THE GREEN'S FUNCTION OF POLYHARMONIC OPERATORS WITH DIVERGING COEFFICIENTS: CONSTRUCTION AND SHARP ASYMPTOTICS.

LORENZO CARLETTI

Abstract. We show existence, uniqueness and positivity for the Green's function of the operator $(\Delta_g + \alpha)^k$ in a closed Riemannian manifold (M, g), of dimension $n > 2k$, $k \in \mathbb{N}$, $k \geq 1$, with Laplace-Beltrami operator $\Delta_g = -\operatorname{div}_g(\nabla \cdot)$, and where $\alpha > 0$. We are interested in the case where α is large : We prove pointwise estimates with explicit dependence on α for the Green's function and its derivatives. We highlight a region of exponential decay for the Green's function away from the diagonal, for large α.

1. Introduction and statement of the result

Let (M, g) be a smooth connected Riemannian manifold of dimension n, compact and without boundary. Let $k \geq 1$ be an integer, we assume that $n > 2k$. Let $\alpha > 0$ be a fixed real number, we consider the elliptic partial differential operator of order $2k$, $(\Delta_g + \alpha)^k$ in M, where we define $\Delta_g := -\operatorname{div}_g(\nabla \cdot)$ the Laplace-Beltrami operator. In this article, relying on the method presented by F. Robert in [22], we construct the Green's function for $(\Delta_g + \alpha)^k$ in M. We show uniqueness and positivity, as well as sharp pointwise asymptotics. The main goal of this paper is to obtain asymptotics that explicitly depend on α to understand the behavior of the Green's function when α is large.

One of the first instances of the construction of a Green's function for a polyharmonic operator is found in [3], where the fundamental solution for $(-\Delta)^k$ on a ball of $\mathbb{R}^n$ with Dirichlet-type boundary conditions is computed. This is also the first example of a positive Green's function for a poly-harmonic operator. Indeed, the question of positivity is highly non-trivial for higher order operators, since the maximum principle no longer holds in general. For instance, see [8, 14] for results on this matter. Estimates on the Green's function have also been studied for polyharmonic problems on the Euclidean space, and sharp bounds from below and above can be obtained, see for instance [4, 12]. Note that in all the previous references, the boundary conditions play an important role and, in the estimates, an explicit dependence on the distance to the boundary is often involved. In the case of this article, these considerations will not be included since we work on a closed manifold. It is also worth mentioning that there exists an extensive literature for the construction of Green's functions for standard operators of second order on common domains of $\mathbb{R}^n$, in particular in $\mathbb{R}^2$, $\mathbb{R}^3$. See for instance [5] for an undergraduate-level textbook on the matter.

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Polyharmonic operators on manifolds have been studied because of their connections with the so-called prescribed Q-curvature equations. These equations involve a special family of conformally invariant operators, called *GJMS operators* [11], for which a Green's function was investigated in [20]. Green's functions for GJMS operators have proven fundamental to obtain existence results for the prescribed Q-curvature equations, we refer to [23] for the conformal Laplacian, see also [19] and the references therein for higher-order Q-curvature equations. Moreover, the operator $(\Delta_g + \alpha)^k$ can be seen as a toy-model for the GJMS operator of order $2k$, which, on an Einstein manifold, can be written as a product of k operators of the form $\Delta_g + c_j$ (see [6]).

In this paper, we study the operator $(\Delta_g + \alpha)^k$, with $\alpha > 0$, in M. Our main motivation to consider these specific operators comes from their importance in the study of the optimal constant for the critical Sobolev embeddings. We refer to [17] in the case $k = 1$ and [15] for the biharmonic case $k = 2$, where the operator $(\Delta_g + \alpha)^k$, as $\alpha \to \infty$, naturally appears in a contradiction argument. In this work, we follow the iterative approach of [22], and obtain sharp pointwise estimates that explicitly depend on α. The present article provides technical results that will be used in future works. Operators of the form $(\Delta_g + \alpha)^k$ also naturally appear in other contexts, see for instance [7, 21] where concentration phenomena for critical nonlinear equations are investigated.

For any $p \geq 1$ and $l \in \mathbb{N}$, let us define the norms

$$\|u\|^p_{\mathcal{H}^{l,p}(M)} := \sum_{m=0}^{l} \left\| \Delta_g^{m/2} u \right\|^p_{L^p(M)} ,$$

where we write $\left| \Delta_g^{m/2} u \right| := \begin{cases} \left|\Delta_g^i u\right| & \text{if } m = 2i \text{ is even,} \\ \left|\nabla \Delta_g^i u\right|_g & \text{if } m = 2i+1 \text{ is odd} \end{cases}$. Let us also define the Sobolev space $\mathcal{H}^{l,p}(M)$ as the closure of $C^\infty(M)$ in $L^p(M)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^{l,p}}$. We write $\mathcal{H}^k(M)$ for the Hilbert space $\mathcal{H}^k(M) = \mathcal{H}^{k,2}(M)$, for $k \geq 1$ integer.

Observe that, for $\alpha > 0$, the operator $(\Delta_g + \alpha)^k$ is coercive, since if $\alpha \geq 1$,

$$\begin{aligned} \|u\|^2_{\mathcal{H}^k(M)} &= \sum_{l=1}^{k} \int_M \left| \Delta_g^{l/2} u \right|^2 dv_g \\ &\leq \sum_{l=1}^{k} \tbinom{k}{l} \alpha^{k-l} \int_M \left| \Delta_g^{l/2} u \right|^2 dv_g = \left\langle (\Delta_g + \alpha)^k u, u \right\rangle_{\mathcal{H}^{-k}, \mathcal{H}^k} , \end{aligned}$$

and if $\alpha < 1$

$$\|u\|^2_{\mathcal{H}^k(M)} \leq \tfrac{1}{\alpha^k} \left\langle (\Delta_g + \alpha)^k u, u \right\rangle_{\mathcal{H}^{-k}, \mathcal{H}^k} .$$

If $\varphi \in C^\infty(M)$, the existence and uniqueness of a solution $u \in C^\infty(M)$ to the linear equation

$$(\Delta_g + \alpha)^k u = \varphi \quad \text{on } M \tag{1.1}$$

follows from the coercivity of the operator, and from standard elliptic theory. See for instance [9] for standard existence and regularity results in the case $k = 1$, which can be iterated in the case of our operator. This allows us to define a Green's function for this operator.

Definition 1.1 (Green's function)**.** Let $\alpha > 0$, $k \geq 1$ and $n > 2k$, and let (M, g) be a connected compact Riemannian manifold of dimension n, without boundary, with Laplace-Beltrami operator $\Delta_g = -\operatorname{div}_g(\nabla \cdot)$. A Green's function for the operator $(\Delta_g + \alpha)^k$ in M is a function $G : M \times M \setminus \{(x,x) : x \in M\} \to \mathbb{R}$ such that, writing $G_x(y) := G(x,y)$ for all $x \neq y$ in M, we have $G_x \in L^1(M)$ for all $x \in M$, and for all $\varphi \in C^\infty(M)$ and all $x \in M$,

$$\int_M G_x (\Delta_g + \alpha)^k \varphi \, dv_g = \varphi(x).$$

This is equivalent to saying that $(\Delta_g + \alpha)^k G_x = \delta_x$ in the distributional sense on M, where δ_x is the Dirac's delta distribution centered at $x \in M$.

We are interested in explicit α-dependent estimates on the function and its derivatives, in particular as α gets large. Our main result is the following Theorem. Note that a closed Riemannian manifold has a positive injectivity radius, $i_g > 0$.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold, of dimension $n \geq 3$, let $k \geq 1$ with $n > 2k$, and $\alpha > 0$. The operator $(\Delta_g + \alpha)^k$ in M has a unique Green's function $G_{g,\alpha}$, which is positive, symmetric, and is in $C^\infty(M \times M \setminus \{(x,x) : x \in M\})$. Moreover, there exists $\alpha_0 \geq 1$ such that we have the following :*

- *There is a constant $C > 0$ such that for all $\alpha \geq \alpha_0$ and all $x \neq y$ in M with $\sqrt{\alpha} d_g(x,y) \leq 1$, we have*

$$G_{g,\alpha}(x,y) = c_{n,k} \, d_g(x,y)^{2k-n} \left(1 + \eta_\alpha(x,y)\right) \tag{1.2}$$

  with

$$|\eta_\alpha(x,y)| \leq C \begin{cases} \sqrt{\alpha}\, d_g(x,y) & n = 2k+1 \\ \alpha\, d_g(x,y)^2 \left(1 + |\log \sqrt{\alpha}\, d_g(x,y)|\right) & n = 2k+2 \\ \alpha\, d_g(x,y)^2 & n \geq 2k+3 \end{cases},$$

  and where $c_{n,k}$ is an explicit positive constant given by (2.1) *below.*
- *For all $0 < \varepsilon < 1$, there is a constant $C_\varepsilon > 0$ such that for all $\alpha \geq \alpha_0$ and all $x, y \in M$ with $\sqrt{\alpha} d_g(x,y) \geq 1$,*

$$G_{g,\alpha}(x,y) \leq C_\varepsilon \begin{cases} d_g(x,y)^{2k-n} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \text{if } d_g(x,y) < i_g/2 \\ e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} & \text{if } d_g(x,y) \geq i_g/2. \end{cases}$$

This Theorem highlights that when $d_g(x,y)$ is small in comparison to $1/\sqrt{\alpha}$, the Green's function for $(\Delta_g + \alpha)^k$ in M behaves to first order as the Green's function for the poly-Laplacian in $\mathbb{R}^n$, $(-\Delta)^k$, up to a remainder term on which we prove explicit bounds. On the other hand, when $d_g(x,y) \geq 1/\sqrt{\alpha}$, we obtain an exponential decay. In particular, any region of M situated at a fixed distance from a given point $x \in M$ will lie in this regime as α becomes large. Note also that most of the construction of the Green's function does not rely on the fact that $\alpha \geq \alpha_0$. It is only at $\alpha \to \infty$, however, that exponential estimates at finite distance are of interest.

Green's function for polyharmonic operators of order $2k$ in n-dimensional domains or manifolds, with $n > 2k$, and with bounded coefficients, have been known to satisfy estimates of the following type: There exists $C > 0$ such that

$$|G(x,y)| \leq C d_g(x,y)^{2k-n}$$

for all $x \neq y$ (see [8, Section 4], [22]). Theorem 1.1 improves these estimates for the specific polyharmonic operator $(\Delta_g + \alpha)^k$ in M, as $\alpha \to \infty$. A new contribution of this work is the derivation of estimates on the decay of $G_{g,\alpha}$ which are explicit in the lower-order term's coefficient. This dependence in α draws parallel to the well-known behavior of the Helmholtz kernel for the operator $-\Delta - \lambda^2$ in $\mathbb{R}^3$ (see for instance [5]).

The article is structured as follows. In Section 2, we construct a fundamental solution in $\mathbb{R}^n$ for $(-\Delta+\alpha)^k$, and prove precise estimates using a modified Giraud's Lemma which is proved in the Appendix A. Section 3 is devoted to the proof of Theorem 1.1. Based on the method of Robert [22], we iteratively construct an approximation of the Green's function in M preserving the estimates of the Euclidean case. We then conclude the proof of the Theorem thanks to a self-improving argument that allows to estimate the remainder term. Moreover, we show estimates on the derivatives of the Green's function, in Proposition 3.9 below.

Remark 1.1 (Notational conventions).

(1) We work on a manifold with fixed metric g. In the following, unless specified otherwise, all constants only depend on (M, g), n, k, they are denoted C, and their explicit value can vary from line to line, sometimes even in the same line.
(2) Let $f : X \times Y \to \mathbb{R}$ be a function, we will write, for any fixed $x \in X$, $f_x : Y \to \mathbb{R}$ with $f_x(y) := f(x, y)$.
(3) We will write $B_x(R)$ for the ball of center x and radius $R > 0$, either in M or in the Euclidean space $\mathbb{R}^n$, without distinction. We also define the diagonal set $Diag := \{(x, y) : x = y\}$ either in M or in $\mathbb{R}^n$, the ambient space will always be clear from context.

2. The Green's function for $(-\Delta + \alpha)^k$ in $\mathbb{R}^n$

In this Section, we prove uniqueness and pointwise bounds for the Green's function of the elliptic polyharmonic operator $(-\Delta + \alpha)^k$ in the Euclidean space $\mathbb{R}^n$.

2.1. Green's function of the poly-Laplacian in $\mathbb{R}^n$.

We start by gathering basic results for the fundamental solution of the poly-Laplacian operator $(-\Delta)^k$ in $\mathbb{R}^n$. Here, we let $\Delta = \sum_{i=1}^n \partial_i^2$ be the Laplace operator in the Euclidean space.

Fix an integer $k \geq 1$, and $n > 2k$, then define

$$c_{n,k} = \frac{1}{4^k \pi^{n/2} (k-1)!} \Gamma\left(\frac{n-2k}{2}\right), \tag{2.1}$$

where $\Gamma(t)$ is the well-known Gamma function. This constant $c_{n,k}$ is chosen such that

$$H^{(k)}(x, y) = c_{n,k} \frac{1}{|x-y|^{n-2k}} \tag{2.2}$$

is a fundamental solution for the poly-Laplacian operator $(-\Delta)^k$ in $\mathbb{R}^n$, see [8, Section 2.6]. This means in particular that

$$(-\Delta)^k H_x^{(k)} = 0 \quad \text{in the weak sense on } \mathbb{R}^n \setminus \{0\}.$$

2.2. **Construction and uniqueness.** Fix $\alpha > 0$, the purpose this Section is to show the existence and study the behavior of the Green's function for $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$. In particular, we are interested in its dependence on the coefficient $\alpha > 0$. We start by observing that the Green's functions for different values of α are related by a simple scaling property. Then, we obtain an exact expression for the Green's function of the operator $-\Delta + 1$. In a second step, we will use properties of the convolutions of distributions to retrieve expressions for the polyharmonic operators $(-\Delta + \alpha)^k$, $k \geq 1$. Finally, we use a modified version of Giraud's Lemma, proved in A.1, to obtain sharp pointwise bounds on the Green's function.

Definition 2.1. Fix $k \geq 1$, $n > 2k$ and $\alpha > 0$, we say that $H_\alpha^{(k)}(x, y)$ is a fundamental solution for the polyharmonic operator $(-\Delta + \alpha)^k$ in $\mathbb{R}^n$ if for all $x \in \mathbb{R}^n$, $H_{\alpha;x}^{(k)} \in L^1_{loc}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} H_\alpha^{(k)}(x,y)(-\Delta+\alpha)^k \varphi(y)\, dy = \varphi(x) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

Remark 2.1. It is straightforward to compute that $H_1^{(k)}(x, y)$ is a fundamental solution for the operator $(-\Delta + 1)^k$ in $\mathbb{R}^n$ if and only if

$$H_\alpha^{(k)}(x,y) := \alpha^{\frac{n-2k}{2}} H_1^{(k)}(\sqrt{\alpha}x, \sqrt{\alpha}y) \tag{2.3}$$

is a fundamental solution for the operator $(-\Delta + \alpha)^k$ in $\mathbb{R}^n$.

With this observation, we only study the Green's function for $(-\Delta + 1)^k$ in a first step, and then retrieve the general case $\alpha > 0$ using relation (2.3).

Recall the definition of the Bessel function of the second kind of order $\nu > 0$, $K_\nu(r)$, which is singular at the origin, and solution to the second order ordinary differential equation

$$u''(r) + \frac{1}{r}u'(r) - \left(1 + \frac{\nu^2}{r^2}\right)u(r) = 0 \qquad \text{on } \mathbb{R}^+ \setminus \{0\}.$$

These are well-known functions with explicit behavior (see [1]).

Proposition 2.1. *Fix $n \geq 3$, then*

$$H_1(x,y) := (2\pi)^{-\frac{n}{2}} |x-y|^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(|x-y|)$$

is a fundamental solution for the operator $(-\Delta + 1)$ in $\mathbb{R}^n$.

Proof. Start by observing that, thanks to the asymptotics for K_ν found in [1], we have the following :

- When $|x - y| \ll 1$,

$$\begin{aligned} H_1(x,y) &= \frac{\pi^{-\frac{n}{2}}}{4}\Gamma\left(\frac{n-2}{2}\right)|x-y|^{-(n-2)}(1+o(1)) \\ &= \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}(1+o(1)) \\ \frac{\partial}{\partial y_i}H_1(x,y) &= \frac{1}{\omega_{n-1}}\frac{(x_i-y_i)}{|x-y|^n}(1+o(1)) \\ \Leftrightarrow \quad |\nabla H_1|(x,y) &= \frac{1}{\omega_{n-1}}|x-y|^{1-n}(1+o(1)); \end{aligned} \tag{2.4}$$

- When $|x-y| \gg 1$,

$$\begin{aligned} H_1(x,y) &= \frac{(2\pi)^{-\frac{n-1}{2}}}{2} |x-y|^{-\frac{n-1}{2}} e^{-|x-y|}(1+o(|x-y|^{-1})) \\ |\nabla H_1|(x,y) &= \frac{(2\pi)^{-\frac{n-1}{2}}}{2} |x-y|^{-\frac{n-1}{2}} e^{-|x-y|}(1+o(|x-y|^{-1})). \end{aligned} \tag{2.5}$$

We now show that

$$(-\Delta+1)H_{1;x}(y) = 0 \qquad \text{for all } y \neq x. \tag{2.6}$$

By the expression of H_1, we write $r = |x-y|$ and define

$$H(r) := H_1(x,y) = (2\pi)^{-\frac{n}{2}} r^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(r).$$

Using the expression of the Laplacian in spherical coordinates on $\mathbb{R}^n$, (2.6) re-writes as

$$\frac{d^2}{dr^2}H(r) + \frac{n-1}{r}\frac{d}{dr}H(r) - H(r) = 0.$$

Now $H(r)$ satisfies this last equation for $r > 0$, by the definition of the Bessel function of the second kind $K_{\frac{n-2}{2}}(r)$. Thus, we conclude that H_1 solves (2.6) for all $x \neq y$.

For the second part of the proof, take $\varphi \in C_c^\infty(\mathbb{R}^n)$, we show that for all $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} H_{1;x}(y)(-\Delta+1)\varphi(y)\,dy = \varphi(x).$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta\varphi+\varphi)H_{1;x}\,dy &= \lim_{\delta\to 0}\int_{B_0(\delta)^c} (-\Delta\varphi+\varphi)H_{1;x}\,dy \\ &= \lim_{\delta\to 0}\big[\,a(\delta)+b(\delta)+c(\delta)\,\big], \end{aligned} \tag{2.7}$$

where

$$a(\delta) := \int_{B_x(\delta)^c} (-\Delta H_{1;x} + H_{1;x})\,\varphi\,dy = 0$$

since $H_{1;x}$ satisfies (2.6) on $\mathbb{R}^n \setminus \{x\}$, and with (2.4),

$$\begin{aligned} b(\delta) &:= \int_{\partial B_x(\delta)} H_{1;x}\partial_\nu\varphi\,d\sigma(y) \sim \delta^{-(n-2)}\delta^{n-1} = \mathcal{O}(\delta), \\ c(\delta) &:= -\int_{\partial B_x(\delta)} \partial_\nu H_{1;x}\varphi\,d\sigma(y) = \frac{\pi^{-\frac{n}{2}}}{2}\Gamma\left(\frac{n}{2}\right)\omega_{n-1}\,\varphi(x) + o(1) \end{aligned} \tag{2.8}$$

as $\delta \to 0$. Using the expression for the surface area of the sphere, we get

$$\int_{\mathbb{R}^n} (-\Delta\varphi+\varphi)H_1\,dy = \varphi(x).$$

We can conclude that $H_1(x,y)$ is a fundamental solution for the operator $(-\Delta+1)$ on $\mathbb{R}^n$. $\square$

Remark 2.2. We additionally observe that, for $|x-y| \ll 1$, the Green's function H_1 and its gradient are equal to first order to the standard Green's function for the Laplacian in $\mathbb{R}^n$ and its gradient, respectively. Moreover, $H_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$ only depends on $|x-y|$.

The following result is technical and establishes improved bounds for fundamental solutions.

Lemma 2.2. *Let $u \in C^{2k}(\mathbb{R}^n \setminus \{0\})$ be a function satisfying $(-\Delta+1)^k u = 0$ on $\mathbb{R}^n \setminus \{0\}$, and such that there exist $C>0$, $\rho \in \mathbb{R}$, with*

$$|u(x)| \leq \begin{cases} C\,|x|^{2k-n} & \text{if } 0<|x|\leq 1 \\ C\,|x|^{\rho}\, e^{-|x|} & \text{if } |x|\geq 1 \end{cases}.$$

Then for $l = 0, \dots 2k$, there is $C_l > 0$ such that

$$\left|\nabla^l u(x)\right| \leq C_l \begin{cases} |x|^{-(n-2k+l)} & 0<|x|\leq 1 \\ |x|^{\rho}\, e^{-|x|} & |x|\geq 1 \end{cases}.$$

Proof. Let $x \neq 0$ be fixed. Notice that there is $C>0$ such that $t^{\rho}e^{-t} \leq Ct^{2k-n}$ for all $t \geq 1$, so that

$$|u(x)| \leq C\,|x|^{2k-n} \qquad \forall\, x \neq 0. \tag{2.9}$$

When $|x| \leq 2$, define $v(y) = u(|x|\,y)$ on a ball $B_{\frac{x}{|x|}}(1/2) \not\ni 0$, and write $\lambda = |x|^2 \neq 0$. We compute

$$(-\Delta+\lambda)^k v(y) = |x|^{2k}\left((-\Delta+1)^k u\right)(|x|\,y) = 0 \qquad \forall\, y \in B_{\frac{x}{|x|}}(1/2).$$

By standard elliptic theory, since $\lambda \leq 4$, there is $C>0$ independent of x such that $v \in C^{2k}(B_{\frac{x}{|x|}}(1/4))$, and for all $y \in B_{\frac{x}{|x|}}(1/4)$

$$\begin{aligned} \left|\nabla^l v(y)\right| &\leq C\Big(\left\|(-\Delta+\lambda)^k v\right\|_{L^\infty(B_{\frac{x}{|x|}}(1/2))} + \|v\|_{L^\infty(B_{\frac{x}{|x|}}(1/2))} \Big) \\ &= C \sup_{z\in B_{\frac{x}{|x|}}(1/2)} |u(|x|\,z)| \leq \sup_{z\in B_{\frac{x}{|x|}}(1/2)} \frac{C}{||x|\,z|^{n-2k}} \leq \frac{C}{|x|^{n-2k}} \end{aligned} \tag{2.10}$$

using (2.9), and since $|z| \geq 1/2$ for all $z \in B_{\frac{x}{|x|}}(1/2)$. Now,

$$\left|\nabla^l v(y)\right| = |x|^l \left|\nabla^l u\right|(|x|\,y),$$

we evaluate inequality (2.10) at $y = \frac{x}{|x|} \in B_{\frac{x}{|x|}}(1/4)$ to obtain

$$\left|\nabla^l u\right|(x) \leq \frac{C}{|x|^{n-2k+l}} \qquad \forall\, |x| \leq 2.$$

On the other hand, when $|x|>2$, we use elliptic theory for u on a ball $B_x(1) \subset \mathbb{R}^n \setminus B_0(1)$. This shows that there is $C>0$ independent of x such that for all $y \in B_x(1/2)$,

$$\left|\nabla^l u\right|(y) \leq C\Big(\left\|(-\Delta+\lambda)^k u\right\|_{L^\infty B_x(1)} + \|v\|_{L^\infty(B_x(1))} \Big) \leq C \sup_{z\in B_x(1)} |z|^{\rho}\, e^{-|z|},$$

where this last inequality follows from the assumption on u. Observe that, when $|x| \geq 2$, $\frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}$ and $|y| \geq |x|-1$ for all $y \in B_x(1)$, so that no matter if ρ is positive or negative,

$$\left|\nabla^l u\right|(y) \leq C\,|x|^{\rho}\, e^{-|x|}.$$

Evaluating this inequality at $y = x \in B_x(\varepsilon)$ gives the result for $|x|>2$.

For the intermediate values $1 \leq |x| \leq 2$, the two regimes coincide, up to a constant. The first part of the proof gives

$$\left|\nabla^l u\right|(x) \leq \frac{C}{|x|^{n-2k+l}}.$$

But now for $l = 0, \dots 2k$, we have C_1, C_2 independent of x such that, if $1 \leq |x| \leq 2$,

$$\begin{aligned} |x|^{-(n-2k+l)} &\leq C_1 \\ C_1 \leq |x|^\rho e^{-|x|} &\leq C_2 \end{aligned},$$

and we conclude. □

Definition 2.2. Fix $k \geq 1$ and $n > 2k$, we define the space $\mathfrak{H}_k$, of all functions $u \in C^{2k}(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$ such that the following holds.

- There exists $C > 0$ such that $|u(x,y)| \leq C\,|x-y|^{-(n-2k)}$ when $|x-y| \leq 1$;
- For all $p \geq 1$, there is $C_p > 0$ such that for $l = 0, \dots 2k$,

$$\left|\nabla^l u(x,y)\right| \leq C_p\,|x-y|^{-p} \quad \text{when } |x-y| \geq 1.$$

Lemma 2.3. *There is a unique fundamental solution of $(-\Delta+1)^k$ in $\mathbb{R}^n$ in the class $\mathfrak{H}_k$.*

Remark 2.3. Note that with this result, the function $H_1 : \mathbb{R}^n \times \mathbb{R}^n \setminus Diag \to \mathbb{R}$ defined in Proposition 2.1 is the unique fundamental solution of the operator $-\Delta+1$ in $\mathbb{R}^n$ in the class $\mathfrak{H}_1$.

Proof. Let $x \in \mathbb{R}^n$, and $H, \tilde{H} \in \mathfrak{H}_k$ be two fundamental solutions for $(-\Delta+1)^k$ in $\mathbb{R}^n$. Define $f_x(y) := H(x,y) - \tilde{H}(x,y)$. Then $f_x \in C^{2k}(\mathbb{R}^n \setminus \{x\})$ with

(1) $|f_x(y)| \leq C\,|x-y|^{2k-n}$ when $|x-y| \leq 1$;
(2) $\left|\nabla^l f_x\right| \leq C_p\,|x-y|^{-p}$ for all p, when $|x-y| \geq 1$, and for $l = 0, \dots 2k$;
(3) f_x satisfies $(-\Delta+1)^k f_x = 0$ in the weak sense on $\mathbb{R}^n$.

We start by proving that the singularity of f_x at x is removable. Note that $f_x \in L^p(B_x(1))$ for all $1 \leq p < \frac{n}{n-2k}$. Elliptic theory gives that $f_x \in \mathcal{H}^{2k,p}(B_x(1/2))$ and there is $C > 0$ independent of x such that

$$\begin{aligned} \|f_x\|_{\mathcal{H}^{2k,p}(B_x(1/2))} &\leq C\left(\|f_x\|_{L^p(B_x(1))} + \left\|(-\Delta+1)^k f_x\right\|_{L^p(B_x(1))}\right) \\ &= C\,\|f_x\|_{L^p(B_x(1))}. \end{aligned}$$

Iterating, and by elliptic theory, we similarly find that $f_x \in \mathcal{H}^{l,p}(B_x(1/2))$ for any $l \geq 0$. By Sobolev embeddings, using a big enough l in the previous argument, then $f_x \in C^{0,\delta}(B_x(1/4))$ for some $\delta > 0$. We conclude that f_x has no singularity at 0, and $f_x \in C^{2k}(\mathbb{R}^n)$ satisfies

$$(-\Delta+1)^k f_x = 0 \qquad \text{on } \mathbb{R}^n \text{ in the classical sense.} \tag{2.11}$$

Using the decay of f_x at infinity, we have $f_x \in L^p(\mathbb{R}^n)$ for all $p \geq 1$. Fix $R > 0$, and take χ_R a cutoff function supported in $B_x(R)$ such that $\chi_R \equiv 1$ on $B_x(R/2)$.

We compute

$$
\begin{aligned}
(-\Delta+1)^k(\chi_R f_x) &= \sum_{l=0}^{k} \binom{k}{l} \left(\chi_R (-\Delta)^l f_x + \mathcal{O}\left(\sum_{m=1}^{2l} |\nabla^m \chi_R| \left|\nabla^{2l-m} f_x\right| \right) \right) \\
&= \mathcal{O}\left(\sum_{l=0}^{k} \sum_{m=1}^{2l} |\nabla^m \chi_R| \left|\nabla^{2l-m} f_x\right| \right)
\end{aligned}
$$

with (2.11). Testing this equation against $\chi_R f_x \in C_c^\infty(\mathbb{R}^n)$, we have by integration by parts on the left-hand side, for all $p>1$ and $R>2$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \sum_{l=0}^{k} \binom{k}{l} \left|(-\Delta)^{l/2}(\chi_R f_x)\right|^2 dy &\le C \sum_{l=0}^{k} \sum_{m=1}^{2l} \int_{\mathbb{R}^n} |\nabla^m \chi_R| \left|\nabla^{2l-m} f_x\right| \chi_R \, |f_x| \, dy \\
&\le C_p \int_{B_x(R)\setminus B_x(R/2)} |x-y|^{-p} \, dy
\end{aligned}
$$

where the latter follows from the decay of f_x at infinity. Now $|x-y|^{-p} \in L^1(\mathbb{R}^n \setminus B_x(1))$ for $p>n$, and thus the right-hand side vanishes as $R\to\infty$ provided we choose a fixed $p>n$. On the other hand since all the terms in the left-hand side are positive,

$$
\begin{aligned}
\|f_x\|_{\mathcal{H}^k(\mathbb{R}^n)} &\le \int_{\mathbb{R}^n} \sum_{l=0}^{k} \binom{k}{l} \left|(-\Delta)^{l/2} f_x\right|^2 dy \\
&\le \liminf_{R\to\infty} \int_{\mathbb{R}^n} \sum_{l=0}^{k} \binom{k}{l} \left|(-\Delta)^{l/2}(\chi_R f_x)\right|^2 dy = 0
\end{aligned}
$$

by Fatou's Lemma. Thus $f=0$ everywhere in $\mathbb{R}^n$. □

Remark 2.4. With the same strategy of proof as in the previous Lemma 2.3, we can show the following. The function $H^{(k)}(x,y)$ defined in (2.2) is the unique fundamental solution for the poly-Laplacian operator $(-\Delta)^k$ in $\mathbb{R}^n$ in the class of functions $u \in C^{2k}(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$ such that there exists $C>0$ and

$$
|u(x,y)| \le C\,|x-y|^{2k-n} \qquad \forall\, x \ne y.
$$

Remark 2.5. A fundamental solution h for $(-\Delta+1)^k$ is always smooth away from its singularity: Let h be a distribution that satisfies

$$
(-\Delta+1)^k h_x = 0 \qquad \text{weakly on } \mathbb{R}^n \setminus \{x\}.
$$

Now for all $\Omega \subset \mathbb{R}^n$ such that $x \notin \overline{\Omega}$, h satisfies

$$
(-\Delta+1)^k h_x = 0 \qquad \text{weakly on } \overline{\Omega}
$$

and by elliptic theory we can conclude $h_x \in C^\infty(U)$ for an open set $U \subset\subset \Omega$. This gives in turn $h_x \in C^\infty(\mathbb{R}^n \setminus \{x\})$.

We have everything we need to construct the Green's function of $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$ and describe its exact behavior. Define $H_1^{(1)} := H_1$ and for $k \ge 1$, iteratively

$$
H_1^{(k+1)}(x,y) := H_1^{(k)} * H_1^{(1)}(x,y) = \int_{\mathbb{R}^n} H_1^{(k)}(x,z) H_1^{(1)}(z,y)\, dz, \tag{2.12}
$$

which is well-defined provided $2k+2<n$, as easily seen by iteratively applying Giraud's Lemma (see Lemma A.1 below).

Theorem 2.4. *Fix $k \geq 1$, $n > 2k$, and $\alpha > 0$, then*

$$H_\alpha^{(k)}(x,y) := \alpha^{\frac{n-2k}{2}} H_1^{(k)}(\sqrt{\alpha}x, \sqrt{\alpha}y)$$

is the unique Green's function for $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$ in the class $\mathfrak{H}_k$ defined in 2.2, where $H_1^{(k)}$ is as defined in (2.12). Moreover, there exists $C > 0$ independent of $\alpha > 0$ such that for all $x \neq y$,

$$H_\alpha^{(k)}(x,y) \leq \begin{cases} C\left|x-y\right|^{2k-n} & \text{when } \sqrt{\alpha}\left|x-y\right| \leq 1 \\ C\alpha^{k\frac{n-3}{4}}\left|x-y\right|^{\frac{(k-2)n+k}{2}} e^{-\sqrt{\alpha}|x-y|} & \text{when } \sqrt{\alpha}\left|x-y\right| \geq 1 \end{cases}. \tag{2.13}$$

Finally, the Green's function is radial, $H_\alpha^{(k)}(x,y)$ only depends on $|x-y|$.

Proof. We begin by showing that $H_\alpha^{(k)}$ is a fundamental solution for the operator $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$. We then get explicit α-dependent estimates for $H_\alpha^{(k)}$, and deduce its uniqueness.

Since $n > 2k$, $H_1^{(l)}$ is defined for all $l = 1, \dots k$. We prove the first statement by induction. First, $H_1^{(1)}(x,y) = H_1(x,y)$ is a Green's function for $-\Delta+1$ in $\mathbb{R}^n$ as we showed in Proposition 2.1. Assume that we have proven that $H_1^{(l)}$ is a Green's function for some $1 \leq l \leq k-1$. Then let $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} H_{1;x}^{(l+1)}(y)(-\Delta+1)^{l+1}\varphi(y)dy \\ &\qquad = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} H_1^{(l)}(x,z)H_1^{(1)}(z,y)dz\right)(-\Delta+1)^{l+1}\varphi(y)dy \\ &\qquad = \int_{\mathbb{R}^n} H_{1;x}^{(l)}(z)\left(\int_{\mathbb{R}^n} H_{1;z}^{(1)}(y)(-\Delta+1)\left[(-\Delta+1)^l\varphi\right](y)dy\right)dz \\ &\qquad = \int_{\mathbb{R}^n} H_{1;x}^{(l)}(z)(-\Delta+1)^l\varphi(z)dz \\ &\qquad = \varphi(x), \end{aligned} \tag{2.14}$$

where the last line is the induction assumption. Now using (2.3), we obtain that $H_\alpha^{(k)}(x,y) = \alpha^{\frac{n-2k}{2}} H_1^{(k)}(\sqrt{\alpha}x, \sqrt{\alpha}y)$ is a Green's function for $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$.

To prove pointwise estimates on $H_1^{(k)}(x,y)$, we use an exponential version of the so-called *Giraud's Lemma*, whose standard proof can be found in [10]. We prove this result in appendix A in the generalized setting of a manifold, following a similar reasoning. With the behavior of $H_1^{(1)}$ in (2.4), (2.5), and Lemma A.1, we get iteratively for $l = 1, \dots k$,

$$H_1^{(l)}(u,v) \leq \begin{cases} C\left|u-v\right|^{2l-n} & \text{when } \left|u-v\right| \leq 1 \\ C\left|u-v\right|^{-l\frac{n-1}{2}+(l-1)n} e^{-|u-v|} & \text{when } \left|u-v\right| \geq 1. \end{cases}$$

We then observe that $H_1^{(k)} \in \mathfrak{H}_k$ using Lemma 2.2, it is thus the only Green's function in this class by Lemma 2.3. Similarly, by relation (2.3), we conclude that $H_\alpha^{(k)}$ is the unique Green's function for $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$ in the class $\mathfrak{H}_k$. The previous estimates now become

$$H_\alpha^{(k)}(x,y) \leq \begin{cases} C\left|x-y\right|^{2k-n} & \sqrt{\alpha}\left|x-y\right| \leq 1 \\ C\alpha^{k\frac{n-3}{4}}\left|x-y\right|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|x-y|} & \sqrt{\alpha}\left|x-y\right| \geq 1. \end{cases}$$

Finally, this Green's function inherits its symmetry from fact that $H_1^{(1)}$ only depends on $|x-y|$ and that the convolution of two radial functions is itself radial. $\square$

Note that the exponent $\frac{(k-2)n+k}{2}$ in (2.13) becomes positive for $k \geq 2$.

2.3. **Refined asymptotics.** We now prove more precise pointwise estimates on the Green's function $H_\alpha^{(k)}$ and its derivatives. When $\sqrt{\alpha}\,|x-y|$ is small, we show that $H_\alpha^{(k)}$ and its first $2k-1$ derivatives are equal to first order to the standard Green's function for the poly-Laplacian in $\mathbb{R}^n$ and its derivatives, respectively.

Proposition 2.5. *Fix $k \geq 1$, $n > 2k$ and $\alpha > 0$. Then $H_\alpha^{(k)} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$, where $H_\alpha^{(k)}$ is defined in Theorem 2.4, and for all $l = 0, \ldots 2k$, there exists $C_l > 0$ independent of α such that for all $x \neq y$ in $\mathbb{R}^n$,*

$$\left|\nabla^l H_{\alpha;x}^{(k)}(y)\right| \leq \begin{cases} C_l\,|x-y|^{-(n-2k+l)} & \sqrt{\alpha}\,|x-y| \leq 1 \\ C_l\,\alpha^{k\frac{n-3}{4}+\frac{l}{2}}\,|x-y|^{\frac{(k-2)n+k}{2}}\,e^{-\sqrt{\alpha}|x-y|} & \sqrt{\alpha}\,|x-y| \geq 1. \end{cases}$$

Proof. The first part $H_\alpha^{(k)} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$ comes from the Remark 2.5. The estimates for $l \geq 1$ are then a direct consequence of Lemma 2.2. For $l = 1, \ldots 2k$ we have

$$\begin{aligned} \left|\nabla^l H_{\alpha;x}^{(k)}(y)\right| &= \alpha^{\frac{n-2k}{2}}\alpha^{\frac{l}{2}}\left(\nabla^l H_{1;\sqrt{\alpha}x}^{(k)}\right)(\sqrt{\alpha}y) \\ &\leq C_l\alpha^{\frac{n-2k}{2}+\frac{l}{2}}\begin{cases} \left(\sqrt{\alpha}\,|x-y|\right)^{-(n-2k+l)} & \sqrt{\alpha}\,|x-y| \leq 1 \\ \left(\sqrt{\alpha}\,|x-y|\right)^{\frac{(k-2)n+k}{2}}e^{-\sqrt{\alpha}|x-y|} & \sqrt{\alpha}\,|x-y| \geq 1 \end{cases} \\ &= C_l\begin{cases} |x-y|^{-(n-2k+l)} & \sqrt{\alpha}\,|x-y| \leq 1 \\ \alpha^{k\frac{n-3}{4}+\frac{l}{2}}\,|x-y|^{\frac{(k-2)n+k}{2}}\,e^{-\sqrt{\alpha}|x-y|} & \sqrt{\alpha}\,|x-y| \geq 1 \end{cases}. \end{aligned}$$

$\square$

We now prove precise estimates for the behavior of $H_\alpha^{(k)}$, when $\sqrt{\alpha}\,|x-y|$ is small. To simplify the notation, define

$$\eta(t) = \begin{cases} t & \text{when } n = 2k+1 \\ t^2\left(1+|\log t|\right) & \text{when } n = 2k+2\,, \\ t^2 & \text{when } n \geq 2k+3 \end{cases} \qquad \text{for } 0 < t \leq 1. \tag{2.15}$$

Proposition 2.6. *Fix $k \geq 1$, $n > 2k$, $\alpha > 0$, and let $H_\alpha^{(k)}$ be the unique Green's function in $\mathfrak{H}_k$ for the operator $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$. Then, when $\sqrt{\alpha}\,|x-y| \leq 1$,*

$$H_\alpha^{(k)}(x,y) = c_{n,k}\,|x-y|^{2k-n}\left(1+\mathcal{O}\big(\eta(\sqrt{\alpha}\,|x-y|)\big)\right),$$

where η is defined in (2.15) *and $c_{n,k}$ is the constant in* (2.1).

Remark 2.6. Here and in the following, the notation $f(x,y) = \mathcal{O}(u(x,y))$, for a positive function u, is used to mean that there is a constant $C > 0$, independent of α, such that for all x, y,

$$\frac{|f(x,y)|}{u(x,y)} \leq C.$$

Proof. We begin by defining $R_\alpha := H_\alpha^{(k)} - H_0^{(k)}$, where we write $H_0^{(k)}(x,y) = c_{n,k}\,|x-y|^{2k-n}$ the Green's function for $(-\Delta)^k$ in $\mathbb{R}^n$. Now, we compute

$$\begin{aligned}(-\Delta)^k R_{\alpha,x} &= (-\Delta+\alpha)^k H_{\alpha;x}^{(k)} - (-\Delta)^k H_{0,x}^{(k)} - \sum_{l=0}^{k-1}\binom{k}{l}\alpha^{k-l}(-\Delta)^l H_{\alpha;x}^{(k)} \\ &= -\sum_{l=0}^{k-1}\binom{k}{l}\alpha^{k-l}(-\Delta)^l H_{\alpha;x}^{(k)}\end{aligned} \tag{2.16}$$

in the distributional sense on $\mathbb{R}^n$. Let $h_\alpha(x,y) := -\sum_{l=0}^{k-1}\binom{k}{l}\alpha^{k-l}(-\Delta)^l H_{\alpha;x}^{(k)}(y)$, straightforward computations with (2.13) then show that

$$|h_\alpha(x,y)| \le C\begin{cases}\alpha\,|x-y|^{-(n-2)} & \sqrt{\alpha}\,|x-y|\le 1\\ \alpha^{k\frac{n+1}{4}}\,|x-y|^{\frac{n(k-2)+k}{2}}\,e^{-\sqrt{\alpha}|x-y|} & \sqrt{\alpha}\,|x-y|\ge 1.\end{cases} \tag{2.17}$$

We now claim that, for all $y \neq x$,

$$R_\alpha(x,y) = \int_{\mathbb{R}^n} h_\alpha(x,z)\,c_{n,k}\,|y-z|^{2k-n}\,dz. \tag{2.18}$$

This follows from the fact that first,

$$|R_\alpha(x,y)| \le \left|H_\alpha^{(k)}(x,y)\right| + \left|H_0^{(k)}(x,y)\right| \le C\,|x-y|^{2k-n} \quad \forall\, x\neq y.$$

Moreover, the right-hand side of (2.18) defines a function in $L^1_{loc}(\mathbb{R}^n)$,

$$Z(x,y) := \int_{\mathbb{R}^n} h_{\alpha,x}(z)\,c_{n,k}\,|y-z|^{2k-n}\,dz,$$

which satisfies $(-\Delta)^k Z_x = h_{\alpha,x}$ in the distributional sense on $\mathbb{R}^n$. By Remark 2.4, we conclude that for all $x\neq y$ in $\mathbb{R}^n$, $R_\alpha(x,y) = Z(x,y)$. We now have, by Lemma A.1 together with (2.17), when $\sqrt{\alpha}\,|x-y|\le 1$,

$$|R_\alpha(x,y)| \le \begin{cases} C\alpha\,|x-y|^{-(n-2k-2)} & \text{when } 2k+2<n\\ C\alpha\big(1+\left|\log\sqrt{\alpha}\,|x-y|\right|\big) & \text{when } 2k+2=n\\ C\alpha^{\frac12} & \text{when } 2k+1=n\end{cases},$$

where the constant $C>0$ does not depend on $\alpha>0$. Finally, coming back to $H_\alpha^{(k)} = H_0^{(k)} + R_\alpha$, we have the conclusion. □

We obtain similar pointwise estimates for the derivatives of $H_\alpha^{(k)}$. The next Corollary shows that the estimates in Proposition 2.6 can be differentiated.

Corollary 2.7. *Fix $k\ge 1$, $n>2k$, $\alpha>0$, and let $H_\alpha^{(k)}$ the unique Green's function in $\mathfrak{H}_k$ for the operator $(-\Delta+\alpha)^k$ in $\mathbb{R}^n$. For $l=1,\dots 2k-1$, there exists $C_l>0$ independent of α such that for all $x\neq y$ with $\sqrt{\alpha}\,|x-y|\le 1$,*

$$\left|\nabla^l\left(|x-y|^{n-2k}\,H_{\alpha;x}^{(k)}(y)\right)\right| \le C_l\,|x-y|^{-l}\,\eta(\sqrt{\alpha}\,|x-y|),$$

where η is defined in (2.15).

Proof. With notations from the previous proof of Proposition 2.6, we have

$$|x-y|^{n-2k}\,H_{\alpha;x}^{(k)}(y) = c_{n,k} + |x-y|^{n-2k}\,R_{\alpha,x}(y),$$

and thus for $l \geq 1$,

$$\left|\nabla^l \left(|x-y|^{n-2k}\ H^{(k)}_{\alpha;x}(y)\right)\right| = \left|\nabla^l \left(|x-y|^{n-2k}\ R_{\alpha,x}(y)\right)\right|. \tag{2.19}$$

To estimate the derivatives of R_α, we go back to (2.18): Since $l \leq 2k-1$, we can differentiate under the integral sign, and we obtain

$$\nabla^l R_{\alpha,x}(y) = \int_{\mathbb{R}^n} h_\alpha(x,z) \nabla^l_{(y)} H_0^{(k)}(y,z)\, dz.$$

By Proposition 2.5, and since $l < 2k$, we can use as before Lemma A.1 to estimate the derivatives of $R_{\alpha,x}$ when $|x-y| \leq 1/\sqrt{\alpha}$:

- When $n-2k = 1$, we have to consider several cases. For $l = 1$ we get

$$\left|\nabla_{(y)} R_{\alpha,x}(y)\right| \leq C\alpha\big(1 + \left|\log \sqrt{\alpha}\,|x-y|\right|\big),$$

  and for $2 \leq l \leq 2k-1$,

$$\left|\nabla^l_{(y)} R_{\alpha,x}(y)\right| \leq C\alpha\,|x-y|^{1-l}.$$

- When $n-2k \geq 2$, we obtain for all $1 \leq l \leq 2k-1$,

$$\left|\nabla^l_{(y)} R_{\alpha,x}(y)\right| \leq C\alpha\,|x-y|^{-(n-2k-2+l)}.$$

By (2.19) and by Leibniz's formula, we now have, for $l = 1, \ldots 2k-1$

$$\begin{aligned}
\left|\nabla^l \left(|x-y|^{n-2k}\ H^{(k)}_{\alpha;x}(y)\right)\right| &\leq C \sum_{m=0}^{l} |x-y|^{n-2k-m} \left|\nabla^{l-m} R_{\alpha,x}(y)\right| \\
&\leq \begin{cases} C\sqrt{\alpha}\,|x-y|^{1-l} & \text{when } n-2k = 1 \\ C\alpha\,|x-y|^{2-l}\left(1 + \left|\log\sqrt{\alpha}\,|x-y|\right|\right) & \text{when } n-2k = 2\ . \\ C\alpha\,|x-y|^{2-l} & \text{when } n-2k \geq 3 \end{cases}
\end{aligned}$$

□

3. Extending the construction to a Riemannian manifold

In this Section we construct the Green's function for $(\Delta_g + \alpha)^k$ on a manifold M. We follow the construction from Robert [22]. We prove uniqueness, positivity, as well as estimates that explicitly depend on α. In the following, we will always consider (M, g) to be a compact Riemannian manifold without boundary, of dimension $n > 2k$ and with injectivity radius $i_g > 0$. We write $\Delta_\xi = -\Delta$ the standard Laplacian on $\mathbb{R}^n$, and $\Delta_g := -\operatorname{div}_g(\nabla\cdot)$ for the Laplace-Beltrami operator in M. The notation $B_x(R)$ will represent a ball of radius $R > 0$ and center x either in $\mathbb{R}^n$ or on the manifold, depending on the context.

Theorem 1.1 is proved in several steps. We first define an approximate fundamental solution for $(\Delta_g+\alpha)^k$ in M which is modelled on the Euclidean fundamental solution of $(-\Delta+\alpha)^k$. It satisfies the equation $(\Delta_g+\alpha)^k G_x = \delta_x$ up to error terms. We then iteratively improve the precision of these terms until we obtain a bounded error, which is finally controlled in subsection 3.3. Subsequently, we prove bounds on the derivatives of the Green's function of the same kind as Proposition 2.5 and Corollary 2.7. We finish this section with a remark on the mass of the operator $(\Delta_g + \alpha)^k$ when the dimension of the manifold $n = 2k + 1$.

We start with an observation.

Lemma 3.1. *Let $\alpha > 0$, if G is a Green's function for the operator $(\Delta_g + \alpha)^k$ in M, as defined in Definition 1.1, then it is unique.*

Proof. Start by noting that, by the same arguments as in Remark 2.5, any Green's function for $(\Delta_g+\alpha)^k$ is smooth away from its singularity, so that $G_x \in C^\infty(M\backslash\{x\})$ for any $x \in M$.

Let $\tilde{G}$ be another Green's function for $(\Delta_g + \alpha)^k$ in M. Take $\varphi \in C^\infty(M)$, and define $u \in C^\infty$ the unique solution to the equation

$$(\Delta_g + \alpha)^k u = \varphi \qquad \text{on } M.$$

We then have, for all $x \in M$,

$$\begin{cases} u(x) = \displaystyle\int_M G(x,y)\varphi(y)\, dv_g(y) \\ u(x) = \displaystyle\int_M \tilde{G}(x,y)\varphi(y)\, dv_g(y) \end{cases}.$$

For all $x \in M$, $\varphi \in C^\infty(M)$, we have obtained

$$\int_M \Big(G(x,y) - \tilde{G}(x,y)\Big)\, \varphi(x)\, dv_g = u(x) - u(x) = 0.$$

Thus, we can conclude that $G_x(y) = \tilde{G}_x(y)$ for almost every $y \in M$, and by continuity $G_x(y) = \tilde{G}_x(y)$ on $M \setminus \{x\}$. ☐

3.1. **Step 1: An approximate fundamental solution.** We start the proof of Theorem 1.1 by pulling back the function $H_\alpha^{(k)}$ onto the manifold. Fix $k \geq 1$, $n > 2k$, and let $0 < \tau_0 < i_g/2$ that will be chosen later. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function with $0 \leq \chi \leq 1$, such that $\chi(t) \equiv 1$ on $[0, \tau_0/2)$ and $\chi(t) \equiv 0$ on $(\tau_0, +\infty)$. We define

$$\tilde{G}_\alpha(x,y) := \chi(d_g(x,y)) H_\alpha^{(k)}(0, \exp_x^{-1}(y)) \tag{3.1}$$

for all $x \neq y$ in M, where $H_\alpha^{(k)}$ is defined in Theorem 2.4. This function only depends on $d_g(x,y)$.

Assume that α is large enough so that $1/\sqrt{\alpha} < \tau_0/2$. Using Theorem 2.4 we obtain

$$\tilde{G}_\alpha(x,y) \leq \begin{cases} C d_g(x,y)^{-(n-2k)} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ C\alpha^{k\frac{n-3}{4}} d_g(x,y)^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \end{cases}. \tag{3.2}$$

In particular, when $\sqrt{\alpha} d_g(x,y) \to 0$, we have that

$$\tilde{G}_\alpha(x,y) = c_{n,k} d_g(x,y)^{2k-n} \left(1 + \mathcal{O}\big(\eta(\sqrt{\alpha} d_g(x,y))\big)\right), \tag{3.3}$$

this function behaves to first order as a Riemannian version of the Green's function for the poly-Laplacian $(-\Delta)^k$ in $\mathbb{R}^n$, where η is defined in (2.15).

The following Proposition estimates the error term between $\tilde{G}_\alpha$ and a true fundamental solution in the distributional sense.

Proposition 3.2. *Let $\tau_0 < i_g/2$ and let $\tilde{G}_\alpha$ be as defined in* (3.1). *There exist $\alpha_0 \geq 1$ and $C > 0$ such that, for all $\alpha \geq \alpha_0$ and $x \in M$, there is $l_{\alpha,x} \in C^0(M \setminus \{x\})$ satisfying*

$$\int_M (\Delta_g + \alpha)^k \varphi\, \tilde{G}_{\alpha;x}\, dv_g = \varphi(x) + \int_M \varphi(y)\, l_{\alpha,x}(y)\, dv_g(y), \tag{3.4}$$

for all $\varphi \in C^\infty(M)$. The function $l_{\alpha,x}$ is $L^1(M)$, has support in $B_x(\tau_0)$, and

$$|l_{\alpha,x}(y)| \le \begin{cases} C d_g(x,y)^{-(n-2)} & \sqrt{\alpha} d_g(x,y) \le 1 \\ C \alpha^{k\frac{n+1}{2}} d_g(x,y)^{\frac{n(k-2)+k+4}{2}} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \ge 1 \end{cases}$$

for all $x \neq y$ in M.

Proof. We compute $(\Delta_g + \alpha)^k \tilde{G}_{\alpha;x}(y)$ and precisely estimate the error terms. Let $\tilde{g} := \exp_x^* g$, it is a metric on $B_0(\tau_0) \subset \mathbb{R}^n$ with bounded geometry since $\tau_0 < i_g$. In particular, for $f \in C^2(\mathbb{R}^n)$,

$$\Delta_{\tilde{g}} f(u) = \Delta_\xi f(u) + \mathcal{O}(|u| \, |\nabla f(u)|) + \mathcal{O}(|u|^2 \left|\nabla^2 f(u)\right|), \tag{3.5}$$

where $u := \exp_x^{-1}(y) \in B_0(\tau_0)$. Since $\tilde{G}_\alpha(x,y) = \chi(d_g(x,y)) H_\alpha^{(k)}(0, \exp_x^{-1}(y))$ is supported in $B_x(\tau_0)$, we can write, when $x \neq y$ with $d_g(x,y) \le \tau_0$,

$$\Delta_g \tilde{G}_{\alpha;x}(y) = \Delta_{\tilde{g}} \left(H_{\alpha;0}^{(k)}(u) \chi(|u|) \right) \Big|_{u = \exp_x^{-1}(y)}.$$

We observe that $\left(H_{\alpha;0}^{(k)} \chi(|\cdot|) \right) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is supported in $B_0(\tau_0)$. Now for $u \neq 0$ such that $|u| < \tau_0/2$, using Proposition 2.5, we have

$$\begin{aligned} \left| \nabla^l \left(\chi(|u|) H_{\alpha;0}^{(k)}(u) \right) \right| &= \left| \nabla^l H_{\alpha;0}^{(k)}(u) \right| \\ &\le C \begin{cases} |u|^{-(n-2k+l)} & \sqrt{\alpha} \, |u| \le 1 \\ \alpha^{k\frac{n-3}{4} + \frac{l}{2}} |u|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|u|} & \sqrt{\alpha} \, |u| \ge 1 \end{cases} \end{aligned}$$

for $l = 0, \dots 2k$. Take α_0 such that $1/\sqrt{\alpha_0} < \tau_0/2$, then we have $\sqrt{\alpha} \, |u| \ge 1$ for all $|u| \ge \tau_0/2$, so that

$$\begin{aligned} \left| \nabla^l \left(\chi(|u|) H_{\alpha;0}^{(k)}(u) \right) \right| &\le \sum_{m=0}^{l} \left| \nabla^{l-m} \chi \right| \left| \nabla^m H_{\alpha;0}^{(k)} \right| \\ &\le C \alpha^{k\frac{n-3}{4} + \frac{l}{2}} |u|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|u|}. \end{aligned}$$

We have thus

$$\left| \nabla^l \left(\chi(|u|) H_{\alpha;0}^{(k)}(u) \right) \right| \le C \begin{cases} |u|^{-(n-2k+l)} & \sqrt{\alpha} \, |u| \le 1 \\ \alpha^{k\frac{n-3}{4} + \frac{l}{2}} |u|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|u|} & \sqrt{\alpha} \, |u| \ge 1 \\ 0 & |u| > \tau_0 \end{cases}. \tag{3.6}$$

We now show that, for $\varphi \in C^\infty(M)$,

$$\int_M (\Delta_g + \alpha)^k \varphi \, \tilde{G}_{\alpha;x} \, dv_g = \varphi(x) + \lim_{\delta \to 0} \int_{M \setminus B_x(\delta)} \varphi \, (\Delta_g + \alpha)^k \tilde{G}_{\alpha;x} \, dv_g. \tag{3.7}$$

Start by observing that since $\tilde{G}_{\alpha;x} \in L^1(M)$,

$$\begin{aligned} \int_M (\Delta_g + \alpha)^k \varphi \, \tilde{G}_{\alpha;x} \, dv_g &= \lim_{\delta \to 0} \int_{M \setminus B_x(\delta)} (\Delta_g + \alpha)^k \varphi \, \tilde{G}_{\alpha;x} \, dv_g \\ &= \lim_{\delta \to 0} \left[\sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l} \int_{M \setminus B_x(\delta)} \Delta_g^l \varphi \, \tilde{G}_{\alpha;x} \, dv_g \right]. \end{aligned}$$

Integration by parts gives, as in (2.7), for $l = 1, \dots k$,

$$\begin{aligned}(3.8)\quad \int_{M\setminus B_x(\delta)} \Delta_g^l \varphi \, \tilde{G}_{\alpha;x}\, dv_g = &\int_{M\setminus B_x(\delta)} \varphi \, \Delta_g^l \tilde{G}_{\alpha;x}\, dv_g \\ &+ \sum_{m=0}^{l-1} \int_{\partial B_x(\delta)} \partial_\nu \Delta_g^{l-1-m} \varphi \Delta_g^m \tilde{G}_{\alpha;x} d\sigma_g \\ &- \sum_{m=0}^{l-1} \int_{\partial B_x(\delta)} \Delta_g^{l-1-m} \varphi \, \partial_\nu \Delta_g^m \tilde{G}_{\alpha;x} d\sigma_g,\end{aligned}$$

where ∂_ν is the covariant derivative along the normal direction to $\partial B_x(\delta)$ in M. Using (3.6), we obtain for $l = 1, \dots k$, $m = 0, \dots l-1$,

$$(3.9)\quad \left| \int_{\partial B_x(\delta)} \partial_\nu \Delta_g^{l-1-m} \varphi \Delta_g^m \tilde{G}_{\alpha;x} d\sigma_g \right| \leq C\delta^{-n+2k-2m} \int_{\partial B_x(\delta)} d\sigma_g = o(1)$$

as $\delta \to 0$. On the other hand, when $l = k$ and $m = k-1$, we compute

$$\begin{aligned}(3.10)\quad &- \int_{\partial B_x(\delta)} \varphi(y) \partial_\nu \Delta_g^{k-1} \tilde{G}_{\alpha;x}(y) d\sigma_g(y) \\ &= \varphi(x) \int_{\partial B_x(\delta)} -\partial_\nu \Delta_g^{k-1} \tilde{G}_{\alpha;x}(y) d\sigma_g(y) + o(1) \quad \text{as } \delta \to 0\end{aligned}$$

since φ is $C^\infty(M)$. Using (3.5) iteratively, we have

$$\begin{aligned}&\int_{\partial B_x(\delta)} -\partial_\nu \Delta_g^{k-1} \tilde{G}_{\alpha;x}(y) d\sigma_g(y) \\ &= \int_{\partial B_0(\delta)} -\partial_\nu \left(\Delta_\xi^{k-1} H_{\alpha;0}^{(k)}(u) \right) d\sigma + \mathcal{O}\left(\int_{\partial B_0(\delta)} \delta^2 \left| \nabla^{2k-1} H_{\alpha;0}^{(k)}(u) \right| d\sigma \right) \\ &\quad + \mathcal{O}\left(\int_{\partial B_0(\delta)} \delta \left| \nabla^{2k-2} H_{\alpha;0}^{(k)}(u) \right| d\sigma \right) + \mathcal{O}\left(\sum_{m=2}^{2k-3} \int_{\partial B_0(\delta)} \left| \nabla^m H_{\alpha;0}^{(k)}(u) \right| d\sigma \right) \\ &= \int_{\partial B_0(\delta)} -\partial_\nu \left(\Delta_\xi^{k-1} H_{\alpha;0}^{(k)}(u) \right) d\sigma + o(1),\end{aligned}$$

as $\delta \to 0$, estimating the terms with (3.6). Here in the right-hand side, ∂_ν is now the derivative normal to the sphere in the Euclidean space. Using Corollary 2.7, one has

$$\int_{\partial B_0(\delta)} -\partial_\nu \left(\Delta_\xi^{k-1} H_{\alpha;0}^{(k)}(u) \right) d\sigma = \int_{\partial B_0(\delta)} -\partial_\nu \left(\Delta_\xi^{k-1} \frac{c_{n,k}}{|u|^{n-2k}} \right) d\sigma + o(1),$$

and thus by definition of $c_{n,k}$ in (2.1), we obtain in the end

$$(3.11)\quad \int_{\partial B_x(\delta)} -\partial_\nu \Delta_g^{k-1} \tilde{G}_{\alpha;x} d\sigma_g = 1 + o(1) \qquad \text{when } \delta \to 0.$$

Finally, all the remaining terms are estimated with (3.6),

$$(3.12)\quad \left| \int_{\partial B_0(\delta)} \Delta_g^{l-1-m} \varphi \, \partial_\nu \Delta_g^m \tilde{G}_{\alpha;x} d\sigma_g \right| \leq C\delta^{2k-m-1} = o(1)$$

as $\delta \to 0$, when $m \neq k-1$, $1 \leq l \leq k$. Coming back to (3.8), putting together (3.10), (3.11) in (3.9), and with (3.12), we obtain (3.7). Note that all the terms $\Delta_g^l \tilde{G}_{\alpha;x}$ are integrable by (3.6) except for the term $\Delta_g^k \tilde{G}_{\alpha;x}$ which is only bounded by $d_g(x,y)^{-n}$ when $d_g(x,y) \leq 1/\sqrt{\alpha}$.

Write $l_{\alpha,x}(y) := (\Delta_g + \alpha)^k \tilde{G}_{\alpha;x}(y)$ for all $x \neq y$, then $l_{\alpha,x} \in C^0(M \setminus \{x\})$ with support in $B_x(\tau_0)$ by the definition of $\tilde{G}_\alpha$. Now for $x \neq y$ such that $d_g(x,y) \leq \tau_0$, write $u := \exp_x^{-1}(y)$, and compute

$$\begin{aligned}
(\Delta_g + \alpha)^k \tilde{G}_{\alpha;x}(\exp_x(u)) &= (\Delta_{\tilde{g}} + \alpha)^k \left(\chi(|u|) H^{(k)}_{\alpha;0}(u)\right) \\
&= \sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l} \Bigg[\chi(|u|) \Delta_\xi^l H^{(k)}_{\alpha;0}(u) + \mathcal{O}\left(\sum_{m=1}^{2l-2} \left| \nabla^m \left(\chi(|u|) H^{(k)}_{\alpha;0}(u) \right) \right| \right) \\
&\quad + \mathcal{O}\left(|u| \left| \nabla^{2l-1} \left(\chi(|u|) H^{(k)}_{\alpha;0}(u) \right) \right| \right) + \mathcal{O}\left(|u|^2 \left| \nabla^{2l} \left(\chi(|u|) H^{(k)}_{\alpha;0}(u) \right) \right| \right) \Bigg],
\end{aligned}$$

using again (3.5) iteratively. By (3.6) and since $H^{(k)}_\alpha$ is a fundamental solution for the operator $(\Delta_\xi + \alpha)^k$ in $\mathbb{R}^n$, we have when $0 < |u| \leq 1/\sqrt{\alpha}$,

$$\begin{aligned}
\left| (\Delta_g + \alpha)^k \tilde{G}_{\alpha;x}(\exp_x(u)) \right| &= \sum_{l=0}^{k} \alpha^{k-l} \Bigg[\mathcal{O}\left(|u|^2 \left| \nabla^{2l} H^{(k)}_{\alpha;0}(u) \right| \right) \\
&\quad + \mathcal{O}\left(|u| \left| \nabla^{2l-1} H^{(k)}_{\alpha;0}(u) \right| \right) + \mathcal{O}\left(\sum_{m=1}^{2l-2} \left| \nabla^m H^{(k)}_{\alpha;0}(u) \right| \right) \Bigg] \\
&\leq C \sum_{l=0}^{k} \alpha^{k-l} |u|^{2k-n-2l+2} \leq |u|^{2-n}.
\end{aligned}$$

Similarly, when $\sqrt{\alpha}\,|u| \geq 1$,

$$\left| (\Delta_g + \alpha)^k \tilde{G}_{\alpha;x}(\exp_x(u)) \right| \leq C \sum_{l=0}^{k} \alpha^{k-l} |u|^2 \alpha^{k\frac{n-3}{4}+l} |u|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|u|}.$$

We have thus shown that, for $x \neq y$,

$$|l_{\alpha,x}(y)| \leq C \begin{cases} d_g(x,y)^{-(n-2)} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ \alpha^{k\frac{n+1}{4}} d_g(x,y)^{\frac{(k-2)n+k+4}{2}} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ 0 & d_g(x,y) \geq \tau_0 \end{cases} . \tag{3.13}$$

In particular, $l_{\alpha,x} \in L^1(M)$ so that (3.7) becomes

$$\int_M (\Delta_g + \alpha)^k \varphi \, \tilde{G}_{\alpha;x} \, dv_g = \varphi(x) + \int_M \varphi \, l_{\alpha,x} \, dv_g$$

for all $\varphi \in C^\infty(M)$. Note that $k\frac{n+1}{4} - \frac{(k-2)n+k+4}{2} = \frac{n-2}{2}$, so that the two regimes in (3.13) are of order $\alpha^{\frac{n-2}{2}}$ when $d_g(x,y) \sim 1/\sqrt{\alpha}$. □

3.2. **Step 2: The induction step.** In this step, we define a sequence of functions to iteratively improve the estimates on the error term.

Proposition 3.3. *There exists $N \in \mathbb{N}$, $\tau_0 > 0$, $\alpha_0 \geq 1$ such that, for all $\alpha \geq \alpha_0$, there is a function $\tilde{G}^*_\alpha \in C^\infty(M \times M \setminus Diag)$ and for all $x \in M$, a function $\gamma_{\alpha,x} \in C^0(M)$ such that*

$$\int_M (\Delta_g + \alpha)^k \varphi(y) \tilde{G}^*_\alpha(x,y)\, dv_g(y) + \int_M \gamma_{\alpha,x}(y)\varphi(y)\, dv_g(y) = \varphi(x) \tag{3.14}$$

for all $\varphi \in C^\infty(M)$. Moreover, both $\tilde{G}^*_{\alpha;x}$ and $\gamma_{\alpha,x}$ are supported in $B_x(N\tau_0)$, $\tilde{G}^*_\alpha$ satisfies

$$\tilde{G}^*_\alpha(x,y) = c_{n,k} d_g(x,y)^{2k-n} \left(1 + \mathcal{O}\left(\eta(\sqrt{\alpha} d_g(x,y))\right)\right) \tag{3.15}$$

for $\sqrt{\alpha} d_g(x,y) \leq 1$, and there is a constant $C > 0$ independent of $\alpha \geq \alpha_0$ such that writing $p_{n,k} := \frac{k(n+1)+4}{2} N - n$,

$$|\gamma_{\alpha,x}(y)| \leq C \alpha^{-N+\frac{n}{2}} \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ (\sqrt{\alpha} d_g(x,y))^{p_{n,k}} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ 0 & d_g(x,y) \geq N\tau_0 \end{cases}, \tag{3.16}$$

for all $x, y \in M$.

Proof. Fix $\tau_0 > 0$ such that $\tau_0 < i_g/(n+2)$, and take $\alpha_0 \geq 1$ given by Proposition 3.2. We define, for all $\alpha \geq \alpha_0$ and all $x \neq y$ in M,

$$\Gamma^1_\alpha(x,y) := -l_{\alpha,x}(y)$$
$$\Gamma^{i+1}_\alpha(x,y) := \int_M \Gamma^i_\alpha(x,z)\Gamma^1_\alpha(z,y)\, dv_g(z) \qquad \forall\, i \geq 2.$$

By the exponential version of Giraud's Lemma, Lemma A.2, and by (3.13) we see that, as long as $2i < n$ and $i\tau_0 < i_g$, we have $\Gamma^i_{\alpha,x} \in L^1(M)$, $\Gamma^i_\alpha \in C^0(M \times M \setminus Diag)$, and

$$\left|\Gamma^i_\alpha(x,y)\right| \leq C_i \begin{cases} d_g(x,y)^{-(n-2i)} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ \alpha^{ki\frac{n+1}{4}} d_g(x,y)^{\frac{k(n+1)+4}{2} i - n} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ 0 & d_g(x,y) \geq i\tau_0 \end{cases}.$$

Now take $N = \left\lfloor \frac{n}{2} \right\rfloor + 1 \in \mathbb{N}$, so that $2N > n$. By the choice of τ_0, we have $N\tau_0 < i_g/2$. Lemma A.2 then shows that $\Gamma^N_\alpha \in C^0(M \times M)$ and that

$$\begin{aligned} \left|\Gamma^N_\alpha(x,y)\right| & \\ \leq C &\begin{cases} \alpha^{-N+\frac{n}{2}} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ \alpha^{kN\frac{n+1}{4}} d_g(x,y)^{\frac{k(n+1)+4}{2} N - n} e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ 0 & d_g(x,y) \geq N\tau_0 \end{cases}. \end{aligned} \tag{3.17}$$

Let, for $i = 1, \dots N-1$ and $x \neq y$,

$$\tilde{G}^i_\alpha(x,y) := \int_M \Gamma^i_\alpha(x,z) \tilde{G}_\alpha(z,y)\, dv_g(z). \tag{3.18}$$

If $x \neq y$ are such that $\sqrt{\alpha} d_g(x,y) \leq 1$, we again have by Lemma A.2

$$\left|\tilde{G}^i_\alpha(x,y)\right| \leq C_i \begin{cases} d_g(x,y)^{-(n-2k-2i)} & \text{when } 2k+2i < n \\ 1 + |\log(\sqrt{\alpha} d_g(x,y))| & \text{if } 2k+2i = n \\ \alpha^{-\frac{2k+2i-n}{2}} & \text{when } 2k+2i > n \end{cases}. \tag{3.19}$$

While, if $\sqrt{\alpha} d_g(x,y) \geq 1$, writing for simplicity

$$p_i := k\frac{i(n+1)+n-3}{4} \qquad \text{and} \qquad \rho_i := \frac{k(n+1)(i+1)-2n+4i}{2} \tag{3.20}$$

satisfying $2p_i - \rho_i = n - 2k - 2i$, we have

$$\left|\tilde{G}^i_\alpha(x,y)\right| \leq C_i \alpha^{p_i} d_g(x,y)^{\rho_i} e^{-\sqrt{\alpha} d_g(x,y)} \tag{3.21}$$

with $\tilde{G}^i_\alpha(x,y) = 0$ when $d_g(x,y) \geq (i+1)\tau_0$.

In some sense, the $\tilde{G}^i_\alpha$ are successive error terms in the expression of the Green's function. Let us define for all $x \neq y$ in M,

$$\tilde{G}^*_\alpha(x,y) := \tilde{G}_\alpha(x,y) + \sum_{i=1}^{N-1} \tilde{G}^i_\alpha(x,y). \tag{3.22}$$

Then we have, for all $\varphi \in C^\infty(M)$ and $x \in M$, by Proposition 3.2 and since $\Gamma^1_{\alpha,x} = -l_{\alpha,x}$,

$$\begin{aligned}
&\int_M (\Delta_g + \alpha)^k \varphi(y) \tilde{G}^*_\alpha(x,y)\, dv_g(y) \\
&\quad = \int_M (\Delta_g+\alpha)^k \varphi\, \tilde{G}_{\alpha;x}\, dv_g \\
&\qquad\quad + \sum_{i=1}^{N-1} \int_M \int_M \Gamma^i_\alpha(x,z) \tilde{G}_\alpha(z,y) (\Delta_g+\alpha)^k \varphi(y)\, dv_g(z)\, dv_g(y) \\
&\quad = \varphi(x) + \int_M l_{\alpha,x}(y)\varphi(y)\, dv_g(y) \\
&\qquad\quad + \sum_{i=1}^{N-1} \int_M \Gamma^i_\alpha(x,z) \left(\int_M \tilde{G}_{\alpha;z} (\Delta_g+\alpha)^k \varphi\, dv_g \right) dv_g(z) \\
&\quad = \varphi(x) + \int_M l_{\alpha,x}\, \varphi\, dv_g + \sum_{i=1}^{N-1} \int_M \Gamma^i_\alpha(x,z) \left[\varphi(z) + \int_M l_{\alpha,z}\, \varphi\, dv_g \right] dv_g(z) \\
&\quad = \varphi(x) - \int_M \Gamma^1_{\alpha,x} \varphi\, dv_g + \sum_{i=1}^{N-1} \int_M \Gamma^i_{\alpha,x} \varphi\, dv_g \\
&\qquad\quad + \sum_{i=1}^{N-1} \int_M \left[\int_M \Gamma^i_\alpha(x,z) l_z(y)\, dv_g(z) \right] \varphi(y)\, dv_g(y) \\
&\quad = \varphi(x) - \int_M \Gamma^N_{\alpha,x} \varphi\, dv_g
\end{aligned}$$

where we used Fubini twice and the definition of Γ^{i+1}_α.

We now let $\gamma_{\alpha,x}(y) := \Gamma^N_\alpha(x,y)$, then (3.14) follows, and $\gamma_{\alpha,x} \in C^0(M)$ satisfies (3.16) thanks to (3.17). Finally, for $\sqrt{\alpha} d_g(x,y) \leq 1$, and again by Lemma A.2, we have

$$\begin{aligned}
\left| \sum_{i=1}^{N-1} \tilde{G}^i_\alpha(x,y) \right| &\leq \begin{cases} C\alpha^{-1/2} & \text{if } n-2k=1, \\ C\left(1 + \left|\log \sqrt{\alpha} d_g(x,y)\right|\right) & \text{if } n-2k=2, \\ C d_g(x,y)^{-(n-2k-2)} & \text{if } n-2k \geq 3, \end{cases} \\
&= C\alpha^{-1} d_g(x,y)^{-(n-2k)} \eta(\sqrt{\alpha} d_g(x,y)),
\end{aligned} \tag{3.23}$$

recalling the definition (2.15) for η. This means that, with (3.3), we still have

$$\tilde{G}^*_\alpha(x,y) = c_{n,k} d_g(x,y)^{-(n-2k)} \left(1 + \mathcal{O}\big(\eta(\sqrt{\alpha} d_g(x,y))\big)\right)$$

when $\sqrt{\alpha} d_g(x,y) \leq 1$. □

3.3. **Step 3: Estimates on the remainder term.** With Proposition 3.3, we have modified the starting function $\tilde{G}_\alpha$ to get closer to a real fundamental solution to the operator $(\Delta_g + \alpha)^k$ in M. The remainder $\gamma_{\alpha,x}$ is now uniformly bounded in α by (3.16) and continuous in $M \times M$.

Fix $\alpha \geq \alpha_0$ given by the Proposition 3.3, and $x \in M$. We let $u_{\alpha,x}$ be the unique solution of

$$(\Delta_g + \alpha)^k u_{\alpha,x} = \gamma_{\alpha,x} \qquad \text{weakly in } M, \tag{3.24}$$

where $\gamma_{\alpha,x} = \Gamma^N_{\alpha,x}$ as in the proof of Proposition 3.3. Such $u_{\alpha,x}$ exists and is unique since $(\Delta_g + \alpha)^k$ is coercive for any $\alpha > 0$, and since $\gamma_{\alpha,x} \in C^0(M)$.

Remark 3.1. By standard elliptic theory and Sobolev's embeddings, we have $u_{\alpha,x} \in C^{2k-1,\theta}(M)$ for all $\theta \in (0,1)$, with

$$\|u_{\alpha,x}\|_{C^l(M)} \leq C_{l,\alpha} \|\gamma_{\alpha,x}\|_{C^0(M)} \leq C_{l,\alpha} \qquad l = 0, \dots 2k-1 \tag{3.25}$$

by (3.16). The constants $C_{l,\alpha}$ depend on α, but we want pointwise estimates on $u_{\alpha,x}$ and its derivatives with an explicit dependence in α, so that (3.25) is not enough. In particular, in this step, we aim at recovering some exponential decay for $u_{\alpha,x}(y)$, when $\sqrt{\alpha} d_g(x,y) \geq 1$.

Proposition 3.4. *Let $\alpha \geq \alpha_0$ given by Proposition 3.3. For $\alpha \geq \alpha_0$ and $x \in M$, let $\tilde{G}^*_\alpha$ be as in* (3.22)*, and let $u_{\alpha,x} \in \mathcal{H}^k(M)$ be the unique weak solution to* (3.24)*. Define*

$$G_{g,\alpha}(x,y) := \tilde{G}^*_\alpha(x,y) + u_{\alpha,x}(y) \tag{3.26}$$

for all $x \neq y$ in M. Then $G_{g,\alpha}$ is a fundamental solution for the operator $(\Delta_g + \alpha)^k$ in M. Moreover, if $f \in L^q(M)$ for some $q > n/2k$, and $\tilde{u} \in \mathcal{H}^{k,q}(M)$ solves $(\Delta_g + \alpha)^k \tilde{u} = f$ in M, then $\tilde{u} \in C^0(M)$ and we have the following representation formula,

$$\tilde{u}(x) = \int_M G_{g,\alpha}(x,z) f(z)\, dv_g(z) \quad \text{for all } x \in M.$$

Proof. For the first part, go back to (3.14). For all $\varphi \in C^\infty(M)$,

$$\begin{aligned}\int_M (\Delta_g + \alpha)^k \varphi(y)\, G_{g,\alpha}(x,y)\, dv_g(y) \qquad\qquad & \\ = \varphi(x) - \int_M \gamma_{\alpha,x} \varphi\, dv_g + \int_M u_{\alpha,x} (\Delta_g + \alpha)^k \varphi\, dv_g & \\ = \varphi(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad &\end{aligned} \tag{3.27}$$

since $u_{\alpha,x}$ is a weak solution to (3.24). By (3.15), and since $u_{\alpha,x}$ is continuous in M, we have $G_{g,\alpha;x} \in L^1(M)$ and $G_{g,\alpha}$ is a fundamental solution for the operator $(\Delta_g + \alpha)^k$ in M.

Now take $f \in L^q(M)$, $q > n/2k$ and $\tilde{u} \in \mathcal{H}^k(M)$ satisfying $(\Delta_g + \alpha)^k \tilde{u} = f$ weakly in M. Standard elliptic theory gives $\tilde{u} \in \mathcal{H}^{2k,q}(M)$, and then with Sobolev's embeddings we have $\tilde{u} \in C^{0,\theta}(M)$ for some $\theta \in (0,1)$. Let $(f_m)_{m \geq 1}$ be a sequence of functions in $C^\infty(M)$ such that $f_m \to f$ in L^q, and take $\tilde{u}_m \in C^\infty(M)$ the respective solutions to $(\Delta_g+\alpha)^k\tilde{u}_m = f_m$. By elliptic estimates and Sobolev's embedding, for any fixed $\alpha \geq 1$, $(\tilde{u}_m)_{m\geq 1}$ is a bounded sequence in $C^{0,\theta}(M)$. Then, by compactness of the inclusion $C^{0,\theta}(M) \subset C^0(M)$ and since the solution is unique, $\tilde{u}_m \to \tilde{u}$ in $C^0(M)$ up to a subsequence. Since $u_m \in C^\infty(M)$ we have, testing (3.27) against u_m,

$$\tilde{u}_m(x) = \int_M G_{g,\alpha}(x,y) f_m(y)\, dv_g(y).$$

And, for all $x \in M$, $\tilde{u}_m(x) \to^{m\to\infty} \tilde{u}(x)$. Finally, for all $1 \leq p < \frac{n}{n-2k}$, we have $G_{g,\alpha;x} \in L^p(M)$ since $\tilde{G}^*_{\alpha;x} \in L^p(M)$ by (3.15) and $u_{\alpha,x} \in C^0(M) \subset L^p(M)$. Thus, since $f_m \to f$ in $L^q(M)$, choosing $1 \leq p < \frac{n}{n-2k}$ such that $\frac{1}{p}+\frac{1}{q}=1$, Hölder's inequality implies that

$$\left| \int_M G_{g,\alpha}(x,y) f_m(y)\, dv_g - \int_M G_{g,\alpha}(x,y) f(y)\, dv_g \right| \leq \|G_{g,\alpha;x}\|_{L^p(M)} \|f_m - f\|_{L^q(M)} \to 0.$$

We have thus shown, for all $x \in M$,

$$\tilde{u}(x) = \int_M G_{g,\alpha}(x,y) f(y)\, dv_g(y).$$

□

Thanks to Proposition 3.4 and (3.24), there is a representation formula for $u_{\alpha,x}$ itself, given by

$$\begin{aligned} u_{\alpha,x}(y) &= \int_M G_{g,\alpha}(y,z)\gamma_{\alpha,x}(z)\, dv_g(z) \\ &= \int_M \tilde{G}^*_\alpha(y,z)\gamma_{\alpha,x}(z)\, dv_g(z) + \int_M u_y(z)\gamma_{\alpha,x}(z)\, dv_g(z), \end{aligned} \tag{3.28}$$

where $\tilde{G}^*_\alpha$ and $\gamma_{\alpha,x}$ are introduced in Proposition 3.3. We now use this formula to self-improve the estimates on $u_{\alpha,x}$. We prove exponential decay on $u_{\alpha,x}$, when $\alpha \geq \alpha_0$ is large enough. This is a striking difference with the case of operators with bounded coefficients.

Proposition 3.5. *There exists $\alpha_0 \geq 1$ such that the following holds. For all $0 < \varepsilon < 1$, there is a constant $C_\varepsilon > 0$ such that for all $\alpha \geq \alpha_0$ and all $x, y \in M$,*

$$|u_{\alpha,x}(y)| \leq C_\varepsilon \alpha^{-k} \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}\, i_g/2} & d_g(x,y) \geq i_g/2 \end{cases}, \tag{3.29}$$

where $u_{\alpha,x}$ is the unique weak solution to (3.24).

Proof. We define, for all $\alpha \geq 1$ and $0 < \varepsilon < 1$, a function $\Psi_{\varepsilon,\alpha} \in L^1(M \times M)$ as

$$\Psi_{\varepsilon,\alpha}(x,y) = \begin{cases} e^{-(1-\varepsilon)} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}\, i_g/2} & d_g(x,y) \geq i_g/2 \end{cases}. \tag{3.30}$$

We prove the Proposition by using the representation formula (3.28) for $u_{\alpha,x}$. The first term is estimated with Lemma A.2, the estimates (3.19), (3.21) and (3.16) give

$$
\begin{aligned}
(3.31)\qquad &\left|\int_M \tilde G^*_\alpha(y,z)\gamma_{\alpha,x}(z)\,dv_g(z)\right| \\
&\qquad\qquad \le C\begin{cases} \alpha^{-N+\frac{n-2k}{2}} & \sqrt{\alpha}d_g(x,y)\le 1 \\ \displaystyle\sum_{i=0}^N \alpha^{\tilde p_i} d_g(x,y)^{\tilde\rho_i} e^{-\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y)\ge 1 \\ 0 & d_g(x,y)\ge 2N\tau_0 \end{cases}
\end{aligned}
$$

where

$$\tilde p_i := k\tfrac{n+1}{4}(N+i) + k\tfrac{n-3}{4} \text{ and } \tilde\rho_i := \tfrac{k(n+1)(N+i+1)}{2} + 2(N+i) - n,$$

with $N\tau_0 < i_g/2$ and $N > n/2$, N and τ_0 are as defined in the proof of Proposition 3.3.

Fix any $0<\varepsilon<1$ and $x\in M$, there is a constant $C_\varepsilon>0$ independent of α and x such that, for α large enough, we can write (3.31) as

$$
\begin{aligned}
(3.32)\qquad &\left|\int_M \tilde G^*_\alpha(y,z)\gamma_{\alpha,x}(z)\,dv_g(z)\right| \\
&\qquad\qquad \le C_\varepsilon \alpha^{-N+\frac{n-2k}{2}}\begin{cases} 1 & \sqrt{\alpha}d_g(x,y)\le 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y)\ge 1 \\ 0 & d_g(x,y)\ge 2\tau \end{cases} \\
&\qquad\qquad \le C_\varepsilon \alpha^{-N+\frac{n-2k}{2}}\Psi_{\varepsilon,\alpha}(x,y),
\end{aligned}
$$

for all $y\in M$. Note that, by integration by parts and the fact that u_x solves (3.24), one obtains

$$
\begin{aligned}
\int_M u_y(z)\gamma_{\alpha,x}(z)\,dv_g(z) &= \int_M u_y(z)\,(\Delta_g+\alpha)^k u_x(z)\,dv_g(z) \\
&= \int_M (\Delta_g+\alpha)^k u_y(z)\,u_x(z)\,dv_g(z) \qquad (3.33)\\
&= \int_M \gamma_{\alpha,y}(z)\,u_x(z)\,dv_g(z).
\end{aligned}
$$

We now claim that $u_{\alpha,x} = o(\Psi_{\varepsilon,\alpha})$ in the sense that

$$\left\|\frac{u_{\alpha,x}(\cdot)}{\Psi_{\varepsilon,\alpha}(x,\cdot)}\right\|_{L^\infty(M)} \to 0 \qquad \text{as } \alpha\to\infty, \text{ when } x\in M \text{ is fixed.}$$

For this, define for all α,

$$\Upsilon_{\alpha,x} := \left\|\frac{u_{\alpha,x}}{\Psi_{\varepsilon,\alpha}(x,\cdot)}\right\|_{L^\infty(M)}$$

and let $y_\alpha\in M$ be such that $\frac{|u_{\alpha,x}(y_\alpha)|}{\Psi_{\varepsilon,\alpha}(x,y_\alpha)} = \Upsilon_{\alpha,x}$. We know $\Upsilon_{\alpha,x}$ and y_α exist since $u_{\alpha,x}$, $\Psi_{\varepsilon,\alpha}(x,\cdot)$ are continuous, and $\Psi_{\varepsilon,\alpha}>0$ in M. Applying (3.28) at the point

y_α and using (3.32) and (3.33) we now have, for all α large enough,

(3.34)
$$\begin{aligned}|u_{\alpha,x}(y_\alpha)| &\le \left|\int_M \tilde G^*_\alpha(y_\alpha,z)\gamma_{\alpha,x}(z)\,dv_g(z)\right| + \left|\int_M u_x(z)\gamma_{\alpha,y_\alpha}(z)\,dv_g(z)\right| \\ &\le C_\varepsilon \alpha^{-N+\frac{n-2k}{2}}\Psi_{\varepsilon,\alpha}(x,y_\alpha) + \Upsilon_{\alpha,x}\int_M \Psi_{\varepsilon,\alpha}(x,z)\,|\gamma_{\alpha,y_\alpha}(z)|\,dv_g(z).\end{aligned}$$

Using the bounds (3.16) on $\gamma_{\alpha,x}$ and Lemma A.3 below, which is a modified version of the exponential Giraud's Lemma, we obtain that

$$\left|\int_M \Psi_{\varepsilon,\alpha}(x,z)\gamma_{\alpha,y_\alpha}(z)\,dv_g(z)\right| \le C'_\varepsilon \alpha^{-N}\Psi_{\varepsilon,\alpha}(x,y_\alpha). \tag{3.35}$$

Going back to (3.34) and dividing by $\Psi_{\varepsilon,\alpha}(x,y_\alpha) > 0$, we have thus proven that

$$\Upsilon_{\alpha,x} \le C_\varepsilon \alpha^{-N+\frac{n-2k}{2}} + C'_\varepsilon \alpha^{-N}\Upsilon_{\alpha,x}, \tag{3.36}$$

with $N > n/2$ by definition of N. Hence, $\Upsilon_{\alpha,x} \to 0$ for all $x \in M$.

We have thus shown, since M is compact, that there exists $\alpha_0 \ge 0$ and a constant $C_\varepsilon > 0$ independent of x such that for all $\alpha \ge \alpha_0$, $x, y \in M$,

$$|u_{\alpha,x}(y)| \le C_\varepsilon \begin{cases} 1 & \sqrt{\alpha}d_g(x,y) \le 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y) \ge 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}i_g/2} & d_g(x,y) \ge i_g/2. \end{cases} \tag{3.37}$$

We now use this estimate to compute again the second term of (3.28): Using Lemma A.3 with (3.37), and with (3.32), we finally get

$$|u_{\alpha,x}(y)| \le C_\varepsilon \alpha^{-k} \begin{cases} 1 & \sqrt{\alpha}d_g(x,y) \le 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y) \ge 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}i_g/2} & d_g(x,y) \ge i_g/2, \end{cases}$$

since $-N + \frac{n-2k}{2} \le -k$, which concludes the proof. □

Remark 3.2. The assumption $\alpha \ge \alpha_0$, with α_0 large, is crucial to obtain (3.37) from (3.36), and thus exponential decay for $u_{\alpha,x}$ when $\sqrt{\alpha}d_g(x,y) \ge 1$.

Remark 3.3. With our approach, we cannot expect to obtain the exact decay $e^{-\sqrt{\alpha}d_g(x,y)}$ for $u_{\alpha,x}$ when $\sqrt{\alpha}d_g(x,y) \ge 1$. Successive convolutions in the second term of the representation formula (3.28) add positive exponents of $d_g(x,y)$ that we cannot get rid of, see Remark A.2. We are thus forced to reduce the exponential decay of $u_{\alpha,x}$ to $e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)}$. This is what allows us to obtain (3.35).

3.4. **Step 4: End of the proof of Theorem 1.1.** We can now proceed to conclude the proof of the main Theorem, putting the several pieces together.

Lemma 3.6. *Fix $\alpha \ge \alpha_0$ and let $G_{g,\alpha}$ be the Green's function for the operator $(\Delta_g + \alpha)^k$ in M defined in Proposition 3.4. Then for all $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ such that for all $x \ne y$, we have*

$$|G_{g,\alpha}(x,y)| \le C_\varepsilon \begin{cases} d_g(x,y)^{-(n-2k)} & \sqrt{\alpha}d_g(x,y) \le 1 \\ d_g(x,y)^{-(n-2k)} e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y) \ge 1\,. \\ e^{-(1-\varepsilon)\sqrt{\alpha}i_g/2} & d_g(x,y) \ge i_g/2 \end{cases} \tag{3.38}$$

Proof. Fix any $0 < \varepsilon < 1$. First, let $x, y \in M$ be such that $d_g(x,y) \le 1/\sqrt{\alpha}$ and $x \ne y$. Going back to the definition of $G_{g,\alpha}$ in (3.26), we use (3.2), (3.19) and the fact that $u_x \in C^0(M)$ satisfies (3.25), and we obtain

$$|G_{g,\alpha}(x,y)| \le C d_g(x,y)^{-(n-2k)}$$

for all $d_g(x,y) \le 1/\sqrt{\alpha}$.

Now let $1\sqrt{\alpha} \le d_g(x,y) < i_g/2$. We use (3.2) and (3.21), there is a constant C_ε independent of α, x, y such that

$$\left|\tilde{G}^i_\alpha(x,y)\right| \le C_\varepsilon \alpha^{-i} d_g(x,y)^{-(n-2k)} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)}$$

for $i = 0, \dots N-1$, writing $\tilde{G}^0_\alpha := \tilde{G}_\alpha$. Up to taking a slightly smaller $0 < \varepsilon' < \varepsilon$, we also have for all $y \in M$

$$\begin{aligned} |u_{\alpha,x}(y)| &\le C_{\varepsilon'} \alpha^{-k} e^{-(1-\varepsilon')\sqrt{\alpha} d_g(x,y)} \\ &\le C_{\varepsilon,\varepsilon'} \alpha^{-\frac{n}{2}} d_g(x,y)^{-(n-2k)} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)}, \end{aligned}$$

using (3.29). We have thus obtained

$$|G_{g,\alpha}(x,y)| \le C_\varepsilon d_g(x,y)^{-(n-2k)} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)}.$$

Finally, when $d_g(x,y) \ge i_g/2$, the estimate follows from (3.29). □

Lemma 3.7. *Fix $\alpha > \alpha_0$ and let $G_{g,\alpha}$ be the Green's function for the operator $(\Delta_g + \alpha)^k$ in M defined in Proposition 3.4. Then for all $x \ne y$ in M, $G_{g,\alpha}(x,y) = G_{g,\alpha}(y,x)$.*

Proof. Let $\varphi \in C^\infty(M)$, define for $y \in M$

$$h(y) := \int_M G_{g,\alpha}(z,y)\varphi(z)\, dv_g(z).$$

Then since $G_{g,\alpha}(\cdot, y) \in C^\infty(M \setminus \{y\}) \cap L^1(M)$, h is well-defined. We claim that h is continuous on M. To prove the latter, fix $y \in M$ and take a sequence of points in M, $y_m \to y$ as $m \to \infty$. Let $\delta_m := d_g(y_m, y)$, we have

$$h(y_m) = \int_{B_{y_m}(\delta_m/2)} G_{g,\alpha}(z,y_m)\varphi(z)\, dv_g(z) + \int_{M \setminus B_{y_m}(\delta_m/2)} G_{g,\alpha}(z,y_m)\varphi(z)\, dv_g(z).$$

On the one hand, by (3.38), we have

$$\int_{B_{y_m}(\delta_m/2)} G(y_m,z)\varphi(z)\, dv_g(z) = o(1)$$

as $m \to \infty$. On the other hand, for $z \in M \setminus B_{y_m}(\delta_m/2)$, we have $d_g(z,y_m) \ge \frac{1}{3} d_g(z,y)$, so that using (3.38),

$$|G(y_m,z)\varphi(z)| \le C d_g(z,y)^{2k-n}.$$

By dominated convergence, we conclude that

$$\begin{aligned} \lim_{m\to\infty} h(y_m) &= \lim_{m\to\infty} \int_{M \setminus B_{y_m}(\delta_m/2)} G_{g,\alpha}(z,y_m)\varphi(z)\, dv_g(z) \\ &= \int_M G_{g,\alpha}(z,y)\varphi(z)\, dv_g(z), \end{aligned}$$

and h is continuous at $y \in M$.

Let now $g \in C^\infty(M)$ be the unique solution to $(\Delta_g+\alpha)^k g = \varphi$ on M. Since $G_{g,\alpha}$ is the Green's function for the operator $(\Delta_g+\alpha)^k$ in M, we have, for all $x \in M$,

$$g(x) = \int_M G_{g,\alpha}(x,y)\varphi(y)\,dv_g(y).$$

Now since $h \in C^0(M)$, $g \in C^\infty(M)$, one has $h-g \in L^\infty(M)$ and there is a unique $\psi_0 \in \mathcal{H}^k(M)$ such that

$$(\Delta_g+\alpha)^k\psi_0 = h-g \quad \text{in the weak sense on } M.$$

Moreover, elliptic theory gives $\psi_0 \in \mathcal{H}^{2k,p}(M)$ for all $p \geq 1$. By Fubini's theorem, and since $G_{g,\alpha;x} \in L^1(M)$ by Proposition 3.4, we have

$$\begin{aligned}\int_M h\,(\Delta_g+\alpha)^k\psi_0\,dv_g &= \int_M \left(\int_M G_{g,\alpha}(x,y)\varphi(x)\,dv_g(x)\right)(\Delta_g+\alpha)^k\psi_0(y)\,dv_g(y)\\ &= \int_M \left(\int_M G_{g,\alpha}(x,y)(\Delta_g+\alpha)^k\psi_0(y)\,dv_g(y)\right)\varphi(x)\,dv_g(x)\\ &= \int_M \psi_0(x)\varphi(x)\,dv_g(x).\end{aligned}$$

Use the definition of g and integrate by parts: Since $\psi_0 \in \mathcal{H}^{2k,p}(M)$ for any $p \geq 1$,

$$\int_M \psi_0\varphi\,dv_g = \int_M \psi_0(\Delta_g+\alpha)^k g\,dv_g = \int_M (\Delta_g+\alpha)^k\psi_0 g\,dv_g.$$

By definition of ψ_0, we have thus shown in the end that

$$\int_M (\Delta_g+\alpha)^k\psi_0\big(h-g\big)\,dv_g = 0 = \int_M \big(h-g\big)^2\,dv_g,$$

so that $h(x) = g(x)$ for almost every $x \in M$, and thus $h \equiv g$ on M by continuity of h, g. This shows that for all $\varphi \in C^\infty(M)$, and all $x \in M$,

$$\int_M \left(G_{g,\alpha}(x,y) - G_{g,\alpha}(y,x)\right)\varphi(y)\,dv_g(y) = 0.$$

Since $G_{g,\alpha}(x,\cdot)$, $G_{g,\alpha}(\cdot,x) \in L^1(M)$ for all $x \in M$, we deduce that

$$G_{g,\alpha}(x,y) = G_{g,\alpha}(y,x)$$

for almost every $y \in M$, and we conclude with the continuity of $G_{g,\alpha}$ in $M \times M \setminus$ *Diag*. □

Proof. (of Theorem 1.1) Recall the definition (3.26) of $G_{g,\alpha}$, we have

$$G_{g,\alpha}(x,y) = \tilde{G}^*_\alpha(x,y) + u_{\alpha,x}(y),$$

where $\tilde{G}^*_\alpha$ was defined in (3.22), and $u_{\alpha,x}$ is the unique solution to (3.24). It is a Green's function for $(\Delta_g+\alpha)^k$ in M as we proved in Proposition 3.4, and we have proved the estimates on $G_{g,\alpha}$ in Lemma 3.6. Moreover, when $d_g(x,y) \leq 1/\sqrt{\alpha}$, by Proposition 3.3, $\tilde{G}^*_\alpha(x,y)$ satisfies (3.15), so that $G_{g,\alpha}(x,y)$ satisfies (1.2), since $u_{\alpha,x}$ is bounded.

Uniqueness was proved in Lemma 3.1, while the symmetry was proved in Lemma 3.7. It remains only to show the positivity of the Green's function.

Let $H^{(1)}_{g,\alpha}$ be the Green's function for the operator $\Delta_g + \alpha$ in M as constructed in Proposition 3.4, and define for all $x \neq y$, as in (2.12),

$$H^{(k)}_{g,\alpha}(x,y) := \int_M H^{(k-1)}_{g,\alpha}(x,z) H^{(1)}_{g,\alpha}(z,y)\, dv_g(z)$$

for $k \geq 2$. This function is well-defined provided that $n > 2k$, and smooth away from the diagonal, thanks to Lemma A.2. The same argument as in (2.14) shows that $H^{(k)}_{g,\alpha}$ is a Green's function for the operator $(\Delta_g + \alpha)^k$ in M. By Lemma 3.1, $H^{(k)}_{g,\alpha} = G_{g,\alpha}$ in $M \times M \setminus Diag$. Now, recall that $H^{(1)}_{g,\alpha}$ is positive by the strong maximum principle and Hopf Lemma (see [22, Theorem 3.]). Thus, $G_{g,\alpha} = H^{(k)}_{g,\alpha}$ is positive in $M \times M \setminus Diag$. □

Remark 3.4. Note that the factorized form of $(\Delta_g + \alpha)^k$ is crucial to obtain positivity. In general, the question of positivity of for higher-order operators is a hard problem. We refer to [8, 13, 14] for positivity results for polyharmonic operators on domains of $\mathbb{R}^n$ with specific boundary conditions. The argument that we describe here for the positivity would work for any polyharmonic operator which is decomposed as the product of coercive operators of order 2, such as the GJMS operator in a Riemannian manifold with Einstein metric (see [6]).

3.5. **Control on the derivatives.** We are now interested in estimates similar to (3.38) for the derivatives of $G_{g,\alpha}$. Let $f \in C^\infty(M)$, then the l^{th} covariant derivative $\nabla^l f \in (T^*M)^l$ is a $(l,0)$-tensor. Fix $x \in M$, on a neighborhood $B_x(\tau)$ with $\tau < i_g$, the exponential map at x allows us to define the metric on $B_0(\tau) \subset \mathbb{R}^n$,

$$\tilde{g}(u) := \exp_x^* g(u). \tag{3.39}$$

Since M is compact, there is a global constant $C > 0$ independent of $x \in M$ such that for $u \in B_0(\tau)$,

$$\begin{cases} |\tilde{g}(u)_{ij} - \delta_{ij}| \leq C\,|u|^2 & 1 \leq i,j \leq n \\ \left|\nabla^l_{\tilde{g}} \tilde{f}(u) - \nabla^l_\xi \tilde{f}(u)\right| \leq C\left(|u|\left|\nabla^{l-1}_\xi \tilde{f}(u)\right| + \displaystyle\sum_{m=1}^{l-2}\left|\nabla^m_\xi \tilde{f}(u)\right|\right) & l \geq 2 \end{cases} \tag{3.40}$$

where $\tilde{f} := f \circ \exp_x$, and $\tilde{g}(u)_{ij}$ are the components of $\tilde{g}$ at u in the exponential chart.

Proposition 3.8. *Let $\alpha_0 \geq 1$ be given by Proposition 3.5. For all $0 < \varepsilon < 1$, there is a constant $C_\varepsilon > 0$ such that for all $\alpha \geq \alpha_0$, for all $x, y \in M$ and for $l = 1, \ldots 2k-1$, we have*

$$\left|\nabla^l u_{\alpha,x}\right|_g (y) \leq C\alpha^{-k+\frac{l}{2}} \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} & d_g(x,y) \geq i_g/2 \end{cases}.$$

Proof. Let us fix $x \in M$ and let $y \in M$. We prove the estimates on the derivatives of $u_{\alpha,x}$ by using elliptic estimates in balls centered at y. Observe that for $z \in B_y(1/\sqrt{\alpha})$,

$$d_g(x,y) - 1/\sqrt{\alpha} \leq d_g(x,z) \leq d_g(x,y) + 1/\sqrt{\alpha}.$$

With this realization, and with Proposition 3.5, we obtain

$$
\begin{aligned}
|u_{\alpha,x}(z)| &\le C_\varepsilon \alpha^{-k} \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \le 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \ge 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} & d_g(x,y) \ge i_g/2 \end{cases} \\
&= C_\varepsilon \alpha^{-k} \Psi_{\varepsilon,\alpha}(x,y)
\end{aligned} \tag{3.41}
$$

for all $z \in B_y(1/\sqrt{\alpha})$, where $\Psi_{\varepsilon,\alpha}$ is as in (3.30). We also have by (3.16) that there exists a constant $C'_\varepsilon > 0$ independent of x, y and $\alpha \ge \alpha_0$ such that

$$
|\gamma_{\alpha,x}(z)| \le C'_\varepsilon \alpha^{-N+\frac{n}{2}} \Psi_{\varepsilon,\alpha}(x,y) \tag{3.42}
$$

for all $z \in B_y(1/\sqrt{\alpha})$.

Set $v := \sqrt{\alpha} \exp_y^{-1}(z)$, then if $z \in B_y(1/\sqrt{\alpha})$, $v \in B_0(1) \subset \mathbb{R}^n$. For $\alpha \ge \alpha_0$, we have $1/\sqrt{\alpha} < i_g$, define

$$
\tilde{g}_{\alpha,y}(v) := \exp_y^* g(v/\sqrt{\alpha}) \qquad \text{and} \qquad \begin{cases} \tilde{u}_{\alpha,x,y}(v) := u_{\alpha,x}(\exp_y(v/\sqrt{\alpha})) \\ \tilde{\gamma}_{\alpha,x,y}(v) := \gamma_{\alpha,x}(\exp_y(v/\sqrt{\alpha})) \end{cases}.
$$

Observe that $\tilde{\gamma}_{\alpha,x,y} \in C^0(B_0(1))$ since $\gamma_{\alpha,x} \in C^0(M)$, and that $\tilde{u}_{\alpha,x,y} \in L^\infty(B_0(1))$. We compute, for all $v \in B_0(1)$,

$$
\Delta_g u_{\alpha,x}(\exp_y(v/\sqrt{\alpha})) = \alpha \Delta_{\tilde{g}_{\alpha,y}} \tilde{u}_{\alpha,x,y}(v)
$$

so that $\tilde{u}_{\alpha,x,y}$ solves the following equation on $B_0(1)$,

$$
\alpha^k (\Delta_{\tilde{g}_{\alpha,y}} + 1)^k \tilde{u}_{\alpha,x,y} = \tilde{\gamma}_{\alpha,x,y}.
$$

Using the fact that when $\alpha \to \infty$, $\tilde{g}_{\alpha,y} \to \xi$ the Euclidean metric in $C^\infty_{loc}(\mathbb{R}^n)$, $(\Delta_{\tilde{g}_{\alpha,y}} + 1)^k$ is an elliptic operator with coefficients bounded independently of $\alpha \ge \alpha_0$. Elliptic theory gives $\tilde{u}_{\alpha,x,y} \in C^{2k-1,\theta}(B_0(1/2))$ for all $\theta \in (0,1)$. There is a constant $C > 0$ that does not depend on α, x, y, such that for $l = 1, \dots 2k-1$ and all $v \in B_0(1/2)$ we have

$$
\begin{aligned}
\left|\nabla^l \tilde{u}_{\alpha,x,y}(v)\right| &\le C(\|\tilde{u}_{\alpha,x,y}\|_{L^\infty(B_0(1))} + \left\|\alpha^{-k}\tilde{\gamma}_{\alpha,x,y}\right\|_{C^0(B_0(1))}) \\
&\le C\alpha^{-k}\Psi_{\varepsilon,\alpha}(x,y),
\end{aligned}
$$

using (3.41) and (3.42). Note that here the gradient and the norm are taken in the Euclidean space $\mathbb{R}^n$. To get the metric-related quantities we use (3.40), for all $v \in B_0(1/2)$, we have

$$
\begin{aligned}
\left|\nabla^l_g u_{\alpha,x}\right|_g (\exp_y(v/\sqrt{\alpha})) \le C \left|\nabla^l_{\tilde{g}_{\alpha,y}} \tilde{u}_{\alpha,x,y}(v)\right| &\le C \sum_{m=1}^{l} \alpha^{\frac{m}{2}} \left|\nabla^m \tilde{u}_{\alpha,x,y}(v)\right| \\
&\le C\alpha^{\frac{l}{2}-k}\Psi_{\varepsilon,\alpha}(x,y).
\end{aligned}
$$

Taking this inequality at $v = 0 \in B_0(1/2)$, then $\exp_y(0) = y$, and we conclude. □

The following Proposition extends Corollary 2.7 to the Riemannian case, showing that the pointwise decomposition of $G_{g,\alpha}$ in (1.2) can be differentiated formally.

Proposition 3.9. *Fix $0 < \varepsilon < 1$, there exists $\alpha_0 \ge 1$ such that for all $\alpha \ge \alpha_0$, the derivatives of $G_{g,\alpha}$ have the following estimates. There exists a constant $C > 0$*

independent of α such that for all $x \neq y \in M$, and for $l = 1, \ldots 2k-1$, $x \neq y$,

$$\left|\nabla^l G_{g,\alpha;x}(y)\right|_g \leq C \begin{cases} d_g(x,y)^{-(n-2k+l)} & \sqrt{\alpha} d_g(x,y) \leq 1 \\ d_g(x,y)^{-(n-2k+l)} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} & d_g(x,y) \geq i_g/2 \end{cases}.$$

Moreover, for $d_g(x,y) \leq 1/\sqrt{\alpha}$ with $x \neq y$,

$$\left|\nabla^l \left(d_g(x,y)^{n-2k} G_{g,\alpha;x}(y)\right)\right|_g \leq C d_g(x,y)^{-l} \eta(\sqrt{\alpha} d_g(x,y)),$$

where η is defined in (2.15).

Proof. We estimate the derivative of each term in the expression 3.26 independently.

First, we compute estimates for the derivatives of $\tilde{G}_\alpha$. Fix $x \in M$, and let $u := \exp_x^{-1}(y)$, then by the definition (3.1), and (3.6), we have for $u \in B_0(\tau_0)$,

$$\begin{aligned} \left|\nabla_g^l \tilde{G}_{\alpha;x}\right|_g (\exp_x(u)) &\leq C \left|\nabla_{\tilde{g}}^l \left(\chi(|u|) H_{\alpha;0}^{(k)}(u)\right)\right| \\ &\leq C \sum_{m=1}^{l} \left|\nabla_\xi^m \left(\chi(|u|) H_{\alpha;0}^{(k)}(u)\right)\right| \\ &\leq C \begin{cases} |u|^{-(n-2k+l)} & \sqrt{\alpha}\,|u| \leq 1 \\ \alpha^{k\frac{n-3}{4}+\frac{l}{2}} |u|^{\frac{n(k-2)+k}{2}} e^{-\sqrt{\alpha}|u|} & \sqrt{\alpha}\,|u| \geq 1 \end{cases}, \end{aligned} \tag{3.43}$$

using (3.40), where $\tilde{g}$ is as in (3.39). By the multiplication by the cutoff, we also have

$$\left|\nabla^l \tilde{G}_{\alpha;x}(y)\right|_g = 0 \qquad \text{for } y \in M \setminus B_x(\tau_0).$$

In a second step, we use (3.18) and the fact that $\left|\nabla^l \tilde{G}_{\alpha;x}\right|_g \in L^1(M)$ for $l = 1, \ldots 2k-1$ thanks to (3.43), and we obtain

$$\left|\nabla^l \tilde{G}_{\alpha;x}^i(y)\right|_g \leq \int_M \left|\Gamma_\alpha^i(x,z)\right| \left|\nabla^l \tilde{G}_{\alpha;z}(y)\right| dv_g(z)_g,$$

for $i = 1, \ldots N-1$. Using Lemma A.2, we then have, when $x \neq y$ with $\sqrt{\alpha} d_g(x,y) \leq 1$,

$$\left|\nabla^l \tilde{G}_{\alpha;x}^i(y)\right|_g \leq C_i \begin{cases} d_g(x,y)^{-(n-2k-2i+l)} & \text{when } 2k+2i-l < n \\ 1 + |\log(\sqrt{\alpha} d_g(x,y))| & \text{if } 2k+2i-l = n \\ \alpha^{-\frac{2k+2i-n-l}{2}} & \text{when } 2k+2i-l > n \end{cases}. \tag{3.44}$$

We also have, when $\sqrt{\alpha} d_g(x,y) \geq 1$,

$$\left|\nabla^l \tilde{G}_{\alpha;x}^i(y)\right|_g \leq C \alpha^{p_i + \frac{l}{2}} d_g(x,y)^{\rho_i} e^{-\sqrt{\alpha} d_g(x,y)}$$

where p_i, ρ_i were defined in (3.20). Finally, $\left|\nabla^l \tilde{G}_{\alpha;x}^i(y)\right|_g = 0$ when $d_g(x,y) \geq (i+1)\tau_0$, where τ_0 is as in Proposition 3.3. There is then a constant $C_\varepsilon > 0$ such that for α large enough and all x, y with $\sqrt{\alpha} d_g(x,y) \geq 1$,

$$\left|\nabla^l \tilde{G}_{\alpha;x}^i(y)\right|_g \leq C_\varepsilon \alpha^{-i} d_g(x,y)^{-(n-2k+l)} e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)}$$

for $i = 1, \ldots N-1$, $l = 1, \ldots 2k-1$.

For the last term $u_{\alpha,x}$, choose $0<\varepsilon'<\varepsilon$, Proposition 3.8 gives

$$\left|\nabla^l u_{\alpha,x}(y)\right|_g \le C\alpha^{-k+\frac{l}{2}}\begin{cases}1 & \sqrt{\alpha}d_g(x,y)\le 1\\ e^{-(1-\varepsilon')\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y)\ge 1\\ e^{-(1-\varepsilon')\sqrt{\alpha}i_g/2} & d_g(x,y)\ge i_g/2\end{cases} \tag{3.45}$$
$$\le C\alpha^{-\frac{n}{2}}d_g(x,y)^{-(n-2k+l)}\begin{cases}1 & \sqrt{\alpha}d_g(x,y)\le 1\\ e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y)\ge 1\ .\\ e^{-(1-\varepsilon)\sqrt{\alpha}i_g/2} & d_g(x,y)\ge i_g/2\end{cases}$$

Putting this together, we obtain for $l=1,\dots 2k-1$,

$$\begin{aligned}\left|\nabla^l G_{g,\alpha;x}(y)\right|_g &\le \left|\nabla^l \tilde G_{\alpha;x}(y)\right|_g+\sum_{i=1}^{N-1}\left|\nabla^l \tilde G^i_{\alpha;x}(y)\right|_g+\left|\nabla^l u_{\alpha,x}(y)\right|_g\\ &\le C\begin{cases}d_g(x,y)^{-(n-2k+l)} & \sqrt{\alpha}d_g(x,y)\le 1\\ d_g(x,y)^{-(n-2k+l)}e^{-(1-\varepsilon)\sqrt{\alpha}d_g(x,y)} & \sqrt{\alpha}d_g(x,y)\ge 1\\ e^{-(1-\varepsilon)\sqrt{\alpha}i_g/2} & d_g(x,y)\ge i_g/2\end{cases}\end{aligned} \tag{3.46}$$

which proves the first part of Proposition 3.9.

For the second part of the proof, we use again (3.26), for $d_g(x,y)\le 1/\sqrt{\alpha}$ we have

$$\begin{aligned}&\left|\nabla^l\left(d_g(x,y)^{n-2k}G_{g,\alpha;x}(y)\right)\right|_g\\ &\qquad=\left|\nabla^l\left(d_g(x,y)^{n-2k}\tilde G_{\alpha;x}(y)\right)\right|_g+\mathcal{O}\left(\tfrac{1}{\alpha}\eta\big(\sqrt{\alpha}d_g(x,y)\big)d_g(x,y)^{-l}\right)\end{aligned} \tag{3.47}$$

using (3.44) and (3.45). Let $F_x(y):=d_g(x,y)^{n-2k}\tilde G_{\alpha;x}(y)$ for $y\in B_x(1/\sqrt{\alpha})$, and $\tilde F(u):=F_x(\exp_x(u))=|u|^{n-2k}H^{(k)}_{\alpha;0}(u)$ for $|u|\le 1/\sqrt{\alpha}$. Then, writing $\tilde g:=\exp_x^* g$ as before,

$$\begin{aligned}\left|\nabla^l_g F_x\right|_g(\exp_x(u))&=\left|\nabla^l_{\tilde g}\tilde F(u)\right|(1+\mathcal{O}(|u|^2))\\ &=\left(\left|\nabla^l_\xi\tilde F(u)\right|+\mathcal{O}\left(|u|\left|\nabla^{l-1}_\xi\tilde F(u)\right|\right)+\mathcal{O}\left(\sum_{m=1}^{l-2}\left|\nabla^m_\xi\tilde F(u)\right|\right)\right)(1+\mathcal{O}(|u|^2))\end{aligned}$$

for all $|u|\le 1/\sqrt{\alpha}$, using (3.40). Each term in this sum is estimated using Corollary 2.7, and we finally obtain in (3.47) that, when $\sqrt{\alpha}d_g(x,y)\le 1$ and $\alpha\ge\alpha_0$,

$$\left|\nabla^l\left(d_g(x,y)^{n-2k}G_{g,\alpha;x}(y)\right)\right|_g\le C\eta\big(\sqrt{\alpha}d_g(x,y)\big)d_g(x,y)^{-l}.$$

□

3.6. **Mass of the operator in dimension $n=2k+1$.** We conclude this paper by a consideration in the case where $n=2k+1$. When $n=2k+1$, the Green's function $G_{g,\alpha}$ can be re-written as

$$G_{g,\alpha}(x,y)=\frac{c_{n,k}}{d_g(x,y)}+\mu_x(y),$$

where $c_{n,k}$ is given by (2.1), and $\mu_x(y)$ is a continuous function for all $x\in M$, as recalled below. It is then standard to define the *mass* of the operator as the quantity $\mu_x(x)$, see [24] for the conformal Laplacian, and [18] for the Paneitz-Branson operator.

Lemma 3.10. *There exists $\alpha_0 \geq 1$ and $C_1, C_2 > 0$ such that for all $x \in M$, $\alpha \geq \alpha_0$,*

$$C_1\sqrt{\alpha} \leq -\mu_x(x) \leq C_2\sqrt{\alpha}.$$

Proof. First, by relation (1.2) we obtain an upper bound on the absolute value of the mass : There exists $\alpha_0 \geq 1$ and $C > 0$ such that for all $x \in M$, $\alpha \geq \alpha_0$,

$$|\mu_x(x)| \leq C\sqrt{\alpha}.$$

Now for the second part, fix $\alpha_0 \geq 1$ given by Theorem 1.1. From the decomposition (3.26), one obtains that

$$|\mu_x(y)| \leq \left|\tilde{G}_\alpha(x,y) - \frac{c_{n,k}}{d_g(x,y)}\right| + \sum_{i=1}^{N} \left|\tilde{G}^i_\alpha(x,y)\right| + |u_{\alpha,x}(y)|,$$

so that

$$\mu_x(y) = \tilde{G}_\alpha(x,y) - \frac{c_{n,k}}{d_g(x,y)} + \mathcal{O}(\alpha^{-1/2})$$

using (3.19), (3.21) and Proposition 3.5. Now with the definition (3.1), we have for $d_g(x,y) \leq \tau_0/2$ given in Proposition 3.3,

$$\mu_x(y) = \left(H^{(k)}_\alpha(0,u) - \frac{c_{n,k}}{|u|}\right)\Big|_{u=\exp_x^{-1}(y)} + \mathcal{O}(\alpha^{-1/2}). \tag{3.48}$$

Using the notations of Proposition 2.6, recall that

$$H^{(k)}_\alpha(0,u) - \frac{c_{n,k}}{|u|} = R_\alpha(0,u),$$

where R_α satisfies (2.18),

$$R_\alpha(0,u) = -\int_{\mathbb{R}^n} c_{n,k}\,|u-z|^{-1}\sum_{l=0}^{k-1}\alpha^{k-l}(-\Delta)^l H^{(k)}_{\alpha;0}(z)dz$$

for all $u \neq 0$. By integration by part, and using Proposition 2.5, we obtain

$$R_\alpha(0,u) = -c_{n,k}\sum_{l=0}^{k-1}\alpha^{k-l}\int_{\mathbb{R}^n}(-\Delta)^l_{(z)}\left(|u-z|^{-1}\right)H^{(k)}_{\alpha;0}(z)dz. \tag{3.49}$$

Simple calculations show that for $l = 0, \dots k-1$, there is a constant $d_{n,k} > 0$ depending only on n, k such that

$$(-\Delta)^l\left(\tfrac{1}{|u|}\right) = d_{n,k}\frac{1}{|u|^{1+2l}}.$$

Since $H^{(k)}_\alpha > 0$, (3.49) gives

$$-R_\alpha(0,u) \geq \tilde{c}_{n,k}\,\alpha\int_{\mathbb{R}^n}\frac{1}{|u-z|^{2k-1}}H^{(k)}_{\alpha;0}(y)\,dy.$$

Let $\delta \in (0,1)$ whose value will be fixed later. Proposition 2.6 shows that there exists a constant $C > 0$ independent of α, such that

$$H^{(k)}_\alpha(x,y) \geq \frac{c_{n,k}}{|x-y|}(1 - C\sqrt{\alpha}\,|x-y|)$$

for all $|x-y| \leq 1/\sqrt{\alpha}$. Thus,

$$\begin{aligned} -R_\alpha(0,u) &\geq \tilde{c}_{n,k}\alpha \int_{B_0(\delta/\sqrt{\alpha})} \frac{1}{|y-u|^{2k-1}} \frac{1}{|y|}(1-C\delta)dy \\ &\geq \tilde{c}_{n,k}\alpha \frac{\delta}{\sqrt{\alpha}}(1-\delta C) \end{aligned}$$

for all $|u| \leq \frac{2\delta}{\sqrt{\alpha}}$. Finally, fix $\delta > 0$ small enough so that $1-\delta C \geq \frac{1}{2}$, we obtain that there exists $C > 0$ independent of α such that for all $|u| \leq \frac{2\delta}{\sqrt{\alpha}}$,

$$-R_\alpha(0,u) \geq C\sqrt{\alpha}. \tag{3.50}$$

The continuity of R_α follows from elliptic theory, since R_α satisfies

$$(-\Delta)^k R_{\alpha,x} = h_{\alpha,x}$$

and $h_{\alpha,x} \in L^p(\mathbb{R}^n)$ for all $p > \frac{n}{n-2}$ using (2.17), with $n-2 < 2k$. This implies the continuity of $\mu_x(y)$, since all the terms $\tilde{G}^i_\alpha$ and $u_{\alpha,x}$ are also continuous by Lemma A.2. Putting (3.48) and (3.50) together and evaluating them at $u = 0$ gives

$$\mu_x(x) = R_\alpha(0,u)\Big|_{u=0} + \mathcal{O}(\alpha^{-1/2}) \leq -C_1\sqrt{\alpha}$$

for some $C_1 > 0$ and for $\alpha \geq \alpha_0$. In particular, $\mu_x(x) \to -\infty$ as $\alpha \to \infty$. □

Note that the terms due to the presence of the metric contribute only in $\mathcal{O}(\alpha^{-1/2})$ whereas the main contribution leading to the divergence of the mass comes from the lower-order terms in the operator $(-\Delta+\alpha)^k$ on the Euclidean space.

Appendix A. Generalized Giraud's Lemmas

In this Appendix, we compute some convolution estimates that are needed in the previous Sections to obtain the bounds on the Green's functions. These are modified versions of results known as Giraud's Lemma in the literature (see [2, Proposition 4.12], [16, Lemma 7.5]), proved in [10, p. 150].

Lemma A.1 (Exponential Giraud's)**.** *Let $X, Y \in C^0(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$. Assume that there exist $\beta, \gamma \in (0,n]$, and $\rho, \nu > -n$ such that*

$$|X(x,y)| \leq \begin{cases} |x-y|^{\beta-n} & \text{if } |x-y| \leq 1 \\ |x-y|^{\rho}\, e^{-|x-y|} & \text{if } |x-y| \geq 1 \end{cases},$$

$$|Y(x,y)| \leq \begin{cases} |x-y|^{\gamma-n} & \text{if } |x-y| \leq 1 \\ |x-y|^{\nu}\, e^{-|x-y|} & \text{if } |x-y| \geq 1 \end{cases}$$

for all $x \neq y$. Let $Z(x,y) := \int_{\mathbb{R}^n} X(x,z)Y(z,y)\,dz$ for $x \neq y$. Then $Z \in C^0(\mathbb{R}^n \times \mathbb{R}^n \setminus Diag)$ and there exists $C > 0$ such that for all $x \neq y$:

- *If $|x-y| \leq 1$,*

$$|Z(x,y)| \leq \begin{cases} C\,|x-y|^{\beta+\gamma-n} & \text{when } \beta+\gamma < n \\ C\big(1+|\log|x-y||\big) & \text{when } \beta+\gamma = n \\ C & \text{when } \beta+\gamma > n. \end{cases}$$

- *If $|x-y| \geq 1$,*

$$|Z(x,y)| \leq C\,|x-y|^{\rho+\nu+n}\, e^{-|x-y|}.$$

Moreover, when $\beta+\gamma>n$, Z is continuous on the whole $\mathbb{R}^n\times\mathbb{R}^n$.

The following Proposition extends this result on a compact Riemannian manifold M with injectivity radius $i_g>0$, and with a scale parameter $\alpha\geq 1$.

Lemma A.2. *Let (M,g) be a closed Riemannian manifold, and $\tau,\sigma>0$ such that $\tau+\sigma<i_g$. Let $X,Y\in C^0(M\times M\setminus Diag)$, such that for all $x,y\in M$, $X(x,\cdot)$ is supported in $B_x(\tau)$ and $Y(\cdot,y)$ in $B_y(\sigma)$. Assume that there exist $\beta,\gamma\in(0,n]$, $p,q\geq 0$ and $\rho,\nu>-n$ satisfying*

$$\begin{cases}2p-\rho\leq n-\beta\\ 2q-\nu\leq n-\gamma\end{cases}\tag{A.1}$$

and such that, for all $x\neq y$, $\alpha\geq 1$,

$$|X(x,y)|\leq\begin{cases}d_g(x,y)^{\beta-n} & \text{if } \sqrt{\alpha}d_g(x,y)\leq 1\\ \alpha^p d_g(x,y)^{\rho}e^{-\sqrt{\alpha}d_g(x,y)} & \text{if } \sqrt{\alpha}d_g(x,y)\geq 1\end{cases}$$

$$|Y(x,y)|\leq\begin{cases}d_g(x,y)^{\gamma-n} & \text{if } \sqrt{\alpha}d_g(x,y)\leq 1\\ \alpha^q d_g(x,y)^{\nu}e^{-\sqrt{\alpha}d_g(x,y)} & \text{if } \sqrt{\alpha}d_g(x,y)\geq 1.\end{cases}$$

Let, for all $x\neq y$, $Z(x,y):=\int_M X(x,z)Y(z,y)\,dv_g(z)$. Then $Z\in C^0(M\times M\setminus Diag)$ and, for all $x\in M$, $Z(x,\cdot)$ is supported in $B_x(\tau+\sigma)$. There exists $\alpha_0\geq 1$ and $C>0$ such that for all $x\neq y$ and $\alpha\geq\alpha_0$, we have the following :

- *If $\sqrt{\alpha}d_g(x,y)\leq 1$,*

$$|Z(x,y)|\leq C\begin{cases}d_g(x,y)^{\beta+\gamma-n} & \text{when } \beta+\gamma<n\\ (1+|\log(\sqrt{\alpha}d_g(x,y))|) & \text{when } \beta+\gamma=n\\ \alpha^{-\frac{\beta+\gamma-n}{2}} & \text{when } \beta+\gamma>n.\end{cases}$$

- *If $\sqrt{\alpha}d_g(x,y)\geq 1$,*

$$|Z(x,y)|\leq C\alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}}d_g(x,y)^{\rho+\nu+n}e^{-\sqrt{\alpha}d_g(x,y)}.$$

Moreover, when $\beta+\gamma>n$, $Z\in C^0(M\times M)$.

Since the proofs of these two Lemmas are the same, we only show the second one. The proof of Lemma A.1 is identical setting formally $\tau,\sigma=\infty$ and $\alpha=1$, and taking the integrals on the Euclidean space.

Proof. Up to choosing α_0 large enough, we can assume that $5/\sqrt{\alpha}<\tau+\sigma<i_g$.

For the first part of the proof, let $x,y\in M$ with $x\neq y$, and assume that $d_g(x,y)\leq 2/\sqrt{\alpha}$. We have

$$\begin{aligned}|Z(x,y)|\leq C\int_{B_x(3/\sqrt{\alpha})}\frac{1}{d_g(x,z)^{n-\beta}}\frac{1}{d_g(z,y)^{n-\gamma}}\,dv_g(z)&\\ +\left|\int_{M\setminus B_x(3/\sqrt{\alpha})}X(x,z)Y(z,y)\,dv_g(z)\right|&,\end{aligned}\tag{A.2}$$

this comes from fact that for $z\in B_x(3/\sqrt{\alpha})$,

$$|X(x,z)|\leq C\alpha^p d_g(x,z)^{\rho}e^{-\sqrt{\alpha}d_g(x,y)}\leq Cd_g(x,z)^{\beta-n}\quad\text{when } d_g(x,z)\geq 1/\sqrt{\alpha}$$

$$|Y(z,y)|\leq C\alpha^q d_g(z,y)^{\nu}e^{-\sqrt{\alpha}d_g(x,y)}\leq Cd_g(z,y)^{\gamma-n}\quad\text{when } d_g(z,y)\geq 1/\sqrt{\alpha}$$

for a constant $C>0$ independent of α,x,y,z, by (A.1).

Write $r := d_g(x,y) \le 2/\sqrt{\alpha}$, we first claim that

$$\text{(A.3)}\quad \int_{B_x(3/\sqrt{\alpha})} \frac{1}{d_g(x,z)^{n-\beta}} \frac{1}{d_g(z,y)^{n-\gamma}}\, dv_g(z) \le C \begin{cases} d_g(x,y)^{-(n-\beta-\gamma)} & \text{when } \beta+\gamma<n \\ 1+|\log(\sqrt{\alpha}d_g(x,y))| & \text{when } \beta+\gamma=n \\ \alpha^{-\frac{\beta+\gamma-n}{2}} & \text{when } \beta+\gamma>n \end{cases}.$$

To prove (A.3), we decompose $B_x(3/\sqrt{\alpha})$ in three parts: $B_x(r/2)$, $B_x(3r/2)\setminus B_x(r/2)$ and $B_x(3/\sqrt{\alpha})\setminus B_x(3r/2)$. First, when $z\in B_x(r/2)$, we have $d_g(y,z)\ge r/2$, so that

$$\text{(A.4)}\quad \begin{aligned} \int_{B_x(r/2)} \frac{1}{d_g(x,z)^{n-\beta}} \frac{1}{d_g(z,y)^{n-\gamma}}\, dv_g(z) &\le Cr^{\gamma-n}\int_{B_x(r/2)} \frac{1}{d_g(x,z)^{n-\beta}}\, dv_g(z) \\ &\le Cr^{\gamma+\beta-n} \end{aligned}$$

since $r/2\le 1/\sqrt{\alpha}<i_g$ and $\beta>0$. Similarly, when $z\in B_x(3r/2)\setminus B_x(r/2)$, we have $d_g(x,z)\ge r/2$, so that

$$\text{(A.5)}\quad \begin{aligned} &\int_{B_x(3r/2)\setminus B_x(r/2)} \frac{1}{d_g(x,z)^{n-\beta}} \frac{1}{d_g(z,y)^{n-\gamma}}\, dv_g(z) \\ &\qquad\le Cr^{\beta-n}\int_{B_x(3r/2)\setminus B_x(r/2)} \frac{1}{d_g(y,z)^{n-\gamma}}\, dv_g(z) \\ &\qquad\le Cr^{\beta+\gamma-n} \end{aligned}$$

since $5r/2\le 5/\sqrt{\alpha}<i_g$ and $\gamma>0$. Finally, when $z\in B_x(3/\sqrt{\alpha})\setminus B_x(3r/2)$, we have $\frac{1}{3}d_g(x,z)\le d_g(z,y)\le \frac{5}{3}d_g(x,z)$, so that

$$\begin{aligned} &\int_{B_x(3/\sqrt{\alpha})\setminus B_x(3r/2)} \frac{1}{d_g(x,z)^{n-\beta}} \frac{1}{d_g(z,y)^{n-\gamma}}\, dv_g(z) \\ &\qquad\le C\int_{B_x(3/\sqrt{\alpha})\setminus B_x(3r/2)} \frac{1}{d_g(x,z)^{2n-\beta-\gamma}}\, dv_g \\ &\qquad\le C\begin{cases} r^{\beta+\gamma-n} & \text{when } \beta+\gamma<n \\ 1+|\log\sqrt{\alpha}r| & \text{when } \beta+\gamma=n \\ \alpha^{-\frac{\beta+\gamma-n}{2}} & \text{when } \beta+\gamma>n \end{cases} \end{aligned}$$

since $3/\sqrt{\alpha}<i_g$. This concludes the proof of (A.3) when $r\le 2\sqrt{\alpha}$, realizing that when $\beta+\gamma>n$, $r^{\beta+\gamma-n}\le C\alpha^{-\frac{\beta+\gamma-n}{2}}$ in (A.4) and (A.5).

We now claim that

$$\text{(A.6)}\quad \int_{M\setminus B_x(3/\sqrt{\alpha})} |X(x,z)|\,|Y(z,y)|\, dv_g(z) \le C\begin{cases} r^{\beta+\gamma-n} & \text{when } \beta+\gamma<n \\ \alpha^{-\frac{\beta+\gamma-n}{2}} & \text{when } \beta+\gamma\ge n \end{cases}.$$

To prove (A.6), note that, by assumption on X,Y, the integral in (A.6) has non-zero contribution only on the support of X and Y, i.e. on $B_x(\tau)\cap B_y(\sigma)$. Moreover, when $z\in M\setminus B_x(3/\sqrt{\alpha})$, we have $d_g(x,z)\ge 1/\sqrt{\alpha}$, $d_g(z,y)\ge 1/\sqrt{\alpha}$ and $\frac{1}{3}d_g(x,z)\le$

$d_g(z,y) \le \frac{5}{3} d_g(x,z)$, so that

$$\begin{aligned}\int_{M\setminus B_x(3/\sqrt{\alpha})} &|X(x,z)|\,|Y(z,y)|\,dv_g(z)\\ &\le C\int_{(B_x(\tau)\cap B_y(\sigma))\setminus B_x(3/\sqrt{\alpha})} \alpha^{p+q} d_g(x,z)^{\rho+\nu} e^{-\frac{4}{3}\sqrt{\alpha}d_g(x,z)}\,dv_g(z)\\ &\le C\alpha^{p+q}\int_{3/\sqrt{\alpha}}^{\tau} t^{\rho+\nu+n-1} e^{-\frac{4}{3}\sqrt{\alpha}t}dt \le C\alpha^{-\frac{\beta+\gamma-n}{2}},\end{aligned}$$

where the last inequality follows from (A.1). This concludes the proof of (A.6) for $r \le 2/\sqrt{\alpha}$, realizing that when $\beta+\gamma<n$, $\alpha^{-\frac{\beta+\gamma-n}{2}} \le Cr^{\beta+\gamma-n}$.

Combining (A.3) and (A.6) with (A.2), we have proven that, for $d_g(x,y) \le 2/\sqrt{\alpha}$,

$$|Z(x,y)| \le \begin{cases} Cd_g(x,y)^{-(n-\beta-\gamma)} & \beta+\gamma<n\\ C(1+|\log(\sqrt{\alpha}d_g(x,y))|) & \beta+\gamma=n\\ C\alpha^{-\frac{\beta+\gamma-n}{2}} & \beta+\gamma>n\end{cases}. \tag{A.7}$$

For the second part of the proof, assume now that $d_g(x,y) \ge 2/\sqrt{\alpha}$. Write again $r := d_g(x,y)$, we split the domain M in the integral that defines $Z(x,y)$ in several parts: $B_x(1/\sqrt{\alpha})$, $B_y(1/\sqrt{\alpha})$, $B_x(3r/2)\setminus(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha}))$ and $M\setminus B_x(3r/2)$. As before, the integral has non-zero contributions only in $\Omega_0 := B_x(\tau)\cap B_y(\sigma)$, and $\overline{\Omega_0} \subset B_x(i_g)$, $\overline{\Omega_0} \subset B_y(i_g)$. It is then clear that $Z(x,\cdot)$ is supported in $B_x(\tau+\sigma)$. Without loss of generality, we can therefore assume that $\Omega_0 \ne \emptyset$, i.e. we can restrict to the case where $d_g(x,y) \le \sigma+\tau$.

For $z \in B_x(1/\sqrt{\alpha})$, we have $d_g(y,z) \ge r - 1/\sqrt{\alpha}$ and $\frac{1}{2}d_g(x,y) \le d_g(y,z) \le \frac{3}{2}d_g(x,y)$, so that

$$\begin{aligned}\left|\int_{B_x(1/\sqrt{\alpha})} X(x,z)Y(z,y)\,dv_g(z)\right| &\le C\int_{B_x(1/\sqrt{\alpha})} \alpha^q \frac{d_g(y,z)^\nu}{d_g(x,z)^{n-\beta}} e^{-\sqrt{\alpha}d_g(z,y)}\,dv_g\\ &\le C\alpha^q r^\nu e^{-\sqrt{\alpha}r}\int_{B_x(1/\sqrt{\alpha})} \frac{1}{d_g(x,z)^{n-\beta}}\,dv_g(z)\\ &\le C\alpha^{q-\frac{\beta}{2}} r^\nu e^{-\sqrt{\alpha}r}\\ &\le C\alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} r^{\rho+\nu+n} e^{-\sqrt{\alpha}r}\end{aligned}$$

since $1/\sqrt{\alpha} < i_g$, where the last inequality follows from (A.1) and $n+\rho \ge 0$. The same arguments on $B_y(1/\sqrt{\alpha})$ similarly show that

$$\left|\int_{B_y(1/\sqrt{\alpha})} X(x,z)Y(z,y)\,dv_g(z)\right| \le C\alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} r^{\rho+\nu+n} e^{-\sqrt{\alpha}r}.$$

Now for $z \in B_x(3r/2)\setminus(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha}))$, we have $d_g(x,z)+d_g(z,y) \ge d_g(x,y)$, so that

$$\begin{aligned}&\left|\int_{B_x(3r/2)\setminus\left(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha})\right)} X(x,z)Y(z,y)\,dv_g(z)\right|\\ &\quad\le C\alpha^{p+q}e^{-\sqrt{\alpha}r}\int_{\Omega_0\cap B_x(3r/2)\setminus\left(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha})\right)} d_g(x,z)^\rho d_g(z,y)^\nu\,dv_g(z).\end{aligned} \tag{A.8}$$

We claim that

$$\int_{\Omega_0\cap B_x(3r/2)\setminus\left(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha})\right)} d_g(x,z)^\rho d_g(z,y)^\nu\, dv_g(z) \le Cr^{\rho+\nu+n} \tag{A.9}$$

for $r \ge 2/\sqrt{\alpha}$. To see this, we decompose the domain of integration in two parts: $B_x(r/2)\setminus B_x(1/\sqrt{\alpha})$ and $\Omega_0\cap B_x(3r/2)\setminus(B_x(r/2)\cup B_y(1/\sqrt{\alpha}))$. When $z\in B_x(r/2)$, we have $r/2 \le d_g(z,y) \le 3r/2$, so that

$$\int_{B_x(r/2)\setminus B_x(1/\sqrt{\alpha})} d_g(x,z)^\rho d_g(z,y)^\nu\, dv_g(z) \le Cr^{\rho+\nu+n},$$

since $r/2 < i_g$ and $\rho+n>0$. For analogous reasons, we have

$$\int_{\Omega_0\cap B_x(3r/2)\setminus\left(B_x(r/2)\cup B_y(1/\sqrt{\alpha})\right)} d_g(x,z)^\rho d_g(z,y)^\nu\, dv_g(z) \le Cr^{\rho+\nu+n}.$$

This concludes the proof of (A.9). Putting (A.8) with (A.9), we have proven that, for $r\ge 2/\sqrt{\alpha}$,

$$\left|\int_{B_x(3r/2)\setminus\left(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha})\right)} X(x,z)Y(z,y)\, dv_g(z)\right| \le C\alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} r^{\rho+\nu+n} e^{-\sqrt{\alpha}r}$$

using (A.1).

Finally, when $z\in M\setminus B_x(3r/2)$, we have as before $\frac{1}{3}d_g(x,z) \le d_g(y,z) \le \frac{5}{3}d_g(x,z)$, so that since $X(x,\cdot)$ is supported in $B_x(\tau)$,

$$\begin{aligned}\left|\int_{M\setminus B_x(3r/2)} X(x,z)Y(z,y)\, dv_g(z)\right| &\\ &\le C\alpha^{p+q}\int_{B_x(\tau)\setminus B_x(3r/2)} d_g(x,z)^{\rho+\nu} e^{-\sqrt{\alpha}\frac{4}{3}d_g(x,z)}\, dv_g(z)\\ &\le C\alpha^{p+q}\,\Gamma(\rho+\nu+n, 2\sqrt{\alpha}r)\end{aligned}$$

where $\Gamma(\delta,t) := \int_t^{+\infty} s^{\delta-1}e^{-s}ds$ is the incomplete Gamma function. Note that this last integral is non-zero only in the case where $r < 2\tau/3$. It is easily seen that $\Gamma(\delta,t)\sim t^\delta e^{-t}$ as $t\to\infty$, so that in the end we have shown that, for $r\ge 2/\sqrt{\alpha}$,

$$\left|\int_{M\setminus B_x(3r/2)} X(x,z)Y(z,y)\, dv_g(z)\right| \le \alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} r^{\rho+\nu+n} e^{-\sqrt{\alpha}r}$$

using (A.1). This concludes the second part of the proof for $d_g(x,y)\ge 2/\sqrt{\alpha}$.

Finally, when $1\le \sqrt{\alpha}d_g(x,y)\le 2$, as before, the two regimes coincide, up to a constant. The first part of the proof shows that, with (A.7),

$$\left|\int_M X(x,z)Y(z,y)\, dv_g\right| \le C\alpha^{\frac{n-\beta-\gamma}{2}}.$$

Moreover, there is a constant $C>0$ independent of α, x,y, such that

$$\frac{1}{C}\alpha^{\frac{n-\beta-\gamma}{2}} \le \alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} d_g(x,y)^{\rho+\nu+n} e^{-\sqrt{\alpha}d_g(x,y)} \le C\alpha^{\frac{n-\beta-\gamma}{2}}.$$

We can then conclude, when $1\le\sqrt{\alpha}d_g(x,y)\le 2$ we have

$$|Z(x,y)| \le C\alpha^{n-\frac{\beta+\gamma}{2}+\frac{\rho+\nu}{2}} d_g(x,y)^{\rho+\nu+n} e^{-\sqrt{\alpha}d_g(x,y)}.$$

Regarding the continuity, fix $x, y \in M$ with $x \neq y$, and take any sequence $((x_m, y_m))_m$ such that $x_m \to x$, $y_m \to y$. Let $\delta_m := d_g(x_m, x)$, $\tilde{\delta}_m := d_g(y_m, y)$, we assume without loss of generality that $d_g(x_m, x) \leq \frac{1}{3} d_g(x, y)$ and $d_g(y_m, y) \leq \frac{1}{3} d_g(x, y)$ for all $m \in \mathbb{N}$. Then we compute

$$
\begin{aligned}
Z(x_m, y_m) = &\int_{B_{x_m}(\delta_m/2)} X(x_m, z) Y(z, y_m)\, dv_g(z) \\
&+ \int_{B_{y_m}(\tilde{\delta}_m/2)} X(x_m, z) Y(z, y_m)\, dv_g(z) \\
&\quad + \int_{M \setminus (B_{x_m}(\delta_m/2) \cup B_{y_m}(\tilde{\delta}/2))} X(x_m, z) Y(z, y_m)\, dv_g(z).
\end{aligned} \tag{A.10}
$$

On the one hand, when $z \in B_{x_m}(\delta_m/2)$, we have $d_g(z, y_m) > r/2$, writing once again $r = d_g(x, y) > 0$, so that

$$
\begin{aligned}
\left| \int_{B_{x_m}(\delta_m/2)} X(x_m, z) Y(z, y_m)\, dv_g(z) \right| &\leq C r^{\gamma - n} \int_{B_{x_m}(\delta_m/2)} d_g(x_m, z)^{\beta - n}\, dv_g(z) \\
&\leq C \delta_m^{\beta} r^{\gamma - n}.
\end{aligned}
$$

Similarly, when $z \in B_{y_m}(\tilde{\delta}_m/2)$, we have $d_g(x_m, z) > r/2$, so that

$$
\begin{aligned}
\left| \int_{B_{y_m}(\tilde{\delta}_m/2)} X(x_m, z) Y(z, y_m)\, dv_g(z) \right| &\leq C r^{\beta - n} \int_{B_{y_m}(\tilde{\delta}_m/2)} d_g(z, y_m)^{\gamma - n}\, dv_g(z) \\
&\leq C \tilde{\delta}_m^{\gamma} r^{\beta - n}.
\end{aligned}
$$

On the other hand, when $z \notin (B_{x_m}(\delta_m/2) \cup B_{x_m}(\tilde{\delta}_m/2))$, we have $d_g(x_m, z) \geq \frac{1}{3} d_g(x, z)$ and $d_g(z, y_m) \geq \frac{1}{3} d_g(z, y)$, so that

$$
\begin{aligned}
|X(x_m, z)| &\leq C d_g(x, z)^{\beta - n}, \\
|Y(z, y_m)| &\leq C d_g(z, y)^{\gamma - n}
\end{aligned}
$$

By dominated convergence, we obtain

$$
\int_{M \setminus (B_{x_m}(\delta_m/2) \cup B_{y_m}(\tilde{\delta}_m/2))} X(x_m, z) Y(z, y_m)\, dv_g(z) \xrightarrow{m \to \infty} \int_M X(x, z) Y(z, y)\, dv_g(z).
$$

Coming back to (A.10), we have shown that for all $x \neq y$ in M,

$$
\lim_{m \to \infty} Z(x_m, y_m) = \int_M X(x, z) Y(z, y)\, dv_g(z) + 0 + 0 = Z(x, y),
$$

and Z is continuous at $(x, y) \in M \times M \setminus Diag$. Additionally, when $\beta + \gamma > n$, for all $w \in M$, and for all $0 < \delta < 1/\sqrt{\alpha}$, there exists $C > 0$ such that for all $x, y \in M$,

$$
\left| \int_{B_w(\delta)} X(x, z) Y(z, y)\, dv_g(z) \right| \leq C \delta^{\beta + \gamma - n},
$$

this holds true even when $x = y$. We conclude that $Z \in C^0(M \times M)$. □

Remark A.1. The assumption (A.1) is a compatibility condition. If $2p - \rho = n - \beta$, we know that the two regimes of X are equivalent when $d_g(x, y) \sim 1/\sqrt{\alpha}$.

Remark A.2. Observe that the convolution Z decreases less quickly than either X and Y at large distances. This is due to the term $d_g(x,y)^{\rho+\nu+n}$ that comes from (A.8). This term becomes larger than $\alpha^{-\frac{\rho+\nu+n}{2}}$ when $d_g(x,y) \gg 1/\sqrt{\alpha}$, which happens at finite distance as $\alpha \to \infty$.

We conclude this Appendix with variant of Lemma A.2, which is used in the proof of Theorem 1.1, where X and Y are allowed to have slightly different exponential decay.

Lemma A.3. *Let $\alpha_0 \geq 1$ and let $X, Y \in C^0(M \times M \setminus Diag)$ be such that $X(x,\cdot)$ is supported in $B_x(\tau)$, $\tau < i_g/2$. Suppose that there are $p \geq 0$ and $\rho > -n$ with $2p - \rho \leq 0$, and $0 < \varepsilon < 1$ such that for all $x \neq y$,*

$$|X(x,y)| \leq \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ \alpha^p d_g(x,y)^\rho e^{-\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \end{cases}$$

$$|Y(x,y)| \leq \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}\, i_g/2} & d_g(x,y) \geq i_g/2. \end{cases}$$

Let, for all $x, y \in M$, $Z(x,y) := \int_M X(x,z) Y(z,y)\, dv_g(z)$. There exists $\alpha_0 \geq 1$ and $C > 0$ such that for all $x, y \in M$ and $\alpha \geq \alpha_0$,

$$|Z(x,y)| \leq C\alpha^{-\frac{n}{2}} \begin{cases} 1 & \sqrt{\alpha} d_g(x,y) \leq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha} d_g(x,y)} & \sqrt{\alpha} d_g(x,y) \geq 1 \\ e^{-(1-\varepsilon)\sqrt{\alpha}\, i_g/2} & d_g(x,y) \geq i_g/2 \end{cases}.$$

Remark A.3. Classically, the exponential decay of Z exactly matches that of the least decreasing term Y.

The proof follows the same steps as the proof of Lemma A.2, and we just explain the modifications.

Proof. Assume first that $d_g(x,y) \leq 2/\sqrt{\alpha}$. Arguing as in the proof of Lemma A.2 in the case $\beta = \gamma = n$, we obtain

$$|Z(x,y)| \leq C\alpha^{-\frac{n}{2}}.$$

Assume now that $2/\sqrt{\alpha} \leq d_g(x,y) \leq i_g/2$. We adapt the proof of Lemma A.2. Write $r = d_g(x,y)$ and split the domain M in the integral that defines $Z(x,y)$ between $B_x(1/\sqrt{\alpha})$, $B_y(1/\sqrt{\alpha})$, and $M \setminus (B_x(1/\sqrt{\alpha}) \cup B_y(1/\sqrt{\alpha}))$.

When $z \in B_x(1/\sqrt{\alpha})$, we have $d_g(y,z) \geq r - 1/\sqrt{\alpha}$, so that

$$\begin{aligned} \left| \int_{B_x(1/\sqrt{\alpha})} X(x,z) Y(z,y)\, dv_g(z) \right| &\leq C e^{-(1-\varepsilon)\sqrt{\alpha} r} \int_{B_x(1/\sqrt{\alpha})} dv_g(z) \\ &\leq C\alpha^{-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} r}. \end{aligned}$$

Now when $z \in B_y(1/\sqrt{\alpha})$, we have

$$|X(x,z)| \leq C e^{-(1-\varepsilon)\sqrt{\alpha} r},$$

since $d_g(x,z) \geq r - 1/\sqrt{\alpha}$ and $p - \rho/2 < 0$. Therefore, we obtain

$$\begin{aligned}\left| \int_{B_y(1/\sqrt{\alpha})} X(x,z)Y(z,y)\, dv_g(z) \right| &\leq C e^{-(1-\varepsilon)\sqrt{\alpha} r} \int_{B_y(1/\sqrt{\alpha})} dv_g(z) \\ &\leq C\alpha^{-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} r}.\end{aligned}$$

Finally, when $z \in M \setminus (B_x(1/\sqrt{\alpha}) \cup B_y(1/\sqrt{\alpha}))$, we have $d_g(x,z) + d_g(z,y) \geq r$ and $d_g(x,z) + i_g/2 \geq i_g/2 \geq r$, so that since $X(x,\cdot)$ is supported in $B_x(\tau)$,

$$\begin{aligned}&\left| \int_{M\setminus(B_x(1/\sqrt{\alpha})\cup B_y(1/\sqrt{\alpha}))} X(x,z)Y(z,y)\, dv_g(z) \right| \\ &\qquad \leq C\alpha^p e^{-(1-\varepsilon)\sqrt{\alpha} r} \int_{B_x(\tau)\setminus B_x(1/\sqrt{\alpha})} d_g(x,z)^\rho e^{-\varepsilon\sqrt{\alpha} d_g(x,z)}\, dv_g(z) \\ &\qquad \leq C\alpha^{p-\frac{\rho}{2}-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} r} \int_\varepsilon^\infty t^{\rho+n-1} e^{-t} dt \\ &\qquad \leq C\alpha^{-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} r}\end{aligned} \tag{A.11}$$

since $p - \rho/2 < 0$. This concludes the proof for the case $2/\sqrt{\alpha} \leq d_g(x,y) \leq i_g/2$.

For the last case, assume that $d_g(x,y) \geq i_g/2$, and split the domain M in the integral that defines $Z(x,y)$ in two parts: $B_x(1/\sqrt{\alpha})$ and $M \setminus B_x(1/\sqrt{\alpha})$. When $z \in B_x(1/\sqrt{\alpha})$, we have $d_g(z,y) \geq d_g(x,y) - 1/\sqrt{\alpha} \geq i_g/2 - 1/\sqrt{\alpha}$, so that

$$\begin{aligned}\left| \int_{B_x(1/\sqrt{\alpha})} X(x,z)Y(z,y)\, dv_g(z) \right| &\leq C e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} \int_{B_x(1/\sqrt{\alpha})} dv_g(z) \\ &\leq C\alpha^{-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2}.\end{aligned}$$

On the other hand, when $z \in M \setminus B_x(1/\sqrt{\alpha})$, we have

$$d_g(x,z) + d_g(z,y) \geq d_g(x,y) \geq i_g/2,$$

so that, as in (A.11),

$$\begin{aligned}&\left| \int_{M\setminus B_x(1/\sqrt{\alpha})} X(x,z)Y(z,y)\, dv_g(z) \right| \\ &\qquad \leq C\alpha^p e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2} \int_{B_x(\tau)\setminus B_x(1/\sqrt{\alpha})} d_g(x,z)^\rho e^{-\varepsilon\sqrt{\alpha} d_g(x,z)}\, dv_g(z) \\ &\qquad \leq C\alpha^{-\frac{n}{2}} e^{-(1-\varepsilon)\sqrt{\alpha} i_g/2}\end{aligned}$$

using $p - \rho/2 < 0$, which concludes the proof. □

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Lorenzo Carletti, Université Libre de Bruxelles, Service d'Analyse, Boulevard du Triomphe - Campus de la Plaine, 1050 Bruxelles, Belgique

Email address: `lorenzo.carletti@ulb.be`