Complex generalized Gauss-Radau quadrature rules for Hankel transforms of integer order

Haiyong Wang*† and Menghan Wu*

March 29, 2024

Abstract

Complex Gaussian quadrature rules for oscillatory integral transforms have the advantage that they can achieve optimal asymptotic order. However, their existence for Hankel transform can only be guaranteed when the order of the transform belongs to $[0, 1/2]$. In this paper we consider the construction of generalized Gauss-Radau quadrature rules for Hankel transform. We show that, if adding certain value and derivative information at the left endpoint, then complex generalized Gauss-Radau quadrature rules for Hankel transform of integer order can be constructed with theoretical guarantees. Orthogonal polynomials that are closely related to such quadrature rules are investigated and their existence for even degrees is proved. Numerical experiments are presented to confirm our findings.

Keywords: Hankel transform, generalized Gauss-Radau quadrature, Abel limit, asymptotic error estimate

AMS classifications: 65R10, 65D32

1 Introduction

Hankel transform of the form

$$(\mathcal{H}_\nu f)(\omega) := \int_0^\infty f(x) J_\nu(\omega x)\,\mathrm{d}x, \tag{1.1}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν and ω is the frequency of oscillations, appears in many physical problems, e.g. the propagation of optical, acoustic and electromagnetic fields and electromagnetic geophysics (see, e.g., [4, 18, 20]). Closed form of this transform is rarely available and numerical methods are generally required. However, conventional numerical methods for the integral (1.1) are quite expensive due to the oscillatory and possibly slowly decaying behaviors of the integrand.

*School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, P. R. China. Email:haiyongwang@hust.edu.cn

†Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, P. R. China

The evaluation of the Hankel transform has received considerable attention due to its practical importance and several methods have been developed, such as the digital linear filter (DLF) method [3, 8], the integration, summation and extrapolation (ISE) method [14, 15, 16], complex Gaussian quadrature rules [1, 21]. Among these methods, the DLF and ISE methods have their own disadvantages that the former requires samples on an exponential grid and the latter requires the evaluation of the zeros of $J_\nu(x)$. Complex Gaussian quadrature rules are of particular interest due to the appealing advantage that their accuracy improves rapidly as ω increases. Specifically, Wong in [21] considered the construction of complex Gaussian quadrature rules for Hankel transforms of the form

$$\int_0^\infty x^\mu f(x) H_\nu^{(s)}(x)\,\mathrm{d}x, \quad s = 1, 2,$$

where $H_\nu^{(s)}(x)$ are the Hankel functions and $\mu \pm \nu > -1$. By rotating the integration path to the imaginary axis such that the integrand is non-oscillatory and decays exponentially, Wong constructed some complex Gaussian quadrature rules for the above transforms. However, when extending Wong's method to the transform (1.1), the resulting integrand will involve a nonintegrable singularity at the origin for $\nu \geq 1$. More recently, Asheim and Huybrechs in [1] studied the construction of complex Gaussian quadrature rule for the Hankel transform (1.1) as well as Fourier and Airy transforms. Their key idea is to construct Gaussian quadrature rules with respect to the oscillatory weight function directly. Taking the Hankel transform (1.1) for example, they introduced the sequence of monic polynomials $\{P_n\}_{n=0}^\infty$ that are orthogonal with respect to $J_\nu(x)$ on $(0, \infty)$, i.e.,

$$\int_0^\infty P_n(x) x^k J_\nu(x)\mathrm{d}x = 0, \quad k = 0, \ldots, n-1, \tag{1.2}$$

and the above improper integrals are defined in terms of their Abel limits (see subsection 2.2). Since $J_\nu(x)$ is a sign-changing function, the existence and uniqueness of the polynomials $P_n(x)$ cannot be guaranteed. Once the polynomials exist, Gaussian quadrature rule for the Hankel transform (1.1) can be derived immediately by using a simple scaling. A remarkable advantage of such rules is that they have the optimal asymptotic order, i.e., their error decays at the fastest algebraic rate as $\omega \to \infty$ among all quadrature rules using the same number of points. In the special case $\nu = 0$, it was shown in [1, Theorem 3.5] that the zeros of $P_n(x)$ with even n are located on the imaginary axis. For $\nu > 0$, based on numerical calculations for $\nu = 1/2, 1, 3/2, 2$, it was observed that the zeros of $P_n(x)$ tend to cluster near the vertical line $\Re(z) = \nu\pi/2$ as $n \to \infty$ (see Figure 1 for more detailed observations). More recently, this observation was proved in [6] for $\nu \in [0, 1/2]$ by using the steepest descent method for the Riemann-Hilbert problem of $P_n(x)$.

In this work, we consider the construction of complex generalized Gauss-Radau quadrature rules for the Hankel transform (1.1) of integer order and the key motivation of this study is to construct some Gaussian quadrature rules with theoretical guarantees. We begin by giving conditions for rotating the integration path of oscillatory integral transforms, including Hankel and Fourier transforms, in the right-half plane under their Abel limits. We then show that, if adding certain value and derivative information of

$f(x)$ at the left endpoint, complex Gaussian quadrature rules for the Hankel transform (1.1) of integer order can be constructed with guaranteed existence. Since such rules involve the value and derivative information of $f(x)$ at the left endpoint and their nodes all lie on the imaginary axis, we refer to them as complex generalized Gauss-Radau quadrature rules. Except for existence, another appealing advantage of the proposed rules is that they also achieve the optimal asymptotic order. On the other hand, let μ be a nonnegative integer and let $\{P_n^{(\mu,\nu)}\}_{n=0}^{\infty}$ be the sequence of monic polynomials that are orthogonal with respect to the weight $x^\mu J_\nu(x)$ on $(0,\infty)$, i.e.,

$$\int_0^\infty P_n^{(\mu,\nu)}(x) x^{k+\mu} J_\nu(x)\,\mathrm{d}x = 0, \quad k = 0, \ldots, n-1. \tag{1.3}$$

It is easily seen that such polynomials include the polynomials in (1.2) as a special case and their existence cannot be guaranteed since the weight $x^\mu J_\nu(x)$ is also a sign-changing function. Unexpectedly, we prove that, if both ν and $\mu-\nu$ are nonnegative integers, such polynomials exist for all degrees when $\mu-\nu$ is even and for only even degrees when $\mu-\nu$ is odd. Moreover, in these cases where the polynomials exist, their zeros are located on the imaginary axis and symmetric with respect to the real axis. These results provide a theoretical rationale for the proposed complex generalized Gauss-Radau quadrature rules.

The rest of this paper is organized as follows. In the next section, we review some basic properties of generalized Gauss-Radau quadrature rules and the regularization of improper Riemann integrals. In section 3, we construct complex generalized Gauss-Radau quadrature rules for the Hankel transform of integer order and study the existence of orthogonal polynomials that are closely related to such quadrature rules. In section 4, we extend the discussion to the Hankel transform of fractional order and Fourier sine transform and give an application to oscillatory Hilbert transform. Finally, we give some conclusions in section 5.

2 Preliminaries

We introduce some basics of generalized Gauss-Radau quadrature rules and the regularization of improper Riemann integrals. Throughout the paper, we denote by $\mathcal{P}_n$ the space of polynomials of degree at most n, i.e., $\mathcal{P}_n = \mathrm{span}\{1, x, \ldots, x^n\}$, and by $\mathbb{N}_0$ the set of nonnegative integers. Moreover, we denote by $\mathcal{K}$ a generic positive constant.

2.1 Generalized Gauss-Radau quadrature rule

In this subsection we review some basic properties of generalized Gauss-Radau quadrature rule. Consider the following integral

$$I(f) = \int_a^b w(x) f(x)\,\mathrm{d}x, \tag{2.1}$$

where $w(x)$ is a weight function on (a,b) and f is a smooth function. Let $r \in \mathbb{N}$, if the left endpoint and its consecutive derivatives up to an order $r-1$ are prescribed, then an interpolatory quadrature rule can be constructed as

$$Q_{n,r}^{\mathrm{GR}}(f) = \sum_{j=0}^{r-1} w_j^a f^{(j)}(a) + \sum_{j=1}^{n} w_j f(x_j). \tag{2.2}$$

If the nodes $\{x_j\}_{j=1}^n$ and the weights $\{w_j^a\}_{j=0}^{r-1} \cup \{w_j\}_{j=1}^n$ in (2.2) are chosen to maximize the degree of exactness, i.e., $(I - Q_{n,r}^{\mathrm{GR}})(f) = 0$ for $f \in \mathcal{P}_{2n+r-1}$, then the quadrature rule (2.2) is known as a generalized Gauss-Radau quadrature rule*.

Generalized Gauss-Radau quadrature rule has received certain attention in the past two decades and it is now known that the nodes $\{x_j\}_{j=1}^n$ are located in (a,b) and the weights $\{w_j^a\}_{j=0}^{r-1} \cup \{w_j\}_{j=1}^n$ are all positive (see, e.g., [9, 11, 13]). In the following we briefly describe the implementation of the generalized Gauss-Radau quadrature rule. Let $\{\psi_k\}_{k=0}^{\infty}$ be the sequence of monic polynomials that are orthogonal with respect to the new weight function $w_r(x) = w(x)(x-a)^r$ on (a,b) and

$$\int_a^b w_r(x)\psi_k(x)\psi_j(x)\mathrm{d}x = \gamma_k \delta_{k,j},$$

where $\delta_{k,j}$ is the Kronecker delta and $\gamma_k > 0$. Moreover, let $\{x_j^{\mathrm{G}}, w_j^{\mathrm{G}}\}_{j=1}^n$ denote the nodes and weights of an n-point Gaussian quadrature with respect to the weight $w_r(x)$ on (a,b), i.e.,

$$\int_a^b w_r(x) f(x)\mathrm{d}x = \sum_{j=1}^{n} w_j^{\mathrm{G}} f(x_j^{\mathrm{G}}), \quad \forall f \in \mathcal{P}_{2n-1}. \tag{2.3}$$

From [10, Theorem 3.9] we know that

$$x_j = x_j^{\mathrm{G}}, \quad w_j = \frac{w_j^{\mathrm{G}}}{(x_j - a)^r}, \quad j = 1, 2, \ldots, n,$$

and thus the interior nodes and weights of $Q_{n,r}^{\mathrm{GR}}(f)$, i.e., $\{x_j\}_{j=1}^n \cup \{w_j\}_{j=1}^n$, can be calculated from the nodes and weights of the Gaussian quadrature rule (2.3). As for the boundary weights of $Q_{n,r}^{\mathrm{GR}}(f)$, i.e., $\{w_j^a\}_{j=0}^{r-1}$, setting $f(x) = (x-a)^{i-1}\psi_n^2(x)$ with $i = 1, 2, \ldots, r$ in (2.2) gives

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ & a_{22} & \cdots & a_{2r} \\ & & \ddots & \vdots \\ & & & a_{rr} \end{pmatrix} \begin{pmatrix} w_0^a \\ w_1^a \\ \vdots \\ w_{r-1}^a \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix},$$

*Strictly speaking, the rule $Q_{n,r}^{\mathrm{GR}}(f)$ is referred to as a Gauss-Radau quadrature rule when $r = 1$ and a generalized Gauss-Radau quadrature rule when $r \geq 2$. Here, we refer to $Q_{n,r}^{\mathrm{GR}}(f)$ as a generalized Gauss-Radau quadrature rule for all $r \geq 1$ for the sake of simplicity.

where

$$a_{ij} = \left[(x-a)^{i-1}\psi_n^2(x)\right]_{x=a}^{(j-1)}, \quad b_i = \int_a^b \omega(x)(x-a)^{i-1}\psi_n^2(x)\,\mathrm{d}x.$$

Therefore, the boundary weights can be derived by solving the above upper triangular system. Note that the diagonal entries $\{a_{ii}\}_{i=1}^r$ will be extremely small for large n and this may result in an underflow problem. However, this issue may be circumvented by rescaling the monic polynomials $\{\psi_n\}$ appropriately (see [11]).

2.2 Regularization of improper Riemann integrals

In this subsection we introduce the regualrization of improper Riemann integrals. For the improper Riemann integral $\int_0^\infty f(x)\,\mathrm{d}x$, its Abel limit is defined by [22, Chapter 4]

$$\lim_{s\to 0^+}\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x. \tag{2.4}$$

When the improper Riemann integral exists, e.g., f is absolutely integrable on $(0,\infty)$, then the Abel limit is simply the improper integral itself. However, the importance of the Abel limit is that it may exist for certain divergent integrals that do not exist in the classical sense. Typical examples include the following two integrals

$$\int_0^\infty x^\mu e^{\mathrm{i}\omega x}\,\mathrm{d}x = \frac{\Gamma(\mu+1)}{\omega^{\mu+1}}e^{(\mu+1)\pi\mathrm{i}/2}, \tag{2.5}$$

and

$$\int_0^\infty x^\mu J_\nu(\omega x)\,\mathrm{d}x = \frac{2^\mu\Gamma((\nu+\mu+1)/2)}{\omega^{\mu+1}\Gamma((\nu-\mu+1)/2)}, \tag{2.6}$$

where $\Gamma(z)$ is the gamma function and $\Re(\mu) > -1$ for (2.5) and $\Re(\mu+\nu) > -1$ for (2.6). Note that (2.5) and (2.6) play an important role in studying the asymptotic behaviors of Fourier and Hankel transforms (see, e.g., [22, Chapter 4]). Moreover, (2.6) was also used in [1] to compute the Gaussian quadrature rule with respect to the weight function $J_\nu(x)$ on $(0,\infty)$.

Below we state the first main result of this work, which gives conditions on the rotation of integration path of Fourier and Hankel transforms in the right-half plane. In the remainder of this paper, all improper integrals are defined using their Abel limits.

Theorem 2.1. *If f is analytic in the right half-plane and $|f(z)| \le \mathcal{K}|z|^\sigma$ for some $\sigma\in\mathbb{R}$, then for $\Re(\mu) > -1$,*

$$\int_0^\infty f(x)x^\mu e^{\mathrm{i}\omega x}\,\mathrm{d}x = e^{(\mu+1)\pi\mathrm{i}/2}\int_0^\infty f(\mathrm{i}x)x^\mu e^{-\omega x}\,\mathrm{d}x, \tag{2.7}$$

and for $\Re(\mu\pm\nu) > -1$,

$$\int_0^\infty f(x)x^\mu J_\nu(\omega x)\,\mathrm{d}x = \int_0^\infty \widehat{f}(\mathrm{i}x)x^\mu K_\nu(\omega x)\,\mathrm{d}x, \tag{2.8}$$

where $\widehat{f}(x) = (e^{(\mu-\nu)\pi\mathrm{i}/2}f(x) + e^{(\nu-\mu)\pi\mathrm{i}/2}f(-x))/\pi$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

Proof. We only sketch the proof of (2.8) and the proof of (2.7) is similar. Let C_ε with $\varepsilon > 0$ denote the circle of radius ε in the first quadrant and let C_λ with $\lambda > \varepsilon$ denote the circle of radius λ in the first quadrant. For $s > 0$, by the identity [17, Equation (10.27.9)], i.e., $\mathrm{i}\pi J_\nu(z) = e^{-\nu\pi\mathrm{i}/2}K_\nu(-\mathrm{i}z) - e^{\nu\pi\mathrm{i}/2}K_\nu(\mathrm{i}z)$ for $|\arg(z)| \le \pi/2$, we have

$$\begin{aligned}\int_\varepsilon^\lambda e^{-sx} f(x) x^\mu J_\nu(\omega x)\,\mathrm{d}x = {}& \frac{e^{-\nu\pi\mathrm{i}/2}}{\mathrm{i}\pi}\int_\varepsilon^\lambda e^{-sx} f(x) x^\mu K_\nu(-\mathrm{i}\omega x)\,\mathrm{d}x \\ & - \frac{e^{\nu\pi\mathrm{i}/2}}{\mathrm{i}\pi}\int_\varepsilon^\lambda e^{-sx} f(x) x^\mu K_\nu(\mathrm{i}\omega x)\,\mathrm{d}x. \end{aligned} \tag{2.9}$$

For the first integral on the right-hand side, by Cauchy's theorem, we know that

$$\begin{aligned}\int_\varepsilon^\lambda e^{-sx} f(x) x^\mu K_\nu(-\mathrm{i}\omega x)\,\mathrm{d}x = {}& \left(\int_{C_\varepsilon} - \int_{C_\lambda}\right) e^{-sz} f(z) z^\mu K_\nu(-\mathrm{i}\omega z)\,\mathrm{d}z \\ & + e^{(\mu+1)\pi\mathrm{i}/2}\int_\varepsilon^\lambda e^{-\mathrm{i}sx} f(\mathrm{i}x) x^\mu K_\nu(\omega x)\,\mathrm{d}x, \end{aligned} \tag{2.10}$$

where the contours are taken in the counterclockwise direction. From [17, Chapter 10] we know that $K_{-\nu}(z) = K_\nu(z)$, and $K_\nu(z) = O(z^{-\nu})$ for $\Re(\nu) > 0$ and $K_0(z) = O(\ln z)$ as $z \to 0$. In the case of $\Re(\nu) = 0$ and $\nu \ne 0$, $K_\nu(z) = O(1)$ as $z \to 0$. Hence,

$$\left|\int_{C_\varepsilon} e^{-sz} f(z) z^\mu K_\nu(-\mathrm{i}\omega z)\,\mathrm{d}z\right| = \begin{cases} O(\varepsilon^{\Re(\mu+1-\nu)}), & \Re(\nu) > 0, \\ O(\varepsilon^{\Re(\mu+1)}\ln\varepsilon), & \Re(\nu) = 0,\ \Im(\nu) = 0, \\ O(\varepsilon^{\Re(\mu+1)}), & \Re(\nu) = 0,\ \Im(\nu) \ne 0, \\ O(\varepsilon^{\Re(\mu+1+\nu)}), & \Re(\nu) < 0, \end{cases}$$

and therefore the contour integral on the left-hand side vanishes as $\varepsilon \to 0^+$. Moreover, parametrizing C_λ by $z = \lambda e^{\mathrm{i}\theta}$ with $0 \le \theta \le \pi/2$ and using [17, Equation (10.25.3)], we have

$$\begin{aligned}\left|\int_{C_\lambda} e^{-sz} f(z) z^\mu K_\nu(-\mathrm{i}\omega z)\,\mathrm{d}z\right| &\le \mathcal{K}\lambda^{\mu+\sigma+1}\int_0^{\pi/2} e^{-s\lambda\cos\theta}|K_\nu(-\mathrm{i}\omega\lambda e^{\mathrm{i}\theta})|\,\mathrm{d}\theta \\ &\le \mathcal{K}\lambda^{\mu+\sigma+1/2}\int_0^{\pi/2} e^{-\lambda(s\cos\theta+\omega\sin\theta)}\,\mathrm{d}\theta, \\ &\le \mathcal{K}\lambda^{\mu+\sigma+1/2}e^{-\lambda s}, \end{aligned}$$

where we have used the fact that the maximum of the integrand in the second inequality is attained at $\theta = 0$ for $\omega \ge s > 0$. Therefore, the contour integral on the left-hand side vanishes as $\lambda \to \infty$. Letting $\varepsilon \to 0^+$ and $\lambda \to \infty$ in (2.10), we obtain

$$\int_0^\infty e^{-sx} f(x) x^\mu K_\nu(-\mathrm{i}\omega x)\,\mathrm{d}x = e^{(\mu+1)\pi\mathrm{i}/2}\int_0^\infty e^{-\mathrm{i}sx} f(\mathrm{i}x) x^\mu K_\nu(\omega x)\,\mathrm{d}x.$$

and similarly,

$$\int_0^\infty e^{-sx} f(x) x^\mu K_\nu(\mathrm{i}\omega x)\,\mathrm{d}x = e^{-(\mu+1)\pi\mathrm{i}/2} \int_0^\infty e^{\mathrm{i}sx} f(-\mathrm{i}x) x^\mu K_\nu(\omega x)\,\mathrm{d}x.$$

Combining the above two equations with (2.9) and letting $s \to 0^+$ gives (2.8). This ends the proof. $\square$

We have the following remarks to Theorem 2.1.

Remark 2.2. Both (2.5) and (2.6) can also be simply derived by setting $f(x) = 1$ in Theorem 2.1. Note that the condition in (2.8) is more restrictive than the condition in (2.6), which implies that the condition $\Re(\mu \pm \nu) > -1$ in (2.8) might be relaxed.

Remark 2.3. Equation (2.8) was actually used in [1, Theorem 3.5] to construct Gaussian quadrature rule with respect to $J_0(x)$ on $(0, \infty)$ with the requirement that f is analytic in the right half-plane and suitably decaying at infinity. However, the validity of this requirement was not proved therein.

3 Complex generalized Gauss-Radau quadrature rules for Hankel transform of integer order

In this section, we consider the construction of complex generalized Gauss-Radau quadrature rules for the Hankel transform (1.1) of integer order, i.e., $\nu \in \mathbb{Z}$. Note that $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$, it is enough to consider the case $\nu \in \mathbb{N}_0$.

Let $\mu \in \mathbb{N}_0$ and $\mu \geq \nu$ and let

$$w_{\mu,\nu}(x) = \frac{K_\nu(\sqrt{x})}{2} \begin{cases} x^{(\mu-1)/2}, & \mu - \nu \text{ even}, \\ x^{\mu/2}, & \mu - \nu \text{ odd}, \end{cases} \tag{3.1}$$

be the weight function on $(0, \infty)$. Let $\{\phi_n\}_{n=0}^\infty$ denote the sequence of monic polynomials that are orthogonal with respect to the weight $w_{\mu,\nu}(x)$ and

$$\int_0^\infty w_{\mu,\nu}(x) \phi_n(x) \phi_m(x)\,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \tau_n, & n = m. \end{cases} \tag{3.2}$$

The n-point Gaussian quadrature rule with respect to $w_{\mu,\nu}(x)$ is

$$\int_0^\infty w_{\mu,\nu}(x) f(x)\,\mathrm{d}x = \sum_{j=1}^n w_j f(x_j), \quad \forall f \in \mathcal{P}_{2n-1}, \tag{3.3}$$

where $\{x_j\}_{j=1}^n$ are the zeros of $\phi_n(x)$ and $w_j = \tau_{n-1}/({\phi_n}'(x_j)\phi_{n-1}(x_j))$. By the properties of orthogonal polynomials and Gaussian quadrature rules, we know that $x_j \in (0, \infty)$ and $w_j > 0$ for $j = 1, \ldots, n$.

Below we state the second main result of this work, which gives complex generalized Gauss-Radau quadrature rules for the Hankel transform (1.1) of integer order.

Theorem 3.1. *Let $\{x_j, w_j\}_{j=1}^n$ be the nodes and weights of the Gaussian quadrature defined in (3.3) and suppose that f is analytic in the right half-plane and $|f(z)| \le \mathcal{K}|z|^\sigma$ for some $\sigma \in \mathbb{R}$. Then, a quadrature rule for the Hankel transform (1.1) is given by*

$$(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega) = \frac{1}{\omega}\left(\sum_{k=0}^{\mu-1} \frac{\hat{w}_k^0}{\omega^k} f^{(k)}(\hat{x}_0) + \sum_{j=1}^{2n} \hat{w}_j f\left(\frac{\hat{x}_j}{\omega}\right)\right), \tag{3.4}$$

where $\{\hat{x}_j\}_{j=1}^{2n} = \{\pm \mathrm{i}\sqrt{x_j}\}_{j=1}^n$ and $\hat{x}_0 = 0$, and

$$\{\hat{w}_j\}_{j=1}^{2n} = \left\{\exp\left(\mp\frac{\nu\pi\mathrm{i}}{2}\right)\frac{w_j x_j^{-\kappa/2}}{\pi}\right\}_{j=1}^n,$$

and for $k = 0, \ldots, \mu-1$,

$$\hat{w}_k^0 = \frac{1}{k!}\left(2^k \frac{\Gamma((\nu+k+1)/2)}{\Gamma((\nu-k+1)/2)} - \frac{2}{\pi}\cos\left(\frac{(k-\nu)\pi}{2}\right)\sum_{j=1}^n w_j x_j^{(k-\kappa)/2}\right),$$

and $\kappa = \mu$ when $\mu-\nu$ is even and $\kappa = \mu+1$ when $\mu-\nu$ is odd. Moreover, the quadrature rule (3.4) is exact for $f \in \mathcal{P}_{4n+\mu-1}$ when $\mu-\nu$ is even and for $f \in \mathcal{P}_{4n+\mu}$ when $\mu-\nu$ is odd.

Proof. Let $\mathcal{T}_\mu$ denote the Taylor expansion of f of degree $\mu-1$ at $\hat{x}_0 = 0$. It follows that

$$(\mathcal{H}_\nu f)(\omega) = \int_0^\infty \mathcal{T}_\mu(x) J_\nu(\omega x)\,\mathrm{d}x + \int_0^\infty \mathcal{R}_\mu(x) x^\mu J_\nu(\omega x)\,\mathrm{d}x,$$

where $\mathcal{R}_\mu(x) = (f(x) - \mathcal{T}_\mu(x))/x^\mu$. For the first integral on the right-hand side, using (2.6) we have

$$\int_0^\infty \mathcal{T}_\mu(x) J_\nu(\omega x)\,\mathrm{d}x = \sum_{k=0}^{\mu-1} \frac{f^{(k)}(\hat{x}_0)}{k!}\int_0^\infty x^k J_\nu(\omega x)\,\mathrm{d}x = \sum_{k=0}^{\mu-1} \frac{f^{(k)}(\hat{x}_0) 2^k}{k!\omega^{k+1}} \frac{\Gamma((\nu+k+1)/2)}{\Gamma((\nu-k+1)/2)}.$$

For the second integral, using Theorem 2.1 we have

$$\int_0^\infty \mathcal{R}_\mu(x) x^\mu J_\nu(\omega x)\,\mathrm{d}x = \int_0^\infty \widehat{\mathcal{R}}_\mu(\mathrm{i}x) x^\mu K_\nu(\omega x)\,\mathrm{d}x,$$

where $\widehat{\mathcal{R}}_\mu(x) = (e^{(\mu-\nu)\mathrm{i}\pi/2}\mathcal{R}_\mu(x) + e^{(\nu-\mu)\mathrm{i}\pi/2}\mathcal{R}_\mu(-x))/\pi$. It is easily verified that $\widehat{\mathcal{R}}_\mu(x)$ is an even function when $\mu-\nu$ is even and is an odd function when $\mu-\nu$ is odd. By the parity of $\widehat{\mathcal{R}}_\mu(x)$ and using the transformation $x \mapsto \sqrt{x}/\omega$ to the integral on the right-hand side yields

$$\int_0^\infty \mathcal{R}_\mu(x)x^\mu J_\nu(\omega x)\,\mathrm{d}x = \frac{1}{\omega^{\mu+1}}\begin{cases} \displaystyle\int_0^\infty \widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x}}{\omega}\right) w_{\mu,\nu}(x)\,\mathrm{d}x, & \mu-\nu \text{ even}, \\ \displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x}}{\omega}\right) w_{\mu,\nu}(x)\,\mathrm{d}x, & \mu-\nu \text{ odd}, \end{cases}$$
$$\approx \frac{1}{\omega^{\mu+1}}\begin{cases} \displaystyle\sum_{j=1}^{n} w_j\widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x_j}}{\omega}\right), & \mu-\nu \text{ even}, \\ \displaystyle\sum_{j=1}^{n} \frac{w_j}{\sqrt{x_j}}\widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x_j}}{\omega}\right), & \mu-\nu \text{ odd}, \end{cases} \tag{3.5}$$

where we have used the n-point Gausssian quadrature rule in (3.3) to evaluate the integrals in the first line. Combining all the above results and after some simplification gives the quadrature rule (3.4). Moreover, the Gaussian quadrature rule in (3.5) is exact for $\widehat{\mathcal{R}}_\mu \in \mathcal{P}_{4n-2}$ when $\mu-\nu$ is even and for $\widehat{\mathcal{R}}_\mu \in \mathcal{P}_{4n-1}$ when $\mu-\nu$ is odd.

Finally, we show the exactness of the quadrature rule (3.4). When $\mu-\nu$ is even, by the Taylor expansion of f, we get

$$\mathcal{R}_\mu(x) = \sum_{j=0}^{\infty}\frac{f^{(j+\mu)}(0)}{(j+\mu)!}x^j \quad\Rightarrow\quad \widehat{\mathcal{R}}_\mu(x) = \frac{2}{\pi}e^{(\mu-\nu)\pi\mathrm{i}/2}\sum_{j=0}^{\infty}\frac{f^{(2j+\mu)}(0)}{(2j+\mu)!}x^{2j}.$$

Recall that the quadrature rule in (3.5) is exact for $\widehat{\mathcal{R}}_\mu \in \mathcal{P}_{4n-2}$, we therefore deduce that the quadrature rule (3.4) is exact for $f \in \mathcal{P}_{4n+\mu-1}$. When $\mu-\nu$ is odd, it can be shown in a similar way that the quadrature rule (3.4) is exact for $f \in \mathcal{P}_{4n+\mu}$. This ends the proof. □

Some remarks on Theorem 3.1 are in order.

Remark 3.2. When ν is an even integer, the weights $\{\hat{w}_j\}_{j=1}^{2n}$ are all real and the weights corresponding to the nodes $\{\mathrm{i}\sqrt{x_j}\}_{j=1}^{n}$ are the same as the weights corresponding to $\{-\mathrm{i}\sqrt{x_j}\}_{j=1}^{n}$. When ν is an odd integer, then the weights $\{\hat{w}_j\}_{j=1}^{2n}$ are all purely imaginary and the weights corresponding to $\{\mathrm{i}\sqrt{x_j}\}_{j=1}^{n}$ are the negative values of the weights corresponding to $\{-\mathrm{i}\sqrt{x_j}\}_{j=1}^{n}$.

Remark 3.3. When $\mu=\nu=0$, the nodes and weights of the quadrature rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ are $\{\hat{x}_j\}_{j=1}^{2n} = \{\pm\mathrm{i}\sqrt{x_j}\}_{j=1}^{n}$ and $\{\hat{w}_j\}_{j=1}^{2n} = \{w_j/\pi\}_{j=1}^{n} \cup \{w_j/\pi\}_{j=1}^{n}$, and $\{x_j, w_j\}_{j=1}^{n}$ are the nodes and weights of the n-point Gaussian quadrature rule with respect to $w_{0,0}(x) = x^{-1/2}K_0(\sqrt{x})/2$ on $(0,\infty)$. In this case, the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ is exactly the Gaussian quadrature rule derived in [1, Theorem 3.5] (Note that w_j/π in $\{\hat{w}_j\}_{j=1}^{2n}$ was mistakenly written as $w_j/2$ in [1, Theorem 3.5]).

Remark 3.4. If $f(x)$ is even and ν is an odd integer or if $f(x)$ is odd and ν is an even integer, then from Remark 3.2 we can deduce that the second sum on the right hand side of (3.4) vanishes and thus the quadrature rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ can be simplified.

Asymptotic error estimates of the quadrature rule (3.4) are given below.

Theorem 3.5. *Under the assumptions of Theorem 3.1, we have*

$$(\mathcal{H}_\nu f)(\omega) - (\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega) = \begin{cases} \mathcal{O}(\omega^{-4n-\mu-1}), & \mu-\nu \text{ even}, \\ \mathcal{O}(\omega^{-4n-\mu-2}), & \mu-\nu \text{ odd}, \end{cases} \qquad \omega \to \infty. \tag{3.6}$$

Proof. Note that the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ is exact for $f \in \mathcal{P}_{4n+\mu-1}$ when $\mu-\nu$ is even and for $f \in \mathcal{P}_{4n+\mu}$ when $\mu-\nu$ is odd, the asymptotic error estimates (3.6) follows immediately from [1, Lemma 1.6]. $\square$

Remark 3.6. For the rules $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ with $\mu = \nu+2k-1$ and $\mu = \nu+2k$ and $k \geq 1$, it is easily verified that their nodes $\{\hat{x}_j\}_{j=1}^{2n}$ and weights $\{\hat{w}_j\}_{j=1}^{2n}$ are the same. Moreover, direct calculation shows that $\hat{w}_{\nu+2j-1}^0 = 0$ for $j \geq 1$, and therefore the boundary weights $\{\hat{w}_j^0\}_{j=0}^{\nu+2k-2}$ for these two rules are also the same. We conclude that the rules $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ with $\mu = \nu+2k-1$ and $\mu = \nu+2k$ and $k \geq 1$ are always the same and this explains why their asymptotic error estimates in (3.6) are the same.

Remark 3.7. Since the exactness of the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ is maximized, we conclude that it achieves the optimal asymptotic order in the sense that their error decays at the fastest algebraic rate with respect to ω^{-1}.

From Theorem 3.1 we see that the nodes of the generalized Gauss-Radau quadrature rule can be derived from the nodes of the Gaussian quadrature rule in (3.3). This implies the existence of the polynomials defined in (1.3) for even degrees. Below we state a more general result, which gives the polynomials of all degrees when $\mu-\nu$ is even. When $\mu-\nu$ is odd, however, the polynomials exist only for even degrees.

Theorem 3.8. *Let $\mu \in \mathbb{N}_0$ and $\mu \geq \nu$ and let $\{\phi_n\}_{n=0}^{\infty}$ be the polynomials defined in* (3.2)*. When $\mu-\nu$ is even, then for each $n \geq 0$,*

$$P_{2n}^{(\mu,\nu)}(x) = (-1)^n \phi_n(-x^2), \tag{3.7}$$

and

$$P_{2n+1}^{(\mu,\nu)}(x) = \frac{(-1)^{n+1}}{x} \left(\phi_{n+1}(-x^2) - \frac{\phi_{n+1}(0)}{\phi_n(0)} \phi_n(-x^2) \right). \tag{3.8}$$

When $\mu-\nu$ is odd, then (3.7) *still holds, but $P_{2n+1}^{(\mu,\nu)}(x)$ does not exist.*

Proof. We first consider the case when $\mu-\nu$ is even. We only consider the proof of (3.7), since the proof of (3.8) is similar. For any $s \in \mathcal{P}_{2n-1}$, by the definition of $P_{2n}^{(\mu,\nu)}(x)$,

$$\int_0^\infty P_{2n}^{(\mu,\nu)}(x) s(x) x^\mu J_\nu(x)\,\mathrm{d}x = 0.$$

For the integral on the left-hand side, let $\Psi(x) = P_{2n}^{(\mu,\nu)}(x)s(x)$ and using Theorem 2.1 we have

$$\begin{aligned}\int_0^\infty \Psi(x)x^\mu J_\nu(x)\,\mathrm{d}x &= \frac{e^{(\mu-\nu)\mathrm{i}\pi/2}}{\pi}\int_0^\infty [\Psi(\mathrm{i}x)+\Psi(-\mathrm{i}x)]\,x^\mu K_\nu(x)\,\mathrm{d}x \\ &= \frac{e^{(\mu-\nu)\mathrm{i}\pi/2}}{\pi}\int_0^\infty \left[\Psi(\mathrm{i}\sqrt{x})+\Psi(-\mathrm{i}\sqrt{x})\right] w_{\mu,\nu}(x)\,\mathrm{d}x,\end{aligned}$$

where we have used the transformation $x \mapsto \sqrt{x}$ in the last equality. If we set $s(x) = x^{2k+1}$, where $k = 0, \ldots, n-1$, then

$$\int_0^\infty \left[\sqrt{x}\left(P_{2n}^{(\mu,\nu)}(\mathrm{i}\sqrt{x}) - P_{2n}^{(\mu,\nu)}(-\mathrm{i}\sqrt{x})\right)\right] x^k w_{\mu,\nu}(x)\,\mathrm{d}x = 0.$$

Note that the term inside the square brackets is a polynomial of degree n and is orthogonal to all polynomial of lower degree with respect to $w_{\mu,\nu}(x)$, we can deduce that

$$\sqrt{x}\left(P_{2n}^{(\mu,\nu)}(\mathrm{i}\sqrt{x}) - P_{2n}^{(\mu,\nu)}(-\mathrm{i}\sqrt{x})\right) = \lambda\phi_n(x),$$

where λ is a constant. Setting $x = 0$ and noting that $\phi_n(0) \neq 0$, we obtain $\lambda = 0$, and thus $P_{2n}^{(\mu,\nu)}(x)$ is even. If we set $s(x) = x^{2k}$, where $k = 0, \ldots, n-1$, then

$$\int_0^\infty \left[P_{2n}^{(\mu,\nu)}(\mathrm{i}\sqrt{x}) + P_{2n}^{(\mu,\nu)}(-\mathrm{i}\sqrt{x})\right] x^k w_{\mu,\nu}(x)\,\mathrm{d}x = 0.$$

Note that the term inside the square brackets is a polynomial of degree n and is orthogonal to all polynomial of lower degree with respect to $w_{\mu,\nu}(x)$, we can deduce that

$$P_{2n}^{(\mu,\nu)}(\mathrm{i}\sqrt{x}) + P_{2n}^{(\mu,\nu)}(-\mathrm{i}\sqrt{x}) = 2(-1)^n\phi_n(x).$$

Recall that $P_{2n}^{(\mu,\nu)}(x)$ is even, it follows that $P_{2n}^{(\mu,\nu)}(x) = (-1)^n\phi_n(-x^2)$. This proves the case where $\mu-\nu$ is even.

When $\mu-\nu$ is odd, (3.7) follows by similar arguments as above. Now we show that $P_{2n+1}^{(\mu,\nu)}(x)$ do not exist. By similar arguments as above we find that

$$\int_0^\infty \left[\frac{P_{2n+1}^{(\mu,\nu)}(\mathrm{i}\sqrt{x}) - P_{2n+1}^{(\mu,\nu)}(-\mathrm{i}\sqrt{x})}{\sqrt{x}}\right] x^k w_{\mu,\nu}(x)\,\mathrm{d}x = 0,$$

where $k = 0, \ldots, n$. Note that the term inside the square brackets is a polynomial of degree n, we deduce that $P_{2n+1}^{(\mu,\nu)}(x)$ is an even function. However, this is impossible since its leading term is x^{2n+1}. We conclude that $P_{2n+1}^{(\mu,\nu)}(x)$ does not exist and this ends the proof. $\square$

Remark 3.9. When $\mu-\nu$ is even, then $P_n^{(\mu,\nu)}(x)$ always exists and is an even function when n is even and is an odd function when n is odd. When $\mu-\nu$ is odd, then $P_n^{(\mu,\nu)}(x)$ exists only for even n and is an even function in this case. Moreover, in these cases that $P_n^{(\mu,\nu)}(x)$ exists, its zeros are all located on the imaginary axis and symmetric with respect to the real axis.

Remark 3.10. It is known that orthogonal polynomials can also be expressed in terms of the associated Hankel determinant [10, Chapter 2]. In fact, when $\mu-\nu$ is odd, it is easily checked that the Hankel determinant associated with $P_n^{(\mu,\nu)}(x)$ vanishes for odd n, which also confirms the nonexistence of the polynomials with odd degrees.

When implementing $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$, one needs to calculate the Gaussian quadrature rule defined in (3.3). By [17, Equation (10.43.19)] we know that the moments of $w_{\mu,\nu}(x)$ can be written explicitly as

$$\int_0^\infty x^k w_{\mu,\nu}(x)\,\mathrm{d}x = \begin{cases} \Gamma\left(k+\dfrac{\mu-\nu+1}{2}\right)\Gamma\left(k+\dfrac{\mu+\nu+1}{2}\right)2^{2k+\mu-1}, & \mu-\nu \text{ even},\\ \Gamma\left(k+\dfrac{\mu-\nu+2}{2}\right)\Gamma\left(k+\dfrac{\mu+\nu+2}{2}\right)2^{2k+\mu}, & \mu-\nu \text{ odd}, \end{cases}$$

and the monic polynomials $\{\phi_n\}_{n=1}^{\infty}$ can be calculated by the Gram–Schmidt orthogonalization procedure. For example, explicit expressions of ϕ_1 and ϕ_2 are given below:

$$\phi_1(x) = \begin{cases} x-(\mu+1)^2+\nu^2, & \mu-\nu \text{ even},\\ x-(\mu+2)^2+\nu^2, & \mu-\nu \text{ odd}, \end{cases} \tag{3.9}$$

and

$$\phi_2(x) = \begin{cases} x^2-2b_{\mu,\nu}x+c_{\mu,\nu}, & \mu-\nu \text{ even},\\ x^2-2b_{\mu+1,\nu}x+c_{\mu+1,\nu}, & \mu-\nu \text{ odd}, \end{cases} \tag{3.10}$$

where $b_{\mu,\nu}$ and $c_{\mu,\nu}$ are given by

$$\begin{aligned} b_{\mu,\nu} &= \frac{(\mu+3)(\mu-\nu+3)(\mu+\nu+3)}{\mu+2},\\ c_{\mu,\nu} &= \frac{(\mu+4)(\mu-\nu+3)(\mu+\nu+3)(\mu-\nu+1)(\mu+\nu+1)}{\mu+2}. \end{aligned}$$

Consequently, the nodes and weights of the Gaussian quadrature rule (3.3) for $n=1,2$ can be calculated immediately using the above expressions.

In Figure 1 we display the zeros of $P_n^{(\mu,\nu)}(x)$ for four integer values of ν and we consider two choices of μ: $\mu=0$ and $\mu=\nu$, which correspond respectively to the nodes of the Gaussian quadrature rule in [1] and the generalized Gauss-Radau quadrature rule in Theorem 3.1. For $\mu=0$, Figure 1 shows that the zeros tend to cluster along the vertical line $\Re(z)=\nu\pi/2$ for $\nu=1,2$, but $\Re(z)=(\nu-2)\pi/2$ for $\nu=4,5$. Note that the vertical line $\Re(z)=\nu\pi/2$ was observed in [1] based on numerical calculations for $\nu=1/2,1,3/2,2$ and was further proved in [6] for $\nu\in[0,1/2]$. However, our calculations suggest that there are transitions in the asymptotic distribution of the zeros and the vertical line becomes $\Re(z)=(\nu-2)\pi/2$ for $\nu\in[3,7]$ and will change further for larger ν. On the other hand, we observe from Figure 1 that the nodes of the generalized Gauss-Radau quadrature rule are always located on the imaginary axis, as expected from Theorems 3.1 and 3.8.

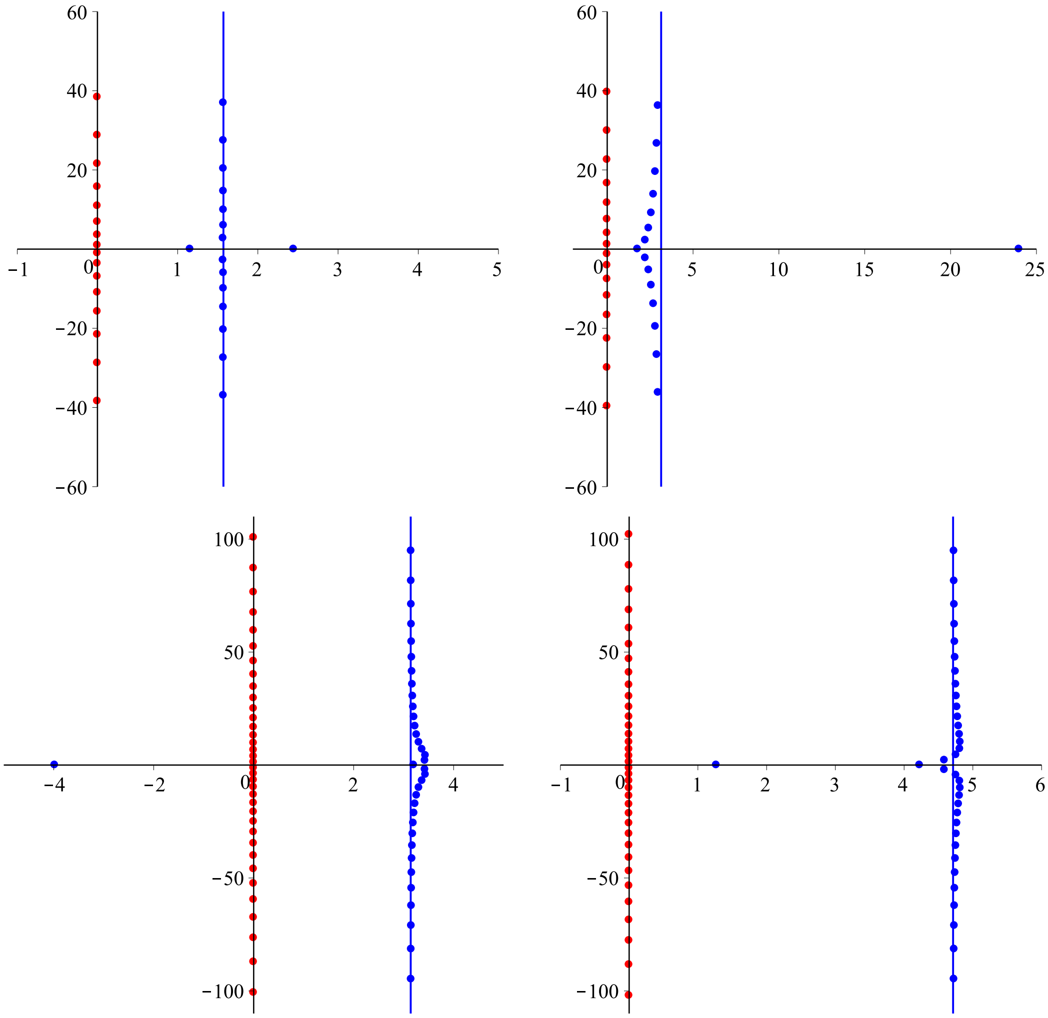

Figure 1: The nodes of the Gaussian qaudrature rule (blue) and the generalized Gauss-Radau quadrature rule (red) with $\mu = \nu$. Top row shows $n = 16$ for $\nu = 1$ (left) and $\nu = 2$ (right) and bottom row shows $n = 36$ for $\nu = 4$ (left) and $\nu = 5$ (right). The vertical lines in the top row are $\Re(z) = \nu\pi/2$ and in the bottom row are $\Re(z) = (\nu - 2)\pi/2$.

In the following we display several examples to demonstrate the performance of the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$. In Figures 2 and 3 we plot the absolute errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ as a function of ω for $n = 1$ and $n = 2$, respectively. For each ν, we consider several different values of μ. We see that the errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ decay at the rate $\mathcal{O}(\omega^{-4n-\mu-1})$ when $\mu - \nu$ is even and the rate $\mathcal{O}(\omega^{-4n-\mu-2})$ when $\mu - \nu$ is odd, which are consistent with our error estimates in Theorem 3.5.

Finally, we mention an interesting superconvergence phenomenon of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ in certain special situations. In Figure 4 we display the absolute error of the rule as a

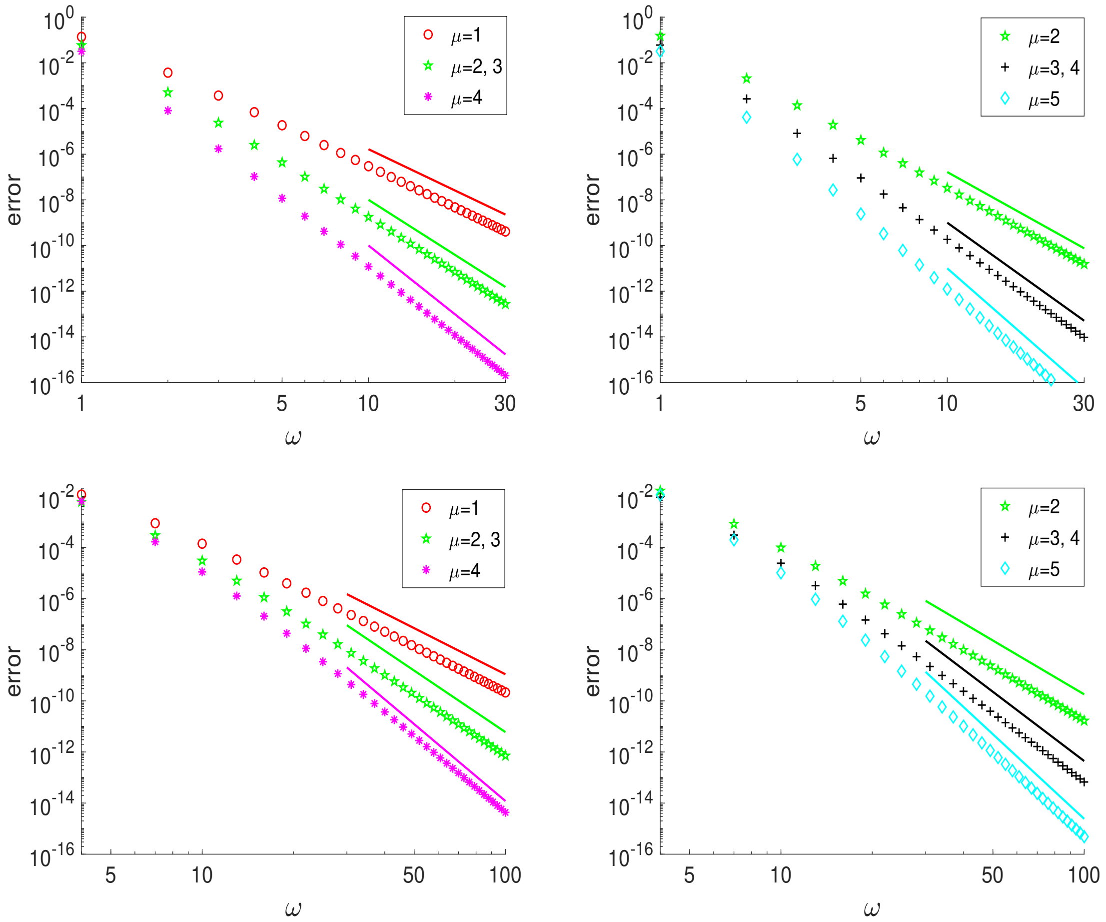

Figure 2: Absolute errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ with $n=1$ as a function of ω for $\nu=1$ (left) and $\nu=2$ (right). Here $f(x)=e^{-x}$ (top) and $f(x)=1/(1+x)^2$ (bottom), we choose different values of μ and the solid lines indicate the predicted rates $\mathcal{O}(\omega^{-4n-\mu-1})$ when $\mu-\nu$ is even and $\mathcal{O}(\omega^{-4n-\mu-2})$ when $\mu-\nu$ is odd.

function of ω for $f(x)=e^{-x^2}$ and $\nu=2$ and the absolute and relative errors of the rule for $\nu=3$ and we choose $\mu=\nu$ in our calculations. We see that $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ converges at the predicted rate when ν is even, but at a much faster rate when ν is odd. In fact, by [12, Equation (6.618,1)] we know for $\nu>-1$ that

$$\int_0^\infty e^{-x^2}J_\nu(\omega x)\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}\exp\left(-\frac{\omega^2}{8}\right)I_{\frac{\nu}{2}}\left(\frac{\omega^2}{8}\right),$$

where $I_\nu(z)$ is the modified Bessel function. When ν is odd, using the above closed form and noting that $f(x)$ is even, we find after some elementary calculations that the convergence rate of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ is $O(e^{-\omega^2}/\omega)$ as $\omega\to\infty$, which explains the superconvergence phenomenon displayed in Figure 4.

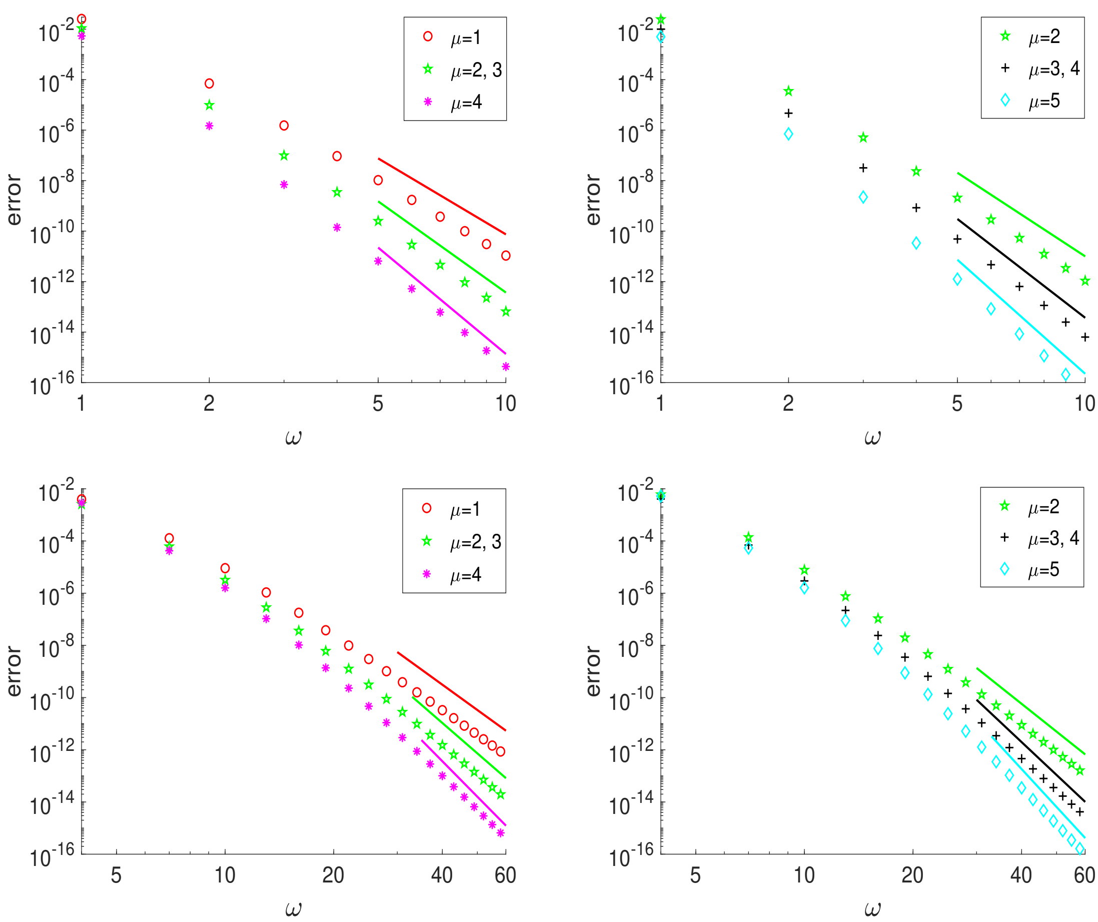

Figure 3: Absolute errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ with $n = 2$ as a function of ω for $\nu = 1$ (left) and $\nu = 2$ (right). Here $f(x) = e^{-x}$ (top) and $f(x) = 1/(1 + x)^2$ (bottom), we choose different values of μ and the solid lines indicate the predicted rates $\mathcal{O}(\omega^{-4n-\mu-1})$ when $\mu - \nu$ is even and $\mathcal{O}(\omega^{-4n-\mu-2})$ when $\mu - \nu$ is odd.

4 Extensions and applications

In this section, we extend our discussion to Hankel transform of fractional order and Fourier sine transform. An application to oscillatory Hilbert transform is also presented.

4.1 Hankel transform of fractional order

In this subsection we consider the Hankel transform of fractional order, i.e., $\nu > -1$ and $\nu \notin \mathbb{N}_0$. It is natural to ask if a generalized Gauss-Radau quadrature rule can be constructed by following the same procedure as in Theorem 3.1. In the following, we will prove a negative result and show that, although a complex quadrature rule can still

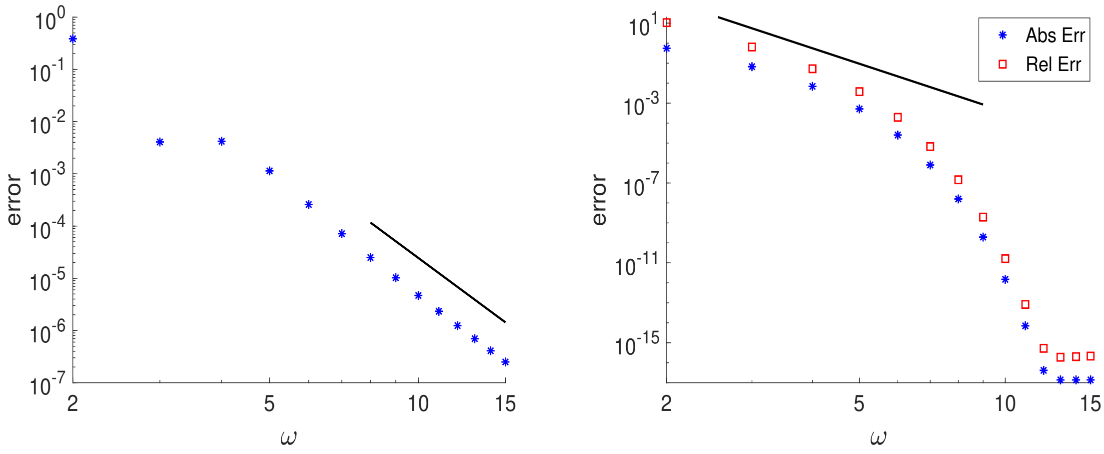

Figure 4: Absolute errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ as a function of ω for $n=1$, $\nu=2$ (left) and absolute and relative errors of $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}}f)(\omega)$ for $n=1$, $\nu=3$ (right). Here $f(x)=e^{-x^2}$, $\mu=\nu$ and the solid lines indicate the predicted rates $\mathcal{O}(\omega^{-4n-\mu-1})$.

be constructed, it will not be a generalized Gauss-radau quadrature rule anymore.

Let $\tilde{w}_{\mu,\nu}(x)=x^{\mu}K_{\nu}(x)$, where $\mu\in\mathbb{N}_0$ when $\nu\in(-1,0)$ and $\mu\in\mathbb{N}_0$ and $\mu>\nu-1$ when $\nu>0$, be the weight function on $(0,\infty)$. The Gaussian quadrature rule is defined by

$$\int_0^{\infty}\tilde{w}_{\mu,\nu}(x)f(x)\,\mathrm{d}x=\sum_{j=1}^{n}w_jf(x_j),\quad \forall f\in\mathcal{P}_{2n-1}. \tag{4.1}$$

Our result is stated as follows.

Theorem 4.1. *Let $\{x_j,w_j\}_{j=1}^{n}$ be the nodes and weights of the Gaussian quadrature rule in (4.1) and suppose that f is analytic in the right half-plane and $|f(z)|\le\mathcal{K}|z|^{\sigma}$ for some $\sigma\in\mathbb{R}$. Then, a quadrature rule for $(\mathcal{H}_{\nu}f)(\omega)$ is given by*

$$(\mathcal{Q}_{2n,\mu}^{\mathrm{HF}}f)(\omega)=\frac{1}{\omega}\left(\sum_{k=0}^{\mu-1}\frac{\hat{w}_k^0}{\omega^k}f^{(k)}(\hat{x}_0)+\sum_{j=1}^{2n}\hat{w}_jf\left(\frac{\hat{x}_j}{\omega}\right)\right), \tag{4.2}$$

where the nodes and weights are defined by

$$\{\hat{x}_j\}_{j=1}^{2n}=\{\pm\mathrm{i}x_j\}_{j=1}^{n},\quad \{\hat{w}_j\}_{j=1}^{2n}=\left\{\exp\left(\mp\frac{\nu\pi\mathrm{i}}{2}\right)\frac{w_jx_j^{-\mu}}{\pi}\right\}_{j=1}^{n},$$

and $\hat{x}_0=0$ and for $k=0,\ldots,\mu-1$,

$$\hat{w}_k^0=\frac{1}{k!}\left(2^k\frac{\Gamma((\nu+k+1)/2)}{\Gamma((\nu-k+1)/2)}-\frac{2}{\pi}\cos\left(\frac{(k-\nu)\pi}{2}\right)\sum_{j=1}^{n}w_jx_j^{k-\mu}\right).$$

Moreover, the quadrature rule (4.2) is exact for $f\in\mathcal{P}_{2n+\mu-1}$.

Proof. Following a line similar to the proof of Theorem 3.1, we have

$$(\mathcal{H}_\nu f)(\omega) = \int_0^\infty \mathcal{T}_\mu(x) J_\nu(\omega x)\,\mathrm{d}x + \int_0^\infty \mathcal{R}_\mu(x) x^\mu J_\nu(\omega x)\,\mathrm{d}x,$$

and the first integral on the right-hand side can be evaluated by (2.6). For the second integral, using Theorem 2.1 we have

$$\int_0^\infty \mathcal{R}_\mu(x) x^\mu J_\nu(\omega x)\mathrm{d}x = \int_0^\infty \widehat{\mathcal{R}}_\mu(\mathrm{i}x) x^\mu K_\nu(\omega x)\mathrm{d}x,$$

where $\widehat{\mathcal{R}}_\mu(x) = (e^{(\mu-\nu)\mathrm{i}\pi/2}\mathcal{R}_\mu(x) + e^{(\nu-\mu)\mathrm{i}\pi/2}\mathcal{R}_\mu(-x))/\pi$. Note that $\widehat{\mathcal{R}}_\mu(x)$ is no longer an even or odd function since $\mu \in \mathbb{N}_0$ and $\nu \notin \mathbb{Z}$. In this case, we evaluate the integral on the right-hand by the Gaussian quadrature rule (4.1):

$$\int_0^\infty \widehat{\mathcal{R}}_\mu(\mathrm{i}x) x^\mu K_\nu(\omega x)\mathrm{d}x = \frac{1}{\omega^{\mu+1}} \int_0^\infty \widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}x}{\omega}\right) \tilde{w}_{\mu,\nu}(x)\mathrm{d}x \approx \frac{1}{\omega^{\mu+1}} \sum_{j=1}^{n} w_j \widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}x_j}{\omega}\right).$$

This gives the quadrature rule (4.2). Note that the quadrature rule in the last step is exact for $\widehat{\mathcal{R}}_\mu \in \mathcal{P}_{2n-1}$, it is easily verified that the rule (4.2) is exact for $f \in \mathcal{P}_{2n+\mu-1}$. This ends the proof. □

Remark 4.2. Note that the quadrature rule (4.2) is exact only for $f \in \mathcal{P}_{2n+\mu-1}$, it is not a generalized Gauss-Radau quadrature rule anymore. Moreover, combining the exactness and [1, Lemma 1.6] gives the asymptotic error estimate

$$(\mathcal{H}_\nu f)(\omega) - (\mathcal{Q}^{\mathrm{HF}}_{2n,\mu} f)(\omega) = \mathcal{O}(\omega^{-2n-\mu-1}), \quad \omega \to \infty. \tag{4.3}$$

Why we fail to construct generalized Gauss-Radau quadrature rules when following the same procedure as in Theorem 3.1? To identify the problem, we plot in Figure 5 the zeros of $P_n^{(\mu,\nu)}(x)$ for $\nu = 3/2, 7/2$ and $\mu = 1, 2$. We see that the zeros of $P_n^{(\mu,\nu)}(x)$ tend to cluster along the vertical line $\Re(z) = (\nu-\mu)\pi/2$, but not the imaginary axis. For the rule (4.2), note that its nodes are all located on the imaginary axis, and hence it is not a generalized Gauss-Radau quadrature rule. In fact, if we construct a generalized Gauss-Radau quadrature rule with respect to the weight $J_\nu(x)$ on $(0, \infty)$, i.e.,

$$\int_0^\infty f(x) J_\nu(x)\,\mathrm{d}x = \sum_{k=0}^{\mu-1} \hat{w}_k^0 f^{(k)}(\hat{x}_0) + \sum_{j=1}^{2n} \hat{w}_j f(\hat{x}_j), \quad \forall f \in \mathcal{P}_{4n+\mu-1}, \tag{4.4}$$

where $\{\hat{x}_j\}_{j=1}^{2n}$ are the zeros of $P_{2n}^{(\mu,\nu)}(x)$, then a generalized Gauss-Radau quadrature rule for the Hankel transform $(\mathcal{H}_\nu f)(\omega)$ follows immediately by using a simple scaling. However, the existence of the polynomial $P_{2n}^{(\mu,\nu)}(x)$ cannot be guaranteed.

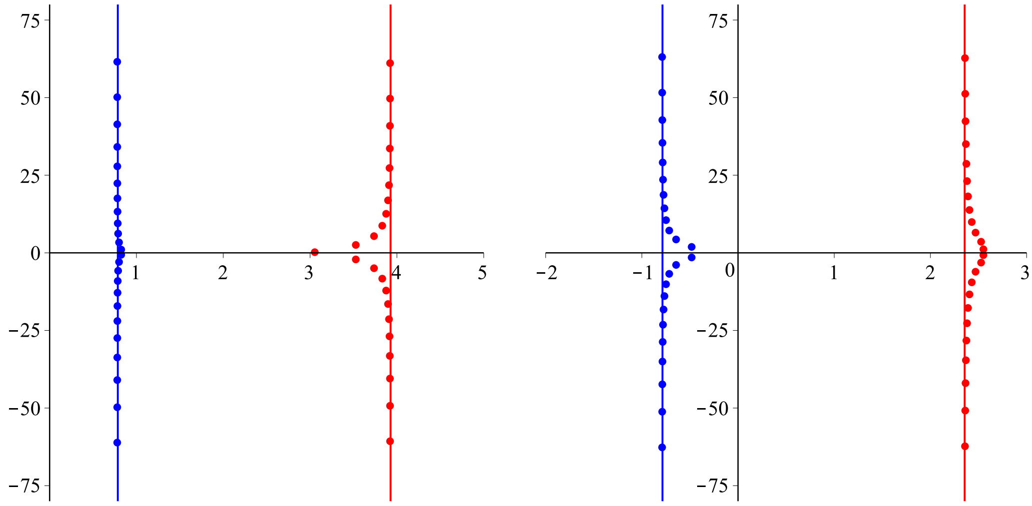

Figure 5: The zeros of $P_n^{(\mu,\nu)}(x)$ for $\mu = 1$ (left) and $\mu = 2$ (right). Here $n = 24$, $\nu = 3/2$ (blue) and $\nu = 7/2$ (red) and the vertical lines are $\Re(z) = (\nu - \mu)\pi/2$.

4.2 Fourier sine transform

In this subsection, we extend the construction of generalized Gauss-Radau quadrature rules to Fourier sine transform of the forms

$$(\mathcal{F}_s f)(\omega) := \int_0^\infty f(x)\sin(\omega x)\mathrm{d}x. \tag{4.5}$$

Let $w_{\sin}(x) = x^{\lfloor(\mu-1)/2\rfloor+1/2}e^{-\sqrt{x}}/2$, where $\mu \in \mathbb{N}_0$, be the weight function on $(0,\infty)$ and let $\{x_j, w_j\}_{j=1}^n$ be the nodes and weights of the n-point Gaussian quadrature rule with respect to $w_{\sin}(x)$ on $(0,\infty)$, i.e.,

$$\int_0^\infty w_{\sin}(x) f(x)\,\mathrm{d}x = \sum_{j=1}^n w_j f(x_j), \quad \forall f \in \mathcal{P}_{2n-1}. \tag{4.6}$$

It is evident that $x_j \in (0,\infty)$ and $w_j > 0$ for all $j = 1,\ldots,n$. The complex generalized gauss-Radau quadrature rule for Fourier sine transform is stated in the following result.

Theorem 4.3. *Let $\{x_j, w_j\}_{j=1}^n$ be the nodes and weights of the Gaussian quadrature rule in (4.6) and suppose that f is analytic in the right half-plane and $|f(z)| \le \mathcal{K}|z|^\sigma$ for some $\sigma \in \mathbb{R}$. Then, a quadrature rule for $(\mathcal{F}_s f)(\omega)$ is given by*

$$(\mathcal{Q}_{2n,\mu}^{\mathrm{FS}} f)(\omega) = \frac{1}{\omega}\left(\sum_{k=0}^{\mu-1}\frac{\hat{w}_k^0}{\omega^k}f^{(k)}(\hat{x}_0) + \sum_{j=1}^{2n}\hat{w}_j f\left(\frac{\hat{x}_j}{\omega}\right)\right), \tag{4.7}$$

where the nodes and weights are defined by

$$\{\hat{x}_j\}_{j=1}^{2n} = \{\pm \mathrm{i}\sqrt{x_j}\}_{j=1}^n, \quad \{\hat{w}_j\}_{j=1}^{2n} = \left\{\frac{w_j}{2}x_j^{-\lfloor(\mu-1)/2\rfloor-1}\right\}\bigcup\left\{\frac{w_j}{2}x_j^{-\lfloor(\mu-1)/2\rfloor-1}\right\}_{j=1}^n,$$

and $\hat{x}_0 = 0$ *and*

$$\hat{w}_k^0 = \cos\left(\frac{k\pi}{2}\right)\left(1 - \frac{1}{k!}\sum_{j=1}^{n} w_j x_j^{k/2-\lfloor(\mu-1)/2\rfloor-1}\right).$$

Moreover, the quadrature rule (4.7) *is exact for* $f \in \mathcal{P}_{4n+2\lfloor(\mu-1)/2\rfloor+1}$.

Proof. The proof is similar to that of Theorem 3.1. Let $\mathcal{T}_\mu(x)$ denote the Taylor expansion of $f(x)$ of degree $\mu-1$ at $\hat{x}_0 = 0$ and let $\mathcal{R}_\mu(x) = (f(x) - \mathcal{T}_\mu(x))/x^\mu$. It follows that

$$(\mathcal{F}_s f)(\omega) = \int_0^\infty \mathcal{T}_\mu(x)\sin(\omega x)\,\mathrm{d}x + \int_0^\infty \mathcal{R}_\mu(x) x^\mu \sin(\omega x)\,\mathrm{d}x.$$

For the first integral on the right-hand side, using (2.5) we have

$$\int_0^\infty \mathcal{T}_\mu(x)\sin(\omega x)\,\mathrm{d}x = \sum_{k=0}^{\mu-1}\frac{f^{(k)}(0)}{k!}\int_0^\infty x^k \sin(\omega x)\,\mathrm{d}x = \sum_{k=0}^{\mu-1}\frac{f^{(k)}(0)}{\omega^{k+1}}\cos\left(\frac{k\pi}{2}\right).$$

For the second integral on the right-hand side, by Euler's formula and Theorem 2.1 we have

$$\int_0^\infty \mathcal{R}_\mu(x) x^\mu \sin(\omega x)\,\mathrm{d}x = \int_0^\infty \widehat{\mathcal{R}}_\mu(\mathrm{i}x) x^\mu e^{-\omega x}\,\mathrm{d}x,$$

where $\widehat{\mathcal{R}}_\mu(x) = (e^{\mu\pi\mathrm{i}/2}\mathcal{R}_\mu(x) + e^{-\mu\pi\mathrm{i}/2}\mathcal{R}_\mu(-x))/2$. Note that $\widehat{\mathcal{R}}_\mu(x)$ is an even function when μ is an even integer and is an odd function when μ is an odd integer. Using the transformation $x \to \sqrt{x}/\omega$ to the integral on the right side of the above equation gives

$$\int_0^\infty \mathcal{R}_\mu(x) x^\mu \sin(\omega x)\,\mathrm{d}x = \frac{1}{\omega^{\mu+1}}\begin{cases} \displaystyle\int_0^\infty \widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x}}{\omega}\right) w_{\sin}(x)\,\mathrm{d}x, & \mu \text{ even}, \\ \displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\widehat{\mathcal{R}}_\mu\left(\frac{\mathrm{i}\sqrt{x}}{\omega}\right) w_{\sin}(x)\,\mathrm{d}x, & \mu \text{ odd}. \end{cases}$$

The quadrature rule (4.7) follows by evaluating the integrals on the right-hand side with the Gaussian quadrature rule (4.6). Finally, the exactness of the rule (4.7) follows from the exactness of the Gaussian quadrature rule (4.6) and the Taylor expansion of $\widehat{\mathcal{R}}_\mu(x)$. This ends the proof. ∎

Remark 4.4. Note that the weights $\{\hat{w}_j\}_{j=1}^{2n}$ are positive and $\hat{w}_k^0 = 0$ for odd k. Moreover, the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{FS}} f)(\omega)$ is exact for $f \in \mathcal{P}_{4n+\mu-1}$ when μ is even and for $f \in \mathcal{P}_{4n+\mu}$ when μ is odd, and its asymptotic error estimate is

$$(\mathcal{F}_s f)(\omega) - (\mathcal{Q}_{2n,\mu}^{\mathrm{FS}} f)(\omega) = \mathcal{O}(\omega^{-4n-2\lfloor(\mu-1)/2\rfloor-3}), \quad \omega \to \infty. \tag{4.8}$$

Remark 4.5. For Fourier cosine transform of the form

$$(\mathcal{F}_c f)(\omega) := \int_0^\infty f(x)\cos(\omega x)\mathrm{d}x, \tag{4.9}$$

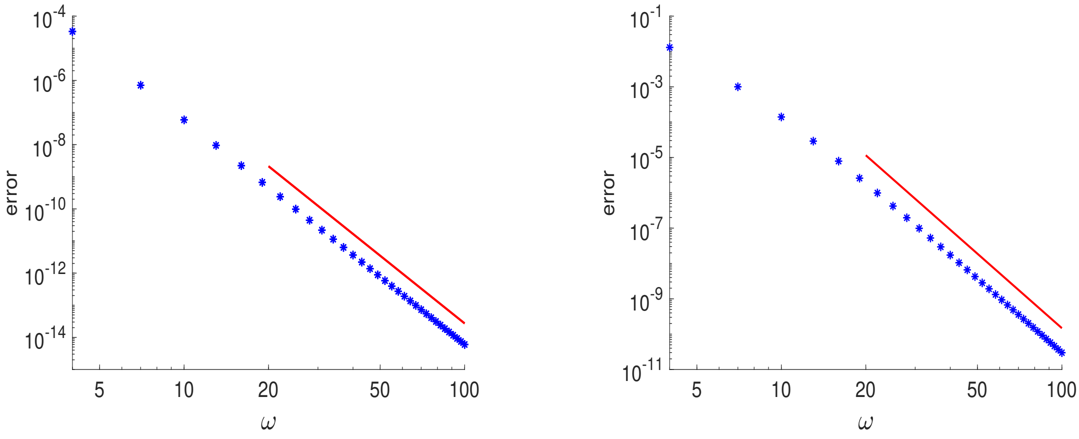

Figure 6: Absolute errors of the rule $(\mathcal{Q}_{2n,\mu}^{\rm FS} f)(\omega)$ for $(\mathcal{F}_s f)(\omega)$ as a function of ω with $n=1$ and $\mu=2$ for $f(x)=e^{-x}$ (left) and $f(x)=1/(1+x)^2$ (right). The red lines show the error estimate $O(\omega^{-7})$.

the generalized Gauss-Radau quadrature rule can be constructed in a similar way. Let $w_{\cos}(x)=x^{\lfloor \mu/2 \rfloor}e^{-\sqrt{x}}/2$ be the weight function on $(0,\infty)$ and let $\{x_j,w_j\}_{j=1}^{n}$ be the nodes and weights of an n-point Gaussian quadrature rule with respect to $w_{\cos}(x)$. Then the generalized Gauss-Radau quadrature rule for $(\mathcal{F}_c f)(\omega)$ is given by

$$(\mathcal{Q}_{2n,\mu}^{\rm FC} f)(\omega)=\frac{1}{\omega}\left(\sum_{k=0}^{\mu-1}\frac{\hat{w}_k^0}{\omega^k}f^{(k)}(\hat{x}_0)+\sum_{j=1}^{2n}\hat{w}_j f\left(\frac{\hat{x}_j}{\omega}\right)\right), \tag{4.10}$$

where the nodes and weights are defined by

$$\{\hat{x}_j\}_{j=1}^{2n}=\{\pm \mathrm{i}\sqrt{x_j}\}_{j=1}^{n}, \quad \{\hat{w}_j\}_{j=1}^{2n}=\left\{\pm \mathrm{i}\frac{w_j}{2}x_j^{-\lfloor \mu/2\rfloor-1/2}\right\}_{j=1}^{n},$$

and $\hat{x}_0=0$ and

$$\hat{w}_k^0=\sin\left(\frac{k\pi}{2}\right)\left(-1+\frac{1}{k!}\sum_{j=1}^{n}w_j x_j^{k/2-\lfloor \mu/2\rfloor-1/2}\right).$$

Moreover, the quadrature rule (4.10) is exact for $f\in\mathcal{P}_{4n+2\lfloor \mu/2\rfloor}$ and it has the asymptotic error estimate $\mathcal{O}(\omega^{-4n-2\lfloor \mu/2\rfloor-2})$ as $\omega\to\infty$. We omit the proof since it is similar to that of Theorem 4.3.

In Figure 6 we display the absolute errors of the rule $(\mathcal{Q}_{2n,\mu}^{\rm FS} f)(\omega)$ as a function of ω for $f(x)=e^{-x}$ and $f(x)=1/(1+x)^2$. As expected, we see that the errors decay at the rate $\mathcal{O}(\omega^{-4n-2\lfloor (\mu-1)/2\rfloor-3})$ as $\omega\to\infty$.

4.3 Oscillatory Hilbert transforms

Oscillatory Hilbert transforms have received growing attention in the last decade (see, e.g., [19, 24]). Consider the following oscillatory integral transform

$$(H_\nu f)(\omega,\tau) := \fint_0^\infty \frac{f(x)}{x-\tau} J_\nu(\omega x)\,\mathrm{d}x, \tag{4.11}$$

where $\tau > 0$ and the bar indicates the Cauchy principal value. To construct efficient methods for such transform, the main difficulties are that the integrand is oscillatory and has a singularity of Cauchy-type.

In the following we pesent a practical method for computing this transform. By the singularity subtraction technique, we have

$$(H_\nu f)(\omega,\tau) := \int_0^\infty \frac{f(x)-f(\tau)}{x-\tau} J_\nu(\omega x)\,\mathrm{d}x + f(\tau)\fint_0^\infty \frac{J_\nu(\omega x)}{x-\tau}\,\mathrm{d}x. \tag{4.12}$$

For the last integral, it can be expressed by the Struve or Meijer G-functions (see, e.g., [19, 24]) and their evaluations can be performed by most modern software, e.g., Maple and Matlab. As for the first integral on the right-hand side of (4.12), it can be evaluated directly by using the generalized Gauss-Radau quadrature rule in Theorem 3.1 when $\nu \in \mathbb{Z}$ and the quadrature rule in Theorem 4.1 otherwise. When the singularity τ is not close to zero, an obvious advantage of using these quadrature rules is that they can avoid the loss of accuracy due to cancellation since their nodes all lie on the imaginary axis.

To show the performance of the above proposed method, we consider the test functions $f(x) = e^{-x}$ and $f(x) = 1/(1+(1+x)^2)$, the orders $\nu = 0, 1$ and the singularities $\tau = 1, 5$. We evaluate the first integral on the right-hand side of (4.12) by the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ with $n = 1$ and the second integral by [19, Corollary 4.2]†

$$\begin{aligned}
\fint_0^\infty \frac{J_0(\omega x)}{x-\tau}\,\mathrm{d}x &= -\frac{\pi}{2}\big[\mathbf{H}_0(\omega\tau) + Y_0(\omega\tau)\big], \\
\fint_0^\infty \frac{J_1(\omega x)}{x-\tau}\,\mathrm{d}x &= \frac{\pi}{2}\big[\mathbf{H}_{-1}(\omega\tau) - Y_1(\omega\tau)\big] - \frac{1}{\omega\tau},
\end{aligned} \tag{4.13}$$

where $\mathbf{H}_\nu(z)$ is the Struve function and $Y_\nu(z)$ is the Bessel function of the second kind. For each ν, we choose $\mu = \nu$ and $\mu = \nu + 1$. Since the errors of computing the integrals (4.13) can be ignored, the error of the proposed method comes only from the rule $(\mathcal{Q}_{2n,\mu}^{\mathrm{HI}} f)(\omega)$ and thus it decays at the rate $\mathcal{O}(\omega^{-4n-\mu-1})$ when $\mu-\nu$ is even and the rate $\mathcal{O}(\omega^{-4n-\mu-2})$ when $\mu-\nu$ is odd. Figure 7 plots the results and we see that the proposed method converge at expected rates.

†We point out that [19, Equation (4.8)] is correct for $\nu = 0$, but is wrong for $\nu = 1$ since the integral in [19, Equation (4.2)] will involve a nonintegrable singularity when setting $f(t) = 1$. However, by setting $f(t) = t$ in [19, Equation (4.2)], it is not difficult to derived the correct result given here.

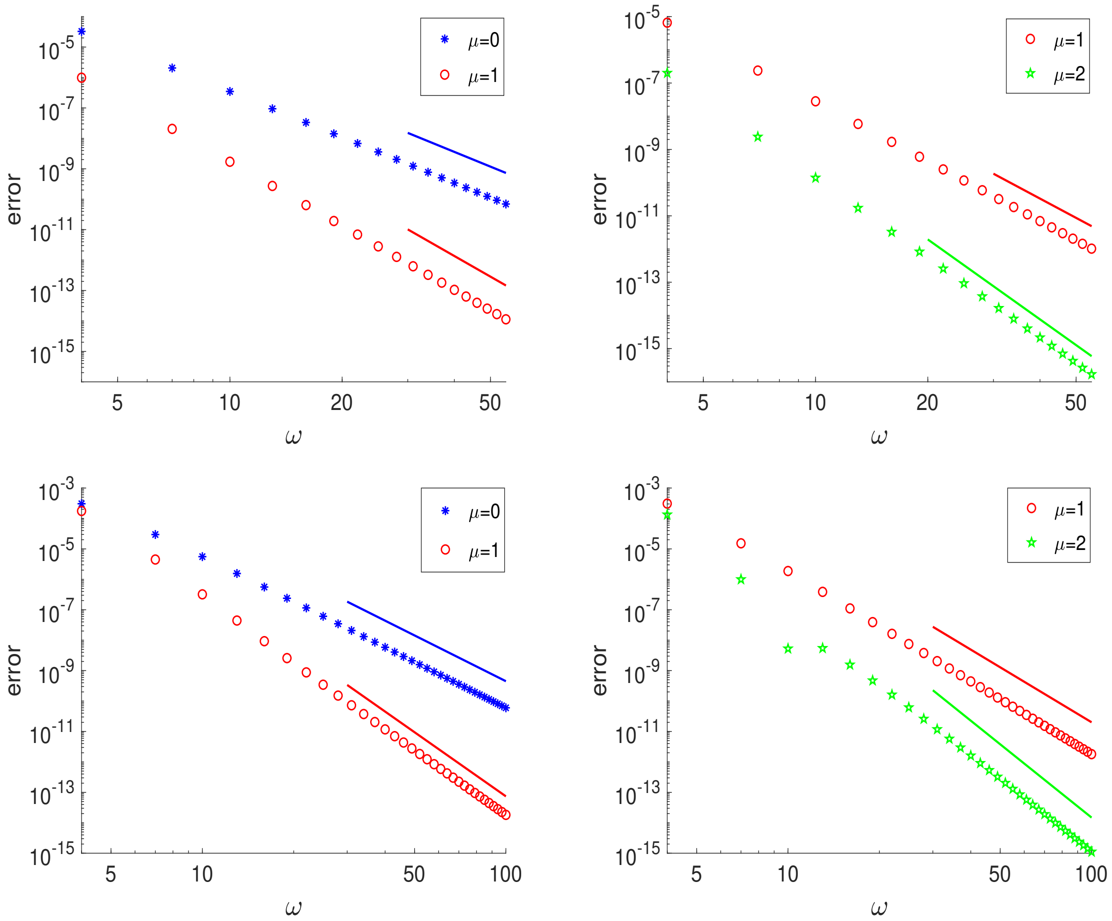

Figure 7: Absolute errors as a function of ω for $\nu = 0$ (left) and $\nu = 1$ (right). Top row shows $f(x) = e^{-x}$ and $\tau = 5$ and bottom row shows $f(x) = 1/(1 + (1 + x)^2)$ and $\tau = 1$ and the solid lines indicate the predicted rates.

5 Conclusion

In this paper, we have studied the construction of complex generalized Gauss-Radau quadrature rules for Hankel transform of integer order. We have shown that, by adding certain value and derivative information at the left endpoint, complex generalized Gauss-Radau quadrature rules can be constructed with guaranteed existence and the nodes are located on the imaginary axis. Except for existence, these rules also have the advantage that they have optimal asymptotic order. Motivated by this finding, we further studied the existence of the polynomials that are orthogonal with respect to the oscillatory weight function $x^\mu J_\nu(x)$ on $(0, \infty)$. When both ν and $\mu - \nu$ are nonnegative integers, we proved that the polynomials always exist for all degrees if $\mu - \nu$ is even, but exist only for even degrees if $\mu - \nu$ is odd. Moreover, in these cases where the polynomials exist, their zeros are located on the imaginary axis and symmetric with respect to the real axis.

Gaussian quadrature rules for highly oscillatory integrals have received increasing attention in recent years due to the fact that they can achieve optimal asymptotic order [1, 2, 7, 23]. A key issue of such rules is that the existence of the orthogonal polynomials with respect to the oscillatory weight function cannot be guaranteed and the proofs are generally quite challenging (see, e.g., [5, 6, 7]). The findings of the present study give a sequence of such polynomials with guaranteed existence, which provide a theoretical rationale for the proposed quadrature rules for Hankel transform of integer order.

References

[1] A. Asheim and D. Huybrechs, Complex Gaussian quadrature for oscillatory integral transforms, IMA J. Numer. Anal., 33(4):1322-1341, 2013.

[2] A. Asheim, A. Deaño, D. Huybrechs and H. Wang, A Gaussian quadrature rule for oscillatory integral on a bounded interval, Disc. Contin. Dyn. Sys., 34(3):883-901, 2014.

[3] W. L. Anderson, Improved digital filters for evaluating Fourier and Hankel transform integrals, US Geological Survey, USGS-GD-75-012, 1975.

[4] P. T. Christopher and K. J. Parker, New approaches to the linear propagation of acoustic fields, J. Acoust. Soc. Am., 90(1):507-521, 1991.

[5] A. F. Celsus, A. Deaño, D. Huybrechs and A. Iserles, The kissing polynomials and their Hankel determinants, Trans. Math. Appl., 6(1):1-66, 2022.

[6] A. Deaño, A. B. J. Kuijlaars and P. Román, Asymptotic behavior and zero distribution of polynomials orthogonal with respect to Bessel functions, Constr. Approx., 43:153-196, 2016.

[7] D. Huybrechs, A. Kuijlaars and N. Lejon, A numerical method for oscillatory integrals with coalescing saddle points, SIAM J. Numer. Anal., 57(6):2707-2729, 2019.

[8] D. P. Ghosh, The application of linear filter theory to the direct interpretation of geoelectrical resistivity sounding measurements, Geophys. Prosp., 19(2):192-217, 1971.

[9] W. Gautschi, Generalized Gauss-Radau and Gauss-Lobatto formulae, BIT Numer. Math., 44(4):711-720, 2004.

[10] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, Oxford University Press, New York, 2004.

[11] W. Gautschi, High-order generalized Gauss-Radau and Gauss-Lobatto formulae for Jacobi and Laguerre weight functions, Numer. Algor., 51(2):143-149, 2009.

[12] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Seventh Edition, Academic Press, 2007.

[13] H. Joulak and B. Beckermann, On Gautschi's conjecture for generalized Gauss-Radau and Gauss-Lobatto formulae, J. Comput. Appl. Math., 233(3):768-774, 2009.

[14] I. M. Longman, Note on a method for computing infinite integrals of oscillatory functions, Math. Proc. Camb. Phi. Soc., 52(4):764-768, 1956.

[15] S. K. Lucas and H. A. Stone, Evaluating infinite integrals involving Bessel functions of arbitrary order, J. Comput. Appl. Math., 64(3):217-231, 1995.

[16] K. A. Michalski, Extrapolation methods for Sommerfeld integral tails, IEEE Trans. Antennas Propag., 46(10):1405-1418, 1998.

[17] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010.

[18] S. E. Sherer, Scattering of sound from axisymetric sources by multiple circular cylinders, J. Acoust. Soc. Am., 115(2):488-496, 2004.

[19] H. Wang, L. Zhang and D. Huybrechs, Asymptotic expansions and fast computation of oscillatory Hilbert transforms, Numer. Math., 123(4):709-743, 2013.

[20] S. H. Ward and G. W. Hohmann, Electromagnetic Theory for Geophysical Applications, in Electromagnetic Methods in Applied Geophysics − Theory, Vol. 1, Edited by M. N. Nabighian, Society of Exploration Geophysicists, Tulsa, 1987.

[21] R. Wong, Quadrature formulas for oscillatory integral transforms, Numer. Math., 39:351-360, 1982.

[22] R. Wong, Asymptotic Approximation of Integrals, SIAM, Philadephia, 2001.

[23] M. Wu and H. Wang, Gaussian quadrature rules for composite highly oscillatory integrals, Math. Comp., 93(346):729-746, 2024.

[24] Z. Xu and G. V. Milovanović, Efficient method for the computation of oscillatory Bessel transform and Bessel Hilbert transforms, J. Comput. Appl. Math., 308:117-137, 2016.