

LIIOVILLE THEOREM FOR k -CURVATURE EQUATION WITH FULLY NONLINEAR BOUNDARY IN HALF SPACE

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ABSTRACT. We obtain the Liouville theorem for constant k -curvature $\sigma_k(A_g)$ in \mathbb{R}_+^n with constant \mathcal{B}_k^g curvature on $\partial\mathbb{R}_+^n$, where \mathcal{B}_k^g is derived from the variational functional for $\sigma_k(A_g)$, and specially represents the boundary term in the Gauss-Bonnet-Chern formula for $k = n/2$.

1. INTRODUCTION

The $\sigma_k(A_g)$ curvature, particularly the $\sigma_2(A_g)$ curvature, has achieved significant advancements in the past three decades. In [9] Chen found a naturally matching boundary curvature \mathcal{B}_k^g on ∂M for $\sigma_k(A_g)$. On locally conformally flat $2k$ -manifolds, the \mathcal{B}_k^g curvature on ∂M is the boundary term in the Gauss-Bonnet-Chern formula:

$$\int_M \sigma_{\frac{n}{2}}(A_g) dv_g + \oint_{\partial M} \mathcal{B}_{\frac{n}{2}}^g d\sigma_g = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2})!} \chi(M, \partial M).$$

Denote

$$\sigma_k(A_g) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \delta \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} A_{g, i_1}^{j_1} \dots A_{g, i_k}^{j_k}$$

and

$$(1.1) \quad \mathcal{B}_k^g = \sum_{i=0}^{k-1} C(n, k, i) \sigma_{2k-i-1, i} (A_g^T, L_g) \quad n \geq 2k,$$

where A_g^T is tangential part of A_g on boundary, L_g is the second fundamental form of boundary with respect to g , $\sigma_{2k-i-1, i} (A_g^T, L_g)$ is the mixed symmetric functions defined in [9] and $C(n, k, i) = \frac{(2k-i-1)!(n-2k+i)!}{(n-k)!(2k-2i-1)!!i!}$.

In [5] Chang-Chen posed the question: Does there exist a metric $g_u \in [g]$ satisfying the equation (1.2)

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$$(1.2) \quad \begin{cases} \sigma_k(A_{g_u}) = c & \text{in } M, \\ \mathcal{B}_k^{g_u} = 0 & \text{on } \partial M, \end{cases}$$

where c is a positive constant. The question is so largely open.

Equation (1.2) from [5, 9] is variational, formulated as:

$$\mathcal{F}'_k[v] = (2k - n) \left[\int_{M^n} \sigma_k(A_g) v dv_g + \oint_{\partial M} \mathcal{B}_k^g v d\sigma_g \right],$$

where

$$\mathcal{F}_k := \int_{M^n} \sigma_k(A_g) dv_g + \oint_{\partial M} \mathcal{B}_k^g d\sigma_g.$$

On locally conformally flat manifolds, the critical point of functional \mathcal{F}_k in $\{g_1 : g_1 \in [g], \text{vol}(M, g_1) = 1\}$ for $n > 2k$ is equation (1.2). For $n = 2k$, the variational functional can be found in Proposition 2.3 in [3] and its corresponding equation is (1.2).

Currently, the only known outcome of the question of Chang-Chen is the case about four-dimensional manifolds with umbilic boundary, in which it is equivalent to investigate the equation with Neumann boundary problem. Specifically, as pointed by Chen [9], when ∂M is umbilic and $g \in \Gamma_k^+$, $\mathcal{B}_k^g = 0$ if and only if $h_g = 0$. Here the well-known Γ_k^+ -cone is defined as below:

$$\Gamma_k^+ := \{g | \sigma_1(A_g) > 0, \dots, \sigma_k(A_g) > 0\}.$$

Chen [9] proved that on manifolds with umbilic boundary, if $Y(M^4, \partial M, [g]) > 0$ and $\int_{M^4} \sigma_2(A_g) dv_g + \oint_{\partial M} \mathcal{B}_2^g d\sigma_g > 0$, then there exists a metric $\hat{g} \in [g]$ such that $\sigma_2(\hat{g}^{-1}A_{\hat{g}})$ is constant and $\mathcal{B}_2^{\hat{g}} = 0$ on ∂M .

However, when the boundary is not umbilic, the boundary curvature \mathcal{B}_k^g is highly fully nonlinear involving second derivatives and first derivatives on boundary. The complexity is rare in the literature even for the uniformly second-order elliptic equation. Consequently, the exploration regarding (1.2) appears distant and challenging to approach.

To comprehend the nature of the boundary curvature \mathcal{B}_k^g , we investigate an alternative equation on manifolds with an umbilic boundary

$$(1.3) \quad \begin{cases} \sigma_k(A_g) = c & \text{in } M^n, \\ \mathcal{B}_k^g = c_0 & \text{on } \partial M, \end{cases}$$

where c_0 is a positive constant, and this equation still has variational structure. This non-vanishing \mathcal{B}_k^g curvature prominently involves second-order derivatives, as evidenced by the following explicit expression for \mathcal{B}_k^g on the umbilic boundary ∂M :

$$(1.4) \quad \mathcal{B}_k^g := \frac{(n-1)!}{(n-k)!(2k-1)!!} h_g^{2k-1} + \sum_{s=1}^{k-1} \frac{(n-1-s)!}{(n-k)!(2k-2s-1)!!} \sigma_s(A_g^T) h_g^{2k-2s-1},$$

where h_g is the mean curvature of boundary with respect to g and A_g^T is tangential part of A_g on boundary.

For simplicity, we denote $g_u^{-1}A_{g_u}$ as A_{g_u} and $g_u^{-1}A_{g_u}^T$ as $A_{g_u}^T$ without confusion. We remark that if $g \in \Gamma_k^+$, then the linearization of the operator \mathcal{B}_k^g is actually elliptic, see (2.1) in Section 2, which is the start point of the whole paper.

In this paper we will build the Liouville theorem for the equation (1.3) in the following:

Theorem 1. *Given a positive constant c_0 , let $g_u = u^{\frac{4}{n-2}}|dx|^2$ in \mathbb{R}_+^n satisfy*

$$(1.5) \quad \begin{cases} \sigma_k(A_{g_u}) = 2^k C_n^k & \text{in } \overline{\mathbb{R}_+^n}, \quad g_u \in \Gamma_k^+, \\ \mathcal{B}_k^{g_u} = c_0 & \text{on } \partial\mathbb{R}_+^n. \end{cases}$$

Assume that $\lim_{x \rightarrow 0} u_{0,1}$ exists, where $u_{0,1}(x) := |x|^{2-n}u\left(\frac{x}{|x|^2}\right)$. Then there exists a positive constant $b \in \mathbb{R}_+$ and $(\bar{x}', \bar{x}_n) \in \mathbb{R}^n$ such that

$$(1.6) \quad u(x', x_n) \equiv \left(\frac{\sqrt{b}}{1 + b|(x', x_n) - (\bar{x}', \bar{x}_n)|^2} \right)^{(n-2)/2} \quad \text{in } \mathbb{R}_+^n,$$

where $h_{g_u} = -\frac{2}{n-2}u_{x_n}u^{-\frac{n}{n-2}} = -2\sqrt{b}\bar{x}_n > 0$ and $A_{g_u}^T = 2\mathbb{I}_{(n-1) \times (n-1)}$ satisfy

$$\begin{aligned} \sum_{s=1}^{k-1} \frac{(n-s)!}{(n-k)!(2k-2s-1)!!n} \sigma_s(2\mathbb{I}_{(n-1) \times (n-1)}) [2\sqrt{b}\bar{x}_n]^{2k-2s-1} \\ + \frac{(n-1)!}{(n-k)!(2k-1)!!} [2\sqrt{b}\bar{x}_n]^{2k-1} = \frac{n+1-k}{n} c_0. \end{aligned}$$

From the proof, we actually know that $\lim_{x \rightarrow 0} u_{0,1}$ is a positive constant. When $\mathcal{B}_k^{g_u} = 0$, the boundary condition becomes $h_{g_u} = 0$ and the corresponding Liouville theorem has been established by Li-Li [20], see more general statement in [20]. As the boundary condition is highly fully nonlinear, to make the boundary equation transparent and elliptic, we consider the metric with positive boundary $\mathcal{B}_k^{g_u}$ curvature in Γ_k^+ cone. From geometric aspects, $u_{0,1}$ is usually C^2 continuous to zero. To prove Theorem 1, we apply the key Lemma

11 in [19] to $W := \frac{2}{n-2} \ln u_{0,1}$ on $\partial\mathbb{R}_+^n \setminus \{0\}$, where $W(x', 0)$ is superharmonic on $\partial\mathbb{R}_+^n$.

With the above theorem, we have the following theorem.

Theorem 2. *Given a positive constant c_0 , let $g_w = w^{\frac{4}{n-2}} |dx|^2$ in \mathbb{B}_1^n satisfy*

$$\begin{cases} \sigma_k(A_{g_w}) = 2^k C_n^k & \text{in } \mathbb{B}_1^n, \quad g_w \in \Gamma_k^+, \\ \mathcal{B}_k^{g_w} = c_0 & \text{on } \partial\mathbb{B}_1^n. \end{cases}$$

Then

$$w = \left(\frac{\sqrt{b}}{1 + b|x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

where $\bar{x} \in \mathbb{R}^n, b \in \mathbb{R}^+$ satisfy $\mathcal{B}_k^{g_w} = c_0$.

The Liouville theorem stands as a fundamental element contributing to the existence of the Yamabe-type equation and holds a central position within the field of partial differential equations and geometry. Extensive studies on Liouville Theorems on $\mathbb{R}^n, \mathbb{S}^n$ have been conducted in the context of the Yamabe-type equation. For the semilinear equation, the Liouville theorem can be traced back to Obata [24], Gidas-Ni-Nirenberg [15] and Caffarelli-Gidas-Spruck [1]. See Li-Zhu [22] and Li-Zhang [23] for the systematic introduction to the method of moving spheres. For fully nonlinear equation, especially σ_k Yamabe equation, Viaclovsky [25, 26] obtained the Liouville theorem under the additional hypothesis that $|x|^{2-n}u(x/|x|^2)$ can be extended to a positive C^2 function near $x = 0$ for $2 \leq k \leq n$. Concerning $k = 2$, Chang-Gursky-Yang [8] utilized Obata's technique to establish the case for $n = 4, 5$ and higher dimensional case under some additional assumptions. For $n = 4$, by constructing a monotone formula with respect to level set of the solution, the author with Fang and Ma [14] introduced an alternative approach and proved a Liouville theorem for some more general σ_2 -type equation, which may not be conformal invariant. Some general cases for conformally invariant equation including σ_k Yamabe operator were established by Li-Li [18, 19] and Li-Lu-Lu [21]. Also Chu-Li-Li [12] derived necessary and sufficient conditions for the validity of Liouville-type theorems in \mathbb{R}^n .

Compared to Liouville theorem in entire space, the results in half space is few as the boundary brings new difficulties. In ball Escobar [13] classified the metric with constant scalar curvature and the constant boundary mean curvature. Concerning the prescribed mean curvature in half space, the Liouville theorem for constant scalar curvature was established by Li-Zhu [22], Chipot-Shafir-Fila [11] and Li-Zhang [23]. And for fully nonlinear cases including σ_k curvature on half space, Li-Li [20] obtained the corresponding Liouville theorem with the constant boundary mean curvature. Case-Wang [1] demonstrated

an Obata-type theorem for $\sigma_k(A_g) = 0$ on \mathbb{S}_+^{n+1} and $\mathcal{B}_k^g = c_0$ on $\mathbb{S}^n = \partial\mathbb{S}_+^{n+1}$ under the additional condition $\sup_{\mathbb{S}^n} h_g \leq (k+1) \inf_{\mathbb{S}^n} h_g$. They [4] classified the local minimizers related with $\sigma_2(A_g)$ and \mathcal{B}_2^g in \mathbb{B}_1^{n+1} , when $n = 3, 4, 5$. Furthermore, with Moreira, in [2] they get some non-uniqueness results in different setting. This paper will focus on non-zero $\sigma_k(A_g)$ curvature and then the method of moving sphere works with appropriate observations on boundary.

We organize the paper as follows. In Section 2 we provide some facts about the linearized operator of \mathcal{B}_k^g . In Section 3 inspired by Li-Li [20], we prove Theorem 1 in stronger assumption (3.2) by the method of moving sphere. In Section 4, due to the super-harmonicity of $\frac{2}{n-2} \ln u_{0,1}$ on the lower dimensional space, we utilize the key lemma in [19] in lower dimensional space, and obtain Theorem 1. In Appendix we list some useful lemmas in [23, 19, 20] for reader's convenience.

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2. PRELIMINARY

In this section we describe the linearization of \mathcal{B}_k^g on $\partial\mathbb{R}_+^n$, which is an elliptic operator on boundary depending on the mean curvature of g and the cone condition.

In \mathbb{R}_+^n , for $g_u = u^{\frac{4}{n-2}} g_{\mathbb{E}}$, then

$$h_{g_u} = -\frac{2}{n-2} u^{-\frac{n}{n-2}} u_n \quad \text{on } \partial\mathbb{R}_+^n,$$

where $u_n = \frac{\partial u}{\partial x_n}$ on $\partial\mathbb{R}_+^n$,

$$\begin{aligned} g_u^{-1} A_{g_u} &= -\frac{2}{n-2} u^{-(n+2)/(n-2)} \nabla^2 u \\ &\quad + \frac{2n}{(n-2)^2} u^{-2n/(n-2)} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-2n/(n-2)} |\nabla u|^2 g_{\mathbb{E}}, \end{aligned}$$

and

$$\begin{aligned} g_u^{-1} A_{g_u}^T &= \left[-\frac{2}{n-2} u^{-(n+2)/(n-2)} \nabla^2 u \right. \\ &\quad \left. + \frac{2n}{(n-2)^2} u^{-2n/(n-2)} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-2n/(n-2)} |\nabla u|^2 g_{\mathbb{E}} \right]^T. \end{aligned}$$

2.1. Linearization of \mathcal{B}_k^g on $\partial\mathbb{R}_+^n$. We first study the linearization of $\mathcal{B}_k^{g_u}$. In [1] Case-Wang showed the conformal linearization of \mathcal{B}_k^g , which they used the symbol H_k instead. Our notation is consistent with Chen [9].

Let $g = u^{\frac{4}{n-2}}g_{\mathbb{E}}$ and $\bar{g} := i^*g_{\mathbb{E}}$, where $i : \partial\mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is the inclusion map, and we use $\bar{\nabla}$ to be the Levi-Civita Connection on $\partial\mathbb{R}_+^n$, α, β, γ range from $1, \dots, n-1$. Now $h_g = -\frac{2}{n-2}u^{-\frac{n}{n-2}}u_n$ on $\partial\mathbb{R}_+^n$, where ∂_{x_n} is the unit inward vector on boundary. Now we state the computation of the boundary operator, which will be frequently used in the following paragraphs.

Lemma 3. *Assume that $g_0 = u_0^{\frac{4}{n-2}}g_{\mathbb{E}}$ and $g_1 = u_1^{\frac{4}{n-2}}g_{\mathbb{E}}$ satisfy $\mathcal{B}_k^{g_0} = \mathcal{B}_k^{g_1} = c_0$. Then, taking $\psi := u_1 - u_0$,*

$$(2.1) \quad \begin{aligned} 0 &= \mathcal{B}_k^{g_1} - \mathcal{B}_k^{g_0} \\ &= -a_{\alpha\beta}\psi_{\alpha\beta} + b_{\alpha}\psi_{\alpha} - b_n\psi_n + c\psi := L\psi, \end{aligned}$$

where b_{α}, c depends on u_1, u_0 . Denote $u = tu_1 + (1-t)u_0$ and $g = u^{\frac{4}{n-2}}g_{\mathbb{E}}$,

$$(2.2) \quad a_{\alpha\beta} := \int_0^1 \frac{2}{n-2} u^{-\frac{n+2}{n-2}} \sum_{s=1}^{k-1} \frac{(n-1-s)!}{(n-k)!(2k-2s-1)!!} h_g^{2k-2s-1} \frac{\partial \sigma_s(A_g^T)}{\partial A_{\alpha\beta}^T} dt,$$

and

$$(2.3) \quad b_n := \frac{2}{n-2} \int_0^1 u^{-\frac{n}{n-2}} \sigma_{k-1}(A_g^T) dt.$$

Proof. For simplicity, denote $C_1(n, k, s) = \frac{(n-1-s)!}{(n-k)!(2k-2s-1)!!}$.

We know that

$$0 = \mathcal{B}_k^{g_1} - \mathcal{B}_k^{g_0} = \int_0^1 \frac{\partial}{\partial t} (\mathcal{B}_k^g) dt,$$

where $g = (tu_1 + (1-t)u_0)^{\frac{4}{n-2}}g_{\mathbb{E}}$.

Define

$$b_n^* := \frac{2}{n-2} u^{-\frac{n}{n-2}} \sigma_{k-1}(A_g^T)$$

and

$$a_{\alpha\beta}^* := \frac{2}{n-2} u^{-\frac{n+2}{n-2}} \sum_{s=1}^{k-1} C_1(n, k, s) h_g^{2k-2s-1} \frac{\partial \sigma_s(A_g^T)}{\partial A_{\alpha\beta}^T}.$$

By (1.4), we have

$$\begin{aligned}
& \frac{\partial}{\partial t}(\mathcal{B}_k^g) \\
&= C_1(n, k, 0)(2k-1)h_g^{2k-2} \frac{\partial h_g}{\partial t} \\
&+ \sum_{s=1}^{k-1} C_1(n, k, s) \frac{\partial}{\partial t} \sigma_s(A_g^T) h_g^{2k-2s-1} \\
&+ \sum_{s=1}^{k-1} C_1(n, k, s)(2k-2s-1) \sigma_s(A_g^T) h_g^{2k-2s-2} \frac{\partial h_g}{\partial t} \\
&= C_1(n, k, 0)(2k-1)h_g^{2k-2} \left(\frac{2}{n-2} \frac{n}{n-2} u^{-\frac{2n-2}{n-2}} u_n \psi - \frac{2}{n-2} u^{-\frac{n}{n-2}} \psi_n \right) \\
&+ \sum_{s=1}^{k-1} C_1(n, k, s) h_g^{2k-2s-1} \frac{\partial}{\partial A_{\alpha\beta}^T} \sigma_s(A_g^T) \left\{ \frac{2}{n-2} \frac{n+2}{n-2} u^{-\frac{2n}{n-2}} u_{\alpha\beta} \psi \right. \\
&- \frac{2}{n-2} u^{-\frac{n+2}{n-2}} \psi_{\alpha\beta} - \frac{2n}{(n-2)^2} \frac{2n}{n-2} u^{-\frac{n+2}{n-2}} u_{\alpha} u_{\beta} \psi \\
&+ \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \psi_{\alpha} u_{\beta} + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \psi_{\beta} u_{\alpha} \\
&+ \left. \frac{2}{(n-2)^2} \frac{2n}{n-2} u^{-\frac{n+2}{n-2}} |\nabla u|^2 \delta_{\alpha\beta} \psi - \frac{4}{(n-2)^2} u^{-\frac{2n}{n-2}} (u_n \psi_n + u_{\gamma} \psi_{\gamma}) \delta_{\alpha\beta} \right\} \\
&+ \sum_{s=1}^{k-1} C_1(n, k, s)(2k-2s-1) \sigma_s(A_g^T) h_g^{2k-2s-2} \left\{ \frac{2}{n-2} \frac{n}{n-2} u^{-\frac{2n-2}{n-2}} u_n \psi \right. \\
&- \left. \frac{2}{n-2} u^{-\frac{n}{n-2}} \psi_n \right\} \\
&=: -b_n^* \psi_n - a_{\alpha\beta}^* \psi_{\alpha\beta} + \sum_{\alpha=1}^{n-1} b_{\alpha}^* \psi_{\alpha} + c^* \psi,
\end{aligned}$$

where b_{α}^*, c^* depends on u_1, u_0 and $\frac{du}{dt} = \psi = u_1 - u_0$.

Collecting all the coefficients of ψ_n , the definition of b_n^* can be deduced as follows.

$$\begin{aligned}
& C_1(n, k, 0)(2k-1)h_g^{2k-2} \left(-\frac{2}{n-2}u^{-\frac{n}{n-2}} \right) \\
& - \frac{2}{n-2}u^{-\frac{n}{n-2}} \sum_{s=1}^{k-1} C_1(n, k, s)(2k-2s-1)\sigma_s(A_g^T) h_g^{2k-2s-2} \\
& - \frac{4}{(n-2)^2}u^{-\frac{2n}{n-2}}u_n \sum_{s=1}^{k-1} C_1(n, k, s) \frac{\partial}{\partial A_{\alpha\beta}^T} \sigma_s(A_g^T) \delta_{\alpha\beta} h_g^{2k-2s-1} \\
& = C_1(n, k, 0)(2k-1)h_g^{2k-2} \left(-\frac{2}{n-2}u^{-\frac{n}{n-2}} \right) \\
& - \frac{2}{n-2}u^{-\frac{n}{n-2}} \sum_{s=1}^{k-1} C_1(n, k, s)(2k-2s-1)\sigma_s(A_g^T) h_g^{2k-2s-2} \\
& + \frac{2}{n-2}u^{-\frac{n}{n-2}} \sum_{s=1}^{k-1} C_1(n, k, s-1)(2k-2s+1)\sigma_{s-1}(A_g^T) h_g^{2k-2s} \\
& = -\frac{2}{n-2}u^{-\frac{n}{n-2}}\sigma_{k-1}(A_g^T) = -b_n^*,
\end{aligned}$$

where the first equality holds because $-\frac{4}{(n-2)^2}u^{-\frac{2n}{n-2}}u_n = \frac{2}{n-2}u^{-\frac{n}{n-2}}h_g$ and

$$\frac{\partial}{\partial A_{\alpha\beta}^T} \sigma_s(A_g^T) \delta_{\alpha\beta} = (n-s)\sigma_{s-1}(A_g^T).$$

Now we have completed the proof. \square

Lemma 4. On $\partial\mathbb{R}_+^n$, $g = u^{\frac{4}{n-2}}g_{\mathbb{E}}$, assume that $g \in \Gamma_k^+$, and \mathcal{B}_k^g has a fixed sign. Then, operator L (2.1) is elliptic on $\partial\mathbb{R}_+^n$.

By $A_g \in \Gamma_k^+$, we know that $\sigma_s(A_g^T) > 0$ from $1 \leq s \leq k-1$ and $\left\{ \frac{\partial \sigma_s(A_g^T)}{\partial A_{\alpha\beta}^T} \right\}_{\alpha \times \beta}$ is positively defined. The sign of \mathcal{B}_k^g is consistent with h_g , making u_n be negative or positive and keeping the ellipticity of L on boundary.

2.2. Conformally invariant boundary condition. Let $z = \varphi_{x,\lambda}(y) = x + \frac{\lambda^2(y-x)}{|y-x|^2}$ and then $\varphi_{x,\lambda}^*(|dz|^2) = \frac{\lambda^4}{|y-x|^4}|dy|^2$. For $g_u = u^{\frac{4}{n-2}}|dz|^2$,

$$\begin{aligned}
\varphi_{x,\lambda}^*(u^{\frac{4}{n-2}}|dz|^2) &= [u \circ \varphi_{x,\lambda}(y)]^{\frac{4}{n-2}} \frac{\lambda^4}{|y-x|^4}|dy|^2 \\
&= \left[\frac{\lambda^{n-2}u \circ \varphi_{x,\lambda}}{|y-x|^{n-2}} \right]^{\frac{4}{n-2}} |dy|^2.
\end{aligned}$$

Now we denote $u_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$ and from above, we obtain

$$\begin{aligned}\varphi_{x,\lambda}^*(\mathcal{B}_k^{g_u}) &= \mathcal{B}_k^{\varphi_{x,\lambda}^*(g_u)} = \mathcal{B}_k^{\varphi_{x,\lambda}^*(u^{\frac{4}{n-2}}|dz|^2)} \\ &= \mathcal{B}_k^{u_{x,\lambda}^{\frac{4}{n-2}}|dy|^2}.\end{aligned}$$

Thus, when $\mathcal{B}_k^{g_u} = c_0$, it holds that $\mathcal{B}_k^{u_{x,\lambda}^{\frac{4}{n-2}}|dy|^2} = c_0$.

3. LIOUVILLE THEOREM UNDER THE STRONGER CONDITION

In this section, we would like to prove a Liouville theorem in a stronger assumption (3.2), which also naturally appears in geometry. Some Lemmas in this section still work without (3.2). In this paper, we use $\overline{\nabla}$ to be the Levi-Civita Connection on $\partial\mathbb{R}_+^n$, α, β, γ range from $1, \dots, n-1$, which are induced from $g_{\mathbb{E}}$.

Theorem 5. *Let $g_u = u^{\frac{4}{n-2}}|dx|^2$ in \mathbb{R}_+^n satisfy*

$$(3.1) \quad \begin{cases} \sigma_k(A_{g_u}) = 2^k C_n^k & \text{in } \overline{\mathbb{R}_+^n}, \quad g_u \in \Gamma_k^+ \\ \mathcal{B}_k^{g_u} = c_0 & \text{on } \partial\mathbb{R}_+^n. \end{cases}$$

Assume that

$$(3.2) \quad u_{0,1}(x) := |x|^{2-n} u\left(\frac{x}{|x|^2}\right) \text{ can be extended to a positive function in } C^2(\overline{\mathbb{B}_1^+}).$$

Then, u is the form of (1.6).

For simplicity $A^u := g_u^{-1}A_{g_u}$, $\mathcal{B}_k^u := \mathcal{B}_k^{g_u}$ and $\partial'B_{r_0}^+ := \partial B_{r_0}^+ \setminus \{x|x_n = 0\}$.

Lemma 6. *Assume that u is the solution to (1.5). For any fixed $r_0 > 0$, when $|y| \geq r_0 > 0$ and $y_n \geq 0$, we have*

$$u(y) \geq (\min_{\partial'B_{r_0}^+} u) r_0^{n-2} |y|^{2-n},$$

and

$$\liminf_{x \in \mathbb{R}_+^n, |x| \rightarrow +\infty} u|x|^{n-2} > 0.$$

Proof. From the assumption, we know that $-\Delta u \geq 0$ in $\overline{\mathbb{R}_+^n} \setminus B_{r_0}^+$, and $-\frac{\partial u}{\partial x_n} > 0$ on $\partial\mathbb{R}_+^n \setminus \overline{B_{r_0}^+}$. With the fact $\Delta|y|^{2-n} = 0$ in \mathbb{R}_+^n and $\frac{\partial|y|^{2-n}}{\partial y_n} = 0$ on $\partial\mathbb{R}_+^n \setminus \{0\}$,

by the maximum principle, we have

$$u(y) \geq (\min_{\partial' B_{r_0}^+} u) r_0^{n-2} |y|^{2-n} \text{ for } |y| \geq r_0 \text{ and } y_n \geq 0.$$

□

For $x \in \partial \mathbb{R}_+^n$, $\lambda > 0$, let $u_{x,\lambda}$ denote the reflection of u with respect to $B_\lambda(x)$, i.e.,

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|} \right)^{n-2} u \left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right).$$

By $g_u \in \Gamma_k^+$, we know that $g_{u_{x,\lambda}} \in \Gamma_k^+$ on $\overline{B_1^+}$ and $\sigma_s(A_{g_{u_{x,\lambda}}}^T) > 0$ for $1 \leq s \leq k-1$ and $\left\{ \frac{\partial \sigma_s(A_{g_{u_{x,\lambda}}}^T)}{\partial A_{\alpha\beta}^T} \right\}_{(n-1) \times (n-1)}$ is positively defined. In this paper, h_{g_u} is positive due to $\mathcal{B}_k^{g_u} > 0$ and $g_u \in \Gamma_k^+$.

For simplicity, we denote $u_\lambda(y) = u_{0,\lambda}(y)$.

For $x \in \partial \mathbb{R}_+^n$, define

$$\bar{\lambda}(x) := \sup \{ \mu > 0 \mid u_{x,\lambda} \leq u \text{ in } \overline{\mathbb{R}_+^n} \setminus B_\lambda(x), \quad \forall 0 < \lambda < \mu \}.$$

Lemma 7. *Assume that u is the solution to (1.5). Then, for any $x \in \partial \mathbb{R}_+^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda} \leq u \quad \text{on } \overline{\mathbb{R}_+^n} \setminus B_\lambda(x), \quad \forall 0 < \lambda < \lambda_0(x).$$

Proof. Without loss of generality, we assume that $x = 0$. As $u \in C^1$, there exists a positive constant r_0 such that for $0 < r < r_0$

$$\begin{aligned} \frac{d}{dr} (r^{(n-2)/2} u(r, \theta)) &= \frac{n-2}{2} r^{\frac{n-4}{2}} u + r^{\frac{n-2}{2}} u_r \\ (3.3) \quad &= r^{\frac{n-4}{2}} \left(\frac{n-2}{2} u + r u_r \right) > 0. \end{aligned}$$

For any λ satisfying $\lambda < |y| < r_0$,

$$\left| \frac{\lambda^2 y}{|y|^2} \right| \leq \frac{\lambda^2}{|y|} < |y| < r_0,$$

and then, by (3.3)

$$|y|^{(n-2)/2} u(|y|, \theta) > \left| \frac{\lambda^2 y}{|y|^2} \right|^{(n-2)/2} u \left(\left| \frac{\lambda^2 y}{|y|^2} \right|, \theta \right).$$

Therefore, for $0 < \lambda < |y| < r_0$,

$$(3.4) \quad u(y) > u_\lambda(y) = \left(\frac{\lambda}{|y|} \right)^{n-2} u \left(\frac{\lambda^2 y}{|y|^2} \right).$$

Then taking $\lambda_0 = \min \left\{ \left(\frac{\min_{\partial' B_{r_0}^+} u}{\max_{B_{r_0}^+} u} \right)^{\frac{1}{n-2}} r_0, r_0 \right\} \leq r_0$, for $0 < \lambda < \lambda_0$ and $|y| \geq r_0$,

$$u_\lambda(y) \leq \left(\frac{\lambda_0}{|y|} \right)^{n-2} \max_{B_{\frac{\lambda_0^2}{r_0}}^+} u \leq \left(\frac{\lambda_0}{|y|} \right)^{n-2} \max_{B_{r_0}^+} u \leq \frac{r_0^{n-2} \min_{\partial' B_{r_0}^+} u}{|y|^{n-2}} \leq u(y),$$

where the last inequality holds due to Lemma 6.

Combining with (3.4), we know that for $0 < \lambda < \lambda_0$ and $y \in \overline{\mathbb{R}_+^n} \setminus B_\lambda(x)$,

$$u_\lambda(y) \leq u(y).$$

□

From Lemma 7, we know $\bar{\lambda}(x) > 0$ and $\bar{\lambda}(x) \leq \infty$.

Lemma 8. *Under the assumption of (3.2), we have $\bar{\lambda}(x) < \infty$ for any $x \in \partial \mathbb{R}_+^n$.*

Proof. As $u_{0,1}$ can be extended to a positive continuous function near zero, we have

$$|x|^{2-n} u \left(\frac{x}{|x|^2} \right) \rightarrow \alpha_0 \quad \text{as } x \rightarrow 0.$$

And then there exists a positive constant α_1, r_1 such that

$$u(y) \leq \frac{\alpha_1}{|y|^{n-2}} \quad \text{for } |y| \geq r_1.$$

Because for any $x \in \partial \mathbb{R}_+^n$, $u(y) \geq u_{x,\lambda}(y)$ for all $\lambda < \bar{\lambda}(x)$ and $|y - x| \geq \lambda$, we have

$$\lambda^{n-2} u(x) = \lim_{y \rightarrow \infty} \lambda^{n-2} u \left(x + \frac{\lambda^2(y - x)}{|y - x|^2} \right) \leq \lim_{y \rightarrow \infty} |y - x|^{n-2} u(y) \leq \alpha_1.$$

Thus,

$$\bar{\lambda}(x) \leq C.$$

□

Lemma 9. *Given the same assumption as Theorem 5, we have that, for all $x \in \partial \mathbb{R}_+^n$,*

$$u_{x,\bar{\lambda}(x)} \equiv u \quad \text{in } \mathbb{R}_+^n \setminus \{x\}.$$

Proof. We argue by a contradiction argument: Without loss of generality, we take $x = 0$ and $u_{0,\bar{\lambda}} \not\equiv u$ on $\mathbb{R}_+^n \setminus \{0\}$. We know that $u_{\bar{\lambda}} \leq u$ on $\overline{\mathbb{R}_+^n} \setminus B_{\bar{\lambda}}$ by the definition of $\bar{\lambda}$.

We know $\sigma_k(A^{u_{\bar{\lambda}}}) = 2^k C_n^k$ on $\overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}^+}$ and $\mathcal{B}_k^{u_{\bar{\lambda}}} = c_0$ on $\partial\mathbb{R}_+^n \setminus B_{\bar{\lambda}}$. Letting $w = tu + (1-t)u_{\bar{\lambda}}$, we know that $u - u_{\bar{\lambda}}$ satisfies

$$\begin{cases} 0 = F(A^u) - F(A^{u_{\bar{\lambda}}}) = \mathbb{L}(u - u_{\bar{\lambda}}) & \text{in } \overline{\mathbb{R}_+^n} \setminus B_{\bar{\lambda}}, \\ 0 = \mathcal{B}_k^u - \mathcal{B}_k^{u_{\bar{\lambda}}} = L(u - u_{\bar{\lambda}}) & \text{on } \partial\mathbb{R}_+^n \setminus B_{\bar{\lambda}}, \end{cases}$$

where

$$\mathbb{L}\varphi := -A_{ij}\partial_{ij}\varphi + B_i\partial_i\varphi + C(x)\varphi,$$

$$A_{ij} := \frac{2}{n-2} \int_0^1 w^{-\frac{n+2}{n-2}} \frac{\partial\sigma_k}{\partial A_{ij}^w} (g_w^{-1}A^w) dt,$$

and $L\varphi$ is defined in (2.1) with $u_0 = u_{\bar{\lambda}}$ and $u_1 = u$ in (2.1).

Claim 1:

$$(3.5) \quad u - u_{\bar{\lambda}} > 0 \text{ on } \overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}}.$$

Proof of Claim 1: We argue by the contradiction and assume $u - u_{\bar{\lambda}}(\bar{x}) = 0$ for some $\bar{x} \in \overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}}$. It holds that

$$(3.6) \quad \begin{cases} \mathbb{L}(u - u_{\bar{\lambda}}) = 0 & \text{in } \overline{\mathbb{R}_+^n} \setminus B_{\bar{\lambda}}, \\ L(u - u_{\bar{\lambda}}) = 0 & \text{on } \partial\mathbb{R}_+^n \setminus B_{\bar{\lambda}}. \end{cases}$$

If $\bar{x} \in \mathbb{R}_+^n \setminus \overline{B_{\bar{\lambda}}}$, then by the strong maximum principle, $u - u_{\bar{\lambda}} = 0$ near \bar{x} , and furthermore, $u = u_{\bar{\lambda}}$ on $\overline{\mathbb{R}_+^n}$, which is contradicted to the assumption. If \bar{x} is on boundary, then by Hopf Lemma, $\partial_n(u - u_{\bar{\lambda}}) > 0$ at \bar{x} .

Moreover, at \bar{x} ,

$$\partial_{\alpha\beta}(u - u_{\bar{\lambda}}) \geq 0, \quad \partial_{\alpha}(u - u_{\bar{\lambda}}) = 0, \quad u - u_{\bar{\lambda}} = 0,$$

and then

$$\begin{aligned} 0 &= L(u - u_{\bar{\lambda}}) \\ &= -a_{\alpha\beta}\partial_{\alpha\beta}(u - u_{\bar{\lambda}}) + b_{\alpha}\partial_{\alpha}(u - u_{\bar{\lambda}}) - b_n\partial_n(u - u_{\bar{\lambda}}) + c(u - u_{\bar{\lambda}}) \\ &\leq -b_n\partial_n(u - u_{\bar{\lambda}}) < 0, \end{aligned}$$

which implies a contradiction. We have proved this claim.

Claim 2:

$$(3.7) \quad \lim_{y \in \overline{\mathbb{R}_+^n}, |y| \rightarrow \infty} |y|^{n-2} (u(y) - u_{\bar{\lambda}}(y)) > 0.$$

Proof of Claim 2: Let $x = y/|y|^2$, we have

$$|y|^{n-2}u(y) = u_{0,1}(x), \quad |y|^{n-2}u_{\bar{\lambda}}(y) = \bar{\lambda}^{n-2}u\left(\frac{\bar{\lambda}^2 y}{|y|^2}\right) = \bar{\lambda}^{n-2}u(\bar{\lambda}^2 x) =: v(x).$$

By (3.5), $u_{0,1} - v > 0$ in $B_{\frac{1}{\bar{\lambda}}}^+$. We prove that $(u_{0,1} - v)(0) > 0$ by a contradiction argument and then the Claim 2 will be proved. Otherwise, as $u_{0,1}(x)$ and $v(x)$ satisfy the equation (1.5), by (3.2) and Hopf Lemma, we know¹

$$\min_{B_{\frac{1}{\bar{\lambda}}}^+}(u_{0,1} - v) = (u_{0,1} - v)(0) = 0, \quad \bar{\nabla}(u_{0,1} - v)(0) = 0, \quad \bar{\nabla}^2(u_{0,1} - v)(0) \geq 0, \quad \partial_n(u_{0,1} - v)(0) > 0.$$

Thus, a contradiction follows from

$$\begin{aligned} 0 &= L(u_{0,1} - v)(0) \\ &= -a_{\alpha\beta}\partial_{\alpha\beta}(u_{0,1} - v) + b_{\alpha}\partial_{\alpha}(u_{0,1} - v) - b_n\partial_n(u_{0,1} - v) + c(u_{0,1} - v) \\ &\leq -b_n\partial_n(u_{0,1} - v) < 0. \end{aligned}$$

We have completed the proof of this claim 2.

As $u - u_{\bar{\lambda}} = 0$ on $\partial B_{\bar{\lambda}} \cap \mathbb{R}_+^n$, by the Hopf Lemma and Claim 1,

$$(3.8) \quad (u - u_{\bar{\lambda}})_{\nu} > 0,$$

where ν denotes the unit outer normal to $\partial B_{\bar{\lambda}} \cap \mathbb{R}_+^n$.

For any $\bar{x} \in \partial B_{\bar{\lambda}} \cap \partial \mathbb{R}_+^n$, taking $\Omega = (\mathbb{R}_+^n \setminus B_{\bar{\lambda}}) \cap B_1(\bar{x})$ and $\sigma = x_n$. For some positive A ,

$$\mathcal{P}(u - u_{\bar{\lambda}}) = \mathbb{L}(u - u_{\bar{\lambda}}) - C(x)(u - u_{\bar{\lambda}}) \geq -A(u - u_{\bar{\lambda}}), \quad \text{in } \Omega$$

and

$$\mathcal{L}(u - u_{\bar{\lambda}}) = L(u - u_{\bar{\lambda}}) - c(x)(u - u_{\bar{\lambda}}) \geq -A(u - u_{\bar{\lambda}}), \quad \text{on } \{\sigma = 0\} \cap \bar{\Omega}.$$

By Lemma 14 in Appendix, we know that $\frac{\partial}{\partial \nu}(u - u_{\bar{\lambda}})|_{x=\bar{x}} > 0$, where ν denotes the unit outer normal to $\partial B_{\bar{\lambda}}$ on $\partial \mathbb{R}_+^n$. Thus, we have

$$(3.9) \quad \frac{\partial}{\partial \nu}(u - u_{\bar{\lambda}}) > 0 \quad \text{on } \partial B_{\bar{\lambda}} \cap \partial \mathbb{R}_+^n,$$

where ν denotes the unit outer normal to $\partial B_{\bar{\lambda}}$ on $\partial \mathbb{R}_+^n$.

Therefore, from (3.8)(3.9), there exists a positive constant $b > 0$ such that

$$\frac{\partial}{\partial \nu}(u - u_{\bar{\lambda}}) > b > 0 \quad \text{on } \partial B_{\bar{\lambda}} \cap \overline{\mathbb{R}_+^n}.$$

¹This has used the assumption that $u_{0,1}(x)$ is C^2 continuous at 0.

By the continuity of ∇u , there exists a $R > \bar{\lambda}$ such that for $\bar{\lambda} \leq \lambda \leq R$, $\lambda \leq |x| \leq R$, we have

$$\left. \frac{\partial}{\partial \nu} (u - u_\lambda) \right|_x > \frac{b}{2},$$

where ν is the unit outer normal to ∂B_λ . For $\bar{\lambda} \leq \lambda \leq R$ and $\lambda < |y| \leq R$, as $u - u_\lambda = 0$ on ∂B_λ , from above,

$$(3.10) \quad u(y) - u_\lambda(y) > 0.$$

By (3.7), we know that there exists a positive constant $C_0 > 0$ such that for $|y| \geq R$ and $y_n \geq 0$,

$$u(y) - u_{\bar{\lambda}}(y) \geq \frac{C_0}{|y|^{n-2}}.$$

Furthermore, for $|y| \geq R$,

$$\begin{aligned} u(y) - u_\lambda(y) &\geq \frac{C_0}{|y|^{n-2}} - (u_\lambda - u_{\bar{\lambda}}) \\ &= \frac{C_0}{|y|^{n-2}} - \frac{1}{|y|^{n-2}} \left(\lambda^{n-2} u \left(\frac{\lambda^2 y}{|y|^2} \right) - \bar{\lambda}^{n-2} u \left(\frac{\bar{\lambda}^2 y}{|y|^2} \right) \right) \\ &\geq \frac{1}{|y|^{n-2}} \left[C_0 - \lambda^{n-2} u \left(\frac{\lambda^2 y}{|y|^2} \right) + \bar{\lambda}^{n-2} u \left(\frac{\bar{\lambda}^2 y}{|y|^2} \right) \right] \\ (3.11) \quad &\geq \frac{\frac{C_0}{2}}{|y|^{n-2}}, \end{aligned}$$

where the last inequality holds as we takes λ close to $\bar{\lambda}$. By (3.10) and (3.11), there exists some small $\varepsilon > 0$ such that for $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon$ and $|y| \geq \lambda$, we have $u(y) - u_\lambda(y) > 0$. It will yield a contradiction to the largest number of $\bar{\lambda}$. Therefore, $u_{\bar{\lambda}} = u$ in $\overline{\mathbb{R}_+^n} \setminus B_{\bar{\lambda}}$ and thus $u_{\bar{\lambda}} = u$ in $\overline{\mathbb{R}_+^n} \setminus \{0\}$. \square

Proof of Theorem 5. Now by Lemma 15 in Appendix and Lemma 9, we know that on $\partial \mathbb{R}_+^n$, there exist $a > 0, d > 0$ and $x'_0 \in \partial \mathbb{R}_+^n$ such that

$$(3.12) \quad u(x', 0) = \left(\frac{a}{d^2 + |x' - x'_0|^2} \right)^{\frac{n-2}{2}}.$$

Let $p = (x'_0, -d)$ and $q = (x'_0, d)$, and define

$$y = (y', y_n) = p + \frac{4d^2(z - p)}{|z - p|^2} : B_{2d}(q) \longrightarrow \mathbb{R}_+^n$$

and

$$v(z) := \left(\frac{2d}{|z - p|} \right)^{n-2} u \left(p + \frac{4d^2(z - p)}{|z - p|^2} \right).$$

Now v satisfies

$$(3.13) \quad \begin{cases} \sigma_k(A_{g_v}) = 2^k C_n^k & \text{in } B_{2d}(q), \\ \mathcal{B}_k^{g_v} = c_0 & \text{on } \partial B_{2d}(q). \end{cases}$$

Note that

$$\begin{aligned} 0 < y_n = -d + \frac{4d^2(z_n + d)}{|z - p|^2} &\Leftrightarrow |z - p|^2 < 4d(z_n + d) \\ &\Leftrightarrow |z' - x'_0|^2 + (z_n - d)^2 < 4d^2 \\ &\text{i.e. } |z - q| < 2d. \end{aligned}$$

On $\partial B_{2d}(q)$,

$$\begin{aligned} p + \frac{4d^2(z - p)}{|z - p|^2} &= (x'_0, -d) + \frac{4d^2(z - (x'_0, -d))}{|z - p|^2} \\ &= \left(x'_0 + \frac{4d^2(z' - x'_0)}{|z - p|^2}, -d + \frac{4d^2(z_n + d)}{|z - p|^2} \right) \\ &= \left(x'_0 + \frac{4d^2(z' - x'_0)}{|z - p|^2}, 0 \right) = (y', 0). \end{aligned}$$

Then, $\varphi(\partial B_{2d}(q)) = \partial \mathbb{R}_+^n$ where $\varphi(z) = p + \frac{4d^2(z - p)}{|z - p|^2}$. Moreover, when $z \in \partial B_{2d}(q)$, we have $y = (y', 0)$ and

$$\begin{aligned} (3.14) \quad v(z) &= \left(\frac{2d}{|z - p|} \right)^{n-2} u \left(p + \frac{4d^2(z - p)}{|z - p|^2} \right) \\ &= \left(\frac{2d}{|z - p|} \right)^{n-2} u(y', 0) \\ &= \left(\frac{2d}{|z - p|} \right)^{n-2} \left(\frac{a}{d^2 + |y' - x'_0|^2} \right)^{\frac{n-2}{2}} \\ &= a^{\frac{n-2}{2}} \left(\frac{2d}{|z - p|} \right)^{n-2} \frac{1}{|y - p|^{n-2}} \\ &= a^{\frac{n-2}{2}} (2d)^{2-n}, \end{aligned}$$

where we use (3.12) in the third equality and the last equality holds due to $|y - p||z - p| = 4d^2$ for $z \in \partial B_{2d}(q)$.

Actually, from (3.14) and $\mathcal{B}_k^{g_v} = c_0$ on $\partial B_{2d}(q)$, we obtain that $\langle \nabla v, \nu \rangle$ is constant on $\partial B_{2d}(q)$, where ν is the unit outer normal vector on $\partial B_{2d}(q)$. With constant Dirichlet and Neumann boundary on $\partial B_{2d}(q)$, the solution v to $\sigma_k(A_{g_v}) = 2^k C_n^k$ on $B_{2d}(q)$ is radial by the standard moving plane argument.

Then, by Theorem 17 in Appendix, we conclude that

$$v(z) = \left(\frac{\sqrt{b}}{1 + b|z - q|^2} \right)^{\frac{n-2}{2}}$$

satisfying (3.13). Thus, we get the form of (1.6) by transforming back to u , which means

$$u(y) = \left(\frac{2d}{|y - p|} \right)^{n-2} v \left(p + \frac{4d^2(y - p)}{|y - p|^2} \right).$$

□

4. PROOF OF THEOREM 1

In this section, we will utilize the beautiful lemma in [19] to relax the assumption of (3.2) and complete the proof of Theorem 1.

Denote $\alpha := \liminf_{x \in \overline{\mathbb{R}_+^n}, |x| \rightarrow +\infty} |x|^{n-2} u$.

Lemma 10. *We have $0 < \alpha < +\infty$, and*

$$\bar{\lambda}(x)^{n-2} u(x) = \alpha, \quad \forall x \in \partial \mathbb{R}_+^n.$$

Proof. By the definition of $\bar{\lambda}(x)$,

$$u_{x,\lambda} \leq u \quad \text{in } \overline{\mathbb{R}_+^n} \setminus B_\lambda(x), \quad \forall 0 < \lambda < \bar{\lambda}(x).$$

For $\forall 0 < \lambda < \bar{\lambda}(x)$,

$$\begin{aligned} \lambda^{n-2} u(x) &= \liminf_{y \rightarrow +\infty, y \in \overline{\mathbb{R}_+^n}} |y|^{n-2} \left(\frac{\lambda}{|y - x|} \right)^{n-2} u \left(x + \frac{\lambda^2(y - x)}{|y - x|^2} \right) \\ &\leq \liminf_{y \rightarrow +\infty, y \in \overline{\mathbb{R}_+^n}} |y|^{n-2} u(y) = \alpha. \end{aligned}$$

Then $\bar{\lambda}(x)^{n-2} u(x) \leq \alpha$ for $\forall x \in \partial \mathbb{R}_+^n$. Now we prove $\bar{\lambda}(x)^{n-2} u(x) = \alpha$ for $\forall x \in \partial \mathbb{R}_+^n$ by a contradiction argument.

We consider the case $\alpha < +\infty$.

Step 1: If there exists a $x \in \partial \mathbb{R}_+^n$ such that $\bar{\lambda}(x)^{n-2} u(x) < \alpha$, then

$$\begin{aligned} (4.1) \quad & \liminf_{y \rightarrow +\infty, y \in \overline{\mathbb{R}_+^n}} |y|^{n-2} (u(y) - u_{x,\bar{\lambda}(x)}(y)) \\ & \geq \alpha - \limsup_{y \rightarrow +\infty, y \in \overline{\mathbb{R}_+^n}} |y|^{n-2} \left(\frac{\bar{\lambda}(x)}{|y - x|} \right)^{n-2} u \left(x + \frac{\bar{\lambda}(x)^2(y - x)}{|y - x|^2} \right) \\ & = \alpha - \bar{\lambda}(x)^{n-2} u(x) > 0. \end{aligned}$$

Step 2: We prove that

$$(4.2) \quad u - u_{x, \bar{\lambda}(x)} > 0 \text{ in } \overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}(x)}}(x).$$

We argue by the contradiction argument and without loss of generality, assume $x = 0$ and $u - u_{\bar{\lambda}}(\bar{x}) = 0$ for some $\bar{x} \in \overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}}$. We know $\sigma_k(A^{u_{\bar{\lambda}}}) = 2^k C_n^k$ on $\overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}}$ and $\mathcal{B}_k^{u_{\bar{\lambda}}} = c_0$ on $\partial\mathbb{R}_+^n \setminus B_{\bar{\lambda}}$.

$$\begin{cases} 0 = F(A^u) - F(A^{u_{\bar{\lambda}}}) = \mathbb{L}(u - u_{\bar{\lambda}}), & \text{in } \overline{\mathbb{R}_+^n} \setminus \overline{B_{\bar{\lambda}}}, \\ 0 = \mathcal{B}_k^u - \mathcal{B}_k^{u_{\bar{\lambda}}} = L(u - u_{\bar{\lambda}}) & \text{on } \partial\mathbb{R}_+^n \setminus B_{\bar{\lambda}}. \end{cases}$$

If $\bar{x} \in \mathbb{R}_+^n \setminus \overline{B_{\bar{\lambda}}}$, then by the strong maximum principle, $u - u_{\bar{\lambda}} = 0$ near \bar{x} and then $u = u_{\bar{\lambda}}$ on $\overline{\mathbb{R}_+^n}$, which is contradicted to (4.1). If \bar{x} is on boundary, then

by Hopf Lemma, $\partial_n(u - u_{\bar{\lambda}}) \Big|_{\bar{x}} > 0$.

At \bar{x} ,

$$\partial_{\alpha\beta}(u - u_{\bar{\lambda}}) \geq 0, \quad \partial_{\alpha}(u - u_{\bar{\lambda}}) = 0, \quad u - u_{\bar{\lambda}} = 0$$

and then

$$\begin{aligned} 0 &= L(u - u_{\bar{\lambda}}) \Big|_{x=\bar{x}} \\ &= -a_{\alpha\beta} \partial_{\alpha\beta}(u - u_{\bar{\lambda}}) + b_{\alpha} \partial_{\alpha}(u - u_{\bar{\lambda}}) - b_n \partial_n(u - u_{\bar{\lambda}}) + c(u - u_{\bar{\lambda}}) \\ &\leq -b_n \partial_n(u - u_{\bar{\lambda}}) < 0, \end{aligned}$$

which yields a contradiction. We have proved (4.2).

With (4.1) and the same argument as the proof of Lemma 9, we know that it will yield a contradiction to the largest number of $\bar{\lambda}$. Thus

$$\bar{\lambda}(x)^{n-2} u(x) = \alpha.$$

We now consider $\alpha = +\infty$ in the following.

Step 3: By similar argument as above, it holds that

$$\bar{\lambda}(x) = +\infty, \quad \forall x \in \partial\mathbb{R}_+^n.$$

Furthermore,

$$u_{x,\lambda} \leq u \text{ on } \mathbb{R}_+^n \setminus B_{\lambda}(x) \quad \text{for } 0 < \lambda < +\infty.$$

By Lemma 16 in Appendix, it holds that $u(x', x_n) = u(0, x_n)$ for $x_n \geq 0$. By $Tr A^T > 0$ on $\partial\mathbb{R}_+^n$, we have the following contradiction:

$$\begin{aligned} 0 &< -\frac{2}{n-2} u^{-(n+2)/(n-2)} \bar{\Delta} u + \frac{2n}{(n-2)^2} u^{-2n/(n-2)} |\bar{\nabla} u|^2 - \frac{2(n-1)}{(n-2)^2} u^{-2n/(n-2)} |\nabla u|^2 \\ &= -\frac{2(n-1)}{(n-2)^2} u^{-2n/(n-2)} u_n^2 \leq 0 \text{ on } \partial\mathbb{R}_+^n. \end{aligned}$$

We have completed the proof of the Lemma. \square

Let us recall the essential lemma in [19].

Lemma 11. *[Li-Li, Acta Math. 2005] For $n \geq 2$, $u \in L^1_{loc}(\mathbb{B}_1^n \setminus \{0\})$ is the solution of $\Delta u \leq 0$ in $\mathbb{B}_1^n \setminus \{0\}$ in the distribution sense. Assume that there exists $a \in \mathbb{R}$ and $p \neq q \in \mathbb{R}^n$ such that*

$$u(x) \geq \max\{a + p \cdot x - \delta(x), a + q \cdot x - \delta(x)\}, \quad x \in \mathbb{B}_1^n \setminus \{0\},$$

where $\delta(x) \geq 0$ satisfies $\lim_{x \rightarrow 0} \frac{\delta(x)}{|x|} = 0$. Then

$$\liminf_{r \rightarrow 0} \inf_{B_r} u > a.$$

For simplicity, we denote $u_\psi := |J_\psi|^{\frac{n-2}{2n}} u \circ \psi$, where J_ψ denotes the Jacobian of ψ . Let us complete the proof of Theorem 1 as the following:

Theorem 12. *(Theorem 1) Assume that u is the solution to (1.5) and $\lim_{x \rightarrow 0} u_{0,1}$ exists. Then u is (1.6) stated in Theorem 1.*

Proof. Let $\varphi^{(x)}(y) = x + \frac{\bar{\lambda}(x)^2(y-x)}{|y-x|^2}$ and $z = \frac{y}{|y|^2} =: \psi(y)$. Note that $u_{x, \bar{\lambda}(x)}(y) = u_{\varphi^{(x)}}(y)$ and

$$u_{\varphi^{(x)} \circ \psi}(y) = \frac{1}{|y|^{n-2}} \frac{\bar{\lambda}(x)^{n-2}}{|\psi(y) - x|^{n-2}} u \left(x + \frac{\bar{\lambda}^2(x)(\psi(y) - x)}{|\psi(y) - x|^2} \right).$$

We have

$$(4.3) \quad \begin{aligned} u_{\varphi^{(x)} \circ \psi}(0) &= \lim_{y \rightarrow 0} u_{\varphi^{(x)} \circ \psi}(y) \\ &= \bar{\lambda}(x)^{n-2} u(x). \end{aligned}$$

Denote $w^{(x)}(y) := u_{\varphi^{(x)} \circ \psi}(y)$. It satisfies

$$\begin{cases} \sigma_k(A^{w^{(x)}}) = 2^k C_n^k & \text{in } \mathbb{R}_+^n, \\ \mathcal{B}_k^{w^{(x)}} = c_0 & \text{on } \partial \mathbb{R}_+^n \setminus \{\frac{x}{|x|^2}\}. \end{cases}$$

Also, $w^{(x)}(0) = \bar{\lambda}(x)^{n-2} u(x) = \alpha$ for $x \in \partial \mathbb{R}_+^n$. Meanwhile, $u_\psi(y) = \frac{1}{|y|^{n-2}} u(\frac{y}{|y|^2})$ satisfies

$$\begin{cases} \sigma_k(A^{u_\psi}) = 2^k C_n^k & \text{in } \mathbb{R}_+^n, \\ \mathcal{B}_k^{u_\psi} = c_0 & \text{on } \partial \mathbb{R}_+^n \setminus \{0\}. \end{cases}$$

Notice that $\lim_{y \rightarrow 0} u_\psi(y) = \alpha$ from the definition of α .

From the definition of $\bar{\lambda}(x)$,

$$u(y) \geq u_{\varphi^{(x)}}(y) \quad \text{on } \overline{\mathbb{R}_+^n} \setminus B_{\bar{\lambda}(x)}(x) \quad \text{for } x \in \partial \mathbb{R}_+^n.$$

Thus, $u_\psi(z) \geq u_{\varphi(x) \circ \psi}(z)$ for $y = \frac{z}{|z|^2} \in \mathbb{R}_+^n \setminus B_{\bar{\lambda}(x)}(x)$. Furthermore, there exists $\delta(x) > 0$ depending on $\bar{\lambda}(x), x$ such that

$$u_\psi(z) \geq w^{(x)}(z) \text{ in } \overline{B_{\delta(x)}^+} \setminus \{0\}.$$

As $w^{(x)}$ is C^2 near 0 and u_ψ is C^2 away from 0 in $\overline{\mathbb{R}_+^n} \setminus \{0\}$, we know that

$$u_\psi(z) \geq w^{(x)}(z) \text{ in } B_{\delta(x)}^\top \setminus \{0\} := \{x | x = (x', 0), |x'| \leq \delta(x)\} \setminus \{0\},$$

and

$$\frac{2}{n-2} \ln u_\psi(z) \geq \frac{2}{n-2} \ln w^{(x)}(z) \text{ in } B_{\delta(x)}^\top \setminus \{0\}.$$

Taking $W := \frac{2}{n-2} \ln u_\psi$, from $A^u \in \Gamma_k^+$, we know that

$$A[W] := -\nabla_{ij}W + \nabla_i W \nabla_j W - \frac{|\nabla W|^2}{2} \delta_{ij} \in \Gamma_2^+.$$

It yields that

$$-\bar{\Delta}W + |\bar{\nabla}W|^2 - (n-1) \frac{|\nabla W|^2}{2} > 0 \quad \text{on } \partial\mathbb{R}_+^n,$$

which is due to $\text{Tr}A[W]^\top > 0$. From above, we know that $-\bar{\Delta}W > 0$ in $B_{\delta(x)}^\top \setminus \{0\}$ for $n \geq 3$.

From Lemma 11, we know that there exists a constant vector $\vec{l} \in \mathbb{R}^{n-1}$ such that

$$\bar{\nabla} \ln w^{(x)}(0) = \vec{l} \text{ for } \forall x \in \partial\mathbb{R}_+^n,$$

otherwise, there exist two points x_1 and x_2 such that $\bar{\nabla} \ln w^{(x_1)}(0) \neq \bar{\nabla} \ln w^{(x_2)}(0)$ and thus, by Lemma 11 and (4.3), $\lim_{z \rightarrow 0, z \in \partial\mathbb{R}_+^n} W > \frac{2}{n-2} \ln \alpha$, which is contradicted to the fact $\lim_{z \rightarrow 0} W = \frac{2}{n-2} \ln \alpha$.

Then there exists a constant vector \vec{V} such that $\bar{\nabla} w^{(x)}(0) = \vec{V}$ for $x \in \partial\mathbb{R}_+^n$. For $|y|$ small,

$$\begin{aligned} w^{(x)}(y) &= \bar{\lambda}(x)^{n-2} (1 + (n-2)x \cdot y + O(|y|^2)) u(x + \bar{\lambda}(x)^2 y + O(|y|^2)) \\ &= \bar{\lambda}(x)^{n-2} (1 + (n-2)x \cdot y) u(x + \bar{\lambda}(x)^2 y) + O(|y|^2), \end{aligned}$$

and by Lemma 10, we have

$$\begin{aligned} \bar{\nabla} w^{(x)}(0) &= (n-2) \bar{\lambda}(x)^{n-2} u(x) x' + \bar{\lambda}(x)^n \bar{\nabla} u(x) \\ &= (n-2) \alpha x' + \alpha^{n/(n-2)} u(x)^{n/(2-n)} \bar{\nabla} u(x) \\ &= \vec{V}. \end{aligned}$$

Therefore, there exists a constant d such that

$$\frac{n-2}{2} \alpha^{n/(n-2)} u^{-\frac{2}{n-2}} = -\vec{V} \cdot x' + \frac{1}{2} (n-2) \alpha |x'|^2 + d \text{ for } x \in \partial\mathbb{R}_+^n.$$

Now

$$\begin{aligned} u(x', 0) &= \left(\frac{-\vec{V} \cdot x' + \frac{1}{2}(n-2)\alpha|x'|^2 + d}{\frac{n-2}{2}\alpha^{n/(n-2)}} \right)^{-\frac{n-2}{2}} \\ &= \alpha \left(\frac{1}{|x' - x'_0|^2 + d_1^2} \right)^{\frac{n-2}{2}}, \end{aligned}$$

where $u > 0$ and d_1 is a constant.

For simplicity, take $x'_0 = 0$. Then

$$u(0) = \alpha d_1^{-(n-2)} = \bar{\lambda}(0)^{n-2} u(0) d_1^{-(n-2)}$$

and now $d_1 = \bar{\lambda}(0)$.

Let $u_{\bar{\lambda}}(y) = \left(\frac{\bar{\lambda}}{|y|}\right)^{n-2} u\left(\frac{\bar{\lambda}^2 y}{|y|^2}\right)$, where $\bar{\lambda} = \bar{\lambda}(0)$ and

$$\begin{aligned} u_{\bar{\lambda}}(x', 0) &= \alpha \left(\frac{\bar{\lambda}}{|x'|} \right)^{n-2} \left(\frac{1}{\bar{\lambda}^4/|x'|^2 + d_1^2} \right)^{\frac{n-2}{2}} \\ &= u(x', 0) \text{ for } x \in \partial\mathbb{R}_+^n. \end{aligned}$$

Also we know that

$$\begin{cases} \sigma_k(A^u) = 2^k C_n^k & \text{in } \overline{\mathbb{R}_+^n}, g_u \in \Gamma_k^+, \\ \sigma_k(A^{u_{\bar{\lambda}}}) = 2^k C_n^k & \text{in } \overline{\mathbb{R}_+^n} \setminus \{0\}, g_{u_{\bar{\lambda}}} \in \Gamma_k^+, \\ \mathcal{B}_k^u = c_0 & \text{on } \partial\mathbb{R}_+^n, \\ \mathcal{B}_k^{u_{\bar{\lambda}}} = c_0 & \text{on } \partial\mathbb{R}_+^n \setminus \{0\}, \\ u_{\bar{\lambda}}(x', 0) = u(x', 0) & \text{on } \partial\mathbb{R}_+^n \setminus \{0\}, \\ u - u_{\bar{\lambda}} \geq 0 & \text{in } \mathbb{R}_+^n \setminus B_{\bar{\lambda}}. \end{cases}$$

By the strong maximum principle, we know that $u = u_{\bar{\lambda}}$ on \mathbb{R}_+^n and now u satisfies (3.2). Then from Theorem 5, we get Theorem 1. \square

Remark 13. As we use Lemma 11 on $\partial\mathbb{R}_+^n$ to obtain a contradiction, we need to know the relationship between $\lim_{z \rightarrow 0, z \in \partial\mathbb{R}_+^n} W$ and $\lim_{z \rightarrow 0, z \in \overline{\mathbb{R}_+^n}} W$. If we assume that

$$\lim_{z \rightarrow 0, z \in \partial\mathbb{R}_+^n} W \leq \lim_{z \rightarrow 0, z \in \overline{\mathbb{R}_+^n}} W (= \frac{2}{n-2} \ln \alpha), \text{ then we can still have the conclusion.}$$

5. APPENDIX

In this appendix, we prove a corner Hopf Lemma for second order boundary condition by mimicking the proof of Li-Zhang [23] and list several lemmas in Li-Li [20] for reader's convenience.

5.1. Corner Hopf Lemma. Let $\{A_{ij}(x)\}_{n \times n}$ and $\{C_{\alpha\beta}(x)\}_{(n-1) \times (n-1)}$ be two positive function matrices such that there exist positive constants $\lambda_1, \lambda_2, A_1, A_2$ such that

$$\begin{aligned}\lambda_1 \delta_{ij} &\leq \{A_{ij}\}_{n \times n} \leq A_1 \delta_{ij}, \\ \lambda_2 \delta_{\alpha\beta} &\leq \{C_{\alpha\beta}\}_{(n-1) \times (n-1)} \leq A_2 \delta_{\alpha\beta}.\end{aligned}$$

For any $\bar{x} \in \partial B_\lambda \cap \partial \mathbb{R}_+^n$, taking $\Omega = (\mathbb{R}_+^n \setminus B_\lambda) \cap B_1(\bar{x})$, $\sigma = x_n$ and $\rho = |x|^2 - \lambda^2$. Let \vec{n} be the unit inward normal vector of the surface $\{x_n = 0\} \cap \partial \Omega$. The proof of the following theorem is almost same as Li-Zhang [23] and we omit it.

Lemma 14. *Let $u \in C^2(\bar{\Omega})$ be a positive function in Ω , $u(0) = 0$ and there exists a positive constant A such that*

$$\begin{cases} \mathcal{P}u := -A_{ij}u_{ij} + B_i u_i \geq -Au & \text{in } \Omega \\ \mathcal{L}u := -C_{\alpha\beta} \partial_{\alpha\beta} u + D_\alpha \partial_\alpha u - C_0 u_n \geq -Au & \text{on } \{\sigma = 0, \rho > 0\}, \end{cases}$$

where C_0 is a positive function and D_α are functions. Then

$$\frac{\partial u}{\partial \nu'}(0) > 0,$$

where ν' is the unit normal vector on $\{\sigma = 0, \rho = 0\}$ entering $\{\sigma = 0, \rho > 0\}$.

5.2. Useful Lemma. For reader's convenience, we list some classical Lemmas in [19, 20, 23].

Lemma 15. [Li-Zhang, J. Anal. Math. 2003] Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1, l > 0$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that

$$\left(\frac{\lambda(x)}{|y-x|} \right)^l f \left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2} \right) = f(y) \quad \text{for } y \in \mathbb{R}^n \setminus \{x\}.$$

Then for some $a \geq 0, d > 0, \bar{x} \in \mathbb{R}^n$,

$$f(x) = \pm \left(\frac{a}{d^2 + |x - \bar{x}|^2} \right)^{\frac{l}{2}}.$$

Lemma 16. [Li-Zhang, J. Anal. Math. 2003] Let $f \in C^1(\mathbb{R}_+^n)$, $n \geq 2, \nu > 0$. Assume that

$$\left(\frac{\lambda}{|y-x|} \right)^\nu f \left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right) \leq f(y), \quad \forall \lambda > 0, x \in \partial \mathbb{R}_+^n, |y-x| \geq \lambda, y \in \mathbb{R}_+^n.$$

Then

$$f(x) = f(x', t) = f(0, t), \quad \forall x = (x', t) \in \mathbb{R}_+^n.$$

Lemma 15 and Lemma 16 can be found in the appendix of [23]. The following lemma is about the classification of radial solution, which can be found in [20] for a more general operator including σ_k .

Theorem 17. [Li-Li, J. Eur. Math. Soc. 2006] For $n \geq 3$, assume that $u \in C^2(\mathbb{B}_1^n)$ is radially symmetric and satisfies

$$\sigma_k^{\frac{1}{k}}(A^u) = 1, \quad A^u \in \Gamma_k^+, \quad u > 0 \quad \text{in } \mathbb{B}_1^n.$$

Then

$$u(x) \equiv \left(\frac{a}{1 + b|x|^2} \right)^{(n-2)/2} \quad \text{in } \mathbb{B}_1^n,$$

where $a > 0, b \geq -1$ and $\sigma_k^{\frac{1}{k}}((2b/a^2)\mathbb{I}_{n \times n}) = 1$.

REFERENCES

- [1] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semi-linear equations with critical Sobolev growth. *Comm. Pure Appl. Math.* 42 (1989), 271–297.
- [2] J. Case, A. C. Moreira, Y. Wang, Nonuniqueness for a fully nonlinear boundary Yamabe-type problem via bifurcation theory. *Calc. Var. Partial Differential Equations* 58 (2019), no.3, Paper No. 106, 32 pp.
- [3] J. Case, Y. Wang, Boundary operators associated to the σ_k -curvature. *Adv. Math.* 337(2018), 83–106.
- [4] J. Case, Y. Wang, Towards a Fully Nonlinear Sharp Sobolev Trace Inequality. *J. Math. Study* 53 (2020), 402–435.
- [5] A. Chang, S. Chen, On a fully non-linear PDE in conformal geometry. *Mat. Enseñ. Univ. (N. S.)* 15 (2007), suppl., 17–36.
- [6] A. Chang, M. Gursky, P. Yang, An a priori estimate for a fully nonlinear equation on four-manifolds. *J. Anal. Math.* 87 (2002), 151–186.
- [7] A. Chang, M. Gursky, P. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature. *Ann. of Math. (2)* 155 (2002), no. 3, 709–787.
- [8] A. Chang, M. Gursky, P. Yang, Entire solutions of a fully nonlinear equation. *Lectures on partial differential equations*, 43–60. *New Stud. Adv. Math.*, Somerville, MA, 2003.
- [9] S. Chen, Conformal deformation on manifolds with boundary. *Geom. Funct. Anal.* 19 (2009), no.4, 1029–1064.
- [10] X. Z. Chen, L. M. Sun, Existence of conformal metrics with constant scalar curvature and constant boundary mean curvature on compact manifolds. *Commun. Contemp. Math.* 21 (2019), no.3, 1850021, 51 pp.
- [11] M. Chipot, I. Shafrir, M. Fila, On the solutions to some elliptic equations with nonlinear Neumann boundary conditions. *Adv. Differential Equations* 1 (1996), 91–110.
- [12] B. Z. Chu, Y. Y. Li, Z. Y. Li, Liouville theorems for conformally invariant fully nonlinear equations. I. *arXiv:2311.07542*.
- [13] J. F. Escobar, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate. *Comm. Pure Appl. Math.* 43 (1990), no.7, 857–883.
- [14] H. Fang, B. Ma, W. Wei, A Liouville’s theorem for some Monge-Ampère type equations. *J. Funct. Anal.* 285 (2023), no. 4, Paper No. 109973, 47 pp.
- [15] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle. *Commun Math. Phys.* 68 (1979), 209–243.

- [16] Q. Jin, Local Hessian estimates for some conformally invariant fully nonlinear equations with boundary conditions. *Differential Integral Equations* 20 (2007), no. 2, 121–132.
- [17] Q. Jin, A. B. Li, Y. Y. Li, Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary. *Calc. Var. Partial Differential Equations* 28 (2007), no. 4, 509–543.
- [18] A. B. Li, Y. Y. Li, On some conformally invariant fully nonlinear equations. *Comm. Pure Appl. Math.* 56 (2003), 1416–1464.
- [19] A. B. Li, Y. Y. Li, On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe. *Acta Math.* 195 (2005), 117–154.
- [20] A. B. Li, Y. Y. Li, A fully nonlinear version of the Yamabe problem on manifolds with boundary. *J. Eur. Math. Soc.* 8 (2006), no.2, 295–316.
- [21] Y.Y. Li, H. Lu, S. Lu, A Liouville theorem for Möbius invariant equations. *Peking Mathematical Journal* 6 (2021), 609–634.
- [22] Y.Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres. *Duke Math* 80 (1995), 383–417.
- [23] Y. Y. Li, L. Zhang, Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations. *J. Anal. Math.* 90 (2003), 27–87.
- [24] M. Obata, The conjecture on conformal transformations of Riemannian manifolds. *J. Differ. Geom.* 6 (1971), 247–258.
- [25] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations. *Duke Math. J.* 101 (2000), no. 2, 283–316.
- [26] J. Viaclovsky, Conformally invariant Monge-Ampère equations: global solutions. *Trans. Amer. Math. Soc.* 352 (2000), no. 9, 4371–4379.

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