Persistent Diagram Estimation of Multivariate Piecewise Hölder-continuous Signals

Hugo Henneuse[1, 2]
hugo.henneuse@universite-paris-saclay.fr

March 29, 2024

Abstract

To our knowledge, the analysis of convergence rates for persistent diagram estimation from noisy signals had remained limited to lifting signal estimation results through sup norm (or other functional norm) stability theorems. We believe that moving forward from this approach can lead to considerable gains. We illustrate it in the setting of Gaussian white noise model. We examine from a minimax perspective, the inference of persistent diagram (for sublevel sets filtration). We show that for piecewise Hölder-continuous functions, with control over the reach of the discontinuities set, taking the persistent diagram coming from a simple histogram estimator of the signal, permit to achieve the minimax rates known for Hölder-continuous functions.

Introduction

Motivation

Inferring information from noisy signals is a central subject in statistics. Specifically, the recovery of the whole signal structure has been extensively studied by the non-parametric statistics community. When the signal is regular (e.g; belonging to a Hölder, Sobolev or Besov space) rigorous minimax study as long as tractable optimal procedures has been provided, forming a nearly exhaustive benchmark. For an overview, see Tsybakov (2008).
When facing more irregular signals, typically signals that are only piecewise continuous, the problem becomes significantly more difficult. Motivated by applications, later works have attempted to explore this case. For an overview, refer to Qiu (2005). However, proposed methods suffer from certain limitations : strong additional knowledge assumptions (e.g. suppose to known the number of jumps, their locations or their magnitudes), restrict to low dimensional cases (only univariate or bivariate signals), high computational costs or lack of rigorous and general statistical guarantees over the risk. Additionally, due to the strong sensibility to point-wise discontinuity of the sup norm, these works only consider L_2 (or sometimes L_p, $p < +\infty$) metric (less sensitive to topology). All these problematic points motivate the exploration of looser descriptors that can be inferred more easily.

In the last two decades, Topological Data Analysis has emerged as a powerful approach, offering new geometric tools for characterizing complex signals. Among these tools, persistent homology has garnered significant attention. Represented through persistent diagrams (or barcodes), it has proven to be a versatile descriptor, valuable from both practical and theoretical standpoints. Recent research has focused on the estimation of such representations, opening up exciting opportunities

[1]Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Orsay, France
[2]DataShape, Inria Saclay, Palaiseau, France

to explore the statistical aspects of Topological Data Analysis. In this context, the model that has received the most attention is the density model, initiated by the work of Bubenik and Kim (2006) in a simple parametric setting. Subsequently, efforts have been made to extend this model to wider, non-parametric settings. Notable contributions include the work of Balakrishnan et al. (2012), which addresses the estimation of Betty numbers for smooth manifolds with different noise models, and Fasy et al. (2014), who provide confidence sets for persistence diagrams in a similar context. Additionally, Chazal et al. (2014) provide a minimax estimator while controlling the regularity of the density support.
The study of non-parametric regression or the Gaussian white noise model remains relatively unexplored in the context of Topological Data Analysis. Advancements in this direction include the works of Bubenik et al. (2009) and Bobrowski et al. (2017), as well as more recent contributions, such as those by Perez (2022), albeit in a different direction.
The general approach followed in these works involves estimating the signal (or density), quantifying the estimation error in sup-norm, Hausdorff distance, or Gromov-Hausdorff distance, and bounding the bottleneck error on the diagram using stability theorems (Cohen-Steiner et al., 2005; Chazal et al., 2009, 2016, 2012). The power and importance of stability theorems are evident as they enable the direct translation of convergence rates in sup-norm (or similar metrics) to convergence rates in bottleneck distance over diagrams (under the assumption that the signal is q-tame). To further underline the significance of stability theorems, some studies, such as Bubenik et al. (2009) and Chazal et al. (2014), demonstrate that these rates are minimax for typical function classes.

However, adopting these approaches may sacrifice efficiency and generality. One of the main interest of the persistent diagram lies in its capacity to provide a more flexible representation compared to the entire signal. Consequently, in certain cases, inferring the persistent diagram should be (strictly) simpler. This observation serves as a crucial motivation to conduct finer analysis of the convergence properties of persistent diagram estimator. Moreover, it highlights the broader appeal of utilizing topological or geometrical descriptors, especially when conventional non-parametric techniques yield unsatisfactory results. As mentioned earlier, such scenarios commonly arise when signals display irregularities.

Framework

Regularity assumptions. For a set $A \subset [0,1]^d$, we denote $\overline{A}$ its adherence, A° its interior, ∂A its boundary and A^c its complement. Let $f : [0,1]^d \to \mathbb{R}$, we make the following assumption over f :

A1. f is a piecewise $(L,\alpha)-$Hölder-continuous function, i.e. there exist $M_1, ..., M_l$ open sets of $[0,1]^d$ such that,

$$\bigcup_{i=1}^{l} \overline{M_i} = [0,1]^d$$

and for all $i \in \{1, ..., l\}$ and $x, y \in M_i$,

$$|f(x) - f(y)| \le L\|x-y\|_2^\alpha.$$

A2. f verifies, $\forall x_0 \in [0,1]^d$,

$$\liminf_{x \in \bigcup_{i=1}^{l} M_i \to x_0} f(x) = f(x_0)$$

In this context, two signals, differing only on a null set, are statistically undistinguishable. And persistent homology is sensitive to point-wise irregularity, two signals differing only on a null set can have very different persistent diagram. Assumption **A2** prevents such scenario. Furthermore, note that for any piecewise Hölder-continuous function f, there exists a modification $\tilde{f}$ verifying Assumption **A2** such that f and $\tilde{f}$ coincide except on a null measure sets.

A3. $\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d$ is a $C^{1,1}$ hypersurface, verifying, for $R > 0$,

$$\text{reach}\left( ]0,1[^d \cap \bigcup_{i=1}^{l} \partial M_i \right) \geq R \text{ and } d_2\left(\cap \bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d, \partial [0,1]^d \right) \geq R$$

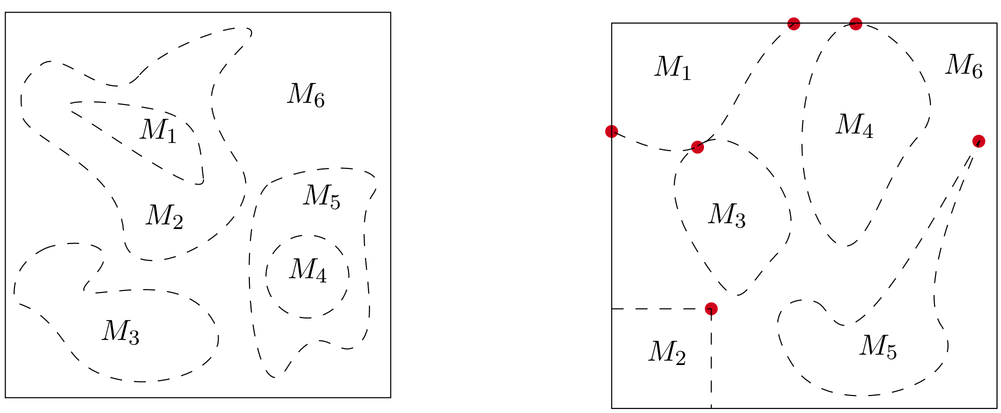
where, for a set $A \subset \mathbb{R}^d$,

$$\text{reach}(A) = \sup \left\{ r \in \mathbb{R} : \forall x \in \mathbb{R}^d \backslash A \text{ with } d_2(\{x\}, A) < r, \exists ! y \in A \text{ s.t. } ||x-y||_2 = d_2(\{x\}, A) \right\}$$

and,

$$d_2(A,B) = \max \left(\sup_{x \in B} \inf_{y \in A} ||x-y||_2, \sup_{x \in A} \inf_{y \in B} ||x-y||_2 \right)$$

The reach is a curvature measure introduced by Federer (1959). An intuitive way to approach it is that if A has a reach R we can roll a ball of radius R along the boundary of A. Positive reach assumptions are fairly common (and sometimes necessary) in statistical TDA (Balakrishnan et al., 2012; Niyogi et al., 2008) and geometric inference (Genovese et al., 2012; Kim et al., 2016; Aamari and Levrard, 2017; Aamari et al., 2019; Berenfeld et al., 2021). Here, the first part of Assumption **A3** gives geometric control over the union of the boundary of the M_i in the interior of $[0,1]^d$, for example it prevents cusps and multiple points to appear. Following the same idea, the second part ensures that the discontinuities not appear too close from the boundary. The combination of Assumptions **A2** and **A3** ensures that the persistence diagram of f is well-defined (see Appendix A, Proposition 6).



(a) Assumption **A3** verified

(b) Assumption **A3** not verified

Figure 1: Illustration of Assumption **A3**

We denote $S_d(L, \alpha, R)$ the set of such functions.

Statistical model. We considered the Gaussian white noise model given by the following stochastic equation,

$$dX_{t_1,...,t_d} = f(t_1,...,t_d)dt_1...dt_d + \theta dW_{t_1,...,t_d} \tag{1}$$

with W a $d-$parameters Wiener field, f a signal in $S_d(L, \alpha, R)$ and $\theta \geq 0$ the level of noise. Model 1 is a classical model in non-parametric statistics.

Estimator. In this context, our goal is to estimate $\mathrm{dgm}(f)$, the persistent diagram of f (considering singular homology with coefficient in a field). The estimation procedures consist of simply taking the persistent diagram induced by the sublevel sets of the signal estimated using histograms.

More formally, let $h > 0$ such that $1/h$ is an integer, consider G_h the regular orthogonal grid over $[0,1]^d$ of step h and C_h the collection of all the closed hypercubes of side h composing G_h. We define, $\forall \lambda \in \mathbb{R}$, the estimator of $\mathcal{F}_\lambda = f^{-1}(]-\infty, \lambda])$, by,

$$\widehat{\mathcal{F}}_{\lambda,h} = \bigcup_{H \in C_{h,\lambda}} H \text{ , with } C_{h,\lambda} = \left\{ H \in C_h \text{ such that } \int_H dX - \int_H \lambda \leq 0 \right\}.$$

It is worth noting that $\widehat{\mathcal{F}}\lambda, h$ represents the sublevel set indexed by λ of the histogram estimator of f. We then consider, for all $s \in \{0, ..., d\}$, $\hat{\mathbb{V}}_{f,s}$ the persistent module induced by the collection of homology groups $\left(H_s\left(\widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}}\right)\right)_{\lambda\in\mathbb{R}}$ equipped with inclusion induced maps and $\widehat{\mathrm{dgm}(f)}$ the associated persistent diagrams. This procedure is illustrated by Figure 2, in the slightly different setting of non-parametric regression with fixed design (see Appendix C), this choice being more convenient for simulations.
A natural question is how to calibrate the window-size h for signals. From the proof of Lemma 2 (see Appendix B), a good choice is taking $h_{\theta,\alpha}$ such that,

$$\frac{h_{\theta,\alpha}^{d+\alpha}}{\sqrt{h_{\theta,\alpha}^{d} \log\left(1 + \frac{1}{h_{\theta,\alpha}^{d}}\right)}} > \theta$$

which implies that we can take,

$$h_{\theta,\alpha} \simeq \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{1}{d+2\alpha}}.$$

Contribution

In this framework, we study the convergence properties of the estimator $\mathrm{dgm}(f)$. We provide a rigorous analysis of the convergence properties for the proposed estimator, showing that it achieves the following rates for the bottleneck distance over the classes $S_d(L, \alpha, R)$.

Theorem 1. *Let $p \geq 1$,*

$$\sup_{f \in S_d(L,\alpha,R)} \mathbb{E}\left(d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)^p\right) \lesssim \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{p\alpha}{d+2\alpha}}.$$

Furthermore, we establish that these rates are optimal, in the minimax sense, over the classes $S_d(L, \alpha, R)$.

Theorem 3. *Let $p \geq 1$,*

$$\inf_{\widehat{\mathrm{dgm}(f)}} \sup_{f \in S_d(L,\alpha,R)} \mathbb{E}\left(d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)^p\right) \gtrsim \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{p\alpha}{d+2\alpha}}.$$

Interestingly, these rates coincide with the well-known minimax rates obtained on Hölder spaces. Up to a multiplicative constant, there is no additional cost for considering signal in $S_d(L, \alpha, R)$. It

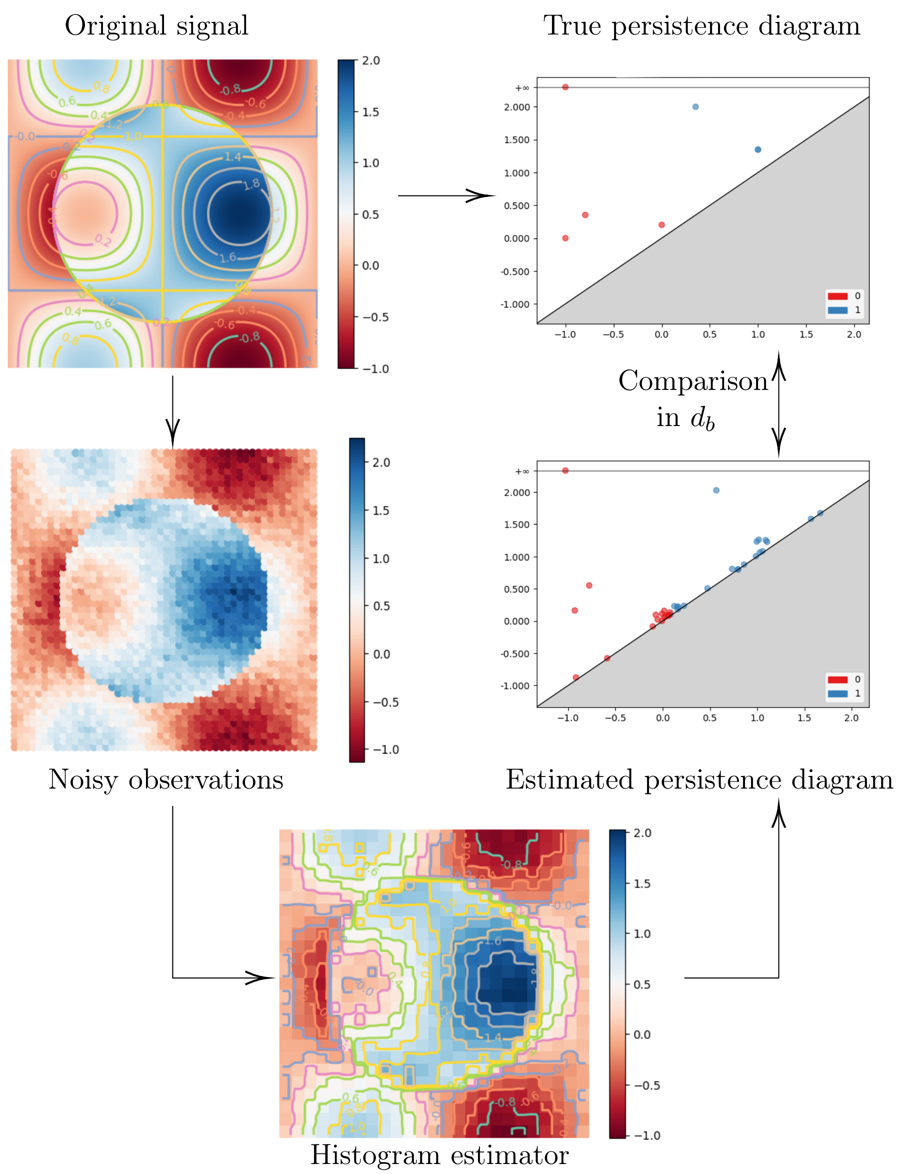


Figure 2: Numerical illustration of the estimation procedures in the setting of the non-parametric regression (see Appendix C). $f(x, y) = \cos(2\pi x)\sin(2\pi x) + \mathbb{1}_{(x-1/2)^2+(y-1/2)^2<1/8}$, $\sigma = 0.1$, $n = 2500$, $h = 1/4(\log(n)/n)^{1/4}$.

demonstrates the gain of breaking free from usual analysis approach in TDA and the robustness to discontinuities of persistent diagram estimation. Also, as such irregularities are challenging to handle for signal estimation, these results promote the use of persistent diagram while processing noisy (irregular) signals.

The adaptivity to regularity is discussed. Applying Lepskii's method (Lepskii, 1991), we derive an adaptive procedure (see Section 2.4) that achieves the established minimax rates.

Following the same idea, we propose an estimator tailored for the non-parametric regression setting, with fixed design. We show that, also in this context, our estimator achieves minimax rates over the classes $S_d(L, \alpha, R)$ (see Appendix C).

1 Background on persistent homology

We first recall the required background on persistent homology, focusing on the case of persistent homology from sublevel sets of real functions. This section does not pretend to give an exhaustive exposition to persistent homology, but simply introduce the essential formalism to follow this paper. For an extensive overview, see Chazal et al. (2016).
The construction introduced here exploited the concept of homology, and especially singular homology. For an introduction to (singular) homology, the reader can refer to Hatcher (2000).

1.1 Filtrations and persistence modules

The idea behind persistence homology is to encode the evolution of the topology (in the homology sense) of a nested family of topological spaces, called filtration. As we are moving along indices, topological features (connected components, cycles, cavities, ...) can appear or die (existing connected components merge, cycle or cavities are filled, ...). Two keys to formalize this idea, that we use along this paper, are the notions of filtration and of persistence module.

Definition 1. Let $\Lambda \subset \mathbb{R}$ be a set of indices. A filtration over Λ is a family $(\mathcal{K}_\lambda)_{\lambda\in\Lambda}$ of topological spaces satisfying, $\forall \lambda, \lambda' \in \Lambda, \lambda \leqslant \lambda'$

$$\mathcal{K}_\lambda \subset \mathcal{K}_{\lambda'}.$$

The typical filtration that we will consider in this paper is, for a function $f : \mathbb{R}^d \to \mathbb{R}$, the family of sublevel sets $(\mathcal{F}_\lambda)_{\lambda\in\mathbb{R}}$.

Definition 2. Let $\Lambda \subset \mathbb{R}$ be a set of indices. A persistence module over λ is a family $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda\in\Lambda}$ of vector spaces equipped with linear application $v_\lambda^{\lambda'} : \mathbb{V}_\lambda \to \mathbb{V}_{\lambda'}$ such that, $\forall \lambda \leqslant \lambda' \leqslant \lambda'' \in \Lambda$,

$$v_\lambda^\lambda = id$$

and

$$v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'} = v_\lambda^{\lambda''}.$$

The typical persistent modules that we will consider in this paper is, for a function $f : \mathbb{R}^d \to \mathbb{R}$ and $s \in \mathbb{N}$, the family of homology groups $\mathbb{V}_{f,s} = (H_s(\mathcal{F}_\lambda))_{\lambda\in\mathbb{R}}$ equipped with $v_\lambda^{\lambda'}$ the linear application induced by the inclusion $\mathcal{F}_\lambda \subset \mathcal{F}_{\lambda'}$. To be more precise, in this paper, $H_s(.)$ is the singular homology functor in degree s with coefficient in a field (typically $\mathbb{Z}/2\mathbb{Z}$). Hence, $H_s(\mathcal{F}_\lambda)$ is a vector space.

1.2 Module decompositions, persistent diagrams and $q-$tameness

Persistent diagram (or equivalently barcode) permits to summarize and represent, discretely, the algebraic structure of a persistent module. Still, this is not possible for all persistent modules. As shown in Chazal et al. (2016), if $\mathbb{V}$ verifies a $q-$tameness assumption, persistent diagrams can be defined. The notion of $q-$tameness is used in this paper to prove that the diagrams we consider are well-defined.

Definition 3. A persistence module $\mathbb{V}$ is said to be q-tame if $\forall \lambda < \lambda' \in \Lambda, \operatorname{rank}\left(v_\lambda^{\lambda'}\right)$ is finite.

By extension, when considering the persistent modules $(\mathbb{V}_{f,s})_{s\in\mathbb{N}}$ coming from the sublevel sets filtration of a real functions f, we say that f is $q-$tame if $\mathbb{V}_{f,s}$ is for all $s \in \mathbb{N}$.

To avoid technical definitions, in a more restrictive but illustrative case, to define persistence diagram. The basic idea being that, if we can then decompose persistent modules as a sum of elementary bricks, called interval modules. The persistent diagram can, in this case, be directly derive from this decomposition.

Definition 4. Let I an interval (possibly unbounded) of $\mathbb{R}$ and $I^{'} = I \cap \Lambda$. A persistence module $\mathbb{V}$ is a interval module on $I^{'}$ if,

- $\mathbb{V}_\lambda = \mathbb{R}$ if $\lambda \in I^{'}$ and $\mathbb{V}_\lambda = \{0\}$ otherwise
- for all $\lambda \leq \lambda^{'}$, $v_\lambda^{\lambda^{'}} = id$ if $\lambda, \lambda^{'} \in I^{'}$ and $v_\lambda^{\lambda^{'}} = 0$ otherwise.

Hence, the structure of interval modules is simple and completely encoded by the extremities of $I^{'} = [b, d](\cap\Lambda)$. Conditions to ensure existence of a decomposition of a persistence module into sum of interval modules,

$$\mathbb{V} \simeq \bigoplus_{j\in J} \mathbb{I}_{[b_j, d_j]} \tag{2}$$

can be found in Chazal et al. (2016) (see theorem 1.4). Assuming we have a decomposition such as 2, the structure of $\mathbb{V}$ is completely described by the extremities (b_j, d_j) of each interval in the decomposition. Thus, the associated persistent diagram can be defined simply as the collection of couples of such extremities. Intuitively, The lower extremity b_j corresponds to the birth time of a topological feature, d_j to its death time, and $d_j - b_j$ represents its lifetime.

Definition 5. Let $\mathbb{V}$ a persistence module that can be decomposed as in 2. The associated persistent diagram is,

$$\operatorname{dgm}(\mathbb{V}) = \{(b_j, d_j), j \in J\} \subset \overline{\mathbb{R}}^2.$$

Barcodes are another popular representation that consider the collection of segments $[b_j, d_j]$ instead of the collection of points (b_j, d_j) in $\overline{\mathbb{R}}^2$. Barcodes and persistent diagrams are equivalent representations.

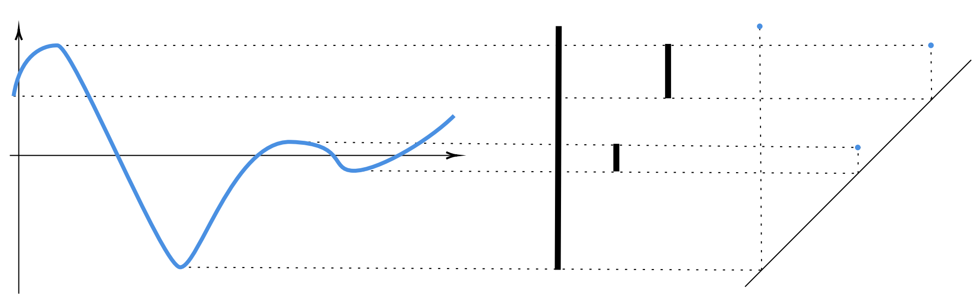


Figure 3: Function filtered by its sublevels, the associated barcode and persistent diagram.

1.3 Bottleneck distance, interleaved modules and stability

In order to compare persistent diagrams, we need a distance. A popular such distance, due to its stability property, is the bottleneck distance. This distance is defined as the infimum over all matching between points in diagrams, of the maximal sup norm distance between two matched points. In order to be able to consider matching between diagrams not containing the same number of points, the diagonal is added to diagrams. This distance will be used in this work to evaluate the quality of our estimation procedures.

Definition 6. The bottleneck distance between two persistent diagrams D_1 and D_2 is,

$$d_b\left(D_1, D_2\right) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1} ||p - \gamma(p)||_\infty$$

with Γ the set of all bijection between D_1 and D_2 (both enriched with the diagonal).

Another notion that will be the key to prove our upper bounds, is the notion of interleaving between persistent modules. We use especially the fact that if two modules are $\varepsilon-$interleaved, then the bottleneck distance between their diagram is upper bounded by ε in bottleneck distance.

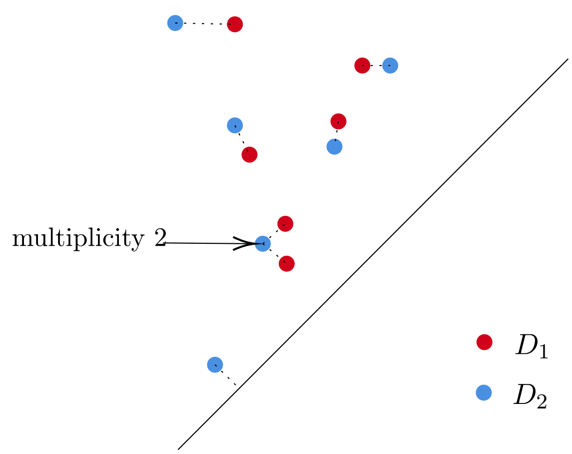


Figure 4: Optimal matching for the bottleneck distance between D_1 and D_2.

Definition 7. Two persistence modules $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in I \subset \mathbb{R}}$ and $\mathbb{W} = (\mathbb{W}_\lambda)_{\lambda \in I \subset \mathbb{R}}$ are said to be ε-interleaved if there exists two families of applications $\phi = (\phi_\lambda)_{\lambda \in I \subset \mathbb{R}}$ and $\psi = (\psi_\lambda)_{\lambda \in I \subset \mathbb{R}}$ where $\phi_\lambda : \mathbb{V}_\lambda \to \mathbb{W}_{\lambda+\varepsilon}$, $\psi_\lambda : \mathbb{W}_\lambda \to \mathbb{V}_{\lambda+\varepsilon}$, and for all $\lambda < \lambda^{'}$ the following diagrams commutes,

$$\begin{tikzcd}
\mathbb{V}_\lambda \arrow[r, "v_\lambda^{\lambda^{'}}"] \arrow[d, "\phi_\lambda"'] & \mathbb{V}_{\lambda'} \arrow[d, "\phi_{\lambda'}"] \\
\mathbb{W}_{\lambda+\varepsilon} \arrow[r, "w_{\lambda+\varepsilon}^{\lambda^{'}+\varepsilon}"'] & \mathbb{W}_{\lambda'+\varepsilon}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}_\lambda \arrow[r, "w_\lambda^{\lambda^{'}}"] \arrow[d, "\psi_\lambda"'] & \mathbb{W}_{\lambda'} \arrow[d, "\psi_{\lambda'}"] \\
\mathbb{V}_{\lambda+\varepsilon} \arrow[r, "v_{\lambda+\varepsilon}^{\lambda^{'}+\varepsilon}"'] & \mathbb{V}_{\lambda'+\varepsilon}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{V}_\lambda \arrow[rr, "v_\lambda^{\lambda+2\varepsilon}"] \arrow[dr, "\phi_\lambda"'] & & \mathbb{V}_{\lambda+2\varepsilon} \\
& \mathbb{W}_{\lambda+\varepsilon} \arrow[ur, "\psi_{\lambda+\varepsilon}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}_\lambda \arrow[rr, "w_\lambda^{\lambda+2\varepsilon}"] \arrow[dr, "\psi_\lambda"'] & & \mathbb{W}_{\lambda+2\varepsilon} \\
& \mathbb{V}_{\lambda+\varepsilon} \arrow[ur, "\phi_{\lambda+\varepsilon}"'] &
\end{tikzcd}$$

Theorem (algebraic stability (Chazal et al., 2009)). *Let $\mathbb{V}$ and $\mathbb{W}$ two $q-$tame persistent modules. If $\mathbb{V}$ and $\mathbb{W}$ are $\varepsilon-$interleaved then,*

$$d_b\left(\mathrm{dgm}(\mathbb{V}), \mathrm{dgm}(\mathbb{W})\right) \leq \varepsilon$$

In the context of sublevel persistence, a direct consequence of this theorem, is the following theorem. This result was already established in particular cases in Cohen-Steiner et al. (2005) and Barannikov (1994).

Theorem (sup norm stability). *Let f and g two real-valued q-tame function, for all $s \in \mathbb{N}$*

$$d_b\left(\mathrm{dgm}\left(\mathbb{V}_{f,s}\right), \mathrm{dgm}\left(\mathbb{V}_{g,s}\right)\right) \leq ||f-g||_\infty.$$

This property is often used to upper bounds the errors (in bottleneck distance) of "plug-in" estimators of persistence diagrams. It is important to note that this sup norm stability is weaker, and adopting such approaches may result in a loss of efficiency and generality.

2 Upper bounds

This section is devoted to the proof of Theorem 1. The strategy to prove this theorem is to construct an interleaving between the estimated and true persistent modules, to then apply the algebraic stability theorem (Chazal et al., 2009). In the case where two filtrations, $\mathcal{F}^1$ and $\mathcal{F}^2$, verify for an $\varepsilon > 0$ and all $\lambda \in \mathbb{R}$, $\mathcal{F}^1_\lambda \subset \mathcal{F}^2_{\lambda+\varepsilon} \subset \mathcal{F}^1_{\lambda+2\varepsilon}$, an $\varepsilon-$interleaving is directly given taking the inclusion induced morphisms between associated modules. Also notes that in this case, if $\mathcal{F}^1$ and $\mathcal{F}^2$ comes from the sublevel sets of functions f_1 and f_2, it implies that f_1 and f_2 are ε close in sup norm. Thus, in our case, doing so is not possible, due to potential arbitrary large errors in neighborhoods of the discontinuity sets. To overcome this difficulty, we investigate the geometric behavior of true and estimated sublevel sets, especially around discontinuities.

2.1 Thicken and shrunk true sublevel sets and retraction properties

This section provides two retraction properties : Propositions 1 and 2. Propositions 1 states that thickened true sublevel sets are included in slightly larger sets that can be retracted by deformation into true sublevel sets (up to on controlled errors on the associated indices). And Proposition 2, that state that the true sublevel sets are included in a slightly larger sets that can be retracted on the shrunken true sublevel sets (up to on controlled errors on the associated indices). As homology is invariant under deformation retracts, these deformation retracts provides some flexibility and margin of error to approximate sublevel "roughly" while preserving the topological information.

Denotes, for a set $A \subset \mathbb{R}^d$ and $b \geq 0$, we denote,

$$A^b = \left\{x \in \mathbb{R}^d \text{ s.t. } d_2(x, A) \leq b\right\}$$

and

$$A^{-b} = \left(\left(A^c\right)^b\right)^c.$$

As, by Assumption **A3**, we have,

$$\mathrm{reach}\left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right) \geq R.$$

Theorem 4.8 of Federer (1959) ensure that for every $x \in [0,1]^d$ at (Euclidean) distance strictly smaller than R of $]0,1[^d \cap \bigcup_{i=1}^{l} \partial M_i$ there exists a unique closest point in $]0,1[^d \cap \bigcup_{i=1}^{l} \partial M_i$, denoted $\xi(x)$. Furthermore, ξ is a continuous function.

To prove Proposition 1, we use the following lemma (see proof in Appendix 2.4).

Lemma 1. *Let $x \in [0,1]^d$. If $d_2\left(x, \partial M_i \cap]0,1[^d\right) < \frac{R}{2}$ then $\xi(x) \in \partial M_i \cap]0,1[^d$.*

Proposition 1. *For all $0 < h < \frac{R}{2}$ there exists a collection of spaces $\mathcal{G} = (\mathcal{G}_{\lambda,h})_{\lambda \in \mathbb{R}}$ such that $\forall \lambda \in \mathbb{R}$, $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda + L(1+3^\alpha)h^\alpha}$ and,*

$$\mathcal{K}_{\lambda,h} := \mathcal{F}_\lambda^h \cup \left(\bigcup_{x \in S_{\lambda,h}} [x, \xi(x)] \right) \subset \mathcal{F}_\lambda^{2h}$$

with

$$S_{\lambda,h} = \left(\left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d \right)^h \setminus \mathcal{F}_{\lambda + Lh^\alpha} \right) \cap \mathcal{F}_\lambda^h$$

retracts by deformation onto $\mathcal{G}_{\lambda,h}$.

Proof. First, note that if x belongs to $\bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ then x is at distance a most h from the union of $(M_i)_{i=1,\dots,l}$, and thus $||x - \xi(x)||_2 \leq h$ which proves that $\mathcal{K}_{\lambda,h} \subset \mathcal{F}_\lambda^{2h}$.

$F_{\lambda,h} : \mathcal{K}_{\lambda,h} \times [0,1] \to \mathcal{K}_{\lambda,h}$ the map define by,

$$F_{\lambda,h}(x,t) = (1-t)x + t\left(\xi(x) + \left(2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}\right)\right)_+ \frac{x - \xi(x)}{||x - \xi(x)||_2} \right)$$

if $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)] \cap M_i$ and $d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}\right) \geq 2h - ||x - \xi(x)||_2$, $i \in \{1, \dots, l\}$, else,

$$F_{\lambda,h}(x,t) = x.$$

We denote $\mathcal{G}_{\lambda,h} = \mathrm{Im}(x \longmapsto F_{\lambda,h}(x,1))$. As $\mathcal{F}_\lambda \subset \left(\bigcup_{x \in S_{\lambda,h}} [x, \xi(x)] \right)^c$, $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda,h}$.

By definition of $\mathcal{G}_{\lambda,h}$,

$$F_{\lambda,h}(x,1) \in \mathcal{G}_{\lambda,h}, \quad \forall x \in \mathcal{K}_{\lambda,h}$$

and by definition of $F_{\lambda,h}$,

$$F_{\lambda,h}(x,0) = x, \quad \forall x \in \mathcal{K}_{\lambda,h}$$

Let $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)] \cap M_i$ verifying $d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}\right) \geq 2h - ||x - \xi(x)||_2$. Remark that $F_{\lambda,h}(x,t) \in [x, \xi(x)]$ for all $t \in [0,1]$, in particular, this implies that $\xi(F_{\lambda,h}(x,1)) = \xi(x)$ thus $F_{\lambda,h}(F_{\lambda,h}(x,1),1) = F_{\lambda,h}(x,1)$. In other cases, by construction $F_{\lambda,h}(x,1) = x$. Hence,

$$F_{\lambda,h}(x,1) = x, \quad \forall x \in \mathcal{G}_{\lambda,h}.$$

The proof for the continuity of $F_{\lambda,h}$ is rather technical and provided in Lemma 6 in Appendix B. Then $F_{\lambda,h}$ is a deformation retract onto $\mathcal{G}_{\lambda,h}$.

Let now prove that $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda + (1+3^\alpha)Lh^\alpha}$. Let $x \in \mathcal{K}_{\lambda,h}$, and suppose $x \in \overline{M}_i \cap [0,1]^d$.

If $x \notin \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$, $F_{\lambda,h}(x,1) = x$ and the definition of $S_{\lambda,h}$ and assumption **A2** ensures that $x \in \mathcal{F}_{\lambda + Lh^\alpha}$.

If $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}\right) \geq 0$, as $F_{\lambda,h}(x,1) \in [x, \xi(x)]$, we have

$$d_2\left(F_{\lambda,h}(x,1), M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}\right) \leq 3h.$$

By Lemma 1, $\xi(x) \in \partial M_i \cap]0,1[^d$, thus, $[x, \xi(x)] \subset \overline{M}_i$ and in particular $F_{\lambda,h}(x,1) \in M_i$. Assumptions **A1** and **A2** then ensures that,

$$F_{\lambda,h}(x,1) \in \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}.$$

If $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) < 0$, then, $F_{\lambda,h}(x,1) = \xi(x)$. Let $\varepsilon > 0$, there exists $j \in \{1, ..., l\}$, $i \neq j$ and $y \in \mathcal{F}_\lambda \cap M_j$, such that $||x - y||_2 \leq h + \varepsilon$. Hence, by Lemma 1, $\xi(x) \in \partial M_j \cap]0,1[^d$ and $||\xi(x) - y||_2 \leq 2h + \varepsilon$. Assumptions **A1** and **A2** then ensures that,

$$\xi(x) \in \mathcal{F}_{\lambda+L(1+(2+\varepsilon)^\alpha)h^\alpha}$$

as it holds for all $\varepsilon > 0$,

$$F_{\lambda,h}(x,1) = \xi(x) \in \mathcal{F}_{\lambda+L(1+2^\alpha)h^\alpha}.$$

Finally, combining cases, $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}$. □

Proposition 2. *For all $0 < h < \frac{R}{2}$ there exists a collection of spaces $\mathcal{M} = (\mathcal{M}_{\lambda,h})_{\lambda\in\mathbb{R}}$ such that $\forall \lambda \in \mathbb{R}$, $\mathcal{F}_\lambda^{-h} \subset \mathcal{M}_{\lambda,h} \subset \mathcal{F}^{-h}_{\lambda+(2+5^\alpha)Lh^\alpha}$ and,*

$$\mathcal{N}_{\lambda,h} := \mathcal{F}_\lambda \cup \left(\bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \right) \subset \mathcal{F}_{\lambda+Lh^\alpha}$$

with

$$P_{\lambda,h} = \left(\left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d \right)^h \setminus \mathcal{F}^{-h}_{\lambda+2Lh^\alpha} \right) \cap \mathcal{F}_\lambda$$

and $\gamma_{\lambda,h}$ the continuous extension over $P_{\lambda,h}$ of,

$$\gamma_h(x) = \begin{cases} x + \dfrac{\left(h - d_2\left(x, \bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right)\right)}{d_2\left(x, \bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right)} (x - \xi(x)), & \text{if } x \in \left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d \right)^h \setminus \bigcup_{i=1}^{l} \partial M_i \\ x, & \text{if } x \notin \left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d \right)^h \end{cases}$$

retracts by deformation onto $\mathcal{M}_{\lambda,h}$.

Proof. The construction of the deformation retract is inspired by Lemma 14 of Kim et al. (2020). Let $x \in P_{\lambda,h} \cap \bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d$. Assumption **A3**, ensures that $P_{\lambda,h} \subset]0,1[^d$. And it does not allow multiple points, thus it ensures that there exists $i, j \in \{1, ..., l\}$ such that $B_2(x,h) \cap \bigcup_{k=1}^{l} \partial M_k \subset \partial M_i \cup \partial M j$. Furthermore, this conditions also ensures that we can roll a ball of radius h along $\partial M_i \cap]0,1[^d$ in $\overline{M}_i$ and along $\partial M_j \cap]0,1[^d$ in $\overline{M}_j$. Hence,

$$B_2(x,h) \subset M_i \cup M_j \cup (\partial M_i \cap \partial M_j).$$

Now, If

$$B_2(x,h) \cap \mathcal{F}_\lambda \cap M_i \neq \emptyset \text{ and } B_2(x,h) \cap \mathcal{F}_\lambda \cap M_j \neq \emptyset$$

then by assumptions **A2** and **A1**, $B_2(x,h) \subset \mathcal{F}_{\lambda+Lh^\alpha}$ and thus $x \in \mathcal{F}^{-h}_{\lambda+Lh^\alpha}$. Hence, $B_2(x,h) \cap \mathcal{P}_{\lambda,h} \cap M_j = \emptyset$ or $B_2(x,h) \cap \mathcal{P}_{\lambda,h} \cap M_i = \emptyset$. Without loss of generality, let suppose $B_2(x,h) \cap \mathcal{P}_{\lambda,h} \cap M_j = \emptyset$.

Assumption **A3** impose that $\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d$ is a $C^{1,1}$ hypersurface and thus ensures that, for all $x \in \overline{M}_i \cap]0,1[^d$, $\lim\limits_{y \in M_i \to x} \gamma_h(y)$ exists. We can then define $\gamma_{\lambda,h}(x) = \lim_{y \in M_i \to x} \gamma_h(y)$. And, doing so for all $x \in P_{\lambda,h} \cap \bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d$ extends continuously $\gamma_h(x)$ to $P_{\lambda,h}$.

Let, $H_{\lambda,h} : \mathcal{N}_{\lambda,h} \times [0,1] \to \mathcal{N}_{\lambda,h}$, defined by,

$$H_{\lambda,h}(x,t) = (1-t)x + t\left(\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2}\right)$$

if $x \in \bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$, $i \in \{1, ..., l\}$, else,

$$H_{\lambda,h}(x,t) = x.$$

and let $\mathcal{M}_{\lambda,h} = \mathrm{Im}(x \longmapsto H_{\lambda,h}(x,1))$. As $\mathcal{F}_\lambda^{-h} \subset \left(\bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)]\right)^c$, $\mathcal{F}_\lambda^{-h} \subset \mathcal{M}_{\lambda,h}$. Note that, by definition of $\mathcal{M}_{\lambda,h}$

$$H_{\lambda,h}(x,1) \in \mathcal{M}_{\lambda,h}, \quad \forall x \in \mathcal{N}_{\lambda,h}$$

and by definition of $H_{\lambda,h}$

$$H_{\lambda,h}(x,0) = x, \quad \forall x \in \mathcal{N}_{\lambda,h}$$

Let $x \in \bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ verifying $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$, by construction $H_{\lambda,h}(x,t) \in [x, \gamma_{\lambda,h}(x)]$ for all $t \in [0,1]$, in particular this implies that $\gamma_{\lambda,x}(H_{\lambda,h}(x,1)) = \gamma_{\lambda,h}(x)$. Thus, $H_{\lambda,h}(H_{\lambda,h}(x,1),1) = H_{\lambda,h}(x,1)$. In other cases $H_{\lambda,h}(x,1) = x$. Hence,

$$H_{\lambda,h}(x,1) = x, \quad \forall x \in \mathcal{M}_{\lambda,\alpha}.$$

The proof of the continuity of $H_{\lambda,h}$ is rather technical and provided by Lemma 7 in Appendix B. Then $H_{\lambda,h}$ is a deformation retract onto $\mathcal{M}_{\lambda,h}$.

Let's now prove that $\mathcal{M}_{\lambda,h} \subset \mathcal{F}^{-h}_{\lambda+L(2+5^\alpha)h^\alpha}$. Let $x \in \mathcal{N}_{\lambda,h}$, and suppose $x \in \overline{M}_i \cap [0,1]^d$.

If $x \notin \bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)]$, directly, $F(1,x) = x \in \mathcal{F}^{-h}_{\lambda+2Lh^\alpha}$.

If $x \in \bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)]$ and $3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 0$, then there exists $z \in M_j \cap \mathcal{F}_{\lambda+2Lh^\alpha}$, $j \neq i$ such that $||x - z||_2 \leq 3h$ thus $||H_{\lambda,h}(x,1) - z||_2 \leq 4h$. Also, by assumption **A3**, $B_2(H_{\lambda,h}(x,1), h) \subset \overline{M}_i \cup \overline{M}_j$. Thus, by assumption **A1** and **A2**, $B_2(H_{\lambda,h}(x,1), h) \cap \overline{M}_j \subset \mathcal{F}_{\lambda+L(2+5^\alpha)h^\alpha}$ and $B_2(H_{\lambda,h}(x,1), h) \cap \overline{M}_i \subset \mathcal{F}_{\lambda+L2^\alpha h^\alpha}$, thus,

$$H_{\lambda,h}(x,1) \in \mathcal{F}^{-h}_{\lambda+L(2+5^\alpha)h^\alpha}.$$

If $x \in \bigcup\limits_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)]$ and $3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) < 0$, then $H_{\lambda,h}(x,1) = \gamma_{\lambda,h}(x)$ and thus $B_2(H_{\lambda,h}(x,1), h) \subset \overline{M}_i$. As $||x - \gamma_{\lambda,h}||_2 \leq h$, it follows that,

$$H_{\lambda,h}(x,1) \in \mathcal{F}^{-h}_{\lambda+L2^\alpha h^\alpha}.$$

From the same reasoning, it also follows that $[x, \gamma_{\lambda,h}(x)] \subset \mathcal{F}_{\lambda+Lh^\alpha}$ and hence $\mathcal{N}_{\lambda,h} \subset \mathcal{F}_{\lambda+Lh^\alpha}$.

Combining all cases, it follows that $\mathcal{M}_{\lambda,h} \subset \mathcal{F}^{-h}_{\lambda+L(2+5^\alpha)h^\alpha}$. □

2.2 Compatibility with the histogram estimator

In this section, we provide two additional key properties : Propositions 3 and 4. These results allow using the deformation retracts from Proposition 1 and Proposition 2 to construct the interleaving we are looking for.

We define,

$$||W||_{cube,h} = \sup_{H \in C_h} \frac{|W(H)|}{\omega(L(H))}$$

with $\omega(r) = \sqrt{r \log(1 + 1/r)}$, $W(H) = \int_H dW$ and L the Lebesgue measure. Before proving Propositions 3, we state the following lemma, which proof can be found in Appendix B.

Lemma 2. *Let $f : [0,1]^d \to \mathbb{R}$. Let $H \subset \mathcal{F}^c_{\lambda+||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}} \cap C_{h_{\theta,\alpha}}$ and $H' \subset \mathcal{F}_{\lambda-||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}} \cap C_{h_{\theta,\alpha}}$. We then have that,*

$$\int_H dX - \int_H \lambda > 0 \text{ and } \int_{H'} dX - \int_H \lambda < 0.$$

Proposition 3. *Let $f : [0,1]^d \to \mathbb{R}$. For all $\lambda \in \mathbb{R}$,*

$$\mathcal{F}^{-\sqrt{d}h_{\theta,\alpha}}_{\lambda-||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}} \subset \widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}} \subset \mathcal{F}^{\sqrt{d}h_{\theta,\alpha}}_{\lambda+||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}}.$$

Proof. Let $x \in \mathcal{F}^{-\sqrt{d}h_{\theta,\alpha}}_{\lambda-||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}}$ and H the hypercube of C_h containing x. We then have,

$$H \subset \mathcal{F}_{\lambda-||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}}.$$

Hence, by Lemma 2, $\int_H dX - \int_H \lambda < 0$, thus,

$$H \subset \widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}}.$$

Now, let $x \in \left(\mathcal{F}^{\sqrt{d}h_{\theta,\alpha}}_{\lambda+||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}}\right)^c$, and H the hypercube of C_h containing x. We then have,

$$H \subset \mathcal{F}^c_{\lambda+||W||_{cube,h_{\theta,\alpha}} h^\alpha_{\theta,\alpha}}.$$

Hence, by Lemma 2, $\int_H dX - \int_H \lambda > 0$, thus,

$$H \subset \widehat{\mathcal{F}}^c_{\lambda,h_{\theta,\alpha}}$$

and Proposition 3 is proved. □

Propositions 3 locate the estimated sublevel sets (up to shifts) between the true shrunken and thickened sublevel sets. It allows building, from the deformation retracts of Proposition 1, a morphism from the estimated persistent modules to the true one. And, It allows building, from the deformation retracts of Proposition 2, a morphism from the true persistent modules to the estimated one.

A key that will ensure that those morphisms induce, in deed, an interleaving, is that the deformation retracts, restricted to the estimated sublevel sets have their supports (again, up to shifts) in estimated sublevel sets. This is a direct consequence of Proposition 4. Before proving it, we provide a technical lemma which proof can also be found in Appendix B.

Lemma 3. *Let $K, h > 0$ such that $Kh < R$. There exists a constant C_2 (depending only on K, d and R) such that for all $i \in \{1, ..., l\}$ and $x \in \left(\partial M_i \cap]0,1[^d\right)^{2Kh} \cap \overline{M}_i$, we have,*

$$d_H\left(B_2\left(\xi(x), Kh\right) \cap M_i, B_2\left(\xi(x), Kh\right) \cap \underline{P}\right) \leq C_2 h^2$$

with

$$\underline{P} = \left\{ z \in [0,1]^d \ s.t. \ \left\langle z, \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \geq \left\langle \xi(x), \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \right\}.$$

Proposition 4. *Let $\lambda \in \mathbb{R}$, $0 < C < \sqrt{d}$, $K \leq \sqrt{d}$ and $x \in \widehat{\mathcal{F}}_\lambda \cap B_2\left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d, Kh_{\theta,\alpha}\right)$. For sufficiently small θ, we have,*

$$[x, \xi(x)] \subset \widehat{\mathcal{F}}^{Ch_{\theta,\alpha}}_{\lambda + (2||W||_{cube,h_{\theta,\alpha}} + (K+\sqrt{d})^\alpha L)h^\alpha_{\theta,\alpha}, h_{\theta,\alpha}} \tag{3}$$

and

$$\left[x, \gamma_{\lambda, Kh_{\theta,\alpha}}(x)\right] \subset \widehat{\mathcal{F}}_{\lambda + (3||W||_{cube,h_{\theta,\alpha}} + (L((K+\sqrt{d})^\alpha + K^\alpha + d^{\alpha/2}))h^\alpha_{\theta,\alpha}, h_{\theta,\alpha}} \tag{4}$$

Proof. Without loss of generality we can suppose $x \in M_i$. Let $H_1 \in C_{h_{\theta,\alpha},\lambda}$ the hypercube containing x and denote x_1 its center. Suppose there exist $y \in [x, \xi(x)]$ such that $y \notin H_1^{Ch_{\theta,\alpha}}$ and without loss of generality, we can suppose that for an arbitrarily small ε, $[y - \varepsilon, y + \varepsilon] \subset [x, \xi(x)]$ is contained in H_2 an hypercube of $C_{h_{\theta,\alpha}}$, we denote x_2 its center. We have,

$$\langle x - y, x_1 - x_2 \rangle = \langle x - x_1, x_1 - x_2 \rangle + ||x_1 - x_2||_2^2 + \langle y - x_2, x_1 - x_2 \rangle$$

As $x \in H_1$, then,

$$\langle x - x_1, x_1 - x_2 \rangle \geq -\frac{||x_1 - x_2||_2^2}{2}$$

As, $y \in H_2 \setminus H_1^{Ch_{\theta,\alpha}}$, for sufficiently small θ,

$$\langle y - x_2, x_1 - x_2 \rangle \geq -\frac{||x_1 - x_2||_2^2}{2} + \frac{Ch^2_{\theta,\alpha}}{2\sqrt{d}}$$

and thus, as $||x - y||_2 \leq ||x - \xi(x)||_2 \leq Kh_{\theta,\alpha}$,

$$\left\langle \frac{x - \xi(x)}{||x - \xi(x)||_2}, x_1 - x_2 \right\rangle = \left\langle \frac{x - y}{||x - y||_2}, x_1 - x_2 \right\rangle \geq \frac{Ch^2_{\theta,\alpha}}{2\sqrt{d}||x - y||_2} \geq \frac{Ch_{\theta,\alpha}}{2K\sqrt{d}}.$$

This implies that, for all $z \in H_1 \cap \underline{P}$,

$$B_2\left(z + (x_1 - x_2), \frac{Ch_{\theta,\alpha}}{2K\sqrt{d}}\right) \subset \underline{P}. \tag{5}$$

Let $z \in H_1^{-C_2 h^2_{\theta,\alpha}} \cap M_i$, by Lemma 3, there exists z' in $H_1 \cap \underline{P}$ such that $||z - z'||_2 \leq C_2 h^2_{\theta,\alpha}$. And by 5,

$$B_2\left(z' + (x_1 - x_2), \frac{Ch_{\theta,\alpha}}{2K\sqrt{d}}\right) \subset \underline{P}.$$

Then, by Lemma 3, for θ sufficiently small for $\frac{Ch_{\theta,\alpha}}{2K\sqrt{d}} h_{\theta,\alpha} > 2C_2 h^2_{\theta,\alpha}$,

$$B_2\left(z + (x_1 - x_2), C_2 h^2_{\theta,\alpha}\right) \subset B_2\left(z' + (x_1 - x_2), \frac{Ch_{\theta,\alpha}}{2K\sqrt{d}} - C_2 h^2_{\theta,\alpha}\right) \subset \overline{M}_i.$$

Consequently, for θ sufficiently small, $L(H_2 \cap M_i) \geq L(H_1 \cap M_i)$. Now, as $H_2 \subset H_1^{(K+\sqrt{d})h_{\theta,\alpha}}$, assumption **A1** and **A2** implies that,

$$\begin{aligned}
\int_{H_2} dX &= \int_{H_2 \cap M_i} f + \int_{H_2 \cap M_i^c} f + \theta \int_{H_2} dW \\
&\leq \int_{H_1 \cap M_i} (f + L((K+\sqrt{d})h_{\theta,\alpha})^\alpha) + \int_{H_1 \cap M_i} (f + L((K+\sqrt{d})h_{\theta,\alpha})^\alpha) \\
&\quad + \theta \int_{H_1} dW + \theta \int_{H_2} dW - \theta \int_{H_1} dW \\
&\leq (\lambda + (K+\sqrt{d})^\alpha L + 2||W||_{cube,h_{\theta,\alpha}}) h_{\theta,\alpha}^\alpha) h_{\theta,\alpha}^d
\end{aligned}$$

by the choice made for $h_{\theta,\alpha}$. Thus, $H_2 \subset \widehat{\mathcal{F}}_{\lambda+(2||W||_{cube,h_{\theta,\alpha}}+(K+\sqrt{d})^\alpha L)h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}}$, and 3 follows.

By construction, $\gamma_{\lambda,Kh_{\theta,\alpha}}(x)$ is a distance $Kh_{\theta,\alpha}$ from $\xi(x)$ and thus at distance at most $Kh_{\theta,\alpha}$ from x. By Proposition 2, $\gamma_{\lambda,Kh_{\theta,\alpha}}(x) \in \mathcal{F}_{\lambda+LK^\alpha h_{\theta,\alpha}^\alpha}$. By construction, $B(\gamma_{\lambda,Kh_{\theta,\alpha}}(x), Kh_{\theta,\alpha}) \subset \overline{M}_i$, and thus, as $K \geq \sqrt{d}$,

$$\gamma_{\lambda,Kh_{\theta,\alpha}}(x) \in \mathcal{F}_{\lambda+L(K^\alpha+d^{\alpha/2})h_{\theta,\alpha}^\alpha}^{-\sqrt{d}h_{\theta,\alpha}}.$$

Proposition 3 then gives,

$$\gamma_{\lambda,Kh_{\theta,\alpha}}(x) \subset \widehat{\mathcal{F}}_{\lambda+L(K^\alpha+d^{\alpha/2}+||W||_{cube,h_{\theta,\alpha}})h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}}.$$

Hence, by 3,

$$[x, \gamma_{\lambda,Kh_{\theta,\alpha}}(x)] \subset [\xi(x), \gamma_{\lambda,Kh_{\theta,\alpha}}(x)] \subset \widehat{\mathcal{F}}_{\lambda+(3||W||_{cube,h_{\theta,\alpha}}+(L((K+\sqrt{d})^\alpha+K^\alpha+d^{\alpha/2}))h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}}$$

which proves 4. $\square$

2.3 Main results

Now equipped with Propositions 1, 2, 3 and 4, we have all the ingredients to establish our main results. We formalize the reasoning describe in the beginning of this section, constructing, for all $s \in \mathbb{N}$, an $Ch_{\theta,\alpha}$-interleaving between $\hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}}$ and the true persistence module $\mathbb{V}_{f,s}$ induced by the filtration $\mathcal{F}$ to provide a concentration bound, Proposition 5, from which follows Theorem 1.

Before proving our main results, we provide concentration results on $||W||_{cube,h}$, used in the proof of Proposition 5.

Lemma 4.

$$\mathbb{P}(||W||_{cube,h} \geq t) \leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2 \log\left(1+\frac{1}{h^d}\right)\right).$$

Consequently, there exists two constants C_0 and C_1 depending only on d such that, for all $h < 1$,

$$\mathbb{P}(||W||_{cube,h} \leq t) \leq C_0 \exp\left(-C_1 t^2\right).$$

Proof. The proof essentially follows from the fact that for all $h > 0$ and H hypercube of side h, $\frac{W(H)}{h^{d/2}}$ is a standard Gaussian.

$$\mathbb{P}\left(\sup_{H \in C_h} \frac{|W(H)|}{\omega(h^d)} > t\right) \leq \left(\frac{1}{h}\right)^d \mathbb{P}\left(\frac{|W(H)|}{\omega(h^d)} > t\right)$$

$$= \left(\frac{1}{h}\right)^d \mathbb{P}\left(\frac{|W(H)|}{h^{d/2}} > t\sqrt{\log\left(1+\frac{1}{h^d}\right)}\right)$$
$$\leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2\log\left(1+\frac{1}{h^d}\right)\right)$$

Now, take $t \geq \sqrt{8}$, then $t^2/4+2 \leq t^2$. Thus,

$$\begin{aligned}\mathbb{P}\left(||W||_{cube,h} \geq t\right) &\leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{1}{2}t^2\log\left(1+\frac{1}{h^d}\right)\right)\\ &\leq 2\left(\frac{1}{h}\right)^d \exp\left(-(t^2/8+1)\log\left(1+\frac{1}{h^d}\right)\right)\\ &\leq 2\exp(-t^2/8).\end{aligned}$$

Hence, for all $t > 0$,

$$\mathbb{P}\left(||W||_{cube,h} \geq t\right) \leq 2e \times \exp(-t^2/8).$$

□

Proposition 5. *There exists $\tilde{C}_0$ and $\tilde{C}_1$ such that, for all $t > 0$,*

$$\mathbb{P}\left(\sup_{f \in S_d(L,\alpha,R)} d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right) \geq t\left(\theta^2\log\left(\frac{1}{\theta}\right)\right)^{\frac{\alpha}{d+2\alpha}}\right) \leq \tilde{C}_0 \exp\left(-\tilde{C}_1 t^2\right).$$

Proof. The strategy is to construct an interleaving between the persistent module $\mathbb{V}_{s,f}$ and the module induced by the filtration $\left(\widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}}\right)_{\lambda\in\mathbb{R}}$. It suffices to show the result for small θ (up to rescaling $\tilde{C}_0$). Hence, suppose that θ is such that $2\sqrt{d}h_{\theta,\alpha} < \frac{R}{2}$ and Proposition 4 holds for $C = 1/4$.

Note that for all $\lambda \in \mathbb{R}$, $\widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}}$ is a union of hypercube of $C_{h_{\theta,\alpha}}$, hence its μ-reach (see definition in Chazal et al. (2006)) is lower bounded by $h_{\theta,\alpha}/2$ for all $\mu < 1/2$. Hence, Theorem 12 of Kim et al. (2020) ensures that $\widehat{\mathcal{F}}^{h/4}_{\lambda,h_{\theta,\alpha}}$ deformation retracts onto $\widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}}$. Then, the module $\mathbb{V}_{s,f}$ can be thought as the module induced by the filtration $\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right)_{\lambda\in\mathbb{R}}$. Let,

$$j_{1,\lambda} : H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right) \to H_s\left(\mathcal{K}_{\lambda+||W||_{cube,h_{\theta,\alpha}}h^{\alpha}_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}\right)$$

the map induced by the inclusion $\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}} \subset \mathcal{K}_{\lambda+||W||_{cube,h_{\theta,\alpha}}h^{\alpha}_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}$,

$$j_{2,\alpha} : H_s\left(\mathcal{K}_{\lambda+||W||_{cube,h_{\theta,\alpha}}h^{\alpha}_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}\right) \to H_s\left(\mathcal{G}_{\lambda+||W||_{cube,h_{\theta,\alpha}}h^{\alpha}_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}\right)$$

induced by the deformation retract of Proposition 1, and,

$$j_{3,\alpha} : H_s\left(\mathcal{G}_{\lambda+||W||_{cube,h_{\theta,\alpha}}h^{\alpha}_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}\right) \to H_s\left(\mathcal{F}_{\lambda+L\left(2^{\alpha}d^{\alpha/2}(1+3^{\alpha})+||W||_{cube,h_{\theta,\alpha}}\right)h^{\alpha}_{\theta,\alpha}}\right)$$

the map induced inclusion following again Proposition 1. We then define,

$$\begin{cases} \overline{\phi}_{\lambda} : H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right) \to H_s\left(\mathcal{F}_{\lambda+L\left(2^{\alpha}d^{\alpha/2}(1+3^{\alpha})+||W||_{cube,h_{\theta,\alpha}}\right)h^{\alpha}_{\theta,\alpha}}\right) \\ \overline{\phi}_{\lambda} = j_{3,\lambda}\circ j_{2,\lambda}\circ j_{1,\lambda} \end{cases}$$

This gives us the first module morphism. Let construct the second one. Let,

$$j_{4,\lambda} : H_s\left(\mathcal{F}_\lambda\right) \to H_s\left(\mathcal{N}_{\lambda,\sqrt{d}h_{\theta,\alpha}}\right)$$

the map induced by the inclusion $\mathcal{F}_\lambda \subset \mathcal{N}_{\lambda,\sqrt{d}h_{\theta,\alpha}}$,

$$j_{5,\lambda} : H_s\left(\mathcal{N}_{\lambda,\sqrt{d}h_{\theta,\alpha}}\right) \to H_s\left(\mathcal{M}_{\lambda,\sqrt{d}h_{\theta,\alpha}}\right)$$

the map induced by the deformation retract of Proposition 2, and,

$$j_{6,\lambda} : H_s\left(\mathcal{M}_{\lambda,\sqrt{d}h_{\theta,\alpha}}\right) \to H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+\left(L\left(2^\alpha d^{\alpha/2}(2+5^\alpha)+d^{\alpha/2}(3^\alpha+2^\alpha+3)\right)+4||W||_{cube,h_{\theta,\alpha}}\right)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right)$$

induced by the inclusion $\mathcal{M}_{\lambda,\sqrt{d}h_{\theta,\alpha}} \subset \widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+\left(L\left(2^\alpha d^{\alpha/2}(2+5^\alpha)+d^{\alpha/2}(3^\alpha+2^\alpha+3)\right)+4||W||_{cube,h_{\theta,\alpha}}\right)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}$, from the combination of Proposition 3 and 2. We then define,

$$\begin{cases} \overline{\psi}_\lambda : H_s\left(\mathcal{F}_\lambda\right) \longrightarrow H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+\left(L\left(2^\alpha d^{\alpha/2}(2+5^\alpha)+d^{\alpha/2}(3^\alpha+2^\alpha+3)\right)+4||W||_{cube,h_{\theta,\alpha}}\right)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right) \\ \overline{\psi}_\lambda = j_{6,\lambda} \circ j_{5,\lambda} \circ j_{4,\lambda} \end{cases}$$

We now show that $\overline{\psi}$ and $\overline{\phi}$ induce an interleaving between $\hat{\mathbb{V}}^{h_{\theta,\alpha}}_{f,s}$ and $\mathbb{V}_{s,f}$. More precisely, we show that the following diagrams commute, for all $\lambda < \lambda^{'}$. For compactness of notation let,

$$K_1 = L\left(2^\alpha d^{\alpha/2}\left(2+5^\alpha\right) + d^{\alpha/2}(3^\alpha+2^\alpha+3)\right) + 4||W||_{cube,h_{\theta,\alpha}}$$

and

$$K_2 = L\left(2^\alpha d^{\alpha/2}\left(1+3^\alpha\right)\right) + ||W||_{cube,h_{\theta,\alpha}}.$$

$$\begin{tikzcd} H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right) \arrow[r, "\hat{v}^{\lambda^{'}}_{\lambda,h_{\theta,\alpha}}"] \arrow[d, "\overline{\phi}_\lambda"] & H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda^{'},h_{\theta,\alpha}}\right) \arrow[d, "\overline{\phi}_{\lambda^{'}}"] \\ H_s\left(\mathcal{F}_{\lambda+K_2h^\alpha_{\theta,\alpha}}\right) \arrow[r, "v^{\lambda^{'}+K_2h^\alpha_{\theta,\alpha}}_{\lambda+K_2h^\alpha_{\theta,\alpha}}"] & H_s\left(\mathcal{F}_{\lambda^{'}+K_2h^\alpha_{\theta,\alpha}}\right) \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} H_s\left(\mathcal{F}_\lambda\right) \arrow[r, "v^{\lambda^{'}}_{\lambda}"] \arrow[d, "\overline{\psi}_\lambda"] & H_s\left(\mathcal{F}_{\lambda^{'}}\right) \arrow[d, "\overline{\psi}_{\lambda^{'}}"] \\ H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+K_1h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right) \arrow[r, "\hat{v}^{\lambda^{'}+K_1h^\alpha_{\theta,\alpha}}_{\lambda+K_1h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}"] & H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda^{'}+K_1h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right) \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right) \arrow[rr, "\hat{v}^{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha}}_{\lambda,h_{\theta,\alpha}}"] \arrow[dr, "\overline{\phi}_\lambda"'] & & H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right) \\ & H_s\left(\mathcal{F}_{\lambda+K_2h^\alpha_{\theta,\alpha}}\right) \arrow[ur, "\overline{\psi}_{\lambda+K_2h^\alpha_{\theta,\alpha}}"'] & \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} H_s\left(\mathcal{F}_\lambda\right) \arrow[rr, "v_\lambda^{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha}}"] \arrow[dr, "\overline{\psi}_\lambda"'] & & H_s\left(\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha}}\right) \\ & H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+K_1h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}\right) \arrow[ur, "\overline{\phi}_{\lambda+K_1h^\alpha_{\theta,\alpha}}"] & \end{tikzcd} \tag{9}$$

- Diagram 6 : We can rewrite the diagram as (unspecified maps are simply induced by set inclusion),

$$\begin{tikzcd} H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right) \arrow[r] \arrow[d] & H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda',h_{\theta,\alpha}}\right) \arrow[d] \\ H_s\left(\mathcal{K}_{\lambda+\|W\|_{cube,h_{\theta,\alpha}}h^\alpha_{\theta,\alpha},\sqrt{d}h_{\theta,\alpha}}\right) \arrow[r] \arrow[d, leftrightarrow, "j_{2,\lambda}"] & H_s\left(\mathcal{K}_{\lambda'+\|W\|_{cube,h_{\theta,\alpha}}h^\alpha_{\theta,\alpha},\sqrt{d}h_{\theta,\alpha}}\right) \arrow[d, leftrightarrow, "j_{2,\lambda'}"] \\ H_s\left(\mathcal{G}_{\lambda+\|W\|_{cube,h_{\theta,\alpha}}h^\alpha_{\theta,\alpha},\sqrt{d}h_{\theta,\alpha}}\right) \arrow[r] \arrow[d] & H_s\left(\mathcal{G}_{\lambda'+\|W\|_{cube,h_{\theta,\alpha}}h^\alpha_{\theta,\alpha},\sqrt{d}h_{\theta,\alpha}}\right) \arrow[d] \\ H_s\left(\mathcal{F}_{\lambda+K_2h^\alpha_{\theta,\alpha}}\right) \arrow[r] & H_s\left(\mathcal{F}_{\lambda'+K_2h^\alpha_{\theta,\alpha}}\right) \end{tikzcd}$$

  By inclusions, the upper and lower faces commute. And, as $j_{2,\lambda}$ and $j_{2,\lambda'}$ comes from deformation retracts, the central face also commutes. Hence, all faces of Diagram 6 commute and consequently Diagram 6 commutes.

- Diagram 7 : it can be decomposed similarly to Diagram 6, one can check that the same reasoning then applies.

- Diagram 8 : Let $C \in C_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right)$ and $[C]$ its classes in $H_s\left(\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda,h_{\theta,\alpha}}\right)$. The morphism $\overline{\phi}_\lambda$ maps $[C]$ to $[C^{'}]$ with $C^{'}$ the retraction of C in $\mathcal{K}_{\lambda+\|W\|_{cube,h_{\theta,\alpha}}h^\alpha_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}$ via the deformation retract constructed in the proof of Proposition 1. And $\overline{\psi}_{\lambda+K_2h^\alpha_{\theta,\alpha}}$ maps $[C^{'}]$ to $[C^{''}]$, with $C^{''}$ the retraction of $C^{'}$ in $\mathcal{N}_{\lambda+(K_2+\|W\|_{cube,h_{\theta,\alpha}})h^\alpha_{\theta,\alpha}),\sqrt{d}h_{\theta,\alpha}}$ given in Proposition 2. Assertion 3 of Proposition 4 ensures that the support of the retraction of C onto $C^{'}$ is included in $\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}$. And Assertion 4 ensures that the support of the retraction of $C^{'}$ onto $C^{''}$ is also included in $\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}$. Hence, C and $C^{''}$ are homologous in $\widehat{\mathcal{F}}^{h_{\theta,\alpha}/4}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha},h_{\theta,\alpha}}$ and Diagram 8 commutes.

- Diagram 9 : Let $C \in C_s\left(\mathcal{F}_\lambda\right)$, $\overline{\psi}_\lambda$ maps $[C]$ to $[C^{'}]$, with $C^{'}$ the retraction of C via the retraction of Proposition 2. And, as $\mathcal{M}_{\lambda,2\sqrt{d}h_{\theta,\alpha}}$ is included in $\mathcal{G}_{\lambda+K_1h^\alpha_{\theta,\alpha},2\sqrt{d}h_{\theta,\alpha}}$, $\overline{\phi}_{\lambda+K_1h^\alpha_{\theta,\alpha}}$, behave as an inclusion induced map, mapping $[C^{'}]$ to $[C^{'}]$. From Proposition 2, the retraction of C on $C^{'}$ has its support included in $\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha}}$. Thus, C and $C^{'}$ are homologous in $\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha_{\theta,\alpha}}$ and Diagram 9 commutes.

The commutativity of diagrams 7,6,8 and 9 means that $\hat{\mathbb{V}}^{h_{\theta,\alpha}}_{f,s}$ and $\mathbb{V}_{f,s}$ are $(K_1+K_2)h^\alpha_{\theta,\alpha}$ interleaved, and thus we get from the algebraic stability theorem (Chazal et al., 2009) that,

$$d_b\left(\mathrm{dgm}\left(\hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}}\right), \mathrm{dgm}\left(\mathbb{V}_{f,s}\right)\right) \leq (K_1+K_2)h_{\theta,\alpha}^{\alpha}$$

and as it holds for all $s \in \mathbb{N}$,

$$\sup_{f \in S_d(L,\alpha,R)} d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right) \leq (K_1+K_2)h_{\theta,\alpha}^{\alpha}.$$

Now, using Lemma 4, this implies that,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{f \in S_d(L,\alpha,R)} d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right) \geq t h_{\theta,\alpha}\right) \\
&\leq \mathbb{P}\left(K_1+K_2 \geq t\right) \\
&= \mathbb{P}\left(||W||_{cube,h_{\theta,\alpha}} \geq \frac{t - L\left(2^{\alpha}d^{\alpha/2}\left(2+5^{\alpha}\right) + d^{\alpha/2}(3^{\alpha}+2^{\alpha}+3)\right) - L\left(2^{\alpha}d^{\alpha/2}\left(1+3^{\alpha}\right)\right)}{5}\right) \\
&\leq C_0 \exp\left(-C_1\left(\frac{t - L\left(d^{\alpha/2}\left(10^{\alpha}+6^{\alpha}+3^{\alpha}+4\times 2^{\alpha}+3\right)\right)}{5}\right)^2\right) \\
&\leq C_0 \exp\left(2C_1\left(\frac{L\left(d^{\alpha/2}\left(10^{\alpha}+6^{\alpha}+3^{\alpha}+4\times 2^{\alpha}+3\right)\right)}{5}t\right)\right) \\
&\quad \times \exp\left(-C_1\left(\frac{L\left(d^{\alpha/2}\left(10^{\alpha}+6^{\alpha}+3^{\alpha}+4\times 2^{\alpha}+3\right)\right)}{5}\right)^2\right) \exp\left(-\frac{C_1}{25}t^2\right)
\end{aligned}$$

and the result follows. $\square$

From this result, we can derive from this result bounds in expectation.

Theorem 1. *Let* $p \geq 1$,

$$\sup_{f \in S_d(L,\alpha,R)} \mathbb{E}\left(d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)^p\right) \lesssim h_{\theta,\alpha}^{p\alpha}$$

Proof. The sub-Gaussian concentration provided by Proposition 5, gives that, for all $t > 0$,

$$\mathbb{P}\left(\frac{d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)}{h_{\theta,\alpha}^{\alpha}} \geq t\right) \leq \tilde{C}_0 \exp\left(-\tilde{C}_1 t^2\right)$$

Now, we have,

$$\begin{aligned}
&\mathbb{E}\left(\frac{d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)^p}{h_{\theta,\alpha}^{p\alpha}}\right) \\
&= \int_0^{+\infty} \mathbb{P}\left(\frac{d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right)^p}{h_{\theta,\alpha}^{p\alpha}} \geq t\right) dt \\
&\leq \int_0^{+\infty} \tilde{C}_0 \exp\left(-\tilde{C}_1 t^{2/p}\right) dt < +\infty.
\end{aligned}$$

$\square$

2.4 Adaptivity

The previous procedure depends strongly on the regularity parameter α as we calibrate the window size h taking account of it. Thus, the procedure is not adaptive to the regularity. In the following, we propose an estimation procedure, based on the previous one, that is adaptive with respect to α. Moreover, we show that this adaptive procedure achieves the same rates as the one given by Theorem 1.

We follow the Lepskii's method (Lepskii, 1991). Suppose that we know an upper bound on the parameter L, denoted $\overline{L}$ and $0 < \alpha_{\min} \leq \alpha \leq \alpha_{\max}$. It is sufficient to work on regular grid $\alpha_{\min} = \alpha_1 < \alpha_2 < ... < \alpha_N = \alpha_{\max}$ with $N \simeq \log\left(\frac{1}{\theta}\right)$, as, for all $1 < j \leq N$,

$$\log\left(\frac{h_{\theta,\alpha_{j-1}}^{\alpha_{j-1}}}{h_{\theta,\alpha_j}^{\alpha_j}}\right) = \left(\frac{\alpha_{j-1}}{2\alpha_{j-1}+d} - \frac{\alpha_j}{2\alpha_j+d}\right)\log\left(\theta\log\left(\frac{1}{\theta}\right)\right) \simeq \frac{\log\left(\theta\log\left(\frac{1}{\theta}\right)\right)}{\log(\theta)} \simeq 1.$$

We consider the Lepskii's estimator defined by,

$$\widehat{\mathrm{dgm}(f)}^* = \widehat{\mathrm{dgm}(f)}_{\hat{\alpha}}$$

with

$$\hat{\alpha} = \max\left\{\alpha \in \{\alpha_1, ..., \alpha_N\} : \frac{d_b(\widehat{\mathrm{dgm}(f)}_\alpha, \widehat{\mathrm{dgm}(f)}_{\alpha'})}{h_{\theta,\alpha}^\alpha} < c_0 \text{ for all } \alpha' \leq \alpha\right\}.$$

c_0 a sufficiently large constant depending on d, $\overline{L}$, $\alpha_{\min}$ and $\alpha_{\max}$. The notation $\widehat{\mathrm{dgm}(f)}_{\hat{\alpha}}$ refer to the estimator $\widehat{\mathrm{dgm}(f)}$ for the window size $h_{\theta,\alpha}$, as it will play a role in this section, we highlight the dependence in α.

Theorem 2. *Let $p \geq 1$,*

$$\sup_{(L,\alpha)\in[0,\overline{L}]\times[\alpha_{\min},\alpha_{\max}]} \quad \sup_{f\in S_d(L,\alpha,R)} \quad \mathbb{E}\left(\frac{d_b\left(\widehat{\mathrm{dgm}(f)}^*, \mathrm{dgm}(f)\right)^p}{h_{\theta,\alpha}^{p\alpha}}\right) \lesssim 1.$$

Proof. We want to apply Corollary 1 of Lepskii (1992), in our case, the only difficulty is to check assumption **A3b**. It then suffices to show that, for all $\alpha' \in [\alpha_{\min}, \alpha_{\max}]$, there exists $c_0 > 0$ such that,

$$\limsup_{\theta\to 0} \frac{\log(1/\theta)^2}{h_{\theta,\alpha_{\max}}^{\alpha_{\max}}} \sup_{\alpha<\alpha'} \mathbb{P}\left(\frac{d_b(\widehat{\mathrm{dgm}(f)}_\alpha, \mathrm{dgm}(f))}{h_{\theta,\alpha}^\alpha} > c_0\right) = 0. \tag{10}$$

Now as shown in the proof of Proposition 5, for sufficiently small θ,

$$d_b(\widehat{\mathrm{dgm}(f)}_\alpha, \mathrm{dgm}(f)) \leq (K_1 + K_2)\, h_{\theta,\alpha}^\alpha.$$

Thus, for sufficiently small θ, using the concentration from Lemma 4,

$$\mathbb{P}\left(\frac{d_b(\widehat{\mathrm{dgm}(f)}_\alpha, \mathrm{dgm}(f))}{h_{\theta,\alpha}^\alpha} > c_0\right)$$

$$\mathbb{P}\left(||W||_{cube,h_{\theta,\alpha}} \geq \frac{c_0 - \overline{L}\left(d^{\alpha/2}\left(10^\alpha + 6^\alpha + 3^\alpha + 4\times 2^\alpha + 3\right)\right)}{5}\right)$$

$$\leq 2\left(\frac{1}{h_{\theta,\alpha}}\right)^d \exp\left(-\frac{1}{2}\left(\frac{c_0-\overline{L}\left(d^{\alpha/2}\left(10^\alpha+6^\alpha+3^\alpha+4\times 2^\alpha+3\right)\right)}{5}\right)^2\log\left(1+\frac{1}{h_{\theta,\alpha}^d}\right)\right).$$

$$\leq 2\left(\frac{1}{h_{\theta,\alpha}}\right)^d \exp\left(-\frac{1}{2}\left(c_0/5-6\overline{L}\sqrt{d}\right)^2\log\left(1+\frac{1}{h_{\theta,\alpha}^d}\right)\right)$$

$$\lesssim \left(\theta^2\log\left(\frac{1}{\theta}\right)\right)^{\frac{d\alpha}{d+2\alpha}\left(\frac{1}{2}\left(c_0/5-6\overline{L}\sqrt{d}\right)^2-1\right)}$$

Thus, for sufficiently big c_0 (depending only on $\alpha_{\min},\alpha_{\max},\overline{L},d$),

$$\mathbb{P}\left(\frac{d_b(\widehat{\mathrm{dgm}(f)}_\alpha,\mathrm{dgm}(f))}{h_{\theta,\alpha}^\alpha}>c_0\right)=o\left(\frac{h_{\theta,\alpha_{\max}}^{\alpha_{\max}}}{\log\left(\frac{1}{\theta}\right)^2}\right).$$

Hence 10 is verified and Corollary 1 of Lepskii (1992) gives the desired result. □

3 Lower bounds

In this section we prove that the rates obtained in the previous section are optimal, in the minimax sense in the non-adaptive and adaptive case, by proving Theorem 3.

Theorem 3. *Let $p\geq 1$*

$$\inf_{\widehat{\mathrm{dgm}(f)}}\sup_{f\in S_d(L,\alpha,R)}\mathbb{E}\left(d_b\left(\widehat{\mathrm{dgm}(f)},\mathrm{dgm}(f)\right)^p\right)\gtrsim\left(\theta^2\log\left(\frac{1}{\theta}\right)\right)^{\frac{p\alpha}{d+2\alpha}}.$$

Where the infimum is taken over all the estimator of $\mathrm{dgm}(f)$.

Proof. The proof follows standard methods to provide minimax lower bounds, as presented in section 2 of Tsybakov (2008). The idea is, for any $r_\theta=o\left(\left(\theta^2\log\left(\frac{1}{\theta}\right)\right)^{\frac{\alpha}{d+2\alpha}}\right)$, to exhibit a finite collection of function in $S_d(L,\alpha,R)$ such that their persistent diagrams are two by two at distance $2r_\theta$ but indistinguishable, with high certainty.

We propose such a collection, let

$$f_0(x_1,...,x_d)=\frac{L}{2}|x_1|^\alpha$$

and for m integer in $[0,\lfloor 1/h\rfloor]$,

$$f_{h,m}(x_1,...,x_d)=f_0-L\left(h^\alpha-||(x_1,...,x_d)-m/\lfloor 1/h\rfloor(1,...,1)||_\infty^\alpha\right)_+$$

f_0 and the $f_{h,m}$ are $(L,\alpha)-$Hölder-continuous and thus belong to $S_d(L,\alpha,R)$ for all $R>0$.

We have $\mathrm{dgm}(f_0)=\{(0,+\infty)\}$ and for all $0<m<\lfloor 1/h\rfloor$, integer,

$$\mathrm{dgm}(f_{h,m})=\left\{(0,+\infty),\left(\frac{L}{2}\left(\frac{m}{\lfloor 1/h\rfloor}\right)^\alpha-Lh^\alpha,\frac{L}{2}\left(\frac{m}{\lfloor 1/h\rfloor}\right)^\alpha-\frac{L}{2}h^\alpha\right)\right\}.$$

Thus, for all $0<m\neq m'<\lfloor 1/h\rfloor$, integers,

$$d_b\left(\mathrm{dgm}(f_0),\mathrm{dgm}(f_{m,h})\right)\geq\frac{Lh^\alpha}{2}\text{ and }d_b\left(\mathrm{dgm}(f_{m,h}),\mathrm{dgm}(f_{m'})\right)\geq\frac{Lh^\alpha}{2}.$$

We set $r_\theta = \frac{Lh^\alpha}{4}$, then,

$$d_b\left(\mathrm{dgm}(f_0), \mathrm{dgm}(f_{d,h,m^{k'},\alpha})\right) \geq 2r_\theta \text{ and } d_b\left(\mathrm{dgm}(f_{h,m}), \mathrm{dgm}(f_{h,m'})\right) \geq 2r_\theta.$$

For a fixed signal f, denote $\mathbb{P}_f^\theta$ the product distribution of the noisy trajectory X define in model 1. From section 2 of Tsybakov (2008), it now suffices to show that if $r_\theta = o\left(\left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{\alpha}{d+2\alpha}}\right)$, then,

$$\frac{1}{\lfloor\frac{1}{h}\rfloor - 2} \sum_{0<m<\lfloor 1/h\rfloor} \chi^2\left(\mathbb{P}^\theta_{f_{h,m}} \mathbb{P}^\theta_{f_0}\right) = \frac{1}{\lfloor\frac{1}{h}\rfloor - 2} \sum_{0<m<\lfloor 1/h\rfloor} \mathbb{E}_{\mathbb{P}^\theta_{f_0}}\left[\left(\frac{d\mathbb{P}^\theta_{f_{h,m}}}{d\mathbb{P}^\theta_{f_0}}\right)^2\right] - 1 \tag{11}$$

converges to zero when θ converges to zero.

By Cameron-Martin formula, for all $0 < m < \lfloor 1/h\rfloor$, integer,

$$\frac{d\mathbb{P}^\theta_{f_{h,m}}}{d\mathbb{P}^\theta_{f_0}} = \exp\left(\int_{[0,1]^d} \theta(f_{h,m} - f_0)(t_1, ..., t_d) dW_{t_1,...,t_d} - \frac{1}{\theta^2}\|f_{h,m} - f_0\|_2^2\right).$$

We denote H_m the hypercube defined by $||(x_1, ..., x_d) - m/\lfloor 1/h\rfloor(1, ..., 1)|| \leq h$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\theta_{f_0}}\left(\left(\frac{d\mathbb{P}^\theta_{f_{h,m}}}{d\mathbb{P}^\theta_{f_0}}\right)^2\right) \\
&= \exp\left(\int_{[0,1]^d} \frac{1}{\theta^2}(f_{h,m} - f_0)^2(t_1, ..., t_d) dt_1...dt_d\right) \\
&= \exp\left(\frac{L^2}{\theta^2}\int_{H_m} (h^\alpha - ||(t_1, ..., t_d) - m/\lfloor 1/h\rfloor(1, ..., 1)||_2^\alpha)^2\, dt_1...dt_d\right) \\
&\leq \exp\left(\frac{L^2}{\theta^2}\left(\int_{H_m} h^{2\alpha} dt_1...dt_d + \int_{H_m} ||(t_1, ..., t_d) - m/\lfloor 1/h\rfloor(1, ..., 1)||_2^{2\alpha} dt_1...dt_d\right)\right) \\
&\leq \exp\left(\frac{2L^2}{\theta^2}\int_{H_m} h^{2\alpha} dt_1...dt_d\right) \\
&\leq \exp\left(\frac{2L^2}{\theta^2} h^{2\alpha+d}\right).
\end{aligned}$$

Hence, if $\left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{1}{d+2\alpha}} \ll h$, we have that 11 converges to zero. Consequently, if $r_\theta = o\left(\left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{\alpha}{d+2\alpha}}\right)$, then $\left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{1}{d+2\alpha}} \ll h$ and we get the conclusion. $\square$

4 Discussion

To date, statistical studies of Topological Data Analysis tools have predominantly relied on lifting known results from signal (or density) estimation using sup norm (or Hausdorff, or Gromov-Hausdorff) stability. However, this work represents a step forward, breaking free from this approach. We provide a finer analysis of the plug-in histogram estimator, showing that it achieves minimax convergence rates on the classes $S_d(L, \alpha, R)$ that coincide with the known ones for Holder-continuous signals. These classes contain irregular functions that pose challenges for conventional non-parametric techniques. Beyond the results shown here, it opens a new path to think and analyze persistent homology inference, showing that it allows relaxation of regularity assumptions over considered signals.

It then raises questions about further relaxations of the regularity assumptions. A first direction is investigating how assumption **A1** can be made more local, controlling the regularity only around the locations of birth and death of topological features. In Dasgupta and Kpotufe (2014) and Jiang and Kpotufe (2017) it is shown that local maxima (and equivalently minima) of a density can be inferred under weak local assumptions. This implies that, for univariate signals, it is possible to achieve the usual convergence rates for persistent diagram inference solely under local regularity assumptions. Understanding how this can be generalized motivates future works. In a parallel vein, we believe that there is room to consider the potential relaxation assumption **A3**. One plausible approach involves controlling the μ-reach, as defined in Chazal et al. (2006), of the discontinuities set. This would extend significantly our results, allowing to handle, for example, signals with sets of discontinuities featuring multiple points and cusps. Still, as illustrated in Figure 5, in this case, a plug-in estimator from histogram will fall short. In this case we may need to move beyond plug-in estimation.

One can also wonder if the methods and convergence rates established here for the Gaussian white noise model extend to other popular and richer models. In this direction, we show in Appendix C how they can be extended to the non-parametric regression model with fixed regular design. Inspired by potential application to modes detection, as sketched for example in Genovese et al. (2015), extending these results to the density model motivates future work in this direction.

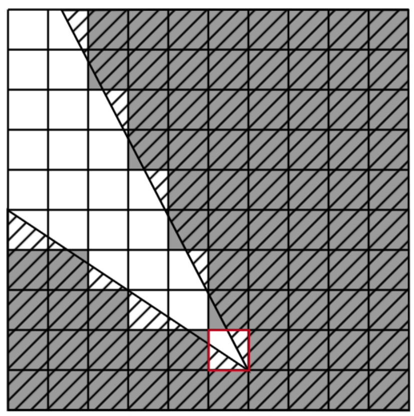


Figure 5: $\lambda-$sublevel cubical approximation for f the function defined as 0 on the hatched area and K outside (for arbitrarily large K) and $\lambda = K/4$. The cycle in red is problematic, as it has a lifetime of CK (C an absolute constant). The discontinuity set has here a positive $\mu-$reach for small μ.

Acknowledgements

The author would like to thank Frédéric Chazal and Pascal Massart for our (many) helpful discussions. The author acknowledge the support of the ANR TopAI chair (ANR–19–CHIA–0001).

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A Proof for $q-$tameness

This section is devoted to prove the claim that the persistent diagrams we consider and estimated persistence diagrams we propose are well-defined, by proving that the underlying persistent modules are $q-$tame.

Lemma 5. *Let $f \in S_d(L, \alpha, R)$. $\forall s \in \mathbb{N}$, $\forall h < \frac{R}{2}$, there exist a morphism ϕ such that, $\forall \lambda \in \mathbb{R}$,*

$$\begin{tikzcd} H_s\left(\mathcal{F}_\lambda\right) \arrow[rr] \arrow[dr] & & H_s\left(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}\right) \\ & H_s\left(\mathcal{F}^h_\lambda\right) \arrow[ur, "\phi_\lambda"'] & \end{tikzcd} \tag{12}$$

is a commutative diagram (unspecified map come from set inclusions).

Proof. Let $\tilde{\phi}_\lambda : H_s\left(\mathcal{K}_{\lambda,h}\right) \to H_s\left(\mathcal{G}_{\lambda,h}\right)$ the morphism associated to the deformation retract from Proposition 1. We also denote $i_{1,\lambda} : H_s\left(\mathcal{F}^h_\lambda\right) \to H_s\left(\mathcal{K}_{\lambda,h}\right)$ the morphism induced by the inclusion $\mathcal{F}^h_\lambda \subset \mathcal{K}_{\lambda,h}$ and $i_{2,\lambda} : H_s\left(\mathcal{G}_{\lambda,h}\right) \to H_s\left(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}\right)$ the morphism induced by the inclusion $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}$, also provided by Proposition 1. We take $\phi_\lambda = i_{2,\lambda} \circ \tilde{\phi}_\lambda \circ i_{1,\lambda}$. Diagram 12 then is (unspecified maps are the one induced by set inclusion),

$$\begin{tikzcd} H_s\left(\mathcal{F}_\lambda\right) \arrow[rrr] \arrow[d] \arrow[dr] \arrow[drr] \arrow[d, phantom, shift left=8ex, "(F1)"] & \arrow[d, phantom, "(F2)"] & \arrow[d, phantom, "(F3)"] & H_s\left(\mathcal{F}_{\lambda+L(1+3^\alpha)h^\alpha}\right) \\ H_s\left(\mathcal{F}^h_\lambda\right) \arrow[r, "i_{1,\lambda}"'] & H_s\left(\mathcal{K}_{\lambda,h}\right) \arrow[r, leftrightarrow, "\tilde{\phi}_\lambda"'] & H_s\left(\mathcal{G}_{\lambda,h}\right) \arrow[ur, "i_{2,\lambda}"'] & \end{tikzcd} \tag{13}$$

Faces $(F1)$ and $(F3)$ simply commutes by inclusion. Face $(F2)$ commutes as $\tilde{\phi}_\lambda$ is induced by a deformation retract. Each faces of diagram 13 are commutative, hence diagram 13 (and equivalently diagram 12) is commutative. $\square$

Proposition 6. *Let $f \in S_d(L, \alpha, R)$ then f is q-tame.*

Proof. Let $s \in \mathbb{N}$ and $\mathbb{V}_{s,f}$ the persistent module (for the $s-$th homology) associated to the sublevel filtration, $\mathcal{F}$ and for fixed levels $\lambda < \lambda'$ let denote $v_\lambda^{\lambda'}$ the associated map. Let $\lambda \in \mathbb{R}$ and $h < \frac{R}{2}$. By Lemma 5, $v_\lambda^{\lambda+L(1+3^\alpha)h^\alpha} = \phi_\lambda \circ \tilde{i}_\lambda$, with $\tilde{i}_\lambda : H_s\left(\mathcal{F}_\lambda\right) \to H_s\left(\mathcal{F}^h_\lambda\right)$. And, due to sublevel thickening by h, $\overline{\mathcal{F}_\lambda} \subset \mathcal{F}^h_\lambda$, and consequently $\tilde{i}_\lambda$ is of finite rank. Thus, $v_\lambda^{\lambda+(\sqrt{d}+1)^\alpha L h^\alpha}$ is of finite rank for all $0 < h < \frac{R}{2}$. As for any $\lambda < \lambda' < \lambda''$, $v_\lambda^{\lambda''} = v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'}$ we then have that $v_\lambda^{\lambda'}$ is of finite rank for all $\lambda < \lambda'$. Hence, f is q-tame. $\square$

Proposition 7. *Let $f \in S_d(L, \alpha, R)$ then, for all $s \in \mathbb{N}$, $\widehat{\mathbb{V}}^h_{s,f}$ is q-tame.*

Proof. Let $h > 0$ and $\lambda \in \mathbb{R}$. $\widehat{\mathcal{F}}_{\lambda,h}$ is a union of hypercubes of the regular grid G_h, thus, $H_s\left(\widehat{\mathcal{F}}_{\lambda,h}\right)$ is finite dimensional. Thus $\widehat{\mathbb{V}}_{s,f}$ is q-tame by Theorem 1.1 of Crawley-Boevey (2012). $\square$

B Proofs of technical lemmas

B.1 Proof of the continuity of $F_{\lambda,h}$

This section is devoted to the proof of the deformation retract $F_{\lambda,h}$, introduced in the proof of Proposition 1.

Lemma 6. *Let $h > 0$ and $\lambda \in \mathbb{R}$, $F_{\lambda,h}$ is continuous.*

Proof. Let $\delta, \delta' > 0$, $x, y \in \mathcal{K}_{\lambda,h}$ such that $||x-y||_2 \leq \delta$ and $t, s \in [0,1]$ a such that $|t-s| \leq \delta'$.

If $x \in \overline{M}_i$ and $y \in \overline{M}_j$, $i \neq j$, then $||x-\xi(x)||_2 \leq \delta$, $||y-\xi(y)||_2 \leq \delta$, and thus, $||\xi(x)-\xi(y)||_2 \leq 2\delta$. Let $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)] \cap M_i$ and $d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq 2h - ||x-\xi(x)||_2$, and if $\left(2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right)\right)_+ > 0$, for sufficiently small δ, we would have,

$$||x-\xi(x)||_2 \leq \delta < \left(2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right)\right)_+$$

which is contradictory. Hence, we can suppose $\left(2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right)\right)_+ = 0$. Then, as $F_{\lambda,h}\left(\xi(x), t\right) = \xi(x)$

$$||F_{\lambda,h}(x,t) - F_{\lambda,h}(\xi(x),t)||_2 = (1-t)||x-\xi(x)||_2 \leq \delta.$$

Otherwise, $F_{\lambda,h}(x,t) = x$, and directly,

$$||F_{\lambda,h}(x,t) - F_{\lambda,h}(\xi(x),t)||_2 = ||x-\xi(x)||_2 \leq \delta.$$

following the same reasoning we also have,

$$||F_{\lambda,h}(y,s) - F_{\lambda,h}(\xi(y),s)||_2 \leq \delta.$$

Then,

$$\begin{aligned}
||F_{\lambda,h}(x,t) - F_{\lambda,h}(y,s)||_2 &\leq ||F_{\lambda,h}(x,t) - F_{\lambda,h}(\xi(x),t)||_2 \\
&\quad + ||F_{\lambda,h}(\xi(x),t) - F_{\lambda,h}(\xi(y),s)||_2 \\
&\quad + ||F_{\lambda,h}(\xi(y),s) - F_{\lambda,h}(y,s)||_2 \\
&\leq 2\delta + ||\xi(x)-\xi(y)||_2 \leq 4\delta
\end{aligned}$$

From now, we suppose that $x, y \in \overline{M}_i$.

If $x \notin \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ or $d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq 2h - ||x-\xi(x)||_2$ and $y \notin \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ or $d_2\left(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq 2h - ||y-\xi(y)||_2$. Then, directly,

$$||F_{\lambda,h}(x,t) - F_{\lambda,h}(y,s)||_2 = ||x-y||_2 \leq \delta.$$

If $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq 2h - ||x-\xi(x)||_2$, and $y \notin \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$. Then, $y \in \mathcal{F}_{\lambda+Lh^\alpha}$, thus,

$$d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \leq ||x-\xi(x)|| + ||x-y||_2 \leq h + \delta$$

and,

$$2h - d_2\left(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq h - \delta \geq ||x-\xi(x)|| - \delta.$$

Consequently,

$$\left\|\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2} - x\right\|_2 \leq \delta$$

Then,

$$\begin{aligned}
&||F_{\lambda,h}(x,t) - F_{\lambda,h}(y,s)||_2 \\
&= \left\|(1-t)x + t\left(\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2}\right) - y\right\|_2 \\
&\leq ||x - y||_2 \quad + \left\|\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2} - x\right\|_2 \\
&\leq 2\delta
\end{aligned}$$

If $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 2h - ||x - \xi(x)||_2$ and $y \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) < 2h - ||y - \xi(y)||_2$. Then,

$$\begin{aligned}
2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) &= 2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\
&\quad + d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\
&\geq 2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) - ||x - y||_2 \\
&\geq ||y - \xi(y)||_2 - ||x - y||_2 \\
&\geq ||x - \xi(x)||_2 - ||\xi(x) - \xi(y)||_2 - 2||x - y||_2.
\end{aligned}$$

Hence,

$$\left\|\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2} - x\right\|_2 \leq 2||x - y||_2 + ||\xi(x) - \xi(y)||_2$$

And thus, we have,

$$\begin{aligned}
&||F_{\lambda,h}(x,t) - F_{\lambda,h}(y,s)||_2 \\
&= \left\|(1-t)x + t\left(\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2}\right) - y\right\|_2 \\
&\leq ||x - y||_2 \quad + \left\|\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2} - x\right\|_2 \\
&\leq 3||x - y||_2 + ||\xi(x) - \xi(y)||_2 \\
&\leq 3\delta + ||\xi(x) - \xi(y)||_2
\end{aligned}$$

and we conclude by continuity of ξ.

Finally, if $x \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 2h - ||x - \xi(x)||_2$ and $y \in \bigcup_{x \in S_{\lambda,h}} [x, \xi(x)]$ and $d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 2h - ||y - \xi(y)||_2$. Then,

$$\begin{aligned}
&||F_{\lambda,h}(x,t) - F_{\lambda,h}(y,s)||_2 \\
&= \left\|(1-t)x + t\left(\xi(x) + (2h - d_2(\xi(x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{x - \xi(x)}{||x - \xi(x)||_2}\right)\right. \\
&\quad \left. -(1-t)y - t\left(\xi(y) + (2h - d_2(\xi(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+ \frac{y - \xi(y)}{||y - \xi(y)||_2}\right)\right\|_2
\end{aligned}$$

and the conclusion follows again in this case by continuity of ξ. □

B.2 Proof of the continuity of $H_{\lambda,h}$

This section is devoted to the proof of the deformation retract $H_{\lambda,h}$, introduced in the proof of Proposition 2.

Lemma 7. *Let $h > 0$ and $\lambda \in \mathbb{R}$, $H_{\lambda,h}$ is continuous.*

Proof. Let $\delta, \delta' > 0$, $x, y \in \mathcal{K}_{\lambda,h}$ such that $||x-y||_2 \leq \delta$. $t, s \in [0,1]$ a such that $|t-s| \leq \delta'$.

if $x \in \overline{M}_i$ and $y \notin \overline{M}_i$, Assumptions **A3** ensures that for sufficiently small δ, there exists $j \in \{1, ..., l\}$, with $y \in M_j$ such that,

$$B_2(x,h) \subset B_2(y,2h) \subset \overline{M_i} \cup \overline{M_j}.$$

By Assumption **A1** and **A2**, this implies that $B_2(x,h) \subset \mathcal{F}_{\lambda+L2^\alpha h^\alpha}$ and thus $x \in \mathcal{F}^{-h}_{\lambda+L2^\alpha h^\alpha}$. From the same reasoning, it follows that $y \in \mathcal{F}^{-h}_{\lambda+L2^\alpha h^\alpha}$. Hence,

$$||H_{\lambda,h}(x,t) - H_{\lambda,h}(y,s)||_2 = ||x-y||_2 \leq \delta.$$

From now, we can suppose that $x, y \in \overline{M}_i$.

If $x \notin \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ or $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$ and $y \notin \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ or $d_2\left(\gamma_{\lambda,h}(y), M_i \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||y - \gamma_{\lambda,h}(y)||_2$, then directly,

$$||H_{\lambda,h}(x,t) - H_{\lambda,h}(y,s)||_2 = ||x-y||_2 \leq \delta.$$

If $x \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$ and $y \notin ((\partial M_i \cap]0,1[^d)^h)^\circ$, then, $d_2\left(x, \partial M_i \cap]0,1[^d\right) \geq h - \delta$ and thus $||x - \gamma_{\lambda,h}(x)||_2 \leq \delta$. As $H_{\lambda,h}(x,1) \in [x, \gamma_{\lambda,h}(x)]$, we have,

$$||H_{\lambda,h}(x,1) - H_{\lambda,h}(y,1)||_2 = ||H_{\lambda,h}(x,1) - y||_2 \leq ||x-y||_2 + ||x - H_{\lambda,h}(x,1)||_2 \leq 2\delta.$$

If $x \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$ and $y \in \left(\bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)]\right)^c \cap \left(\left(\partial M_i \cap]0,1[^d\right)^h\right)^\circ$, then $y \in \mathcal{F}^{-h}_{\lambda+2Lh^\alpha}$. Thus, for sufficiently small δ there exists $z \in (\overline{M}_i)^c \cap B_2(y,h)$ and,

$$d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) - 2h \leq ||x-y||_2 \leq \delta.$$

Hence,

$$\left\|\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2} - x\right\|_2 \leq \delta$$

and,

$$\begin{aligned}
&||H_{\lambda,h}(x,t) - H_{\lambda,h}(y,s)||_2 \\
&= \left\|(1-t)x + t\left(\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2}\right) - y\right\|_2 \\
&\leq \left\|\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2} - x\right\|_2 + ||x-y||_2 \\
&\leq 2\delta
\end{aligned}$$

If $x \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$ and $y \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) < 3h - ||y - \gamma_{\lambda,h}(y)||_2$, then,

$$\begin{aligned} 3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) &\geq ||y - \gamma_{\alpha,h}(y)||_2 - ||x - y||_2 \\ &\geq ||x - \gamma_{\alpha,h}(x)||_2 - ||\gamma_{\lambda,h}(x) - \gamma_{\alpha,h}(y)||_2 - 2||x - y||_2 \end{aligned}$$

Thus,

$$\begin{aligned} &||H_{\lambda,h}(x,t) - H_{\lambda,h}(y,s)||_2 \\ &= \left\| (1-t)x + t\left(\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2}\right) - y \right\|_2 \\ &\leq \left\| \gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2} - x \right\|_2 + ||x - y||_2 \\ &\leq ||\gamma_{\lambda,h}(x) - \gamma_{\alpha,h}(y)||_2 + 3||x - y||_2 \\ &\leq 3\delta + ||\gamma_{\lambda,h}(x) - \gamma_{\alpha,h}(y)||_2 \end{aligned}$$

and we conclude, in this case, by continuity of $\gamma_{\lambda,h}$.

Finally, if $x \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right) \geq 3h - ||x - \gamma_{\lambda,h}(x)||_2$ and $y \in \bigcup_{x \in P_{\lambda,h}} [x, \gamma_{\lambda,h}(x)] \cap M_i$ and $d_2\left(\gamma_{\lambda,h}(y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}\right) \geq 3h - ||y - \gamma_{\lambda,h}(y)||_2$, then,

$$\begin{aligned} &||H_{\lambda,h}(x,t) - H_{\lambda,h}(y,s)||_2 \\ &= \Bigg\| (1-t)x + t\left(\gamma_{\lambda,h}(x) + \left(3h - d_2\left(\gamma_{\lambda,h}(x), (\overline{M}_i)^c \cap \mathcal{F}_{\lambda+2Lh^\alpha}\right)\right)_+ \frac{x - \gamma_{\lambda,h}(x)}{||x - \gamma_{\lambda,h}(x)||_2}\right) \\ &\quad -(1-t)y - t\left(\gamma_{\lambda,h}(y) + \left(3h - d_2\left(\gamma_{\lambda,h}(y), (\overline{M}_i)^c \cap \mathcal{F}^{-h}_{\lambda+Lh^\alpha}\right)\right)_+ \frac{y - \gamma_{\lambda,h}(y)}{||y - \gamma_{\lambda,h}(y)||_2}\right) \Bigg\|_2 \end{aligned}$$

and again the conclusion, follows in this case, by continuity of $\gamma_{\lambda,h}$. □

B.3 Proof of Lemma 1

Proof. Let $x \in [0,1]^d$ such that $d\left(x, \partial M_i \cap]0,1[^d\right) = r < R/2$. Let denote $B_{x,h}$, the Euclidean ball centered in $\xi(x) + h\frac{x - \xi(x)}{||x - \xi(x)||_2}$ of radius h. By theorem 4.8 of Federer (1959), $\forall h < R$,

$$B_{x,h} \subset \left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right)^c \cup \{\xi(x)\}\,.$$

By definition of the closest point $||x - \xi(x)||_2 \leq d_2\left(x, \partial M_i \cap]0,1[^d\right) = r$, and thus $x \in B_{x,r}$. Consequently, $B_2(x,r) \subset B_{x,2r}$. Now, as $2r < R$,

$$B_{x,2r} \subset \left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right)^c \cup \{\xi(x)\}$$

and thus,

$$B_2(x,r) \subset \left(\bigcup_{i=1}^{l} \partial M_i \cap]0,1[^d\right)^c \cup \{\xi(x)\}\,.$$

By assumption $B_2(x,r) \cap \partial M_i \cap]0,1[^d \neq \emptyset$, hence, $\xi(x) \in \partial M_i \cap]0,1[^d$. □

B.4 Proof of Lemma 2

Proof. Let consider here the case where in $H^{'} \subset \mathcal{F}_{\lambda - ||W||_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}}$ (The proof being the same in both cases). Note that,

$$\begin{aligned}
&\int_{H'} dX - \int_{H'} \lambda \\
&= \int_{H'} (f - \lambda) + \theta \int_{H'} dW \\
&\leq -||W||_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha} |H^{'}| + ||W||_{cube,h_{\theta,\alpha}} \theta \omega(h_{\theta,\alpha}^{d}) \\
&\leq ||W||_{cube,h_{\theta,\alpha}} \left(-h_{\theta,\alpha}^{d+\alpha} + \theta \omega(h_{\theta,\alpha}^{d}) \right) < 0
\end{aligned}$$

by the choice made for $h_{\theta,\alpha}$. □

B.5 Proof of Lemma 3

Proof. Let B_1 the Euclidean closed ball centered in $\xi(x) + R\frac{x-\xi(x)}{\|x-\xi(x)\|_2}$ of radius R and B_2 the Euclidean closed ball centered in $\xi(x) - R\frac{x-\xi(x)}{\|x-\xi(x)\|_2}$ of radius R. By Assumption **A3**, $B_1 \subset \overline{M_i}$ and $B_2 \subset \overline{M_i^c}$. Then, the Hausdorff distance between $B_2(\xi(x), Kh) \cap M_i$ and $B_2(\xi(x), Kh) \cap \underline{P}$ is upper bounded by the Hausdorff distance between union of sphere $\partial B_1 \cup \partial B_2$ intersected with $B_2(\xi(x), Kh)$ and the intersection with $B_2(\xi(x), Kh)$ of the hyperplane,

$$P = \left\{ z \in [0,1]^d \text{ s.t. } \left\langle z, \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle = \left\langle \xi(x), \frac{x - \xi(x)}{\|x - \xi(x)\|_2} \right\rangle \right\}.$$

By symmetry, this distance is equal to the Hausdorff distance between $\partial B_1 \cap B_2(\xi(x), Kh)$ and $P \cap B_2(\xi(x), Kh)$.

Now, let $x \in \partial B_1 \setminus \{\xi(x)\}$, and $p(x)$ its projection on P. Let Q the plane containing x, $p(x)$ and $\xi(x)$, Q intersects ∂B_1 into a circle C of radius R and intersects P into a line D tangent to C. The problem then simplify to upper bounding the distance between a circle and a tangent line around the intersection point. Without loss of generality, we can suppose that we are in $\mathbb{R}^2$, C being the circle of radius R centered at $(0, R)$ and D the line $y = 0$ (tangent to C at $(0, 0)$). In $B((0,0), Ch)$, as $Ch < R$, C can be described as,

$$C = \left\{ (x, y) \in B((0,0), Ch) \text{ s.t. } y = R - \sqrt{R^2 - x^2} \right\}.$$

Hence the distance between C and D in $B((0,0), Ch)$ is upper bounded by,

$$R - \sqrt{R^2 - (Ch)^2} = \frac{C^2}{2R} h^2 + O(h^3)$$

and the result follows. □

C Extension to non-parametric regression

The model 1 proves to be valuable for establishing theoretical results. However, it has a limitation as it assumes the observation of a complete trajectory, making it less popular for practical applications. In this section, we focus on proposing extensions to another essential non-parametric model with greater practical interest: non-parametric regression. The proofs of the main results are essentially the same, we detail only the few differences.

We consider the classical non-parametric regression setting (with fixed regular design), observing $n = N^d$ points,

$$X_i = f(x_i) + \sigma\varepsilon_i$$

with x_i a point on the regular N^d grid G_n over $[0,1]^d$, σ the level of noise and ε_i a standard Gaussian variable. In this context, we define,

$$\widehat{\mathcal{F}}_{\lambda,h} = \bigcup_{H \in C_{h,\lambda}} H, \text{ with } C_{h,\lambda} = \left\{ H \in C_h \text{ such that } \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} X_i \leq \lambda \right\}.$$

The key here to lift the convergence results established in Section 2 in this context is to show an analogous inclusion from the one obtained in Proposition 3, then the exact same reasoning applies. All we have to provide is similar noise control. For $h > 0$, let denote the variable,

$$N_h = \frac{\max_{H \in C_h} \left| \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma\varepsilon_i \right|}{\sqrt{2\sigma^2 \frac{\log(1/h^d)}{\lfloor Nh \rfloor^d}}}$$

Lemma 4 bis. *Let $h > {}^1/_N$,*

$$\mathbb{P}\left(N_h \geq t\right) \leq 2\left(\frac{1}{h}\right)^d \exp\left(-t^2 \log\left(1/h^d\right)\right).$$

Proof. Let $h > {}^1/_N$ and $H \subset [0,1]^d$ be a closed hypercube of side h. As the $(\varepsilon_i)_{i=1,\dots,n}$ are i.i.d and standard Gaussian variables, we have, for all $H \in C_h$,

$$\mathbb{P}\left(\left| \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma\varepsilon_i \right| \geq t\right) \leq 2\exp\left(-\frac{|\{x_i \in H\}| t^2}{2\sigma^2}\right).$$

And thus, as the number of point in any $H \in C_h$ is at least to $\lfloor hN \rfloor^d$,

$$\mathbb{P}\left(\left| \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma\varepsilon_i \right| \geq t\right) \leq 2\exp\left(-\frac{\lfloor hN \rfloor^d t^2}{2\sigma^2}\right).$$

Now, by union bound, using $|C_h| = 1/h$,

$$\mathbb{P}\left(\max_{H \in C_h} \left| \frac{1}{|\{x_i \in H\}|} \sum_{x_i \in H} \sigma\varepsilon_i \right| \geq t\right) \leq 2\left(\frac{1}{h}\right)^d \exp\left(-\frac{\lfloor hN \rfloor^d t^2}{2\sigma^2}\right).$$

and the result follows. □

In particular, as in Lemma 4, it follows that N_h is sub-Gaussian, more precisely there exists C_0 and C_1 depending only on d such that, for all h,

$$\mathbb{P}\left(N_h \geq t\right) \leq C_0 \exp(-C_1 t^2).$$

Let now choose, $h_{n,\alpha}$ such that,

$$h_{n,\alpha}^{\alpha} > \sqrt{\frac{\log\left(1/h_{n,\alpha}^d\right)}{\lfloor N h_{n,\alpha} \rfloor^d}}$$

thus, we can choose,

$$h_{n,\alpha} \simeq \left(\frac{\log(n)}{n}\right)^{\frac{1}{d+2\alpha}}.$$

With this choice we obtain the following key lemma.

Lemma 2 bis. *Let $f : [0,1]^d \mapsto \mathbb{R}$. Let $H \subset \mathcal{F}^c_{\lambda+\sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha}} \cap C_{h_{n,\alpha}}$ and $H^{'} \subset \mathcal{F}_{\lambda-\sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha}} \cap C_{h_{n,\alpha}}$. We then have that,*

$$\frac{1}{|\{x_i \in H\}|}\sum_{x_i \in H} X_i > \lambda \text{ and } \frac{1}{|\{x_i \in H'\}|}\sum_{x_i \in H'} X_i < \lambda.$$

Proof. Let consider here the case where in $H^{'} \subset \mathcal{F}_{\lambda-\sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha}}$ (The proof being the same in both cases). We have,

$$\begin{aligned}
&\frac{1}{|\{x_i \in H'\}|}\sum_{x_i \in H'} X_i \\
&= \frac{1}{|\{x_i \in H'\}|}\sum_{x_i \in H'} f(x_i) + \sigma\varepsilon_i \\
&\leq \lambda - \sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha} + N_{h_{n,\alpha}}\sqrt{2\sigma^2}\sqrt{\frac{\log\left(1/h^d_{n,\alpha}\right)}{\lfloor N h_{n,\alpha}\rfloor^d}} \\
&< \lambda
\end{aligned}$$

By the choice made for $h_{n,\alpha}$. ☐

Using Lemma the Lemma 2 bis instead of Lemma 2 in the proof of Proposition 3, we obtain the following analogous proposition.

Proposition 3 bis. *Let $f : [0,1]^d \to \mathbb{R}$. For all $\lambda \in \mathbb{R}$,*

$$\mathcal{F}^{-\sqrt{d}h_{\theta,\alpha}}_{\lambda-\sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha}} \subset \widehat{\mathcal{F}}_{\lambda,h_{n,\alpha}} \subset \mathcal{F}^{\sqrt{d}h_{n,\alpha}}_{\lambda+\sqrt{2\sigma^2}N_{h_{n,\alpha}}h^\alpha_{n,\alpha}}$$

We define $\hat{\mathbb{V}}^h_{f,s}$ and $\widehat{\mathrm{dgm}(f)}$ in the exact same way we did for the Gaussian White Noise model. Again, we can show that this module is q-tame applying the same ideas used in the proofs of Proposition 7.

Having the inclusion given by Proposition 3 bis, the reasoning from the proof of Proposition 5 gives,

Proposition 5 bis. *There exists $\tilde{C}_0$ and $\tilde{C}_1$ such that, for all $t > 0$,*

$$\mathbb{P}\left(\sup_{f \in S_d(L,\alpha,R)} d_b\left(\widehat{\mathrm{dgm}(f)}, \mathrm{dgm}(f)\right) \geq t\left(\frac{\log(n)}{n}\right)^{\frac{\alpha}{d+2\alpha}}\right) \leq \tilde{C}_0 \exp\left(-\tilde{C}_1 t^2\right).$$

From this, we obtain, as in Section 2, upper bounds for estimation. This bound can be shown to be minimax also in this setting (adapting the proof of Theorem 3). Adaptivity also follows as in Section 2.4.