Carleman estimates for space semi-discrete approximations of one-dimensional stochastic parabolic equation and its applications

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Abstract

In this paper, we study discrete Carleman estimates for space semi-discrete approximations of one-dimensional stochastic parabolic equation. As applications of these discrete Carleman estimates, we apply them to study two inverse problems for the spatial semi-discrete stochastic parabolic equations, including a discrete inverse random source problem and a discrete Cauchy problem. We firstly establish two Carleman estimates for a one-dimensional semi-discrete stochastic parabolic equation, one for homogeneous boundary and the other for non-homogeneous boundary. Then we apply these two estimates separately to derive two stability results. The first one is the Lipschitz stability for the discrete inverse random source problem. The second one is the Hölder stability for the discrete Cauchy problem.

Keywords: Carleman estimates, inverse random source problem, Cauchy problem, spatial semi-discrete stochastic parabolic equation.

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $B(\cdot)$, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $T>0$, $G=(0,L)$ and $Q=G\times(0,T)$. Consider the following one-dimensional stochastic parabolic equation

$$\begin{cases} \mathrm{d}y - y_{xx}\mathrm{d}t = (ay+by_x+f)\mathrm{d}t + (cy+g)\mathrm{d}B(t), & (x,t)\in Q,\\ y(0,t)=y(L,t)=0, & t\in(0,T),\\ y(x,0)=y^0(x), & x\in G \end{cases} \tag{1.1}$$

with suitable coefficients a, b and c. Physically, f and g are source terms, g stands for the intensity of a random force of the white noise type.

The main objective of this paper is to establish discrete Carleman estimates for the finite-difference space discretization of stochastic parabolic equation (1.1), and to give its applications in two kinds of typical inverse problems, namely the inverse random source problem and the Cauchy problem for spatial semi-discrete stochastic parabolic equation. To do this, let us consider $N\in\mathbb{N}^*$, a step $h=\frac{L}{N+1}$, and an equidistant mesh of the interval $(0,L), 0=x_0<x_1<\cdots<x_N<x_{N+1}=L$, with $x_i=ih, 0\leq i\leq N+1$. The finite-difference approximation of the space derivatives leads to the following semi-discretization of (1.1):

$$\begin{cases} \mathrm{d}y_i(t) - \frac{y_{i+1}(t)-2y_i(t)+y_{i-1}(t)}{h^2}\mathrm{d}t = \left(a_iy_i(t)+b_i\frac{y_{i+1}(t)-y_i(t)}{h}+f_i\right)\mathrm{d}t & \\ \qquad\qquad\qquad\qquad\qquad\qquad +(c_iy_i(t)+g_i)\mathrm{d}B(t), & 1\leq i\leq N,\ t>0,\\ y_0(t)=y_{N+1}(t)=0, & t>0,\\ y_i(0)=y_i^0, & 1\leq i\leq N, \end{cases} \tag{1.2}$$

where $y_i(t)$ stands for $y(x_i,t)$, analogous definitions are for a_i, b_i, f_i, g_i and the other functions in the sequel.

Throughout this paper, we denote the discrete domains and discrete boundaries by the following notations

$$\begin{aligned} &G_h=\{x_1,x_2,\cdots,x_N\}, && Q_h=G_h\times(0,T),\\ &\overline{G}_h=\{x_0,x_1,x_2,\cdots,x_N,x_{N+1}\}, && \overline{Q}_h=\overline{G}_h\times(0,T),\\ &\partial G_h=\{x_0,x_{N+1}\}, && \Sigma_h=\partial G_h\times(0,T),\\ &G_h^-=\{x_0,x_1,x_2,\cdots,x_N\}, && Q_h^-=G_h^-\times(0,T). \end{aligned}$$

We denote by $\mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}}$ the set of discrete functions defined on G_h and $\overline{G}_h$, respectively. Furthermore, for $u_h=(u_0,u_1,\cdots,u_{N+1})^{\mathrm{T}}\in\mathbb{R}^{\overline{\mathfrak{M}}}$, we define the averaging operators and difference operators as follows

$$\begin{aligned} &(\mathbf{m}_h^+u_h)_i=\frac{u_{i+1}+u_i}{2}, \quad (\mathbf{m}_h^-u_h)_i=\frac{u_i+u_{i-1}}{2}, \quad (\mathbf{m}_hu_h)_i=\frac{u_{i+1}+2u_i+u_{i-1}}{4},\\ &(\mathbf{D}_h^+u_h)_i=\frac{u_{i+1}-u_i}{h}, \quad (\mathbf{D}_h^-u_h)_i=\frac{u_i-u_{i-1}}{h}, \quad (\mathbf{D}_hu_h)_i=\frac{u_{i+1}-u_{i-1}}{2h},\\ &(\Delta_hu_h)_i=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}. \end{aligned}$$

These notations allow us to express semi-discretization (1.2) in a more compact way, which is necessary to formulate our inverse problems.

Discrete inverse random source problem. Determine the random source term $g_h \in \mathbb{R}^{\mathfrak{M}}$ in the following semi-discrete stochastic parabolic equation:

$$\begin{cases} \mathrm{d}y_h - \Delta_h y_h \mathrm{d}t = (a_h y_h + b_h \mathbf{D}_h^+ y_h)\mathrm{d}t + g_h \mathrm{d}B(t), & (x_h, t) \in Q_h, \\ y_h = 0, & (x_h, t) \in \Sigma_h, \\ y_h(0) = y_h^0, & x_h \in G_h, \end{cases} \tag{1.3}$$

by the observation data

$$(\mathbf{D}_h^- y_h)_{N+1} \quad \text{and} \quad y_h(T),$$

where $y_h \in \mathbb{R}^{\overline{\mathfrak{M}}}$ and $a_h, b_h, y_h^0 \in \mathbb{R}^{\mathfrak{M}}$.

Let $G_0 \subset G$ such that $\overline{G_0} \subset G \cup \{x_{N+1}\}$ and $\partial G_0 \cap \partial G = \{x_{N+1}\}$. We set $G_{0,h} = G_0 \cap G_h$. The discrete Cauchy problem is described as follows.

Discrete Cauchy problem. For any $\epsilon > 0$, determine the solution y_h in $G_{0,h} \times (\epsilon, T - \epsilon)$ of the following semi-discrete stochastic parabolic equation:

$$\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t = (a_h y_h + b_h \mathbf{D}_h^+ y_h)\mathrm{d}t + c_h y_h \mathrm{d}B(t), \quad (x_h, t) \in Q_h, \tag{1.4}$$

by the discrete lateral boundary data

$$(y_h)_{N+1} = \xi(t) \quad \text{and} \quad \left(\mathbf{D}_h^- y_h\right)_{N+1} = \eta(t), \quad t \in (0, T). \tag{1.5}$$

Carleman estimate is a class of weighted energy estimates related to some differential operator, which can be applied to many aspects, such as inverse problems [18, 20, 21, 33], control theory [14, 19, 29] and so on. There are rich references on discrete Carleman estimates for deterministic partial differential equations, which can be divided into three main categories: space-discrete, time-discrete and full-discrete results. We refer to [4, 5, 7, 9, 10, 12, 23, 27] for space-discrete Carleman estimates, [6, 16] for time-discrete Carleman estimates and [8, 11] for full-discrete Carleman estimates. These discrete Carleman estimates have been successfully applied to prove the uniform controllability and the stability of inverse problems for various discrete deterministic partial differential equations [22, 28, 36, 37]. To the best of our knowledge, there is no paper considering discrete Carleman estimates for stochastic partial differential equations.

Carleman estimate is a powerful tool to study inverse problems related to various stochastic partial differential equations [15, 24, 31, 32, 34]. For inverse source problem related to stochastic parabolic equations, we refer to [25] for the uniqueness of an inverse problem of determining source function f in (1.1). The inverse source problem of determining two kinds of sources f and g simultaneously in (1.1) was studied in [35]. Moreover, we also refer to [1] or [2] for applications of regularization techniques in the numerical methods for inverse random source problems. The Cauchy problem aims to recover the solution with observed data from the lateral boundary. In [33], a conditional stability was proved for the

Cauchy problem of deterministic parabolic equations. Recently, this method is extended to the stochastic case. The conditional stability and convergence rate of the Tikhonov regularization method for the Cauchy problem of stochastic parabolic equations was obtained [13]. It is worth mentioning that in [26] the authors gave a detailed review on inverse problems for stochastic partial differential equations. However, these results were all obtained within a continuous framework. Discrete inverse problems for the stochastic differential equations have not been studied thoroughly yet.

In this paper, we firstly focus on Carleman estimates for discrete stochastic parabolic equation. More precisely, we will prove two Carleman estimates for spatial semi-discretization (1.2) of one-dimensional stochastic parabolic equation. We apply the first Carleman estimate to study the discrete inverse random source problem. Unlike the deterministic counterparts, the solution of a stochastic differential equation is not differentiable with respect to time variable. We have to choose a regular weight function to put the random source term on the left-hand side of discrete Carleman estimate. Since the second parameter λ plays an important role in the proof of the uniform stability result with respect to the mesh size, we need to carefully decouple λ from the constant C in our Carleman estimate. Secondly, in order to handle the discrete Cauchy problem, we need a slight revision to present the second discrete Carleman estimate with non-homogeneous boundary. Applying this Carleman estimate, we obtain a Hölder stability for the discrete Cauchy problem. In comparison with the deterministic discrete Carleman estimates, there are additional terms arising from stochastic effects to be considered.

The rest of this paper is organized as follows. In section 2, we present discrete settings and our main results. In section 3, we show two Carleman estimates for space semi-discrete approximations of one-dimensional stochastic parabolic equation. In next two sections, based on these two Carleman estimates we study the discrete inverse random source problem and the discrete Cauchy problem, respectively.

2 Discrete settings and main results

In this section, we introduce fundamental concepts of discrete calculus, including discrete function spaces, some useful discrete identities, and integration by parts for discrete operators, which will be used in the proofs of our main results. Subsequently, we present our main results in this paper. We first give two discrete Carleman estimates for the semi-discrete stochastic parabolic equation, namely, one for homogeneous boundary and the other for non-homogeneous boundary. Then we address two stability results. The first one pertains to the discrete inverse random source problem, while the second one focuses on the discrete Cauchy problem.

2.1 Discrete settings

By analogy with the continuous case, for $u_h = (u_0, u_1, \cdots, u_{N+1})^{\mathrm{T}} \in \mathbb{R}^{\overline{\mathfrak{M}}}$ we define the discrete integrals:

$$\int_{G_h} u_h = h \sum_{x_i \in G_h} u_i = h \sum_{i=1}^{N} u_i, \quad \int_{G_h^-} u_h = h \sum_{x_i \in G_h^-} u_i = h \sum_{i=0}^{N} u_i.$$

For $u_h, v_h \in \mathbb{R}^{\mathfrak{M}}$, we define the following L^2-inner product on $\mathbb{R}^{\mathfrak{M}}$

$$(u_h, v_h)_{L^2(G_h)} = h \sum_{i=1}^{N} u_i v_i.$$

The associated norm is denoted by $\|u_h\|_{L^2(G_h)}$. Analogously, we define the L^∞-norm on $\mathbb{R}^{\mathfrak{M}}$

$$||u_h||^2_{L^\infty(G_h)} = \max_{1\le i\le N} |u_i|.$$

Also, we introduce

$$\|u_h\|^2_{H^1(G_h)} = \int_{G_h^-} |\mathbf{D}_h^+ u_h|^2 + \int_{G_h} |u_h|^2$$

and

$$\|u_h\|_{H^2(G_h)} = \int_{G_h} |\Delta_h u_h|^2 + \int_{G_h^-} |\mathbf{D}_h^+ u_h|^2 + \int_{G_h} |u_h|^2.$$

Furthermore, for a discrete Banach space $\mathcal{X}_h$ defined on G_h, we denote by $L^2(0,T;\mathcal{X}_h)$ the set of discrete functions endowed with the norm

$$||u_h||^2_{L^2(0,T;\mathcal{X}_h)} = \int_0^T \|u_h\|^2_{\mathcal{X}_h} \mathrm{d}t.$$

Now we introduce some notations for stochastic analysis on discrete space meshes. For a Banach space $\mathcal{Y}$, we denote by $L^2_{\mathcal{F}}(\Omega;\mathcal{Y})$ the space of all progressively measurable stochastic process ζ such that $\mathbb{E}(\|\zeta\|^2_{\mathcal{Y}}) < \infty$. For a discrete Banach space $\mathcal{X}_h$ on G_h, we denote by $L^2_{\mathcal{F}}(0,T;\mathcal{X}_h)$ the Banach space consisting of all $\mathcal{X}_h$-valued $\{\mathcal{F}_t\}_{t\ge0}$-adapted processes $\zeta(\cdot)$ such that $\mathbb{E}(\|\zeta(\cdot)\|^2_{L^2(0,T;\mathcal{X}_h)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;\mathcal{X}_h)$ the Banach space consisting of all $\mathcal{X}_h$-valued $\{\mathcal{F}_t\}_{t\ge0}$-adapted bounded processes.

For the space-discrete operators, we also need to provide several preliminary identities and discrete integration by parts formula that will be extensively used in the sequel. We present the results without the proof and refer the readers to [7] (see also [5]) for a detailed discussion.

Lemma 2.1. *Let u and v be discrete functions defined on $\overline{G}_h$. Then for the averaging operators and difference operators, we have the following identities:*

$$\mathbf{m}_h^+ u = u + \frac{h}{2}\mathbf{D}_h^+ u, \;\; \mathbf{m}_h u = u + \frac{h^2}{4}\Delta_h u, \;\; \mathbf{D}_h u = \mathbf{m}_h^- \mathbf{D}_h^+ u, \;\; \Delta_h u = \mathbf{D}_h^+\mathbf{D}_h^- u, \tag{2.1}$$

$$\mathbf{m}_h^+(uv) = \mathbf{m}_h^+ u \mathbf{m}_h^+ v + \frac{h^2}{4}\mathbf{D}_h^+ u \mathbf{D}_h^+ v, \qquad \mathbf{D}_h^+(uv) = \mathbf{D}_h^+ u \mathbf{m}_h^+ v + \mathbf{m}_h^+ u \mathbf{D}_h^+ v, \tag{2.2}$$

$$\Delta_h(uv) = \Delta_h u \mathbf{m}_h v + 2\mathbf{D}_h u \mathbf{D}_h v + \mathbf{m}_h u \Delta_h v. \tag{2.3}$$

Lemma 2.2. *Let u and v be discrete functions defined on $\overline{G}_h$ such that $v_0 = v_{N+1} = 0$. Then we have the following identities:*

$$2\int_{G_h} uv\mathbf{D}_h v = -\int_{G_h} \mathbf{D}_h u|v|^2 + \frac{h^2}{2}\int_{G_h^-} \mathbf{D}_h^+ u|\mathbf{D}_h^+ v|^2, \tag{2.4}$$

$$\int_{G_h} u\Delta_h v = -\int_{G_h^-} \mathbf{D}_h^+ u\mathbf{D}_h^+ v - u_0(\mathbf{D}_h^+ v)_0 + u_{N+1}(\mathbf{D}_h^- v)_{N+1}, \tag{2.5}$$

$$\int_{G_h} uv\Delta_h v = -\int_{G_h^-} \mathbf{m}_h^+ u|\mathbf{D}_h^+ v|^2 + \frac{1}{2}\int_{G_h} \Delta_h u|v|^2, \tag{2.6}$$

$$2\int_{G_h} u\mathbf{D}_h v\Delta_h v = -\int_{G_h^-} \mathbf{D}_h^+ u|\mathbf{D}_h^+ v|^2 + u_{N+1}|(\mathbf{D}_h^- v)_{N+1}|^2 - u_0|(\mathbf{D}_h^+ v)_0|^2. \tag{2.7}$$

Remark 2.1. According to the detailed proof of discrete integration by parts (2.5) in [5], we know that it also holds for v without $v_0 = v_{N+1} = 0$. So we can use (2.5) to obtain (3.60) in next section.

2.2 Main results

To state our first result, we introduce several weight functions which will be used in our discrete Carleman estimate. Let λ and s be two large parameters. For some $x^* < 0$, $t_0 \in (0,T)$ and a small parameter $\beta > 0$, we define the regular weight functions by

$$\varphi(x,t) = e^{\lambda\psi(x,t)}, \quad \theta(x,t) = e^{s\varphi(x,t)}, \quad r(x,t) = \theta^{-1}(x,t) \tag{2.8}$$

with

$$\psi(x,t) = |x - x^*|^2 - \beta|t - t_0|^2. \tag{2.9}$$

Moreover, in the following, we will use C to denote generic positive constants depending on x^*, L, T, β, but independent of s and λ. Similarly, $C(\lambda)$ denote constants also depending on λ. Moreover, we use the notation $\mathcal{O}_\lambda(\gamma)$, which satisfies $|\mathcal{O}_\lambda(\gamma)| \le C(\lambda)|\gamma|$ with a constant $C(\lambda)$. All of these notations may vary from line to line and are independent of h.

According to Proposition 2.9 and Lemma 2.12 in [5], we have the following asymptotic expansion properties:

$$\theta\mathbf{D}_h r = -s\lambda A_1, \quad \theta\Delta_h r = s^2\lambda^2 A_2 - s\lambda^2 A_3 - s\lambda A_4 \tag{2.10}$$

where

$$A_j = f_j + \mathcal{O}_\lambda(sh), \quad j = 1,2,3,4 \tag{2.11}$$

with

$$f_1 = \varphi\partial_x\psi, \quad f_2 = \varphi^2|\partial_x\psi|^2, \quad f_3 = \varphi|\partial_x\psi|^2, \quad f_4 = \varphi\partial_{xx}\psi.$$

Further, we also have for $j = 1,2,3,4$ that

$$\begin{cases} \mathbf{m}_h^\pm A_j = f_j + \mathcal{O}_\lambda(sh), & \partial_t\mathbf{m}_h^+ A_j = \partial_t f_j + \mathcal{O}_\lambda(sh), \\ \partial_t A_j = \partial_t f_j + \mathcal{O}_\lambda(sh), & \mathbf{D}_h A_j = \partial_x f_j + \mathcal{O}_\lambda(sh) = \mathbf{D}_h^\pm A_j + \mathcal{O}_\lambda(sh), \\ \Delta_h A_j = \partial_{xx} f_j + \mathcal{O}_\lambda(sh), & \partial_t\mathbf{D}_h^\pm A_j = \partial_{xt} f_j + \mathcal{O}_\lambda(sh). \end{cases} \tag{2.12}$$

The first main result in this paper is the following uniform Carleman estimate for the semi-discrete stochastic parabolic equation.

Theorem 2.3. *Let $f_h \in L^2_{\mathcal{F}}(0,T;L^2(G_h))$, $g_h \in L^2_{\mathcal{F}}(0,T;H^1(G_h))$. For the parameter $\lambda \geq 1$ sufficiently large, there exist positive constant C depending on x^*, L, T, β, and positive constants s_0, $\varepsilon, h_0, C(\lambda)$ also depending on λ, all of which are independent of h such that*

$$\begin{aligned}
&\mathbb{E}\int_{Q_h} \frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|\partial_h^+ y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|y_h|^2\mathrm{d}t \\
&+\mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|g_h|^2\mathrm{d}t \\
\leq & C\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2|\partial_h^+ g_h|^2\mathrm{d}t + C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- y_h)_{N+1}|^2\mathrm{d}t \\
&+ C(\lambda)s^2e^{C(\lambda)s}||y_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}
\end{aligned} \tag{2.13}$$

for all $h \in (0,h_0)$, $s \in (s_0, \sqrt{\varepsilon/h})$ and all $y_h \in L^2_{\mathcal{F}}(0,T;H^2(G_h))$ satisfying

$$\begin{cases}
\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t = f_h\mathrm{d}t + g_h\mathrm{d}B(t), & (x_h,t)\in Q_h,\\
y_h = 0, & (x_h,t)\in\Sigma_h,\\
y_h(0) = 0, & x_h \in G_h.
\end{cases} \tag{2.14}$$

Remark 2.2. In comparison with the existing discrete Carleman estimates for parabolic equations [6, 8, 11], we introduce a regular weight function in our Carleman estimate. The reason is that the solution of a stochastic parabolic equation is not differentiable with respect to time variable. This leads to that the method by applying Carleman estimate to u_t could not directly employed for the inverse random source problem. Consequently, we have to choose the regular weight function to put the random source term on the left-side of the Carleman estimate. Unfortunately, there is still a first-order difference term left on the right-hand side of the Carleman estimate. This means that the unknown random source function has to satisfy condition (2.17) in the proof of the stability result.

Remark 2.3. In this Carleman estimate, the boundary term at x_{N+1} could be replaced by the one at x_0. In fact, if we choose x^* in weight function (2.9) such that $x^* > L$, we can obtain $\psi_x < 0$ for all $x \in G$. Then the boundary term left in (3.43) is the one at x_0.

Remark 2.4. In this Carleman estimate, we use a special function $|x - x^*|^2$ in ψ. In fact, we can choose a general form

$$\psi(x,t) = d(x) - \beta|t-t_0|^2$$

with function d such that $|d_x| > 0$ in G to guarantee (3.39) and (3.40).

In order to deal with the discrete Cauchy problem, we need the following discrete Carleman estimate with non-homogeneous boundary condition.

Theorem 2.4. *Let $f_h \in L^2_{\mathcal{F}}(0,T;L^2(G_h))$, $g_h \in L^2_{\mathcal{F}}(0,T;H^1(G_h))$ and $\gamma_1,\gamma_2 \in L^2_{\mathcal{F}}(\Omega;H^1(0,T))$ satisfying compatibility condition $\gamma_1(0)=\gamma_2(0)=0$. For the parameter $\lambda \geq 1$ sufficiently large, there exist positive constant C depending on x^*, L, T, β, and positive constants s_0, $\varepsilon, h_0, C(\lambda)$ also depending on λ, all of which are independent of h such that*

$$\begin{aligned}
&\mathbb{E}\int_{Q_h} \frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|\partial_h^+ y_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|y_h|^2 \mathrm{d}t \\
&\quad + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|g_h|^2 \mathrm{d}t \\
\leq & C\mathbb{E}\int_{Q_h} \theta^2|f_h|^2 \mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2|\partial_h^+ g_h|^2 \mathrm{d}t + C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- y_h)_{N+1}|^2 \mathrm{d}t \\
&\quad + C(\lambda)s^3 e^{C(\lambda)s}\sum_{i=1}^2 \|\gamma_i\|^2_{L^2(\Omega;H^1(0,T))} + C(\lambda)s^2 e^{C(\lambda)s}||y_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}
\end{aligned} \tag{2.15}$$

for all $h \in (0,h_0)$, $s \in (s_0, \sqrt{\varepsilon/h})$ and all $y_h \in L^2_{\mathcal{F}}(0,T;H^2_h(G_h))$ satisfying

$$\begin{cases}
\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t = f_h \mathrm{d}t + g_h \mathrm{d}B(t), & (x_h,t) \in Q_h, \\
(y_h)_0 = \gamma_1,\ (y_h)_{N+1} = \gamma_2, & t \in (0,T), \\
y_h(0) = 0, & x_h \in G_h.
\end{cases} \tag{2.16}$$

Remark 2.5. The duality argument introduced in [17] is a main tool to handle non-homogeneous boundary conditions when proving Carleman estimates. It seems that by employing this duality argument, we could obtain a weaker Carleman estimate with $\gamma_1,\gamma_2 \in L^2_{\mathcal{F}}(\Omega;L^2(0,T))$. In this case, the second derivative term could not be included in the left-hand side of (2.15). However, as mentioned in [33], this regularity of $\gamma_1,\gamma_2 \in L^2_{\mathcal{F}}(\Omega;H^1(0,T))$ is necessary to establish the stability for the Cauchy problem. Therefore, we use a simple method to make the boundary conditions homogeneous, and then use Theorem 2.3 to prove (2.15).

Based on the first Carleman estimate, we obtain the uniform stability result with respect to the mesh size h for our discrete inverse random source problem.

Theorem 2.5. *Let $a_h, b_h \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G_h))$ and $g_h^{(j)} \in L^2_{\mathcal{F}}(0,T;H^1(G_h))$ for $j=1,2$ such that*

$$\left|\mathbf{D}_h^+\left(g_h^{(1)} - g_h^{(2)}\right)_i\right| \leq C\left|\left(g_h^{(1)} - g_h^{(2)}\right)_i\right|, \quad i = 0,1,2,\cdots,N. \tag{2.17}$$

Then there exists a positive constant C depending on x^*, L, T and β, but independent of h such that

$$\begin{aligned}
\left\|g_h^{(1)} - g_h^{(2)}\right\|_{L^2_{\mathcal{F}}(0,T;L^2(G_h))} \leq & C\left\|\left(\mathbf{D}_h^- y_h^{(1)}\right)_{N+1} - \left(\mathbf{D}_h^- y_h^{(2)}\right)_{N+1}\right\|_{L^2_{\mathcal{F}}(\Omega;L^2(0,T))} \\
& + C\left\|y_h^{(1)}(T) - y_h^{(2)}(T)\right\|_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))},
\end{aligned} \tag{2.18}$$

where $y_h^{(j)}$ is the solution to (1.3) corresponding to $g_h^{(j)}$ for $j=1,2$, respectively.

Remark 2.6. A special form of unknown source function is $g_h = r(t)R_h$ with known R_h such that

$$\left|\left(\mathbf{D}_h^+ R_h\right)_i\right| \le R_1 \quad \text{and} \quad |(R_h)_i| \ge R_0 > 0, \quad \mathbb{P}-a.s.$$

with positive constants R_0 and R_1. Then we have

$$\left|\mathbf{D}_h^+\left(g_h^{(1)} - g_h^{(2)}\right)_i\right| = \left|r^{(1)} - r^{(2)}\right| \left|\left(\mathbf{D}_h^+ R_h\right)_i\right| \le \frac{R_1}{R_0}\left|\left(g_h^{(1)} - g_h^{(2)}\right)_i\right|.$$

Compared to the continuous inverse problem addressed in [31, 35], a similar condition is imposed on the random source function in the continuous setting to investigate the corresponding inverse problem.

Remark 2.7. There is an additional term depending on the mesh size h in the stability result for discrete inverse problem related to hyperbolic equations [4, 5]. This is because of the appearance of the term $h\partial_h^+\partial_t y_h$ on the right-hand side of the Carleman estimate, which can not be removed according to [5]. However, for parabolic equations, there is no such an additional term in Carleman estimate (2.13). This means that our stability result is uniform with respect to h.

Remark 2.8. The uniqueness is a direct result from Theorem 2.5. More precisely, under the same assumptions as in Theorem 2.5 and if

$$\left(\mathbf{D}_h^- y_h^{(1)}\right)_{N+1} = \left(\mathbf{D}_h^- y_h^{(2)}\right)_{N+1} \quad \textit{and} \quad y_h^{(1)}(T) = y_h^{(2)}(T), \quad \mathbb{P}-a.s.,$$

then $g_h^{(1)} = g_h^{(2)}$ in Q_h, $\mathbb{P}$−a.s.

The last main result is the stability result for our discrete Cauchy problem, which is an application of the second discrete Carleman estimate.

Theorem 2.6. *Let $a_h, b_h \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G_h))$, $c_h \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G_h))$, $\xi \in L^2_{\mathcal{F}}(\Omega;H^1(0,T))$ and $\eta \in L^2_{\mathcal{F}}(\Omega;L^2(0,T))$. Then for any $\epsilon > 0$, there exist positive constants C, h^* and $\kappa \in (0,1)$ depending on x^*, L, T, β and ϵ, such that*

$$\|y_h\|_{L^2_{\mathcal{F}}(\epsilon,T-\epsilon;H^2(G_{0,h})} \le CM^\kappa \left(\|\xi\|_{L^2(\Omega;H^1(0,T))} + \|\eta\|_{L^2(\Omega;L^2(0,T))}\right)^{1-\kappa} \tag{2.19}$$

for all $h \in (0,h^*)$ and all $y_h \in L^2_{\mathcal{F}}(0,T;H^2(G_h))$ satisfying

$$\|y_h\|_{L^2_{\mathcal{F}}(0,T;H^2(G_h))} \le M.$$

Remark 2.9. The stability result also holds for discrete Cauchy problem with the lateral boundary data at x_0. Since $\epsilon > 0$ is arbitrary, (2.19) immediately implies the uniqueness of the solution of (1.4) in Q_h with $\xi(t) = \eta(t) = 0$ for $t \in (0,T)$.

3 Discrete Carleman estimates

In this section, we prove two discrete Carleman estimates for semi-discrete stochastic parabolic equation, i.e. Theorem 2.3 and Theorem 2.4. The proof follows as close as possible the ideas presented in the classical continuous setting (see e.g. [3, 30]), where Carleman estimates for stochastic parabolic equations are obtained in the continuous setting.

3.1 Proof of Theorem 2.3

Let $l = s\varphi, \theta = e^l, r = \theta^{-1}$ and $Y_h = \theta y_h$. By Itô formula, we obtain

$$\theta \mathrm{d}y_h = \mathrm{d}Y_h - \partial_t l Y_h \mathrm{d}t. \tag{3.1}$$

By (2.1) and (2.3) in Lemma 2.1, we have

$$\begin{aligned}\theta\Delta_h y_h \mathrm{d}t =& \theta\left(\Delta_h r \mathbf{m}_h Y_h + 2\mathbf{D}_h r \mathbf{D}_h Y_h + \mathbf{m}_h r \Delta_h Y_h\right)\mathrm{d}t\\ =& \theta\Delta_h r\left(Y_h + \frac{h^2}{4}\Delta_h Y_h\right)\mathrm{d}t + 2\theta\mathbf{D}_h r\mathbf{D}_h Y_h\mathrm{d}t + \theta\left(r + \frac{h^2}{4}\Delta_h r\right)\Delta_h Y_h \mathrm{d}t\\ =& \theta\Delta_h r Y_h \mathrm{d}t + \left(1 + \frac{h^2}{2}\theta\Delta_h r\right)\Delta_h Y_h\mathrm{d}t + 2\theta\mathbf{D}_h r\mathbf{D}_h Y_h\mathrm{d}t.\end{aligned} \tag{3.2}$$

Then from (3.1) and (3.2), it follows

$$\begin{aligned}&\theta\left(\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t\right)\\ =& \mathrm{d}Y_h - \partial_t l Y_h \mathrm{d}t - \theta\Delta_h r Y_h \mathrm{d}t - \left(1 + \frac{h^2}{2}\theta\Delta_h r\right)\Delta_h Y_h \mathrm{d}t - 2\theta\mathbf{D}_h r \mathbf{D}_h Y_h \mathrm{d}t.\end{aligned} \tag{3.3}$$

According to (2.10), we rewrite (3.3) as the following form:

$$\theta(\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t) = I_1 + I\mathrm{d}t, \tag{3.4}$$

where

$$\begin{aligned}&I_1 = \mathrm{d}Y_h + 2s\lambda A_1 \mathbf{D}_h Y_h \mathrm{d}t + \Psi Y_h \mathrm{d}t,\\ &I = -(1 + A_0)\Delta_h Y_h - s^2\lambda^2 A_2 Y_h + (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)Y_h.\end{aligned}$$

with

$$A_0 = \frac{h^2}{2}\left(s^2\lambda^2 A_2 - s\lambda^2 A_3 - s\lambda A_4\right), \quad \Psi = \tau\partial_{xx} l.$$

Here τ is a positive constant such that $\tau \in (3/2, 3)$. By (2.12) we have the following properties of A_0:

$$A_0 = \frac{h^2}{2}\left(s^2\lambda^2 f_2 - s\lambda^2 f_3 - s\lambda f_4 + s^2\mathcal{O}_\lambda(sh)\right) = \mathcal{O}_\lambda(sh). \tag{3.5}$$

Similarly,

$$\mathbf{m}_h^+ A_0 = \mathcal{O}_\lambda(sh), \quad \mathbf{D}_h^+ A_0 = \mathcal{O}_\lambda(sh), \quad \Delta_h A_0 = \mathcal{O}_\lambda(sh). \tag{3.6}$$

Furthermore, multiplying I in both sides of (3.4), integrating the result equality over Q_h and taking mathematical expectation, we find

$$\mathbb{E}\int_{Q_h} \theta I(\mathrm{d}y_h - \Delta_h y_h \mathrm{d}t) = \mathbb{E}\int_{Q_h} II_1 + \mathbb{E}\int_{Q_h} I^2 \mathrm{d}t. \tag{3.7}$$

Next, we provide a detailed calculation of the term involving II_1 and then find positive lower bounds for the terms related to $\mathrm{d}Y_h$ and $\mathrm{d}t$, respectively. Finally, we combine these estimates to complete our proof. To do this, we divide the subsequent proof into the following several steps. For clarity, we first split the term of II_1 into a sum of nine terms

$$\mathbb{E}\int_{Q_h} II_1 = \sum_{i,j=1}^{3} I_{ij}, \tag{3.8}$$

where I_{ij} is the inner product of i-th term of I_1 and the j-th term of I.

3.1.1 Estimates involving the terms of $\mathrm{d}Y_h$

By discrete identities (2.1) and discrete integration by parts (2.5), together with

$$(\mathrm{d}Y_h)_0 = (\mathrm{d}Y_h)_{N+1} = 0, \quad t \in [0,T]$$

due to $y_h = 0$ on Σ_h, we obtain

$$\begin{aligned}
I_{11} =& -\mathbb{E}\int_{Q_h} (1+A_0)\Delta_h Y_h \mathrm{d}Y_h \\
=&\mathbb{E}\int_{Q_h^-} \mathbf{D}_h^+ Y_h \mathbf{D}_h^+(\mathrm{d}Y_h + A_0\mathrm{d}Y_h) + \mathbb{E}\int_0^T (\mathbf{D}_h^+ Y_h)_0(\mathrm{d}Y_h + A_0\mathrm{d}Y_h)_0 \\
&- \mathbb{E}\int_0^T (\mathbf{D}_h^- Y_h)_{N+1}(\mathrm{d}Y_h + A_0\mathrm{d}Y_h)_{N+1} \\
=&\mathbb{E}\int_{Q_h^-} \left(1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0\right) \mathbf{D}_h^+ Y_h \mathbf{D}_h^+ \mathrm{d}Y_h + \mathbb{E}\int_{Q_h^-} \mathbf{D}_h^+ A_0 \mathbf{D}_h^+ Y_h \mathrm{d}Y_h.
\end{aligned} \tag{3.9}$$

By using Itô formula, we further obtain

$$I_{11} = X_1 + Y_1 + Z_1 + \mathbb{E}\int_{Q_h^-} B_1 |\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t, \tag{3.10}$$

where

$$\begin{aligned}
X_1 &= \frac{1}{2}\mathbb{E}\int_{Q_h^-} \mathrm{d}\left(\left(1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0\right)|\mathbf{D}_h^+ Y_h|^2\right), \\
Y_1 &= -\frac{1}{2}\mathbb{E}\int_{Q_h^-} \left(1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0\right)|\mathbf{D}_h^+ \mathrm{d}Y_h|^2, \\
Z_1 &= \mathbb{E}\int_{Q_h^-} \mathbf{D}_h^+ A_0 \mathbf{D}_h^+ Y_h \mathrm{d}Y_h, \\
B_1 &= -\frac{1}{2}\partial_t\left(1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0\right).
\end{aligned}$$

Using Itô formula again, we obtain

$$I_{12} = -\mathbb{E}\int_{Q_h} s^2\lambda^2 A_2 Y_h \mathrm{d}Y_h = X_2 + Y_2 + \mathbb{E}\int_{Q_h} D_1 |Y_h|^2 \mathrm{d}t, \tag{3.11}$$

where

$$
\begin{aligned}
X_2 =& -\frac{1}{2}\mathbb{E}\int_{Q_h} \mathrm{d}(s^2\lambda^2 A_2|Y_h|^2),\\
Y_2 =& \frac{1}{2}\mathbb{E}\int_{Q_h} s^2\lambda^2 A_2|\mathrm{d}Y_h|^2,\\
D_1 =& \frac{1}{2}s^2\lambda^2\partial_t A_2.
\end{aligned}
$$

Similarly,

$$
I_{13} = \mathbb{E}\int_{Q_h} (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)Y_h \mathrm{d}Y_h = X_3 + Y_3 + \mathbb{E}\int_{Q_h} D_2|Y_h|^2\mathrm{d}t, \tag{3.12}
$$

where

$$
\begin{aligned}
X_3 =& \frac{1}{2}\mathbb{E}\int_{Q_h} \mathrm{d}\left(\left(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi\right)|Y_h|^2\right),\\
Y_3 =& -\frac{1}{2}\mathbb{E}\int_{Q_h} (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)|\mathrm{d}Y_h|^2,\\
D_2 =& -\frac{1}{2}(-\partial_{tt} l + s\lambda^2\partial_t A_3 + s\lambda\partial_t A_4 - \partial_t\Psi).
\end{aligned}
$$

Therefore, we have

$$
\sum_{j=1}^{3} I_{1j} = \sum_{i=1}^{3} X_i + \sum_{i=1}^{3} Y_i + Z_1 + \sum_{i=1}^{2}\mathbb{E}\int_{Q_h} D_i|Y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} B_1|\mathbf{D}_h^+ Y_h|^2\mathrm{d}t. \tag{3.13}
$$

Using (3.6), we obtain

$$
1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0 = 1 + \mathcal{O}_\lambda(sh). \tag{3.14}
$$

If we choose $\varepsilon = \varepsilon(\lambda)$ sufficiently small such that $|\mathcal{O}_\lambda(sh)| \leq \frac{1}{2}$, we further have

$$
1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0 \geq \frac{1}{2}. \tag{3.15}
$$

On the other hand, we have

$$
\begin{aligned}
&-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi = -s\partial_t\varphi + s\lambda^2 A_3 + s\lambda A_4 - \tau\partial_{xx} l\\
=& -s\lambda\varphi\psi_t + (1-\tau)(s\lambda^2\varphi|\partial_x\psi|^2 + s\lambda\varphi\partial_{xx}\psi) + s\mathcal{O}_\lambda(sh).
\end{aligned} \tag{3.16}
$$

By (3.15) and (3.16), together with $Y_h(0) = 0$ due to $y_h(0) = 0$, we then use integration by parts with respect to time t to yield

$$
\begin{aligned}
\sum_{i=1}^{3} X_i =& \frac{1}{2}\mathbb{E}\int_{G_h^-} \left(\left(1 + \frac{h}{2}\mathbf{D}_h^+ A_0 + \mathbf{m}_h^+ A_0\right)|\mathbf{D}_h^+ Y_h|^2\right)\Big|_{t=T}\\
&- \frac{1}{2}\mathbb{E}\int_{G_h} (s^2\lambda^2 A_2|Y_h|^2)\Big|_{t=T}\\
&+ \frac{1}{2}\mathbb{E}\int_{G_h} \left(\left(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi\right)|Y_h|^2\right)\Big|_{t=T}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{4}\mathbb{E}\int_{G_h^-}\left(|\mathbf{D}_h^+ Y_h|^2\right)\Big|_{t=T} - C(\lambda)s^2(1+|\mathcal{O}_\lambda(sh)|)\,\mathbb{E}\int_{G_h}(|Y_h|^2)\Big|_{t=T}\\
&\quad - C(\lambda)s\,(1+|\mathcal{O}_\lambda(sh)|)\,\mathbb{E}\int_{G_h}\left(|Y_h|^2\right)\Big|_{t=T}\\
&\geq -C(\lambda)s^2\,\mathbb{E}\int_{G_h}\left(\theta^2|y_h|^2\right)\Big|_{t=T}.
\end{aligned}
\tag{3.17}
$$

Next we estimate the terms involving Y_i in (3.13). By (3.6), we obtain

$$
\begin{aligned}
\sum_{i=1}^{3} Y_i = &-\frac{1}{2}\mathbb{E}\int_{Q_h^-}(1+\mathcal{O}_\lambda(sh))|\mathbf{D}_h^+\mathrm{d}Y_h|^2\\
&+\frac{1}{2}\mathbb{E}\int_{Q_h}\left(s^2\lambda^2 A_2+\partial_t l - s\lambda^2 A_3 - s\lambda A_4+\Psi\right)|\mathrm{d}Y_h|^2.
\end{aligned}
\tag{3.18}
$$

Obviously,

$$
\begin{aligned}
|\mathbf{D}_h^+\mathrm{d}Y_h|^2 =&|\mathbf{D}_h^+(\theta\mathrm{d}y_h)|^2 = |\mathbf{D}_h^+\theta\mathbf{m}_h^+\mathrm{d}y_h+\mathbf{m}_h^+\theta\mathbf{D}_h^+\mathrm{d}y_h|^2\\
=&|\mathbf{D}_h^+\theta\mathrm{d}y_h + h\mathbf{D}_h^+\theta\mathbf{D}_h^+\mathrm{d}y_h+\theta\mathbf{D}_h^+\mathrm{d}y_h|^2\\
=&|\mathbf{D}_h^+\theta|^2|\mathrm{d}y_h|^2+\left(h\mathbf{D}_h^+\theta+\theta\right)^2|\mathbf{D}_h^+\mathrm{d}y_h|^2+2\left(h|\mathbf{D}_h^+\theta|^2+\theta\mathbf{D}_h^+\theta\right)\mathrm{d}y_h\mathbf{D}_h^+\mathrm{d}y_h.
\end{aligned}
\tag{3.19}
$$

By using Young's inequality, we further obtain

$$
\begin{aligned}
|\mathbf{D}_h^+\mathrm{d}Y_h|^2 \leq&|\mathbf{D}_h^+\theta|^2|\mathrm{d}y_h|^2+2h^2|\mathbf{D}_h^+\theta|^2|\mathbf{D}_h^+\mathrm{d}y_h|^2+2\theta^2|\mathbf{D}_h^+\mathrm{d}y_h|^2\\
&+\frac{1}{2}h^2\partial_{xx}l|\mathbf{D}_h^+\theta|^2|\mathrm{d}y_h|^2+\frac{2}{\partial_{xx}l}|\mathbf{D}_h^+\theta|^2|\mathbf{D}_h^+\mathrm{d}y_h|^2\\
&+\frac{1}{2}\partial_{xx}l\theta^2|\mathrm{d}y_h|^2+\frac{2}{\partial_{xx}l}|\mathbf{D}_h^+\theta|^2|\mathbf{D}_h^+\mathrm{d}y_h|^2\\
\leq&\left(|\mathbf{D}_h^+\theta|^2+\frac{1}{2}h^2\partial_{xx}l|\mathbf{D}_h^+\theta|^2+\frac{1}{2}\partial_{xx}l\theta^2\right)|\mathrm{d}y_h|^2\\
&+\left(2h^2|\mathbf{D}_h^+\theta|^2+2\theta^2+\frac{4}{\partial_{xx}l}|\mathbf{D}_h^+\theta|^2\right)|\mathbf{D}_h^+\mathrm{d}y_h|^2.
\end{aligned}
\tag{3.20}
$$

By Taylor formula, we have

$$
\left(\mathbf{D}_h^+\theta\right)_j=\int_0^1\partial_x\theta(x_j+\sigma h,t)\mathrm{d}\sigma=\left(\partial_x\theta\right)_j+s\mathcal{O}_\lambda(sh)\theta_j.
\tag{3.21}
$$

Similarly,

$$
\left(\mathbf{\Delta}_h\theta\right)_j=\left(\partial_{xx}\theta\right)_j+s^2\mathcal{O}_\lambda(sh)\theta_j.
\tag{3.22}
$$

Therefore, we obtain

$$
\begin{aligned}
&|\mathbf{D}_h^+\theta|^2+\frac{1}{2}h^2\partial_{xx}l|\mathbf{D}_h^+\theta|^2+\frac{1}{2}\partial_{xx}l\theta^2\\
\le&\left(1+\frac{1}{2}h^2(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}\psi)\right)\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+s^2\left|\mathcal{O}_\lambda(sh)\right|\right)\theta^2\\
&+\frac{1}{2}\left(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}\psi\right)\theta^2\\
\le&\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+\frac{1}{2}s\lambda^2\varphi|\partial_x\psi|^2+\frac{1}{2}s\lambda\varphi\partial_{xx}\psi+s^2\left|\mathcal{O}_\lambda(sh)\right|\right)\theta^2
\end{aligned}\tag{3.23}
$$

and

$$
\begin{aligned}
2h^2|\mathbf{D}_h^+\theta|^2+2\theta^2+\frac{4}{\partial_{xx}l}|\mathbf{D}_h^+\theta|^2\le&2h^2\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+s^2\left|\mathcal{O}_\lambda(sh)\right|\right)\theta^2+2\theta^2\\
&+\frac{4\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+s^2\left|\mathcal{O}_\lambda(sh)\right|\right)\theta^2}{s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}\psi}\\
\le&\left(2+4s\varphi+s|\mathcal{O}_\lambda(sh)|\right)\theta^2.
\end{aligned}\tag{3.24}
$$

Combining (3.23), (3.24) and (3.20) we obtain

$$
\begin{aligned}
|\mathbf{D}_h^+\mathrm{d}Y_h|^2\le&\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+\frac{1}{2}s\lambda^2\varphi|\partial_x\psi|^2+\frac{1}{2}s\lambda\varphi\partial_{xx}\psi+s^2\left|\mathcal{O}_\lambda(sh)\right|\right)\theta^2|\mathrm{d}y_h|^2\\
&+C\left(s\varphi+s|\mathcal{O}_\lambda(sh)|\right)\theta^2|\mathbf{D}_h^+\mathrm{d}y_h|^2.
\end{aligned}\tag{3.25}
$$

On the other hand, from (2.11) and (3.16), it follows that

$$
\begin{aligned}
&s^2\lambda^2A_2+\partial_tl-s\lambda^2A_3-s\lambda A_4+\Psi\\
=&s^2\lambda^2\varphi^2|\partial_x\psi|^2+s\lambda\varphi\psi_t+(\tau-1)(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}\psi)+s\mathcal{O}_\lambda(sh).
\end{aligned}\tag{3.26}
$$

Then, noticing that $\mathrm{d}y_h=0$ at $x=x_0$ due to $(y_h)_0=0$ and substituting (3.25), (3.26) into (3.18), we obtain the following estimate

$$
\begin{aligned}
\sum_{i=1}^{3}Y_i\ge&-\frac{1}{2}\mathbb{E}\int_{Q_h^-}(s^2\lambda^2\varphi^2|\partial_x\psi|^2+\frac{1}{2}s\lambda^2\varphi|\partial_x\psi|^2+\frac{1}{2}s\lambda\varphi\partial_{xx}\psi)\theta^2|\mathrm{d}y_h|^2\\
&-s^2\mathbb{E}\int_{Q_h^-}|\mathcal{O}_\lambda(sh)|\theta^2|\mathrm{d}y_h|^2-C\mathbb{E}\int_{Q_h^-}\left(s\varphi+s|\mathcal{O}_\lambda(sh)|\right)\theta^2|\mathbf{D}_h^+\mathrm{d}y_h|^2\\
&+\frac{1}{2}\mathbb{E}\int_{Q_h^-}\left(s^2\lambda^2\varphi^2|\partial_x\psi|^2+s\lambda\varphi\psi_t+(\tau-1)(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}\psi)\right)\theta^2|\mathrm{d}y_h|^2\\
\ge&\frac{1}{4}\mathbb{E}\int_{Q_h^-}\left(\tau-\frac{3}{2}\right)s\lambda^2\varphi|\partial_x\psi|^2\theta^2|\mathrm{d}y_h|^2-s^2\mathbb{E}\int_{Q_h^-}|\mathcal{O}_\lambda(sh)|\theta^2|\mathrm{d}y_h|^2\\
&-C\mathbb{E}\int_{Q_h^-}\left(s\varphi+s|\mathcal{O}_\lambda(sh)|\right)\theta^2|\mathbf{D}_h^+\mathrm{d}y_h|^2.
\end{aligned}\tag{3.27}
$$

where we choose λ sufficiently small such that

$$
\frac{1}{4}\left(\tau-\frac{3}{2}\right)\lambda|\psi_x|^2\ge\frac{1}{2}\left(\tau-\frac{3}{2}\right)|\partial_{xx}\psi|+\frac{1}{2}|\psi_t|,\quad t\in[0,T].
$$

Since $s \in (s_0, \sqrt{\varepsilon/h})$, there exists $\varepsilon(\lambda)$ sufficiently small such that

$$s^2|\mathcal{O}_\lambda(sh)| \le \frac{\varepsilon}{h}C(\lambda)sh \le \frac{1}{8}s. \tag{3.28}$$

Thus, together with $|\mathrm{d}y_h|^2 = |g_h|^2\mathrm{d}t$, we obtain

$$\sum_{i=1}^{3} Y_i \ge \frac{1}{4}\mathbb{E}\int_{Q_h^-} (\tau-2)\, s\lambda^2\varphi|\partial_x\psi|^2\theta^2|g_h|^2\mathrm{d}t - C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2|\mathbf{D}_h^+ g_h|^2\mathrm{d}t. \tag{3.29}$$

Now we deal with the cross term Z_1 in (3.13). By (2.1) and Itô's inequality, we obtain

$$\begin{aligned}
&\mathbf{D}_h^+ A_0 \mathbf{D}_h^+ Y_h \mathrm{d}Y_h\\
=&\mathbf{D}_h^+ A_0(\mathbf{D}_h^+\theta \mathbf{m}_h^+ y_h + \mathbf{m}_h^+\theta \mathbf{D}_h^+ y_h)(\theta \mathrm{d}y_h + \partial_t\theta y_h\mathrm{d}t)\\
=&\mathbf{D}_h^+ A_0\left(\mathbf{D}_h^+\theta y_h + \theta\mathbf{D}_h^+ y_h + h\mathbf{D}_h^+\theta\mathbf{D}_h^+ y_h\right)(\theta\mathrm{d}y_h + \partial_t\theta y_h\mathrm{d}t)\\
=&\frac{1}{2}\mathrm{d}\left(\mathbf{D}_h^+ A_0\theta\mathbf{D}_h^+\theta|y_h|^2\right) - \frac{1}{2}\mathbf{D}_h^+ A_0\theta\mathbf{D}_h^+\theta|\mathrm{d}y_h|^2 - \frac{1}{2}\partial_t(\mathbf{D}_h^+ A_0\theta\mathbf{D}_h^+\theta)|y_h|^2\mathrm{d}t\\
&+\mathbf{D}_h^+ A_0\partial_t\theta\mathbf{D}_h^+\theta|y_h|^2\mathrm{d}t + \mathbf{D}_h^+ A_0\left(\theta + h\mathbf{D}_h^+\theta\right)\partial_t\theta y_h\mathbf{D}_h^+ y_h\mathrm{d}t\\
&+\mathbf{D}_h^+ A_0\left(\theta + h\mathbf{D}_h^+\theta\right)\theta\mathbf{D}_h^+ y_h\mathrm{d}y_h.
\end{aligned} \tag{3.30}$$

By (3.6) and (3.21), we have

$$\begin{cases}
\mathbf{D}_h^+ A_0\theta\mathbf{D}_h^+\theta = \mathcal{O}_\lambda(sh)(s\lambda\varphi\partial_x\psi + s\mathcal{O}_\lambda(sh))\theta^2 = s\mathcal{O}_\lambda(sh)\theta^2,\\
\mathbf{D}_h^+ A_0\left(\theta + h\mathbf{D}_h^+\theta\right)\theta = \mathcal{O}_\lambda(sh)\left(1 + h(s\lambda\varphi\partial_x\psi + s\mathcal{O}_\lambda(sh))\right)\theta^2 = \mathcal{O}_\lambda(sh)\theta^2.
\end{cases} \tag{3.31}$$

Then, using (3.30) and (3.31) we obtain the following estimate for Z_1:

$$\begin{aligned}
Z_1 \ge &-\mathbb{E}\int_{G_h^-} s|\mathcal{O}_\lambda(sh)|\left(\theta^2|y_h|^2\right)\Big|_{t=T} - \mathbb{E}\int_{Q_h^-} s|\mathcal{O}_\lambda(sh)|\theta^2|g_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} D_3|Y_h|^2\mathrm{d}t\\
&+\mathbb{E}\int_{Q_h^-} K y_h\mathbf{D}_h^+ y_h\mathrm{d}t + \mathbb{E}\int_{Q_h^-}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+ y_h\mathrm{d}y_h,
\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
D_3 &= \left(-\frac{1}{2}\partial_t(\mathbf{D}_h^+ A_0\theta\mathbf{D}_h^+\theta) + \mathbf{D}_h^+ A_0\partial_t\theta\mathbf{D}_h^+\theta\right)\theta^{-2},\\
K &= \mathbf{D}_h^+ A_0\left(\theta + h\mathbf{D}_h^+\theta\right)\partial_t\theta.
\end{aligned}$$

By the equation of y_h in (2.16), we have

$$\mathbf{D}_h^+ y_h\mathrm{d}y_h = \mathbf{D}_h^+ y_h\Delta_h y_h\mathrm{d}t + f_h\mathbf{D}_h^+ y_h\mathrm{d}t + g_h\mathbf{D}_h^+ y_h\mathrm{d}B(t). \tag{3.33}$$

Then, using (3.33) and

$$\mathbb{E}\int_{Q_h^-}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+ y_h\mathrm{d}y_h = \mathbb{E}\int_{Q_h}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+ y_h\mathrm{d}y_h$$

duet to $y_h=0$ at $x=x_0$, we obtain the following estimate for the last term on the right-hand side of (3.32):

$$
\begin{aligned}
&\mathbb{E}\int_{Q_h^-}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+y_h\mathrm{d}y_h\\
=&\mathbb{E}\int_{Q_h}\mathcal{O}_\lambda(sh)\theta^2\left(\mathbf{D}_h^+y_h\Delta_hy_h\mathrm{d}t+f_h\mathbf{D}_h^+y_h\mathrm{d}t+g_h\mathbf{D}_h^+y_h\mathrm{d}B(t)\right)\\
=&\mathbb{E}\int_{Q_h}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+y_h\Delta_hy_h\mathrm{d}t+\mathbb{E}\int_{Q_h}\mathcal{O}_\lambda(sh)\theta^2f_h\mathbf{D}_h^+y_h\mathrm{d}t,
\end{aligned}
$$

where we have used

$$
\mathbb{E}\int_{Q_h}\mathcal{O}_\lambda(sh)\theta^2g_h\mathbf{D}_h^+y_h\mathrm{d}B(t)=0.
$$

Young's inequality with ϵ further yields that

$$
\begin{aligned}
&\mathbb{E}\int_{Q_h^-}\mathcal{O}_\lambda(sh)\theta^2\mathbf{D}_h^+y_h\mathrm{d}y_h\\
\geq&-\epsilon\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_hy_h|^2\mathrm{d}t-C(\epsilon)\mathbb{E}\int_{Q_h}s|\mathcal{O}_\lambda(sh)|\varphi\theta^2|\mathbf{D}_h^+y_h|^2\mathrm{d}t\\
&-\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t,
\end{aligned}
\tag{3.34}
$$

Therefore, we deduce from (3.34) and (3.32) that

$$
\begin{aligned}
Z_1\geq&-\mathbb{E}\int_{G_h^-}s|\mathcal{O}_\lambda(sh)|\left(\theta^2|y_h|^2\right)\Big|_{t=T}-\mathbb{E}\int_{Q_h^-}s|\mathcal{O}_\lambda(sh)|\theta^2|g_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}D_3|y_h|^2\mathrm{d}t\\
&-\epsilon\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_hy_h|^2\mathrm{d}t-C(\epsilon)\mathbb{E}\int_{Q_h}s|\mathcal{O}_\lambda(sh)|\varphi\theta^2|\mathbf{D}_h^+y_h|^2\mathrm{d}t-\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t\\
&+\mathbb{E}\int_{Q_h^-}Ky_h\mathbf{D}_h^+y_h\mathrm{d}t.
\end{aligned}
\tag{3.35}
$$

Substituting (3.17), (3.29) and (3.35) into (3.13), and noticing that $s|\mathcal{O}_\lambda(sh)|\leq 1/8$ we obtain

$$
\begin{aligned}
\sum_{j=1}^3I_{1j}\geq&\frac{1}{4}\mathbb{E}\int_{Q_h^-}(\tau-2)\,s\lambda^2\varphi|\partial_x\psi|^2\theta^2|g_h|^2\mathrm{d}t-C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\mathbf{D}_h^+g_h|^2\mathrm{d}t\\
&-\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t-C(\lambda)s^2e^{C(\lambda)s}||y_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}\\
&-\epsilon\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_hy_h|^2\mathrm{d}t-C(\epsilon)\mathbb{E}\int_{Q_h}\varphi\theta^2|\mathbf{D}_h^+y_h|^2\mathrm{d}t+\sum_{i=1}^3\mathbb{E}\int_{Q_h}D_i|Y_h|^2\mathrm{d}t\\
&+\mathbb{E}\int_{Q_h^-}B_1|\mathbf{D}_h^+Y_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}Ky_h\mathbf{D}_h^+y_h\mathrm{d}t.
\end{aligned}
\tag{3.36}
$$

3.1.2 Estimates involving the terms of $\mathrm{d}t$

Proceeding as done in [5] or [10], we apply discrete integrations by parts in Lemma 2.2 to yield

- $$\begin{aligned} I_{21} =& -2\mathbb{E}\int_{Q_h} s\lambda A_1(1+A_0)\mathbf{D}_h Y_h \Delta_h Y_h \mathrm{d}t \\ =& \mathbb{E}\int_{Q_h^-} s\lambda \mathbf{D}_h^+ (A_1(1+A_0))|\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t + \mathbb{E}\int_0^T s\lambda(A_1(1+A_0))(x_0,t)|(\mathbf{D}_h^+ Y_h)_0|^2 \mathrm{d}t \\ & - \mathbb{E}\int_0^T s\lambda(A_1(1+A_0)(x_{N+1},t))|(\mathbf{D}_h^- Y_h)_{N+1}|^2 \mathrm{d}t, \end{aligned}$$

- $$\begin{aligned} I_{22} =& -2\mathbb{E}\int_{Q_h} s^3\lambda^3 A_1 A_2 Y_h \mathbf{D}_h Y_h \mathrm{d}t \\ =& \mathbb{E}\int_{Q_h} s^3\lambda^3 \mathbf{D}_h(A_1A_2)|Y_h|^2 \mathrm{d}t - \frac{h^2}{2}\mathbb{E}\int_{Q_h^-} s^3\lambda^3 \mathbf{D}_h^+(A_1A_2)|\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t, \end{aligned}$$

- $$\begin{aligned} I_{23} =& 2\mathbb{E}\int_{Q_h} s\lambda A_1(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi) Y_h \mathbf{D}_h Y_h \mathrm{d}t \\ =& -\mathbb{E}\int_{Q_h} s\lambda \mathbf{D}_h\left(A_1(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\right)|Y_h|^2 \mathrm{d}t \\ & + \frac{h^2}{2}\mathbb{E}\int_{Q_h^-} s\lambda \mathbf{D}_h^+\left(A_1(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\right)|\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t, \end{aligned}$$

- $$\begin{aligned} I_{31} =& -\mathbb{E}\int_{Q_h} (1+A_0)\Psi Y_h \Delta_h Y_h \mathrm{d}t \\ =& \mathbb{E}\int_{Q_h^-} \mathbf{m}_h^+\left((1+A_0)\Psi\right)|\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t - \frac{1}{2}\mathbb{E}\int_{Q_h} \Delta_h((1+A_0)\Psi)|Y_h|^2 \mathrm{d}t, \end{aligned}$$

- $$I_{32} = -\mathbb{E}\int_{Q_h} s^2\lambda^2 A_2 \Psi |Y_h|^2 \mathrm{d}t,$$

- $$I_{33} = \mathbb{E}\int_{Q_h} (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\Psi|Y_h|^2 \mathrm{d}t.$$

Then we find that

$$\sum_{i=2}^{3}\sum_{j=1}^{3} I_{ij} = \mathbb{E}\int_{Q_h} D_4|Y_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h^-} B_2|\mathbf{D}_h^+ Y_h|^2 \mathrm{d}t + \mathbb{E}\int_0^T R_1 \mathrm{d}t, \tag{3.37}$$

where

$$\begin{aligned}
D_4 =& s^3\lambda^3\mathbf{D}_h(A_1A_2) - s\lambda\mathbf{D}_h\left(A_1(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\right) - \frac{1}{2}\Delta_h((1+A_0)\Psi)\\
&- s^2\lambda^2 A_2\Psi + (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\Psi,\\
B_2 =& s\lambda\mathbf{D}_h^+(A_1(1+A_0)) - \frac{h^2}{2}s^3\lambda^3\mathbf{D}_h^+(A_1A_2) + \frac{h^2}{2}s\lambda\mathbf{D}_h^+\left(A_1(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\right)\\
&+ \mathbf{m}_h^+\left((1+A_0)\Psi\right),\\
R_1 =& s\lambda(A_1(1+A_0)(x_0,t))|(\mathbf{D}_h^+ Y_h)_0|^2 - s\lambda(A_1(1+A_0)(x_{N+1},t))|(\mathbf{D}_h^- Y_h)_{N+1}|^2.
\end{aligned}$$

Therefore, from (3.36) and (3.37) it follows that

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij} \geq& \frac{1}{4}\mathbb{E}\int_{Q_h^-}(\tau-2)\,s\lambda^2\varphi|\partial_x\psi|^2\theta^2|g_h|^2\mathrm{d}t - C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\mathbf{D}_h^+ g_h|^2\mathrm{d}t\\
&- \mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t - C(\lambda)s^2e^{C(\lambda)s}||y_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}\\
&- \epsilon\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t - C(\epsilon)\mathbb{E}\int_{Q_h}\theta^2|\mathbf{D}_h^+ y_h|^2\mathrm{d}t + \sum_{i=1}^{4}\mathbb{E}\int_{Q_h}D_i|Y_h|^2\mathrm{d}t\\
&+ \sum_{i=1}^{2}\mathbb{E}\int_{Q_h^-}B_i|\mathbf{D}_h^+ Y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-}Ky_h\mathbf{D}_h^+ y_h\mathrm{d}t + \mathbb{E}\int_0^T R_1\mathrm{d}t. \qquad (3.38)
\end{aligned}$$

3.1.3 Positive lower bounds for the terms of $|Y_h|^2$ and $|\mathbf{D}_h^+ Y_h|^2$

A direct calculation gives

$$\begin{aligned}
D_1 =& \frac{1}{2}s^2\lambda^2(\partial_t f_2 + \mathcal{O}_\lambda(sh)) = s^2\mathcal{O}_\lambda(1),\\
D_2 =& -\frac{1}{2}(-s\varphi_{tt} + s\lambda^2(\partial_t f_3 + \mathcal{O}_\lambda(sh)) + s\lambda(\partial_t f_4 + \mathcal{O}_\lambda(sh)) - \tau s\varphi_{xxt}) = s\mathcal{O}_\lambda(1),\\
D_3 =& -\frac{1}{2}\left(\partial_t(\mathbf{D}_h^+ A_0)\theta\mathbf{D}_h^+\theta - \mathbf{D}_h^+ A_0\partial_t\theta\mathbf{D}_h^+\theta + \mathbf{D}_h^+ A_0\theta\partial_t(\mathbf{D}_h^+\theta)\right)\theta^{-2} = s^2\mathcal{O}_\lambda(sh),\\
D_4 =& s^3\lambda^3(\mathbf{D}_h A_1\mathbf{m}_h A_2 + \mathbf{m}_h A_1\mathbf{D}_h A_2) - s^2\lambda^2 A_2\Psi\\
&+ (-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\Psi - s\lambda\mathbf{D}_h A_1\mathbf{m}_h(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi)\\
&- s\lambda\mathbf{m}_h A_1\mathbf{D}_h(-\partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi) - \frac{1}{2}\Delta_h\Psi - \frac{1}{2}\Delta_h(A_0\Psi)\\
=& s^3\lambda^3\left(\partial_x(\varphi\partial_x\psi)(\varphi^2|\partial_x\psi|^2) + \varphi\partial_x\psi\partial_x(\varphi^2|\partial_x\psi|^2) + \mathcal{O}_\lambda(sh)\right) - \tau s^3\lambda^2\varphi^2\partial_{xx}\varphi|\partial_x\psi|^2\\
&+ s^3\mathcal{O}_\lambda(sh) + s^2\mathcal{O}_\lambda(1)\\
=& (3-\tau)\left(s^3\lambda^4\varphi^3|\partial_x\psi|^4 + s^3\lambda^3\varphi^3\partial_{xx}\psi|\partial_x\psi|^2\right) + s^3\mathcal{O}_\lambda(sh) + s^2\mathcal{O}_\lambda(1)
\end{aligned}$$

and

$$\begin{aligned}
B_1 =& h\mathcal{O}_\lambda(sh) + \mathcal{O}_\lambda(sh) = \mathcal{O}_\lambda(sh),\\
B_2 =& \mathbf{m}_h^+\Psi + \mathbf{m}_h^+ A_0\mathbf{m}_h^+\Psi + \frac{h^2}{4}\mathbf{D}_h^+ A_0\mathbf{D}_h^+\Psi + s\lambda\left(\mathbf{D}_h^+ A_1(1+\mathbf{m}_h^+ A_0) + \mathbf{m}_h^+ A_1\mathbf{D}_h^+ A_0\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{h^2}{2}s^3\lambda^3(\mathbf{D}_h^+ A_1\mathbf{m}_h^+ A_2+\mathbf{m}_h^+ A_1\mathbf{D}_h^+ A_2)+\frac{h^2}{2}s\lambda\mathbf{D}_h^+\left(A_1(-\partial_t l+s\lambda^2 A_3+s\lambda A_4-\Psi)\right)\\
=&\tau\left(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\psi_{xx}+s\mathcal{O}_\lambda(sh)\right)+s\mathcal{O}_\lambda(sh)+sh^2 O_\lambda(sh)\\
&+\left(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\partial_{xx}+s\mathcal{O}_\lambda(sh)\right)+s^3h^2\mathcal{O}_\lambda(1)\\
=&(\tau+1)\left(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\psi_{xx}\right)+s\mathcal{O}_\lambda(sh),
\end{aligned}
$$

where we have used

$$
\begin{cases}
\partial_t(\mathbf{D}_h^+ A_0)\theta\mathbf{D}_h^+\theta=\mathcal{O}_\lambda(sh)(s\lambda\varphi\partial_x\psi+s\mathcal{O}_\lambda(sh))\theta^2=s\mathcal{O}_\lambda(sh)\theta^2,\\
\mathbf{D}_h^+ A_0\partial_t\theta\mathbf{D}_h^+\theta=\mathcal{O}_\lambda(sh)(s\lambda\varphi\partial_x\psi+s\mathcal{O}_\lambda(sh))s\lambda\psi_t\theta^2=s^2\mathcal{O}_\lambda(sh)\theta^2,\\
\mathbf{D}_h^+ A_0\theta\partial_t(\mathbf{D}_h^+\theta)=\mathcal{O}_\lambda(sh)\left(s\lambda\partial_t\varphi\partial_x\psi+s^2\lambda^2\varphi^2\partial_t\psi\partial_x\psi+s^2\mathcal{O}_\lambda(sh)\right)\theta^2=s^2\mathcal{O}_\lambda(sh)\theta^2.
\end{cases}
$$

Since $\tau\in(3/2,3)$ and $|\partial_x\psi|>0$ for all $x\in\overline{G}$, we can choose λ sufficiently large to satisfy

$$
\begin{aligned}
&(3-\tau)\left(s^3\lambda^4\varphi^3|\partial_x\psi|^4+s^3\lambda^3\varphi^3\partial_{xx}\psi|\partial_x\psi|^2\right)+s^3\mathcal{O}_\lambda(sh)+s^2\mathcal{O}_\lambda(1)\\
\geq&\frac{1}{2}(3-\tau)s^3\lambda^4\varphi^3|\partial_x\psi|^4\geq Cs^3\lambda^4\varphi^3
\end{aligned}
\tag{3.39}
$$

and

$$
(\tau+1)\left(s\lambda^2\varphi|\partial_x\psi|^2+s\lambda\varphi\psi_{xx}\right)+s\mathcal{O}_\lambda(sh)\geq\frac{\tau+1}{2}s\lambda^2\varphi|\partial_x\psi|^2\geq Cs\lambda^2\varphi. \tag{3.40}
$$

Then, we have

$$
\begin{aligned}
&\sum_{i=1}^4\mathbb{E}\int_{Q_h}D_i|Y_h|^2\mathrm{d}t+\sum_{i=1}^2\mathbb{E}\int_{Q_h^-}B_i|\mathbf{D}_h^+Y_h|^2\mathrm{d}t\\
\geq&C\mathbb{E}\int_{Q_h}s^3\lambda^4\varphi^3|Y_h|^2\mathrm{d}t+C\int_{Q_h^-}s\lambda^2\varphi|\mathbf{D}^+Y_h|^2\mathrm{d}t.
\end{aligned}
\tag{3.41}
$$

3.1.4 The remainder of the proof of Theorem 2.3

Since $\partial_x\psi=2(x-x^*)>0$ for all $x\in\overline{G}$, we obtain

$$
R_1=s\lambda(\varphi\partial_x\psi+\mathcal{O}_\lambda(sh))\left(|(\mathbf{D}_h^+Y_h)_0|^2-|(\mathbf{D}_h^-Y_h)_{N+1}|^2\right)\geq-Cs\lambda\varphi|(\mathbf{D}_h^-Y_h)_{N+1}|^2, \tag{3.42}
$$

which leads to

$$
\mathbb{E}\int_0^T R_1\mathrm{d}t\geq-C\mathbb{E}\int_0^T s\lambda\varphi|(\mathbf{D}_h^-Y_h)_{N+1}|^2\mathrm{d}t. \tag{3.43}
$$

Then, we substitute (3.41) and (3.43) into (3.38) to yield

$$
\begin{aligned}
&\mathbb{E}\int_{Q_h}s^3\lambda^4\varphi^3|Y_h|^2\mathrm{d}t+\int_{Q_h^-}s\lambda^2\varphi|\mathbf{D}^+Y_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|g_h|^2\mathrm{d}t\\
\leq&C\mathbb{E}\int_{Q_h}II_1+C\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t+C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\mathbf{D}_h^+g_h|^2\mathrm{d}t+C\mathbb{E}\int_0^T s\lambda\varphi|(\mathbf{D}_h^-Y_h)_{N+1}|^2\mathrm{d}t\\
&+C(\lambda)s^2e^{C(\lambda)s}\|y_h(T)\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}+\epsilon C\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t\\
&+C(\epsilon)\mathbb{E}\int_{Q_h}\theta^2|\mathbf{D}_h^+y_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}Ky_h\mathbf{D}_h^+y_h\mathrm{d}t.
\end{aligned}
\tag{3.44}
$$

On the other hand, by (3.7) we have

$$\begin{aligned}\mathbb{E}\int_{Q_h} II_1 =&\mathbb{E}\int_{Q_h} \theta I\left(f_h \mathrm{d}t + g_h \mathrm{d}B(t)\right) - \mathbb{E}\int_{Q_h} I^2 \mathrm{d}t\\ \le&\frac{1}{2}\mathbb{E}\int_{Q_h}\theta^2|f_n|^2\mathrm{d}t - \frac{1}{2}\mathbb{E}\int_{Q_h} I^2\mathrm{d}t. \end{aligned}\tag{3.45}$$

Moreover, we have

$$\begin{aligned}&\mathbf{D}_h^+ Y_h = \mathbf{m}_h^+\theta\mathbf{D}_h^+ y_h + \mathbf{D}_h^+\theta\mathbf{m}_h^+ y_h\\ =&(1+\mathcal{O}_\lambda(sh))\theta\mathbf{D}_h^+ y_h + (s\lambda\varphi\psi_x + s\mathcal{O}_\lambda(sh))\theta\left(y_h + \frac{h}{2}\mathbf{D}_h^+ y_h\right)\\ =&\theta\mathbf{D}_h^+ y_h + \mathcal{O}_\lambda(sh)\theta\mathbf{D}_h^+ y_h + sO_\lambda(1)\theta y_h. \end{aligned}\tag{3.46}$$

Therefore, by (3.44)-(3.46) and choosing $\mathcal{O}_\lambda(sh)$ sufficiently small and s sufficiently large to absorb the last two terms of $\mathbf{D}_h^+ y_h$ and y_h on the right-hand side of (3.46), we find that

$$\begin{aligned}&\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|y_h|^2\mathrm{d}t + \int_{Q_h^-} s\lambda^2\varphi\theta^2|\mathbf{D}^+ y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi|\theta^2|g_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h} I^2\mathrm{d}t\\ \le& C\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2|\mathbf{D}_h^+ g_h|^2\mathrm{d}t + C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- y_h)_{N+1}|^2\mathrm{d}t\\ &+ C(\lambda)s^2e^{C(\lambda)s}\|y_h(T)\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))} + \epsilon C\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t\\ &+ C(\epsilon)\mathbb{E}\int_{Q_h}\theta^2|\mathbf{D}_h^+ y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} K y_h\mathbf{D}_h^+ y_h\mathrm{d}t. \end{aligned}\tag{3.47}$$

The first-order difference term of $\mathbf{D}_h^+ y_h$ on the right-hand side of (3.47) can be absorbed by the first-order difference term on the left-hand side of (3.47) if we choose λ sufficiently large such that $\lambda \ge C(\epsilon)$. To handle the second-order central difference term, we have to express $\Delta_h Y_h$ in terms of I and provide an estimate by the terms on the left-hand side of (3.47). The definition of I gives

$$(1+A_0)\Delta_h Y_h = -I + (-s^2\lambda^2 A_2 - \partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi_h)Y_h.$$

Then, we obtain

$$\begin{aligned}(1+A_0)^2|\Delta_h Y_h|^2 \le& |I|^2 + \left(-s^2\lambda^2 A_2 - \partial_t l + s\lambda^2 A_3 + s\lambda A_4 - \Psi\right)^2|Y_h|^2\\ \le& |I|^2 + C\left(s^4\lambda^4\varphi^4|\partial_x\psi|^4 + s^4\mathcal{O}_\lambda(sh) + s^2\mathcal{O}_\lambda(1)\right)|Y_h|^2, \end{aligned}$$

which implies

$$\mathbb{E}\int_{Q_h}(1+A_0)^2\frac{1}{s\varphi}|\Delta_h Y_h|^2\mathrm{d}t \le \mathbb{E}\int_{Q_h} I^2\mathrm{d}t + C\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3|Y_h|^2\mathrm{d}t \tag{3.48}$$

for sufficiently large s such that $s\varphi \ge 1$. If we choose h sufficiently small such that $|A_0| = |\mathcal{O}_\lambda(sh)| \le \frac{1}{2}$, we have $(1+A_0)^2 \ge \frac{1}{4}$. Then by (3.48), we obtain

$$\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}|\Delta_h Y_h|^2\mathrm{d}t \le C\mathbb{E}\int_{Q_h} I^2\mathrm{d}t + C\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3|Y_h|^2\mathrm{d}t. \tag{3.49}$$

We use (2.1), (2.3) and (3.22) to obtain

$$\begin{aligned}
\Delta_h Y_h =&\Delta_h \theta \mathbf{m}_h y_h + \mathbf{m}_h \theta \Delta_h y_h + 2\mathbf{D}_h \theta \mathbf{D}_h y_h \\
=&\Delta_h \theta y_h + \left(\theta + \frac{h^2}{2}\Delta_h \theta\right) \Delta_h y_h + 2\mathbf{D}_h \theta \left(\mathbf{D}_h^+ y_h + \frac{h}{2}\Delta_h y_h\right) \\
=& \left(s^2\lambda^2\varphi^2|\psi_x|^2 + s\lambda\partial_x(\varphi\partial_x\psi) + s^2\mathcal{O}_\lambda(sh)\right)\theta y_h + \left(1 + s^2h^2\mathcal{O}_\lambda(1)\right)\theta\Delta_h y_h \\
&+ 2(s\lambda\varphi\psi + s\mathcal{O}_\lambda(sh))\theta\left(\mathbf{D}_h^+ y_h + \frac{h}{2}\Delta_h y_h\right) \\
=&\theta\Delta_h y_h + \mathcal{O}_\lambda(sh)\theta\Delta_h y_h + s^2\lambda^2\varphi^2\mathcal{O}(1)\theta y_h + s\lambda\varphi\mathcal{O}(1)\theta\mathbf{D}_h^+ y_h. \qquad (3.50)
\end{aligned}$$

Then, combining (3.49) and (3.50) we obtain

$$\begin{aligned}
&\epsilon\mathbb{E}\int_{Q_h} \frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t \\
\leq&\epsilon C\mathbb{E}\int_{Q_h} \frac{1}{s\varphi}|\Delta_h Y_h|^2\mathrm{d}t + \epsilon C\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|y_h|^2\mathrm{d}t + \epsilon C\int_{Q_h^-} s\lambda^2\varphi\theta^2|\mathbf{D}^+ y_h|^2\mathrm{d}t \\
\leq&\epsilon C\mathbb{E}\int_{Q_h} I^2\mathrm{d}t + \epsilon C\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|y_h|^2\mathrm{d}t + \epsilon C\int_{Q_h^-} s\lambda^2\varphi\theta^2|\mathbf{D}^+ y_h|^2\mathrm{d}t. \qquad (3.51)
\end{aligned}$$

Next we deal with the last term on the right-hand side of (3.47). By

$$K = \mathbf{D}_h^+ A_0\theta\partial_t\theta + h\mathbf{D}_h^+ A_0\mathbf{D}_h^+\theta\partial_t\theta = s\mathcal{O}_\lambda(sh)\theta^2 + s^2hO_\lambda(sh)\theta^2 = s\mathcal{O}_\lambda(sh)\theta^2,$$

we have

$$\begin{aligned}
&\mathbb{E}\int_{Q_h^-} Ky_h\mathbf{D}_h^+ y_h\mathrm{d}t = \mathbb{E}\int_{Q_h^-} s\mathcal{O}_\lambda(sh)\theta^2 y_h\mathbf{D}_h^+ y_h\mathrm{d}t \\
\leq&\mathbb{E}\int_{Q_h} s\mathcal{O}_\lambda(sh)\theta^2|y_h|^2\mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\theta^2|\mathbf{D}_h^+ y_h|^2\mathrm{d}t. \qquad (3.52)
\end{aligned}$$

Therefore, by substituting (3.51) and (3.52) into (3.47) we obtain

$$\begin{aligned}
&\mathbb{E}\int_{Q_h} \left((1-\epsilon C)s^3\lambda^4 - s\mathcal{O}_\lambda(sh)\right)\varphi^3\theta^2|y_h|^2\mathrm{d}t + \int_{Q_h^-} \left((1-\epsilon C)s\lambda^2 - s\right)\varphi\theta^2|\mathbf{D}^+ y_h|^2\mathrm{d}t \\
&+ \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|g_h|^2\mathrm{d}t + \int_{Q_h} (1-\epsilon C)I^2\mathrm{d}t \\
\leq& C\mathbb{E}\int_{Q_h} \theta^2|f_h|^2\mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2|\mathbf{D}_h^+ g_h|^2\mathrm{d}t + C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- y_h)_{N+1}|^2\mathrm{d}t \\
&+ C(\lambda)s^2e^{C(\lambda)s}\|y_h(T)\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}. \qquad (3.53)
\end{aligned}$$

Finally, choosing ϵ sufficiently small, we can obtain the desired estimate (2.13) and then complete the proof of Theorem 2.3.

3.2 Proof of Theorem 2.4

In order to apply Theorem 2.3 to prove (2.15), we need to make the boundary conditions in (2.16) homogeneous. To do this, we introduce u_h satisfying

$$
\begin{cases}
\mathrm{d}u_h - \Delta_h u_h \mathrm{d}t = 0, & (x_h, t) \in Q_h, \\
(u_h)_0 = \gamma_1, \ (u_h)_{N+1} = \gamma_2, & t \in (0, T), \\
u_h(0) = 0, & x \in G_h.
\end{cases}
\tag{3.54}
$$

Obviously, we know for a.e. $\omega \in \Omega$ that $u_h \in L^2(0, T; H^2(G_h)) \cap C([0, T]; L^2(G_h))$ satisfies

$$
\|u_h\|_{L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^2(G_h)))} + \|u_h\|_{L^2_{\mathcal{F}}(\Omega; C([0,T];L^2(G_h)))} \leq C \sum_{i=1}^{2} \|\gamma_i\|_{L^2_{\mathcal{F}}(\Omega; H^1(0,T))}. \tag{3.55}
$$

Additionally, we can also obtain the following estimate for $\mathbf{D}_h^- u_h$ at boundary x_{N+1}:

$$
\mathbb{E} \int_0^T |(\mathbf{D}_h^- u_h)_{N+1}|^2 \mathrm{d}t \leq C \sum_{i=1}^{2} \|\gamma_i\|^2_{L^2_{\mathcal{F}}(\Omega; H^1(0,T))}. \tag{3.56}
$$

Indeed, we introduce a cut-off function $\chi \in C^\infty[0, L]$ such that $0 \leq \chi(x) \leq 1$ and

$$
\chi(x) = \begin{cases} 0, & x \in [x_0, x_1], \\ 1, & x \in [x_N, x_{N+1}]. \end{cases} \tag{3.57}
$$

Letting $\hat{u}_h = \chi u_h$, we easily see that

$$
\begin{cases}
\mathrm{d}\hat{u}_h - \Delta_h \hat{u}_h \mathrm{d}t = \hat{f} \mathrm{d}t, & (x_h, t) \in Q_h, \\
(\hat{u}_h)_0 = 0, \ (\hat{u}_h)_{N+1} = \gamma_2, & t \in (0, T), \\
\hat{u}_h(0) = 0, & x \in G_h,
\end{cases}
\tag{3.58}
$$

where

$$
\hat{f} = -\Delta_h \chi \mathbf{m}_h u_h - 2\mathbf{D}_h \chi \mathbf{D}_h u_h - \frac{h^2}{4} \Delta_h \chi \Delta_h u_h
$$

satisfies

$$
|\hat{f}| \leq C \left(|u_h| + |\mathbf{D}_h^+ u_h| + h |\Delta_h u_h| \right). \tag{3.59}
$$

We multiply the equation of $\Delta_h \hat{u}_h$ by $\mathbf{D}_h^- \hat{u}_h$ and integrate over Q_h. Then by using discrete integration by parts (2.5) and (3.57), we obtain

$$
\int_{Q_h} \partial_t \hat{u}_h \mathbf{D}_h^- \hat{u}_h \mathrm{d}t + \int_{Q_h^-} \Delta_h \hat{u}_h \mathbf{D}_h^+ \hat{u}_h \mathrm{d}t - \int_0^T \left| (\mathbf{D}_h^- u_h)_{N+1} \right|^2 \mathrm{d}t = \int_{Q_h} \hat{f} \mathbf{D}_h^- \hat{u}_h \mathrm{d}t, \tag{3.60}
$$

which implies

$$
\mathbb{E} \int_0^T \left| (\mathbf{D}_h^- u_h)_{N+1} \right|^2 \mathrm{d}t \leq C \left(\|\hat{u}_h\|^2_{L^2(0,T;H^2(G_h))} + \|\hat{f}\|^2_{L^2(0,T;L^2(G_h))} \right). \tag{3.61}
$$

Therefore, from (3.55), (3.59) and (3.61) we deduce (3.56).

Letting $z_h = y_h - u_h$, we then obtain

$$
\begin{cases}
\mathrm{d}z_h - \Delta_h z_h \mathrm{d}t = f_h \mathrm{d}t + g_h \mathrm{d}B(t), & (x_h, t) \in Q_h, \\
z_h = 0, & (x_h, t) \in \Sigma_h, \\
z_h(0) = 0, & x_h \in G_h.
\end{cases}
\tag{3.62}
$$

Applying (2.13) to z_h, we have

$$\begin{aligned}
&\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_h z_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|\partial_h^+ z_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h}s^3\lambda^4\varphi^3\theta^2|z_h|^2\mathrm{d}t\\
&+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|g_h|^2dt\\
\leq& C\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t+C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\partial_h^+ g_h|^2\mathrm{d}t+C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- z_h)_{N+1}|^2\mathrm{d}t\\
&+C(\lambda)s^2e^{C(\lambda)s}||z_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))},
\end{aligned}\tag{3.63}$$

which implies

$$\begin{aligned}
&\mathbb{E}\int_{Q_h}\frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|\partial_h^+ y_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h}s^3\lambda^4\varphi^3\theta^2|y_h|^2\mathrm{d}t\\
&+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|g_h|^2dt\\
\leq& C\mathbb{E}\int_{Q_h}\theta^2|f_h|^2\mathrm{d}t+C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\partial_h^+ g_h|^2\mathrm{d}t+C\mathbb{E}\int_0^T s\lambda\theta^2(x_{N+1},t)|(\mathbf{D}_h^- y_h)_{N+1}|^2\mathrm{d}t\\
&+C(\lambda)s^2e^{C(\lambda)s}||y_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}+C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^- u_h)_{N+1}|^2\mathrm{d}t\\
&+C(\lambda)s^3e^{C(\lambda)s}\left(||u_h||^2_{L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^2(G_h)))}+||u_h(T)||^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}\right).
\end{aligned}\tag{3.64}$$

Substituting (3.55) and (3.56) into (3.64), we obtain the desired estimate (2.15) and then complete the proof of Theorem 2.4.

4 Proof of Theorem 2.5

In this section, we will prove the stability of our discrete inverse random source problem, i.e. Theorem 2.5.

Proof of Theorem 2.5. Letting $\tilde{y}_h=y_h^{(1)}-y_h^{(2)}$ and $\tilde{g}_h=g_h^{(1)}-g_h^{(2)}$, we obtain

$$\begin{cases}
\mathrm{d}\tilde{y}_h-\Delta_h\tilde{y}_h\mathrm{d}t=\left(a_h\tilde{y}_h+b_h\mathbf{D}_h^+\tilde{y}_h\right)\mathrm{d}t+\tilde{g}_h\mathrm{d}B(t), & (x_h,t)\in Q_h,\\
\tilde{y}_h=0, & (x_h,t)\in\Sigma_h,\\
\tilde{y}_h(0)=0, & x_h\in G_h.
\end{cases}\tag{4.1}$$

Then applying Theorem 2.3 to $\tilde{y}_h$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|\partial_h^+\tilde{y}_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h}s^3\lambda^4\varphi^3\theta^2|\tilde{y}_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-}s\lambda^2\varphi\theta^2|\tilde{g}_h|^2dt\\
\leq& C\mathbb{E}\int_{Q_h}\theta^2\left(|\tilde{y}_h|^2+|\mathbf{D}_h^+\tilde{y}_h|^2\right)\mathrm{d}t+C\mathbb{E}\int_{Q_h^-}s\varphi\theta^2|\partial_h^+\tilde{g}_h|^2\mathrm{d}t\\
&+C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^-\tilde{y}_h)_{N+1}|^2\mathrm{d}t+C(\lambda)s^2e^{C(\lambda)s}\|\tilde{y}_h(T)\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}.
\end{aligned}\tag{4.2}$$

By means of (2.17), and choosing λ sufficiently large to absorb the first and second terms on the right-hand side of (4.2), we obtain

$$\begin{aligned}&\mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|\partial_h^+\tilde{y}_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|\tilde{y}_h|^2\mathrm{d}t+\mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|\tilde{g}_h|^2dt\\ \leq & C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1},t)|(\mathbf{D}_h^-\tilde{y}_h)_{N+1}|^2\mathrm{d}t+C(\lambda)s^2e^{C(\lambda)s}\|\tilde{y}_h(T)\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P};L^2(G_h))}.\end{aligned} \tag{4.3}$$

From (4.3), we immediately deduce (2.18) and then complete the proof of Theorem 2.5. □

5 Proof of Theorem 2.6

In this section, we prove the conditional stability for the discrete Cauchy problem, i.e. Theorem 2.6. In the proof we borrow some ideas from [33].

Proof of Theorem 2.6. In order to estimate the solution of (1.4) in $G_{0,h}\times(\epsilon,T-\epsilon)$ by Cauchy data at x_{N+1}, we need to choose a suitable weight function φ. Let $\tilde{G}=(0,L+\delta)$ with $\delta>0$, and let $\hat{G}\subset\tilde{G}\setminus\overline{G}$. It is easy to find a function $d\in C^2(\tilde{G})$ such that

$$\begin{cases} d(x)>0, & x\in\tilde{G},\\ d(x)=0, & x\in\partial\tilde{G},\\ |\partial_x d(x)|>0, & x\in G\subset\tilde{G}\setminus\hat{G}.\end{cases} \tag{5.1}$$

A typical form of function d is $d(x)=x((L+\delta)-x)$ with $\delta>L$. Then, since $G_0\subset\subset\tilde{G}$, we can choose a sufficiently large $N>1$ such that

$$G_0\subset\left\{x\mid x\in\tilde{G},\ d(x)>\frac{4}{N}\|d\|_{L^\infty(\tilde{G})}\right\}\cap G. \tag{5.2}$$

Moreover, we choose a positive number β such that

$$\beta\epsilon^2<\|d\|_{L^\infty(\tilde{G})}<2\beta\epsilon^2. \tag{5.3}$$

We arbitrarily fix $t_0\in[\sqrt{2}\epsilon,T-\sqrt{2}\epsilon]$. Meanwhile, we denote

$$\psi(x,t)=d(x)-\beta(t-t_0)^2,\quad \varphi(x,t)=e^{\lambda\psi(x,t)}$$

with fixed large parameter $\lambda>0$. Let

$$\mu_k=e^{\lambda\left(\frac{k}{N}\|d\|_{L^\infty(\tilde{G})}-\frac{\beta\epsilon^2}{N}\right)}$$

and

$$Q^{(k)}=\left\{(x,t)\mid x\in\overline{G},\ \varphi(x,t)>\mu_k\right\},\quad k=1,2,3,4. \tag{5.4}$$

Then, we can verify that

$$G_0\times\left(t_0-\frac{\epsilon}{\sqrt{N}},t_0+\frac{\epsilon}{\sqrt{N}}\right)\subset Q^{(k)}\subset\overline{G}\times(t_0-\sqrt{2}\epsilon,t_0+\sqrt{2}\epsilon),\quad k=1,2,3,4. \tag{5.5}$$

Let $Q_h^{(k)} = Q^{(k)} \cap Q_h$. In order to apply Theorem 2.4, we introduce a cut-off function $\tilde{\chi} \in C^\infty(Q)$ such that $0 \le \tilde{\chi} \le 1$ and

$$\tilde{\chi}(x,t) = \begin{cases} 1, & \varphi(x,t) > \mu_3, \\ 0, & \varphi(x,t) < \mu_2. \end{cases} \tag{5.6}$$

Then letting $z_h = \tilde{\chi} y_h$, we obtain

$$\mathrm{d}z_h - \Delta_h z_h \mathrm{d}t = (a_h z_h + b_h \mathbf{D}_h^+ z_h + F_h)dt + c_h z_h \mathrm{d}B(t), \quad (x_h, t) \in Q_h, \tag{5.7}$$

where

$$\begin{aligned} F_h =& \partial_t \tilde{\chi} y_h - \Delta_h \tilde{\chi} \mathbf{m}_h y_h - 2\mathbf{D}_h \tilde{\chi} \mathbf{D}_h y_h - \frac{h^2}{4} \Delta_h \tilde{\chi} \Delta_h y_h \\ & - b_h \mathbf{D}_h^+ \tilde{\chi} \mathbf{m}_h^+ y_h - \frac{h}{2} b_h \mathbf{D}_h^+ \tilde{\chi} \mathbf{D}_h^+ y_h. \end{aligned}$$

From the definition of $\tilde{\chi}$, together with $d(x_0) = 0$, we have $(x_0, t) \in Q \setminus Q^{(1)}$ and then $\tilde{\chi}(x_0, t) = 0$. We further have

$$(z_h)_0 = 0, \quad t \in (0, T). \tag{5.8}$$

Since $t_0 \in \left[\sqrt{2}\epsilon, T - \sqrt{2}\epsilon\right]$, we have

$$\max\{\psi(x,0), \psi(x,T)\} \le d(x) - 2\beta\epsilon^2 \le 0,$$

which implies $(G \times \{t = 0\}) \cup (G \times \{t = T\}) \subset \mathrm{Supp}(\tilde{\chi})$ and then

$$z_h(0) = 0, \quad \text{and} \quad z_h(T) = 0. \tag{5.9}$$

Applying (2.15) to z_h and noting (5.8) and (5.9), we obtain

$$\begin{aligned} & \mathbb{E}\int_{Q_h} \frac{1}{s\varphi}\theta^2 |\Delta_h z_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2 |\mathbf{D}_h^+ z_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2 |z_h|^2 \mathrm{d}t \\ \le & C\mathbb{E}\int_{Q_h} \theta^2 |a_h z_h + b_h \mathbf{D}_h^+ z_h + F_h|^2 \mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2 |\mathbf{D}_h^+ (c_h z_h)|^2 \mathrm{d}t \\ & + C\mathbb{E}\int_0^T s\lambda\varphi\theta^2(x_{N+1}, t)|\eta|^2 \mathrm{d}t + C(\lambda)s^3 e^{C(\lambda)s} \|\xi\|^2_{L^2(\Omega; H^1(0,T))} \\ \le & C\mathbb{E}\int_{Q_h} \theta^2 |F_h|^2 \mathrm{d}t + C\mathbb{E}\int_{Q_h^-} s\varphi\theta^2 \left(1 + \frac{h}{2}\right)^2 \left(|z_h|^2 + |\mathbf{D}_h^+ z_h|^2\right) \mathrm{d}t \\ & + C(\lambda)s^3 e^{C(\lambda)s} \left(\|\xi\|^2_{L^2(\Omega; H^1(0,T))} + \|\eta\|^2_{L^2(\Omega; L^2(0,T))}\right) \end{aligned} \tag{5.10}$$

Firstly, we notice that the second term on the right-hand side of (5.10) can be absorbed by the second and third terms on the left-side of (5.10). Secondly, by (5.5), we have $\overline{G_0} \times \left(t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}\right) \subset Q^{(4)}$ and then

$$\overline{G_{0,h}} \times \left(t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}\right) \subset Q_h^{(4)} \subset Q_h \tag{5.11}$$

for sufficiently small $h \in (0, h_1)$. Then, noting (5.6) and (5.11) we see that

$$
\begin{aligned}
&\mathbb{E}\int_{Q_h} \frac{1}{s\varphi}\theta^2|\Delta_h z_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h^-} s\lambda^2\varphi\theta^2|\mathbf{D}_h^+ z_h|^2 \mathrm{d}t + \mathbb{E}\int_{Q_h} s^3\lambda^4\varphi^3\theta^2|z_h|^2 \mathrm{d}t \\
\geq &\mathbb{E}\int_{t_0-\frac{\epsilon}{\sqrt{N}}}^{t_0+\frac{\epsilon}{\sqrt{N}}} \left(\int_{G_{0,h}} \frac{1}{s\varphi}\theta^2|\Delta_h y_h|^2 + \int_{G_{0,h}^-} s\lambda^2\varphi\theta^2|\mathbf{D}_h^+ y_h|^2 + \int_{G_{0,h}} s^3\lambda^4\varphi^3\theta^2|y_h|^2\right) \mathrm{d}t \\
\geq &C(\lambda)\frac{1}{s}e^{2\mu_4 s}\mathbb{E}\int_{t_0-\frac{\epsilon}{\sqrt{N}}}^{t_0+\frac{\epsilon}{\sqrt{N}}} \left(\int_{G_{0,h}} |\Delta_h y_h|^2 + \int_{G_{0,h}^-} |\mathbf{D}_h^+ y_h|^2 + \int_{G_{0,h}} |y_h|^2\right) \mathrm{d}t. \qquad (5.12)
\end{aligned}
$$

On the other hand, we can verify that there exists a sufficiently small h_2 such that

$$
\partial_t\tilde{\chi}(x_h,t) = \mathbf{D}_h^+\tilde{\chi}(x_h,t) = \mathbf{D}_h\tilde{\chi}(x_h,t) = \Delta_h\tilde{\chi}(x_h,t) = 0, \quad (x_h,t) \in Q_h' \qquad (5.13)
$$

for all $h \in (0, h_2)$, where

$$
Q_h' = \left\{(x_h,t) \in Q_h \mid \varphi(x_h,t) > \mu_3 + \frac{1}{2}(\mu_4-\mu_3)\right\}.
$$

Indeed, we only need to prove

$$
\tilde{\chi}(x_h \pm h, t) \equiv 1, \quad (x_h,t) \in Q_h'. \qquad (5.14)
$$

Since $\varphi(x_h,t) > \mu_3 + \frac{1}{2}(\mu_4-\mu_3)$, we have

$$
\varphi(x_h \pm h, t) = \varphi(x_h,t)\left(1+\mathcal{O}_\lambda(h)\right) > \mu_3 + \frac{1}{2}(\mu_4-\mu_3) + \mathcal{O}_\lambda(h).
$$

Then we can choose h_2 sufficiently small such that $\varphi(x_h \pm h, t) > \mu_3$ for all $h \in (0, h_2)$. By the definition of $\tilde{\chi}$ we obtain (5.14). We apply (2.1) and (5.13) to yield

$$
\begin{aligned}
&\mathbb{E}\int_{Q_h} \theta^2|F_h|^2 \mathrm{d}t \\
\leq &Ce^{2s\left(\mu_3+\frac{1}{2}(\mu_4-\mu_3)\right)}\mathbb{E}\int_{Q_h\setminus Q_h'} \left(|y_h|^2 + |\mathbf{m}_h y_h|^2 + |\mathbf{m}_h^+ y_h|^2 + |\mathbf{D}_h y_h|^2\right) \mathrm{d}t \\
&+ Ce^{2s\left(\mu_3+\frac{1}{2}(\mu_4-\mu_3)\right)}\mathbb{E}\int_{Q_h\setminus Q_h'} \mathcal{O}(h)\left(|\mathbf{D}_h^+ y_h|^2 + |\Delta_h y_h|^2\right) \mathrm{d}t \\
\leq &Ce^{s(\mu_3+\mu_4)}\mathbb{E}\int_{Q_h\setminus Q_h'} \left(|y_h|^2 + |\mathbf{D}_h^+ y_h|^2 + |\Delta_h y_h|^2\right) \mathrm{d}t. \qquad (5.15)
\end{aligned}
$$

Therefore, from (5.10), (5.12) and (5.15) it follows that

$$
\begin{aligned}
&C(\lambda)\frac{1}{s}e^{2\mu_4 s}\mathbb{E}\int_{t_0-\frac{\epsilon}{\sqrt{N}}}^{t_0+\frac{\epsilon}{\sqrt{N}}} \left(\int_{G_{0,h}} |\Delta_h y_h|^2 + \int_{G_{0,h}^-} |\mathbf{D}_h^+ y_h|^2 + \int_{G_{0,h}} |y_h|^2\right) \mathrm{d}t \\
\leq &Ce^{s(\mu_3+\mu_4)}\|y_h\|^2_{L^2_{\mathcal{F}}(0,T;H^2(G_h))} + C(\lambda)s^3e^{C(\lambda)s}\left(\|\xi\|^2_{L^2(\Omega;H^1(0,T))} + \|\eta\|^2_{L^2(\Omega;L^2(0,T))}\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\mathbb{E}\int_{t_0-\frac{\epsilon}{\sqrt{N}}}^{t_0+\frac{\epsilon}{\sqrt{N}}} \left(\int_{G_{0,h}} |\Delta_h y_h|^2 + \int_{G_{0,h}^-} |\mathbf{D}_h^+ y_h|^2 + \int_{G_{0,h}} |y_h|^2\right) \mathrm{d}t \\
\leq &C(\lambda)se^{-s(\mu_4-\mu_3)}\|y_h\|^2_{L^2_{\mathcal{F}}(0,T;H^2(G_h))} + C(\lambda)s^4e^{C(\lambda)s}\left(\|\xi\|^2_{L^2(\Omega;H^1(0,T))} + \|\eta\|^2_{L^2(\Omega;L^2(0,T))}\right) \\
\leq &e^{-\frac{1}{2}s(\mu_4-\mu_3)}\|y_h\|^2_{L^2_{\mathcal{F}}(0,T;H^2(G_h))} + e^{2C(\lambda)s}\left(\|\xi\|^2_{L^2(\Omega;H^1(0,T))} + \|\eta\|^2_{L^2(\Omega;L^2(0,T))}\right)
\end{aligned}
$$

for all sufficiently large s. Noticing that $\mu_4 - \mu_3 > 0$, for fixed λ by the standard argument we obtain

$$\|y_h\|_{L^2_{\mathcal{F}}\left(t_0-\frac{\epsilon}{\sqrt{N}},t_0+\frac{\epsilon}{\sqrt{N}};H^2(G_{0,h})\right)} \leq CM^{\kappa}\left(\|\xi\|_{L^2(\Omega;H^1(0,T))} + \|\eta\|_{L^2(\Omega;L^2(0,T))}\right)^{1-\kappa} \tag{5.16}$$

with

$$\kappa = \frac{C(\lambda)}{C(\lambda)+\frac{1}{4}(\mu_4-\mu_3)} \in (0,1).$$

Finally, in (5.16) taking $t_0 = \sqrt{2}\epsilon + \frac{j\epsilon}{\sqrt{N}}$, $j = 0, 1, 2, \cdots, m$ such that

$$\sqrt{2}\epsilon + \frac{m\epsilon}{\sqrt{N}} \leq T - \sqrt{2}\epsilon \leq \sqrt{2}\epsilon + \frac{m\epsilon}{\sqrt{N}}$$

and summing up over j, we obtain the desired estimate (2.19) with replacing ϵ by $\sqrt{2}\epsilon$ and then complete the proof of Theorem 2.6 due to ϵ is arbitrary. $\square$

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