THE ℓ_1 DOUBLE-BUBBLE PROBLEM IN THREE DIMENSIONS

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Abstract. We characterize the unique minimizer of the three-dimensional double-bubble problem with respect to the ℓ_1-norm for volume ratios between 1/2 and 2.

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1. Introduction

The double-bubble problem consists in determining the optimal pair of sets of given volumes minimizing the total surface. In the classical Euclidean setting, optimal configurations are pairs of regions enclosed by three spherical caps, meeting at a $2\pi/3$ angle. This was first proved in the planar case in [11], then extended in [15] to three dimensions, and finally to all dimensions in [22]. Besides the Euclidean case, double-bubble problems have been considered in a variety of different settings, including hyperbolic spaces [4, 6, 7, 18], hyperbolic surfaces [2] and cones [16, 20], the three-dimensional torus [3, 5], the Gauß space [4, 19], and the anisotropic Grushin plane [12].

This note is concerned with the three-dimensional double-bubble problem for the ℓ_1-norm. Given $v \in \mathbb{R}^3$, we denote by $|v|_1$ its ℓ_1-norm

$$|v|_1 = |v_1| + |v_2| + |v_3|$$

(we will later use the same notation for the ℓ_1-norm of vectors in $\mathbb{R}^2$). For any $\mathcal{H}^2$-rectifiable subset $F \subset \mathbb{R}^3$, we denote the corresponding ℓ_1-surface by

$$\ell_1(F) = \int_F |\nu_F|_1 \, \mathrm{d}\mathcal{H}^2,$$

where ν_F denotes the measure-theoretical normal to F and $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. We consider sets of finite perimeter $G \subset \mathbb{R}^3$ [1] and use the fact that their so-called *reduced boundary* $\partial^* G$ is a $\mathcal{H}^2$-rectifiable set. To each configuration (A, B) consisting of two three-dimensional sets of finite perimeter, we associate the energy

$$E(A, B) := \ell_1(\partial^* A) + \ell_1(\partial^* B) - \ell_1(\partial^* A \cap \partial^* B).$$

This corresponds to the ℓ_1-surface of the set $A \cup B$ together with the ℓ_1-interface between the sets A and B. In the following, we let the volumes $V_A := \mathcal{L}^3(A)$ and $V_B := \mathcal{L}^3(A)$ be fixed, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. Our main assumption is that the ratio V_B/V_A belongs to $[1/2, 2]$.

The *double-bubble problem* hence corresponds to

$$\min \left\{ E(A, B) : \quad A,\, B \subset \mathbb{R}^3 \text{ of finite perimeter},\ A \cap B = \emptyset,\ \mathcal{L}^3(A) = V_A,\ \mathcal{L}^3(B) = V_B \right\}. \tag{1.1}$$

Our main result reads as follows.

Theorem 1.1 (Characterization of the minimizer)**.** *Letting $V_B/V_A \in [1/2, 2]$, the unique minimizer of the double-bubble problem (1.1) are two cuboids sharing a square face. Up to translation and axis-preserving isometries, the minimizer can be specified as*

$$A = \Big[-\frac{V_A}{(2(V_A+V_B)/3)^{2/3}}, 0 \Big] \times \Big[0, (2(V_A+V_B)/3)^{1/3} \Big]^2,$$
$$B = \Big[0, \frac{V_B}{(2(V_A+V_B)/3)^{2/3}} \Big] \times \Big[0, (2(V_A+V_B)/3)^{1/3} \Big]^2.$$

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The minimal energy is given by

$$E(A,B) = \left(3\left(\frac{2}{3}\right)^{2/3} + 4\left(\frac{3}{2}\right)^{1/3}\right)(V_A+V_B)^{2/3}.$$

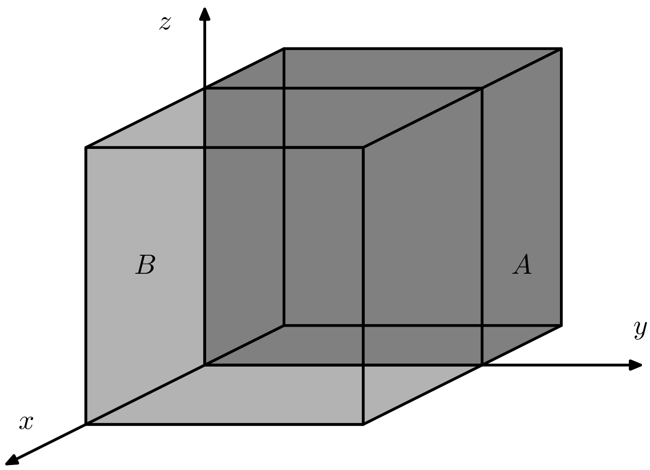

FIGURE 1. The unique minimizer of the double-bubble problem (1.1).

The minimality of the configuration in Theorem 1.1 for this specific volume-ratio range has already been conjectured by WECHT, BARBER, & TICE [23]. In fact, by reducing the problem to cuboids, the optimality of (A,B) from Theorem 1.1 easily follows. Our aim here is to provide a proof of this conjecture, starting from the most general setting of disjoint sets of finite perimeter. Note that, for volume ratios r smaller than $1/2$ or bigger than 2, the configuration in Figure 1 can be easily proved to be not optimal and the occurrence of different optimal configurations is conjectured [23].

In the *planar case*, the characterization of optimal double-bubble configurations with respect to the ℓ_1-norm in $\mathbb{R}^2$ is already well-known. The emergence of three different minimizers, depending on the volume ratio, has been discussed by MORGAN, FRENCH, & GREENLEAF [21]. A new proof of these results, based on different tools, has been recently presented by DUNCAN, O'DWYER, & PROCACCIA [8]. The reach of the theory has been extended to the general setting of finite perimeter sets and to arbitrary interaction intensity in [14]. The continuous problem in $\mathbb{R}^2$ is naturally connected with its discrete analogue on the $\mathbb{Z}^2$-lattice, which has also been studied [9, 13]. We further refer to [10] for an analogous problem in the hexagonal norm.

To our knowledge, our result is the first rigorous one for the ℓ_1 double-bubble problem in three dimensions. In fact, our arguments build on the available understanding of the planar case by means of a *slicing* argument. We slice the minimizing configuration with respect to a specific axis direction and we bound the 3D energy E in terms of an integral of the planar energies of the slices. Moving from the knowledge of the exact value of the 2D minimal energy, see Proposition 2.2, this slicing approach allows us to obtain an estimate of the minimal 3D energy E, see Proposition 2.3. This eventually turns out to completely characterize optimal configurations.

The remainder of the paper is devoted to proving Theorem 1.1. In particular, the proof of Theorem 1.1 is given in Section 2, based on a few technical lemmas. These lemmas are then proved in Section 3.

2. PROOF OF THE MAIN RESULT

As mentioned in the Introduction, the core step of the proof of Theorem 1.1 is that of estimating from below the minimal value of the energy E by taking advantage of the characterization of minimizers in the planar case. We hence start by recalling the 2D result in Subsection 2.1. We then collect some notation and present a crucial optimal bound in Subsection 2.2. After stating some technical lemmas in Subsection 2.3, the actual proof of Theorem 1.1 is given in Subsection 2.4. Eventually, the technical lemmas from Subsection 2.3 are proved in Section 3.

2.1. **The planar case.** Let us start by recalling the 2D result. Given a planar finite perimeter set $F_{2D} \subset \mathbb{R}^2$, we denote by $\partial^*_{2D} F_{2D} \subset \mathbb{R}^2$ its *planar reduced boundary*, and by $\nu = (\nu_1, \nu_2)$ the corresponding (measure-theoretic) *planar outer unit normal*. For all $\mathcal{H}^1$-rectifiable subsets $\varphi \subset \partial^*_{2D} F_{2D}$, we denote by

$$\ell_{1,2D}(\varphi) = \int_\varphi (|\nu_1| + |\nu_2|) \, \mathrm{d}\mathcal{H}^1$$

its length with respect to the ℓ_1-norm in the plane.

We indicate the *minimal energy* of a planar double bubble with regions of fixed areas a, $b > 0$ as

$$\begin{aligned} E_{2D}(a,b) := \min \Big\{ & \ell_{1,2D}(\partial^*_{2D} A_{2D}) + \ell_{1,2D}(\partial^*_{2D} B_{2D}) - \ell_{1,2D}(\partial^*_{2D} A_{2D} \cap \partial^*_{2D} B_{2D}) : \\ & A_{2D},\, B_{2D} \subset \mathbb{R}^2 \text{ of finite perimeter with} \\ & A_{2D} \cap B_{2D} = \emptyset,\ \mathcal{L}^2(A_{2D}) = a,\ \mathcal{L}^2(B_{2D}) = b \Big\}. \end{aligned} \tag{2.1}$$

Define now the value

$$r_* = \left(\frac{4(\sqrt{2}-1)}{1+2\sqrt{2}} \right)^2 \sim 0.1872957155. \tag{2.2}$$

The main result in the planar case is the following [9, 14].

Proposition 2.1 (Characterization of the planar minimizer)**.** *Up to translations and axis-preserving isometries, the configurations (A_{2D}, B_{2D}) realizing the minimum in (2.1) are given by*

- *Case $a/b \in [1/2, 1]$*

$$A_{2D} = [-a/c, 0] \times [0, c], \quad B_{2D} = [0, b/c] \times [0, c] \quad \textit{with} \ \ c = \sqrt{\frac{2(a+b)}{3}},$$

  and corresponding energy $E_{2D}(a,b) = 2\sqrt{6}\sqrt{a+b}$;
- *Case $a/b \in [r_*, 1/2]$*

$$A_{2D} = [-a/c, 0] \times [0, c] + (0, \lambda), \quad B_{2D} = [0, \sqrt{b}]^2 \quad \textit{with} \ \ c = \sqrt{2a}$$

  for some $\lambda \in [0, \sqrt{b} - c]$, and corresponding energy $E_{2D}(a,b) = 2\sqrt{2a} + 4\sqrt{b}$;
- *Case $a/b \in (0, r_*]$*

$$A_{2D} = [0, \sqrt{a}]^2, \quad B_{2D} = [0, \sqrt{a+b}]^2 \setminus A_{2D},$$

  with corresponding energy $E_{2D}(a,b) = 4\sqrt{a+b} + 2\sqrt{a}$.

An illustration of the three types of minimizers of the planar double-bubble problem is given in Figure 2.

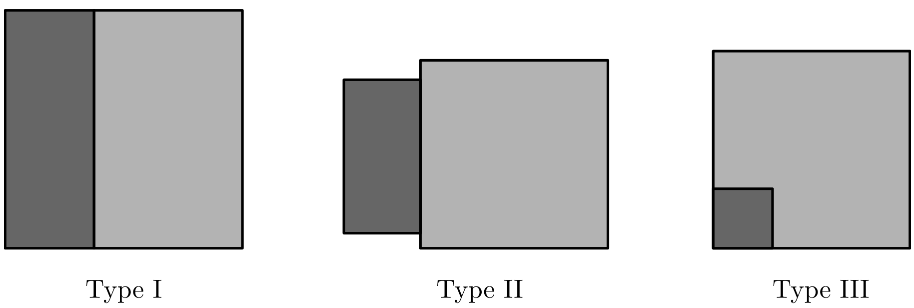

FIGURE 2. Minimizers for the planar double-bubble problem (2.1).

2.2. **Notation and optimal lower bound.** We start by introducing some notation used throughout the rest of the paper.

At first, let us consider the specific geometry of Figure 1. It consists of the union of two cuboids sharing a square face. Indicating by M the area of such a shared face, the cuboids have a square cross section of sidelength $\sqrt{M}$ and have heigth V_A/M and V_B/M, respectively. Within this specific class of configurations, one can identify the minimal value of the energy by simply computing

$$E_{\min} := \min_{M>0} \Big(3M + \frac{4(V_A + V_B)}{\sqrt{M}} \Big). \tag{2.3}$$

Indeed, such configuration features seven faces: three squares with area M and four rectangular faces with sidelengths $\sqrt{M}$ and $(V_A + V_B)/M$.

Let now (A, B) be a pair of disjoint sets of finite perimeter in $\mathbb{R}^3$, not necessarily being cuboids, satisfying

$$r := \frac{V_B}{V_A} \in [1/2, 2].$$

Choose a plane spanned by two coordinate directions, and denote the area of the orthogonal projection of $A \cup B$ onto the plane by m, as well as the area of the projection of the two sets A and B by m_A and m_B, respectively. We further set

$$p := \frac{m_A + m_B}{m} - 1. \tag{2.4}$$

The value p describes the size of the overlap of the projections of A and B onto the chosen plane. Up to redefining the axes, in the following we assume that such plane is given by $\mathbb{R}^2 \times \{0\}$, so that the projection occurs in the z direction.

We consider horizontal slices $\mathbb{R}^2 \times \{t\}$ and set

$$a(t) = \mathcal{L}^2\big(A \cap (\mathbb{R}^2 \times \{t\})\big), \quad b(t) = \mathcal{L}^2\big(B \cap (\mathbb{R}^2 \times \{t\})\big).$$

Fubini's Theorem ensures that

$$V_A = \int_{\mathbb{R}} a(t)\,\mathrm{d}t, \qquad V_B = \int_{\mathbb{R}} b(t)\,\mathrm{d}t. \tag{2.5}$$

Let $\mathcal{T}_0 = \{t\colon ra(t) = b(t) > 0\}$, $\mathcal{T}_A = \{t\colon ra(t) > b(t)\}$ and $\mathcal{T}_B = \{t\colon ra(t) < b(t)\}$. For convenience, we define

$$U_A = \int_{\mathcal{T}_A} (a(t) + b(t))\,\mathrm{d}t, \quad U_B = \int_{\mathcal{T}_B} (a(t) + b(t))\,\mathrm{d}t, \quad U_0 = \int_{\mathcal{T}_0} (a(t) + b(t))\,\mathrm{d}t.$$

Clearly, $V_A + V_B = U_A + U_B + U_0$. For $t \in \mathcal{T}_A$, we set $\alpha(t) = b(t)/a(t) \in [0, r)$ and for $t \in \mathcal{T}_B$, we set $\beta(t) = a(t)/b(t) \in [0, 1/r)$. Since $r\mathcal{L}^3(A) = rV_A = V_B = \mathcal{L}^3(B)$, we have by the definition of $\mathcal{T}_0$

$$r \int_{\mathcal{T}_A \cup \mathcal{T}_B} a(t)\,\mathrm{d}t = \int_{\mathcal{T}_A \cup \mathcal{T}_B} b(t)\,\mathrm{d}t$$

and therefore

$$\int_{\mathcal{T}_A} a(t)(r - \alpha(t))\,\mathrm{d}t = \int_{\mathcal{T}_A} (ra(t) - b(t))\,\mathrm{d}t = \int_{\mathcal{T}_B} (b(t) - ra(t))\,\mathrm{d}t = \int_{\mathcal{T}_B} b(t)(1 - r\beta(t))\,\mathrm{d}t. \tag{2.6}$$

These definitions allow us to restate the result in the planar case from Proposition 2.1 as follows.

Proposition 2.2 (Minimal planar energy)**.** *Recall the definition of r_* in formula (2.2) and define the function $f\colon [0,\infty) \to \mathbb{R}$ by*

$$f(x) = \Big(4 + 2\sqrt{\frac{x}{x+1}}\Big)\chi_{[0,r_*]} + \frac{4 + 2\sqrt{2x}}{\sqrt{x+1}}\chi_{(r_*,1/2]} + 2\sqrt{6}\chi_{(1/2,1]}$$

for $x \in [0, 1]$ and by $f(x) = f(1/x)$ for $x > 1$. Then, for $t \in \mathcal{T}_A$ we have

$$E_{2D}(a(t), b(t)) = \sqrt{a(t) + b(t)}\,f(\alpha(t)),$$

for $t \in \mathcal{T}_B$ we have

$$E_{2D}(a(t), b(t)) = \sqrt{a(t) + b(t)}\,f(\beta(t)),$$

and for $t \in \mathcal{T}_0$ we have

$$E_{2D}(a(t), b(t)) = 2\sqrt{6}\sqrt{a(t) + b(t)}.$$

Along the proof of Theorem 1.1, we make use of the explicit values of the minimal energy in the planar case in order to estimate E by considering horizontal slices with respect to a well-chosen coordinate direction. We assume that the parameter p given by (2.4) describing the overlap between projections of the two sets A and B in the z direction satisfies $p \leq 1/3$. In fact, Lemma 2.5 shows that this is not restrictive, up to possibly relabeling the axes. The core of the proof of Theorem 1.1 consists in the following claim.

Proposition 2.3 (Optimal lower bound)**.** *Let (A, B) be any configuration of disjoint sets with finite perimeter, with $r\mathcal{L}^3(A) = rV_A = V_B = \mathcal{L}^3(B)$ for $r \in [1/2, 2]$. Suppose that $p \leq 1/3$. Then,*

$$E(A, B) \geq (2 + p)m + \frac{4\sqrt{6}}{\sqrt{4 + 2p}\sqrt{m}}(U_A + U_B) + \frac{2\sqrt{6}}{\sqrt{m}}U_0. \tag{2.7}$$

Moreover, the equality in (2.7) *is attained if and only if $\mathcal{L}^1(\mathcal{T}_A) = \mathcal{L}^1(\mathcal{T}_B) = 0$ and $a(t) + b(t) = m$ for $t \in \mathcal{T}_0 \setminus \mathcal{N}$, where $\mathcal{N}$ is a set of negligible $\mathcal{L}^1$-measure.*

Moving from the optimal lower bound (2.7), the proof of Theorem 1.1 follows by proving that the configuration in Figure 1 is the only one (up to translations and axis-preserving isometries) realizing the equality case. This in particular follows by checking that actually $p = 0$ for the minimizer, so that the two sets A and B have disjoint projections.

Proposition 2.3 is proved in Subsection 2.4 below. As a preparation, in the next subsection we state some auxiliary results whose proofs are postponed to Section 3.

2.3. **Statements of auxiliary results.** First, we will state a slicing result for the double-bubble energy. To this end, we will assume that the vertical direction corresponds to the last coordinate. Letting $G \subset \mathbb{R}^3$ be any set of finite perimeter, we indicate by $G_t = G \cap (\mathbb{R}^2 \times \{t\})$ the horizontal slice at level t in the z direction and denote by $\pi_3 G$ the orthogonal projection of G on $\mathbb{R}^2 \times \{0\}$.

Lemma 2.4 (Slicing lemma)**.** *Suppose that A and B are disjoint bounded sets of finite perimeter. Then,*

$$E(A, B) \geq \int_{\mathbb{R}} E_{2D}(a(t), b(t))\,\mathrm{d}t + 2\mathcal{H}^2(\pi_3 A \cup \pi_3 B) + \mathcal{H}^2(\pi_3 A \cap \pi_3 B).$$

Let us state a result showing that the assumption $p \leq 1/3$ is not restrictive, up to relabeling the axes. Indeed, we have the following.

Lemma 2.5 (Upper bound on p)**.** *Suppose that (A, B) is an optimal configuration. Then, we can pick a coordinate direction such that $p \leq 1/3$, with p defined in* (2.4) *on the basis of the projections m, m_A, and m_B along that coordinate direction.*

The last technical lemma concerns the properties of some auxiliary functions depending on the function f defined in Proposition 2.2. We separated it from the proof of the main result in order to simplify the argument, as this algebraic calculation simply follows from the very definition of f.

Lemma 2.6 (Functions g_A^r and g_B^r)**.** *Suppose that*

$$m_A \geq \frac{2 + p}{3}m > \frac{1 + 2p}{3}m \geq m_B, \tag{2.8}$$

where $p \in [0, 1/3]$. Define

$$g_A^r(\alpha) := \frac{1 + \alpha}{r - \alpha}\Big(\frac{f(\alpha)}{\min\{\sqrt{m}, \sqrt{(1 + \alpha)m_A}\}} - \frac{4\sqrt{6}}{\sqrt{4 + 2p}\sqrt{m}}\Big),$$

$$g_B^r(\beta) := \frac{1 + \beta}{1 - r\beta}\Big(\frac{f(\beta)}{\min\{\sqrt{m}, \sqrt{(1 + \beta)m_B}\}} - \frac{4\sqrt{6}}{\sqrt{4 + 2p}\sqrt{m}}\Big). \tag{2.9}$$

Then, we have

(a) $\min_{\alpha \in [0, r]} (g_A^r(\alpha)) = g_A^r(0) \leq 0,$ (b) $\min_{\beta \in [0, \frac{1}{r}]} g_B^r(\beta) = g_B^r(0) > 0.$

2.4. **Proof of Theorem 1.1.** Moving from the discussion in the beginning of Section 2, the first step in the proof of Theorem 1.1 is to show that inequality (2.7) holds for all admissible configurations (A,B). Therefore, we first prove Proposition 2.3. Recall that the proofs of the auxiliary Lemmas 2.4–2.6 are postponed to Section 3.

Proof of Proposition 2.3. **Step 1.** By Lemma 2.4 and the definition of the projections m and p, a first lower bound for the energy of a configuration (A,B) is given by

$$E(A,B) \geq (2+p)m + \int_{\mathbb{R}} E_{2D}(a(t),b(t))\,\mathrm{d}t.$$

We now use the expression for $E_{2D}(a(t),b(t))$ from Proposition 2.2 to get

$$E(A,B) \geq (2+p)m + \int_{\mathcal{T}_A} \sqrt{a(t)+b(t)}f(\alpha(t))\,\mathrm{d}t + \int_{\mathcal{T}_B} \sqrt{a(t)+b(t)}f(\beta(t))\,\mathrm{d}t + \int_{\mathcal{T}_0} 2\sqrt{6}\sqrt{a(t)+b(t)}\,\mathrm{d}t. \tag{2.10}$$

By definition, we have $a(t)+b(t) \leq m$ for all $t \in \mathbb{R}$. We first estimate the last addend in (2.10) by

$$\int_{\mathcal{T}_0} 2\sqrt{6}\sqrt{a(t)+b(t)}\,\mathrm{d}t \geq \int_{\mathcal{T}_0} \frac{2\sqrt{6}}{\sqrt{m}}(a(t)+b(t))\,\mathrm{d}t = \frac{2\sqrt{6}}{\sqrt{m}}U_0, \tag{2.11}$$

with strict inequality if and only if $a(t)+b(t) < m$ on a subset of $\mathcal{T}_0$ with positive $\mathcal{L}^1$-measure.

Now, we estimate the second addend in (2.10). We again use that $a(t)+b(t) \leq m$ for all $t \in \mathbb{R}$, this time together with

$$a(t)+b(t) \leq (1+\alpha(t))m_A \quad \text{for } t \in \mathcal{T}_A,$$

to get

$$\int_{\mathcal{T}_A} \sqrt{a(t)+b(t)}f(\alpha(t))\,\mathrm{d}t \geq \int_{\mathcal{T}_A} (a(t)+b(t))\frac{f(\alpha(t))}{\min\{\sqrt{m},\sqrt{((1+\alpha(t))m_A}\}}\,\mathrm{d}t \tag{2.12}$$

with strict inequality if and only if

$$a(t)+b(t) < \min\{m,((1+\alpha(t))m_A\}$$

on a subset of $\mathcal{T}_A$ with positive $\mathcal{L}^1$-measure. One can verify that the function $f(x)/\sqrt{1+x}$ is increasing on the interval $[0, 2/\sqrt{3}-1]$ and decreasing on the interval $[2/\sqrt{3}-1, 1/2]$. Therefore, the minimum is exactly attained in 0 or $\frac{1}{2}$, and one can check that its values at 0 and 1/2 are equal, namely

$$f(0) = 4 = \frac{f(1/2)}{\sqrt{1+1/2}}.$$

Moreover, we have $f(x) = 2\sqrt{6}$ for $1/2 \leq x \leq 2$. Therefore,

$$\int_{\mathcal{T}_A} \sqrt{a(t)+b(t)}f(\alpha(t))\,\mathrm{d}t \geq \int_{\mathcal{T}_A} (a(t)+b(t))\min\left\{\frac{2\sqrt{6}}{\sqrt{m}},\frac{4}{\sqrt{m_A}}\right\}\mathrm{d}t, \tag{2.13}$$

with strict inequality if $m_A \leq 2m/3$ and $a(t)+b(t) < m$ on a subset of $\mathcal{T}_A$ with positive $\mathcal{L}^1$-measure. In fact, if $m \leq ((1+\alpha(t))m_A$, we have strict inequality already in (2.12), and if $m > ((1+\alpha(t))m_A$, we have strict inequality between the right-hand sides of (2.12) and (2.13) since $m_A \leq 2m/3$ implies $\min\{\frac{2\sqrt{6}}{\sqrt{m}},\frac{4}{\sqrt{m_A}}\} = \frac{2\sqrt{6}}{\sqrt{m}}$.

We similarly estimate the third addend in (2.10): since $a(t)+b(t) \leq m$ for all $t \in \mathbb{R}$ and

$$a(t)+b(t) \leq (1+\beta(t))m_B \quad \text{for } t \in \mathcal{T}_B,$$

we have

$$\int_{\mathcal{T}_B} \sqrt{a(t)+b(t)}f(\beta(t))\,\mathrm{d}t \geq \int_{\mathcal{T}_B} (a(t)+b(t))\frac{f(\beta(t))}{\min\{\sqrt{m},\sqrt{((1+\beta(t))m_B}\}}\,\mathrm{d}t, \tag{2.14}$$

so in particular

$$\int_{\mathcal{T}_B} \sqrt{a(t)+b(t)}f(\beta(t))\,\mathrm{d}t \geq \int_{\mathcal{T}_B} (a(t)+b(t))\min\left\{\frac{2\sqrt{6}}{\sqrt{m}},\frac{4}{\sqrt{m_B}}\right\}\mathrm{d}t, \tag{2.15}$$

with strict inequality if $m_B \le 2m/3$ and $a(t)+b(t) < m$ on a subset of $\mathcal{T}_B$ with positive $\mathcal{L}^1$-measure. We collect these estimates and deduce from (2.10) that

$$E(A,B) \ge (2+p)m + \int_{\mathcal{T}_A} (a(t)+b(t)) \min\left\{\frac{2\sqrt{6}}{\sqrt{m}}, \frac{4}{\sqrt{m_A}}\right\} \mathrm{d}t \\ + \int_{\mathcal{T}_B} (a(t)+b(t)) \min\left\{\frac{2\sqrt{6}}{\sqrt{m}}, \frac{4}{\sqrt{m_B}}\right\} \mathrm{d}t + \frac{2\sqrt{6}}{\sqrt{m}} U_0. \tag{2.16}$$

To simplify a later argument in Step 2 below, we rewrite the above inequality in an equivalent form. Let us introduce the notation

$$\tilde{p} = \sqrt{6}\sqrt{1+p/2} - \sqrt{6}$$

and

$$\hat{U} = \frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}(U_A + U_B) + \frac{2\sqrt{6}}{\sqrt{m}} U_0,$$

where $\hat{U}$ corresponds to the last two terms in the right-hand side of (2.7). By adding and subtracting $\frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}(a(t)+b(t))$ to both integrals, and recalling that integration of $a(t)+b(t)$ over $\mathcal{T}_A$ recovers U_A (and similarly for B), we can write inequality (2.16) as

$$E(A,B) \ge (2+p)m + \int_{\mathcal{T}_A} \frac{4(a(t)+b(t))}{\sqrt{4+2p}\sqrt{m}\sqrt{m_A}} \min\left\{\tilde{p}\sqrt{m_A}, \sqrt{4+2p}\sqrt{m} - \sqrt{6m_A}\right\} \mathrm{d}t \\ + \int_{\mathcal{T}_B} \frac{4(a(t)+b(t))}{\sqrt{4+2p}\sqrt{m}\sqrt{m_B}} \min\left\{\tilde{p}\sqrt{m_B}, \sqrt{4+2p}\sqrt{m} - \sqrt{6m_B}\right\} \mathrm{d}t + \hat{U}. \tag{2.17}$$

Step 2. In order to proceed with the proof of (2.7), we shall check that the integrals on the right-hand side of (2.17) are not negative. We separately consider the following mutually exclusive alternatives:

(1) At least one of the sets $\mathcal{T}_A$ and $\mathcal{T}_B$ has zero measure. Then, using (2.6) we get that both $\mathcal{T}_A$ and $\mathcal{T}_B$ have zero measure, and (2.7) follows. Moreover, the inequality is strict if and only if $a(t)+b(t) < m$ on a subset of $\mathcal{T}_0$ of positive $\mathcal{L}^1$-measure, see (2.11).

(2) Both sets $\mathcal{T}_A, \mathcal{T}_B$ have positive measure and

$$\sqrt{4+2p}\sqrt{m} > \sqrt{6m_A} \quad \text{and} \quad \sqrt{4+2p}\sqrt{m} > \sqrt{6m_B}. \tag{2.18}$$

The inequality (2.7) follows from (2.17) since the integrands are positive. Furthermore, the inequality is strict whenever $p > 0$. If $p = 0$, given that $m_A < 2m/3$ and $m_B < 2m/3$, the separate estimates for $\mathcal{T}_0$, $\mathcal{T}_A$ and $\mathcal{T}_B$ which are given in inequalities (2.11), (2.13), and (2.15) respectively, imply that the inequality is strict whenever $a(t)+b(t) < m$ on a subset of $\mathcal{T}_0 \cup \mathcal{T}_A \cup \mathcal{T}_B$ with positive $\mathcal{L}^1$-measure. Since in this case we have $m = m_A + m_B$, equality in (2.7) shows $a(t) = m_A$ and $b(t) = m_B$ for a.e. t. In view of (2.5), this gives $r = V_B/V_A = m_B/m_A$. Thus, $ra(t) = b(t)$ for a.e. t, which implies $\mathcal{L}^1(\mathcal{T}_A) = \mathcal{L}^1(\mathcal{T}_B) = 0$, and we are back in case (1).

(3) Both sets $\mathcal{T}_A, \mathcal{T}_B$ have positive measure and condition (2.18) fails, i.e., we have

$$\sqrt{4+2p}\sqrt{m} \le \sqrt{6m_A} \quad \text{or} \quad \sqrt{4+2p}\sqrt{m} \le \sqrt{6m_B},$$

which we rewrite as

$$\max\{m_A, m_B\} \ge \frac{2+p}{3} m. \tag{2.19}$$

The rest of the proof (including Step 3) concerns this case. Recall the definition of the functions g_A^r and g_B^r in Lemma 2.6. Using these definitions, we can rewrite estimate (2.12) as

$$\int_{\mathcal{T}_A} \sqrt{a(t)+b(t)} f(\alpha(t)) \,\mathrm{d}t \ge \int_{\mathcal{T}_A} a(t)(r-\alpha(t)) g_A^r(t) \,\mathrm{d}t + \frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}} U_A,$$

and similarly we can rewrite (2.14) as

$$\int_{\mathcal{T}_B} \sqrt{a(t)+b(t)}f(\beta(t))\,\mathrm{d}t \geq \int_{\mathcal{T}_B} b(t)(1-r\beta(t))g_B^r(t)\,\mathrm{d}t + \frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}U_B.$$

By plugging this into (2.10), and using (2.11) for the last addend, we get

$$E(A,B) \geq (2+p)m + \int_{\mathcal{T}_A} a(t)(r-\alpha(t))g_A^r(t)\,\mathrm{d}t + \int_{\mathcal{T}_B} b(t)(1-r\beta(t))g_B^r(t)\,\mathrm{d}t + \hat{U}.$$

Step 3. From now on, we assume without restriction that $m_A \geq m_B$. Hence, $m_B = (1+p)m - m_A$, and by (2.19) we get $m_B \leq \frac{1+2p}{3}m$. From Lemma 2.6 we have

$$\text{(i)} \quad \min_{\beta\in[0,1/r)} g_B^r(\beta) = g_B^r(0) > 0, \quad \text{(ii)} \quad \max_{\alpha\in[0,r)} (-g_A^r(\alpha)) = -g_A^r(0) \geq 0,$$

and thus

$$\inf_{t\in\mathcal{T}_B} g_B^r(\beta(t)) - \sup_{t\in\mathcal{T}_A}(-g_A^r(\alpha(t))) \geq g_B^r(0) + g_A^r(0).$$

As $g_A^r(0) \leq 0$ for each r, we get by monotonicity in r that

$$\inf_{t\in\mathcal{T}_B} g_B^r(\beta(t)) - \sup_{t\in\mathcal{T}_A}(-g_A^r(\alpha(t))) \geq g_B^{1/2}(0) + g_A^{1/2}(0) = \frac{4}{\sqrt{m_B}} + \frac{8}{\sqrt{m_A}} - \frac{12\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}.$$

By the definition of p, we have $m_B = m(1+p) - m_A$, and thus

$$\inf_{t\in\mathcal{T}_B} g_B^r(\beta(t)) - \sup_{t\in\mathcal{T}_A}(-g_A^r(\alpha(t))) \geq \frac{1}{\sqrt{m}}\Big(\frac{4}{\sqrt{1+p-\frac{m_A}{m}}} + \frac{8}{\sqrt{\frac{m_A}{m}}} - \frac{12\sqrt{6}}{\sqrt{4+2p}}\Big). \tag{2.20}$$

We minimize the sum of the first two addends in terms of $m_A/m \in [0,1]$ and get

$$\inf_{t\in\mathcal{T}_B} g_B^r(\beta(t)) - \sup_{t\in\mathcal{T}_A}(-g_A^r(\alpha(t))) \geq \frac{1}{\sqrt{m}}\Big(\frac{4(1+2^{2/3})^{3/2}}{\sqrt{1+p}} - \frac{12\sqrt{6}}{\sqrt{4+2p}}\Big). \tag{2.21}$$

Optimizing with respect to $p \in [0,1/3]$ we conclude

$$c_2 := \inf_{t\in\mathcal{T}_B} g_B^r(\beta(t)) > \sup_{t\in\mathcal{T}_A}(-g_A^r(\alpha(t))) =: c_1.$$

Then, denoting the value of the integrals in (2.6) by U_*, using that $r-\alpha \geq 0$ on $\mathcal{T}_A$ and $1-r\beta \geq 0$ on $\mathcal{T}_B$ we get from Step 2 that

$$\begin{aligned} E(A,B) &\geq (2+p)m + \int_{\mathcal{T}_A} a(t)(r-\alpha(t))g_A^r(t)\,\mathrm{d}t + \int_{\mathcal{T}_B} b(t)(1-r\beta(t))g_B^r(t)\,\mathrm{d}t + \hat{U} \\ &= (2+p)m - \int_{\mathcal{T}_A} a(t)(r-\alpha(t))(-g_A^r(t))\,\mathrm{d}t + \int_{\mathcal{T}_B} b(t)(1-r\beta(t))g_B^r(t)\,\mathrm{d}t + \hat{U} \\ &\geq (2+p)m + c_2\int_{\mathcal{T}_B} b(t)(1-r\beta(t))\,\mathrm{d}t - c_1\int_{\mathcal{T}_A} a(t)(r-\alpha(t))\,\mathrm{d}t + \hat{U} \\ &= (2+p)m + (c_2-c_1)U_* + \hat{U}. \end{aligned}$$

Recall that the term $\hat{U}$ is exactly the one appearing in (2.7). Therefore, whenever $U_* > 0$, the inequality in (2.7) is strict. Hence, in the case of equality, we have $U_* = 0$, but by the definition of U_* this implies that $\mathcal{L}^1(\mathcal{T}_A) = \mathcal{L}^1(\mathcal{T}_B) = 0$, and we are back to case (1) from Step 2 above. This concludes the proof. □

Having established Proposition 2.3, we can proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. As $U_A + U_B + U_0 = V_A + V_B$ and $p \geq 0$, we use (2.7) in order to estimate

$$E(A,B) \geq (2+p)m + \frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}(V_A+V_B),$$

where an equality is possible only if $p = 0$. Then, by the change of variables $M = (2+p)m/3$, and optimizing with respect to all possible values $M > 0$ and $p \in [0,1/3]$, we find $E(A,B) \geq E_{\min}$ (see (2.3)) with equality only if $\mathcal{L}^1(\mathcal{T}_A) = \mathcal{L}^1(\mathcal{T}_B) = 0$ and $a(t)+b(t) = m$ for $t \in \mathcal{T}_0 \setminus \mathcal{N}$. Optimizing with respect to M in (2.3) indeed gives the minimal energy given in Theorem 1.1. Moreover, we observe that this energy is attained by the configuration indicated in Theorem 1.1.

We can characterize ground states uniquely as follows: the above argument shows that any minimizer necessarily has $U_A = U_B = 0$, $U_0 = V_A + V_B$, and $p = 0$. This yields that the projection of A and B must have empty intersection and each slice with $t \in \mathcal{T}_0 \setminus \mathcal{N}$ has the same geometry. By the planar double-bubble result of Proposition 2.1, this geometry is then given by two specific rectangles joined at one face, namely the configuration from Theorem 1.1. □

3. Proofs of the auxiliary results

3.1. **Proof of Lemma 2.4.** Let us recall a classical slicing result for rectifiable sets, see for instance [17, Section 18.3]. Suppose that $F \subset \mathbb{R}^3$ is a rectifiable set with $\mathcal{H}^2(F) < +\infty$. Recall that F_t is the horizontal slice of the set F at level t in direction x_3, i.e.,

$$F_t = F \cap \{(x_1, x_2, t) : (x_1, x_2) \in \mathbb{R}^2\}.$$

In a similar fashion, we let

$$F^{(x_1,x_2)} := F \cap \{(x_1, x_2, y) : y \in \mathbb{R}\},$$

and get that $\mathcal{H}^0(F^{(x_1,x_2)})$ is finite for almost every (x_1, x_2). For every Borel function $g \colon \mathbb{R}^3 \to [-\infty, \infty]$ with $g \geq 0$ or $g \in L^1(\mathbb{R}^3)$ we have

$$\int_F g \sqrt{1 - (\nu_F \cdot e_3)^2} \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{R}} \int_{F_t} g \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}t, \tag{3.1}$$

where ν_F denotes the unit normal to F, as well as

$$\int_F |\nu_F \cdot e_3| \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{R}^2} \mathcal{H}^0\big(F^{(x_1,x_2)}\big) \, \mathrm{d}(x_1, x_2). \tag{3.2}$$

We are now ready to prove the slicing Lemma 2.4.

Proof of Lemma 2.4. Let $G := A \cup B$ and $F := \partial^* A \cup \partial^* B$. We split the ℓ_1-perimeter of F into a 'vertical' and 'horizontal' part. To be exact, we write

$$E(A, B) = \int_F |\nu_F|_1 \, \mathrm{d}\mathcal{H}^2 = \int_F (|\nu'_F|_1 + |(\nu_F)_3|) \, \mathrm{d}\mathcal{H}^2,$$

where we denote $x = (x', x_3)$ with $x' \in \mathbb{R}^2$. Recall that we use the notation $|\cdot|_1$ to denote the ℓ_1-norm of a vector both in two and three dimensions. The 'vertical' part and the 'horizontal' part is given by the integration of $|\nu'_F|_1 = |(\nu_F)_1| + |(\nu_F)_2|$ and $|(\nu_F)_3|$ over F, respectively. To this end, we introduce the function

$$\bar{g} = \frac{|(\nu_F)_1| + |(\nu_F)_2|}{\sqrt{1 - (\nu_F \cdot e_3)^2}}.$$

We use (3.1) with $g = \bar{g}$ on F (and 0 otherwise) and get

$$\int_F |\nu'_F|_1 \, \mathrm{d}\mathcal{H}^2 = \int_F \bar{g} \sqrt{1 - (\nu_F \cdot e_3)^2} \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{R}} \int_{F_t} \bar{g} \, \mathrm{d}\mathcal{H}^1 \mathrm{d}t = \int_{\mathbb{R}} \ell_1(F_t) \, \mathrm{d}t,$$

where in the last step we used the fact that $\frac{1}{\sqrt{1-(\nu_F \cdot e_3)^2}} \nu'_F \in \mathbb{R}^2$ is a unit normal to F_t. Since for a.e. $t \in \mathbb{R}$ we have $F_t = (\partial^* A \cup \partial^* B)_t$, the value $\ell_1(F_t)$ corresponds to the double-bubble energy of the configuration (A_t, B_t), and consequently we get

$$\ell_1(F_t) \geq E_{2D}(a(t), b(t)),$$

because the areas of A_t and B_t are $a(t)$ and $b(t)$, respectively. Therefore,

$$\int_F |\nu'_F|_1 \, \mathrm{d}\mathcal{H}^2 \geq \int_{\mathbb{R}} E_{2D}(a(t), b(t)) \, \mathrm{d}t. \tag{3.3}$$

On the other hand, the 'horizontal' term can be estimated in terms of the area of the largest horizontal slice. By (3.2) we get

$$\int_F |(\nu_F)_3| \, \mathrm{d}\mathcal{H}^2 = \int_{\mathbb{R}^2} \mathcal{H}^0\big(F^{(x_1,x_2)}\big) \, \mathrm{d}(x_1, x_2).$$

For $\mathcal{H}^2$-a.e. $(x_1,x_2)\in(\pi_3 A\cup\pi_3 B)\setminus(\pi_3 A\cap\pi_3 B)$ we have $\mathcal{H}^0((\partial^* G)^{(x_1,x_2)})\geq 2$, and for $\mathcal{H}^2$-a.e. $(x_1,x_2)\in(\pi_3 A\cap\pi_3 B)$ we have $\mathcal{H}^0((\partial^* G)^{(x_1,x_2)})\geq 3$. This shows

$$\int_F |(\nu_F)_3|\,\mathrm{d}\mathcal{H}^2=\int_{\mathbb{R}^2}\mathcal{H}^0\big(F^{(x_1,x_2)}\big)\,\mathrm{d}(x_1,x_2)\geq 2\mathcal{H}^2(\pi_3 A\cup\pi_3 B)+\mathcal{H}^2(\pi_3 A\cap\pi_3 B). \tag{3.4}$$

Combining (3.3) and (3.4) concludes the proof. □

3.2. **Proof of Lemma 2.5.** Denote by (m_1,m_2,m_3) the areas of projections of $A\cup B$ on all coordinate directions, by (m_1^A,m_2^A,m_3^A) the areas of projections of A, and by (m_1^B,m_2^B,m_3^B) the areas of projections of B. Then, for $i=1,2,3$ let $p_i=(m_i^A+m_i^B)/m_i-1$ and suppose by contradiction that $p_i>1/3$ for all i.

Letting $F=\partial^* A\cup\partial^* B$, by arguing as in the proof of inequality (3.4) we get

$$E(A,B)=\int_F|\nu_F|_1\,\mathrm{d}\mathcal{H}^2=\sum_{i=1}^3\int_F|(\nu_F)_i|\,\mathrm{d}\mathcal{H}^2\geq\sum_{i=1}^3\big(2\mathcal{H}^2(\pi_i A\cup\pi_i B)+\mathcal{H}^2(\pi_i A\cap\pi_i B)\big),$$

where π_i denotes the orthogonal projection on the plane with normal vector e_i. Then, we get

$$E(A,B)\geq\sum_{i=1}^3(2m_i+p_i m_i)>\sum_{i=1}^3(2m_i+\frac{1}{3}m_i)=\frac{7}{3}\sum_{i=1}^3 m_i.$$

Now, if we let $\overline{m}=(m_1+m_2+m_3)/3$, we have

$$E(A,B)>\frac{7}{3}\sum_{i=1}^3 m_i=7\overline{m}.$$

But this is the double-bubble energy of the following configuration $(\hat{A},\hat{B})$: the set $\hat{A}\cup\hat{B}$ is a cube with area of each side equal to $\overline{m}$, both sets $\hat{A}$ and $\hat{B}$ are cuboids, and the interface between them is a square of area $\overline{m}$ which is parallel to one of the sides of the original cube. The placement of the interface is such that the volume ratio is preserved. By the isoperimetric inequality for the ℓ_1-norm, the volume of $\hat{A}\cup\hat{B}$ is greater or equal to the volume of $A\cup B$. Thus, since $E(A,B)>E(\hat{A},\hat{B})$, the original configuration (A,B) was not optimal: a contradiction.

3.3. **Proof of Lemma 2.6.** We start by observing that $g_A^r(0)\leq 0$ and $g_B^r(0)>0$. In fact, using that $m_A/m\geq(2+p)/3$ (see assumption (2.8)), we get

$$\frac{4\sqrt{6}}{\sqrt{4+2p}}-\frac{4}{\sqrt{m_A/m}}\geq\frac{4\sqrt{6}}{\sqrt{4+2p}}-\frac{4\sqrt{3}}{\sqrt{2+p}}=0$$

all $p\in[0,1/3]$ which shows that $g_A^r(0)\leq 0$ since $f(0)=4$. In a similar fashion, $g_B^r(0)>0$ follows from

$$\frac{4}{\sqrt{m_B/m}}-\frac{4\sqrt{6}}{\sqrt{4+2p}}\geq\frac{4\sqrt{3}}{\sqrt{1+2p}}-\frac{4\sqrt{6}}{\sqrt{4+2p}}>0$$

for all $p\in[0,1/3]$, where we used $m_B/m\leq(1+2p)/3$, see (2.8).

The main part of the proof consists now in checking that g_A^r and g_B^r attain their minima at 0. The proof is structured as follows. In Step 1 we first show that the problem can be reduced to the cases $r=1/2$ and $r=1$. In Step 2 we introduce several auxiliary functions and use their specific properties to prove the statement. The proof of these properties is then given in Steps 3–8.

Step 1. Let us reduce the problem to specific values of r: we claim that it suffices to show

(i) $\min_{\alpha\in[0,1]} g_A^1(\alpha)=g_A^1(0)$,
(ii) $\min_{\alpha\in[0,1/2]} g_A^{1/2}(\alpha)=g_A^{1/2}(0)$,
(iii) $\min_{\beta\in[0,2]} g_B^{1/2}(\beta)=g_B^{1/2}(0)$.

We assume for the moment that the conditions (i)–(iii) hold, and show the statement of Lemma 2.6. For simplicity, we use the abbreviation

$$v(x,y)=\frac{f(x)}{\min\{\sqrt{m},\sqrt{(1+x)y}\}}-\frac{4\sqrt{6}}{\sqrt{4+2p}\sqrt{m}}.$$

We start by proving (a) of Lemma 2.6. As $g_A^r(0) \le 0$, it is not restrictive to consider only $\alpha \in [0, r]$ with $g_A^r(\alpha) \le 0$. Suppose first that $r \ge 1$. We write

$$g_A^r(\alpha) = \frac{1}{r}\frac{1+\alpha}{1-\frac{\alpha}{r}}v(\alpha, m_A) = \frac{1}{r}\frac{1-\alpha}{1-\frac{\alpha}{r}}\frac{1+\alpha}{1-\alpha}v(\alpha, m_A) = \frac{1}{r}\frac{1-\alpha}{1-\frac{\alpha}{r}}g_A^1(\alpha).$$

If $\alpha \in [0, 1]$, we have $0 \le \frac{1-\alpha}{1-\frac{\alpha}{r}} \le 1$. This, along with the above relation and (i), implies that

$$0 \ge g_A^1(\alpha) \ge g_A^1(0),$$

and consequently

$$g_A^r(\alpha) \ge \frac{1}{r}g_A^1(0) = g_A^r(0).$$

If instead $1 \le \alpha \le r \le 2$, by Proposition 2.2 and assumption (2.8) we have

$$g_A^r(\alpha) = \frac{1+\alpha}{r-\alpha}\Big(\frac{2\sqrt{6}}{\sqrt{m}} - \frac{4\sqrt{6}}{\sqrt{4+2p\sqrt{m}}}\Big) \ge 0.$$

Thus, the minimum of g_A^r is attained at $\alpha = 0$ with $g_A^r(0) \le 0$.

On the other hand, if $1/2 \le r < 1$, we first write

$$g_A^r(\alpha) = \frac{1}{r}\frac{1+\alpha}{1-\frac{\alpha}{r}}v(\alpha, m_A) = \frac{1}{r}\frac{\frac{1}{2}-\alpha}{1-\frac{\alpha}{r}}\frac{1+\alpha}{\frac{1}{2}-\alpha}v(\alpha, m_A) = \frac{1}{r}\frac{\frac{1}{2}-\alpha}{1-\frac{\alpha}{r}}g_A^{1/2}(\alpha).$$

If $\alpha \in [0, 1/2]$, we have $0 \le \frac{1/2-\alpha}{1-\alpha/r} \le 1/2$. This, together with the above relation and (ii), shows

$$0 \ge g_A^{1/2}(\alpha) \ge g_A^{1/2}(0),$$

and then also

$$g_A^r(\alpha) \ge \frac{1}{2r}g_A^{1/2}(0) = g_A^r(0).$$

If instead $1/2 \le \alpha \le r \le 1$, by Proposition 2.2 and assumption (2.8) we have

$$g_A^r(\alpha) = \frac{1+\alpha}{r-\alpha}\Big(\frac{2\sqrt{6}}{\sqrt{m}} - \frac{4\sqrt{6}}{\sqrt{4+2p\sqrt{m}}}\Big) \ge 0,$$

and thus the minimum of g_A^r is attained at $\alpha = 0$ with $g_A^r(0) \le 0$.

Let us now come to the proof of (b) of Lemma 2.6. We first write

$$g_B^r(\beta) = \frac{1+\beta}{1-r\beta}v(\beta, m_B) = \frac{1-\frac{\beta}{2}}{1-r\beta}\frac{1+\beta}{1-\frac{\beta}{2}}v(\beta, m_B) = \frac{1-\frac{\beta}{2}}{1-r\beta}g_B^{1/2}(\beta).$$

Recall that $g_B^{1/2}(0) > 0$. Then, for $\beta \in [0, 1/r]$, by $\frac{1-\beta/2}{1-r\beta} \ge 1$ and $g_B^{1/2}(\beta) \ge g_B^{1/2}(0) > 0$ (see (iii)), we conclude

$$g_B^r(\beta) \ge g_B^{1/2}(0) = g_B^r(0),$$

i.e., g_B^r attains its minimum at $\beta = 0$.

Step 2. We now proceed with the proof of the properties (i)–(iii). To this end, we define the auxiliary functions

$$h_1^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{f(x)}{2}\Big),$$

$$h_2^r(x) = \frac{1+x}{1-rx}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{f(x)}{2}\Big),$$

$$h_3^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_A}}\frac{f(x)}{2\sqrt{1+x}}\Big),$$

and

$$h_4^r(x) = \frac{1+x}{1-rx}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{f(x)}{2\sqrt{1+x}}\Big).$$

Recalling the definition of the function g_A^r in (2.9), we have

$$-g_A^r(\alpha) = \frac{2}{\sqrt{m}}\min(h_1^r(\alpha), h_3^r(\alpha)).$$

In other words, it is given by $h_3^r(\alpha)$ for $\alpha \in [0, m/m_A - 1]$ and by $h_1^r(\alpha)$ for $\alpha \in [m/m_A - 1, r]$. In Steps 3–5 below we show for $r = 1/2$ or $r = 1$ that h_1^r is decreasing and h_3^r achieves its maximum on the interval $[0, m/m_A - 1]$ at 0. This will show that $-g_A^r$ is maximized (and thus g_A^r is minimized) at 0 for $r = 1/2$ or $r = 1$. In a similar fashion, we have

$$g_B^{1/2}(\beta) = \frac{2}{\sqrt{m}} \max(-h_2^{1/2}(\beta), -h_4^{1/2}(\beta)).$$

In other words, it is given by $-h_4^{1/2}(\beta)$ for $\beta \in [0, m/m_B - 1]$ and by $-h_2^{1/2}(\beta)$ for $\beta \in [m/m_B - 1, 1/r]$. In Steps 6–8 below we show that $h_2^{1/2}$ is decreasing, so that $-h_2^{1/2}$ is increasing, and that $h_4^{1/2}$ attains its maximum at 0. This is enough to conclude that $g_B^{1/2}$ is minimized at 0.

Step 3. We first check that h_1^r is decreasing on $[0, r]$. For later purposes, we derive this property not only for $r = 1/2$ and $r = 1$ but also for $r = 2$. Recall the value r_* defined in (2.2). We will check separately the cases $x \in [0, r_*]$, $x \in [r_*, 1/2]$, and $x \in [1/2, r]$, where the case $x \in [1/2, r]$ is only necessary for $r = 1$ or $r = 2$. For $x \in [0, r_*]$, we have $f(x) = 4 + 2\sqrt{x/(1+x)}$, so

$$h_1^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - 2 - \frac{\sqrt{x}}{\sqrt{1+x}}\Big).$$

Let $C_1 = \frac{\sqrt{6}}{\sqrt{1+p/2}} - 2 \leq \sqrt{6} - 2$. For $r = 1/2, 1$, or 2, we compute the derivative and get

$$(h_1^r)'(x) = \frac{2C_1(1+r)\sqrt{x(x+1)} - (1+2r)x - r}{2(x-r)^2\sqrt{x(x+1)}}.$$

A direct calculation yields

$$2C_1(1+r)\sqrt{x(x+1)} - (1+2r)x - r < 0$$

for all $x \in [0, r_*]$ and $r = 1/2, 1$, or 2, so that h_1^r is decreasing on the interval $[0, r_*]$.

Similarly, for $x \in [r_*, 1/2]$, we have $f(x) = (4 + 2\sqrt{2x})/\sqrt{1+x}$, so

$$h_1^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{2}{\sqrt{1+x}} - \frac{\sqrt{2x}}{\sqrt{1+x}}\Big).$$

Denote by $C_2 = \frac{\sqrt{6}}{\sqrt{1+p/2}} \leq \sqrt{6}$. For $r \in \{1/2, 1, 2\}$, we directly calculate the derivative and get

$$\begin{aligned}(h_1^r)'(x) &= \frac{(2+2r)C_2\sqrt{x(x+1)} - 2\sqrt{x}(x+2+r) - \sqrt{2}((1+2r)x+r)}{2(x-r)^2\sqrt{x(x+1)}} \\ &\leq \frac{(2+2r)\sqrt{6}\sqrt{x(x+1)} - 2\sqrt{x}(x+2+r) - \sqrt{2}((1+2r)x+r)}{2(x-r)^2\sqrt{x(x+1)}} \leq 0\end{aligned}$$

with equality if and only if $x = 1/2$. Hence, h_1^r is also decreasing on the interval $[r_*, 1/2]$, and consequently we have shown that h_1^r is decreasing on the whole interval $[0, 1/2]$.

Finally, for $x \in [1/2, r]$, we just need to consider the cases $r = 1$ and $r = 2$. We have $f(x) = 2\sqrt{6}$, so

$$h_1^1(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \sqrt{6}\Big).$$

We compute the derivative and get

$$(h_1^1)'(x) = \Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \sqrt{6}\Big)\frac{1+r}{(r-x)^2} \leq 0.$$

Hence, h_1^r is decreasing on $[0, r]$ for $r = 1/2, 1$, or 2.

Step 4. Now, we focus on h_3^r. By (2.8), we have $m/m_A - 1 \leq 1/2$. In this step we show that h_3^r is maximized on $[0, m/m_A - 1]$ at one of the three points 0, $m/m_A - 1$, or r_* (and that the latter is only possible if $r_* \leq m/m_A - 1$). The values at the three points will then be compared in Step 5. To this end, we will analyze the monotonicity of h_3^r. More precisely, we check that in the intervals $[0, r_*]$ and $[r_*, 1/2]$ the function $(h_3^r)'$ changes sign at most once and, if it does, it changes from minus to plus (as x increases). This indeed shows that the maximum is attained at 0, $m/m_A - 1$, or r_*.

Let us come to the details. For $x \in [0, r_*]$, we have $f(x) = 4 + 2\sqrt{x/(1+x)}$, so

$$h_3^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{2}{\sqrt{1+x}} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{\sqrt{x}}{1+x}\Big).$$

For $r = 1/2$ or $r = 1$, we directly compute the first derivative and get

$$\begin{aligned}
(h_3^r)'(x) &= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1+x}{r-x}\Big)' - \frac{\sqrt{m}}{\sqrt{m_A}}\Big(\frac{1+x}{r-x} \cdot \frac{2}{\sqrt{1+x}}\Big)' - \frac{\sqrt{m}}{\sqrt{m_A}}\Big(\frac{1+x}{r-x} \cdot \frac{\sqrt{x}}{1+x}\Big)' \\
&= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} \cdot \frac{1+r}{(r-x)^2} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{r+x+2}{(r-x)^2(1+x)^{1/2}} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{r+x}{2\sqrt{x}(r-x)^2} \\
&= (r-x)^{-2}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}(1+r) - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{r+x+2}{(1+x)^{1/2}} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{r+x}{2\sqrt{x}}\Big).
\end{aligned}$$

Note that

$$x \mapsto -\frac{r+x+2}{(1+x)^{1/2}} - \frac{r+x}{2\sqrt{x}}$$

is increasing on $[0, r_*]$, and thus $(h_3^r)'$ can change sign at most once (from negative to positive as x increases). For $x \in [r_*, 1/2]$, we have $f(x) = (4 + 2\sqrt{2x})/\sqrt{1+x}$, and thus

$$h_3^r(x) = \frac{1+x}{r-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{2}{1+x} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{\sqrt{2x}}{1+x}\Big).$$

For $r = 1/2$ or $r = 1$, we calculate the first derivative and obtain

$$\begin{aligned}
(h_3^r)'(x) &= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1+x}{r-x}\Big)' - \frac{\sqrt{m}}{\sqrt{m_A}}\Big(\frac{1+x}{r-x} \cdot \frac{2}{1+x}\Big)' - \frac{\sqrt{m}}{\sqrt{m_A}}\Big(\frac{1+x}{r-x} \cdot \frac{\sqrt{2x}}{1+x}\Big)' \\
&= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} \cdot \frac{1+r}{(r-x)^2} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{2}{(r-x)^2} - \frac{\sqrt{m}}{\sqrt{m_A}} \cdot \frac{r+x}{\sqrt{2}(r-x)^2x^{1/2}} \\
&= (r-x)^{-2}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}(1+r) - 2\frac{\sqrt{m}}{\sqrt{m_A}} - \frac{\sqrt{m}}{\sqrt{m_A}}\frac{r+x}{\sqrt{2x}}\Big).
\end{aligned}$$

We note that $x \mapsto -2 - (r+x)/\sqrt{2x}$ is increasing on $[r_*, 1/2]$, i.e., $(h_3^r)'$ can change sign at most once.

Step 5. As seen in Step 4, h_3^r attains its maximum on the interval $[0, m/m_A - 1]$ at one of the points 0, r_*, and $m/m_A - 1$, where r_* is only possible if $r_* \le m/m_A - 1$. Using $f(r_*) = \frac{20}{41}(7+2\sqrt{2})$, we compute explicitly the three values and get

$$h_3^r(0) = \frac{\sqrt{6}}{r\sqrt{1+\frac{p}{2}}} - 2\frac{\sqrt{m}}{r\sqrt{m_A}},$$

$$h_3^r(r_*) = \frac{1+r_*}{r-r_*}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_A}}\frac{10(7+2\sqrt{2})}{41\sqrt{1+r_*}}\Big),$$

and

$$\begin{aligned}
h_3^r\Big(\frac{m}{m_A} - 1\Big) &= \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_A}}\frac{f(\frac{m}{m_A}-1)}{2\sqrt{\frac{m}{m_A}}}\Big) \\
&= \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{f(\frac{m}{m_A}-1)}{2}\Big).
\end{aligned}$$

To see that $h_3^r(0) \geq h_3^r(r_*)$ in the case $r_* \leq m/m_A - 1$, it suffices to observe that, for $r = 1/2$ or $r = 1$,

$$\Big(\frac{10(7+2\sqrt{2})}{41}\frac{\sqrt{1+r_*}}{r-r_*} - \frac{2}{r}\Big)\frac{\sqrt{m}}{\sqrt{m_A}} \geq \Big(\frac{10(7+2\sqrt{2})}{41}\frac{\sqrt{1+r_*}}{r-r_*} - \frac{2}{r}\Big)\sqrt{r_*+1}$$
$$\geq \sqrt{6}\Big(\frac{1+r_*}{r-r_*} - \frac{1}{r}\Big) \geq \Big(\frac{1+r_*}{r-r_*} - \frac{1}{r}\Big)\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}},$$

where in the first inequality we used that $m/m_A \geq r_*+1$, and the second inequality can be checked by an elementary computation. In a second step, we now check that the number

$$h_3^r(0) - h_3^r\Big(\frac{m}{m_A} - 1\Big) = \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Bigg(\frac{1}{r} - \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Bigg) - 2\frac{\sqrt{m}}{r\sqrt{m_A}} + \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\frac{f(\frac{m}{m_A}-1)}{2}$$

is nonnegative. Recall that by assumption (2.8) we have that $m/m_A \in [1,3/2]$. We distinguish two cases depending on whether $m/m_A - 1$ lies in $[0,r_*]$ or $[r_*,1/2]$. First, for $m/m_A - 1 \in [0,r_*]$, using the explicit formula for the minimal two-dimensional energy given in Proposition 2.2, we have

$$h_3^r(0) - h_3^r\Big(\frac{m}{m_A} - 1\Big) = \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Bigg(\frac{1}{r} - \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Bigg) - 2\frac{\sqrt{m}}{r\sqrt{m_A}} + \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Bigg(2 + \frac{\sqrt{\frac{m}{m_A}-1}}{\sqrt{\frac{m}{m_A}}}\Bigg).$$

We can look at the above expression as a function of a single parameter m/m_A, i.e., define

$$h_3^*(x) = \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1}{r} - \frac{x}{1+r-x}\Big) - \frac{2\sqrt{x}}{r} + \frac{x}{1+r-x}\Big(2 + \frac{\sqrt{x-1}}{\sqrt{x}}\Big),$$

so that $h_3^*(m/m_A) = h_3^r(0) - h_3^r(m/m_A - 1)$. Since $\frac{1}{r} \leq \frac{x}{1+r-x}$ for $x \geq 1$, we get

$$h_3^*(x) \geq \sqrt{6}\Big(\frac{1}{r} - \frac{x}{1+r-x}\Big) - \frac{2\sqrt{x}}{r} + \frac{x}{1+r-x}\Big(2 + \frac{\sqrt{x-1}}{\sqrt{x}}\Big).$$

This function is positive on $(1,3/2]$ and equal to zero at 1 in both cases $r = 1/2$ and $r = 1$. Hence,

$$h_3^r(0) \geq h_3^r\Big(\frac{m}{m_A} - 1\Big).$$

In the second case, i.e., for $m/m_A - 1 \in [r_*,1/2]$, again using the explicit formula given in Proposition 2.2 we have that $h_3^r(0) - h_3^r(m/m_A - 1)$ can be written as

$$\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Bigg(\frac{1}{r} - \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Bigg) - 2\frac{\sqrt{m}}{r\sqrt{m_A}} + \frac{\frac{m}{m_A}}{1+r-\frac{m}{m_A}}\Bigg(\frac{2}{\sqrt{\frac{m}{m_A}}} + \frac{\sqrt{2(\frac{m}{m_A}-1)}}{\sqrt{\frac{m}{m_A}}}\Bigg).$$

Again, we can look at the above expression as a function of a single parameter m/m_A, i.e., we define

$$h_3^{**}(x) = \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1}{r} - \frac{x}{1+r-x}\Big) - \frac{2\sqrt{x}}{r} + \frac{x}{1+r-x}\Big(\frac{2}{\sqrt{x}} + \frac{\sqrt{2(x-1)}}{\sqrt{x}}\Big),$$

so that $h_3^{**}(m/m_A) = h_3^r(0) - h_3^r(m/m_A - 1)$. Since $\frac{1}{r} \leq \frac{x}{1+r-x}$ for $x \geq 1$, we find

$$h_3^{**}(x) \geq \sqrt{6}\Big(\frac{1}{r} - \frac{x}{1+r-x}\Big) - \frac{2\sqrt{x}}{r} + \frac{x}{1+r-x}\Big(\frac{2}{\sqrt{x}} + \frac{\sqrt{2(x-1)}}{\sqrt{x}}\Big).$$

For $r = 1/2$ or $r = 1$, this function is positive on $(1,3/2)$ and equal to zero at 1 and 3/2. Hence, we have $h_3^r(0) \geq h_3^r(m/m_A - 1)$, so h_3^r attains its maximum on the interval $[0, m/m_A - 1]$ at 0.

Step 6. To see that $h_2^{1/2}$ is decreasing, it suffices to note that

$$h_2^{1/2}(x) = \frac{1+x}{1-\frac{x}{2}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{f(x)}{2}\Big) = 2\frac{1+x}{2-x}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{f(x)}{2}\Big) = 2h_1^2(x),$$

and to use that h_1^2 is decreasing, see Step 3.

Step 7. In this step we show that $h_4^{1/2}$ attains its maximum on $[0, m/m_B - 1]$ at one of the points 0, r_*, 1/2, or 2. The values at the three points will then be compared in Step 8. Similarly to the argument used in Step 4, here the argument relies on the fact that in the intervals $[0, r_*]$, $[r_*, 1/2]$, and $[1/2, 2]$ the function $(h_4^{1/2})'$ changes sign at most once (from negative to positive as x increases). This indeed shows that the maximum is attained at 0, r_*, 1/2, or 2. As in Step 4, we check separately the cases $x \in [0, r_*]$, $x \in [r_*, 1/2]$, and $x \in [1/2, 2]$.

For $x \in [0, r_*]$, we have $f(x) = 4 + 2\sqrt{x/(1+x)}$, so

$$h_4^{1/2}(x) = \frac{1+x}{1-\frac{x}{2}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}} \cdot \frac{2}{\sqrt{1+x}} - \frac{\sqrt{m}}{\sqrt{m_B}} \cdot \frac{\sqrt{x}}{1+x}\Big).$$

We directly compute the derivative and get

$$\begin{aligned}(h_4^{1/2})'(x) &= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1+x}{1-\frac{x}{2}}\Big)' - \frac{\sqrt{m}}{\sqrt{m_B}}\Big(\frac{1+x}{1-\frac{x}{2}}\frac{2}{\sqrt{1+x}}\Big)' - \frac{\sqrt{m}}{\sqrt{m_B}}\Big(\frac{1+x}{1-\frac{x}{2}}\frac{\sqrt{x}}{1+x}\Big)' \\ &= (2-x)^{-2}\Big(\frac{6\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{2x+8}{\sqrt{1+x}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{x+2}{\sqrt{x}}\Big).\end{aligned}$$

We note that

$$x \mapsto -\frac{2x+8}{\sqrt{1+x}} - \frac{x+2}{\sqrt{x}}$$

is increasing on $[0, r_*]$ and thus $(h_4^{1/2})'$ can change sign at most once. For $x \in [r_*, 1/2]$, we have $f(x) = (4 + 2\sqrt{2x})/\sqrt{1+x}$, and thus

$$h_4^{1/2}(x) = \frac{1+x}{1-\frac{x}{2}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{2}{1+x} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{\sqrt{2x}}{1+x}\Big).$$

We directly compute the derivative and get

$$\begin{aligned}(h_4^{1/2})'(x) &= \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}\Big(\frac{1+x}{1-\frac{x}{2}}\Big)' - \frac{\sqrt{m}}{\sqrt{m_B}}\Big(\frac{1+x}{1-\frac{x}{2}}\frac{2}{1+x}\Big)' - \frac{\sqrt{m}}{\sqrt{m_B}}\Big(\frac{1+x}{1-\frac{x}{2}}\frac{\sqrt{2x}}{1+x}\Big)' \\ &= (2-x)^{-2}\Big(\frac{6\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - 4\frac{\sqrt{m}}{\sqrt{m_B}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{\sqrt{2}(x+2)}{\sqrt{x}}\Big).\end{aligned}$$

We observe that

$$x \mapsto -4 - \frac{\sqrt{2}(x+2)}{\sqrt{x}}$$

is increasing on $[r_*, 1/2]$, i.e., $(h_4^{1/2})'$ can change sign at most once.

Eventually, for $x \in [1/2, 2]$, we have

$$h_4^{1/2}(x) = \frac{1+x}{1-\frac{x}{2}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{\sqrt{6}}{\sqrt{1+x}}\Big)$$

and the derivative reads as

$$(h_4^{1/2})'(x) = (2-x)^{-2}\Big(\frac{6\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{x+4}{\sqrt{1+x}}\Big).$$

We observe that $x \mapsto -(x+4)/\sqrt{1+x}$ is increasing on $[1/2, 2]$.

Step 8. In this step, we compare the values of $h_4^{1/2}$ at 0, r_*, 1/2, and 2 in order to conclude that the maximum in $[0, m/m_B - 1]$ is indeed at 0.

We recall that $f(r_*) = \frac{20}{41}(7 + 2\sqrt{2})$ and we compute explicitly

$$h_4^{1/2}(0) = \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - 2\frac{\sqrt{m}}{\sqrt{m_B}},$$

$$h_4^{1/2}(r_*) = \frac{1+r_*}{1-r_*/2}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{10(7+2\sqrt{2})}{41\sqrt{1+r_*}}\Big),$$

and

$$h_4^{1/2}\Big(\frac{1}{2}\Big) = 2\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - 2\frac{\sqrt{m}}{\sqrt{m_B}}\Big).$$

As $0 \leq p \leq 1/3$ and $m_B/m \leq \frac{1+2p}{3}$ (see (2.8)), we note that

$$h_4^{1/2}(0) \leq \frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - 2\frac{\sqrt{3}}{\sqrt{1+2p}} < 0.$$

This directly shows $h_4^{1/2}(0) \geq h_4^{1/2}(\frac{1}{2})$. To see that $h_4^{1/2}(0) \geq h_4^{1/2}(r_*)$, use again the assumption $p \leq 1/3$ and (2.8) to see $m/m_B \geq \frac{3}{1+2p} \geq \frac{9}{5}$. Therefore, it is elementary to check

$$\begin{aligned}\Big(\frac{10(7+2\sqrt{2})}{41}\frac{\sqrt{1+r_*}}{1-r_*/2} - 2\Big)\frac{\sqrt{m}}{\sqrt{m_B}} &\geq \Big(\frac{10(7+2\sqrt{2})}{41}\frac{\sqrt{1+r_*}}{1-r_*/2} - 2\Big)\sqrt{\frac{9}{5}} \geq \sqrt{6}\Big(\frac{1+r_*}{1-r_*/2} - 1\Big)\\ &\geq \Big(\frac{1+r_*}{1-r_*/2} - 1\Big)\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}}.\end{aligned}$$

Eventually, we show that $h_4^{1/2}(0) \geq \limsup_{x\to 2} h_4^{1/2}(x)$, which is a bit more delicate since $h_4^{1/2}$ is not defined at $x = 2$. Recalling that for $x \in [1/2, 2]$ we have $f(x) = 2\sqrt{6}$, it holds

$$h_4^{1/2}(x) = \frac{1+x}{1-\frac{x}{2}}\Big(\frac{\sqrt{6}}{\sqrt{1+\frac{p}{2}}} - \frac{\sqrt{m}}{\sqrt{m_B}}\frac{\sqrt{6}}{\sqrt{1+x}}\Big). \tag{3.5}$$

Since $x \leq m/m_B - 1$ and thus $m/m_B \geq 1 + x$, we get $h_4^{1/2}(x) \leq 0$ with strict inequality for $p > 0$. In particular, for $p > 0$, we get $\lim_{x\to 2} h_4^{1/2}(x) = -\infty$. Now, suppose that $p = 0$, and recall by (2.8) that $m_B/m \leq 1/3$. If $m_B/m < 1/3$, the term on the right-hand side of (3.5) is again negative for x close to 2 leading to $\lim_{x\to 2} h_4^{1/2}(x) = -\infty$. If $p = 0$ and $m_B/m = 1/3$ we calculate

$$\lim_{x\to 2} h_4^{1/2}(x) = \lim_{x\to 2} \frac{1+x}{1-\frac{x}{2}}\Big(\sqrt{6} - \frac{\sqrt{3}\sqrt{6}}{\sqrt{1+x}}\Big) = -\sqrt{6}.$$

In this case, we also have $h_4^{1/2}(0) = \sqrt{6} - 2\sqrt{3}$. This shows $h_4^{1/2}(0) \geq \limsup_{x\to 2} h_4^{1/2}(x)$ and concludes the proof.

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