Ordering Collective Unit Tasks: from Scheduling to Computational Social Choice

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Abstract

We study the collective schedules problem, which consists in computing a one machine schedule of a set of tasks, knowing that a set of individuals (also called voters) have preferences regarding the order of the execution of the tasks. Our aim is to return a consensus schedule. We consider the setting in which all tasks have the same length – such a schedule can therefore also be viewed as a ranking. We study two rules, one based on a distance criterion, and another one based one a binary criterion, and we show that these rules extend classic scheduling criteria. We also consider time constraints and precedence constraints between the tasks, and focus on two cases: the preferences of the voters fulfill these constraints, or they do not fulfill these constraints (but the collective schedule should fulfill them). In each case, either we show that the problem is NP-hard, or we provide a polynomial time algorithm which solves it. We also provide an analysis of a heuristic, which appears to be a 2 approximation of the Spearman's rule.

Keywords: Computational social choice, Rank aggregation, Scheduling

The collective schedules problem (Durand & Pascual, 2022; Pascual, Rzadca, & Skowron, 2018) consists in scheduling a set of n tasks shared by v individuals, also called voters. The tasks may represent talks of a conference that will be done in a same room, works to be done sequentially by co-workers, or events that will occur in one of the weekly meetings of an association. Each voter has his or her own preferences regarding the order in which theses tasks will be executed. We consider two models. In the first one, introduced by Pascual et al. (2018) and that we will call *Order Preferences*, each voter gives his or her preferred order – a permutation of the tasks. In the second model, that we introduce in this paper, and that we call *Interval Preferences*, each voter gives for each task the interval in which he or she would like the task to be done. In this paper, we focus on situations in which all the tasks have the same duration (a time slot per task). Our aim is, given the preferences of each individual, to compute a good compromise schedule of the n tasks.

Note here that, since the tasks are unit tasks, a schedule of the n tasks can be seen as a ranking of n tasks (or candidates). In this paper, we will use several concepts – such as precedence constraints – from the scheduling field: we will therefore use the term schedule and not ranking for a permutation of the n tasks. Note however that several results of this paper are interesting not only in the context of scheduling but also in the context of ranking candidates.

The dissatisfaction of a voter if the returned schedule is schedule S is measured thanks to two families of criteria, coming from the scheduling theory field. One is a binary criterion, which says that a voter is satisfied if a task is not scheduled too late (or not too early) in S with respect to the preferences of the voters (expressed as a order preferences or interval preferences). The other family of criteria is a distance criterion, which says that the closer the returned schedule is to the voter's preferences, the more satisfied a voter is.

We measure the quality of a compromise schedule S for all the voters by summing up the sum of the dissatisfaction of the voters for schedule S. This sum, divided by v, represents the average dissatisfaction of a voter with solution S. We focus on an utilitarian criterion: our aim is to compute a schedule with the smallest sum of dissatisfaction.

An assignment problem. Without additional constraints, this problem can be solved polynomially, both for Order and Interval Preferences, as it is an assignment problem. Indeed, the returned schedule being a permutation of the n tasks, we know that there will be n time slots, between 0 and n, one for each task. We create a complete bipartite graph with the tasks on the left and the time slots on the right. For each couple (task t, time slot s), the cost of the edge (t, s) is the sum of the dissatisfaction caused by task t to all the voters if t is scheduled at time slot s. Therefore, a schedule that minimizes the total dissatisfaction corresponds to a minimum cost matching in such a graph. The graph can be built in $O(vn^2)$, and a minimum cost matching can be found with Hungarian algorithm in $O(n^3)$ (Edmonds & Karp, 1972; Tomizawa, 1971), leading to a $O(vn^2 + n^3)$ algorithm.

Additional constraints. Our aim is to study this problem by adding the main constraints in scheduling: time constraints and precedence constraints. Time constraints mean that to each task is associated a release date and a due date (or deadline), and that in the returned schedule each task should be scheduled between its release date and its deadline. Precedence constraints mean that there is a precedence graph of the n tasks: if there is an edge from task i to task j in this graph, this means that in the returned schedule task i should be scheduled before task j. We will study both the case where these constraints are imposed, and the case where they are inferred from the preferences of the voters. Before presenting our results and the map of the paper, we review related works.

Related work. Our work is at the boundary between computational social choice (Brandt, Conitzer, Endriss, Lang, & Procaccia, 2016) and scheduling (Brucker, 1999), two major domains in operational research.

As mentioned above, the collective schedule problem generalizes the collective ranking problem, which is an active field in computational social choice (see e.g. (Asudeh, Jagadish, Stoyanovich, & Das, 2019; Biega, Gummadi, & Weikum, 2018; Celis, Straszak, & Vishnoi, 2018; Dwork, Kumar, Naor, & Sivakumar, 2001; Geyik, Ambler, & Kenthapadi, 2019; Narasimhan, Cotter, Gupta, & Wang, 2020; Singh & Joachims, 2018; Skowron, Lackner, Brill, Peters, & Elkind, 2017)). In this field, authors often design rules (i.e. algorithms) which return fair rankings, and they often focus on fairness in the beginning of the rankings. If the items (or candidates) to be ranked are recommendations (of restaurants, web pages, etc.) for users, the beginning of the ranking is indeed probably the most important part. Note that this does not hold for our problem since all the planned tasks will be executed – only their order matters. This means that rules designed for the collective ranking problem are not suitable not only because they do not consider duration for the items, but also because they focus on the beginning of the ranking. This also means that the rules that we will study can be relevant for consensus ranking problems where the whole ranking is of interest.

The collective schedule problem has been introduced in (Pascual et al., 2018) for Order Preferences and when tasks have different processing times (lengths). In this paper, the authors introduced a weighted variant of the Condorcet principle, called the PTA Condorcet principle (where PTA stands for "Processing Time Aware"), and they adapted previously known Condorcet consistent rules when tasks have different processing times. They also introduced a new rule, which computes a schedule which minimizes the sum of the tardiness of tasks between the preferred schedules of the voters and the schedule which is returned. They show that the optimization problem solved by this rule is NP-hard but that it can be solved for reasonable size instances with a linear program.

Durand and Pascual (2022) also study the collective schedule problem with Order Preferences and when the tasks have different lengths. They propose an axiomatic study of three rules: a generalization of the Kemeny rule, and the two rules which minimizes the sum of the deviations of the tasks between the preferred schedules and the returned schedule (rule ΣDev, optimal for the total deviation criterion), and the sum of the tardiness (rule $\Sigma\mathrm{T}$, optimal for the total tardiness criterion). They show that these rules solve NP-hard problems, but that it is possible to solve optimally these problems for reasonable size instances.

Multi agent scheduling problems also aim at returning consensus schedules, but they focus on cases where (usually two) agents own their *own* tasks, that are scheduled on shared machines: the aim is to find a Pareto-optimal and/or a fair schedule of the tasks of the agents, each agent being interested by her own tasks only (Agnetis, Billaut, Gawiejnowicz, Pacciarelli, & Soukhal, 2014; Saule & Trystram, 2009).

Last but not least, the work of Elkind, Kraiczy, and Teh (2022) focus on the assignment of shared unit size tasks to specific unit size time slots. This latter work focus on fairness notions extended from the multi-winner voting problem and does not use scheduling notions such as precedence constraints or time constraints.

Our results and map of the paper.

- We first start by introducing notations in Section 1, as well as formal definition of the binary and distance criteria studied in this paper. As we will see, these criteria generalize the other criteria studied before (total tardiness, and total deviation), and also allow us to model famous scheduling criteria, as the minimization of the total earliness of the tasks, or also the minimization of the number of late tasks. Rules that return optimal solutions of these criteria will be studied in the sequel.
- In Section 2, we focus on the algorithm which, in the Order Preference setting, computes the median start time of each task, and then schedules the tasks by increasing median start times (rule EMD – for Earliest Median Date). We show that, interestingly, this rule returns a schedule which is a 2-approximation of the total tardiness criterion.
- We then focus in Section 3 on time constraints: we show that it is still possible to get an optimal solution in polynomial time with time constraints on the tasks. We focus on the rules optimizing the binary and distance criteria (without time constraints), as well as the EMD rule, and we present an axiomatic study of these rules when time constraints are induced by the preferences of the voters (e.g. if all the voters schedule, in their preferred schedules, a task t at time X, is this task t necessarily started exactly at time X in the returned schedule ? If task t is always started after time X in the preferred schedules, is it always scheduled after time X in an optimal solution ?).
- In Section 4, we focus on precedence constrains between the tasks. We show that the previously studied rules, which could be run in polynomial time without precedence constraints, can still be used (with an additional polynomial time step) when the precedence constraints are inferred by the preference of the voters. On the contrary, we show that we have to solve NP-hard problem when the precedence constraints are not fulfilled by the preferred schedules of the voters. This is true both for the distance and the binary criterion, and in particular in the cases where we wish to minimize the total deviation, the total tardiness, or the number of late tasks.
- We conclude this paper in Section 5 by an overview of our results and a few research directions.

1 Preliminaries

1.1 Definitions and notations

Order Preferences and Interval Preferences.

We consider a set $J = \{1, \ldots, n\}$ of n tasks of interest for a set $V = \{1, \ldots, v\}$ of v voters. Each task has a processing time of 1. The preferences of voter i are denoted by $\mathcal{V}_i$, and depend of the setting used.

In the *Order Preferences* setting, each voter indicates its preferred schedule, as a permutation of the n tasks (we do not consider idle times between the tasks). Therefore, $\mathcal{V}_i$ is the preferred schedule of voter i. We denote by $C_j(\mathcal{V}_i)$ the completion time of task j in the preferred schedule of voter i. More generally, given a schedule S of tasks of J, we denote by $C_j(S)$ the completion time of task j in S.

In the *Interval Preferences* setting, each voter indicates for each task the interval in which he or she wishes to see the task scheduled. More precisely, for each task $j \in J$, voter $i \in V$ indicate a release date – that will be denoted by $r_j(\mathcal{V}_i)$ – , and which means that voter i would like task j to be started at the soonest at time $r_j(\mathcal{V}_i)$. Likewise, voter $i \in V$ indicates a due date (also called deadline) – that will be denoted by $d_j(\mathcal{V}_i)$ –, and which means that voter i would like task j to be completed at the latest at time $d_j(\mathcal{V}_i)$. Therefore, $\mathcal{V}_i$ is the set of the n couples (release date, due date) that voter i sets for the n tasks. Note that this setting generalizes the Order Preferences settings, since it is possible for a voter to set for each task a release date (resp. a due date) equal to its start (resp. its completion time) in its preferred schedule. The only constraint we impose is that there exists a feasible schedule that fulfills the time constraint given by a voter (i.e. in which each task j is scheduled in the interval $[r_j(\mathcal{V}_i), d_j(\mathcal{V}_i)]$).

In the sequel, we will penalize schedules in which tasks are scheduled out of the intervals given by the voters. The Interval Preferences setting allows voters to express pretty precise preferences. Indeed, if a voter wants a task to be done before a given date t, and has no preference on the starting date of a task then she can indicate a release date of 0 and a due date of t. If her only wish is that a task starts after a given time t', then she can indicate a release date of t' and a due date of n. Finally, if a voter wants a task to start exactly at time t'', then she can give a release date of t'' and a due date of $t'' + 1$. This flexibility in the preferences allow voters to express situations in which they have different expectations regarding the task, from having no interest in a task to wanting it to be completed exactly at a given time.

Let us now present the two general objective functions that we will consider in this paper: the binary criterion, and the distance criterion.

Binary criterion.

The first criterion, called *Binary Criterion*, measures whether a task is executed in the time interval indicated by a voter or not (the penalty is 0 if the tasks is scheduled in the desired interval, and is 1 if the task is not scheduled in the desired interval). The dissatisfaction of voter $i \in V$ concerning task $j \in J$ is thus:

$$b_j(S, \mathcal{V}_i) = \begin{cases} 1 & \text{if } C_j(S) > d_j(\mathcal{V}_i) \text{ or } C_j(S) \leq r_j(\mathcal{V}_i) \\ 0 & \text{otherwise} \end{cases}$$

The dissatisfaction of a voter i concerning a schedule S with the binary criterion is then:

$$B(S, \mathcal{V}_i) = \sum_{j \in J} b_j(S, \mathcal{V}_i)$$

Distance criterion.

The second criterion, called *Distance Criterion*, also does not count any penalty when a task is scheduled in its time interval, but otherwise it counts a penalty which expresses how far from its interval the task is. The dissatisfaction of voter i concerning task j in schedule S is:

$$d_j(S, \mathcal{V}_i) = \begin{cases} C_j(S) - d_j(\mathcal{V}_i) & \text{if } C_j(S) > d_j(\mathcal{V}_i) \\ r_j(\mathcal{V}_i) - (C_j(S) - 1) & \text{if } C_j(S) \leq r_j(\mathcal{V}_i) \\ 0 & \text{otherwise} \end{cases}$$

The dissatisfaction of a voter i concerning a schedule S with the distance criterion is then:

$$D(S, \mathcal{V}_i) = \sum_{j \in J} d_j(S, \mathcal{V}_i)$$

Aggregation function.
As said in the introduction, we will study the utilitarian utility function. Our aim will be to minimize $\Sigma_{i\in\{1,\dots,v\}} B(S, \mathcal{V}_i)$ with the binary criterion, or $\Sigma_{i\in\{1,\dots,v\}} D(S, \mathcal{V}_i)$ with the distance criterion. The *Binary Criterion rule* is an algorithm that returns a schedule minimizing binary criterion, while the *Distance Criterion rule* is an algorithm that returns a schedule minimizing distance criterion.

1.2 Generalization of classical scheduling criteria.

The two above defined criteria generalize the main criteria already studied in the Order Preferences setting (Durand & Pascual, 2022; Pascual et al., 2018). Assume indeed that voters have expressed their preferences using the Order Preferences setting (i.e. each voter indicates his or her preferred schedule). Let P be the preference profile, i.e. the set of all the preferences expressed by the voters.

- *Total deviation.* The total deviation of a schedule S is $Dev(S, P) = \Sigma_{i\in\{1,\dots,v\}} Dev(S, \mathcal{V}_i)$, where $Dev(S, \mathcal{V}_i) = \sum_{j\in J} |C_j(S) - C_j(\mathcal{V}_i)|$. If our aim is to compute a schedule of minimal total deviation, as does rule ΣDev (Durand & Pascual, 2022; Pascual et al., 2018), then we should use the Distance Criterion by setting the release date of task j for voter i at $C_j(\mathcal{V}_i) - 1$, and the due date of task j for voter i at $C_j(\mathcal{V}_i)$.
- *Total tardiness.* The total tardiness of a schedule S is $T(S, P) = \Sigma_{i\in\{1,\dots,v\}} T(S, \mathcal{V}_i)$, where $T(S, \mathcal{V}_i) = \sum_{j\in J} \max(0, C_j(S) - C_j(\mathcal{V}_i))$. If our aim is to compute a schedule of minimal total tardiness, as does rule ΣT (Durand & Pascual, 2022; Pascual et al., 2018), then we should use the Distance Criterion by setting the release date of task j for voter i at 0, and the due date of task j for voter i at $C_j(\mathcal{V}_i)$.
- *Total earliness.* The total earliness of a schedule S is $E(S, P) = \Sigma_{i\in\{1,\dots,v\}} E(S, \mathcal{V}_i)$, where $E(S, \mathcal{V}_i) = \sum_{j\in J} \max(0, C_j(\mathcal{V}_i) - C_j(S))$. If our aim is to minimize the total earliness, a classic criterion in scheduling (Brucker, 1999), then we should use the Distance Criterion by setting the release date of task j for voter i at $C_j(\mathcal{V}_i) - 1$, and the due date of task j for voter i at n.
- *Total number of late tasks.* The total number of late tasks of a schedule S is $U(S, P) = \Sigma_{i\in\{1,\dots,v\}} U(S, \mathcal{V}_i)$, where $U(S, \mathcal{V}_i)$ is the number of tasks of J such that $C_j(S) > C_j(\mathcal{V}_i)$ (such tasks are called *late* tasks). This a classic criterion, denoted

by ΣU (for "Unit Penalties"), in scheduling (Brucker, 1999). We can solve this optimization problem by using the Binary Criterion by setting the release date of task j for voter i at 0, and the due date of task j for voter i at $C_j(\mathcal{V}_i)$.

- *Total number of tasks not well positioned.* If our aim is to maximize the number of tasks scheduled at the exact position given by the voters, then we should use the Binary Criterion by setting the release date of task j for voter i at $C_j(\mathcal{V}_i) - 1$, and the due date of task j for voter i at $C_j(\mathcal{V}_i)$.

In the next section, we introduce the EMD rule and show that it is a 2 approximation of the total tardiness (ΣT) and total earliness (ΣE) criteria.

2 An analysis of the EMD rule

The EMD rule, introduced as a heuristic by Durand and Pascual (2022) in the Order Preferences setting, schedules the tasks by increasing median completion times, where the median time of a task j is the median of the set $\{C_j(\mathcal{V}_1), \ldots, C_j(\mathcal{V}_v)\}$. If several tasks have the same median completion time, any tie breaker mechanism can be used.

It was shown previously (Pascual et al., 2018) that, for unit size tasks, and for any preference profile P and any schedule S, we have $Dev(S, P) = 2T(S, P)$, and thus that $T(S, P) = E(S, P)$ since $Dev(S, P) = E(S, P) + T(S, P)$. Therefore, a α-approximate algorithm[1] for the total tardiness criterion will also be an α-approximate algorithm for the earliness criteria, and an α-approximate algorithm for the total deviation criterion.

We consider that we are in the Order Preferences setting. Before showing that EMD is 2-approximate for the total tardiness criterion (and thus also for the deviation and earliness criterion), we introduce a way to see the instance that will facilitate the analysis.

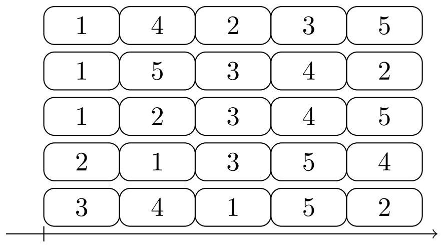
Breaking down the preference profile. We "break down" the preference profile not by looking at voters individually, but by looking at time slots. Note that this does not change the preference profile: it is just another way of looking at the preference profile. For each time slot between 1 and n, each voter i selected a task that she has scheduled in this time slot in her preferred schedule. We call *choice* a triplet $(\mathcal{V}_x, j, t)$ indicating that voter x schedules task j between time $t - 1$ and t in her preference $\mathcal{V}_x$. We can thus express a preference profile as a set of choices, such that there are v choices for each time slot and there are n choices for each voter, each task and each slot being chosen exactly once by each voter. We denote by $\mathcal{C}$ the set of all the choices and, for each $y \in \{1 \ldots n\}$, we denote by $\mathcal{C}_y$ the set of all choices $(\mathcal{V}_x, j, t)$ such that $t \leq y$.

Iterative breakdown of the tardiness criterion. As we have seen, the total tardiness of a schedule S given a preference profile is the sum, over all voters, of the tardiness of each task in S in comparison to its completion time in the preference of the voter. By breaking down the set of preferences into choices, it is possible to express the tardiness in another way, that will facilitate the analysis of the algorithm.

[1] An α-approximate algorithm, for a minimization problem (as are our criterion), returns, for each instance I, a solution (schedule) of cost at most $\alpha OPT(I)$, where $OPT(I)$ is the minimal cost that can be obtained on a solution of instance I.

If a task j has been scheduled by a voter i at time slot t, then, if it is not scheduled in a solution S by time t then we count a penalty; if it is not scheduled by time $t+1$, then we count another penalty; and so forth. We can then split the tardiness by looking at the tasks scheduled by S at each time slot: for each slot between $C_j(\mathcal{V}_i)$ to n, if task j has not been scheduled yet, then we count 1 tardiness penalty (for voter i). We sum this over all the voters. By this way, we compute for each slot t the number of penalties caused by the decision taken in t – there will be 1 penalty of each couple (voter i, task j) if task i has not been completed at time t whereas $C_j(\mathcal{V}_i) \leq t$.

Example 1. *Let us consider an instance with 5 voters and 5 tasks as follows. Each line represents the preferred schedule of a voter – e.g. the preferred schedule of the first voter is made of task 1, then task 4, followed by task 2, then task 3 and finally task 5 (such a schedule can be written as:* $1 \prec 4 \prec 2 \prec 3 \prec 5$*):*



Looking at time slot 1 *(between dates* 0 *and* 1*), task* 1 *has been scheduled three times, task* 2 *once and task* 3 *once. In a solution* S*, scheduling task* 1 *at slot* 1 *causes a (total) tardiness of* 2 *since task* 2 *and* 3 *which were chosen by two voters will not be scheduled on time. Scheduling task* 2 *or task* 3 *causes a tardiness of* 4*, and scheduling task* 4 *or task* 5 *creates a tardiness of* 5*.*

In the proof of the following proposition, to compute the sum of the tardiness (also called the total tardiness) of a schedule S, we will look a time slots, starting from the first one, between dates 0 and 1, to the last one, between dates $n-1$ and n. When looking at time slot y, for each choice $(\mathcal{V}_x, j, t) \in \mathcal{C}_y$, we will count a penalty if task j has not been scheduled at time slot y or before. We denote by k_y the number of late tasks at time slot y: $k_y(S,P) = \sum_{(\mathcal{V}_x,j,t)\in\mathcal{C}_y} \mathbb{1}_{C_j(S)>y}$. The total tardiness can be expressed as follows: $T(S,P) = \sum_{y=1}^{n} k_y(S,P)$.

Proposition 1. *The EMD rule is* 2*-approximate for the total tardiness criterion.*

Proof. Let us consider a preference profile P. Let S be the schedule returned by the EMD rule and let S^* be a schedule minimizing the total tardiness with respect to preference profile P. We prove this result by showing that for all $i \geq 0$, $k_i(S,P) \leq 2k_i(S^*,P)$.

For $i = 0$, we have $k_i(S,P) = k_i(S^*,P) = 0$, since no task is scheduled before the first time slot. For any time slot from 1 to n, we express $k_i(S,P)$ as the difference between $i \times v$, the total number of choices from time slot 1 to time slot i, and the number of choices $(\mathcal{V}_x, j, t)$ such that $t \leq i$ and j has been scheduled at the latest at time slot i in S. We denote by q_i the number of tasks with median completion time smaller than or equal to i. There are two cases:

1. $q_i \leq i$: in this case, the EMD rule schedules the q_i tasks with median completion time smaller than or equal to i in the i first time slots. Let q_i^* be the number of tasks with median completion time smaller than or equal to i that are scheduled in S^* at the latest at date i. These q_i^* tasks are necessarily scheduled before date i by the EMD rule as well. Let $\mathcal{Q}_i^*$ be the set of the q_i^* tasks of median completion time smaller than or equal to i and that are scheduled in S^* before or at time i.

   Finally, we denote by Q_i^* the number of choices $(\mathcal{V}_x, j, y)$ of $\mathcal{C}_i$ such that $j \in \mathcal{Q}_i^*$. These choices are removed from the $i \times v$ choices for both the solutions S and S^*. There are $q_i - q_i^*$ tasks of median completion time smaller than or equal to i that are scheduled in S before or at time i and that are scheduled after i in S^*. For each of these tasks, there are at least $v/2$ choices among the $i \times v$ which are removed by scheduling the task before date i. There are also $(i - q_i)$ tasks with median completion time strictly larger than i that are scheduled at the latest at date i in S, but we have no guarantee that scheduling these tasks remove any choice. We therefore have:
$$k_i(S, P) \leq i \times v - Q_i^* - (q_i - q_i^*)v/2$$
   In S^*, there are $(i - q_i^*)$ tasks of median strictly larger than i, at most, scheduling these tasks before or at time i removes $v/2$ choices. We then have:
$$k_i(S^*, P) \geq i \times v - Q_i^* - (i - q_i^*)v/2$$
   We then compute:
$$2k_i(S^*, P) - k_i(S, P) \geq 2i \times v - 2Q_i^* - (i - q_i^*)v - i \times v + Q_i^* + (q_i - q_i^*)v/2$$
$$2k_i(S^*, P) - k_i(S, P) \geq q_i^* v - Q_i^* + (q_i - q_i^*)v/2$$
   We know that $Q_i^* \leq q_i^* v$ since each task in $\mathcal{Q}_i^*$ is scheduled at most v times in the preference profile, once per voter. We also know that $q_i \geq q_i^*$. We then have:
$$2k_i(S^*, P) - k_i(S, P) \geq 0$$

2. $q_i > i$: in this case, the EMD rule schedules, from dates 0 to i, exactly i tasks of median completion time smaller than or equal to i. There remains $(q_i - i)$ tasks of median completion time smaller than or equal to i that are not scheduled by date i in S, the schedule returned by the EMD rule. Each of these tasks can appear in at most v choices in $\mathcal{C}_i$.

   Let $r_i \geq 0$ be the number of tasks with median completion time strictly larger than i that are not scheduled by date i in S. Let $\mathcal{R}_i$ the set of these r_i tasks, and let R_i the set of choices $(\mathcal{V}_x, j, t)$ in $\mathcal{C}_i$ such that $j \in \mathcal{R}_i$. We have:
$$k_i(S, P) \leq (q_i - i)v + |R_i|$$

In S^*, the i tasks scheduled by time slot i are split between the q_i tasks of median completion time smaller than or equal to i and the r_i tasks of median completion time strictly larger than i. We denote by $\mathcal{R}_i^*$ the tasks of $\mathcal{R}_i$ scheduled by S^* before or at time i. We call $r_i^* = |\mathcal{R}_i^*|$, and R_i^* the set of choices $(\mathcal{V}_x, j, t)$ of $\mathcal{C}_i$ such that $j \in \mathcal{R}_i^*$. There are $(q_i - (i - r_i^*))$ tasks of median completion time smaller than or equal to i that are not scheduled by S^* by time i. Each of these tasks is at least in $v/2$ choices in $\mathcal{C}_i$. We can then write:

$$k_i(S^*, P) \geq (q_i - i + r_i^*)v/2 + |R_i| - |R_i^*|$$

We then have:

$$2k_i(S^*, P) - k_i(S, P) \geq (q_i - i + r_i^*)v + 2|R_i| - 2|R_i|^* - (q_i - i)v - |R_i|$$

$$2k_i(S^*, P)^* - k_i(S, P) \geq r_i^* \times v + |R_i| - 2|R_i^*|$$

Since $|R_i| \geq |R_i^*|$ and $r_i^* \times v > |R_i^*|$, we have: $2k_i(S^*, P) - k_i(S, P) \geq 0$.

In both cases, we have $k_i(S, P) \leq 2k_i(S^*, P)$ for all $i \geq 1$. Therefore, we have: $\sum_{i=1}^{n} k_i(S, P) \leq 2\sum_{i=1}^{n} k_i(S^*, P)$ and then $\Sigma T(S, P) \leq 2\Sigma T(S^*, P)$. □

As seen above, since, with unitary tasks, $T(S, P) = E(S, P)$ and $Dev(S, P) = 2T(S, P)$, for any schedule S and preference profile P, we get the following corollary.
Corollary 1. *The EMD rule is 2-approximate for the ΣDev criterion, and for the ΣE criterion.*

Additional results in voting theory

As noted by Pascual et al. (2018), the ΣDev criterion is equivalent to the Spearman ranking correlation coefficient when task are of unit size, this gives us the following corrolary. Although this minimization problem is polynomially solvable, it is still an interesting property to have for the EMD rule.
Corollary 2. *The EMD rule is 2-approximate for the minimization of the total Spearman correlation coefficient to the preference profile.*

We can show that EMD is 4-approximate for the well-studied Kemeny rule Kemeny (1959). The Kemeny rule returns a ranking R^* that minimizes the Kendall-Tau distance $\delta(R, P)$ of a ranking R to the preference profile P where this distance is defined as the number of ordered pairs on which the ranking disagrees with the voters in P.
Proposition 2. *The EMD rule is 4-approximate for the minimization of the Kendall-Tau distance to the preference profile.*

Proof. Diaconis and Graham (1977) showed that for any ranking R and any preference profile P, the Spearman correlation coefficient ρ fulfills the following property: $\delta(R, P) \leq \rho(R, P) \leq 2\delta(R, P)$. We call S the solution returned by EMD, S_{KT}^* a solution minimizing the Kendall-Tau distance to the preference profile and S^* a solution minimizing the total Spearman correlation coefficient with the preference profile.

For the sake of contradiction, let us assume that:

$$\delta(S, P) > 4\delta(S_{KT}^*, P)$$

We then have

$$\rho(S,P) \geq \delta(S,P) > 4\delta(S^*_{KT},P) \geq 2\rho(S^*_{KT},P)$$

And since S^* is optimal for the Spearman, rule we have:

$$\rho(S,P) \geq \delta(S,P) > 4\delta(S^*_{KT},P) \geq 2\rho(S^*_{KT},P) \geq 2\rho(S^*,P)$$

A contradiction, given the result from Proposition 1. We therefore have:

$$\delta(S,P) \leq 4\delta(S^*_{KT},P)$$

□

In the next section, we focus on release time and due dates constraints.

3 Scheduling tasks with time constraints

We first show that it is still possible to compute in polynomial time an optimal solution of the total dissatisfaction of the voters with both the Binary criterion and the Distance criterion presented in Section 1.

3.1 Getting optimal solutions with time constraints

Let us consider that each task $j \in J$ has a release date r_j and a due date d_j. These dates can be imposed, for example when the tasks represent events that cannot occur before a date r_j or after a date d_j. They can also be inferred from the preferences of the voters (by setting $r_j = \min_{i\in\{1,\ldots,v\}}\{r_j(\mathcal{V}_i)\}$ and $d_j = \max_{i\in\{1,\ldots,v\}}\{d_j(\mathcal{V}_i)\}$). In this case, we want no task to be scheduled earlier than in the preferred interval of any voter, or later than in the preferred interval of any voter. This case is particularly interesting if voters are aware of real time constraints on the events that are represented by the tasks, and if the scheduler does not necessarily know these constraints.

Returning an optimal schedule for both the Binary criterion and the Distance criterion is, as without any time constraints, an assignment problem. In the bipartite graph with the tasks on the left and the time slots on the right, for each couple (task j, time slot s), we just put an edge between j and s if and only if $r_j \leq s \leq d_j - 1$. The costs of the edges are equal to the sum of the dissatisfaction of the v voters if task j is scheduled between $s-1$ and s. An optimal solution which fulfills time constraints – if there is one feasible solution – is a minimum cost matching. Such a matching, if it exists, can be found with Hungarian algorithm in $O(n^3)$ (Edmonds & Karp, 1972; Tomizawa, 1971).

In the next section, we study to which extent the rules presented earlier propagate constraints fulfilled by the preferences of the voters. For example, if all the voters schedule a task after a given time, it may be because this task is not available before. This is particularly interesting in contexts in which the preferences given are not necessarily votes but feasible solutions for a problem (potentially optimizing different aspects like cost, employee satisfaction, inventory management . . .). In this case, the question becomes: given several feasible solutions satisfying a set of constraints, do the aggregation rule ensures that the returned solution fulfills the same constraints?

3.2 Axiomatic study of rules with inferred time constraints

3.2.1 Release dates and deadlines consistencies.

The idea of the two following axioms is the following one: if a task j consistently starts after (resp. ends before) a given date t, we can interpret it as t being a firm release date (resp. deadline) for task j. In this case, we would like the rule to return a solution in which j starts after (resp. ends before) t.

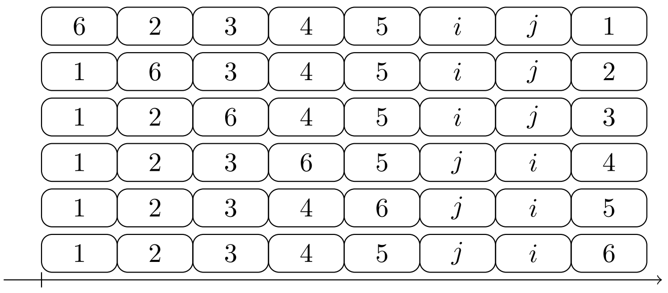
Definition 1. *Let V be a set of voters and j a task such that for each preference $\mathcal{V}_i$ expressed by voter $i \in \{1, \ldots, v\}$, we have $C_j(\mathcal{V}_i) \geq t$, with t a constant. An aggregation rule fulfills* release date consistency *if it always returns a schedule S in which $C_j(S) \geq t$.*

Definition 2. *Let V be a set of voters and j a task such that for each preference $\mathcal{V}_i$ expressed by voter $i \in \{1, \ldots, v\}$, we have $C_j(\mathcal{V}_i) \leq t$, with t a constant. An aggregation rule fulfills* deadline consistency *if it always returns a schedule S in which $C_j(S) \leq t$.*

We show that the Distance Criterion rule, the Binary Criterion rule, and the EMD rule do not fulfill these axioms.

Proposition 3. *The Distance Criterion rule does not fulfill the deadline consistency nor the release date consistency, even when preferences are expressed as schedules.*

Proof. Let us consider an instance with 8 tasks and 6 voters and the following preferences:

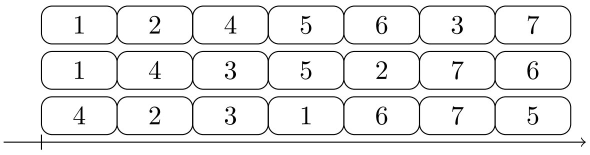


The schedules $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec i \prec j)$ and $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec j \prec i)$ are the only two optimal schedules, with a total distance of 54. They do not fulfill release date consistency since all the voters have completed tasks i and j at time 7 in their preferred schedules, whereas in the returned solution, one of these two tasks is completed at time 8. The best solutions fulfilling release date consistency are $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec j \prec i \prec 6)$, and $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec i \prec j \prec 6)$, with a total distance of 56. Therefore, the Distance Criterion rule does not fulfill deadline consistency.

By reversing the preferences (e.g. , when a preferred schedule $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec j \prec i \prec 6)$ becomes $(6 \prec i \prec j \prec 5 \prec 4 \prec 3 \prec 2 \prec 1)$), we obtain an instance in which tasks i and j always start after or at 1, but in which the optimal solutions are $(i \prec j \prec 6 \prec 5 \prec 4 \prec 3 \prec 2 \prec 1)$ and $(j \prec i \prec 6 \prec 5 \prec 4 \prec 3 \prec 2 \prec 1)$ (the previous optimal solutions but reversed). Either task i or j starts at time 0 in these solutions, whereas no voter schedule theses tasks before time 1. Therefore, the Distance Criterion rule does not fulfill release date consistency. □

Proposition 4. *The Binary Criterion rule does not fulfill the deadline consistency nor the release date consistency.*

Proof. Let us consider the following preferences of 3 voters over 7 tasks:



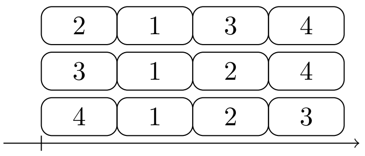
We first consider that the deadlines are the one given in the above schedules, and that the release dates are 0. The binary criterion this corresponds to the ΣU criterion (which minimizes the number of late tasks).

The only optimal solution for ΣU is $(1 \prec 2 \prec 3 \prec 5 \prec 6 \prec 7 \prec 4)$. The number of late tasks in this solution is 3, task 4 being considered late by the three voters. In a solution fulfilling the deadline consistency property, task 4 has to be completed at most at time 3. This implies that either task 1, task 2 or task 3 has to end after time 3 and will therefore be late for two voters. Additionally, if task 1 is delayed, because of deadline consistency, it has to end at time 4, meaning that task 5 has to be delayed and will therefore be considered late for 2 voters, which amounts to 4 late tasks, more than the optimum. The same line of reasoning can be applied for tasks 2 and 3 if they are delayed after time 3, causing delay to task 5, 6 or 7 if they are scheduled at time 4, 5 or 6. Any solution respecting the deadline consistency property has therefore a number of late tasks of at least 4: the ΣU rule does not fulfill deadline consistency.

We can show similarly that the Binary Criterion rule does not fulfill the release date consistency. To this end, we consider that the the deadlines are n, and that the release dates are the one given in the above schedules, once they have been reversed. The binary criterion in this case minimizes the number of early tasks). □

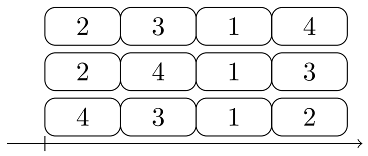
Proposition 5. *The EMD rule does not fulfill deadline consistency nor release date consistency.*

Proof. Let us consider the following preferences of 3 voters over 4 tasks:



With such preferences, the median completion times are as follows: $m_1 = 2$, $m_2 = m_3 = 3$ et $m_4 = 4$. The EMD rule returns a schedule in which task 1 is scheduled first and therefore completes at time 1, which is before its completion time in all the preferences of the voters. Therefore, the EMD rule does not fulfill release date consistency.

Let us now consider the following preferences of 3 voters over 4 tasks:



With such preferences, the median completion times are as follows: $m_1 = 3$, $m_2 = 1$, $m_3 = 2$ et $m_4 = 2$. The EMD rule returns a schedule in which task 1 is scheduled last and therefore completes at time 4, which is after its completion time in all the preferences of the voters: the EMD rule does not fulfill deadline consistency. ☐

Since our three rules do no fulfill release date nor deadline consistency, we propose a weaker, yet meaningful, property called *temporal unanimity.*

3.2.2 Temporal unanimity

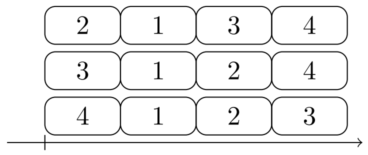
An aggregation rule satisfies temporal unanimity if, when all voters agree on the time interval during which a task i is scheduled, then i is scheduled during this time interval in the solution returned by the rule. When preferences are given as schedules, this property means that if all voters schedule task i at the same time slot in their preferred schedules, then i should be scheduled at the same time slot in the returned solution. When preferences are expressed as time intervals, it means that if all voters agree on the same release date and deadline for i, then i should be scheduled in this given interval in the returned solution.

Definition 3. *Let V be a set of voters, and let j be a task such that for each voter i in V, we have $d_j(\mathcal{V}_i) = d$, with d a constant, and $r_j(\mathcal{V}_i) = r$, with r a constant strictly smaller than d. An aggregation rule fulfills* temporal unanimity *if it returns a schedule in which task j is executed between r and d.*

We show that EMD does not fulfill this property, whereas the Binary and the Distance Criterion rules do fulfill this axiom.

Proposition 6. *The EMD rule does not fulfill the temporal unanimity property.*

Proof. Let us consider the following instance:



The median completion times are as follows: $m_1 = 2$, $m_2 = m_3 = 3$ and $m_4 = 4$. The EMD rule returns a schedule in which task 1 is scheduled first and thus completes at time 1 even though it completed at time 2 in all the schedules expressed by the voters. $\square$

Proposition 7. *The Binary Criterion rule fulfills temporal unanimity.*

Proof. Let us consider a task $s(1)$ such that, for all voter i, $d_{s(1)}(\mathcal{V}_i) = d$ and $r_{s(1)}(\mathcal{V}_i) = r$ with r and d two constants such that $0 \leq r < d \leq n$. Let us now consider an optimal schedule S^* for the Binary Criterion minimization in which task $s(1)$ is not scheduled between r and d. Since each of the preferences has to be compatible with a feasible schedule, in each preferences there are at most $d - r$ tasks with release dates and deadlines included in the $[r, d]$ interval. Since task $s(1)$ is always included in this interval in the preferences of the voters, there is in the $[r, d]$ interval of S^* at least one task $s(2)$ is scheduled in the preferences of the voters at least once before r or after d. We distinguish two sub-cases.

- If this task $s(2)$ does not have a unique release date r' given by the voters and a unique due date d' given by the voters, then we can simply perform the swap between the positions of $s(1)$ and $s(2)$ to obtain a new solution S' in which there is one less task out of its unique time interval and which is at least as good as S^* since we decrease the binary criterion cost for $s(1)$ by the number of voters v and we increase it for $s(2)$ by at most v.
- If this task $s(2)$ has a unique release date r' and due date d' then we consider two subcases:
  - If task $s(2)$ is scheduled in S^* before r' or after d', and therefore not in its unique time interval, or if its time interval covers the position of $s(1)$ in S^*, we can perform the swap between the positions of $s(1)$ and $s(2)$ as in the above mentioned case.
  - Otherwise, we consider the other tasks, if any, scheduled in S^* between r and d and which are not always scheduled between r and d in the preferences. If none of these tasks fulfill any of the two previous conditions then we consider the set $\mathcal{T}_{s(1)}$ of all these tasks scheduled between r and d in S^* and which have a unique time interval in which they are scheduled. For each of these tasks, its time interval r', d' is either as $r' < r$ or $d' > d$, or both. We now consider the interval from the smallest unique release date of a task in $\mathcal{T}_{s(1)}$ to the maximum unique deadline of a task in $\mathcal{T}_{s(1)}$. We then repeat the same reasoning as above:
    - ∗ if there is a task $s(3)$ which is in the time interval of a task $s(2)$ from $\mathcal{T}_{s(1)}$, and which does not have a unique time interval or which has a time interval containing the position of $s(1)$ in S^*, then we perform the following circular exchange: task $s(1)$ takes the time slot of $s(2)$, which takes the time slot of $s(3)$, which takes the time slot of $s(1)$. Such a circular exchange does not increase the binary criterion, since the cost relative to $s(1)$ is decreased by v, the cost relative to $s(2)$ does not increase, since $s(2)$ stays in its interval, and the cost of $s(3)$ increases by at most v.

* If no such task $s(3)$ exists, then there is at least one task which has a unique release date $r'' < r'$, or a unique deadline $d'' > d'$, or both. We then consider the set $\mathcal{T}_{s(2)}$ of such tasks and expand the considered interval. Since at each of these steps we extend the considered interval by at least one unit of time, the interval will necessarily include the position of $s(1)$ at some point and we will be able to perform a swap.

□

Proposition 8. *The Distance Criterion rule fulfills temporal unanimity when preferences are schedules.*

Proof. Let us consider that a task l is always scheduled between time k and time $k+1$ in the preferences of the voters. Let S^* be an optimal solution for the Distance Criterion minimization, and let us assume, by contradiction, that task l is not scheduled between k and $k+1$ in S^*. Let S be a schedule obtained from S^* by swapping the positions of task l and the task j scheduled between k and $k+1$ in S^*. Note that the distance of any task other than l or j is the same in S and S^*. The distance of task l is decreased by the absolute value of the difference between its position in S and its position in S^*, times the number of voters (since all voters scheduled it between k and $k+1$), while the distance of task j is increased by at most this value. Therefore S is an optimal schedule as well.

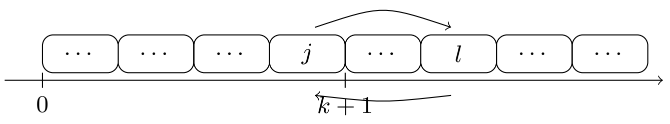


Fig. 1 Schedule S^* and the swap performed to obtain S.

Let us now examine the case in which the distance of task j has increased by $|C_l(S^*) - C_j(S^*)|$ – we will actually show that this cannot happen. Note that if task j was scheduled before task l in S^* (case 1) then the distance of j increased by $C_l(S^*) - C_j(S^*)$: it means that j has been scheduled before its completion time in S^* by all voters. Likewise, if task j was scheduled after task l in S^* (case 2) then the distance of j increased by $C_j(S^*) - C_l(S^*)$: it means that j has been scheduled after its completion time in S^* by all the voters.

In case 1, let b be the maximum completion time of task j in the preference profile, and let k be the task which is completed at time b in S^*. We build schedule S' from S by swapping the position of task j and task k. The distance of j is decreased by the difference between the position of j and k for all voters. If the distance of k increases by the same value it means that task k always completes before b in the preferences of the voters. By repeating such swaps, the date b is decreased each time and we will necessarily reach a point where we either find a task for which the distance increase is smaller than the distance decrease when doing the swap or find a b of 1.

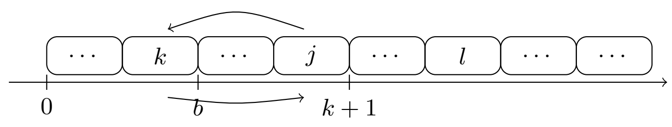


Fig. 2 Schedule S^* and a preliminary swap (case 1) ensuring that the final swap of task l will strictly decrease the total distance.

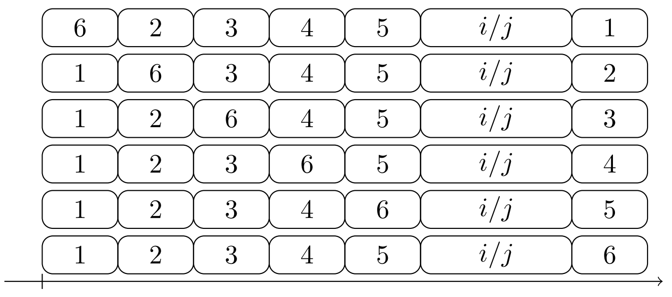
The same thing can be done in case 2, by defining b as the minimum completion time of j in the preference profile, and k as the task which is completed at time b in S^*. The distance of j is decreased by the difference between the position of j and k for all voters. If the distance of k increases by the same value than the distance of j is decreased, it means that task k always completes after b in the preferences of the voters. By repeating such swaps, the date b is increased each time and we will necessarily reach a point where we either find a task for which the distance increase is smaller than the distance decrease when doing the swap or find a b of n.

If we did not find $b = 1$, or $b = n$, it means that we have found a schedule of cost (sum of the distances) better than S^*, a contradiction. If we ended with $b = 1$ or $b = n$, then a task that would always complete before (resp. after) or at time $b = 1$ (resp. $b = n$) would always be scheduled first (resp. last), and doing the swap will always be strictly better. The solution obtained after doing the swaps is strictly better than the solution S^* supposed to be optimal, a contradiction.

□

Proposition 9. *The Distance Criterion rule does not fulfill temporal unanimity when preferences are expressed as release dates and deadlines.*

Proof. Let us consider an instance with 8 tasks and 6 voters and the following preferences:



In this profile each voter gives a time interval of 1 for all tasks except for i and j which have a time interval of 2. For example the first voter indicates that task 6 has a release date of 0 and a due date of 1, while both tasks i and j have a release date of 5 and a due date of 7. The two optimal solutions for the Distance Criterion minimization are $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec i \prec j)$ and $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec j \prec i)$, with a total distance of 48. These optimal schedule do not fulfill temporal unanimity since either i or j is scheduled outside of the time interval agreed on by all the voters. The best solutions fulfilling temporal unanimity are $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec j \prec i \prec 6)$, and $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec i \prec j \prec 6)$, with a total distance of 50. □

4 Precedence constraints

In this section, we focus on precedence constraints between the tasks. We will consider two settings.

Firstly, we consider a setting in which the precedence constraints are known by the voters. In this setting, that we call *inferred precedences*, if a task a has to be scheduled before a task b, then, in the preference $\mathcal{V}_x$ of any voter x, we have $C_a(\mathcal{V}_x) < C_b(\mathcal{V}_x)$. Our aim is to determine whether or not a given aggregation rule guarantees that task a will be scheduled before task b in the schedule returned by the rule. Note that in voting theory this property is called *unanimity*.

The second setting corresponds to the case in which the precedence constraints are not known by the voters, and therefore preferences do not necessarily fulfill these precedence constraints: the precedence constraints only exist for the schedule that has to be returned. This setting is called *precedence graph*.

We define a family of optimization problems of form $\alpha - Prec$ where α is an optimization criterion and $Prec$ is a setting for the precedence constraints: it is INFERRED when precedence constraints are fulfilled by the preferences, or GRAPH when the preference constraints only apply to the returned solution. For example, the problem ΣDev-GRAPH has the following input: a set J of n tasks ; an acyclic directed graph G which represents the precedence constraints between the tasks in J ; a set V of v preferred schedules (permutation of tasks) – these schedules do not necessarily fulfill the precedence constraints. The aim is to output a schedule which fulfills the precedence constraints and, among these feasible schedules, which minimizes the sum of the deviations with respect to the preferences of the voters: $\sum_{i \in V} \sum_{j \in J} D_j(S, \mathcal{V}_i)$.

Our aim is to study the complexity of problems mentioned before (total deviation, total tardiness, number of late tasks), when there are inferred or given precedence constraints. In Section 4.1, we study the case in which precedence constraints are inferred by the preferences of the voters, and we show that problems ΣDev-INFERRED and ΣT-INFERRED can be solved in polynomial time. In Section 4.2, we study the case in which precedence constraints are given and are not necessarily fulfilled by the preferred schedules of the voters. We will show that problems ΣDev-GRAPH, ΣT-GRAPH and ΣU-GRAPH are NP-hard.

4.1 Inferred precedence constraints

Proposition 10. *Problems ΣDev-INFERRED and ΣT-INFERRED can be solved in $O(vn^2 + n^3)$.*

Proof. Durand and Pascual (2022) showed that when two tasks a and b are of same length, if task a is scheduled before b in all the preferences then there exists an optimal solution for the ΣDev rule and the ΣT rule such that a is scheduled before b. Additionally, for any optimal solution in which b would be scheduled before a, it is possible to swap the position of a and b without increasing the deviation (or the tardiness). Therefore by doing successive permutations from an optimal solution, we can find another optimal solution in which precedence constraints are fulfilled. We now show that the number of permutations needed is bounded by n^2. Problems ΣDev-INFERRED and ΣT-INFERRED can thus be solved in polynomial time by

1. computing an optimal solution of ΣDev or ΣT (without the precedence constraints) through an assignment problem, as seen in the introduction. This can been done in $O(vn^2 + n^3)$;
2. swapping couple of tasks (a, b) that do not fulfill precedence constraints in the returned schedule of Step 1. As we will show now, there will be at most n^2 swaps.

We create a precedence directed graph with n vertices, one for each task, and in which there is an edge from vertex a to vertex b if the task corresponding to vertex a is always scheduled before the task corresponding to vertex b in the preferences of the voters. There are at most n^2 edges so it is possible to create this graph in $O(n^2)$ operations. Note that this precedence relation is transitive: if a is always scheduled before b and b is always scheduled before c then a is always scheduled before c. This implies that this graph has no cycle. This also implies that there exists at least one vertex with no successor.

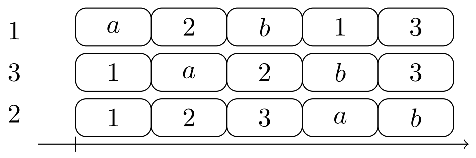
We choose a vertex x among the vertices with no successor in the above mentioned precedence graph. For readability, we will in the sequel denote the task corresponding to vertex x as task x. We look whether among the predecessors of x there exist vertices corresponding to a task scheduled after x in the optimal schedule returned by ΣDev or ΣT. If such vertices exist, we swap the position of the task x with the task corresponding to its predecessor scheduled after it and as close as possible to x in the returned schedule. We repeat this step until all the tasks corresponding to predecessors of x are scheduled before x. By swapping x with its closest predecessor scheduled after it, we make sure that we do not create any violation of the precedence constraints. Studying all the vertices takes n operations, consisting in at most n swaps: the total number of swaps is then bounded by n^2. $\square$

Note that the previous proof is a constructive proof: we can compute an optimal solution for ΣDev-INFERRED (or ΣT-INFERRED) by solving an assignment problem for ΣDev (or ΣT), and then swapping tasks which do not fulfill the precedence constraints as explained in the proof of Proposition 10.

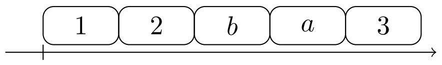
We cannot take the same approach for ΣU-INFERRED. Indeed, there are instances in which no optimal solution for the minimization of the total number of late tasks criterion fulfills the inferred precedence constraints, as shown by the following proposition.

Proposition 11. *There exist instances for which no optimal solution for the ΣU criterion fulfills the inferred precedence constraints.*

Proof. Let us consider the following instance of 5 tasks and 6 voters. The number at the left of each schedule indicates the number of voters whose schedule is the preferred schedule (e.g. the favorite schedule of three voters is the second schedule).



The only optimal solution for the ΣU criterion is the following one:



In this solution, b is scheduled before a, whereas all the voters have scheduled a before b in their favorite schedules: this violates the inferred precedence constraints. □

This last proposition means that we cannot proceed like in Proposition 10, by computing an optimal solution for ΣU and then swapping tasks which would not be in the right order. Whether problem ΣU-INFERRED is NP-hard or not is an open question.

We end this section by noting that the rule EMD computes schedules which fulfill all the inferred precedence constraints. Indeed, if all the voters schedule a task a before a task b in their preferred schedules, then the median completion time of a will be smaller than the median completion time of b and thus a will be scheduled before b in the schedule returned by EMD.

4.2 Imposed precedence graph

We start by proving that this problem is strongly NP-hard for the total tardiness criterion.

Proposition 12. *The ΣT-GRAPH problem is strongly NP-hard, even when the precedence graph consists in chains.*

We prove this proposition by doing a polynomial time reduction from the scheduling problem denoted by $(1|chains, p_j = 1|\Sigma T_j)$ using the Graham's notation, a classical way to denote problems in scheduling theory (Brucker, 1999). An instance of this problem is:

- a set J of n unit tasks, each task j having a due date d_j. Without loss of generality, we assume that $d_j \leq n$ for all j.
- a precedence graph, modeling precedence constraints between the tasks. We assume that and this graph is made of chains in chains (i.e. each task has at most one successor and one predecessor, and there are no cycle).

The optimization version of this problem consists in minimizing the sum of the tardiness of the tasks. The decision version of this problem consists in answering the following question: given an integer K, is there a schedule S of the tasks in J on a single machine, such that the precedence constraints are fulfilled, and such that the total tardiness of the tasks, $\sum_{j\in J} \max(0, C_j(S) - d_j)$, is smaller than or equal to K ? This problem is known to be NP-hard (Leung & Young, 1990).

We create an instance of ΣDev-GRAPH from the instance from $(1|chains, p_j = 1|\Sigma T_j)$ as follows.

- For each task j in J we create a task t_j and a task dum_j. These tasks are split into two sets $J_t = \{j_1, \dots, j_n\}$ and $J_{dum} = \{dum_1, \dots, dum_n\}$. The set J' of the tasks of the instance of ΣT-GRAPH is the union of J_t and J_{dum}.
- For each precedence relation in the $(1|chains, p_j = 1|\Sigma T_j)$ problem between tasks i and j, we create a precedence constraint between t_i and t_j in the precedence graph of problem ΣT-GRAPH.
- For each task j in J we also create three voters. Their preferred schedules, that we will describe now, are represented on Figure 3. The first two voters, of type T, schedule t_j first, followed by t_{j+1} and so forth until t_n, then t_1 to t_{j-1} by increasing index. They then schedule the dum tasks following the same pattern: dum_j first, then dum_{j+1} to dum_n, followed by dum_1 to dum_{j-1} by increasing index (see top schedule in Figure 3). The last voter, of type D, schedules task t_j between time $d_j - 1$ and d_j. Before that, she schedules $(d_j - 1)$ dum tasks from dum_j by increasing index (using again a circular order of the tasks, where task dum_1 follows task dum_n). The remaining dum tasks are scheduled after t_j by increasing indexes. The schedule is completed with tasks t_{j+1}, t_{j+2}, $\dots$, until task t_{j-1} if $j \neq 1$, or task t_n if $j = 1$ (see bottom schedule in Figure 3).

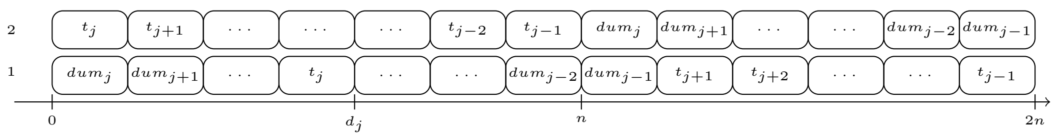


Fig. 3 Preferred schedules of the 3 voters generated for task j.

In order to prove Proposition 12, we start by proving the following lemma.

Lemma 1. *For the instance of the ΣT-GRAPH problem described above, there is an optimal solution in which all the t tasks are scheduled before all the dum tasks.*

Proof. Let us assume by contradiction that there is no optimal solution in which all t tasks are scheduled before all dum tasks. Let S be such an optimal solution: there is at least one dum task completing just before a task t. Let us call dum_i the first dum task scheduled before a t task, and t_j be the t task scheduled just after dum_i. Let k be the completion time of dum_i in S: we have $C_{dum_i}(S) = k$ and $C_{t_j}(S) = k+1$ (note that $1 \leq k < 2n$).

We call S' the schedule obtained from S by swapping the position of dum_i and t_j. The total tardiness of S' is similar to S except for the tardiness of dum_i and t_j. We then have $C_{dum_i}(S') = k+1$ and $C_{t_j}(S') = k$. Note that if the precedence constraints over the t tasks are satisfied by S, they are also satisfied by S' since the order on the

t tasks has not changed. Therefore, since S is a feasible solution, S' is also a feasible solution. We distinguish two sub-cases:

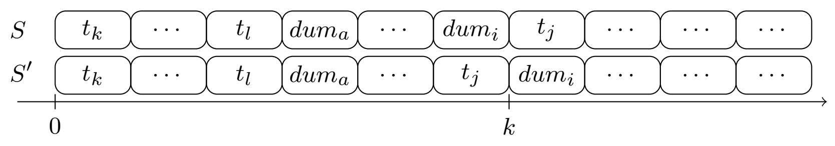


Fig. 4 Schedules S and S'. The first *dum* task to be scheduled just before a t task in S is dum_i.

- $k \leq n$. In this case, the tardiness relative to task t_j is reduced in S' in comparison to S by at least $2k$. Indeed, there are $2k$ voters scheduling t_j at time k or before. Therefore, moving t_j from $k+1$ to k reduces the tardiness of t_j by one for each of these voters, giving a total of $2k$. The tardiness of task dum_i is increased in S' in comparison to S. There are at most k voters of type D scheduling dum_i at time k or before: for each of these voters, the tardiness is increased by one. The sum of the tardiness of S' is decreased by at least $2k$ (due to t_j), and increased by at most k (due to dum_i), in comparison to the sum of the tardiness of S: the total tardiness of S' is thus smaller than the tardiness of S. Since S minimizes the sum of the tardiness, there is a contradiction.
- $k > n$. In this case, the tardiness relative to task t_j is reduced in S' in comparison to S by $2n+1+(k-n)$. Indeed, there are $2n$ voters of type T scheduling t_j before k, and one voter of type D scheduling t_j so that this task is completed at date d_j with $d_j \leq n$. This makes a total of $2n+1$. Additionally, there are $k-n$ voters of type D scheduling t_j at time $n+1$, $n+2$ up to k. For each of these voters, the tardiness of t_j is reduced by one.
  On the other hand, the tardiness of dum_i is increased in S' in comparison to S by $n+2(k-n)$. The n voters of type D scheduled dum_i so it that it is completed at most at time $n+1$, meaning that delaying dum_i from k to $k+1$ increases the tardiness by one for each of these n voters. Additionally, there are $2(k-n)$ voters of type T scheduling dum_i so that it completes at dates $n+1$, $n+2$ to k: the tardiness is increased by one for each of these voters. If we compare the increase in tardiness for task dum_i, $n+2(k-n)$, to the decrease of the tardiness for task t_j, $2n+1+(k-n)$, we see that the sum of the tardiness in S' is decreased by $2n+1-k$. Since $k < 2n$, this value is always strictly positive. This means that the total tardiness of S' is strictly smaller than the tardiness of S, an optimal solution: a contradiction.

□

We can now start the proof of Proposition 12.

Proof. From Lemma 1, we know that there exists an optimal schedule S in which t tasks are scheduled before *dum* tasks. We analyze the sum of the tardiness in such a schedule. We first show that the sum of the tardiness of *dum* tasks is the same in any schedule fulfilling the property of Lemma 1 (first item below), and we then analyze the sum of the tardiness due to t tasks (second item below).

- We show that in any schedule in which *dum* tasks are scheduled after t tasks fulfilling this property, the tardiness due to *dum* tasks is always the same.
  Voters of type T schedule each *dum* task twice between n and $n+1$, twice between $n+1$ and $n+2$ and so on until $2n-1$ and $2n$. In schedule S, the *dum* task scheduled between n and $n+1$ in not late for any voter of type T, the task scheduled between $n+1$ and $n+2$ is late of one unit of time for 2 voters of type T, and so on. Overall, the total tardiness of *dum* tasks for T voters is then $2\sum_{i=1}^{n}\sum_{j=1}^{i}(j-1)$, a constant number.
  Let us now show that the sum of the tardiness of *dum* tasks for D voters will be the same in any schedule S in which t tasks are scheduled before *dum* tasks. Indeed, for each D voter j, and for each task dum_i, the completion time of dum_i in the preferred schedule of j is at most $n+1$, whereas the completion time of dum_i in S is at least $n+1$. Therefore, the sum of the tardiness due to *dum* tasks for D voters is equal to the sum of the distances between completion times of *dum* tasks in the preferred schedules of voters D to date $n+1$ – which is a constant, since preferred schedules are fixed –, plus the sum of the distances of *dum* tasks between date $n+1$ and the completion time of *dum* tasks in S – this is also a constant since the completion times of *dum* tasks in S are the set of times $\{n+1,\dots,2n\}$. Therefore, the the sum of the tardiness of *dum* tasks for D voters is a constant.
  We have seen that the sum of the tardiness of *dum* tasks is value is the same for any schedule S which fulfills Lemma 1. Let T_{dum} denote this value, which, as we have seen, is a constant.
- Regarding tasks t, voters of type T schedule them such that each task t_i is completed twice at time 1, twice at time 2 and so on. So, regardless of the order of tasks t in S, the first task of S is not late for any voter, the second task of S is late by 1 unit of time for 2 voters, the third task is late by 1 unit of time for 2 voters, by 2 units of time for two voters and so on. Therefore, the sum of the tardiness of t tasks for voters of type T is also the same for each schedule S in which t tasks precede *dum* tasks. Let T_t denote this sum of tardiness.
  Voters D schedule all tasks t after $n+1$ except one task t_j (for the j-th voter of type D), and this task is completed at time d_j. Therefore in S, each task t_j is always early for all voters D except one, the j-th voter of type D, and its tardiness for this voter is equal to $\max(0, C_{t_j}(S) - d_j)$.

The sum of the tardiness $T(S)$ in schedule S is thus equal to:

$$T(S) = T_{dum} + T_t + \sum_{t_j \in J_t} \max(0, C_{t_j}(S) - d_j)$$

Since T_{dum} and T_t do not depend on the order of the tasks in S as long as all tasks t are scheduled first and all tasks *dum* are scheduled afterwards, the tardiness of schedule S only depends on the position of tasks t relatively to the due dates of the $(1|chains, p_j=1|\Sigma T_j)$ problem.

We will now prove that there exists a solution S for the instance of the ΣT-GRAPH problem described above such that $T(S) \leq T_{dum} + T_t + K$, if and only if there exists a schedule S' for $(1|chains, p_j=1|\Sigma T_j)$ problem such that the tardiness is smaller than or equal to K. In other words, the answer to the question of ΣT-GRAPH problem is then "yes" if and only if the answer to the question of the corresponding instance of $(1|chains, p_j = 1|\Sigma T_j)$ is "yes".

Let us assume first that there is a solution S of ΣT-GRAPH problem such that $T(S) \leq T_{dum} + T_t + K$. It means that $\sum_{t_j \in J} \max(0, C_{t_j}(S) - d_j) \leq K$. Let S' be a schedule of tasks of $(1|chains, p_j = 1|\Sigma T_j)$ such that the completion time of task j in S' is equal to the completion time of t_j in S. We have $\sum_{j \in J} \max(0, C_j(S') - d_j) \leq K$, and this solution is feasible since the precedence constraints between the tasks of the $(1|chains, p_j = 1|\Sigma T_j)$ problem are the same than between the t tasks. The answer to the question of the $(1|chains, p_j = 1|\Sigma T_j)$ is then "yes".

Let us now assume that there is a feasible solution (schedule) S' of $(1|chains, p_j = 1|\Sigma T_j)$ such that the total tardiness is smaller than or equal to K. If we create solution S such that the completion time of task t_j in S is equal to the completion time of j in S', we then have $\sum_{t_j \in J_t} \max(0, C_{t_j}(S) - d_j) \leq K$. The *dum* tasks are then scheduled in any order. Such a solution has then a total tardiness of $T_t + T_{dum} + K$. This solution is feasible since the precedence constraints between tasks of the $(1|chains, p_j = 1|\Sigma T_j)$ problem are the same than between the t tasks. This implies that the answer to the ΣT-GRAPH problem is thus "yes".

There is a polynomial time reduction from decision problem $(1|chains, p_j = 1|\Sigma T_j)$, which is strongly NP-complete, to the decision version of our problem ΣT-GRAPH. Problem ΣT-GRAPH is thus strongly NP-hard. $\square$

Since, as we have seen before, with unit tasks graphs, and for any profile P and any schedule S, the sum of the deviations in S with respect to profile P is equal to twice the sum of the tardiness in S, a schedule minimizing the sum of the deviations among schedules which fulfill the precedence constraints will also minimize the sum of the tardiness. Given Proposition 12, we deduce the following corollary.

Corollary 3. *The ΣDev-GRAPH problem is strongly NP-hard, even when the precedence graph consists in chains.*

We now show that problem ΣU-GRAPH, which aims at minimizing the number of late tasks in the returned schedule, with respect to the preferred schedules of the voters, is also a strongly NP-hard problem.

Proposition 13. *The ΣU-GRAPH problem is strongly NP-hard, even when the precedence graph only consists in chains.*

We prove this results by doing a polynomial time reduction from the $(1|chains, p_j = 1|\Sigma U_j)$ problem. The decision version of this problem is the following one. An instance of this problem is:

- A set $J' = \{1, \ldots, n\}$ of n unit tasks. Each task i has a deadline d_i.
- A a acyclic precedence graph of n vertices $\{1, \ldots, n\}$: there is one edge from vertex i to vertex j if task i has to be scheduled before task j. This graph can be only a set of chains between some tasks.
- An integer K'

The aim of optimization problem is to compute a schedule which fulfills the precedence constraints and which minimizes the number of late tasks (i.e. tasks which are completed after their deadlines). The question of the corresponding decision problem is the following one: is there a schedule S which fulfills the precedence constraints and in which at most K' tasks are late ?

Garey and Johnson (1976) have shown that this problem is strongly NP-hard with general precedence constraints, even with unit time tasks. Lenstra and Rinnooy Kan (1980) have sharpened this result by showing that this problem remains strongly NP-hard, even if the set of precedence constraints is a set of chains.

Without loss of generality we assume that $d_i \leq n$ (tasks with deadlines larger than n will never be late in a schedule of n unit tasks without idle time). We create an instance of ΣU-GRAPH as follows.

For each task i of J', we create a task t_i and a task dum_i. For each task i we also create $(n+1)$ voters as shown in Figure 5. There are n voters "of type T" scheduling task t_i first, then t_{i+1} and so forth until t_n and then scheduling tasks t_1 to t_{i-1} by increasing index. They then schedule tasks dum_1, dum_2, $\ldots dum_n$. The last voter, "of type D" schedules task t_i so that it is completed at time d_i, and, if $d_i \neq 1$, she schedules task dum_2 to dum_{d_i-2} by increasing index from time 0 to time d_i-2. From time d_i, she schedules tasks dum_{d_i-1} to dum_n until time n, by increasing index, and she schedules dum_1 so that this tasks is completed at time $n+1$. She completes the schedule with tasks $t_{i+1}, \ldots, t_n$ by increasing index, followed by tasks t_1 to t_{i-1} by increasing index. For any precedence relation between tasks i and j in $(1|chains, p_j=1|\Sigma U_j)$, we create the same preference relation between tasks t_i and t_j of our ΣU-GRAPH instance.

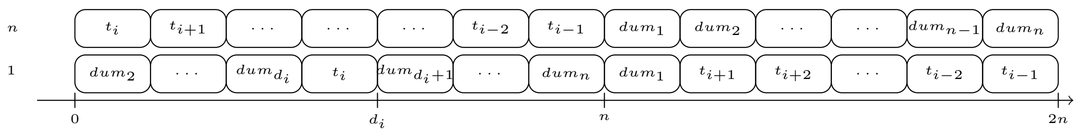


Fig. 5 Preferred schedules of the $n+1$ voters generated for task i.

In order to prove Proposition 13, we introduce several lemmas which describe an optimal schedule for the above described instance. As in the proof of Proposition 12, we will see that computing such an optimal solution allow us the associated NP-complete scheduling problem (problem $(1|chains, p_j=1|\Sigma U_j)$ in our case).

Lemma 2. *There exists an optimal solution for* ΣU-GRAPH *in which task* dum_1 *completes at time* $n+1$.

Proof. All voters schedule dum_1 so that it is completed at time $n+1$. Let S be a schedule in which dum_i does not complete at time $n+1$. We distinguish two sub-cases:

1. Task dum_1 completes before time $n+1$: we create schedule S' from S by scheduling dum_i so that it is completed at time $n+1$. We decrease from 1 unit of time any task scheduled in S between dum_1 and time $n+1$. Task dum_1 is not late in S' for any voter, just like in S and the task that have been scheduled before cannot become late in S' if they were not in S. Therefore the number of late tasks cannot increase from S to S'.
2. Task dum_1 completes after $n+1$. We distinguish two sub-cases:

- If the task j completing at time $n+1$ in S is a *dum* task, we create S' from S by swapping the position of dum_1 with the task j. The unit time penalty for all tasks but j and dum_1 are identical between S and S'. Task dum_1 is in S' on time for the $n(n+1) = n^2 + n$ voters, whereas it was late in S. On the other hand the unit time cost for task j is increased, but at most by n^2 voters, since the n voters of type D already considered it late since they scheduled it before time $n+1$. Overall the unit time penalty is reduced in S' in comparison to S.
- If the task j completing at time $n+1$ in S is a t task, we create a new schedule S' by scheduling dum_1 so that it completes at time $n+1$. We then perform consecutive swaps such that the order on the t tasks is the same in S, which is a feasible solution, and S'. If there is at least one t task scheduled between $n+1$ and, $C_{dum_1}(S)$, the completion time of dum_1 in S, we schedule task j at the time slot occupied by the first t task scheduled after $n+1$ in S. Let t_i be such a task. This task t_i is then scheduled at the time slot of the following t task which is completed before $C_{dum_1}(S)$, and so on until there is no t task left before $C_{dum_1}(S)$. The final t task moved that way goes on the time slot occupied by dum_1 in S (i.e. is completed at time $C_{dum_1}(S)$).

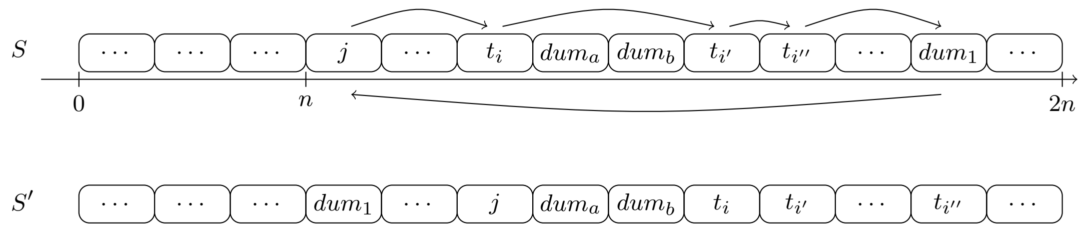


Fig. 6 Schedule S and the swaps performed to obtain S'.

Note that if the precedence constraints between the t tasks are fulfilled by S, they are also fulfilled by S' since the order on the t tasks do not change, just their positions.
The t tasks which have been moved in S' were considered late in S by all T voters: delaying them do not increase unit time penalty for T voters. Since D voters schedule t task in a cyclic fashion, each t task completes once at time $n+2$, once at time $n+3$ and so on. Therefore delaying a t task by one unit of time between $n+1$ and $2n$ increases its unit time penalty by 1 (since one additional voter will consider it late). Therefore, when delaying these tasks, the cumulative delay is at most n. On the other hand, scheduling dum_1 at time $n+1$ decreases the number of late tasks by n^2+n since it is late for all voters in S and on time for all voters in S'. This means that the total unit time penalty is smaller in S' than in S.

In all the cases, we managed to generate a solution S' in which dum_1 is scheduled between time n and time $n+1$ with the total number of late tasks of S' smaller than or equal to the number of late tasks in S. Therefore there always exist an optimal solution in which dum_1 is scheduled between n and $n+1$.

□

Lemma 3. *There exists an optimal solution of* ΣU-GRAPH *that fulfills Lemma 2 and such that there is in this solution a set of successive dum tasks scheduled by increasing index from time* n*, and none of these tasks are considered late by any voter of type* T*.*

Proof. Let S be an optimal solution fulfilling the property of Lemma 2: task dum_1 is completed at time $n+1$.

Let us assume that in S some tasks of the dum set starting at time n are not scheduled by increasing index. Let dum_a and dum_b be the two tasks scheduled the earliest in this set and such that dum_a is scheduled before dum_b with $a > b$. Since they are the first two tasks fulfilling this condition any task of this set scheduled before dum_a in S has a smaller index than a and is also scheduled before dum_a in the preferences of voters T.

Let us consider the solution S' obtained from S by swapping the positions of dum_a and dum_b. Since $b < a$, dum_b is scheduled before dum_a in the preferences of T voters and since all tasks of the set scheduled before dum_a in S are also scheduled before dum_a in preferences of voters T, dum_a cannot be late for voters of type T. This means that task dum_a does not become late for T voters in S'. This tasks is late for D voters in both S and S' since it is scheduled after $n+1$. Since the completion time of dum_b is reduced in S' in comparison to S, it cannot be late in S' whereas it was not late in S. Therefore, the number of late tasks in S' is not larger than the number of mate tasks in S.

Repeating these swaps until all the dum tasks of the set are scheduled by increasing index, we obtain a new solution in which all of these tasks are on time for T voters and in scheduled by increasing index and such that number of late tasks is not increased in comparison to S. □

Lemma 4. *There exists an optimal solution for* ΣU-GRAPH *which fulfills Lemma 3, and such that all* t *tasks scheduled between time* 0 *and time* n *are scheduled before any* dum *tasks scheduled between time* 0 *and time* n*.*

Proof. Let us consider a solution S fulfilling the properties of Lemmas 2 and 3 and such that there is a task dum_i scheduled between time 0 and time n and such that there is a task t_j scheduled just after dum_i in S. Since the task scheduled between time n and $n+1$ is dum_1 in S, task t_j completes at most at time n.

We create a schedule S' from S by swapping the positions of dum_i and t_j. For each date k between 1 and n, there are n voters of type T scheduling t_j so that it is completed at time k. Therefore, advancing t_j by one unit of time between 1 and n, decreases the number of late tasks by n. On the other hand, task dum_i is delayed by one unit of time. This does not impact the T voters since they schedule dum_i after time $n+1$. Voters of type D might have an increased unit time penalty for task dum_i. Since there are n voters of type D, this increases the number of late tasks by at most n. Therefore, the number of late tasks in S' is smaller than or equal to the the number of late tasks in S. □

Lemma 5. *There exists an optimal solution for* ΣU-GRAPH *which fulfills Lemma 4, and in which all the* t *tasks are scheduled before all* dum *tasks. Moreover, in this solution, the* dum *tasks are scheduled in order of increasing indexes.*

Proof. Let S be an optimal solution satisfying the properties of Lemma 4 and such that all tasks t are not scheduled before all dum tasks. Let dum_i be the first dum task to be scheduled in S. This implies that $C_{dum_i} \leq n$. Because of Lemma 4, task dum_i has to be scheduled after a series of t tasks, and all tasks scheduled after dum_i and before dum_1 are dum tasks as well. Let t_j be the first t task scheduled after dum_i. As we have seen, dum_i is scheduled after $n+1$ and after a set of dum tasks scheduled by increasing index.

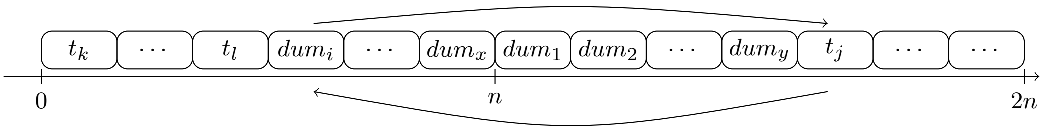


Fig. 7 Schedule S with the swap performed to obtain S_{tmp}.

Let S_{tmp} be the schedule obtained by swapping from S the position of dum_i and t_j, and let S' be the schedule obtained by swapping from S the position of dum_i and t_j and in which dum_i is re-positioned in the dum set so that the tasks in the set are scheduled by increasing indexes (therefore that S' can also be obtained from S_{tmp} by repositioning dum_i at the right place in the set of dum tasks that follow it in S).

Note that since S fulfills the precedence constraints on the t tasks, then they are fulfilled by S_{tmp} and S' as well since the order on the t tasks does not change. In its new position in S_{tmp} and S', task dum_i is not late for voters of type T (who schedule dum_i after time n). It may be late for some voters of type D whereas it was not necessarily late for these voters in schedule S. Therefore, since there are n voters of type D, the number of late tasks due to dum_i is increased by at most n in S_{tmp} and in S'. Let us now focus on the total number of late tasks tardiness in S_{tmp}. The only tasks whose time slot has changed (compared to is time slot in S) is t_j, which was late for all voters of type T in S but now completes at most at time n. It thus in S_{tmp} on time for at least n voters of type T (the ones scheduling it between time $n-1$ and n). Overall the number of late tasks does not increase in S_{tmp} in comparison to S. Since all the voters schedule, in their preferred schedules, the dum tasks by increasing order, the number of late tasks in S' is not larger than the number of late tasks in S_{tmp}. Therefore, the number of late tasks does not increase in S' compared to in S.

By repeating, if needed, this type of swaps, we obtain an optimal solution in which all t tasks are scheduled before all dum tasks, and in which dum tasks are scheduled by increasing indices. ☐

Starting from any optimal solution S and applying the successive swaps described in Lemmas 2 to 5, we obtain an optimal solution in which tasks t are scheduled first and are followed by dum tasks which are scheduled between time n and $2n$ by increasing indices. Let us now prove Proposition 13.

Proof. We show that there exists a solution with a total number of late task smaller than or equal to K' for $(1|chains, p_j = 1|\Sigma U_j)$ if and only if there exists a solution with a total number of late tasks for ΣU-GRAPH smaller than or equal to $K = K' + n(n+1) + \sum_{i=1}^{n}(i-1)n$.

Let us first assume that there exists a solution with at most K late tasks for ΣU-GRAPH. Thanks to Lemmas 5, we know that there exists an optimal solution in which tasks of type t are scheduled before dum tasks, which are scheduled by increasing indices. In such a solution S, the number of late tasks can be split into two parts, one independent from the order of the t tasks, and one depending on this order.

Regardless of the order of the t tasks, the dum tasks are all on time for the voters of type T, and all (except dum_1) late for the voters of type D. There are therefore $n-1$ dum tasks late for each of the n voters of type D, which amounts to $n(n-1)$ late tasks. Furthermore, t task completes n times at time 1, n times at time 2, and so on until time n. The t task completing at time 1 will be on time for all voters of type T, the t task completing at time 2 will be late for n voters of type T, the third task will be late for $2n$ voters and so on. This amounts to $\sum_{i=1}^{n}(i-1)n$. For each $i \in \{1, \dots, n\}$, task t_i is on time for D voters, except for the i-th D voter, who scheduled task t_i so that it is completed at time d_i.

This means that the total number of late task in S is $U_t(S)+n(n-1)+\sum_{i=1}^{n}(i-1)n$, where $U_t(S)$ denotes the number of late t tasks in S for voters of type D. Since S is an optimal solution and since the answer to the ΣU-GRAPH problem is 'yes', this means that $U_t(S) \leq K'$.

We label the t tasks according to their position in schedule S, which can be described as follows: $t_{S(1)}, t_{S(2)}, \dots, t_{S(n)}, dum_1, dum_2, \dots, dum_n$, where $S(i)$ denotes the index of the task scheduled in position i in S. We consider the schedule S' of tasks of $(1|chains, p_j = 1|\Sigma U_j)$: $S(1), S(2), \dots, S(n)$. Note that since S is a feasible solution of ΣU-GRAPH and since the precedence constraints on t tasks are the same than on the tasks of the $(1|chains, p_j = 1|\Sigma U_j)$ instance, S' is a feasible solution of $(1|chains, p_j = 1|\Sigma U_j)$. In S', task $S(i)$ is completed at the same time than task $t_{S(i)}$ in S, therefore task $S(i)$ is late if and only if $S(i)$ is late for the voter scheduling $S(i)$ at time $d_{S(i)}$. Therefore if $U_t(S) \leq K'$, the total number of late tasks in S' is also smaller than or equal to K', which means that the answer to the $(1|chains, p_j = 1|\Sigma U_j)$ problem is also 'yes'.

Reciprocally, if the answer to the $(1|chains, p_j = 1|\Sigma U_j)$ problem is 'yes', then there exists a schedule S' for $(1|chains, p_j = 1|\Sigma U_j)$ such that the total number of late tasks in S' is smaller than or equal to K'. We consider S the schedule $t_{S'(1)}, \dots, t_{S'(n)}, dum_1, dum_2, \dots, dum_n$ for the ΣU-GRAPH problem. Schedule S fulfills the precedence constraints of the ΣU-GRAPH instance since these precedence constraints are the same than the precedence constraints on the t tasks of the corresponding instance of ΣU-GRAPH.

Since S fulfills the property of Lemma 5, there is a constant number of late task $n(n-1) + \sum_{i=1}^{n}$ for voters of type T. The number of late t tasks for voters of type D

depends on whether task t_i is scheduled before or after time d_i since only one D voter schedules task t_i before time n (she schedules t_i between times $d_i - 1$ and d_i). Task t_i is completed in S at the same time than task i in S'. Therefore task t_i completes after d_i in S if and only if task i is late in S'. Therefore the number of late t tasks in S for voters of type D is equal to the number of late tasks in S'. Since the number of late tasks in S' is smaller than or equal to K', the total number of late tasks in S is smaller than or equal to $K' + n(n+1) + \sum_{i=1}^{n}(i-1)n$ and the answer to ΣU-GRAPH is then 'yes'.

The answer to the ΣU-GRAPH problem is 'yes' if and only if the answer to the $(1|chains, p_j = 1|\Sigma U_j)$ problem is 'yes'. Since the decision version of problem $(1|chains, p_j = 1|\Sigma U_j)$ is strongly NP-complete (Lenstra & Rinnooy Kan, 1980), we conclude that the decision version or problem ΣU-GRAPH is also strongly NP-complete. □

Proposition 12 shows that problem ΣT-GRAPH is strongly NP-hard, while Proposition 13 shows that problem ΣU-GRAPH is strongly NP-hard, even if the precedence graphs are only made of chains of tasks. Since, as we have seen in Section 1.2, problem ΣT is a special case of the Distance Criterion, and problem ΣU is a special case of the Binary Criterion, we get the following corollary.

Corollary 4. *Returning an optimal solution for the Distance Criterion or the Binary Criterion are strongly NP-hard problems when there are imposed precedence constraints. This is true even with precedence graphs only made of chains.*

5 Conclusion

In this paper, we studied the collective scheduling problem with unit size tasks, which can also be seen as a collective ranking problem since tasks of length 1 can be considered as items and preferred schedules as preferred rankings.

We introduced two general objective functions, one based on a distance, and the other one on a binary criterion. The distance based function minimizes the average distance between the returned schedule (or ranking) and the preferences of the voters (expressed as preferred schedules or preferred intervals for each task). It generalizes already known rules that minimize of the average deviation (ΣDev), or the average tardiness (ΣT). The binary function generalizes the rule ΣU that minimizes the average number of late tasks. These rules can be applied in polynomial time even if we add release dates and deadlines constraints on the tasks.

We studied these two general rules from an axiomatic point of view when we infer release dates and deadlines from the preferences of the voters, showing that they do not fulfills release date or deadline consistency, but that they fulfill temporal unanimity, three axioms that we have introduced in this paper.

We have also shown that the rule EMD which schedules the tasks by increasing median completion time (or by increasing median place in a ranking if we consider rankings instead of schedules), is a 2-approximation for the sum of the deviation (or the sum of the tardiness) minimization. Note that the rule which minimizes the sum of the deviations between a collective ranking and a set of preferred ranking, is known as Spearman's rule: interestingly, EMD is thus a 2-approximation of Spearman's rule.

Last but not least, we studied the case where there are precedence constraints between the tasks. Note that precedence constraints also make sense in the context of rankings if items, a constraint between two items a and b saying that item a has to be ranked higher than item b. We showed that if the precedence constraints are fulfilled by the preferred schedules (or rankings) of the voters, then it is easy to get an optimal schedule (ranking) which fulfills the precedence constraints while minimizing the average deviation (or the average tardiness). When the preferred schedules do not necessarily fulfill the constraints, we showed that on the contrary, it is NP-hard to find a schedule that fulfills the precedence constraints while minimizing the average deviation (or the average tardiness, or the average number of late tasks).

Whether the minimization of the number of late tasks is an NP-hard problem with inferred precedence constraints, remains an open question. Looking for exact but efficient, or approximate, algorithms to compute solutions for the above mentioned NP-hard problems is also an interesting research direction. Finally, in context in which release date or deadline consistency are important axioms, finding a rule which fulfills these axioms while returning good solutions with respect to the Distance or the Binary criterion would also be a promising research direction.

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