

On the k -anti-traceability Conjecture

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Abstract

An oriented graph is called k -anti-traceable if the subdigraph induced by every subset with k vertices has a hamiltonian anti-directed path. In this paper, we consider an anti-traceability conjecture. In particular we confirm this conjecture holds when $k \leq 4$. We also show that every sufficiently large k -anti-traceable oriented graph admits an anti-path that contains $n - o(n)$ vertices.

1 Introduction

One of the fundamental and extensively studied problems in digraph theory is finding sufficient conditions for a digraph to contain a hamiltonian oriented path of a certain kind. A well-known result of Rédei [14] asserts that every *tournament* is *traceable*, that is, every orientation of the complete graph contains a hamiltonian directed path. Extending Rédei's result, Chen and Manalastas [9] proved that every strongly connected digraph with independence number two is traceable. Havet further strengthened this result in [12]. A digraph D is k -*traceable* if its order is at least k and each of its induced subdigraphs of order k is traceable. Note that every 2-traceable *oriented graph* (i.e., a digraph which can be obtained by orienting edges of an undirected graph)

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is a tournament. In [6] van Aardt, Dunbar, Frick, Nielsen and Oellermann formulated the following conjecture called the *Traceability Conjecture* (TC for short).

Conjecture 1.1. ([6]) *For every integer $k \geq 2$, every k -traceable oriented graph of order at least $2k - 1$ is traceable.*

Despite considerable interest, the TC remains unsolved. Improving on previous results in [3, 6] it was shown in [4] that every k -traceable oriented graph of order at least $6k - 20$ is traceable for $k \geq 4$. For $2 \leq k \leq 6$ and $k = 8$ it is known [1, 2] that every k -traceable oriented graph of order at least k is traceable. For $k = 7$ the situation is different as there exists a 7-traceable graph of order 8 that is not traceable (but every 7-traceable graph of order at least 9 is traceable) [2].

The TC has also been studied for several classes of digraphs without specific subdigraphs, see, e.g., [5, 15, 8].

An oriented path $P = x_1x_2 \dots x_p$ in a digraph D is called *anti-directed* (or, AT for short) if every two consecutive arcs of P have opposite orientations, i.e., for every $1 \leq i \leq p - 2$ we have either $x_i x_{i+1}$ and $x_{i+2} x_{i+1}$ are arcs in D (also denoted by $x_i \rightarrow x_{i+1}$ and $x_{i+2} \rightarrow x_{i+1}$) or $x_{i+1} x_i$ and $x_{i+1} x_{i+2}$ are arcs in D . Note that in an anti-directed path P , unlike in a directed path, the first arc of P may be oriented from or to the initial vertex of P . A *hamiltonian anti-directed path* of a digraph D includes all vertices of D . A digraph is *anti-traceable* if it contains a hamiltonian anti-directed path.

It is well-known that anti-directed paths behave very differently to directed paths. For example not all tournaments have a hamiltonian anti-directed path as we will see shortly. To do so we need the following definitions. Let q be a prime power with $q \equiv 3 \pmod{4}$. The *Paley tournament* of order q , denoted by PT_q , is the tournament with vertex set $V(PT_q) = GF(q)$ and arc set $A(PT_q) = \{(i, j) : j - i \text{ is a nonzero square in } GF(q)\}$. Let RT_n be the *rotational tournament* of order $n = 2k + 1$, that is, $V(RT_n) = [2k + 1] = \{1, 2, \dots, 2k + 1\}$ and $A(RT_n) = \{(i, j) : j = i + t \pmod{2k + 1}, 1 \leq t \leq k\}$, where $(i + t \pmod{2k + 1}) \in [2k + 1]$. Grünbaum [11] proved the following:

Theorem 1.2. ([11]) *Every tournament is anti-traceable unless it is isomorphic to one of the tournaments in $\{PT_3, RT_5, PT_7\}$.*

In [17], Rosenfeld strengthened Theorem 1.2 by showing that for every tournament T_n of order $n \geq 9$ and every vertex v of T_n , there is a hamiltonian anti-directed path in T_n starting at v . The reader is referred to [13, 10, 18] for more related interesting results with respect to hamiltonian oriented paths in tournaments.

A digraph D is k -anti-traceable (or, k -AT for short) if its order is at least k and each of its induced subdigraphs of order k is anti-traceable. Not surprisingly k -anti-traceable oriented graphs also behave very differently to k -traceable graphs. For example, it was shown in [3] that all k -traceable graphs on $6k - 20$ vertices have independence number at most 2. As the following family of example shows the independence number of a k -anti-traceable graph can be as high as $\lceil k/2 \rceil$. Let $n \geq k \geq 16$. We construct an oriented graph D on n vertices by having an independent set X of size $\lceil k/2 \rceil$ and a set Y that forms a tournament on $n - \lceil k/2 \rceil$ vertices. In addition there is an arc from every vertex in Y to every vertex in X . Clearly the independence number of D satisfies $\alpha(D) = |X| = \lceil k/2 \rceil$. We now show that D is k -anti-traceable. Consider a subset K of $V(D)$ of size k . If $K \subseteq Y$ then by Theorem 1.2 it contains an anti-directed path. If K intersects X in $t \geq 1$ vertices then we reserve $t - 1$ vertices in $K \cap Y$. If $k - 2t + 1$ is even then by Theorem 1.2 there is an anti-directed path $P = x_1 x_2 \dots x_{k-2t+1}$ covering all non-reserved vertices in $K \cap Y$ and moreover, either $x_1 \rightarrow x_2$ or $x_{k-2t+1} \rightarrow x_{k-2t}$. If $x_1 \rightarrow x_2$, then let $u = x_1$ and $v = x_2$, otherwise, let $u = x_{k-2t+1}$ and $v = x_{k-2t}$. Hence we can extend P to an anti-directed path containing all vertices of K by starting with path P in such a direction that the last vertex is u and then alternating between the vertices in $X \cap K$ and the reserved vertices. If $k - 2t + 1$ is odd then chose a non-reserved vertex a in $Y \cap K$ and observe that by Theorem 1.2 there is an anti-directed path $Q = x_1 x_2 \dots x_{k-2t}$ consisting of all non-reserved vertices apart from a . Define u and v in Q similarly to the way we defined them in P . Now, construct an anti-directed path by starting at a , using all vertices of $K \cap X$ and all reserved vertices and then using Q starting at u .

Inspired by the above results, we will focus on finding sufficient conditions for more general digraphs to be anti-traceable.

Open Problem 1.3. *For any positive integer k , what is the minimum integer $f(k) \geq k$ such that every k -anti-traceable oriented graph with order at least $f(k)$ has a hamiltonian anti-directed path? (If no such finite $f(k)$ exists then we set $f(k) = \infty$.)*

Recall that any 2-anti-traceable oriented graph is a tournament. Thus, it follows from Theorem 1.2 that $f(2) = 8$. In the next section we prove that $f(3) = 3$ and $f(4) = 8$. In Section 3, using Szemerédi's regularity lemma, we will show that every large k -anti-traceable oriented graph admits an anti-directed path that contains all but $o(n)$ vertices.

2 Values of $f(3)$ and $f(4)$

We first prove that $f(3) = 3$ and then that $f(4) = 8$.

2.1 $f(3) = 3$

To characterise 3-traceable graphs we need the following definition. For a digraph D and a set $X \subseteq V(D)$, $D[X]$ denotes the subdigraph of D induced by X . For a digraph D with vertices v_1, \dots, v_n and a sequence of n digraphs H_1, \dots, H_n , the *composition* $\hat{D} = D[H_1, \dots, H_n]$ has $V(\hat{D}) = \bigcup_{i=1}^n V(H_i)$ and

$$A(\hat{D}) = \bigcup_{i=1}^n A(H_i) \cup \{x_i x_j : v_i v_j \in A(D), x_i \in V(H_i), x_j \in V(H_j), 1 \leq i \neq j \leq n\}.$$

For two integers $p \leq q$, let $[p, q] = \{i : p \leq i \leq q\}$. If $p = 1$, then $[q] = [p, q]$.

Theorem 2.1. *An oriented graph D is 3-anti-traceable if and only if D is an extended transitive tournament $T[I_1, I_2, \dots, I_t]$ for some positive integer t , where I_i is an independent set with size at most two for all $i \in [t]$.*

Proof. We first show the following two claims.

Claim 2.2. *D is transitive.*

Proof. If there are three vertices v_1, v_2 and v_3 in D such that $v_1 \rightarrow v_2 \rightarrow v_3$, then one can see that $v_1 \rightarrow v_3$ since otherwise $D[\{v_1, v_2, v_3\}]$ has no anti-directed path of length two. \square

Claim 2.3. *For any pair of non-adjacent vertices u and v in D , we have $V(D) \setminus \{u, v\} \subseteq (N^+(u) \cap N^+(v)) \cup (N^-(u) \cap N^-(v))$.*

Proof. For any vertex $w \in V(D) \setminus \{u, v\}$, since $D[\{u, v, w\}]$ is anti-traceable, w is either a common out- or in-neighbour of u and v . \square

We first prove the necessity. Let u, v, w be any three different vertices of D . Suppose u, v, w belong to three different independent sets. Then $D[\{u, v, w\}]$ is anti-traceable as $D[\{u, v, w\}]$ is a transitive tournament. Suppose u, v, w belong to two different independent sets. Without loss of generality, we assume u and v belong to the same independent set. Then w is either a common out- or in-neighbour of u and v . Therefore, $D[\{u, v, w\}]$ is anti-traceable.

Let $|V(D)| = n$. We now prove the sufficiency by induction on n . It is trivially true when $n \leq 2$. We can assume that $n \geq 3$. By Claim 2.2, we have that D is acyclic. Let v_1, v_2, \dots, v_n be an acyclic ordering of $V(D)$. If $N^-(v_n) = V(D) \setminus \{v_n\}$, then by induction hypothesis $D - v_n$ is a required extended transitive tournament, which means D is also a required one. If there exists v_s with $1 \leq s < n$ such that v_s and v_n are non-adjacent, then $s = n - 1$, as otherwise v_{n-1} should be either a common out- or in-neighbour of v_s and v_n by Claim 2.3, a contradiction to the fact that v_1, v_2, \dots, v_n is an acyclic ordering of $V(D)$. Thus, v_{n-1} and v_n are non-adjacent. Again,

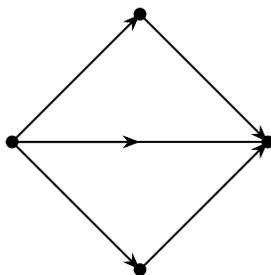


Figure 1: T_4

by Claim 2.3 and the fact that they are the last two vertices in the acyclic ordering, all vertices are common in-neighbours of v_n and v_{n-1} . Thus, we are done by applying the induction hypothesis to $D - \{v_{n-1}, v_n\}$. \square

The following results proved $f(3) = 3$ in a slightly stronger form. One can observe from Theorem 2.1 that the following holds.

Observation 2.4. *If D is a 3-AT oriented graph with four vertices, then D is anti-traceable. Moreover, D has an hamiltonian anti-directed cycle unless $D \cong T_4$ (See Fig. 1).*

Proof. We can directly check that D is anti-traceable when $D \cong T_4$. Let D be 3-AT with four vertices and $D \not\cong T_4$. By Theorem 2.1, one can observe that the last two vertices in the acyclic order are common out-neighbours of the first two vertices. Thus, D is anti-traceable and has an hamiltonian anti-directed cycle. \square

Theorem 2.5. *If D is 3-AT and $|V(D)| \geq 5$, then D has an hamiltonian anti-directed cycle when $|V(D)|$ is even, and an anti-cycle with length $|V(D)| - 1$ and an hamiltonian anti-directed path when $|V(D)|$ is odd.*

Proof. We first prove the even case by induction on $|V(D)|$. The base case is $|V(D)| = 6$. Let u be a source of D and v be a sink. By Theorem 2.1, $u \rightarrow v$ and $D - u - v$ is also 3-AT. Therefore, by Observation 2.4, $D - u - v$ has an hamiltonian anti-directed path $P = v_1 v_2 v_3 v_4$, and we may assume without loss of generality that $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4$. By using Theorem 2.1 as well as the fact that u is a source and v is a sink, we have $u \rightarrow v_4$ and $v_1 \rightarrow v$. Thus, $u v v_1 v_2 v_3 v_4 u$ is an hamiltonian anti-directed cycle in D . We now assume that $|V(D)| \geq 8$. By Propostion 2.1, we see that $D = T[I_1, I_2, \dots, I_t]$ for some positive integer t . Let u be a source and v be a sink of D . Evidently, $u \rightarrow v$. By the induction hypothesis, $D - u - v$ has an hamiltonian anti-directed cycle C . Let $u'v'$ be an arc on C . Since u' is not a sink, we get $u' \notin I_t$, which yields

that $u' \rightarrow v$. Similarly, since $v' \notin I_1$, we have $u \rightarrow v'$. Thus, $uvu'Cv'u$ is an hamiltonian anti-directed cycle of D . This completes the proof of the even case.

Now, we show the second statement. For $|V(D)| \geq 7$, we arbitrarily pick a vertex v of D . Clearly, $|V(D - v)| \geq 6$ is even and $D - v$ is also 3-AT. Hence, $D - v$ has an hamiltonian anti-directed cycle C , which is a required cycle in D . For $|V(D)| = 5$, we choose a vertex v , either a sink or a source, such that $D - v \not\cong T_4$ (such vertex must exist due to Theorem 2.1). Thus, by Observation 2.4, $D - v$ has an hamiltonian anti-directed cycle which is also a required cycle.

Now, add a new vertex v^* and all arcs from $V(D)$ to v^* . Denote the new digraph by D^* . Note that $|V(D^*)|$ is even and $|V(D^*)| \geq 6$. By Theorem 2.1, D^* is 3-AT and thus contains an hamiltonian anti-directed cycle C^* . Therefore, $C^* - v^*$ is a required hamiltonian anti-directed path in D . \square

2.2 $f(4) = 8$

Proposition 2.6. $f(4) \geq 8$.

Proof. As mentioned in Theorem 1.2, PT_7 does not contain an anti-directed path. Thus, we have $f(4) \geq 8$ since PT_7 is 4-AT. \square

To make the following proofs easier to follow, we first give two simple observations.

Observation 2.7. *Let D be a 4-AT oriented graph and a, b, c, d be four different vertices in D . Then*

- (i) *if $A(D[\{a, b, c, d\}]) \setminus \{ad, da\} \subseteq \{ab, bc, ca, cd, db\}$, then $a \sim d$;*
- (ii) *if $D[\{a, b, c\}]$ is a subdigraph of a directed triangle and $a \not\sim d$, then $b \rightarrow d \leftarrow c$ or $b \leftarrow d \rightarrow c$;*
- (iii) *if $A(D[\{a, b, c, d\}]) \setminus \{ac, ca, bd, db\}$ forms a subdigraph of a directed 4-cycle, then $a \sim c$ and $b \sim d$.*

Observation 2.8. *Let D be a 4-AT oriented graph and $P = v_1v_2 \dots v_s$ be a longest anti-directed path in D . Let w be a vertex in $V(D) \setminus V(P)$. Then*

- (i) $\{w, v_2\} \not\subseteq N^+(v_1)$ and $\{w, v_2\} \not\subseteq N^-(v_1)$; $\{w, v_{s-1}\} \not\subseteq N^+(v_s)$ and $\{w, v_{s-1}\} \not\subseteq N^-(v_s)$;
- (ii) *If $w \sim v_1$, then $D[\{v_1, v_2, w\}]$ is a subdigraph of a directed triangle. Similarly, if $w \sim v_s$, then $D[\{v_{s-1}, v_s, w\}]$ is a subdigraph of a directed triangle.*

Proof. (i). If $\{w, v_2\} \subseteq N^+(v_1)$ or $\{w, v_2\} \subseteq N^-(v_1)$, then $wv_1v_2 \dots v_s$ is a longer anti-directed path, a contradiction. If $\{w, v_{s-1}\} \subseteq N^+(v_s)$ or $\{w, v_2\} \subseteq N^-(v_1)$, then $v_1v_2 \dots v_s w$ is a longer anti-directed path, a contradiction.

(ii) We assume without loss of generality that $v_1 \rightarrow v_2$. If $w \sim v_1$, then $w \rightarrow v_1$. If $w \rightarrow v_2$, then $v_1wv_2 \dots v_s$ is a longer anti-directed path, a contradiction. Thus, we have $v_2 \rightarrow w$ or $v_2 \not\sim w$. Then, $D[\{v_1, v_2, w\}]$ is a subdigraph of a directed triangle. By a similar discussion, if $w \sim v_s$, then we also get $D[\{v_{s-1}, v_s, w\}]$ is a subdigraph of a directed triangle. \square

Lemma 2.9. *Let D be a 4-AT oriented graph and $P = v_1v_2 \dots v_s$ be a longest anti-directed path in D . If $|V(P)| < |V(D)|$, then $v_1 \sim v_s$.*

Proof. Let w be a vertex in $V(D) \setminus V(P)$. Suppose to the contrary that $v_1 \not\sim v_s$. Since $D[\{w, v_1, v_2, v_s\}]$ is anti-traceable, we have $w \sim v_1$ or $w \sim v_s$ (or both). We assume without loss of generality that $w \sim v_1$. We may further assume that $v_1 \rightarrow v_2$ as the discussion for the case $v_2 \rightarrow v_1$ is similar. By Observation 2.8, we have $w \rightarrow v_1$ and $D[\{v_1, v_2, w\}]$ is a subdigraph of directed triangle. Since $v_1 \not\sim v_s$, we $v_2 \rightarrow v_s \leftarrow w$ or $v_2 \leftarrow v_s \rightarrow w$ by Observation 2.7(ii). In particular, $w \sim v_s$ and therefore $D[\{v_{s-1}, v_s, w\}]$ is a subdigraph of directed triangle by Observation 2.8(ii). Thus, by Observation 2.7(ii), we have $v_{s-1} \rightarrow v_1$ since $v_1 \not\sim v_s$ and $w \rightarrow v_1$.

Recall that $D[\{v_{s-1}, v_s, w\}]$ is a subdigraph of a directed triangle and $v_2 \rightarrow v_s \leftarrow w$ or $v_2 \leftarrow v_s \rightarrow w$. Thus, we have $v_s \rightarrow v_{s-1}$ if $v_2 \rightarrow v_s \leftarrow w$ and $v_{s-1} \rightarrow v_s$ if $v_2 \leftarrow v_s \rightarrow w$. If $v_2 \rightarrow v_s \leftarrow w$, then we deduce that $v_1 \rightarrow v_2 \rightarrow v_s \rightarrow v_{s-1} \rightarrow v_1$. Since $v_1 \not\sim v_s$, by Observation 2.7(iii), $D[\{v_1, v_2, v_{s-1}, v_s\}]$ cannot be anti-traceable, contradicting the fact that D is 4-AT. If $v_2 \leftarrow v_s \rightarrow w$, then $wv_1v_{s-1}v_{s-2} \dots v_2v_s$ is a longer anti-directed path, a contradiction. \square

Lemma 2.10. *Let D be a 4-AT oriented graph and $P = v_1v_2 \dots v_s$ be a longest anti-directed path in D . If $|V(P)| < |V(D)|$, then $|V(P)|$ is even.*

Proof. Suppose to the contrary that s is odd. Let w be a vertex in $V(D) \setminus V(P)$. By Lemma 2.9, we have $v_1 \sim v_s$. Notice that when s is odd, each vertex on $C = v_1v_2v_3 \dots v_s v_1$ is a source or a sink except for exactly one vertex. Thus, we may assume without loss of generality that $v_1 \rightarrow v_2 \leftarrow v_3 \dots v_{s-1} \leftarrow v_s \rightarrow v_1$.

By Observation 2.8(i), we have $v_1 \not\sim w$ and $v_s \not\sim w$. If $w \rightarrow v_1$, then $wv_1v_s v_{s-1} \dots v_2$ is an anti-directed path longer than P , which leads to a contradiction. Thus, $w \not\sim v_1$. Since $D[\{w, v_1, v_2, v_s\}]$ is anti-traceable, we have $w \sim v_2$. By Observation 2.7(iii) and the fact that $v_s \not\sim v_1$ and $w \not\sim v_1$, we have $w \rightarrow v_2$. But now $wv_2v_3 \dots v_s v_1$ is a longer anti-directed path than P , a contradiction. \square

Proposition 2.11. *If D is a 4-AT oriented graph, then for any integer $k \in [2, \frac{|V(D)|}{2}]$, we have D is $2k$ -AT.*

Proof. We prove it by induction on k . When $k = 2$, there is nothing left to prove. Suppose for any integer ℓ with $3 \leq \ell < k$, D is 2ℓ -AT. Now let $\ell = k$ and S be an arbitrary subset of $V(D)$ with size $2k$. We only need to show that $D[S]$ is anti-traceable.

If $D[S]$ is a tournament, then we are done by Theorem 1.2. Thus, we may assume that $D[S]$ has two non-adjacent vertices u and v . By the induction hypothesis, $D[S - u - v]$ is anti-traceable. Let $P = v_1 v_2 \dots v_{2k-2}$ be a hamiltonian anti-directed path in $D[S - u - v]$. If P is not a longest path in $D[S]$, then by Lemma 2.10, $D[S]$ is anti-traceable.

In the remaining proof, we always assume that P is a longest anti-directed path in $D[S]$. By symmetry, we may further assume that $P = v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow \dots \leftarrow v_{2k-3} \rightarrow v_{2k-2}$. Since $D[\{u, v, v_{2k-3}, v_{2k-2}\}]$ is anti-traceable, one of u and v must be adjacent to v_{2k-2} . Assume without loss of generality that $v \sim v_{2k-2}$. By Observation 2.8, we have $v_{2k-2} \rightarrow v$ and $D[\{v, v_{2k-3}, v_{2k-2}\}]$ is a subdigraph of a directed triangle. Since $u \not\sim v$ and $D[\{u, v, v_{2k-3}, v_{2k-2}\}]$ is anti-traceable, by Observation 2.7(ii), u must be either a common out- or in-neighbour of v_{2k-3} and v_{2k-2} . But now either $v_{2k-2} u v_{2k-3} \dots v_1$ or $u v_{2k-2} v_{2k-3} \dots v_1$ is a longer anti-directed path, which leads to a contradiction. \square

Theorem 2.12. $f(4) = 8$.

Proof. Let D be a 4-AT oriented graph with $|V(D)| = n \geq 8$. Let $P = v_1 v_2 v_3 \dots v_s$ be a longest anti-directed path in D . Suppose to a contrary that $s < n$. By Lemma 2.10 and Proposition 2.11, we have s that is even and $s = n - 1 \geq 7$, which implies that $s \geq 8$. Let w be the only vertex in $V(D) \setminus V(P)$. Since s is even, we may assume by the symmetry that $v_1 \rightarrow v_2 \leftarrow v_3 \dots v_{s-1} \rightarrow v_s$. By Lemma 2.9, v_1 and v_s are adjacent. We consider the following two cases.

Case 1. $v_1 \rightarrow v_s$.

All subscripts are taken modulo s in the following. Note that for any $1 \leq i \leq s$, v_i and v_{i+1} are the endpoints of a longest anti-directed path. Thus, by Observation 2.8(i), we have $w \rightarrow v_i$ if $w \sim v_i$ and i is odd, and $v_i \rightarrow w$ if $w \sim v_i$ and i is even. Since $D[\{w, v_i, v_{i+1}, v_{i+2}\}]$ is anti-traceable, by Observation 2.7(i), we have $v_i \sim v_{i+2}$ for any $1 \leq i \leq s$. Without loss of generality, we may assume $v_1 \rightarrow v_3$ (otherwise we can switch the subscripts). Then, $w \rightarrow v_3$ for otherwise $D[\{w, v_1, v_2, v_3\}]$ is not anti-traceable (Observation 2.7(iii)). One can deduce that $v_4 \rightarrow v_2$ for otherwise $w v_3 v_1 v_s v_{s-1} \dots v_4 v_2$ is an hamiltonian anti-directed path. Then, we have $v_4 \rightarrow w$ for otherwise $D[\{w, v_2, v_3, v_4\}]$ is not anti-traceable (Observation 2.7(iii)). Similar to the foregoing discussion, we get $w \rightarrow v_i$ and $v_i \rightarrow v_{i+2}$ when i is odd, $v_i \rightarrow w$ and $v_i \leftarrow v_{i+2}$ when i is even.

Recall that $|V(P)| = s \geq 8$. If $D[\{V(P) \cup \{w\}\}]$ is a tournament, then by Theorem 1.2, $D[\{V(P) \cup \{w\}\}]$ is anti-traceable. Thus, without loss of generality, we assume $d_D(v_1) < n - 1$. Let k be the smallest subscript such that $v_1 \not\sim v_k$. We have $4 \leq k \leq s - 2$. Notice that for any $1 \leq i \leq s$, $D[\{w, v_i, v_{i+1}\}]$ is a directed triangle. Now consider $D[\{w, v_1, v_{k-1}, v_k\}]$, $D[\{w, v_1, v_k, v_s\}]$ and $D[\{w, v_1, v_2, v_k\}]$, by Observation 2.7(ii), We can determine the relationship between each of these three vertex pairs $\{v_{k-1}, v_1\}$, $\{v_k, v_s\}$ and $\{v_k, v_2\}$. When $v_k \rightarrow w$, we have $v_{k-1} \rightarrow v_1$, $v_k \rightarrow v_s$ and $v_k \rightarrow v_2$. Then $wv_1v_{k-1}v_{k-2} \dots v_2v_kv_s v_{s-1} \dots v_{k+1}$ is a hamiltonian anti-directed path, a contradiction. When $w \rightarrow v_k$, we have $v_{k-1} \rightarrow v_1$ and $v_2 \rightarrow v_k$. Then $D[\{v_1, v_2, v_{k-1}, v_k\}]$ is not anti-traceable (Observation 2.7(iii)), a contradiction.

Case 2. $v_s \rightarrow v_1$.

We first present the following two claims.

Claim 2.13. $w \rightarrow v_1$ and $v_s \rightarrow w$.

Proof. By Observation 2.8(i), we have $v_1 \not\rightarrow w$ and $w \not\rightarrow v_s$. Assume to the contrary that $w \not\sim v_1$. Note that $D[\{v_1, v_2, w, v_s\}]$ is anti-traceable. If $w \not\sim v_2$, then the anti-directed hamiltonian path in $D[\{v_1, v_2, w, v_s\}]$ is $v_1 \rightarrow v_2 \leftarrow v_s \rightarrow w$. If $w \rightarrow v_2$, then consider another longest path $P' = wv_2v_3 \dots v_s$. By Lemma 2.9, $w \sim v_s$ and therefore $v_s \rightarrow w$ as $w \not\rightarrow v_s$. If $v_2 \rightarrow w$, then the hamiltonian anti-directed path in $D[\{v_1, v_2, w, v_s\}]$ is $v_1 \rightarrow v_2 \leftarrow v_s \rightarrow w$ or $v_2 \rightarrow w \leftarrow v_s \rightarrow v_1$. Thus, in all cases we have $v_s \rightarrow w$. Since $v_s \rightarrow w$, by Observation 2.8(ii), $D[\{w, v_{s-1}, v_s\}]$ is a subdigraph of a directed triangle and then by Observation 2.7(ii), we have $v_{s-1} \rightarrow v_1$. It follows that $wv_s v_1 v_{s-1} v_{s-2} \dots v_2$ is a hamiltonian anti-directed path, a contradiction. Thus $w \rightarrow v_1$.

Suppose to the contrary that $w \not\sim v_s$. Since $w \rightarrow v_1$, by Observation 2.8(ii), we have $D[\{w, v_1, v_2\}]$ is a subdigraph of a directed triangle. Then by Observation 2.7(ii), we have $v_s \rightarrow v_2$. It follows that $wv_1v_s v_2 v_3 \dots v_{s-1}$ is a longer anti-directed path when $v_2 \rightarrow w$, a contradiction. Thus, $v_s \rightarrow w$. \square

Claim 2.14. $v_2 \rightarrow w$, $w \rightarrow v_{s-1}$, $v_1 \rightarrow v_{s-1}$, $v_2 \rightarrow v_s$, $w \rightarrow v_3$, $v_{s-2} \rightarrow w$, $v_1 \rightarrow v_3$ and $v_{s-2} \rightarrow v_s$.

Proof. By Claim 2.13, if $v_s \rightarrow v_2$ or $v_{s-1} \rightarrow v_1$, then $wv_1v_s v_2 v_3 \dots v_{s-1}$ or $wv_s v_1 v_{s-1} v_{s-2} \dots v_2$ is an hamiltonian anti-directed path. Thus, we have $v_2 \rightarrow v_s$ when $v_2 \sim v_s$ and $v_1 \rightarrow v_{s-1}$ when $v_1 \sim v_{s-1}$. By Observation 2.8(ii), we have $v_2 \rightarrow w$ when $v_2 \sim w$ and $w \rightarrow v_{s-1}$ when $w \sim v_{s-1}$. Considering $D[\{w, v_1, v_2, v_s\}]$, since $D[\{v_1, v_2, v_s\}]$ is a subdigraph of a directed triangle and $v_s \rightarrow w \rightarrow v_1$, by Observation 2.7(ii), we have $v_2 \sim w$ and therefore $v_2 \rightarrow w$. Similarly, considering $D[\{w, v_1, v_{s-1}, v_s\}]$, we get $w \rightarrow v_{s-1}$.

Next, we will show that $v_2 \rightarrow v_s$ and $v_1 \rightarrow v_{s-1}$. Suppose $v_2 \not\rightarrow v_{s-1}$ or $v_2 \rightarrow v_{s-1}$. Since $D[\{v_1, v_2, v_{s-1}, v_s\}]$ is anti-traceable, by Observation 2.7(iii), we have $v_2 \sim v_s$ and $v_1 \sim v_{s-1}$ and therefore $v_2 \rightarrow v_s$ and $v_1 \rightarrow v_{s-1}$. Suppose $v_{s-1} \rightarrow v_2$. Considering $D[\{w, v_1, v_2, v_{s-1}\}]$, since $D[\{w, v_1, v_2\}]$ is a directed triangle and $w \rightarrow v_{s-1} \rightarrow v_2$, by Observation 2.7(ii), we have $v_1 \sim v_{s-1}$ and so $v_1 \rightarrow v_{s-1}$. Similarly, considering $D[\{w, v_2, v_{s-1}, v_s\}]$, we get $v_2 \rightarrow v_s$.

Since both $wv_2v_sv_{s-1}\dots v_3$ and $wv_{s-1}v_1v_2\dots v_{s-2}$ are longest anti-directed paths, we have $w \rightarrow v_3$ and $v_{s-2} \rightarrow w$ as otherwise we may consider these two paths and are done by **Case 1**. As $D[\{w, v_1, v_2, v_3\}]$ and $D[\{w, v_{s-2}, v_{s-1}, v_s\}]$ are anti-traceable, by Observation 2.7(i), we have $v_1 \sim v_3$ and $v_{s-2} \sim v_s$. If $v_3 \rightarrow v_1$ (or $v_s \rightarrow v_{s-2}$), then $wv_2v_sv_{s-1}\dots v_3v_1$ (or $wv_{s-1}v_1v_2\dots v_{s-2}v_s$) is a longer anti-directed path than P , a contradiction. We thus deduce that $v_1 \rightarrow v_3$ and $v_{s-2} \rightarrow v_s$. \square

Note that Claim 2.13 and Claim 2.14 are true for any anti-directed path $Q = u_1u_2\dots u_s$ of D satisfying that $|Q| = |P|$, $u_1 \rightarrow u_2$ and $u_s \rightarrow u_1$.

Claim 2.15. *For any anti-directed path $Q = u_1u_2\dots u_s$ of D satisfying that $|Q| = |P|$, $u_1 \rightarrow u_2$ and $u_s \rightarrow u_1$, we have $u_{s-2(t+1)} \rightarrow u_{s-2t}$ for all $0 \leq t \leq \frac{s}{2} - 2$.*

Proof. We prove this claim by induction on integer t . By Claim 2.14, it is true for $t = 0$. Assume that $t \geq 1$ it and is true for all integers less than t . Now we prove it is true for t . Let $Q = u_1u_2\dots u_s$ be an anti-directed path of D satisfying that $|Q| = |P|$, $u_1 \rightarrow u_2$ and $u_s \rightarrow u_1$. We need to prove $u_{s-2t-2} \rightarrow u_{s-2t}$. Let v be the vertex in $V(D) \setminus V(Q)$. By Claim 2.13 and Claim 2.14, we have $Q' = u'_1u'_2\dots u'_s = vv_{s-1}u_1u_2u_3\dots u_{s-4}u_{s-3}u_{s-2}$ is an anti-directed path satisfying that $|Q'| = |P|$, $u'_1 \rightarrow u'_2$ and $u'_s \rightarrow u'_1$. Then for Q' , by the induction hypothesis, we have $u'_{s-2t} \rightarrow u'_{s-2(t-1)}$, which means that $u_{s-2t-2} \rightarrow u_{s-2t}$ and thus we are done. \square

Let $P' = v'_1v'_2\dots v'_s = wv_{s-1}v_1v_2v_3\dots v_{s-4}v_{s-3}v_{s-2}$. Note that P' is an anti-directed path satisfying that $v'_1 \rightarrow v'_2$, $v'_s \rightarrow v'_1$ and $|P'| = |P|$. For P' , by Claim 2.15, we get $v'_2 \rightarrow v'_4$ and $v'_6 \rightarrow v'_8$, which implies $v_{s-1} \rightarrow v_2$ and $v_4 \rightarrow v_6$. Since $v_{s-1} \rightarrow v_2$, we get an anti-directed path $P'' = v''_1v''_2\dots v''_s = wv_3v_1v_2v_{s-1}v_{s-2}\dots v_6v_5v_4$. Note that P'' satisfies $|P''| = |P|$ and $v''_1 \rightarrow v''_2$ as $w \rightarrow v_3$. Due to **Case 1**, we may assume $v''_s \rightarrow v''_1$. Thus, by Claim 2.14, we have $v''_{s-2} \rightarrow v''_s$. This implies that $v_6 \rightarrow v_4$, contradicting the fact that D is an oriented graph.

This completes the proof of Theorem 2.12. \square

3 Almost spanning anti-directed paths

In this section we show that for every $\varepsilon > 0$ every sufficiently large dense oriented graph G has an anti-directed path with at least $(1 - \varepsilon)|V(G)|$ vertices. Note that every oriented graph on n vertices that is k -antitractable has minimum degree at least $n - k + 1$, since if there were a vertex v of degree less than $n - k + 1$ then v and $k - 1$ non-neighbours of v would form a set of size k without an anti-directed path. To prove the result we will use the celebrated regularity lemma. To do so we need to introduce the following notation and well-known results.

Definition 3.1. Let V be the vertex set of a digraph G , and let X, Y be disjoint subsets of V . Furthermore, let $(X, Y)_G$ be the set of arcs that start in X and end in Y . Then the arc density of X and Y is

$$d(X, Y) = \frac{|(X, Y)_G|}{|X||Y|}.$$

Definition 3.2. Let G be a digraph with vertex set V and let $\varepsilon > 0$. An ordered pair (X, Y) of disjoint subsets of V is ε -regular if for all subsets $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon|X|, |B| \geq \varepsilon|Y|$,

$$|d(X, Y) - d(A, B)| \leq \varepsilon.$$

Definition 3.3. Let G be a digraph with vertex set V and let $\mathcal{P} = \{V_0, V_1, \dots, V_k\}$ be a partition of V into $k + 1$ sets. Then, \mathcal{P} is called an ε -regular partition of G if it satisfies the following conditions:

- $|V_0| \leq \varepsilon|V|$,
- $|V_1| = \dots = |V_k|$,
- all but at most εk^2 of ordered pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ε -regular.

We are now able to state Szemerédi's regularity lemma for oriented graphs which was proved by Alon and Shapira [7].

Theorem 3.4. If $\varepsilon \geq 0$ and $m \in \mathbb{N}$, then there exists $M \in \mathbb{N}$ such that if G is an oriented graph of order at least M then there exists an ε -regular partition of G with k parts $\{V_1, V_2, \dots, V_k\}$, where $m \leq k \leq M$.

It is well known that ε -regularity of a pair (X, Y) implies conditions on the degrees of vertices. For example, by considering the set X' in X of all vertices of out-degree less than $(d(X, Y) - \varepsilon)|Y'|$ into a set $Y' \subseteq Y$ with $|Y'| \geq \varepsilon|Y|$, it follows immediately from the definition that $|X'| \leq \varepsilon|X|$. We state this observation as a lemma for easier reference.

Lemma 3.5. *Let (X, Y) be a ε -regular pair with density d and let $Y' \subseteq Y$ ($X' \subseteq X$, resp.) be of size at least $\varepsilon|Y|$ ($\varepsilon|X|$, resp.). Then for all but at most $\varepsilon|X|$ ($\varepsilon|Y|$, resp.) vertices $v \in X$ ($v \in Y$, resp.), the inequality $|N_{Y'}^+(v)| \geq (d - \varepsilon)|Y'|$ ($|N_{X'}^-(v)| \geq (d - \varepsilon)|X'|$, resp.) holds.*

The following lemma shows that one can find long anti-directed paths in an ε -regular pair.

Lemma 3.6. *Let $\varepsilon > 0$ and let $d \geq 5\varepsilon$. Let (X, Y) be an ε -regular pair of density at least d with $|X| = |Y| = n$, and let $X' \subset X$ and $Y' \subset Y$ satisfy $|X'|, |Y'| < \varepsilon n$. Then for all $x \in X \setminus X'$ and all $y \in Y \setminus Y'$ of degree at least $(d - \varepsilon)n$ there exists an anti-directed path in $X \setminus X' \cup Y \setminus Y'$ starting at x and ending in y of length at least $(1 - \frac{\varepsilon}{d - \varepsilon} - 3\varepsilon)(|X| + |Y|)$.*

Proof. We first fix a set $X^* \subset X \setminus (X' \cup \{x\})$ of size $\varepsilon|X|$ in the neighbourhood of y . Note that such a set exists as y has degree at least $(d - \varepsilon)n - |X'| - 1 \geq \varepsilon n$ in $X \setminus (X' \cup \{x\})$. We now build the path iteratively. To do so we set $\bar{X}_0 = X' \cup X^* \cup \{x\}$, $\bar{Y}_0 = Y' \cup \{y\}$, and $p_0 = x$. At step t we assume that we have built a path from x to p_t using only vertices in $\bar{X}_t \cup \bar{Y}_t$ and that p_t has in-degree more than εn into $X \setminus \bar{X}_t$ if $p_t \in Y$ and out-degree more than εn into $Y \setminus \bar{Y}_t$ if $p_t \in X$. Note that these conditions are satisfied if $t = 0$ as the degree of x into $Y \setminus \bar{Y}_0$ is at least $(d - \varepsilon)n - |\bar{Y}_0| > \varepsilon n$. We stop the process when $X \setminus \bar{X}_t$ is smaller than $\varepsilon n / (d - \varepsilon) + 2$. (It will turn out that $|X \setminus \bar{X}_t|$ is always smaller than or equal to $|Y \setminus \bar{Y}_t|$ so this is a bound on both sets).

At step $t + 1$, we consider p_t . First assume that $p_t \in X$. By Lemma 3.5, there are at most εn vertices in Y with degree less than $(d - \varepsilon)|X \setminus \bar{X}_t|$ into $X \setminus \bar{X}_t$. As we have not stopped the process, $(d - \varepsilon)|X \setminus \bar{X}_t| > \varepsilon n$. As the degree of p_t is bigger than εn we can choose p_{t+1} as a neighbour of p_t that has εn neighbours in $X \setminus \bar{X}_t$. We then add p_{t+1} to \bar{Y}_t to obtain \bar{Y}_{t+1} and set $\bar{X}_{t+1} := \bar{X}_t$. If $p_t \in Y$ then we proceed analogously: By Lemma 3.5 there are at most εn vertices in X with degree less than $(d - \varepsilon)|Y \setminus \bar{Y}_t|$ into $Y \setminus \bar{Y}_t$. As we have not stopped the process $(d - \varepsilon)|Y \setminus \bar{Y}_t| > \varepsilon n$. As the degree of p_t is bigger than εn we can choose p_{t+1} as a neighbour of p_t that has εn neighbours in $Y \setminus \bar{Y}_t$. We then add p_{t+1} to \bar{X}_t to obtain \bar{X}_{t+1} and set $\bar{Y}_{t+1} := \bar{Y}_t$. When this process has stopped at vertex $x^* \in X$ we have an anti-directed path of length at least $(1 - \frac{\varepsilon}{d - \varepsilon} - 3\varepsilon)(|X| + |Y|)$ from x to x^* as we started with $X \setminus \bar{X}_0$ of size at least $n - 2\varepsilon n - 1$ and reduced the size of this set by 1 at every second step while adding a vertex to the path at every step until we reached a set of size $\varepsilon n / (d - \varepsilon) + 1$. We continue this path to y by choosing a neighbour of x^* that is not part of the anti-directed path that is a neighbour of a vertex in X^* . As the neighbourhood of x^* that is not part of the anti-directed path is of size at least εn and X^* is of size εn the density of this set is at least $d - \varepsilon$ and thus there are many possibilities. \square

Theorem 3.7. *Let $\varepsilon > 0$, $n, k \in \mathbb{N}$. For every sufficiently large n and every $k = o(n)$, every oriented graph D on n vertices with minimum degree $n - k$, has an anti-directed path of length at least $(1 - \varepsilon)n$.*

Proof. Let $\varepsilon' < \varepsilon/10$. We apply Theorem 3.4 with ε' (and $m = 1$) to obtain an ε' -regular partition V_0, \dots, V_ℓ . We consider the undirected simple reduced graph R which has a vertex for each V_1, \dots, V_ℓ and an edge between V_i, V_j if (V_i, V_j) or (V_j, V_i) form an ε' -regular pair of density at least $1/2 - \varepsilon'$. Note that there are at least $|V_i|(|V_j| - k)$ arcs between V_i and V_j in either direction and it follows that for sufficiently large n at least one of the pairs (V_i, V_j) of (V_j, V_i) have density at least $d = 1/2 - \varepsilon'$. Thus R is a complete graph with at most $\varepsilon'l^2$ edges missing. It is well known that a complete graph can be decomposed into (nearly) perfect matchings (in fact, a complete graph with an even number of vertices can be decomposed into perfect matchings and an old result attributed to Walecki by Lucas in 1892 [16] says that every complete graph with an odd number of vertices can be decomposed into hamiltonian cycles). It follows that R must have one matching with at least $\frac{\binom{\ell}{2} - \varepsilon'l^2}{\ell} = \frac{(1 - 2\varepsilon')\ell - 1}{2}$ many edges. Consider one of these matchings. Each edge of this matching corresponds to an ε' -regular (directed) pair (V_i, V_j) of density at least d . Let $(A_1, B_1), \dots, (A_t, B_t)$ be these ε' -regular pairs of density at least d .

We know from Lemma 3.6 that (A_i, B_i) contain long anti-directed paths and we now choose the starting and end vertex for each pair in a way that we can connect the anti-paths in (A_i, B_i) and (A_{i+1}, B_{i+1}) to get a almost hamiltonian anti-path in D . For $a_1 \in A_1$ we choose any vertex with out-degree into B_1 at least $(d - \varepsilon')|V(B_1)|$. By Lemma 3.5 many such vertices exist. For $i = 1, \dots, t - 1$, we consider two cases: Either there exists an arc (v, u) between a vertex u of B_i of in-degree at least $(d - \varepsilon')|V(A_i)|$ from A_i and a vertex v of A_{i+1} of out-degree at least $(d - \varepsilon')|V(B_{i+1})|$ into B_{i+1} , in which case we set $a_{i+1} = v$ and $b_i = u$; or all arcs between vertices of sufficiently large degree are directed from B_i to A_{i+1} . In the latter case, we choose an arc (u, v) in B_i between vertices of in-degree at least $(d - \varepsilon')|V(A_i)|$ from A_i . And then we choose an arc (x, y) within the subgraph induced by u 's out-neighbours in vertices in A_{i+1} with out-degree at least $(d - \varepsilon')|V(B_{i+1})|$ into B_{i+1} . Note that such arcs exist as by Lemma 3.5 the set of these vertices is much larger than k and the minimum degree is $n - k$. In addition, the path $v \leftarrow u \rightarrow x \leftarrow y$ can connect any (a_i, v) anti-path in (A_i, B_i) obtained by Lemma 3.6 and any anti-path in (A_i, B_i) starts at y . Thus, we choose $b_i = v$, $a_{i+1} = x$ and set $B'_i = \{u\}$ and $A'_{i+1} = \{y\}$ when applying Lemma 3.6. Finally we choose b_t any vertex with in-degree into A_t at least $(d - \varepsilon')|V(A_t)|$. Again by Lemma 3.5 many such vertices exist.

Therefore, we formed an anti-path with order at least

$$\frac{(1 - 2\varepsilon')\ell - 1}{2} \times \left(1 - \frac{\varepsilon'}{d - \varepsilon'} - 3\varepsilon'\right) \left(\frac{2(1 - \varepsilon')n}{\ell}\right) \geq (1 - 9\varepsilon')(1 - \varepsilon')n > (1 - \varepsilon)n,$$

which completes the proof. \square

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