Σ). We note that while Π depends on Σ, we will surpress this dependence in writing for reasons that will become clear.

Corollary 2.12. *The Lie algebra $\mathfrak{g}_{\overline{0}}$ is generated by $\mathfrak{m}_{\overline{0}}$, $\mathfrak{a}$, and $(\mathfrak{g}_\alpha)_{\overline{0}}, (\mathfrak{g}_{-\alpha})_{\overline{0}}$, where α runs over all principal roots.*

Proof. This follows from Corollary 2.10 applied to the pair $(\mathfrak{g}_{\overline{0}}, \mathfrak{k}_{\overline{0}})$. □

2.5. **Properties of restricted roots.**

Definition 2.13. For a root $\alpha \in \Delta$, write $m_\alpha = (m_{\alpha,\overline{0}}|m_{\alpha,\overline{1}})$, where $m_{\alpha,\overline{0}} := \dim(\mathfrak{g}_\alpha)_{\overline{0}}$, $m_{\alpha,\overline{1}} = \dim(\mathfrak{g}_\alpha)_{\overline{1}}$.

Lemma 2.14. *If $\alpha \in \Delta$, then if $k\alpha \in \Delta$ we must have $k \in \{\pm 1, \pm 2, \pm 1/2\}$. Further, if 2α is a root, then $m_{2\alpha,\overline{1}} = 0$.*

Proof. This follows from the classification presented in Section 3.6. □

Definition 2.15. We say a root α is *real* if the subalgebra generated by $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ contains $\mathfrak{sl}(2)$. We say a root is *singular* if it is not real. Note that if a root α is real then $(\alpha,\alpha)\neq 0$, but the converse need not hold. If α is real, set $\varepsilon(\alpha)$ to be 0 if 2α is not a root, and otherwise set $\varepsilon(\alpha)=1$.

We clearly have a decomposition $\Delta=\Delta_{re}\sqcup\Delta_{sing}$ of roots into real and singular roots.

Definition 2.16. For a nonisotropic root $\alpha\in\Delta$, set $h_\alpha := \frac{2(\alpha,-)}{(\alpha,\alpha)}\in\mathfrak{a}$.

Lemma 2.17. *Let α be a real root, and suppose that $\beta\in\Delta$. Then $\beta(h_\alpha)\in 2^{\varepsilon(\alpha)}\mathbb{Z}$.*

Proof. If $\varepsilon(\alpha)=0$, then the statement follows from the representation theory of $\mathfrak{sl}(2)$. If $\varepsilon(\alpha)=1$, we see that $\mathfrak{g}_{-2\alpha},\mathfrak{g}_{2\alpha}$ will generate a copy of $\mathfrak{sl}(2)$ by Lemma 2.14, so that $\beta(h_{2\alpha})=\frac{1}{2}\beta(h_\alpha)\in\mathbb{Z}$. This forces $\beta(h_\alpha)\in 2\mathbb{Z}$. □

2.6. **Rank one sub-pairs.** For $\alpha\in\Delta$, set

$$\mathfrak{g}\langle\alpha\rangle := \mathfrak{c}(\mathfrak{a}) + \bigoplus_{n\in\mathbb{Q}_{\neq 0}}\mathfrak{g}_{n\alpha}$$

Then θ stabilizes $\mathfrak{g}\langle\alpha\rangle$, and we write $\mathfrak{k}\langle\alpha\rangle$ for the fixed subalgebra. This will be a supersymmetric pair of rank 1.

2.7. **Reflections in simple roots.** Let Σ be a base, and for a simple root $\alpha\in\Sigma$, define $r_\alpha\Sigma$ to be the base of the positive system $(\Delta_\Sigma^+\setminus\{\alpha\})\sqcup\{-\alpha\}$. Note that the latter is indeed a positive system, as we may define

$$\phi_{\Sigma,\alpha}(\alpha)=\epsilon,\quad \phi_{\Sigma,\alpha}(\beta)=1\ \text{ for } \beta\in\Sigma\setminus\{\alpha\} \tag{2.1}$$

where ϵ is a small negative number. Then the positive system obtained from $\phi_{\Sigma,\alpha}$ will be $(\Delta_\Sigma^+\setminus\{\alpha\})\sqcup\{-\alpha\}$.

Definition 2.18. We call a reflection r_α a singular reflection if α is singular.

The following is clear:

Lemma 2.19. *Any two bases obtained from one another by a sequence of singular reflections have the same principal roots.*

If α is real, then r_α is simply the usual reflection coming from the baby Weyl group. However, if α is singular then we show that we can view it as a bijection $r_\alpha:\Sigma\to r_\alpha\Sigma$, which works as follows:

Lemma 2.20. *Suppose that $\alpha\in\Sigma$ simple root (singular or real). Then $r_\alpha\Sigma$ consists of the roots $r_\alpha\alpha=-\alpha$ along with $r_\alpha\beta$ for $\beta\in\Sigma\setminus\{\alpha\}$, with $r_\alpha\beta=\beta+k_{\alpha\beta}\alpha$, where $k_{\alpha\beta}$ is the maximal non-negative integer k such that $\beta+k\alpha\in\Delta$.*

Proof. Indeed, it is clear that with $\phi_{\Sigma,\alpha}$ as defined in 2.1, the roots $-\alpha$ along with $\beta+k_{\alpha\beta}\alpha$ for $\beta\in\Sigma\setminus\{\alpha\}$ will be both linearly independent and will take the minimum positive values under $\phi_{\Sigma,\alpha}$ as required. □

Remark 2.21. If $\alpha\in\Sigma$ is singular and $\beta\in\Sigma$, then $r_\alpha\beta=\beta+k_{\alpha\beta}\alpha$ for $k_{\alpha\beta}\in\{0,1,2\}$.

Lemma 2.22. *Any two bases Σ, Σ' may be obtained from one another by a sequence of simple reflections.*

Proof. If $\Sigma \neq \Sigma'$, then there exists $\alpha \in \Sigma \cap \Delta_{\Sigma'}^{-}$. Thus we have $|\Delta_{r_\alpha\Sigma}^{+} \cap \Delta_{\Sigma'}^{+}| < |\Delta_{\Sigma}^{+} \cap \Delta_{\Sigma'}^{+}|$, and we may conclude by induction. □

2.8. **Equivalent positive systems.** We partition the collection of bases into equivalence classes, declaring that $\Sigma \sim \Sigma'$ if there exists a sequence of singular reflections $r_{\alpha_1}, \dots, r_{\alpha_k}$ such that $\Sigma' = r_{\alpha_k} \cdots r_{\alpha_1}\Sigma$. Note that a given equivalence class $\mathcal{S}$ of bases has a well-defined set Π of principal roots by Lemma 2.19.

Lemma 2.23. *For all $\gamma \in \Pi$, there exists some base Σ for which $\Pi \subseteq \Delta_{\Sigma}^{+}$ and either $\gamma \in \Sigma$ or $\gamma/2 \in \Sigma$.*

Proof. We may write $\mathbb{Z}\Delta = Q \oplus P$, Q, P are free $\mathbb{Z}$-modules and $\mathbb{Z}\Pi \subseteq Q$ is finite index. Let $\psi : \mathbb{Z}\Pi \to \mathbb{R}$ be such that $\psi(\gamma) = \epsilon > 0$ is very small and positive, and $\psi(\gamma') \gg 0$ for $\gamma' \in \Pi \setminus \{\gamma\}$. Define $\phi : \mathbb{Z}\Delta \to \mathbb{R}$ by extending ψ to Q via the injectivity of $\mathbb{R}$, and letting ϕ be very large in absolute value on all non-zero projections of elements of Δ to P. Then base obtained from ϕ will have the desired properties. □

The following is entirely analogous to Cor. 4.5 of [13], and follows from the same proof as Lemma 2.22

Lemma 2.24. *If Σ, Σ' are bases with the same principal roots, then they are equivalent.*

Corollary 2.25. *Let $\mathcal{S}$ be an equivalence class of bases with principal roots Π. Then for every $\gamma \in \Pi$, there exists some base $\Sigma \in \mathcal{S}$ for which either γ or $\gamma/2$ is simple.*

Proof. This follows immediately from Lemmas 2.23 and 2.24. □

3. Spherical weights

In this section, we introduce spherical weights and study their behavior under singular reflections. At this point we chose to abuse the classification of supersymmetric pairs for $\mathfrak{g}$ indecomposable and Kac-Moody, as avoiding it seems rather difficult and lacking sufficient payoff.

3.1. **Σ-spherical weights.** Let $\Sigma \subseteq \Delta$ be a choice of simple roots, and consider the parabolic subalgebra:

$$\mathfrak{p}_\Sigma = \mathfrak{c}(\mathfrak{a}) \oplus \mathfrak{n}_\Sigma = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_\Sigma^{+}} \mathfrak{g}_\alpha.$$

Definition 3.1. Define $P_\Sigma^{+} \subseteq \mathfrak{a}^*$ to be the those weights $\lambda \in \mathfrak{a}^*$ for which there exists a highest weight (with respect to $\mathfrak{p}_\Sigma$), finite-dimensional $\mathfrak{g}$-module V of highest weight λ, such that $(V^*)^{\mathfrak{k}} \neq 0$. We call elements $\lambda \in P_\Sigma^{+}$ the Σ-*spherical weights*. Note that P_Σ^{+} is a submonoid of $\mathfrak{a}^*$ by Cor. 5.10 of [19].

3.2. **Parabolic Verma module.** For $\lambda \in \mathfrak{a}^*$, consider the one dimensional, purely even $\mathfrak{c}(\mathfrak{a})$-module
$\Bbbk_\lambda = \Bbbk\langle v_\lambda\rangle$ on which $\mathfrak{a}$ acts via λ and $\mathfrak{m}$ acts by 0. Inflate $\Bbbk_\lambda$ to a module over $\mathfrak{p}_\Sigma$, and set

$$M_\Sigma(\lambda) := \operatorname{Ind}_{\mathfrak{p}_\Sigma}^{\mathfrak{g}} \Bbbk_\lambda.$$

The following lemma is standard, and follows from Prop. 5.5.4 and Prop. 5.5.8 of Diximier.

Lemma 3.2. *There exists a one-dimensional space of $\mathfrak{k}$-coinvariants on $M_\Sigma(\lambda)$. In particular, there exists a minimal quotient $V_\Sigma(\lambda)$ of $M_\Sigma(\lambda)$ which continues to admit a nonzero $\mathfrak{k}$-coinvariant.*

Caution: $V_\Sigma(\lambda)$ need not be irreducible!

Remark 3.3. Note that if V is a finite-dimensional, indecomposable highest weight $\mathfrak{g}$-module such that $(V^*)^{\mathfrak{k}} \neq 0$, then there exists $\lambda \in \mathfrak{a}^*$ such that $V_\Sigma(\lambda)$ is a quotient of V. Indeed this follows from the classical situation, and is shown in [1].

Remark 3.4. Let $\mathcal{G}$ be a quasireductive supergroup which is a global form of $\mathfrak{g}$ and is such that θ lifts to an involution of $\mathcal{G}$. Let $\mathcal{K} \subseteq \mathcal{G}$ be a subgroup satisfying $(\mathcal{G}^\theta)^\circ \subseteq \mathcal{K} \subseteq \mathcal{G}^\theta$, where $(\mathcal{G}^\theta)^\circ$ denotes the connected component of the identity of $\mathcal{G}^\theta$. Notice that $\mathcal{K}$ will also be quasireductive.

Then we call $\mathcal{G}/\mathcal{K}$ a supersymmetric space (see [7] for the construction of homogeneous spaces). Given an Iwasawa Borel subalgebra $\mathfrak{b}$, that is, a Borel subalgebra containing $\mathfrak{a} \oplus \mathfrak{n}$, we may consider the $\mathfrak{b}$-eigenfunctions in $\Bbbk[\mathcal{G}/\mathcal{K}]$. This set will exactly be those $\lambda \in P_\Sigma^+$ for which $V_\Sigma(\lambda)$ integrates to a representation of the group $\mathcal{G}$ and for which $\mathcal{K}_0$ acts trivially on the $\mathfrak{k}$-coinvariant on $V_\Sigma(\lambda)$. Further, for exactly such λ we will have an embedding $V_\Sigma(\lambda) \subseteq \Bbbk[\mathcal{G}/\mathcal{K}]$.

3.3. **Properties of the $\mathfrak{k}$-coinvariant.**

Lemma 3.5. *Let v_λ denote the highest weight vector of $M_\Sigma(\lambda)$. Then we have $\mathcal{U}\mathfrak{k}\cdot v_\lambda = M_\Sigma(\lambda)$. Thus:*

$$M_\Sigma(\lambda) = \Bbbk\langle v_\lambda\rangle \oplus \mathfrak{k}\mathcal{U}\mathfrak{k} \cdot v_\lambda.$$

In particular, if $\varphi : M_\Sigma(\lambda) \to \Bbbk$ is a nontrivial $\mathfrak{k}$-coinvariant, then $\varphi(v) = 0$ if and only if $v \in \mathfrak{k}\mathcal{U}\mathfrak{k}v_\lambda$.

Proof. This follows from the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_\Sigma$, and the fact that $\mathfrak{a} \oplus \mathfrak{n}_\Sigma \subseteq \mathfrak{p}_\Sigma$. □

Lemma 3.6. *If $\alpha \in \Delta$, then $\mathcal{U}\mathfrak{g}\langle\alpha\rangle \cdot v_\lambda \cong M_{\{\alpha\}}(\lambda)$ as $\mathfrak{g}\langle\alpha\rangle$-modules. Thus the following are equivalent for a vector $v \in \mathcal{U}\mathfrak{g}\langle\alpha\rangle \cdot v_\lambda$:*

(1) *$v \in \mathfrak{k}\langle\alpha\rangle\mathcal{U}\mathfrak{k}\langle\alpha\rangle v_\lambda$;*
(2) *$v \in \mathfrak{k}\mathcal{U}\mathfrak{k}v_\lambda$;*
(3) *the nontrivial $\mathfrak{k}$-coinvariant on $M_\Sigma(\lambda)$ vanishes on v.*

Proof. This follows from Lemma 3.5 applied to both pairs $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}\langle\alpha\rangle, \mathfrak{k}\langle\alpha\rangle)$. □

3.4. **Integrability.**

Definition 3.7. For $\alpha \in \Delta$, we say that a $\mathfrak{g}$-module V is α-integrable if both $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ act locally nilpotently on V, i.e. if for each $v \in V$ there exists $N > 0$ such that $\operatorname{ad}(\mathfrak{g}_\alpha)^N v = 0$ and $\operatorname{ad}(\mathfrak{g}_{-\alpha})^N v = 0$.

Lemma 3.8. *For $\lambda \in \mathfrak{a}^*$, the following are equivalent:*

(1) $\lambda \in P_\Sigma^+$;
(2) $V_\Sigma(\lambda)$ *is finite-dimensional;*
(3) *for every real root* $\alpha \in \Delta$, *there exists an* $N > 0$ *such that* $\mathrm{ad}(\mathfrak{g}_\alpha)^N v_\lambda = 0$;
(4) *for every* $\alpha \in \Pi$, *there exists an* $N > 0$ *such that* $\mathrm{ad}(\mathfrak{g}_{-\alpha})^N v_\lambda = 0$.

Proof. Here we use that $\mathcal{U}\mathfrak{g}$ is a α-integrable for all $\alpha \in \Delta$. For the last equivalence, we use Corollary 2.12. □

3.5. **Detecting singular subspaces.** Recall that for two distinct simple roots $\alpha, \beta \in \Sigma$, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}] = 0.$$

Thus, if $W \subseteq \mathcal{U}\mathfrak{g}\langle\alpha\rangle \cdot v_\lambda$ is a $\mathfrak{c}(\mathfrak{a})$-stable subspace annihilated by $\mathfrak{g}_\alpha$, then we will have $\mathfrak{n}_\Sigma W = 0$ and $\mathfrak{p}_\Sigma W \subseteq W$, i.e. we obtain a singular subspace with respect to $\mathfrak{p}_\Sigma$.

If further we have that the $\mathfrak{k}$-coinvariant φ on $M_\Sigma(\lambda)$ vanishes on W, then the quotient map $M_\Sigma(\lambda) \to V_\Sigma(\lambda)$ will necessarily vanish on W, and thus on $\mathcal{U}\mathfrak{g} \cdot W = \mathcal{U}\mathfrak{k} \cdot W$, again using the Iwasawa decomposition.

3.6. **Classification of rank one supersymmetric pairs.** For the following classification result, we simply refer to the classification of supersymmetric pairs, which is described in Sec. 5.2 of [16], and is based off [15].

Lemma 3.9. *Suppose that* $(\mathfrak{g}, \mathfrak{k})$ *is rank one, so that* $\Sigma = \{\alpha\}$. *Then up to split factors fixed by* θ, *we have the following possibilites for* $(\mathfrak{g}, \mathfrak{k})$. *In the following,* $\mathfrak{a}'$ *denotes a complimentary abelian subalgebra on which* θ *acts by* (-1):

(1) α *is singular:*
 (i) $(\alpha, \alpha) \neq 0$, $m_\alpha = (0|2n)$ *for* $n \geq 1$, *and* $m_{2\alpha} = (0|0)$:
$$\mathfrak{g} = \mathfrak{osp}(2|2n) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{osp}(1|2n);$$
 (ii) $(\alpha, \alpha) = 0$, $m_\alpha = (0|2)$, *and* $m_{2\alpha} = (0|0)$:
$$\mathfrak{g} = \mathfrak{gl}(1|1) \times \mathfrak{gl}(1|1) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{gl}(1|1);$$
(2) α *is real:*
 (i) $m_\alpha = (m-2|2n)$ *for* $m \geq 3$, $n \geq 1$, *and* $m_{2\alpha} = (0|0)$:
$$\mathfrak{g} = \mathfrak{osp}(m|2n) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{osp}(m-1|2n);$$
 (ii) $m_\alpha = (4(n-2)|2m)$ *for* $m \geq 1$, $n \geq 2$, *and* $m_{2\alpha} = (3|0)$, *for* $n \geq 2$:
$$\mathfrak{g} = \mathfrak{osp}(m|2n) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{osp}(m|2n-2) \times \mathfrak{sp}(2);$$
 (iii) $m_\alpha = (2(m-2)|2n)$ *for* $m \geq 2$, $n \geq 1$, *and* $m_{2\alpha} = (1|0)$:
$$\mathfrak{g} = \mathfrak{gl}(m|n) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{gl}(m-1|n) \times \mathfrak{gl}(1);$$
 (iv) $m_\alpha = (2|0)$ *and* $m_{2\alpha} = (0|0)$:
$$\mathfrak{g} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{sl}(2);$$
 (v) $m_\alpha = (0|2)$ *and* $m_{2\alpha} = (2|0)$:
$$\mathfrak{g} = \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \times \mathfrak{a}', \quad \mathfrak{k} = \mathfrak{osp}(1|2).$$

Corollary 3.10. *If* $(\mathfrak{g}, \mathfrak{k})$ *is any supersymmetric pair, and* $\alpha \in \Delta$, *then* $(\mathfrak{g}\langle\alpha\rangle, \mathfrak{k}\langle\alpha\rangle)$ *is isomorphic to one of the pairs above after quotienting by split,* θ*-fixed ideals.*

3.7. **Rank one integrability conditions.**

Lemma 3.11. *If Σ is a base and $\alpha\in\Sigma$ is a real root, then $V_{\Sigma}(\lambda)$ is α-integrable if and only if $\lambda\in P_{\{\alpha\}}^{+}$ with respect to the rank one root system determined by $(\mathfrak{g}\langle\alpha\rangle,\mathfrak{k}\langle\alpha\rangle)$.*

Proof. This follows immediately from Remark 3.5 and Lemma 3.6. □

Theorem 3.12. *Let $\alpha\in\Sigma$ be a real, simple root. Then a weight $\lambda\in\mathfrak{a}^*$ is α-integrable if and only if $\lambda(h_\alpha)\in 2^{\varepsilon(\alpha)}\cdot 2\mathbb{Z}_{\geq 0}$*

Proof. Suppose that λ is α-integrable, so that $V_{\{\alpha\}}(\lambda)$ is a finite-dimensional $\mathfrak{g}\langle\alpha\rangle$-module. Then λ will be a spherical weight for the underlying even symmetric pair of rank one, which by Thm. 3.12 of [6] implies that $\lambda(h_\alpha)\in 2^{\varepsilon(\alpha)}\cdot 2\mathbb{Z}_{\geq 0}$.

For the converse, we use the classification given in Section 3.6 to construct explicit highest weight representations $V_{\Sigma}(\lambda)$ satisfying $\lambda(h_\alpha)=2^{\varepsilon(\alpha)}\cdot 2$, which is enough because P_{Σ}^{+} is a monoid.

(1) For $(\mathfrak{osp}(m|2n),\mathfrak{osp}(m-1|2n))$ we use the standard representation $\mathbb{k}^{m|2n}$.
(2) For $(\mathfrak{osp}(m|2n),\mathfrak{osp}(m|2n-2)\times\mathfrak{sp}(2))$ we use $(S^2\mathbb{k}^{m|2n})/\mathbb{k}$.
(3) For $(\mathfrak{gl}(m|n),\mathfrak{gl}(m-1|n))$ we use the adjoint representation.
(4) The diagonal cases $(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$ are easy.

□

3.8. **Spherical weights for $\Delta=\Delta_{re}$ or $\Pi\subseteq\Sigma$.** In the case $\Delta=\Delta_{re}$, we have that for all $\alpha\in\Pi$, either $\alpha\in\Sigma$ or $\alpha/2\in\Sigma$. Thus in this case we obtain:

Theorem 3.13. *Suppose that $\Delta_{re}=\Delta$. Then for a base Σ, we have $\lambda\in\mathfrak{a}^*$ is Σ-spherical if and only if for all $\alpha\in\Sigma$ we have $\lambda(h_\alpha)\in 2^{\varepsilon(\alpha)}\cdot 2\mathbb{Z}$.*

Theorem 3.13 applies to the following pairs:

$$(\mathfrak{gl}(m|n),\mathfrak{gl}(r)\times\mathfrak{gl}(m-r|n)),\quad r\leq m/2,\quad (\mathfrak{osp}(m|2n),\mathfrak{osp}(r|2n)\times\mathfrak{so}(r)),\quad r<m/2,$$

$$(\mathfrak{osp}(m|2n),\mathfrak{osp}(m|2n-2s)\times\mathfrak{sp}(2s)),\quad s\leq n/2,\quad (\mathfrak{ag}(1|2),\mathfrak{d}(2,1;3)).$$

The following is also clear.

Proposition 3.14. *Suppose that Σ is a set of simple roots such that $\Pi\subseteq\Sigma$. Then a weight $\lambda\in\mathfrak{a}^*$ is Σ-spherical if and only if for all $\alpha\in\Pi$ we have $\lambda(h_\alpha)\in 2^{\varepsilon(\alpha)}\cdot 2\mathbb{Z}_{\geq 0}$.*

Proposition 3.14 applies to the following pairs for particular choices of base Σ:

$$(\mathfrak{gl}(m|2n),\mathfrak{osp}(m|2n)),\quad (\mathfrak{osp}(2|2n),\mathfrak{osp}(1|2r)\times\mathfrak{osp}(1|2n-2r)).$$

3.9. **Singular reflections of highest weights.**

Lemma 3.15. *Let $\alpha\in\Sigma$ be a singular root with $(\alpha,\alpha)=0$, and let $\lambda\in\mathfrak{a}^*$. Then $V_{\Sigma}(\lambda)\cong V_{r_\alpha\Sigma}(r_\alpha\lambda)$, where $r_\alpha\lambda\in\mathfrak{a}^*$ is given by the following formula:*

$$r_\alpha\lambda=\begin{cases}\lambda-2\alpha & \text{if } (\lambda,\alpha)\neq 0\\ \lambda & \text{if } (\lambda,\alpha)=0.\end{cases}$$

Proof. Recall that $\mathfrak{g}\langle\alpha\rangle = \mathfrak{gl}(1|1)\times\mathfrak{gl}(1|1)\times\mathfrak{a}'$, up to split factors fixed by θ, and the involution swaps the two factors of $\mathfrak{gl}(1|1)$, and is (-1) on $\mathfrak{a}'$,. Thus we may write $\lambda = (\lambda_0, -\lambda_0, \lambda')$, where λ_0 is a weight of $\mathfrak{gl}(1|1)$ and λ' is a weight of $\mathfrak{a}'$. We may similarly write $\alpha = (\alpha', -\alpha', 0)/2$, where α' is a root of $\mathfrak{gl}(1|1)$ with coroot $h_{\alpha'}$. Now we see that

$$(\lambda, \alpha) = \lambda_0(h_{\alpha'}).$$

First suppose that $(\lambda,\alpha) = 0$, which is equivalent to $\lambda_0(h_{\alpha'}) = 0$. Let e_1, e_2 be the raising operators and f_1, f_2 the lowering operators of the two copies of $\mathfrak{gl}(1|1)$. Thus $\mathfrak{g}_{-\alpha} = \Bbbk\langle f_1, e_2\rangle$. Then if we set $W := \Bbbk\langle f_1 v_\lambda, e_2 v_\lambda\rangle$, we see that it is $\mathfrak{m}$-stable and $\mathfrak{g}_\alpha W = 0$. Further, W is purely odd so that the $\mathfrak{k}$-coinvariant vanishes on it. Thus Section 3.5 implies that the map $M_\Sigma(\lambda) \to V_\Sigma(\lambda)$ factors through the quotient by $\mathcal{U}\mathfrak{g}\cdot W$. In particular, $\mathfrak{g}\langle\alpha\rangle$ stabilizes $v_\lambda \in V_\Sigma(\lambda)$. From this we see that $V_\Sigma(\lambda)$ is a quotient of $M_{r_\alpha\Sigma}(\lambda)$, and so by universality we have $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda)$.

On the other hand, $(\lambda, \alpha) \neq 0$ is equivalent to $\lambda_0(h'_\alpha) \neq 0$. In this case we see that $f_1 e_2 v_\lambda \in M_\Sigma(\lambda)$ will be of weight $\lambda - 2\alpha$ with respect to $\mathfrak{a}$, and will be a $\mathfrak{p}_{r_\alpha\Sigma}$-singular vector. Thus we obtain a map $M_{r_\alpha\Sigma}(\lambda-2\alpha) \to M_\alpha(\lambda)$. Surjectivity is easy to check, and so we get an isomorphism $M_{r_\alpha\Sigma}(\lambda - 2\alpha) \cong M_\alpha(\lambda)$. From this we easily obtain $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda - 2\alpha)$. $\square$

We now state what happens in our 'best-behaved' case. Note the following proposition is effectively a generalization of Thm. 10.5 of [13].

Proposition 3.16. *Suppose that every singular root $\alpha \in \Delta_{sing}$ is isotropic. Let $\Pi \subseteq \Delta_\Sigma^+$ be the set of principal roots and let $\lambda \in \mathfrak{a}^*$. Then $\lambda \in P_\Sigma^+$ if and only if for each $\gamma \in \Pi$ there exists some base $\Sigma' \sim \Sigma$ such that $\gamma \in \Sigma'$ and the corresponding reflected weight $\lambda_{\Sigma'}$ satisfies $\lambda_{\Sigma'}(h_\gamma) \in 2^{\varepsilon(\gamma)}\cdot 2\mathbb{Z}_{\geq 0}$.*

Proof. This follows from Corollary 2.25 and 3.15. $\square$

Proposition 3.16 applies to the following pairs:

$$(\mathfrak{g}\times\mathfrak{g}, \mathfrak{g}), \qquad (\mathfrak{gl}(m|n), \mathfrak{gl}(r|s)\times\mathfrak{gl}(m-r|n-s))$$

3.10. **Reflections in nonisotropic, singular roots.**

Lemma 3.17. *Suppose that $\alpha \in \Sigma$ is a singular, nonisotropic root of multiplicity $(0|2n)$, and $\lambda \in \mathfrak{a}^*$.*

1. *If $\lambda(h_\alpha)/2 \notin \{0, 1, \dots, n-1, n+1, \dots, 2n\}$, then $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda - 2n\alpha)$;*
2. *if $k := \lambda(h_\alpha)/2 \in \{0, 1, \dots, n-1\}$ then*
$$V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda - 2k\alpha);$$
3. *if $\lambda(h_\alpha)/2 = n + k \in \{n+1, n+2, \dots, 2n\}$, then $V_\Sigma(\lambda)$ contains $V_\Sigma(\lambda - 2k\alpha)$ as a submodule. In particular, if $\lambda \in P_\Sigma^+$ then $\lambda - 2k\alpha \in P_\Sigma^+$.*

Proof. Observe that $\mathfrak{g}_\alpha$ will be an irreducible, purely odd, $\mathfrak{m}$-module with a symplectic form $\omega \in \Lambda^2\mathfrak{g}_{-\alpha}$. Thus $\omega^n \in \Lambda^{2n}\mathfrak{g}_{-\alpha} = \Lambda^{top}\mathfrak{g}_{-\alpha}$ is non-zero.

In case (1), λ will be a typical weight for $\mathfrak{osp}(2|2n) \subseteq \mathfrak{g}\langle\alpha\rangle$. Thus $\omega^n v_\lambda$ will generate $M_\Sigma(\lambda)$, is of weight $\lambda - 2n\alpha$, and is annihilated by $\mathfrak{m}$ and $\mathfrak{g}_{-\alpha}$. Hence we obtain a map $M_{r_\alpha\Sigma}(\lambda - 2n\alpha) \to M_\Sigma(\lambda)$, and it is easy to see it is an isomorphism. In this way, we obtain an isomorphism $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda - 2n\alpha)$, as desired.

In case (2), we use the computations in Sec. 10 of [17] to see that $\mathcal{U}\mathfrak{g}\langle\alpha\rangle v_\lambda$ contains a $\mathfrak{p}_\Sigma$-stable subspace W on which the $\mathfrak{k}$-coinvariant vanishes. Taking the quotient of $M_\Sigma(\lambda)$ by the submodule generated by W, we obtain a module M' for which $\mathcal{U}\mathfrak{g}\langle\alpha\rangle v_\lambda$ is irreducible, with lowest weight vector $\omega^k v_\lambda$ (again applying computations in [17]). Thus we obtain a surjective map $M_{r_\alpha\Sigma}(\lambda - 2k\alpha) \to M'$, from which it easily follows that $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda - 2k\alpha)$.

Finally, for case (3), the computations of Sec. 10 in [17] once again show that $\omega^k v_\lambda$ is an $\mathfrak{m}$-fixed, $\mathfrak{p}$-singular vector on which the $\mathfrak{k}$-coinvariant does not vanish. Therefore we get a map $V_\Sigma(\lambda - k\alpha) \to V_\Sigma(\lambda)$, and injectivity is by definition. □

Definition 3.18. Let $\lambda \in \mathfrak{a}^*$ and let $\alpha \in \Sigma$ be a singular, nonisotropic root. Then we say that λ is an α-critical weight if $\lambda(h_\alpha)/2 \in \{n+1, \dots, 2n\}$.

Definition 3.19. Let $\lambda \in \mathfrak{a}^*$, and let $\alpha \in \Sigma$ be a simple, singular, non-isotropic root. If λ is not α-critical, then set:

$$r_\alpha\lambda := \begin{cases} \lambda - \lambda(h_\alpha)\alpha & \text{if } \lambda(h_\alpha)/2 \in \{0, \dots, n-1\}; \\ \lambda - 2n\alpha & \text{otherwise.} \end{cases}$$

In particular, $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(r_\alpha\lambda)$.

Using singular reflections, it is clear that we can understand when a weight $\lambda \in \mathfrak{a}^*$ lies in P_Σ^+ if no reflection of λ is critical for simple, singular, non-isotropic root. For weights that are critical for some singular, nonisotropic $\alpha \in \Sigma$, we may use the following to understand something.

Lemma 3.20. *Let $\alpha \in \Sigma$ be a nonisotropic, singular root, and let β be a simple real root of $r_\alpha\Sigma$. Then for an α-critical weight $\lambda \in \mathfrak{a}^*$, $V_\Sigma(\lambda)$ is β-integrable if and only if both $V_{r_\alpha\Sigma}(\lambda)$ and $V_{r_\alpha\Sigma}(\lambda - 2n\alpha)$ are.*

Caution: we do *not* have an isomorphism $V_\Sigma(\lambda) \cong V_{r_\alpha\Sigma}(\lambda)$.

Proof. Let $\mathfrak{p}_{\alpha,\Sigma}$ be the parabolic subalgebra of $\mathfrak{g}$ containing both $\mathfrak{p}_\Sigma$ and $\mathfrak{g}_{-\alpha}$. Let $V_{\lambda,\alpha}$ be the finite-dimensional Kac-module over $\mathfrak{g}\langle\alpha\rangle \cong \mathfrak{osp}(2|2n) \times \mathfrak{a}' \times (...)$ of highest weight λ. Note that $V_{\lambda,\alpha}$ is indecomposable with composition series $0 \to L_{\lambda-2k,\alpha} \to V_{\lambda,\alpha} \to L_{\lambda,\alpha} \to 0$, where $L_{(-),\alpha}$ is the corresponding simple module over $\mathfrak{g}\langle\alpha\rangle$.

Then we may inflate $V_{\lambda,\alpha}$ to $\mathfrak{p}_{\alpha,\Sigma}$. Observe that $V_\Sigma(\lambda)$ is a quotient of $\mathrm{Ind}_{\mathfrak{p}_{\alpha,\Sigma}}^{\mathfrak{g}} V_{\lambda,\alpha}$. As a module over $\mathfrak{p}_{r_\alpha\Sigma}$, it is generated by $V_{\lambda,\alpha}^{\mathfrak{m}} = \Bbbk\langle v_\lambda, \omega v_\lambda, \cdots, \omega^{2n} v_\lambda\rangle$. Thus $V_\Sigma(\lambda)$ is β-integrable if and only if each vector $\omega^j v_\lambda$ is β-integrable, i.e. $(\mathfrak{g}_{-\beta})^N \omega^j v_\lambda$ for $N \gg 0$.

However, by Lemma 3.12, this is in turn is equivalent to:

$$(\lambda - 2j\alpha)(h_\beta) \in 2^{\varepsilon(\beta)} \cdot 2\mathbb{Z}_{\geq 0} \quad \text{for all } j = 0, \dots, 2n.$$

Since evaluation at h_β is a linear function in j, the result is now clear from Lemma 2.17. □

We note that in principal one could use the idea of Lemma 3.20 to try and understand what happens after performing multiple reflections. However, we don't see at this moment how to understand this picture in a simple way in any generality.

4. Explicit computations of spherical weights

In the final section, we explicitly describe the sets P_Σ^+ for convenient choices of Σ when $(\mathfrak{g}, \mathfrak{k})$ is a supersymmetric pair with $\mathfrak{g}$ indecomposable. In the problematic cases, i.e. those for which nonisotropic singular roots are present, we choose Σ so that all but the last singular reflection will have critical weights, allowing us to rely on Lemma 3.20.

We stress that we have made choices of seemingly convenient bases Σ. However the techniques we employ have the potential to work for other bases as well. If one is interested in a description of P_Σ^+ for a Σ not used here, one may attempt to apply our techniques in their case. In particular, our technique will work when Σ has the property that every principal root either lies in Σ or lies in $r_\alpha\Sigma$ for some singular root $\alpha\in\Sigma$. Such Σ can often be constructed by making them consists of as many singular roots as possible.

We begin by working abstractly with restricted root systems, for simplicity.

Example 4.1. Let $r,s\in\mathbb{Z}_{\geq 0}$, $k\in\Bbbk^\times$, and define $BC_k(r,s)$ to be the weak generalized root system with two real components $\Delta_{re}=BC_r\sqcup BC_s$ and singular roots $\Delta_{sing}=W(\omega_1^{(1)}+\omega_1^{(2)})$. Here $\omega_i^{(j)}$ denotes the ith fundamental weight of the jth irreducible component of Δ_{re}, as in [14]. The bilinear form is normalized such that $(\omega_1^{(1)},\omega_1^{(1)})=1$ and $(\omega_1^{(2)},\omega_1^{(2)})=k$. Define $C_k(r,s)\subseteq BC_k(r,s)$ to be the same as $BC_k(r,s)$ but without the short roots.

Let us write $\gamma_1,\dots,\gamma_r,\nu_1,\dots,\nu_s$ for a basis of the underlying vector space given by mutually orthogonal short roots, where $\gamma_1,\dots,\gamma_r\in BC_r$ and $\nu_1,\dots,\nu_s\in BC_s$. Then one base Σ for $BC_k(r,s)$ is given by:

$$\gamma_1-\gamma_2,\dots,\gamma_{r-1}-\gamma_r,\gamma_r-\nu_1,\dots,\nu_{s-1}-\nu_s,\nu_s.$$

A set of principal roots is given by $\Pi=\{\gamma_1-\gamma_2,\dots,\gamma_{r-1}-\gamma_r,\gamma_r,\nu_1-\nu_2,\dots,\nu_{s-1}-\nu_s,\nu_s\}$. Therefore we have that $\{\gamma_r\}=\Pi\setminus(\Pi\cap\Sigma)$.

For $C_k(r,s)$, we may take the same base and principal roots, only we need to multiply ν_s and γ_r by 2. However this minor difference does not affect any of the computations of P_Σ^+ below.

Let $\lambda=\sum\limits_i a_i\gamma_i+\sum\limits_j b_j\nu_j$. Then a necessary condition for $\lambda\in P_\Sigma^+$ is that $\lambda(h_\alpha)\in 2^{\varepsilon(\alpha)}\cdot 2\mathbb{Z}_{\geq 0}$ for all $\alpha\in\Pi$, i.e.

$$a_i-a_{i+1},a_r,b_i-b_{i+1},b_s\in 2\mathbb{Z}_{\geq 0}.$$

To obtain γ_r as a simple root, we must apply the simple reflections $r_{\gamma_r-\nu_1},\dots,r_{\gamma_r-\nu_s}$. We see that for $k\neq -1$:

$$\lambda(h_{\gamma_r-\nu_1})/2=\frac{a_r-kb_1}{1+k}.$$

Case I, $k=-1$: In this case all singular roots are isotropic, so in fact Proposition 3.16 applies. We see that $(\lambda,\gamma_r-\nu_1)=a_r+b_1$, which is non-negative, and zero if and only if $a_r=b_1=0$. Thus either $b_i=0$ for all i and $\lambda\in P_\Sigma^+$, or $r_{\gamma_r-\nu_1}\lambda=\lambda-2\gamma_r+2\nu_1$, meaning we must have $a_r\geq 2$. If $b_2=0$ then again it is clear that $\lambda\in P_\Sigma^+$, and otherwise we need $a_r\geq 4$. Continuing like this, we learn that

$$\lambda\in P_\Sigma^+\iff a_r/2\geq|\{i:b_i\neq 0\}|.$$

Case II, $k=-1/2$, $m_{\gamma_r-\nu_1}=(0|2)$: We have $\gamma_i-\nu_j$ is a non-isotropic singular root. Observe that $\lambda(h_{\gamma_r-\nu_1})/2=2a_r+b_1$ which is an even, non-negative integer. If this quantity is 0, then $b_i=0$ for all i and $a_r=0$, so that $\lambda\in P_\Sigma^+$.

If $a_r=0$ and $b_1>0$ then by Lemma 3.17, $\lambda\in P_\Sigma^+$ only if $\lambda-2\gamma_r+2\nu_1$ is integrable. But it is clearly not since the coefficient of γ_r would become negative. Thus $a_r=0\Rightarrow b_1=0$, and for all other values of (a_r,b_1) we have $2a_r+b_1>2$. Hence there are no critical weights for $\gamma_r-\nu_1$. Because $m_{\gamma_r-\nu_1}=(0|2)$, in these cases we will have $r_{\gamma_r-\nu_1}\lambda=\lambda-2\gamma_r+2\nu_1$, meaning we must have $a_r\geq 2$.

Using inductive reasoning as in the first case, we once again find that:

$$\lambda \in P_\Sigma^+ \iff a_r/2 \geq |\{i : b_i \neq 0\}|.$$

Case III: $r = s = 1$**,** $m_{\gamma_1-\nu_1} = (0|2)$**,** $k \neq -1$

Recall that

$$\lambda(h_{\gamma_1-\nu_1})/2 = \frac{a_1 - kb_1}{1+k}.$$

If the above quantity is equal to 0, then we must have $a_1 = kb_1$, implying that either $a_1 = b_1 = 0$ or $a_1, b_1 > 0$. If the above quantity is equal to 2, then we must have $a_1 \neq 2$. However if $a_1 = 0$ then $\lambda - 2\gamma_1 + 2\nu_1$ is not dominant, so λ can't be either by Lemma 3.17.

In all other cases, $r_\alpha\lambda = \lambda - 2\gamma_1 + 2\nu_1$ must be integrable, meaning that $a_1 \geq 2$. Therefore we obtain that:

$$\lambda = a_1\gamma_1 + b_1\nu_1 \in P_\Sigma^+ \iff a_1 = 0 \Rightarrow b_1 = 0.$$

4.1. **Tables with P_Σ^+.** In Table 1, we describe Δ and make a choice of base Σ for each pair. In Tables 2 and 3 which follow, we explicitly describe P_Σ^+ for the given choice of Σ. We use the presentations of generalized root systems given in Sec. 5.2 of [16].

In the next section we will justify our computations.

Table 1.

$(\mathfrak{g}, \mathfrak{k})$	Δ Σ
$(\mathfrak{gl}(m\|2n), \mathfrak{osp}(m\|2n))$	$A(m-1, n-1)$ $\epsilon_1 - \epsilon_2, \ldots, \epsilon_m - \nu_1, \nu_1 - \nu_2, \ldots, \nu_{n-1} - \nu_n$ $\nu_i := (\delta_{2i-1} + \delta_{2i})/2$
$(\mathfrak{gl}(m\|n), \mathfrak{gl}(r\|s) \times \mathfrak{gl}(m-r\|n-s))$ $r \leq m/2, s \leq n/2$	$(B)C_{-1/2}(r, s)$ $\gamma_1 - \gamma_2, \ldots, \gamma_r - \nu_1, \nu_1 - \nu_2, \ldots, \nu_{s-1} - \nu_s, (2)\nu_s$ $\gamma_i := (\epsilon_i - \epsilon_{m-i+1})/2$, $\nu_i := (\delta_i - \delta_{n-i+1})/2$
$(\mathfrak{osp}(2m\|2n), \mathfrak{gl}(m\|n))$	$(B)C_{-1/2}(n, m)$ $\delta_1 - \delta_2, \ldots, \delta_n - \gamma_1, \gamma_1 - \gamma_2, \ldots, \gamma_{m-1} - \gamma_m, (2)\gamma_m$ $\gamma_i = (\epsilon_{2i-1} + \epsilon_{2i})/2$
$(\mathfrak{osp}(m\|2n), \mathfrak{osp}(r\|2s) \times \mathfrak{osp}(m-r\|2n-2s))$ $r < m/2, s \leq n/2$	$BC_{-1/2}(r, s)$ $\epsilon_1 - \epsilon_2, \ldots, \epsilon_r - \nu_1, \nu_1 - \nu_2, \ldots, \nu_{s-1} - \nu_s, \nu_s$ $\nu_i = (\delta_{2i-1} + \delta_{2i})/2$
$(\mathfrak{osp}(2r\|2n), \mathfrak{osp}(r\|2s) \times \mathfrak{osp}(r\|2n-2s))$ $0 < s < n/2$	$\Delta_{re} = D_r \sqcup C_s$, $\Delta_{sing} = W(\omega_1^{(1)} + \omega_2^{(2)}) \sqcup W\omega_1^{(1)}$ $\epsilon_r + \nu_s, -\epsilon_r + \nu_s, -\nu_s + \epsilon_{r-1}, -\epsilon_{r-1} + \nu_{s-1}, \ldots$ $\nu_i = (\delta_{2i-1} + \delta_{2i})/2$
$(\mathfrak{osp}(2r\|4s), \mathfrak{osp}(r\|2s) \times \mathfrak{osp}(r\|2s))$	$D(r, s)$ $\nu_1 - \nu_2, \ldots, \nu_s - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{r-1} - \epsilon_r, \epsilon_{r-1} + \epsilon_r$ $\nu_i = (\delta_{2i-1} + \delta_{2i})/2$
$(\mathfrak{osp}(2r\|2n), \mathfrak{osp}(r\|2n) \times \mathfrak{so}(r))$	$\Delta_{re} = D_r$, $\Delta_{sing} = W\omega_1$ $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{r-1} - \epsilon_r, \epsilon_r$
$(\mathfrak{d}(2,1;a), \mathfrak{osp}(2\|2) \times \mathfrak{so}(2))$	$C_a(1,1)$ $\alpha - \beta, 2\beta$
$(\mathfrak{ab}(1\|3), \mathfrak{gosp}(2\|4))$	$C_{-3}(1,1)$ $\epsilon/2 - \delta/2, \delta$
$(\mathfrak{ab}(1\|3), \mathfrak{sl}(1\|4))$	$\Delta_{re} = B_2 \sqcup C_1$, $\Delta_{sing} = W(\omega_2^{(1)} + \omega_1^{(2)})$ $\epsilon_2, (\epsilon_1 - \epsilon_2 - \delta)/2, \delta$
$(\mathfrak{ab}(1\|3), \mathfrak{d}(2,1;2) \times \mathfrak{sl}(2))$	$\Delta_{re} = B_3$, $\Delta_{sing} = W\omega_3$ $\epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2, (-\epsilon_1 + \epsilon_2 + \epsilon_3)/2$
$(\mathfrak{ag}(1\|2), \mathfrak{d}(2,1;3))$	G_2 any base

Table 2.

$(\mathfrak{g}, \mathfrak{k})$	P_Σ^+
$(\mathfrak{gl}(m\|2n), \mathfrak{osp}(m\|2n))$	$a_1\epsilon_1 + \cdots + a_m\epsilon_m + b_1\nu_1 + \cdots + b_n\nu_n$ $a_i - a_{i+1}, b_i - b_{i+1} \in 2\mathbb{Z}_{\geq 0}$
$(\mathfrak{gl}(m\|n),$ $\mathfrak{gl}(r\|s) \times \mathfrak{gl}(m-r\|n-s))$ $r \leq m/2,\ s \leq n/2$	$a_1\gamma_1 + \cdots + a_r\gamma_r + b_1\nu_1 + \cdots + b_s\gamma_s$ $a_i - a_{i+1}, a_r, b_i - b_{i+1}, b_s \in 2\mathbb{Z}_{\geq 0},\ a_r/2 \geq \|\{i : b_i \neq 0\}\|$
$(\mathfrak{osp}(2m\|2n), \mathfrak{gl}(m\|n))$	$a_1\gamma_1 + \cdots + a_m\gamma_m + b_1\delta_1 + \cdots + b_n\delta_n$ $a_i - a_{i+1}, a_m, b_i - b_{i+1}, b_n \in 2\mathbb{Z}_{\geq 0},\ b_n/2 \geq \|\{i : a_i \neq 0\}\|$
$(\mathfrak{osp}(m\|2n),$ $\mathfrak{osp}(r\|2s) \times \mathfrak{osp}(m-r\|2n-2s))$ $r < m/2, s \leq n/2$	$a_1\epsilon_1 + \cdots + a_r\epsilon_r + b_1\nu_1 + \cdots + b_s\nu_s$ $a_i - a_{i+1}, a_r, b_i - b_{i+1}, b_s \in 2\mathbb{Z}_{\geq 0},\ a_r/2 \geq \|\{i : b_i \neq 0\}\|$
$(\mathfrak{osp}(2r\|2n),$ $\mathfrak{osp}(r\|2s) \times \mathfrak{osp}(r\|2n-2s))$ $0 < s < n/2$	$a_1\epsilon_1 + \cdots + a_r\epsilon_r + b_1\nu_1 + \cdots + b_s\nu_s$ $a_i - a_{i+1}, a_{r-1} + a_r, b_i - b_{i+1}, b_s \in 2\mathbb{Z}_{\geq 0}$. If $s < r$ then either there exists $i = 0, \ldots, s$ such that $a_1 \geq a_2 \geq \cdots \geq a_{r-s} > a_{r-s+1} > \cdots > a_{r-s+i} = 0$ and $b_1 > b_2 > \cdots > b_i \geq b_{i+1} = 0$, otherwise $a_1 \geq a_2 \geq \cdots \geq a_{r-s} > a_{r-s+1} > \cdots > \|a_r\|$ and $b_1 > b_2 > \cdots > b_{s-1} > b_s \geq 0$. If $s \geq r$ then either there exists $i = 0, \ldots, r$ such that $a_1 > a_2 > \cdots > a_i = 0$ and $b_1 \geq \cdots \geq b_{s-r+1} > b_{s-r+2} > \cdots > b_{s-r+i} \geq b_{s-r+i+1} = 0$, otherwise $a_1 > a_2 > \cdots > a_{r-1} > \|a_r\|$ and $b_1 \geq \cdots \geq b_{s-r+1} > b_{s-r+2} > \cdots > b_s \geq 0$
$(\mathfrak{osp}(2r\|4s),$ $\mathfrak{osp}(r\|2s) \times \mathfrak{osp}(r\|2s))$	$a_1\epsilon_1 + \cdots + a_r\epsilon_r + b_1\nu_1 + \cdots + b_s\nu_s$ $a_i - a_{i+1}, b_i - b_{i+1}, b_s \in 2\mathbb{Z}_{\geq 0}$, $a_{r-1} \geq \|a_r\|,\ \ b_s/2 \geq \|\{i : a_i \neq 0\}\|$
$(\mathfrak{osp}(2r\|2n), \mathfrak{osp}(r\|2n) \times \mathfrak{so}(r))$	$a_1\epsilon_1 + \cdots + a_r\epsilon_r$ $a_i - a_{i+1} \in 2\mathbb{Z}_{\geq 0}$, and either $a_r \geq 0$ or $a_{r-1} \geq \|a_r - 2n\|$

Table 3. Continuation of Table 2

$(\mathfrak{g}, \mathfrak{k})$	P_Σ^+
$(\mathfrak{d}(2,1;a), \mathfrak{osp}(2\|2)\times\mathfrak{so}(2))$	$a\alpha + b\beta$ $a, b \in 2\mathbb{Z}_{\geq 0}$, $b = 0 \Rightarrow a = 0$
$(\mathfrak{ab}(1\|3), \mathfrak{gosp}(2\|4))$	$a\epsilon + b\delta$ $a, b \in \mathbb{Z}_{\geq 0}$ and either $a = b = 0$ or $a \geq 2$
$(\mathfrak{ab}(1\|3), \mathfrak{sl}(1\|4))$	$a_1\epsilon_1 + a_2\epsilon_2 + b\delta$ $a_1 - a_2 \in 2\mathbb{Z}_{\geq 0}$, $a_2, b \in \mathbb{Z}_{\geq 0}$ either $a_1 = a_2 = b = 0$ or $a_1 \geq 0$
$(\mathfrak{ab}(1\|3), \mathfrak{d}(2,1;2)\times\mathfrak{sl}(2))$	$a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$ $a_i - a_{i+1} \in 2\mathbb{Z}_{\geq 0}$, $a_3 \in \mathbb{Z}_{\geq 0}$ and either $a_3 = 0$ or $a_1 > a_2$
$(\mathfrak{ag}(1\|2), \mathfrak{d}(2,1;3))$	$a_1\omega_1 + a_2\omega_2$ $a_1, a_2 \in 2\mathbb{Z}_{\geq 0}$, ω_1, ω_2 the fundamental dominant weights.

4.2. **Computations of P_Σ^+ for supersymmetric pairs.** We now justify the computations presented in the tables above by going through the pairs case by case. We once again remind that we use the presentations of generalized root systems given in Sec. 5.2 of [16].

(1) $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$: For the choice of Σ given we have $\Pi \subseteq \Sigma$, so we may apply Proposition 3.14 to obtain our description.
(2) $(\mathfrak{gl}(m|n), \mathfrak{gl}(r|s)\times\mathfrak{gl}(m-r|n-s))$, $r \leq m/2$, $s \leq n/2$: In this case we obtain the restricted root system $BC_{-1}(r,s)$ if $r < n/2$ or $s < m/2$, and otherwise we get $C_{-1}(m/2, n/2)$. Thus we may apply Case I of Example 4.1 to obtain our description.
(3) $(\mathfrak{osp}(2m|2n), \mathfrak{gl}(m|n))$: After renormalizing the form, we obtain $C_{-1/2}(n,m)$ if m is even and $BC_{-1/2}(n,m)$ otherwise. Thus we may apply Case II of Example 4.1 to obtain our description.
(4) $(\mathfrak{osp}(m|2n), \mathfrak{osp}(m-r|2n-2s)\times\mathfrak{osp}(r|2s))$, $r < m/2$, $s \leq n/2$: We obtain restricted root system $BC_{-1/2}(r,s)$, so we may apply Case II of Example 4.1.
(5) $(\mathfrak{osp}(2r|2n), \mathfrak{osp}(r|2s)\times\mathfrak{osp}(r|2n-2s))$, $0 < s < n/2$: let us be more explicit about the base we choose. If $s < r$, we take the base to be

$$\epsilon_1 - \epsilon_2, \dots, \epsilon_{r-s} - \nu_1, \nu_1 - \epsilon_{r-s+1}, \dots, \epsilon_{r-1} - \nu_s, \nu_s \pm \epsilon_r.$$

Thus we would like to independently apply the singular reflections $r_{\epsilon_{r-s}-\nu_1}, r_{\epsilon_{r-s+1}-\nu_2}, \dots, r_{\epsilon_{r-1}-\nu_s}, r_{\epsilon_r+\nu_s}$ in order to obtain the remaining real roots.

In this case, this tells us that for $i = 0, \dots, s-1$ either $a_{r-s+i} = b_{i+1} = 0$ or $a_{r-s+i} > |a_{r-s+i+1}|$ and $b_i > b_{i+1}$, where if $i = 0$ the condition $b_i > b_{i+1}$ is empty. Further, either $2a_r = b_s$ or $a_{r-1} > |a_r - 2|$ and $b_{s-1} > b_s$. In other words, either for some $i = 0, \dots, s$ we have

$$a_1 \geq a_2 \geq \cdots \geq a_{r-s} > a_{r-s+1} > \cdots > a_{r-s+i} = 0$$

and

$$b_1 > b_2 > \cdots > b_i \geq b_{i+1} = 0.$$

or

$$a_1 \geq a_2 \geq \cdots \geq a_{r-s} > a_{r-s+1} > \cdots > |a_r|$$

and

$$b_1 > b_2 > \cdots > b_{s-1} > b_s \geq 0.$$

If, on the other hand, $s \geq r$, we have base:

$$\nu_1 - \nu_2, \ldots, \nu_{s-r+1} - \epsilon_1, \epsilon_1 - \nu_{s-r+2}, \ldots, \epsilon_{r-1} - \nu_s, \nu_s \pm \epsilon_r.$$

Thus we would like to independently apply the singular reflections $r_{\epsilon_1-\nu_{s-r+2}}, r_{\epsilon_2-\nu_{s-r+3}}, \ldots, r_{\epsilon_{r-1}-\nu_s}, r_{\epsilon_r+\nu_s}$. This forces either $a_i = b_{s-r+i+1} = 0$ or $a_i > |a_{i+1}|$ and $b_{s-r+i} > b_{s-r+i+1}$. Further, either $2a_r = b_s$ or $a_{r-1} \geq |a_r - 2|$ and $b_{s-1} > b_s$. Stated more succinctly, we have that either for some $i = 1, \ldots, r$:

$$a_1 > a_2 > \cdots > a_i = 0$$

and

$$b_1 \geq b_2 \geq \cdots \geq b_{s-r+1} > b_{s-r+2} > \cdots > b_{s-r+i} \geq b_{s-r+i+1} = 0.$$

or

$$a_1 > a_2 > \cdots > a_{r-1} > |a_r|$$

and

$$b_1 \geq b_2 \geq \cdots \geq b_{s-r+1} > b_{s-r+2} > \cdots > b_{s-1} > b_s \geq 0.$$

(6) $(\mathfrak{osp}(2r|4s), \mathfrak{osp}(r|2s) \times \mathfrak{osp}(r|2s))$: take base $\gamma_1 - \gamma_2, \ldots, \gamma_{s-1} - \gamma_s, \gamma_s - \epsilon_1, \ldots, \epsilon_{r-1} \pm \epsilon_r$. Then we want to apply simple reflections $r_{\gamma_s-\epsilon_1}, r_{\gamma_s-\epsilon_2}, \ldots, r_{\gamma_s-\epsilon_r}$. Let $\lambda = \sum_j b_j\gamma_j + \sum_i a_i\epsilon_i$. Then if $b_s = 2k$ with $k < r-1$, we are forced to have $a_{k+1} = \cdots = a_r = 0$, and we will have $\lambda \in P_\Sigma^+$.

Suppose instead that $b_s = 2k$ with $k \geq r-1$. Then

$$\lambda' := r_{\gamma_s-\epsilon_{r-1}} \ldots r_{\gamma_s-\epsilon_1}\lambda = b_1\gamma_1 + \cdots + b_{s-1}\gamma_{s-1} + (b_s - 2(r-1))\gamma_s + (a_1+2)\epsilon_1 + \cdots + (a_{r-1}+2)\epsilon_{r-1} + a_r\epsilon_r.$$

We see that

$$\lambda'(h_{\gamma_s-\epsilon_r}) = -(b_s - 2(r-1)) - 2a_r.$$

If this quantity is 0, then $\lambda \in P_\Sigma^+$. Otherwise, the following weight must be integrable:

$$b_1\gamma_1 + \cdots + b_{s-1}\gamma_{s-1} + (b_s - 2r)\gamma_s + (a_1+2)\epsilon_1 + \cdots + (a_{r-1}+2)\epsilon_{r-1} + (a_r+2)\epsilon_r,$$

meaning we need $b_s \geq 2r$.

(7) $(\mathfrak{osp}(2r|2n), \mathfrak{osp}(r|2n) \times \mathfrak{so}(r))$ with $n > 0$: in this case we have base $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{r-1} - \epsilon_r, \epsilon_r$. So we are missing the principal root $\epsilon_{r-1} + \epsilon_r$, which is obtained by r_{ϵ_r}. Notice that $m_{\epsilon_r} = (0|2n)$. If $\lambda = \sum_{i=1}^r a_i\epsilon_i$, then $\lambda \in P_\Sigma^+$ if and only if $a_i - a_{i+1} \in 2\mathbb{Z}_{\geq 0}$, $a_{r-1} \geq |a_r|$, and either $a_r \geq 0$ or $a_{r-1} \geq |a_r - 2n|$.

(8) $(\mathfrak{d}(2,1;a), \mathfrak{osp}(2|2) \times \mathfrak{so}(2))$: Here we obtain restricted root system $C_a(1,1)$ where non-isotropic singular roots have mulitplicity $(0|2)$. Thus we may apply Case III of Example 4.1.

(9) $(\mathfrak{ab}(1|3), \mathfrak{gosp}(2|4))$: $C(1,1)$ deformed: $\mathfrak{a}^*$ has basis ϵ, δ where $(\epsilon,\epsilon) = 1/3$, $(\delta,\epsilon) = 0$, $(\delta,\delta) = -1$. Then Σ can be taken as $(\epsilon-\delta)/2, \delta$ and $\Pi = \{\epsilon, \delta\}$. We have $r_{\epsilon-\delta}\Sigma = \{(\delta-\epsilon)/2, \epsilon\}$. Then for $\lambda = a\epsilon + b\delta$, we have $\lambda \in P_\Sigma^+$ if and only if $a, b \in \mathbb{Z}$ and either $a = b = 0$ or $a \geq 2$.

(10) $(\mathfrak{ab}(1|3), \mathfrak{sl}(1|4))$: $B_2 \sqcup C_1$, deformed: $\mathfrak{a}^*$ has basis $\epsilon_1, \epsilon_2, \delta$ where $(\epsilon_i, \epsilon_j) = \delta_{ij}/3$, $(\epsilon_i, \delta) = 0$, and $(\delta,\delta) = -1$. Then we may take Σ to be $\epsilon_2, (\epsilon_1 - \epsilon_2 - \delta)/2, \delta$, and we have $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2, \delta\}$. Then $r_{(\epsilon_1-\epsilon_2-\delta)/2}\Sigma$ is $(\epsilon_1+\epsilon_2-\delta)/2, (-\epsilon_1+\epsilon_2+\delta)/2, \epsilon_1 - \epsilon_2$.

Now if $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + b\delta$ then $\lambda \in P_\Sigma^+$ if and only if $a_2, b \in \mathbb{Z}_{\geq 0}$, $a_1 - a_2 \in 2\mathbb{Z}_{\geq 0}$, and either $a_1 = a_2 = b = 0$ or $a_1 \geq 1$.

(11) $(\mathfrak{ab}(1|3), \mathfrak{d}(2,1;2))$: B_3 with small orbit: $\mathfrak{a}^*$ has basis $\epsilon_1, \epsilon_2, \epsilon_3$, where $(\epsilon_i, \epsilon_j) = \delta_{ij}/3$. Then for Σ we may take $\epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2, (-\epsilon_1 + \epsilon_2 + \epsilon_3)/2$, and $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3\}$. Then we have $r_{(-\epsilon_1+\epsilon_2+\epsilon_3)/2}\Sigma$ is given by $\epsilon_2 - \epsilon_3, \epsilon_3, (\epsilon_1 - \epsilon_2 - \epsilon_3)/2$.

Let $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 \in \mathfrak{a}^*$. Then a necessary condition that $\lambda \in P_\Sigma^+$ is that $a_1 - a_2, a_2 - a_3 \in 2\mathbb{Z}_{\geq 0}$, $a_3 \in \mathbb{Z}_{\geq 0}$. We see that

$$\lambda(h_{(-\epsilon_1+\epsilon_2+\epsilon_3)/2})/2 = \frac{2}{3}(-a_1 + a_2 + a_3).$$

Thus any weight $\lambda = (a+b)\epsilon_1 + a\epsilon_2 + b\epsilon_3$ such that $a, b, a-b \in 2\mathbb{Z}_{\geq 0}$ will lie in P_Σ^+. Otherwise, we need that $a_1 > a_2$.

(12) $(\mathfrak{ag}(1|2), \mathfrak{d}(2,1;3))$: G_2: In this case the restricted root system is just G_2, and all simple roots are real. We present $\mathfrak{a}^*$ with basis ν_1, ν_2, the fundamental weights for G_2. Then for $\lambda = a_1\nu_1 + a_2\nu_2 \in \mathfrak{a}^*$, we have $\lambda \in P_\Sigma^+$ if and only if $a_1 - a_2, a_2 \in 2\mathbb{Z}_{\geq 0}$.

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Alexander Sherman, School of Mathematics and Statistics, University of Sydney, Camperdown NSW 2006

Email address: xandersherm@gmail.com