# POINTS OF BOUNDED HEIGHT ON CERTAIN SUBVARIETIES OF TORIC VARIETIES

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Abstract. We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of toric varieties.

Contents



## 1. Introduction

We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of smooth split proper toric varieties. This line of research has been initiated by Blomer and Brüdern [BB18] in the setting of diagonal hypersurfaces in products of projective spaces. Other results in this direction include hypersurfaces and complete intersections in products of projective spaces [Sch16], improvements for bihomogeneous hypersurfaces for degree $(2,2)$ and $(1,2)$ by Browning and Hu [BH19] and Hu [Hu20], as well as generalisations to hypersurfaces in certain toric varieties by Mignot [Mig15, Mig16, Mig18].

The versions of hyperbola method used in all of these articles are rather close to the original one [BB18] for products of projective spaces. In our recent work [PS24] we established a very general form of the hyperbola method for split toric varieties, in which the height condition can also globally be given by the maximum of several monomials. The goal of this article is to show applications of our new hyperbola method. We develop a refined framework for the split torsor method on split smooth proper toric varieties and show that counting results for subvarieties

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of projective spaces can be carried over to toric varieties by a direct application of the hyperbola method [PS24]. With this we can prove new cases of Manin's conjecture [FMT89, BM90] on the number of rational points of bounded height on Fano varieties for certain subvarieties in toric varieties.

The split torsor method provides a parametrisation of rational points on Fano varieties via Cox rings [Sal98, DP20]. The Cox ring of a smooth proper toric variety $X$ is a polynomial ring endowed with a grading by the Picard group of the toric variety [Cox95]. Subvarieties of toric varieties are intersections of hypersurfaces, which are defined by $\mathrm{Pic}(X)$-homogeneous polynomials in the Cox ring of $X$. The subvarieties considered in this paper are defined by homogeneous elements in the Cox ring of the toric variety such that each polynomial involves only variables of the same degree. With the split torsor method parametrisation the height is given by the maximum of a set of monomials and with this in the correct shape to apply our generalised version of the hyperbola method [PS24]. The hyperbola method reduces the counting problem to counting functions over boxes of different shapes. An advantage of our method is that it is already adapted to the shape of height functions appearing. Also compared to earlier versions of the hyperbola method, we do not need estimates for lower dimensional boxes and with this our proofs are relatively short.

We now illustrate our approach on a number of examples. In a similar fashion, it is possible to apply counting results such as [Bir62, HB96, RM18, RM19, BHB17] and many others, to subvarieties of toric varieties defined by elements of the Cox ring each involving only variables of the same degree.

1.1. **Results.** Let $X$ be a smooth split complete toric $\mathbb{Q}$-variety with open torus $T$. Let $\mathbf{D}_1, \dots, \mathbf{D}_s \in \mathrm{Pic}(X)$ be the pairwise distinct classes of the torus invariant prime divisors on $X$. For $i \in \{1, \dots, s\}$, let $n_i = \dim_{\mathbb{Q}} H^0(X, \mathbf{D}_i)$. Let $H_L$ be the height associated to a semiample torus invariant divisor $L$ on $X$ as in Section 2.2.

Our first result concerns subvarieties of toric varieties defined by linear forms.

**Theorem 1.1.** *Let $V \subseteq X$ be a complete intersection of hypersurfaces $H_{i,l}$ with $1 \le i \le s$, $1 \le l \le t_i$ such that $[H_{i,l}] = \mathbf{D}_i$ in $\mathrm{Pic}(X)$ for $i \in \{1, \dots, s\}$ with $t_i \neq 0$. Assume that $V \cap T \neq \emptyset$ and $t_i \le n_i - 2$ for all $i \in \{1, \dots, s\}$. Assume that $L = -(K_X + \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}])$ is ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Q}$-rational points on $V \cap T$ of height $H_L$ at most $B$. Then*

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

*where $b = \mathrm{rk}\,\mathrm{Pic}(V)$, and $c$ is a positive constant, which is defined by* (3.7) *with $k = b - 1$, $C_{M,\mathbf{d}}$ given by* (4.1)*, and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$.*

We use this result as a toy example to show how to combine the hyperbola method with the universal torsor method in the context of rather general smooth split toric varieties. We now move on to results which require deeper understanding of the underlying Diophantine problems via methods from Fourier analysis.

We start with a result that concerns subvarieties of toric varieties defined by bihomogeneous polynomials. It is obtained combining the framework developed in this paper with the hyperbola method [PS24] and preliminary counting results in boxes of different side lengths from [Sch16].

**Theorem 1.2.** *Let $V \subseteq X$ be a smooth complete intersection of hypersurfaces $H_1, \dots, H_t$ of the same degree $e_1\mathbf{D}_1 + e_2\mathbf{D}_2$ in $\mathrm{Pic}(X)$. Assume that $V \cap T \neq \emptyset$, that $n_i - te_i \ge 2$ for $i \in \{1, 2\}$, and that $n_1 + n_2 > \dim V_1^* + \dim V_2^* + 3 \cdot 2^{e_1+e_2} e_1 e_2 t^3$, where $V_1^*, V_2^* \subseteq \mathbb{A}^{n_1+n_2}$ are affine varieties defined in §5. Assume that $L = -([K_X] + [H_1 + \cdots + H_t])$ is ample. Then there is an open subset $W \subseteq X$ such that the number*

*$N_{V,W}(B)$ of $\mathbb{Q}$-rational points on $V \cap W \cap T$ of height $H_L$ at most $B$ satisfies*

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s)$$

*for $B > 0$, where $b = \operatorname{rk}\operatorname{Pic}(V)$, and $c$ is defined in (3.7) with $k = b-1$, $C_{M,\mathbf{d}}$ given by (5.1), $\varpi_i = n_i - te_i$ for $i \in \{1,2\}$, and $\varpi_i = n_i$ for $i \in \{3,\dots,s\}$. The constant $c$ is positive if $V(\mathbb{Q}_v) \neq \emptyset$ for all places $v$ of $\mathbb{Q}$.*

Theorems 1.1 and 1.2 are compatible with Manin's conjecture [FMT89], as $L|_V = -K_V$ by adjunction. The proofs in Sections 4 and 5 yield an asymptotic formula even if we drop the ampleness assumption on $L$.

Theorem 1.2 as well as work of Mignot [Mig16, Mig18] include the case of certain hypersurfaces in products of projective spaces. However, in comparison to Mignot's work we do not require the condition that the effective cone of the toric variety is simplicial. An example of a split toric variety with non-simplicial effective cone, where our theorem applies, is the blow-up at a torus-invariant point of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times Y$ where $n_1, n_2$ are sufficiently large and $Y$ is a split del Pezzo surface of degree 6.

Our last result concerns sets of Campana points in the sense of [PSTVA21] for subvarieties defined by diagonal equations. We introduce the following integral models. Let $\mathscr{X}$ be the $\mathbb{Z}$-toric scheme defined by the fan of $X$. For $i \in \{1,\dots,s\}$, let $\mathscr{D}_{i,1},\dots,\mathscr{D}_{i,n_i}$ be the torus invariant prime divisors on $\mathscr{X}$ of class $\mathbf{D}_i$.

**Theorem 1.3.** *Let $V \subseteq X$ be an intersection of hypersurfaces $H_1,\dots,H_t$ such that $H_i$ is defined by a homogeneous diagonal polynomial in the Cox ring of $X$ of degree $e_i\mathbf{D}_i$ in $\operatorname{Pic}(X)$ and with none of the coefficients equal to zero. Let $\mathscr{V}$ be the closure of $V$ in $\mathscr{X}$. For $i \in \{1,\dots,s\}$, fix integers $2 \leq m_{i,1} \leq \dots \leq m_{i,n_i}$. Let $\mathscr{D}_{\mathbf{m}} = \sum_{i=1}^{s}\sum_{j=1}^{n_i}(1-\frac{1}{m_{i,j}})\mathscr{D}_{i,j}$. Assume that $V \cap T \neq \emptyset$, that $n_1,\dots,n_t \geq 2$, and for $i \in \{1,\dots,s\}$, that $\sum_{j=1}^{n_i}\frac{1}{m_{i,j}} > 3$, that $\sum_{j=1}^{n_i-1}\frac{1}{e_i m_{i,j}(e_i m_{i,j}+1)} \geq 1$ if $e_i = 1$, and $\sum_{j=1}^{n_i}\frac{1}{2s_0(e_i m_{i,j})} > 1$ if $e_i \geq 2$, where $s_0(e_i m_{i,j})$ is defined in Lemma 6.1. Let $L = -(K_X + \mathscr{D}_{\mathbf{m}}|_X + H_1 + \dots + H_t)$ be ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Z}$-Campana points on $(\mathscr{V}, \mathscr{D}_{\mathbf{m}}|_{\mathscr{V}})$ that lie in $T$ and have height $H_L$ at most $B$. Then*

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

*where $b = \operatorname{rk}\operatorname{Pic}(V)$, and $c$ is defined in (3.7) with $k = b-1$, $C_{M,\mathbf{d}}$ given by (6.11), and $\varpi_1,\dots,\varpi_s$ given by (6.10).*

The order of growth in Theorem 1.3 is compatible with the Manin-type conjecture for Campana points [PSTVA21], as $L|_V$ is the log anticanonical divisor of the pair $(V, \mathscr{D}_{\mathbf{m}}|_V)$ by adjunction.

Due to the range of application of the circle method, Theorems 1.2 and 1.3 require the Cox ring of the toric variety to have a large number of variables of the same degree. This holds for toric varieties with several torus invariant prime divisors of the same degree and for products of such toric varieties. Here are some examples: the Cox ring of the projective space $\mathbb{P}^n$ has $n+1$ variables of the same degree, the Cox ring of the blow-up of the projective space $\mathbb{P}^n$ at $l < n+1$ torus invariant points has $n+1-l$ variables of the same degree. Another example is given by blow-ups of products of toric varieties each with several torus invariant prime divisors of the same degree. Indeed, if $X$ and $Y$ are smooth split toric varieties and the Cox ring of $X$ has $n_X$ variables of the same degree $d_X$, and the Cox ring of $Y$ has $n_Y$ variables of the same degree $d_Y$, and $P \in X \times Y$ is a point where $m_X \leq n_X$ variables of degree $d_X$ vanish and $m_Y \leq n_Y$ variables of degree $d_Y$ vanish, then the Cox ring of the blow-up of $X \times Y$ at $P$ has $m_X$ variables of the same degree $d_X - e$ and $m_Y$ variables of the same degree $d_Y - e$, where $e$ is the class of the exceptional divisor.

The structure of this article is as follows. In Section 2 we reformulate the height function and the multiplicative function $\mu$ for Möbius inversion according to the principle of grouping variables of the same degree. In Section 3 we combine the new framework with the hyperbola method developed in [PS24] to obtain a general counting tool for points of bounded height on subvarieties of toric varieties. Theorems 1.1, 1.2, and 1.3 are proven in Sections 4, 5, and 6, respectively.

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## 2. Toric varieties setting

Let $\Sigma$ be the fan of a complete smooth split toric variety $X$ over a number field $\mathbb{K}$. We denote by $\{\mathbf{D}_1, \dots, \mathbf{D}_s\} \subseteq \operatorname{Pic}(X)$ the set of degrees of prime torus invariant divisors of $X$. For each $i \in \{1, \dots, s\}$ we denote by $D_{i,1}, \dots, D_{i,n_i}$ the torus invariant divisors of degree $\mathbf{D}_i$, and by $\rho_{i,1}, \dots, \rho_{i,n_i}$ the corresponding rays of $\Sigma$. Let $\mathcal{I} := \{(i,j) \in \mathbb{N}^2 : 1 \leq i \leq s, 1 \leq j \leq n_i\}$. Let $\Sigma_{\max}$ be the set of maximal cones of $\Sigma$. For each maximal cone $\sigma$ of $\Sigma$, let $\mathcal{J}_\sigma := \{(i,j) \in \mathcal{I} : \rho_{i,j} \subseteq \sigma\}$, let $\mathcal{I}_\sigma = \mathcal{I} \smallsetminus \mathcal{J}_\sigma$, and let $I_\sigma$ be the set of indices $i \in \{1, \dots, s\}$ such that $\{(i,1), \dots, (i, n_i)\} \cap \mathcal{I}_\sigma \neq \emptyset$.

Let $\mathscr{X}$ be the toric scheme defined by $\Sigma$ over $\mathcal{O}_\mathbb{K}$, and for each $(i,j) \in \mathcal{I}$, let $\mathscr{D}_{i,j}$ be the closure of $D_{i,j}$ in $\mathscr{X}$.

Let $R$ be the polynomial ring over $\mathcal{O}_\mathbb{K}$ with variables $x_{i,j}$ for $(i,j) \in \mathcal{I}$ and endowed with the $\operatorname{Pic}(X)$-grading induced by assigning degree $\mathbf{D}_i$ to the variable $x_{i,j}$ for all $(i,j) \in \mathcal{I}$. For every torus invariant divisor $D = \sum_{i=1}^s \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ on $X$ and every vector $\mathbf{x} = (x_{i,j})_{(i,j) \in \mathcal{I}} \in \mathbb{C}^\mathcal{I}$, we write

$$\mathbf{x}^D := \prod_{i=1}^s \prod_{j=1}^{n_i} x_{i,j}^{a_{i,j}}.$$

By [Sal98, §8], $\mathscr{X}$ has a unique universal torsor $\pi : \mathscr{Y} \to \mathscr{X}$, and $\mathscr{Y} \subseteq \mathbb{A}^{\sharp\mathcal{I}}_{\mathcal{O}_\mathbb{K}}$ is the open subset whose complement is defined by $\mathbf{x}^{D_\sigma} = 0$ for all maximal cones $\sigma$ of $\Sigma$, where $D_\sigma := \sum_{(i,j) \in \mathcal{I}_\sigma} D_{i,j}$ for all $\sigma \in \Sigma_{\max}$.

Let $r$ be the rank of $\operatorname{Pic}(X)$. Let $\mathcal{C}$ be a set of ideals of $\mathcal{O}_\mathbb{K}$ that form a system of representatives for the class group of $\mathbb{K}$. As in [PS24, §6.1], we fix a basis of $\operatorname{Pic}(X)$, and for every divisor $D$ on $X$ we write $\mathfrak{c}^{[D]} := \prod_{i=1}^r \mathfrak{c}_i^{b_i}$ where $[D] = (b_1, \dots, b_r)$ with respect to the fixed basis of $\operatorname{Pic}(X)$. Then, as in [Pie16, §2],

$$X(\mathbb{K}) = \mathscr{X}(\mathcal{O}_\mathbb{K}) = \bigsqcup_{\mathfrak{c} \in \mathcal{C}^r} \pi^\mathfrak{c} \left(\mathscr{Y}^\mathfrak{c}(\mathcal{O}_\mathbb{K})\right),$$

where $\pi^\mathfrak{c} : \mathscr{Y}^\mathfrak{c} \to \mathscr{X}$ is the twist of $\pi$ defined in [FP16, Theorem 2.7]. The fibers of $\pi|_{\mathscr{Y}^\mathfrak{c}(\mathcal{O}_\mathbb{K})}$ are all isomorphic to $(\mathcal{O}_\mathbb{K}^\times)^r$, and $\mathscr{Y}^\mathfrak{c}(\mathcal{O}_\mathbb{K}) \subseteq \mathcal{O}_\mathbb{K}^\mathcal{I}$ is the subset of points $\mathbf{x} \in \bigoplus_{(i,j) \in \mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$ that satisfy

$$\sum_{\sigma \in \Sigma_{\max}} \mathbf{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \mathcal{O}_\mathbb{K}. \tag{2.1}$$

Let $N$ be the lattice of cocharacters of $X$. Then $\Sigma \subseteq N \otimes_\mathbb{Z} \mathbb{R}$. For every $(i,j) \in \mathcal{I}$, let $\nu_{i,j}$ be the unique generator of $\rho_{i,j} \cap N$. For every torus invariant $\mathbb{Q}$-divisor $D = \sum_{i=1}^s \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ of $X$ and for every $\sigma \in \Sigma_{\max}$, let $u_{\sigma,D} \in \operatorname{Hom}_\mathbb{Z}(N, \mathbb{Q})$

be the character determined by $u_{\sigma,D}(\nu_{i,j}) = a_{i,j}$ for all $(i,j) \in \mathcal{J}_\sigma$, and define $D(\sigma) := D - \sum_{i=1}^{s} \sum_{j=1}^{n_i} u_{\sigma,D}(\nu_{i,j}) D_{i,j}$. Then $D$ and $D(\sigma)$ are linearly equivalent.

2.1. **Torus invariant divisors.** Here we collect some properties of toric varieties and their torus invariant divisors.

**Lemma 2.1.**

*(i) Let $\sigma \in \Sigma_{\max}$.*
   *(a) For $i \in I_\sigma$, there is a unique index $j_{i,\sigma} \in \{1, \ldots, n_i\}$ such that $(i, j_{i,\sigma}) \in \mathcal{I}_\sigma$. So $\sharp I_\sigma = \sharp \mathcal{I}_\sigma = r$.*
   *(b) For $i \in I_\sigma$, $(i, j') \in \mathcal{J}_\sigma$ for all $j' \in \{1, \ldots, n_i\} \smallsetminus \{j_{i,\sigma}\}$.*
   *(c) For $i \in \{1, \ldots, s\} \smallsetminus I_\sigma$, $\{(i, 1), \ldots, (i, n_i)\} \subseteq \mathcal{J}_\sigma$.*

*Let $D$ be a torus invariant $\mathbb{Q}$-divisor on $X$. For $\sigma \in \Sigma_{\max}$, write*

$$D(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j}.$$

*For $i \in \{1, \ldots, s\}$, let $\alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$.*

*(ii) Let $\sigma \in \Sigma_{\max}$. Then $D(\sigma) = \sum_{i \in I_\sigma} \alpha_{i,\sigma} D_{i, j_{i,\sigma}}$.*
*(iii) Let $\sigma, \sigma' \in \Sigma_{\max}$. If there are $i \in I_\sigma$ and $j \in \{1, \ldots, n_i\}$ such that $\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'} = \mathcal{J}_\sigma \smallsetminus \{(i,j)\}$, then $I_\sigma = I_{\sigma'}$ and $\alpha_{i',\sigma} = \alpha_{i',\sigma'}$ for all $i' \in \{1, \ldots, s\}$.*
*(iv) Let $\sigma \in \Sigma_{\max}$ and for every $i \in I_\sigma$, let $j_i \in \{1, \ldots, n_i\}$. Then there exists a unique $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for $i \in \{1, \ldots, s\}$.*
*(v) The relation $\sigma \sim \sigma'$ if and only if $I_\sigma = I_{\sigma'}$ defines an equivalence relation on $\Sigma_{\max}$, and the equivalence class of $\sigma$ has cardinality $\prod_{i \in I_\sigma} n_i$.*
*(vi) Let $\mathcal{J} \subseteq \mathcal{I}$ minimal for inclusion and such that $\mathcal{J} \cap \mathcal{I}_\sigma \neq \emptyset$ for all $\sigma \in \Sigma_{\max}$. Let $i \in \{1, \ldots, s\}$ such that $\{(i,1), \ldots, (i, n_i)\} \cap \mathcal{J} \neq \emptyset$. Then $\{(i,1), \ldots, (i, n_i)\} \subseteq \mathcal{J}$.*

*Proof.* Part (i) follows from the fact that $[D_{i,j}] = \mathbf{D}_i$ for all $j \in \{1, \ldots, n_i\}$ and that the set $\{\mathbf{D}_i : i \in I_\sigma\}$ is a basis of $\mathrm{Pic}(X)$ by [CLS11, Theorem 4.2.8] as $X$ is smooth and proper.

Part (ii) follows from part (i) and the fact that by construction $\alpha_{i,j,\sigma} = 0$ whenever $(i,j) \in \mathcal{J}_\sigma$.

For part (iii) we observe that if $\sigma \neq \sigma'$, then $\mathcal{J}_\sigma = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i,j)\}$ and $\mathcal{J}_{\sigma'} = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i, j_{i,\sigma})\}$, where $j_{i,\sigma}$ is the index defined in part (i). Thus $i \in I_\sigma \cap I_{\sigma'}$, and for every index $i' \in \{1, \ldots, s\}$ with $i' \neq i$ we have $\mathcal{J}_\sigma \cap \{(i', 1), \ldots, (i', n_{i'})\} = \mathcal{J}_{\sigma'} \cap \{(i', 1), \ldots, (i', n_{i'})\} \subseteq \mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}$. Recall that $[D(\sigma)] = \sum_{i \in I_\sigma} \alpha_{i,\sigma} \mathbf{D}_i$ and $[D(\sigma')] = \sum_{i \in I_{\sigma'}} \alpha_{i,\sigma'} \mathbf{D}_i$. Now the result follows as $[D(\sigma)] = [D(\sigma')]$ in $\mathrm{Pic}(X)$ and $\{\mathbf{D}_i : i \in I_\sigma\}$ is a basis of $\mathrm{Pic}(X)$.

For part (iv), write $I_\sigma = \{i_1, \ldots, i_r\}$. We construct by induction $\sigma_1, \ldots, \sigma_r$ such that for each $l \in \{1, \ldots, r\}$, $(i_1, j_{i_1}), \ldots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}$, $I_{\sigma_l} = I_\sigma$ and $\alpha_{i,\sigma_l} = \alpha_{i,\sigma}$ for all $i \in \{1, \ldots, s\}$. If $(i_1, j_{i_1}) \in \mathcal{I}_\sigma$, let $\sigma_1 = \sigma$. Otherwise, $(i_1, j_{i_1}) \in \mathcal{J}_\sigma$ and by [Sal98, Lemma 8.9] there is $\sigma_1 \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_1} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \smallsetminus \{(i_1, j_{i_1})\}$. Since $i_1 \in I_\sigma$, by part (iii) we have $I_{\sigma_1} = I_\sigma$ and $\alpha_{i,\sigma_1} = \alpha_{i,\sigma}$ for all $i \in \{1, \ldots, s\}$. Assume that we have constructed $\sigma_{l-1}$ for given $l \leq r$. If $(i_l, j_{i_l}) \in \mathcal{I}_{\sigma_{l-1}}$, let $\sigma_l = \sigma_{l-1}$. Otherwise, $(i_l, j_{i_l}) \in \mathcal{J}_{\sigma_{l-1}}$ and by [Sal98, Lemma 8.9] there is $\sigma_l \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_l} \cap \mathcal{J}_{\sigma_{l-1}} = \mathcal{J}_{\sigma_{l-1}} \smallsetminus \{(i_l, j_{i_l})\}$. Since $i_l \in I_{\sigma_{l-1}}$, by part (iii) we have $I_{\sigma_l} = I_{\sigma_{l-1}} = I_\sigma$ and $\alpha_{i,\sigma_l} = \alpha_{i,\sigma_{l-1}} = \alpha_{i,\sigma}$ for all $i \in \{1, \ldots, s\}$. Since $(i_1, j_{i_1}), \ldots, (i_{l-1}, j_{i_{l-1}}) \in \mathcal{I}_{\sigma_{l-1}}$ and $\mathcal{J}_{\sigma_l} = (\mathcal{J}_{\sigma_{l-1}} \cap \mathcal{J}_{\sigma_l}) \cup \{(i_l, j_{i_l, \sigma_{l-1}})\}$, where $j_{i_l, \sigma_{l-1}}$ is the index defined in part (i), we conclude that $(i_1, j_{i_1}), \ldots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}$. Take $\sigma' = \sigma_r$. The uniqueness of $\sigma'$ follows from part (i), as $\sigma'$ is completely determined by $\mathcal{I}_{\sigma'}$.

Part (v) is a direct consequence of part (iv).

For part (vi), let $j \in \{1, \dots, n_i\}$ such that $(i,j) \in \mathcal{J}$. By minimality of $\mathcal{J}$, there exists $\sigma \in \Sigma_{\max}$ such that $\mathcal{J} \cap \mathcal{I}_\sigma = \{(i,j)\}$. If $n_i > 1$, let $j' \in \{1, \dots, n_i\} \smallsetminus \{j\}$. By [Sal98, Lemma 8.9] there is $\sigma' \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma'} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \smallsetminus \{(i,j')\}$. Hence, $\mathcal{I}_{\sigma'} = (\mathcal{I}_\sigma \smallsetminus \{(i,j)\}) \cup \{(i,j')\}$. Since $\mathcal{J} \cap (\mathcal{I}_\sigma \smallsetminus \{(i,j)\}) = \emptyset$ and $\mathcal{J} \cap \mathcal{I}_{\sigma'} \neq \emptyset$, we conclude that $(i,j') \in \mathcal{J}$. □

2.2. **Heights.** Let $L$ be a semiample torus invariant $\mathbb{Q}$-divisor on $X$. Let $H_L$ be the height on $X$ defined by $L$ as in [PS24, §6.3]. For $\sigma \in \Sigma_{\max}$, write $L(\sigma) = \sum_{i=1}^s \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j}$ and $\alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$ for all $i \in \{1, \dots, s\}$. Let $\Omega_\mathbb{K}$ be the set of places of $\mathbb{K}$.

**Lemma 2.2.** *For every $\nu \in \Omega_\mathbb{K}$ and every $\mathbf{x} \in \mathscr{Y}(\mathbb{K})$, we have*

$$\sup_{\sigma \in \Sigma_{\max}} |x^{L(\sigma)}|_\nu = \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^s \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}.$$

*Proof.* Fix $\nu \in \Omega_\mathbb{K}$ and $\mathbf{x} \in \mathscr{Y}(\mathbb{K})$. By Lemma 2.1(ii), we have

$$\sup_{\sigma \in \Sigma_{\max}} |x^{L(\sigma)}|_\nu = \sup_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} |x_{i,j_{i,\sigma}}|_\nu^{\alpha_{i,\sigma}}.$$

For every $i \in \{1, \dots, s\}$, let $j_i \in \{1, \dots, n_i\}$ such that $|x_{i,j_i}|_\nu = \sup_{1 \le j \le n_i} |x_{i,j}|_\nu$. Let $\sigma \in \Sigma_{\max}$. By Lemma 2.1(iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for all $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Then

$$|x^{L(\sigma')}|_\nu = \prod_{i \in I_{\sigma'}} |x_{i,j_i}|_\nu^{\alpha_{i,\sigma'}} = \prod_{i \in I_\sigma} \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}} = \prod_{i=1}^s \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}. \qquad \square$$

Thus $H_L(\mathbf{x}) = \prod_{\nu \in \Omega_\mathbb{K}} \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^s \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}$ for all $\mathbf{x} \in \mathscr{Y}(\mathbb{K})$.

2.3. **Coprimality conditions.** We now rewrite the coprimality condition (2.1) in terms of the notation introduced in this paper.

**Lemma 2.3.** *For all $\mathbf{x} \in \bigoplus_{(i,j) \in \mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$,*

$$\sum_{\sigma \in \Sigma_{\max}} \mathbf{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \sum_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} (x_{i,1}, \dots, x_{i,n_i}) \mathfrak{c}^{-\mathbf{D}_i}.$$

*Proof.* For $\sigma \in \Sigma_{\max}$, let $X_\sigma = \{\prod_{i \in I_\sigma} x_{i,j_i} : j_i \in \{1, \dots, n_i\} \ \forall i \in \{1, \dots, s\}\}$. The inclusion $\subseteq$ is clear as $\mathbf{x}^{D_\sigma} \in X_\sigma$ and $\mathfrak{c}^{-[D_\sigma]} = \prod_{i \in I_\sigma} \mathfrak{c}^{-\mathbf{D}_i}$ for all $\sigma \in \Sigma_{\max}$. For the converse inclusion, fix $\sigma \in \Sigma_{\max}$ and $x \in X_\sigma$. For every $i \in I_\sigma$, let $j_i \in \{1, \dots, n_i\}$ such that $x = \prod_{i \in I_\sigma} x_{i,j_i}$. By Lemma 2.1(iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$ and $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$ . Then $\mathbf{x}^{D_{\sigma'}} = x$. □

2.4. **Möbius function.** Let $\mathcal{I}_\mathbb{K}$ be the set of nonzero ideals of $\mathcal{O}_\mathbb{K}$. Let $\chi : \mathcal{I}_\mathbb{K}{}^s \to \{0,1\}$ be the characteristic function of the subset

$$\left\{ \mathfrak{b} \in \mathcal{I}_\mathbb{K}{}^s : \sum_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} \mathfrak{b}_i = \mathcal{O}_\mathbb{K} \right\}. \tag{2.2}$$

For every $\mathfrak{d} \in \mathcal{I}_\mathbb{K}{}^s$, let $\chi_\mathfrak{d} : \mathcal{I}_\mathbb{K}{}^s \to \{0,1\}$ be the characteristic function of the subset

$$\left\{ \mathfrak{b} \in \mathcal{I}_\mathbb{K}{}^s : \mathfrak{b}_i \subseteq \mathfrak{d}_i \ \forall i \in \{1, \dots, s\} \right\}.$$

As in [Pey95, Lemme 8.5.1] there exists a unique multiplicative function $\mu : \mathcal{I}_\mathbb{K}{}^s \to \mathbb{Z}$ such that

$$\chi = \sum_{\mathfrak{d} \in \mathcal{I}_\mathbb{K}{}^s} \mu(\mathfrak{d}) \chi_\mathfrak{d}.$$

Note that if $X = \mathbb{P}^n_{\mathbb{Q}}$, the function $\mu$ defined above coincides with the classical Möbius function.

**Remark 2.4.** Let $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ be a prime ideal. The function $\mu$ is defined recursively by the formula $\mu(\mathfrak{b}) = \chi(\mathfrak{b}) - \sum_{\mathfrak{b} \subsetneq \mathfrak{d}} \mu(\mathfrak{d})$ for every $\mathfrak{b} \in {\mathcal{I}_{\mathbb{K}}}^s$, and satisfies the following properties.

(i) $\mu(\mathbf{1}) = \chi(\mathbf{1}) = 1$.

(ii) If $e_i \geq 2$ for some $i \in \{1, \dots, s\}$, then $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$, as in that case $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = \chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s})$ for $e'_i = e_i - 1$ and $e'_l = e_l$ for all $l \neq i$.

(iii) By induction one shows that $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ whenever $(e_1, \dots, e_s) \neq \mathbf{0}$ and there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$, as $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 1$ if and only if there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$.

(iv) Let
$$\tilde{f} := \min \{\sharp J : J \subseteq \{1, \dots, s\}, J \cap I_\sigma \neq \emptyset \ \forall \sigma \in \Sigma_{\max}\}.$$
By property (iii), if $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0$, then there are at least $\tilde{f}$ indices $i$ with $e_i = 1$. Let $J \subseteq \{1, \dots, s\}$ be smallest with respect to inclusion and such that $J \cap I_\sigma \neq \emptyset$ for all $\sigma \in \Sigma_{\max}$. Let $J' = J \smallsetminus \{j\}$ for some $j \in J$. Let $e_i = 1$ for $i \in J$ and $e_i = 0$ for $i \notin J$. Let $e'_i = e_i$ for $i \neq j$ and $e'_j = 0$. Then $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ and $\chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s}) = 1$ by minimality of $J$. Thus $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = -1 \neq 0$. Hence,
$$\tilde{f} = \min \left\{ \sum_{i=1}^s e_i : (e_1, \dots, e_s) \neq \mathbf{0}, \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0 \right\}. \tag{2.3}$$

For $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{R}^s_{\geq 0}$, let
$$f_\beta := \min \left\{ \sum_{i=1}^s \beta_i e_i : (e_1, \dots, e_s) \neq \mathbf{0}, \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0 \right\}.$$

**Lemma 2.5.** *(i) The series*
$$\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^s \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}}$$
*converges absolutely if $f_\beta > 1$.*

*(ii) If $f_\beta > 1$ and $\beta_1, \dots, \beta_s \in \mathbb{Z}_{>0}$, then $\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^s \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} > 0$.*

*Proof.* For part (i) we follow the proof of [Sal98, Lemma 11.15] and [Pie16, Proposition 4]. For $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ prime ideal, let $S(\mathfrak{p}) = \sum_{(e_1, \dots, e_s) \in \mathbb{Z}^s_{\geq 0}} \frac{|\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s})|}{\prod_{i=1}^s \mathfrak{N}(\mathfrak{p})^{\beta_i e_i}}$. As in [Sal98, Lemma 11.15] and [Pie16, Proposition 4],
$$\lim_{b \to \infty} \sum_{\substack{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s \\ \prod_{i=1}^s \mathfrak{N}(\mathfrak{d}_i) \leq b}} \frac{|\mu(\mathfrak{d})|}{\prod_{i=1}^s \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} = \prod_{\mathfrak{p}} S(\mathfrak{p}).$$
By Remark 2.4(ii) the sum $S(\mathfrak{p})$ is finite. By definition of $f_\beta$, if $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0$ and $(e_1, \dots, e_s) \neq \mathbf{0}$, then $f_\beta \leq \sum_{i=1}^s \beta_i e_i$. Thus
$$S(\mathfrak{p}) = 1 + \frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right),$$
where $Q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a monotone increasing function. Since $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s})$ is independent of the choice of $\mathfrak{p}$, the function $Q$ is independent of the choice of $\mathfrak{p}$. Thus
$$\sum_{\mathfrak{p}} \frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right) \leq [\mathbb{K} : \mathbb{Q}] Q(1) \sum_{n \in \mathbb{Z}_{>0}} \frac{1}{n^{f_\beta}}.$$

In part (ii) the series is absolutely convergent by part (i), hence it suffices to show that each factor of its Euler product $\prod_{\mathfrak{p}} S_{\mathfrak{p}}$ is positive. For a prime ideal $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$, let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers of the completion $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$ at the valuation $v_{\mathfrak{p}}$ defined by $\mathfrak{p}$. Endow $\mathbb{K}_{\mathfrak{p}}$ with the Haar measure normalized such that $\mathcal{O}_{\mathfrak{p}}$ has volume 1. Then $\int_{\mathfrak{p}^j \mathcal{O}_{\mathfrak{p}}} dy = \mathfrak{N}(\mathfrak{p})^{-j}$ for all $j \geq 0$ by [CLNS18, §1.1.13] and [Neu99, Proposition II.4.3]. We denote by $\chi$ the characteristic function of (2.2) where ideals of $\mathcal{O}_{\mathbb{K}}$ are replaced by ideals of $\mathcal{O}_{\mathfrak{p}}$. By Remark 2.4(ii),

$$\begin{aligned} S_{\mathfrak{p}} &= \sum_{\boldsymbol{e} \in \{0,1\}^s} \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \prod_{i=1}^{s} \mathfrak{N}(\mathfrak{p})^{-e_i \beta_i} \\ &= \sum_{\boldsymbol{e} \in \{0,1\}^s} \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \prod_{i=1}^{s} \prod_{j=1}^{\beta_i} \int_{\mathfrak{p}^{e_i}} \mathrm{d}y_{i,j} \\ &= \int_{\mathcal{O}_{\mathfrak{p}}^{\sum_{i=1}^s \beta_i}} \chi((y_{1,1}, \dots, y_{1,\beta_1}), \dots, (y_{s,1}, \dots, y_{s,\beta_s})) \prod_{i=1}^{s} \prod_{j=1}^{\beta_i} \mathrm{d}y_{i,j} \\ &\geq \int_{(\mathcal{O}_{\mathfrak{p}}^{\times})^{\sum_{i=1}^s \beta_i}} \prod_{i=1}^{s} \prod_{j=1}^{\beta_i} \mathrm{d}y_{i,j} = \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{\sum_{i=1}^s \beta_i} > 0, \end{aligned}$$

as $\chi$ is a nonnegative function with $\chi(\mathcal{O}_{\mathfrak{p}}, \dots, \mathcal{O}_{\mathfrak{p}}) = 1$. □

**Definition 2.6.** A function $A : \mathbb{Z}_{>0}^s \to \mathbb{R}$ is compatible with Möbius inversion on $X$ if there are $\beta_1, \dots, \beta_s \in \mathbb{R}^s$ such that $A(\mathbf{d}) \ll \prod_{i=1}^s d_i^{-\beta_i}$ with $f_{(\beta_1,\dots,\beta_s)} > 1$.

**Remark 2.7.** (i) The inequality $f_{\beta} > 1$ holds whenever $\beta_1, \dots, \beta_s > 1$.

(ii) If $\beta_1 = \dots = \beta_s = 1$, then $f_{\beta} = \tilde{f}$ by (2.3).

(iii) (Case $\beta_1 = n_1, \dots, \beta_s = n_s$) As in [Sal98, Lemma 11.15(d)], let $f$ be the smallest positive integer such that there are $f$ rays of the fan $\Sigma$ that are not contained in a maximal cone. Then $f \geq 2$, as $X$ is proper. Moreover,

$$f = \min \left\{ \sharp \mathcal{J} : \mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \cap \mathcal{I}_{\sigma} \neq \emptyset \ \forall \sigma \in \Sigma_{\max} \right\},$$

and Remark 2.4 combined with Lemma 2.1(vi) gives

$$\begin{aligned} f &= \min \left\{ \sum_{i \in J} n_i : J \subseteq \{1, \dots, s\}, J \cap I_{\sigma} \neq \emptyset \ \forall \sigma \in \Sigma_{\max}, \sharp J = \tilde{f} \right\} \\ &= \min \left\{ \sum_{i=1}^{s} n_i e_i : (e_1, \dots, e_s) \neq \mathbf{0}, \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0 \right\}. \end{aligned}$$

## 3. Subvarieties

Here we want to count rational points or Campana points of bounded height in subvarieties of toric varieties.

From now on $\mathbb{K} = \mathbb{Q}$. Let $X$ be a complete smooth split toric variety as in Section 2. Assume that $\operatorname{rk} \operatorname{Pic}(X) \geq 2$, that is $X$ is not a projective space. Let $L$ be a semiample toric invariant $\mathbb{Q}$-divisor on $X$ that satisfies [PS24, Assumption 6.3]. The latter holds, for example, if $L$ is ample.

Let $g_1, \dots, g_t \in R$ be $\operatorname{Pic}(X)$-homogeneous elements. Let $V \subseteq X$ be the schematic intersection of the $t$ hypersurfaces defined by $g_1, \dots, g_t$. Let $T \subseteq X$ be the torus. Without loss of generality, we can assume that $V \cap T \neq \emptyset$. Otherwise, $V$ is contained in a complete smooth split toric subvariety $X'$ of $X$, and we can replace $X$ by $X'$. Fix $m_{i,j} \in \mathbb{Z}_{\geq 1}$ for each $(i,j) \in \mathcal{I}$. Let $\mathbf{m} = (m_{i,j})_{(i,j) \in \mathcal{I}}$, and $\mathscr{D}_{\mathbf{m}} = \sum_{i=1}^s \sum_{j=1}^{n_i} \left(1 - \frac{1}{m_{i,j}}\right) \mathscr{D}_{i,j}$. Let $\mathscr{V}$ be the Zariski closure of $V$ in $\mathscr{X}$. Let

$(\mathscr{V}, \mathscr{D}_{\mathbf{m}}|_{\mathscr{V}})(\mathbb{Z})$ be the set of Campana $\mathbb{Z}$-points on the Campana orbifold $(\mathscr{V}, \mathscr{D}_{\mathbf{m}}|_{\mathscr{V}})$ as in [PSTVA21, Definition 3.4], where if $V$ is singular, the intersection multiplicity of a point $\mathbf{x} : \operatorname{Spec} \mathcal{O}_{\mathbb{K}} \to \mathscr{V}$ with $\mathscr{D}_i|_{\mathscr{V}}$ at a place $v$ of $\mathbb{K}$ is defined as the colength of the ideal of the fiber product of $\operatorname{Spec} \mathcal{O}_{\mathbb{K}} \times_{\mathscr{V}} \mathscr{D}_i|_{\mathscr{V}}$ after base change to the completion of $\mathcal{O}_{\mathbb{K}}$ at $v$.

Let $N_V(B)$ be the number of points in $(\mathscr{V}, \mathscr{D}_{\mathbf{m}}|_{\mathscr{V}})(\mathbb{Z}) \cap T(\mathbb{Q})$ of height $H_L$ at most $B$. If $m_{i,j} = 1$ for all $(i,j) \in \mathcal{I}$, then $N_V(B)$ is the set of $\mathbb{Q}$-rational points on $V \cap T$ of height $H_L$ at most $B$.

For $i \in \{1, \ldots, s\}$ and $\mathbf{x} \in \mathscr{Y}(\mathbb{Z})$, let $y_i = \sup_{1 \le j \le n_i} |x_{i,j}|$. For $\sigma \in \Sigma_{\max}$, write $L(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j}$ and $\alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$ for all $i \in \{1, \ldots, s\}$. Then by [PS24, Proposition 6.10] and Lemma 2.2,

$$H_L(\mathbf{x}) = \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} y_i^{\alpha_{i,\sigma}}.$$

By construction, $(\mathscr{V}, \mathscr{D}_{\mathbf{m}}|_{\mathscr{V}})(\mathbb{Z}) = (\mathscr{X}, \mathscr{D}_{\mathbf{m}})(\mathbb{Z}) \cap V(\mathbb{Q})$. We use the torsor parameterization of $(\mathscr{X}, \mathscr{D}_{\mathbf{m}})(\mathbb{Z})$ from [PS24, §6.4]. For $B > 0$ and $\mathbf{d} \in (\mathbb{Z}_{>0})^s$, let $A(B, \mathbf{d})$ be the set of points $\mathbf{x} = (x_{i,j})_{1 \le i \le s, 1 \le j \le n_i} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}}$ such that

$$H(\mathbf{x}) \le B, \tag{3.1}$$
$$d_i \mid x_{i,j} \ \forall i \in \{1, \ldots, s\}, \forall j \in \{1, \ldots, n_i\}, \tag{3.2}$$
$$x_{i,j} \text{ is } m_{i,j}\text{-full } \forall i \in \{1, \ldots, s\}, \forall j \in \{1, \ldots, n_i\}, \tag{3.3}$$
$$g_1 = \cdots = g_t = 0. \tag{3.4}$$

We observe that $A(B, \mathbf{d})$ is a finite set by [PS24, Lemma 6.11]. Then

$$N_V(B) = \frac{1}{2^r} \sum_{\mathbf{d} \in (\mathbb{Z}_{>0})^s} \mu(d) \sharp A(B, \mathbf{d}) \tag{3.5}$$

by Lemma 2.3 and the definition of $\mu$ in Section 2.4.

Write

$$\sharp A(B, \mathbf{d}) = \sum_{\substack{y_1, \ldots, y_s \in \mathbb{Z}_{>0} \\ \prod_{i=1}^{s} y_i^{\alpha_{i,\sigma}} \le B, \ \forall \sigma \in \Sigma_{\max}}} f_{\mathbf{d}}(y_1, \ldots, y_s),$$

where

$$f_{\mathbf{d}}(y_1, \ldots, y_s) = \sharp\{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}} : (3.2), (3.3), (3.4), y_i = \sup_{1 \le j \le n_i} |x_{i,j}| \ \forall i \in \{1, \ldots, s\}\}.$$

Let

$$F_{\mathbf{d}}(B_1, \ldots, B_s) = \sum_{1 \le y_i \le B_i, 1 \le i \le s} f_{\mathbf{d}}(y_1, \ldots, y_s).$$

**Lemma 3.1.** *Assume that*

$$F_{\mathbf{d}}(B_1, \ldots, B_s) = C_{M,\mathbf{d}} \prod_{i=1}^{s} B_i^{\varpi_i} + O\left(C_{E,\mathbf{d}} \left(\min_{1 \le i \le s} B_i\right)^{-\epsilon} \prod_{i=1}^{s} B_i^{\varpi_i}\right) \tag{3.6}$$

*with* $C_{M,\mathbf{d}}, C_{E,\mathbf{d}}, \varpi_1, \ldots, \varpi_s, \epsilon > 0$ *such that* $C_{M,\mathbf{d}}$, $C_{E,\mathbf{d}}$ *are compatible with Möbius inversion on* $X$ *as functions of the variables* $\mathbf{d}$.

*Let* $a$ *be the maximal value of* $\sum_{i=1}^{s} \varpi_i u_i$ *on the polytope* $\mathcal{P} \subseteq \mathbb{R}^s$ *defined by*

$$\sum_{i=1}^{s} \alpha_{i,\sigma} u_i \le 1 \ \forall \sigma \in \Sigma_{\max}, \qquad u_i \ge 0 \ \forall i \in \{1, \ldots, s\}.$$

*Let* $F$ *be the face of* $\mathcal{P}$ *where* $\sum_{i=1}^{s} \varpi_i u_i = a$. *Let* $k$ *be the dimension of* $F$.

(i) *If $F$ is not contained in a coordinate hyperplane of $\mathbb{R}^s$, then*

$$N_V(B) = cB^a(\log B)^k + O(B^a(\log B)^{k-1}(\log\log B)^s),$$

*where $k$ is the dimension of $F$, and*

$$c = (s-1-k)!c_{\mathcal{P}}2^{-r}\sum_{\mathbf{d}\in\mathbb{Z}^s_{>0}}\mu(\mathbf{d})C_{M,\mathbf{d}}. \tag{3.7}$$

*Here, $c_{\mathcal{P}} = \lim_{\delta\to 0}\delta^{k+1-s}\operatorname{meas}_{s-1}(H_\delta\cap\mathcal{P})$, where $H_\delta\subseteq\mathbb{R}^s$ is the hyperplane defined by $\sum_{i=1}^s \varpi_i u_i = a-\delta$, and $\operatorname{meas}_{s-1}$ is the $(s-1)$-dimensional measure on $H_\delta$ given by $\prod_{1\le i\le s, i\neq\tilde{i}}(\varpi_i \mathrm{d}u_i)$ for any choice of $\tilde{i}\in\{1,\dots,s\}$.*

(ii) *If $L$ is ample, then*

$$a = \inf\left\{t\in\mathbb{R} : t[L] - \left[\sum_{i=1}^s \varpi_i\mathbf{D}_i\right] \textit{ is effective}\right\},$$

*and $k+1$ is the codimension of the minimal face of the effective cone of $X$ containing $a[L]-[\sum_{i=1}^s\varpi_i\mathbf{D}_i]$.*

(iii) *If $[L]=\sum_{i=1}^s\varpi_i\mathbf{D}_i$ is ample, then the face $F$ is not contained in a coordinate hyperplane, $a=1$ and $k=\operatorname{rk}\operatorname{Pic}(X)-1$.*

*Proof.* (i) Let $t_i=\varpi_i u_i$ for all $i\in\{1,\dots,s\}$. By the assumptions on $L$, the polytope $\mathcal{P}$ is bounded and nondegenerate by [PS24, Remark 6.2]. Applying [PS24, Theorem 1.1] to $\sharp A(B,\mathbf{d})$ gives

$$N_V(B) = cB^a(\log B)^k + O\left(B^a(\log B)^{k-1}(\log\log B)^s\sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s}\mu(d)C_{E,\mathbf{d}}\right).$$

The sums $\sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s}$ in the leading constant $c$ and in the error term converge absolutely by Lemma 2.5 as $C_{M,\mathbf{d}}$, $C_{E,\mathbf{d}}$ are compatible with Möbius inversion on $X$.

(ii) Let

$$\mathbb{R}^r\hookrightarrow\mathbb{R}^s\hookrightarrow\mathbb{R}^{\mathcal{I}}$$

be the sequence of injective linear maps dual to

$$d:\bigoplus_{(i,j)\in\mathcal{I}} D_{i,j}\mathbb{Z}\twoheadrightarrow\bigoplus_{i=1}^s\mathbf{D}_i\mathbb{Z}\twoheadrightarrow\operatorname{Pic}(X).$$

Here,

$$\mathbb{R}^s\hookrightarrow\mathbb{R}^{\mathcal{I}},\qquad \sum_{i=1}^s u_i e_i\mapsto\sum_{i=1}^s\sum_{j=1}^{n_i}u_i e_{i,j},$$

where $\{e_1,\dots,e_s\}$ denotes the dual basis to $\{\mathbf{D}_1,\dots,\mathbf{D}_s\}$, and $\{e_{i,j} : (i,j)\in\mathcal{I}\}$ denotes the dual basis to $\{D_{i,j} : (i,j)\in\mathcal{I}\}$. Let $\widetilde{P}$ be the polytope defined by

$$\sum_{(i,j)\in\mathcal{I}}\alpha_{i,j,\sigma}u_{i,j}\le 1\quad\forall\sigma\in\Sigma_{\max},\qquad u_{i,j}\ge 0\quad\forall(i,j)\in\mathcal{I}.$$

Then $\widetilde{P}\cap\mathbb{R}^s = P$ and $\sum_{(i,j)\in\mathcal{I}}\frac{\varpi_i}{n_i}u_{i,j}\Big|_P = \sum_{i=1}^s\left(\sum_{j=1}^{n_i}\frac{\varpi_i}{n_i}\right)u_i$. By [PS24, Lemma 6.7], the face $F$ of $\widetilde{P}$ where the maximal value of the function

$$\sum_{(i,j)\in\mathcal{I}}\frac{\varpi_i}{n_i}u_{i,j} \tag{3.8}$$

is attained is contained in $\widetilde{P} \cap \mathbb{R}^r$, and hence also in $P$. Then $a$ is the maximal value of the function (3.8) on $P$. The dual linear programming problem is given by minimizing $\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$ on the polytope given by

$$\sum_{\sigma \in \Sigma_{\max}} \alpha_{i,j,\sigma} \lambda_\sigma \geq \frac{\varpi_i}{n_i} \quad \forall (i,j) \in \mathcal{I}, \qquad \lambda_\sigma \geq 0, \ \forall \sigma \in \Sigma_{\max}.$$

The arguments in [PS24, §6.5.1] show that $a$ is the smallest real number such that $a[L] - \sum_{i=1}^s \sum_{j=1}^{n_i} \frac{\varpi_i}{n_i} \mathbf{D}_i$ is effective. As in [PS24, Proposition 6.13], the smallest face of $\mathrm{Eff}(X)$ that contains $a[L] - \sum_{i=1}^s \varpi_i \mathbf{D}_i$ is dual to the cone generated by $F$ in $\mathbb{R}^r$, and the latter is defined by $a \sum_{i=1}^s \alpha_{i,\sigma} u_i - \sum_{i=1}^s \varpi_i u_i = 0$ for any $\sigma \in \Sigma_{\max}$ such that $F \subseteq \{\sum_{i=1}^s \alpha_{i,j,\sigma} u_{i,j} = 1\}$. Thus the minimal face of $\mathrm{Eff}(X)$ containing $a[L] - [\sum_{i=1}^s \varpi_i \mathbf{D}_i]$ has codimension $k+1$.

(iii) We argue as in the proof of [PS24, Lemma 6.7(ii)]. Let $\widetilde{H} \subseteq \mathbb{R}^s$ be the inclusion dual to the surjection $\bigoplus_{i=1}^s \mathbb{R}\mathbf{D}_i \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\sum_{i=1}^s \alpha_{i,\sigma} u_i = \sum_{i=1}^s \varpi_i u_i$ for all $\mathbf{u} \in \widetilde{H}$ and all $\sigma \in \Sigma_{\max}$. Thus $\mathcal{P} \cap \widetilde{H}$ is the set of elements $\mathbf{u}$ of $\widetilde{H}$ such that $u_1, \ldots, u_s \geq 0$ and $\sum_{i=1}^s \varpi_i u_i \leq 1$. Since $F \subseteq \widetilde{H}$ by [PS24, Lemma 6.7(i)], we have $F = \widetilde{H} \cap \{\sum_{i=1}^s \varpi_i u_i = 1\}$. As in in the proof of [PS24, Lemma 6.7(ii)] we conclude that $F$ is not contained in a coordinate hyperplane of $\mathbb{R}^s$.

□

## 4. Rational points on linear complete intersections

*Proof of Theorem 1.1.* For $1 \leq i \leq s$ and $1 \leq l \leq t_i$, let $g_{i,l} \in R$ be a linear polynomial defining $H_{i,j}$. Then

$$g_{i,l} = \sum_{j=1}^{n_i} c_{i,j,l} x_{i,j}, \quad l \in \{1, \ldots, t_i\},$$

with $c_{i,j,l} \in \mathbb{Z}$, and $g_{i,1}, \ldots, g_{i,t_i}$ are linearly independent for all $i \in \{1, \ldots, s\}$. Let $m_{i,j} = 1$ for all $(i,j) \in \mathcal{I}$. Then

$$F_{\mathbf{d}}(B_1, \ldots, B_s) = \prod_{i=1}^s F_{i,d_i}(B_i),$$

where for $i \in \{1, \ldots, s\}$, $d \in \mathbb{Z}_{>0}$ and $B > 0$,

$$\begin{aligned} F_{i,d}(B) = \sharp\{(x_{i,1}, \ldots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B, \\ d \mid x_{i,j} \ \forall j \in \{1, \ldots, n_i\}, \ g_{i,1} = \cdots = g_{i,t_i} = 0\}. \end{aligned}$$

For $i \in \{1, \ldots, s\}$, let $W_i \subseteq \mathbb{R}^{n_i}$ be the linear space defined by $g_{i,1} = \cdots = g_{i,t_i} = 0$, and let $\Lambda_i \subseteq W_i$ be restriction of the standard lattice $\mathbb{Z}^{n_i} \subseteq \mathbb{R}^{n_i}$ to $W_i$. Then by [BG06, Lemma 11.10.15] for every $T \geq 1$,

$$\begin{aligned} \sharp(\mathbb{Z}^{n_i} \cap [-T,T]^{n_i} \cap W_i) &= \sharp(\Lambda_i \cap (T([-1,1]^n \cap W_i)) \\ &= T^{n_i - t_i} \frac{\mathrm{meas}_{n_i - t_i}([-1,1]^{n_i} \cap W_i)}{\det \Lambda_i} + O\left(T^{n_i - t_i - 1}\right), \end{aligned}$$

where $\mathrm{meas}_{n_i - t_i}$ is the $(n_i - t_i)$-dimensional measure induced by the Lebesgue measure on $\mathbb{R}^{n_i}$. Let

$$c_i = \frac{\mathrm{meas}_{n_i - t_i}([-1,1]^{n_i} \cap W_i)}{\det \Lambda_i}.$$

Then applying this estimate with $T = B/d$ gives

$$F_{i,d}(B) = c_i (B/d)^{n_i - t_i} + O((B/d)^{n_i - t_i - 1})$$

whenever $d \leq B$. If $d > B$, then $F_{i,d}(B) = 0$ and the same estimate holds. Hence, for $\delta > 0$,

$$F_{\mathbf{d}}(B_1, \dots, B_s) = C_{M,\mathbf{d}} \prod_{i=1}^{s} B_i^{n_i - t_i} + O\left(C_{E,\mathbf{d}} \left(\prod_{i=1}^{s} B_i^{n_i-t_i}\right)\left(\min_{1\leq i\leq s} B_i\right)^{-\delta}\right),$$

where

$$C_{M,\mathbf{d}} = \prod_{i=1}^{s} \frac{c_i}{d_i^{n_i-t_i}}, \tag{4.1}$$

$$C_{E,\mathbf{d}} = \prod_{i=1}^{s} d_i^{-(n_i-t_i)+\delta}. \tag{4.2}$$

We show that for $\delta > 0$ sufficiently small, the assumptions of Lemma 3.1 are satisfied. Since $n_i - t_i \geq 2$ for all $i \in \{1, \dots, s\}$ such that $t_i \neq 0$, if $f_{(n_1-t_1,\dots,n_s-t_s)} < 2$, by Remark 2.4(iv) there is $\tilde{i} \in \{1, \dots, s\}$ such that $t_{\tilde{i}} = 0$, $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$, and $n_{\tilde{i}} = 1$. Then $\rho_{\tilde{i},1}$ is not contained in any maximal cone of $\Sigma$, contradicting the fact that $X$ is proper. Thus $f_{(n_1-t_1,\dots,n_s-t_s)} \geq 2$. By definition and by Remark 2.4(ii), $f_{(n_1-t_1-\delta,\dots,n_s-t_s-\delta)} \geq f_{(n_1-t_1,\dots,n_s-t_s)} - s\delta$.

Since $V$ is a smooth complete intersection of smooth divisors, by adjunction [MR192, Proposition 16.4] we have $K_V = K_X + \sum_{i=1}^{s}\sum_{l=1}^{t_i}[H_{i,l}]$. Since

$$\sum_{i=1}^{s}(n_i - t_i)\mathbf{D}_i = -[K_X] - \sum_{i=1}^{s} t_i \mathbf{D}_i = -[K_X] - \sum_{i=1}^{s}\sum_{l=1}^{t_i}[H_{i,l}],$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \operatorname{rk}\operatorname{Pic}(X)$, and $c$ is defined in (3.7) with $k = b-1$, $C_{M,\mathbf{d}}$ given by (4.1) and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$. The restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $t_i \leq n_i - 2$ for all $i \in \{1, \dots, s\}$. The leading constant $c$ is positive by Lemma 2.5(ii). $\square$

## 5. Bihomogeneous hypersurfaces

*Proof of Theorem 1.2.* In the setting of Theorem 1.2, the hypersurfaces $H_1, \dots, H_t$ are defined by bihomogeneous polynomials $g_1, \dots, g_t$ of degree $(e_1, e_2)$ in the two sets of variables $\{x_{1,j} : 1 \leq j \leq n_1\}$ and $\{x_{2,j} : 1 \leq j \leq n_2\}$. Let $m_{i,j} = 1$ for all $(i,j) \in \mathcal{I}$.

We will apply [Sch16, Theorem 4.4] with $R = t$, $F_i = g_i$, $\mathscr{B}_i = [-1,1]^{n_i}$, $P_i = B_i/d_i$, $d_i = e_i$. In order to apply [Sch16, Theorem 4.4] we need to restrict the points to an open set. Let $U \subseteq \mathbb{A}^{n_1+n_2}$ be the open set in [Sch16, Theorem 4.4]. Since the complement of $U$ is the zero set of homogeneous polynomials by [Sch16, Theorems 4.1, 4.2], the set $W := \pi(\{\mathbf{x} \in Y : (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in U\})$ is an open subset of $X$. Then

$$N_{V,W}(B) = \frac{1}{2^r} \sum_{\mathbf{d}\in(\mathbb{Z}_{>0})^s} \mu(\mathbf{d}) \sharp A^W(B, \mathbf{d}),$$

with

$$A^W(B,\mathbf{d}) = \sum_{\substack{y_1,\dots,y_s\in\mathbb{Z}_{>0} \\ \prod_{i=1}^{s} y_i^{\alpha_{i,\sigma}} \leq B,\ \forall\sigma\in\Sigma_{\max}}} f_{\mathbf{d}}^W(y_1, \dots, y_s)$$

and

$$f_{\mathbf{d}}^W(y_1,\dots,y_s)=\sharp\{\mathbf{x}\in(\mathbb{Z}_{\neq 0})^{\mathcal{I}}:(x_{1,1},\dots,x_{1,n_1},x_{2,1},\dots,x_{2,n_2})\in U(\mathbb{Q}),\\ (3.2),(3.3),(3.4), y_i=\sup_{1\le j\le n_i}|x_{i,j}|\ \forall i\in\{1,\dots,s\}\}.$$

Let

$$F_{\mathbf{d}}^W(B_1,\dots,B_s)=\sum_{1\le y_i\le B_i, 1\le i\le s} f_{\mathbf{d}}^W(y_1,\dots,y_s).$$

Then

$$F_{\mathbf{d}}^W(B_1,\dots,B_s)=\widetilde{F}_{d_1,d_2}^W(B_1,B_2)\prod_{i=3}^s F_{i,d_i}(B_i),$$

where

$$\widetilde{F}_{d_1,d_2}^W=\sharp\{(x_{1,1},\dots,x_{1,n_1},x_{2,1},\dots,x_{2,n_2})\in(\mathbb{Z}_{\neq 0})^{n_1+n_2}\cap U(\mathbb{Q}):\\ \sup_{1\le j\le n_i}|y_{i,j}|\le B_i/d_i\ \forall i\in\{1,2\},\ g_1=\dots=g_t=0\}.$$

and for $d\in\mathbb{Z}_{>0}$ and $B>0$,

$$F_{i,d}(B)=\sharp\{(x_{i,1},\dots,x_{i,n_i})\in(\mathbb{Z}_{\neq 0})^{n_i}:\sup_{1\le j\le n_i}|x_{i,j}|\le B,\\ d\mid x_{i,j}\ \forall j\in\{1,\dots,n_i\}\}.$$

If $d\le B_i$, then

$$F_{i,d}(B)=2^{n_i}(B/d)^{n_i}+O((B/d)^{n_i-\delta})$$

with $0<\delta\le 1$. If $d>B$, then $F_{i,d}(B)=0$, and the same estimate holds.

To compute $\widetilde{F}_{d_1,d_2}^W(B_1,B_2)$, write $x_{i,j}=d_iy_{i,j}$ for all $(i,j)\in\mathcal{I}$. Then

$$\widetilde{F}_{d_1,d_2}^W(B_1,B_2)=\sharp\{(y_{1,1},\dots,y_{1,n_1},y_{2,1},\dots,y_{2,n_2})\in(\mathbb{Z}_{\neq 0})^{n_1+n_2}\cap U(\mathbb{Q}):\\ \sup_{1\le j\le n_i}|y_{i,j}|\le B_i/d_i\ \forall i\in\{1,2\},\ g_1=\dots=g_t=0\},$$

as the complement of $U$ is the zero set of homogeneous polynomials by [Sch16, Theorems 4.1, 4.2]. Let $V_i^*\subseteq\mathbb{A}^{n_1+n_2}$ be the locus where the matrix $\left(\frac{\partial g_l}{\partial x_{i,j}}\right)_{\substack{1\le l\le t\\ 1\le j\le n_i}}$ does not have full rank. If $n_1+n_2>\dim V_1^*+\dim V_2^*+3\cdot 2^{e_1+e_2}e_1e_2t^3$, then by [Sch16, Theorem 4.4] there is $\delta>0$ such that

$$\begin{aligned}\widetilde{F}_{d_1,d_2}^W(B_1,B_2)&=C\prod_{i=1}^2(B_i/d_i)^{n_i-te_i}+O\left(\left(\min_{i=1,2}B_i/d_i\right)^{-\delta}\prod_{i=1}^2(B_i/d_i)^{n_i-te_i}\right)\\ &=C\prod_{i=1}^2(B_i/d_i)^{n_i-te_i}+O\left(\left(\prod_{i=1}^2 d_i^{-(n_i-te_i)+\delta}\right)\left(\min_{i=1,2}B_i\right)^{-\delta}\prod_{i=1}^2 B_i^{n_i-te_i}\right)\end{aligned}$$

with $C\in\mathbb{R}_{\ge 0}$, and $C>0$ whenever $V$ has nonsingular $\mathbb{Q}_v$-points for all places $v$ of $\mathbb{Q}$. Thus

$$F_{\mathbf{d}}^W(B_1,\dots,B_s)=C_{M,\mathbf{d}}B_1^{n_1-te_1}B_2^{n_2-te_2}\prod_{i=3}^s B_i^{n_i}\\ +O\left(C_{E,\mathbf{d}}\left(\min_{1\le i\le s}B_i\right)^{-\delta}B_1^{n_1-te_1}B_2^{n_2-te_2}\prod_{i=3}^s B_i^{n_i}\right),$$

where

$$C_{M,\mathbf{d}} = C d_1^{-(n_1-te_1)} d_2^{-(n_2-te_2)} \prod_{i=3}^{s} d_i^{-n_i}, \tag{5.1}$$

$$C_{E,\mathbf{d}} = d_1^{-(n_1-te_1)+\delta} d_2^{-(n_2-te_2)+\delta} \prod_{i=3}^{s} d_i^{-n_i+\delta}. \tag{5.2}$$

Recall that $n_i - te_i \geq 2$ for $i \in \{1,2\}$. For $\delta > 0$ sufficiently small, if

$$f_{n_1-te_1-\delta, n_2-te_2-\delta, n_3-\delta,\dots,n_s-\delta} \leq 1,$$

then by Remark 2.4(iv) there is $\tilde{i} \in \{3,\dots,s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of $\Sigma$, contradicting the fact that $X$ is proper.

Since $V$ is a smooth complete intersection, the adjunction formula [MR192, Proposition 16.4] gives $K_V = K_X + H_1 + \cdots + H_t$. Let $\varpi_i = n_i - te_i$ for $i \in \{1,2\}$, and $\varpi_i = n_i$ for $i \in \{3,\dots,s\}$. Since

$$\sum_{i=1}^{s} \varpi_i \mathbf{D}_i = -[K_X] - t(e_1\mathbf{D}_1 + e_2\mathbf{D}_2) = -[K_X] - [H_1 + \cdots + H_t],$$

Lemma 3.1 applied to $F_{\mathbf{d}}^W(B_1,\dots,B_s)$ and $N_{V,W}(B)$ gives

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B^a(\log B)^{b-2}(\log\log B)^s)$$

for $B > 0$, where $b = \operatorname{rk}\operatorname{Pic}(X)$, and $c$ is defined in (3.7) with $k = b-1$, $C_{M,\mathbf{d}}$ given by (5.1). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism, as $t \leq \min\{n_1, n_2\} - 2$. By Lemma 2.5(ii) the leading constant $c$ is positive if $V(\mathbb{Q}_v) \neq \emptyset$ for all places $v$ of $\mathbb{Q}$ as $C$ is positive under the same conditions by [Sch16, Theorems 4.3 and 4.4]. □

## 6. Campana points on certain diagonal complete intersections

*Proof of Theorem 1.3.* In the setting of Theorem 1.3, the hypersurfaces $H_1,\dots,H_t$ are defined by homogeneous diagonal polynomials $g_1,\dots,g_t \in R$ with $\deg g_i = e_i\mathbf{D}_i$ in $\operatorname{Pic}(X)$ for all $i \in \{1,\dots,t\}$. Then

$$g_i = \sum_{j=1}^{n_i} c_{i,j} x_{i,j}^{e_i},$$

with $c_{i,j} \in \mathbb{Z}_{\neq 0}$, and

$$F_{\mathbf{d}}(B_1,\dots,B_s) = \prod_{i=1}^{s} F_{i,d_i}(B_i),$$

where for $i \leq t$,

$$F_{i,d}(B) = \sharp\{(x_{i,1},\dots,x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : d \mid x_{i,j},\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1,\dots,n_i\}, \\ \sup_{1\leq j\leq n_i} |x_{i,j}| \leq B,\ g_i = 0\},$$

and for $i > t$,

$$F_{i,d}(B) = \sharp\{(x_{i,1},\dots,x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \sup_{1\leq j\leq n_i} |x_{i,j}| \leq B,\ d \mid x_{i,j}, \\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1,\dots,n_i\}\}. \tag{6.1}$$

For $i \leq t$, we estimate $F_{i,d}(B)$ via the following lemma.

**Lemma 6.1.** *Let $n, e, m_1,\dots,m_n \in \mathbb{Z}_{>0}$. Let $c_1,\dots,c_n \in \mathbb{Z}_{\neq 0}$. Let $d$ be a square-free positive integer. Assume that $n \geq 2$ and $2 \leq m_1 \leq \cdots \leq m_n$.*

*(1) If $e=1$, assume that $\sum_{j=1}^n \frac{1}{m_j} > 3$, and $\sum_{j=1}^{n-1} \frac{1}{em_j(em_j+1)} \geq 1$.*
*(2) If $e \geq 2$ assume that $\sum_{j=1}^n \frac{1}{em_j} > 3$, $\sum_{j=1}^n \frac{1}{2s_0(em_j)} > 1$, where*

$$s_0(m) = \min\left\{2^{m-1}, \frac{1}{2}m(m-1) + \lfloor\sqrt{2m+2}\rfloor\right\},\ m \in \mathbb{Z}_{\geq 0}.$$

*For $B>0$, let*

$$F_d(B) = \sharp\{(x_1,\dots,x_n) \in (\mathbb{Z}_{\neq 0})^n : d \mid x_j,\ x_j \text{ is } m_j\text{-full } \forall j \in \{1,\dots,n\}, \sup_{1\leq j\leq n} |x_j| \leq B,\ \sum_{j=1}^n c_j x_j^e = 0\}.$$

*Then there is $\eta>0$ such that*

$$F_d(B) = c_{e,d}B^\Gamma + O(d^{-1-\eta}B^{\Gamma-\eta}),$$

*where $\Gamma = \sum_{j=1}^n \frac{1}{m_j} - e$ and $c_{e,d}$ is defined in (6.5) and satisfies $0 \leq c_{e,d} \ll d^{-1-\eta}$.*

*Proof.* For every $j \in \{1,\dots,n\}$ and $x_j \in \mathbb{Z}_{\neq 0}$ that is $m_j$-full, there exist unique $u_j, v_{j,1}, \dots, v_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$|x_j| = u_j^{m_j} \prod_{r=1}^{m_j-1} v_{j,r}^{m_j+r},$$

$$\mu^2(v_{j,r}) = 1, \quad \gcd(v_{j,r}, v_{j,r'}) = 1 \quad \forall r,r' \in \{1,\dots,m_j-1\}, r \neq r'.$$

For every choice of $u_j$ and $v_{j,r}$ as above, if $d \mid x_j$ with $d \in \mathbb{Z}_{>0}$ squarefree, then there exist unique $s_j, t_{j,1}, \dots, t_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$d = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \quad \mu^2(s_j) = \mu^2(t_{j,r}) = 1\ \forall r \in \{1,\dots,m_j-1\}$$
$$\gcd(s_j, v_{j,r}) = \gcd(s_j, t_{j,r}) = \gcd(t_{j,r}, t_{j,r'}) = 1\ \forall r,r' \in \{1,\dots,m_j-1\}, r \neq r'$$
$$s_j \mid u_j,\ t_{j,r} \mid v_{j,r}\ \forall r \in \{1,\dots,m_j-1\}.$$

Write $u_j = s_j\tilde{u}_j$ and $v_{j,r} = t_{j,r}\tilde{v}_{j,r}$ for all $r \in \{1,\dots,m_{j-1}\}$. Write $\mathbf{s} = (s_1,\dots,s_n)$, $\mathbf{t} = (t_{j,r})_{1\leq j\leq n, 1\leq r\leq m_j-1}$. For $j \in \{1,\dots,n\}$, write

$$\sigma_j = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \qquad \tau_j = s_j^{m_j} \prod_{r=1}^{m_j-1} t_{j,r}^{m_j+r}, \qquad w_j = \prod_{r=1}^{m_j-1} \tilde{v}_{j,r}^{m_j+r}.$$

Let $\mathcal{T}_d(B)$ be the set of pairs $(\mathbf{s},\mathbf{t}) \in \mathbb{Z}_{>0}^n \times \mathbb{Z}_{>0}^{\sum_{j=1}^n (m_j-1)}$ that satisfy

$$\mu^2(\sigma_j) = 1, \quad d = \sigma_j, \quad \tau_j \leq B \quad \forall j \in \{1,\dots,s\}.$$

Note that the first two conditions imply

$$\sharp\mathcal{T}_d(B) \leq \prod_{j=1}^n m_j^{\omega(d)} \ll d^\epsilon, \tag{6.2}$$

where $\omega(d)$ is the number of distinct prime divisors of $d$. Let $\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)$ be the set of $\tilde{\mathbf{v}} = (\tilde{v}_{j,r})_{1\leq j\leq n, 1\leq r\leq m_j-1} \in \mathbb{Z}_{>0}^{\sum_{j=1}^n (m_j-1)}$ such that

$$\mu^2(t_{j,r}\tilde{v}_{j,r}) = 1,\ \gcd(s_j, \tilde{v}_{j,r}) = 1\ \forall j \in \{1,\dots,n\}, r \in \{1,\dots,m_j-1\},$$
$$\gcd(t_{j,r}\tilde{v}_{j,r}, t_{j,r'}\tilde{v}_{j,r'}) = 1\ \forall j \in \{1,\dots,n\}, r,r' \in \{1,\dots,m_j-1\}, r \neq r',$$
$$\tau_j w_j \leq B\ \forall j \in \{1,\dots,n\}.$$

Let $\mathcal{T}_d(\infty) = \bigcup_{B>0} \mathcal{T}_d(B)$ and $\mathcal{V}_{\mathbf{s},\mathbf{t}}(\infty) = \bigcup_{B>0} \mathcal{V}_{\mathbf{s},\mathbf{t}}(B)$.

Then

$$F_d(B) = \begin{cases} \sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^n} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} M_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is odd,} \\ 2^n \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} M_{\mathbf{c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is even,} \end{cases} \tag{6.3}$$

where $\mathbf{c} = (c_1, \dots, c_n)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, $\boldsymbol{\varepsilon}\mathbf{c} = (\varepsilon_1 c_1, \dots, \varepsilon_n c_n)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ with

$$\gamma_j = s_j^{em_j} \prod_{r=1}^{m_j-1} t_{j,r}^{e(m_j+r)} \tilde{v}_{j,r}^{e(m_j+r)} \quad \forall j \in \{1, \dots, n\},$$

and

$$M_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}}(B^e) = \sharp \left\{ (\tilde{u}_1, \dots, \tilde{u}_n) \in \mathbb{Z}_{>0}^n : \max_{1\le j\le n} \gamma_j \tilde{u}_j^{em_j} \le B^e,\ \sum_{j=1}^n \varepsilon_j c_j \gamma_j \tilde{u}_j^{em_j} = 0 \right\}.$$

An estimate for $M_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}}(B^e)$ is proven in [BY21, Theorem 2.7] in the case where $\sum_{j=1}^{n-1} \frac{1}{em_j(em_j+1)} \ge 1$. The subsequent paper [BBK$^+$23, Theorem 5.3] extends the range of applicability of [BY21, Theorem 2.7] to the case where $\sum_{j=1}^n \frac{1}{em_j} > 3$, $\sum_{j=1}^n \frac{1}{2s_0(em_j)} > 1$.

Let

$$\Theta_e = \begin{cases} \frac{1}{m_n(m_n+1)} & \text{if } e = 1, \\ \sum_{j=1}^n \frac{1}{2s_0(em_j)} - 1 & \text{if } e \ge 2. \end{cases}$$

For $0 < \delta < \frac{1}{(2(n-1)+5)em_n(em_n+1)}$ and $\epsilon > 0$, [BY21, Theorem 2.7] and [BBK$^+$23, Theorem 5.3] give

$$\sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} M_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}}(B^e) = \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \frac{\mathfrak{S}_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}} \mathfrak{I}_{\boldsymbol{\varepsilon}\mathbf{c}}}{\prod_{j=1}^n \gamma_j^{\frac{1}{em_j}}} B^\Gamma + O\left(B^\Gamma(F_1 + F_2 + F_3)\right), \tag{6.4}$$

where

$$\mathfrak{S}_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}} = \sum_{q=1}^\infty \frac{1}{q^n} \sum_{\substack{a (\mathrm{mod}\ q) \\ \gcd(a,q)=1}} \prod_{j=1}^n \sum_{r=1}^q \exp(2\pi i a \varepsilon_j c_j \gamma_j r^{em_j}/q),$$

$$\mathfrak{I}_{\boldsymbol{\varepsilon}\mathbf{c}} = \int_{-\infty}^\infty \prod_{j=1}^n \left( \int_0^1 \exp(2\pi i \lambda \varepsilon_j c_j \xi^{em_j}) \mathrm{d}\xi \right) \mathrm{d}\lambda,$$

$$F_1 = B^{e((2(n-1)+5)\delta-1)-\Gamma} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \left( \prod_{j=1}^n \frac{B^{\frac{1}{m_j}}}{\gamma_j^{\frac{1}{em_j}}} \right) \sum_{l=1}^n \frac{\gamma_l^{\frac{1}{em_l}}}{B^{\frac{1}{m_l}}},$$

$$F_2 = B^{-e\delta} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \sum_{q=1}^\infty q^{1-\Gamma/e+\epsilon} \prod_{j=1}^n \gcd(\gamma_j, q)^{\frac{1}{em_j}} \gamma_j^{-\frac{1}{em_j}},$$

$$F_3 = \begin{cases} B^{-e\delta\Theta_e+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \prod_{j=1}^n \gamma_j^{-\frac{1}{m_j+1}} & \text{if } e = 1, \\ B^{-e\delta\Theta_e+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \prod_{j=1}^n \gamma_j^{-\frac{1}{em_j}+\frac{1}{2s_0(em_j)}} & \text{if } e \ge 2. \end{cases}$$

Let

$$c_{e,d} = \begin{cases} \sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^n} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(\infty)} \frac{\mathfrak{S}_{\boldsymbol{\varepsilon}\mathbf{c},\boldsymbol{\gamma}} \mathfrak{I}_{\boldsymbol{\varepsilon}\mathbf{c}}}{\prod_{j=1}^n \gamma_j^{\frac{1}{em_j}}} & \text{if } e \text{ is odd,} \\ 2^n \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(\infty)} \frac{\mathfrak{S}_{\mathbf{c},\boldsymbol{\gamma}} \mathfrak{I}_{\mathbf{c}}}{\prod_{j=1}^n \gamma_j^{\frac{1}{em_j}}} & \text{if } e \text{ is even.} \end{cases} \tag{6.5}$$

For $T > 0$, let

$$f_1(q) = \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\right)^{\frac{1}{em_j}},$$

$$f_2(q) = \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{1},\mathbf{1}}(\infty)} \prod_{j=1}^{n} \left(\frac{\gcd(w_j^e, q)}{w_j^e}\right)^{\frac{1}{em_j}},$$

and

$$f_2(q, T, \mathbf{s}, \mathbf{t}) = \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(\infty)\smallsetminus\mathcal{V}_{\mathbf{s},\mathbf{t}}(T)} \prod_{j=1}^{n} \left(\frac{\gcd(w_j^e, q)}{w_j^e}\right)^{\frac{1}{em_j}}.$$

Note that for $\sum_{j=1}^{n} \frac{1}{em_j} > 1$ we have

$$|\mathfrak{I}_{\boldsymbol{\varepsilon}\mathbf{c}}| \ll 1.$$

Similarly as in [BY21, (2.8), (2.9), (2.12)], the difference between $c_{e,d}B^{\Gamma}$ and the main term obtained combining (6.3) and (6.4) is bounded by

$$B^{\Gamma} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(\infty)\smallsetminus\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \sum_{q=1}^{\infty} q^{1-\sum_{j=1}^{n}\frac{1}{em_j}} \prod_{j=1}^{n} \gamma_j^{-\frac{1}{em_j}} \gcd(\gamma_j, q)^{\frac{1}{em_j}} \tag{6.6}$$

$$\ll B^{\Gamma} \sum_{q=1}^{\infty} q^{-\Gamma/e+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\right)^{\frac{1}{em_j}} f_2(q, B, \mathbf{s}, \mathbf{t}), \tag{6.7}$$

and $c_{e,d} \ll \sum_{q=1}^{\infty} q^{-\Gamma/e+\epsilon} f_1(q) f_2(q)$.

By [BY21, (3.10)] and the arguments used to prove [BY21, (3.9)], we have

$$f_2(q) \ll q^{\epsilon}, \tag{6.8}$$

and

$$f_2(q, T, \mathbf{s}, \mathbf{t}) \ll \sum_{i_0=1}^{n} \sum_{\substack{\tilde{v}_{i,r},\ 1\le i\le n,\ 1\le r\le m_i-1 \\ \prod_{r=1}^{m_i-1} \tilde{v}_{i,r}^{m_i+r} > \frac{T}{\tau_i} \text{ if } i=i_0}} \prod_{i=1}^{n} \prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r}) \gcd(\tilde{v}_{i,r}^{e(m_i+r)}, q)^{\frac{1}{em_i}}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}}$$

$$\ll q^{\epsilon} \sum_{i=1}^{n} \sum_{\substack{\tilde{v}_{i,r},\ 1\le r\le m_i-1 \\ \prod_{r=1}^{m_i-1} \tilde{v}_{i,r}^{m_i+r} > \frac{T}{\tau_i}}} \prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r}) \gcd(\tilde{v}_{i,r}^{e(m_i+r)}, q)^{\frac{1}{em_i}}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}}.$$

Our next goal is to provide an upper bound for sums of the type occurring in this estimate for $f_2(q, T, \mathbf{s}, \mathbf{t})$.

**Lemma 6.2.** *Let $m \in \mathbb{N}_{\geq 2}$, $e \in \mathbb{N}$ and $A > 0$ a real parameter. Then, for every $0 < \epsilon < \frac{1}{m(m+1)}$ we have*

$$\sum_{\substack{v_r\in\mathbb{N},\ 1\le r\le m-1 \\ \prod_{r=1}^{m-1} v_r^{m+r} > A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r) \gcd(v_r^{e(m+r)}, q)^{\frac{1}{em}}}{v_r^{(m+r)/m}} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}.$$

*Proof.* We first consider the sum

$$S_1 := \sum_{\substack{v_r\in\mathbb{N},\ 1\le r\le m-1 \\ \prod_{r=1}^{m-1} v_r^{m+r} > A}} \prod_{r=1}^{m-1} \frac{1}{v_r^{(m+r)/m}}$$

for $A > 1$. A dyadic decomposition for each of the variables $v_r$, $1 \leq r \leq m-1$, leads us to the upper bound

$$S_1 \ll \sum_{\substack{l_1,\dots,l_{m-1}\in\mathbb{N}\\ 2^{(m+1)l_1+\dots+(2m-1)l_{m-1}}>A}} 2^{-\frac{1}{m}l_1-\dots-\frac{m-1}{m}l_{m-1}}$$

Note that for each $k \in \frac{1}{m}\mathbb{N}$ we have

$$\sharp\{l_1,\dots,l_{m-1}\in\mathbb{N}:\ \frac{1}{m}l_1+\dots+\frac{m-1}{m}l_{m-1}=k\} \ll_m k^{m-2}.$$

We deduce that

$$S_1 \ll_m \sum_{\substack{k\in\frac{1}{m}\mathbb{N}\\ r(k)>0}} k^{m-2}2^{-k}$$

where $r(k)$ is the number of $(l_1,\dots,l_{m-1}) \in \mathbb{N}^{m-1}$ such that both

$$\frac{1}{m}l_1+\dots+\frac{m-1}{m}l_{m-1}=k \quad \text{and} \quad 2^{(m+1)l_1+\dots+(2m-1)l_{m-1}} > A$$

hold. Observe that if $r(k) > 0$, then there exists $(l_1,\dots,l_{m-1}) \in \mathbb{N}^{m-1}$ with $\frac{1}{m}l_1+\dots+\frac{m-1}{m}l_{m-1}=k$ and

$$\begin{aligned} m(m+1)k &= (m+1)\left(l_1+\dots+(m-1)l_{m-1}\right)\\ &\geq (m+1)l_1+\frac{m+2}{2}2l_2+\dots+\frac{2m-1}{m-1}(m-1)l_{m-1} > \log A/\log 2, \end{aligned}$$

i.e.

$$S_1 \ll_m \sum_{\substack{k\in\frac{1}{m}\mathbb{N}\\ m(m+1)k>\log A/\log 2}} k^{m-2}2^{-k} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon}.$$

Note that the upper bound for $S_1$ also holds for $A \leq 1$ and $\epsilon < \frac{1}{m(m+1)}$.

We now turn to the sum in the statement of the lemma. If $v_r$ is a square-free natural number and $d_r = \gcd(v_r^{e(m+r)}, q)$, then we can write

$$d_r = d_{r,1}d_{r,2}^2\dots d_{r,e(m+r)}^{e(m+r)}, \quad \mu^2(d_{r,j}) = 1,\ \forall 1\leq j\leq e(m+r), \quad \gcd(d_{r,j}, d_{r,j'}) = 1,\ \forall j\neq j'.$$

Writing $v_r = v_r'\prod_{j=1}^{e(m+r)} d_{r,j}$ and $d_r' = \prod_{j=1}^{e(m+r)} d_{r,j}$ we find that

$$\begin{aligned} S_2 &:= \sum_{\substack{v_r\in\mathbb{N},\ 1\leq r\leq m-1\\ \prod_{r=1}^{m-1} v_r^{m+r}>A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r)\gcd(v_r^{e(m+r)},q)^{\frac{1}{em}}}{v_r^{(m+r)/m}}\\ &\ll \sum_{d_{r,1}d_{r,2}^2\dots d_{r,e(m+r)}^{e(m+r)}|q,\ 1\leq r\leq m-1} \ \sum_{\substack{v_r'\in\mathbb{N},\ 1\leq r\leq m-1\\ \prod_{r=1}^{m-1}(d_r'v_r')^{m+r}>A}} \prod_{r=1}^{m-1} \frac{d_r^{\frac{1}{em}}}{(d_r'v_r')^{(m+r)/m}}\\ &\ll \sum_{d_{r,1}d_{r,2}^2\dots d_{r,e(m+r)}^{e(m+r)}|q,\ 1\leq r\leq m-1} \prod_{r=1}^{m-1}\left(\frac{d_r^{\frac{1}{em}}}{d_r'^{\frac{m+r}{m}}}\right) \sum_{\substack{v_r'\in\mathbb{N},\ 1\leq r\leq m-1\\ \prod_{r=1}^{m-1}(d_r'v_r')^{m+r}>A}} \prod_{r=1}^{m-1}\frac{1}{(v_r')^{(m+r)/m}} \end{aligned}$$

By using the upper bound for $S_1$ we find that for $\epsilon > 0$ sufficiently small

$$S_2 \ll_{\epsilon,m} \sum_{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}|q,\ 1\le r\le m-1} \prod_{r=1}^{m-1}\left(\frac{d_r^{\frac{1}{em}}}{d_r'^{\frac{m+r}{m}}}\right) A^{-\frac{1}{m(m+1)}+\epsilon}\left(\prod_{r=1}^{m-1}(d_r')^{m+r}\right)^{\frac{1}{m(m+1)}}$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}|q,\ 1\le r\le m-1} \prod_{r=1}^{m-1}\left(d_r^{\frac{1}{em}}(d_r')^{-\frac{m+r}{m+1}}\right)$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}|q,\ 1\le r\le m-1} \prod_{r=1}^{m-1} d_r^{\frac{1}{em}-\frac{m+r}{e(m+r)(m+1)}}$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{d_r|q,\ 1\le r\le m-1} \prod_{r=1}^{m-1} d_r^{\frac{1}{em(m+1)}} \ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}. \quad \square$$

Lemma 6.2 shows that we can bound $f_2(q,T,\mathbf{s},\mathbf{t})$ by

$$f_2(q,T,\mathbf{s},\mathbf{t}) \ll \sum_{i=1}^{n}\left(\frac{T}{\tau_i}\right)^{-\frac{1}{m_i(m_i+1)}+\epsilon} q^{\frac{m_i-1}{em_i(m_i+1)}+\epsilon}.$$

In the following we write $\Delta_i = \frac{m_i-1}{em_i(m_i+1)}$. Then (6.7) is bounded by

$$S_3 := B^{\Gamma}\sum_{i=1}^{n}\sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \prod_{j=1}^{n}\left(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\right)^{\frac{1}{em_j}} \left(\frac{B}{\tau_i}\right)^{-\frac{1}{m_i(m_i+1)}+\epsilon}$$

$$\ll B^{\Gamma}\sum_{i=1}^{n} B^{-\frac{1}{m_i(m_i+1)}+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon}\tau_i^{\frac{1}{m_i(m_i+1)}} \prod_{j=1}^{n}\left(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\right)^{\frac{1}{em_j}}.$$

As we will encounter similar expressions in our further analysis, we introduce for $E, D > 0$ and $d$ squarefree the following sum

$$S_d(D,E) := d^E \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n}\left(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\right)^{\frac{1}{em_j}}.$$

We write $q = q_1q_2$ with $\gcd(q_1,d) = 1$ and such that all prime divisors of $q_2$ divide $d$. With this we obtain that

$$S_d(D,E) \ll d^E \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} \sum_{q_1=1}^{\infty} q_1^{-\Gamma/e+D+\epsilon} \sum_{\substack{q_2=1\\ p|q_2\Rightarrow p|d}}^{\infty} q_2^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n}\left(\frac{\gcd(\tau_j^e,q_2)}{\tau_j^e}\right)^{\frac{1}{em_j}}.$$

If we assume $-\Gamma/e + D < -1$, then the sum over $q_1$ is absolutely convergent. For a given vector $(\mathbf{s},\mathbf{t}) \in \mathcal{T}_d(\infty)$ and a prime $p$ we write $\tau_{j,p}$ for the power of $p$ which exactly divides $\tau_j$. We find that

$$S_d(D,E) \ll \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} d^E \prod_{p|d}\left(\sum_{l=0}^{\infty} p^{l(-\Gamma/e+D+\epsilon)} \prod_{j=1}^{n}\left(\frac{\gcd(\tau_{j,p}^e,p^l)}{\tau_{j,p}^e}\right)^{\frac{1}{em_j}}\right).$$

We now split the summation over $l$ into the term $l = 0$, where we use the inequality $\tau_{j,p} \ge p^{m_j}$, and we bound the rest by a geometric sum for $l \ge 1$ using $\gcd(\tau_{j,p}^e, p^l) \le$

$\tau_{j,p}^e$,

$$S_d(D,E) \ll_D \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(\infty)} d^{E+\epsilon} \prod_{p\mid d} \left(p^{-n} + p^{-\Gamma/e+D+\epsilon}\right)$$
$$\ll_D d^{\epsilon} \prod_{p\mid d} \left(p^{E-n} + p^{-\Gamma/e+D+E+\epsilon}\right)$$

If $-\Gamma/e + D + E < -1$, then we deduce that

$$S_d(D,E) \ll_D d^{-1-\eta} \tag{6.9}$$

for some $\eta > 0$.

Applying equation (6.9) to $S_3$ with $D = \Delta_i$ and $E = \frac{2m_i-1}{m_i(m_i+1)}$ we obtain $S_3 \ll B^{\Gamma-\eta} d^{-1-\eta}$, for some $\eta > 0$. Hence,

$$F_d(B) = c_{e,d}B^{\Gamma} + O(B^{\Gamma}(d^{-1-\eta}B^{-\eta} + F_1 + F_2 + F_3)).$$

We use the bound in (6.8) and apply equation (6.9) with $D = E = 0$ to get $c_{e,d} \ll d^{-1-\eta}$.

It remains to estimate the error terms $F_1, F_2, F_3$. We rewrite $F_1$ as follows:

$$F_1 = B^{e\delta(2(n-1)+5)} \sum_{l=1}^{n} B^{-\frac{1}{m_l}} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\mathbf{v}}\in\mathcal{V}_{\mathbf{s},\mathbf{t}}(B)} \prod_{\substack{1\le j\le n\\ j\ne l}} \gamma_j^{-\frac{1}{em_j}}.$$

As in [BY21, §3] and [BBK$^+$23, §6], we have

$$F_1 \ll B^{-\frac{1}{m_n(m_n+1)}+e\delta(2(n-1)+5)} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}}$$
$$\ll d^{-\sum_{j=1}^n \frac{m_j}{m_j+1}+\varepsilon} B^{-\frac{1}{m_n(m_n+1)}+e\delta(2(n-1)+5)},$$

where the last estimate follows from

$$\sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \le \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \prod_{j=1}^{n} \sigma_j^{-\frac{m_j}{m_j+1}}$$
$$\le d^{-\sum_{j=1}^n \frac{m_j}{m_j+1}} \sharp\mathcal{T}_d(B) \ll d^{-\sum_{j=1}^n \frac{m_j}{m_j+1}+\varepsilon}.$$

by (6.2). Combining the arguments for $F_3$ in [BY21, §3] and in [BBK$^+$23, §6] and the estimate above we have

$$F_3 \ll B^{-e\delta\Theta_e+\epsilon} \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \ll d^{-\sum_{j=1}^n \frac{m_j}{m_j+1}+\epsilon} B^{-e\delta\Theta_e+\epsilon}.$$

Since $\sum_{j=1}^n \frac{m_j}{m_j+1} \ge \frac{2}{3}n > 1$ is satisfied for $n \ge 2$, we have $F_1, F_3 \ll d^{-1-\eta}B^{-\eta}$ for a suitable $\eta > 0$. Since

$$F_2 = B^{-e\delta} \sum_{q=1}^{\infty} q^{1-\Gamma/e+\epsilon} f_1(q) f_2(q),$$

the estimate (6.8) combined with (6.9) for $D = 1$ and $E = 0$ yields $F_2 \ll d^{-1-\eta}B^{-e\delta}$, as $\Gamma/e > 2$. □

By Lemma 6.1 and [PS24, Lemma 5.6]

$$F_{\mathbf{d}}(B_1, \dots, B_s) = \prod_{i=1}^{s} \left(c_{M,i} B_i^{\varpi_i} + O(d_i^{\nu_i+\varepsilon} B_i^{\varpi_i-\delta})\right)$$

where

$$\varpi_i = \begin{cases} \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} - e_i & \text{if } i \le t, \\ \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} & \text{if } i > t, \end{cases} \tag{6.10}$$

$\nu_i < -1$ for $i \le t$, $\nu_i = -\frac{2}{3} n_i$ if $i > t$, $c_{M,i}$ is the constant $c_{e_i,d_i}$ defined in (6.5) if $i \le t$, and $c_{M,i} = 2^{n_i} \left( \prod_{j=1}^{n_i} c_{m_{i,j},d_i} \right)$, where $c_{m_{i,j},d_i}$ is the constant defined in [PS24, (5.11)].

Thus

$$F_{\mathbf{d}}(B_1, \ldots, B_s) = C_{M,\mathbf{d}} \prod_{i=1}^{s} B_i^{\varpi_i} + O\left( C_{E,\mathbf{d}} (\min_{1 \le i \le s} B_i)^{-\delta} \prod_{i=1}^{s} B_i^{\varpi_i} \right),$$

where

$$C_{M,\mathbf{d}} = \prod_{i=1}^{s} c_{M,i}. \tag{6.11}$$

Lemma 6.1 and [PS24, (5.14), (5.15)] give $C_{M,\mathbf{d}}, C_{E,\mathbf{d}} \ll \prod_{i=1}^{s} d_i^{-\beta_i}$ with $\beta_i > 1$ whenever $n_i \ge 2$, and $\beta_i > \frac{2}{3} - \varepsilon$ otherwise. For $\varepsilon > 0$ sufficiently small, $\beta_i + \beta_j > 1$ for every $i, j \in \{1, \ldots, s\}$. Thus, if $f_{\beta_1, \ldots, \beta_s} \le 1$, by Remark 2.4(iv) then there exists an index $\tilde{i} \in \{1, \ldots, s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of $\Sigma$, contradicting the fact that $X$ is proper.

Since $c_{i,j} \neq 0$ for all $i \in \{1, \ldots, t\}$, $j \in \{1, \ldots, n_i\}$, the adjunction formula [MR192, Proposition 16.4] gives $K_V = (K_X + H_1 + \cdots + H_t)|_V$. Since

$$\sum_{i=1}^{s} \varpi_i \mathbf{D}_i = -K_X - \sum_{i=1}^{s} \sum_{j=1}^{n_i} (1 - \frac{1}{m_{i,j}}) \mathbf{D}_i + \sum_{i=1}^{t} e_i \mathbf{D}_i = -(K_X + [\mathscr{D}_{\mathbf{m}}|_X] + [H_1 + \cdots + H_t]),$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2} (\log\log B)^s),$$

where $b = \operatorname{rk} \operatorname{Pic}(X)$, and $c$ is defined in (3.7) with $k = b-1$, $C_{M,\mathbf{d}}$ given by (6.11), and $\varpi_1, \ldots, \varpi_s$ given by (6.10). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $n_i \ge 3$ for $1 \le i \le t$. □

## References


[BB18] V. Blomer and J. Brüdern. Counting in hyperbolic spikes: The diophantine analysis of multihomogeneous diagonal equations. *J. Reine Angew. Math.*, 737:255–300, 2018.

[BBK+23] F. Balestrieri, J. Brandes, M. Kaesberg, J. Ortmann, M. Pieropan, and R. Winter. Campana points on diagonal hypersurfaces. arXiv:2302.08164, 2023.

[BG06] E. Bombieri and W. Gubler. *Heights in Diophantine Geometry.* New Mathematical Monographs. Cambridge University Press, 4 edition, 2006.

[BH19] T. D. Browning and L. Q. Hu. Counting rational points on biquadratic hypersurfaces. *Adv. Math.*, 349:920–940, 2019.

[BHB17] T. Browning and R. Heath-Brown. Forms in many variables and differing degrees. *J. Eur. Math. Soc. (JEMS)*, 19(2):357–394, 2017.

[Bir62] B. J. Birch. Forms in many variables. *Proc. Roy. Soc. Ser. A*, 265:245–263, 1961/1962.

[BM90] V. V. Batyrev and Y. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.

[BY21] T. Browning and S. Yamagishi. Arithmetic of higher-dimensional orbifolds and a mixed Waring problem. *Math. Z.*, 299(1-2):1071–1101, 2021.

[CLNS18] A. Chambert-Loir, J. Nicaise, and J. Sebag. *Motivic integration*, volume 325 of *Progress in Mathematics.* Birkhäuser/Springer, New York, 2018.

[CLS11] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.

[Cox95] D. A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.

[DP20] U. Derenthal and M. Pieropan. The split torsor method for Manin's conjecture. *Trans. Amer. Math. Soc.*, 373(12):8485–8524, 2020.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[FP16] C. Frei and M. Pieropan. O-minimality on twisted universal torsors and Manin's conjecture over number fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):757–811, 2016.

[HB96] D. R. Heath-Brown. A new form of the circle method, and its application to quadratic forms. *J. Reine Angew. Math.*, 481:149–206, 1996.

[Hu20] L. Q. Hu. Counting rational points on biprojective hypersurfaces of bidegree $(1,2)$. *J. Number Theory*, 214:312–325, 2020.

[Mig15] Teddy Mignot. Points de hauteur bornée sur les hypersurfaces lisses de l'espace triprojectif. *Int. J. Number Theory*, 11(3):945–995, 2015.

[Mig16] T. Mignot. Points de hauteur bornée sur les hypersurfaces lisses des variétés toriques. *Acta Arith.*, 172(1):1–97, 2016.

[Mig18] Y. Mignot. Points de hauteur bornée sur les hypersurfaces lisses des variétés toriques. cas général. arXiv:1802.02832, 2018.

[MR192] *Flips and abundance for algebraic threefolds.* Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[Pey95] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.

[Pie16] M. Pieropan. Imaginary quadratic points on toric varieties via universal torsors. *Manuscripta Math.*, 150(3-4):415–439, 2016.

[PS24] M. Pieropan and D. Schindler. Hyperbola method on toric varieties. *J. Éc. polytech. Math.*, 11:107–157, 2024.

[PSTVA21] M. Pieropan, A. Smeets, S. Tanimoto, and A. Várilly-Alvarado. Campana points of bounded height on vector group compactifications. *Proc. Lond. Math. Soc. (3)*, 123(1):57–101, 2021.

[RM18] S. L. Rydin Myerson. Quadratic forms and systems of forms in many variables. *Invent. Math.*, 213(1):205–235, 2018.

[RM19] S. L. Rydin Myerson. Systems of cubic forms in many variables. *J. Reine Angew. Math.*, 757:309–328, 2019.

[Sal98] P. Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque*, (251):91–258, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[Sch16] D. Schindler. Manin's conjecture for certain biprojective hypersurfaces. *J. Reine Angew. Math.*, 714:209–250, 2016.



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