

# FISHER-RAO GRADIENT FLOWS OF LINEAR PROGRAMS AND STATE-ACTION NATURAL POLICY GRADIENTS

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**ABSTRACT.** Kakade’s natural policy gradient method has been studied extensively in the last years showing linear convergence with and without regularization. We study another natural gradient method which is based on the Fisher information matrix of the state-action distributions and has received little attention from the theoretical side. Here, the state-action distributions follow the Fisher-Rao gradient flow inside the state-action polytope with respect to a linear potential. Therefore, we study Fisher-Rao gradient flows of linear programs more generally and show linear convergence with a rate that depends on the geometry of the linear program. Equivalently, this yields an estimate on the error induced by entropic regularization of the linear program which improves existing results. We extend these results and show sublinear convergence for perturbed Fisher-Rao gradient flows and natural gradient flows up to an approximation error. In particular, these general results cover the case of state-action natural policy gradients.

**Keywords:** Fisher-Rao metric, linear program, entropic regularization, multi-player game, Markov decision process, natural policy gradient

**MSC codes:** 65K05, 90C05, 90C08, 90C40, 90C53

## 1. INTRODUCTION

Natural policy gradient (NPG) methods and their proximal and trust region formulations known as PPO and TRPO are among the most popular policy optimization techniques in modern reinforcement learning (RL). As such they serve as a cornerstone of many recent RL success stories including celebrated advancements in computer games [48, 49, 11] and the recent development of large language models like ChatGPT [1]. This has motivated a quickly growing body of work studying the theoretical aspects such as the convergence properties and statistical efficacy of natural policy gradient methods. Almost all of these works consider a specific model geometry where the Fisher-Rao metrics of the individual rows of the policy are mixed according to their state distribution or slight modifications of this [24, 9, 34, 26]. However, other choices for the model geometry are possible. In particular, the Fisher metric on the state-action distributions has been used to design a natural gradient method as well as actor-critic and a trust-region variant known as relative entropy search (REPS) [35, 36, 42]. This alternative natural policy gradient has been found to have the potential to reduce the severity of plateaus [35] and improve the performance of actor-critic methods [36].

Working with the state-action distributions has the significant benefit that the reward optimization problem becomes a linear program [25]. In particular, for rich enough parametric policy models, the state-action natural policy gradient flows follow the Fisher-Rao gradient flows of the state-action linear program [39]. This motivates us to study the convergence properties of Fisher-Rao gradient flows of general linear programs. These flows coincide with the solutions of entropy-regularized linear programs and thus by studying the convergence of the flow we also bound the error introduced by entropic regularization in linear programming.

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**1.1. Contributions.** We provide an explicit convergence analysis of Fisher-Rao gradient flows of general linear programs and natural (policy) gradient flows with and without function approximation. More precisely, our main contributions can be summarized as follows:

- We study Fisher-Rao flows of general linear programs and show linear convergence both in KL-divergence and in function value with an exponential rate depending on the geometry of the linear program, see Theorem 3.2.
- In the case of non-unique optimizers, we show that the Fisher-Rao gradient flow converges to the information projection of the initial condition to the set of optimizers thereby characterizing its *implicit bias*, see Corollary 3.12.
- We obtain an estimate on the regularization error in entropy regularized linear programming improving known convergence rates, see Corollary 3.3.
- We study natural gradients for parametric measures, and show sublinear convergence under inexact gradient evaluations up to an approximation error and a distribution mismatch measured in the  $\chi^2$ -divergence, see Corollary 4.7.
- In a multi-player game with a specific payoff structure, we show linear convergence of the natural gradient flow, see Theorem 4.9.
- In the context of Markov decision processes, we study state-action natural policy gradients and provide a sublinear convergence result for general policy parametrizations, see Corollary 5.4, and a linear convergence guarantee gradient for regular parametrizations, see Corollary 5.8. In particular, this covers tabular softmax, escort, and log-linear parametrizations.

**1.2. Related works.** State-action natural policy gradients were recently studied with and without state-action entropy regularization in [39]. For regularization strength  $\lambda > 0$  that work showed  $O(e^{-\lambda t})$  convergence, but in the unregularized case the precise exponential rate was not characterized.

A mirror descent variant of the state-action natural policy gradients was shown to achieve an optimal  $O(\sqrt{T})$  regret in an online setting in [61, 20, 41].

There has been a recent surge of works studying the natural policy gradient method proposed by Kakade. The initial results of [2] showed sublinear convergence rate  $O(t^{-1})$  for unregularized problems. This was subsequently improved to a linear rate for step sizes found by exact line search [12] and constant step sizes [27, 3, 57]. For regularized problems, the method converges linearly for small step sizes, locally quadratically for Newton-like step sizes, and linearly with linear function approximation [16, 30]. The linear convergence of NPG has been extended to the function approximation regime and more general problem geometries, where these results either require geometrically increasing step sizes [56, 3, 57, 4] or entropy regularization [15, 28, 58, 30, 4]. However, these geometries do not cover the state-action geometries. Apart from the works on convergence rates for policy gradient methods for standard MDPs, a primal-dual NPG method with sublinear global convergence guarantees has been proposed for constrained MDPs [21, 22]. Where all of these results work in discrete time, the gradient flows corresponding to this type of natural policy gradient have been shown to converge linearly under entropy regularization for Polish state and action spaces [26].

Hessian geometries, which provide a rich generalization of the Fisher-Rao metric, have been studied in convex optimization both from a continuous time perspective and via a discrete-time mirror descent analysis [5, 53]. In the context of linear programming, linear convergence of the Fisher-Rao gradient flow was shown in [5] albeit without a characterization of the convergence rate.

In the case of a linear program, the Fisher-Rao gradient flow parametrized by time corresponds to the trajectory of solutions of the entropy-regularized program parametrized by the inverse regularization strength, which has been studied in several works. An exponential convergence result

was obtained in [18] and subsequently the rate was characterized as  $O(e^{-\delta t})$  for a constant  $\delta$  depending on the geometry of the linear program [55, 51]. The results obtained in this article follow an alternative proof strategy and provide exponential convergence  $O(e^{-\Delta t})$ , where  $\Delta \geq \delta$ . This improvement can be strict for the linear programs encountered in Markov decision processes under standard assumptions. Whereas existing works study convergence in function value, our results also cover convergence in the KL-divergence. Finally, the geometry of Fisher-Rao gradient flows or equivalently the entropic central path was recently described as the intersection of the feasible region with a toric variety [50].

**1.3. Notation and terminology.** For a finite set  $\mathbb{X}$ , we denote the *free vector space* over  $\mathbb{X}$  by  $\mathbb{R}^{\mathbb{X}} = \{\mu: \mathbb{X} \rightarrow \mathbb{R}\}$ . Its elements can be identified with vectors  $(\mu_x)_{x \in \mathbb{X}}$ . Similarly, we denote the vectors with non-negative entries and positive entries by  $\mathbb{R}_{\geq 0}^{\mathbb{X}}$  and  $\mathbb{R}_{> 0}^{\mathbb{X}}$ , respectively. For two elements  $\mu, \nu \in \mathbb{R}^{\mathbb{X}}$  we denote the *Hadamard product*, i.e., the entrywise product, between  $\mu$  and  $\nu$  by  $\mu \odot \nu \in \mathbb{R}^{\mathbb{X}}$ , so that  $\mu \odot \nu(x) := \mu(x)\nu(x)$ . The *total variation* norm  $\|\cdot\|_{\text{TV}}: \mathbb{R}^{\mathbb{X}} \rightarrow \mathbb{R}$  is given by  $\|\mu\|_{\text{TV}} := \frac{1}{2} \sum_x |\mu_x|$ . Finally, with  $\mathbf{1}_{\mathbb{X}} \in \mathbb{R}^{\mathbb{X}}$  we denote the all-one vector.

A *polyhedron* is a set  $P = \{\mu \in \mathbb{R}^{\mathbb{X}} : \ell_i(\mu) \geq 0 \text{ for } i = 1, \dots, k\} \subseteq \mathbb{R}^{\mathbb{X}}$ , where  $\ell_i: \mathbb{R}^{\mathbb{X}} \rightarrow \mathbb{R}$  are affine linear functions for  $i = 1, \dots, k$ . A bounded (and thus compact) polyhedron is called a *polytope*. A polytope can be shown to be the convex hull of finitely many extreme points, which are called *vertices* and which we denote by  $\text{Vert}(P)$ . Two vertices  $\mu_1, \mu_2 \in \text{Vert}(P)$  are called *neighbors* if the subspace  $\{c \in \mathbb{R}^{\mathbb{X}} : c^\top \mu_1 = c^\top \mu_2 = \max_{\mu \in P} c^\top \mu\}$  has dimension  $|\mathbb{X}| - 1$ . We denote the set of all neighbors of a vertex  $\mu$  by  $N(\mu) \subseteq \text{Vert}(P)$ . The *affine space*  $\text{aff span}(P)$  of a polytope  $P \subseteq \mathbb{R}^{\mathbb{X}}$  is the smallest affine subspace of  $\mathbb{R}^{\mathbb{X}}$  containing  $P$ . The relative interior  $\text{int}(P)$  and boundary  $\partial P$  of  $P$  are the interior and boundary of  $P$  in its affine hull. Finally, the *tangent space*  $TP$  of  $P$  is given by the linear part of  $\text{aff span}(P)$ .

We call  $\Delta_{\mathbb{X}} := \{\mu \in \mathbb{R}_{\geq 0}^{\mathbb{X}} : \sum_x \mu_x = 1\}$  the *probability simplex*. We say that  $\mu \in \Delta_{\mathbb{X}}$  is absolutely continuous with respect to  $\nu \in \Delta_{\mathbb{X}}$  if  $\nu(x) = 0$  implies  $\mu(x) = 0$  and write  $\mu \ll \nu$ . We denote the expectation with respect to  $\mu \in \Delta_{\mathbb{X}}$  by  $\mathbb{E}_{\mu}$  and call  $\chi^2(\mu, \nu) := \mathbb{E}_{\nu} \left[ \frac{(\mu(x) - \nu(x))^2}{\nu(x)^2} \right]$  the  $\chi^2$ -divergence between  $\mu$  and  $\nu$ . If  $\mathbb{Y}$  is another finite set, we call the Cartesian product  $\Delta_{\mathbb{X}}^{\mathbb{Y}} = \Delta_{\mathbb{X}} \cdot \dots \cdot \Delta_{\mathbb{X}}$  the *conditional probability polytope* and associate its elements with stochastic matrices  $P \in \mathbb{R}_{\geq 0}^{\mathbb{X} \times \mathbb{Y}}$  with  $\sum_x P(x|y) = 1$ .

For a differentiable function  $f: \Omega \rightarrow \mathbb{R}$  on an open subset  $\Omega \subseteq \mathbb{R}^{\mathbb{X}}$  we denote the Euclidean gradient and Hessian of  $f$  at  $\mu \in \mathbb{R}^{\mathbb{X}}$  by  $\nabla f(\mu) \in \mathbb{R}^{\mathbb{X}}$  and  $\nabla^2 f(\mu) \in \mathbb{R}^{\mathbb{X} \times \mathbb{X}}$ .

## 2. PRELIMINARIES ON FISHER-RAO GRADIENT FLOWS

Here, we provide a self-contained review of the properties of Fisher-Rao gradient flows that we require later. The results in this section can be generalized to a large class of Hessian geometries and – apart from the central path property – also to other objectives albeit with different proofs, for which we refer to [5, 37].

The *Fisher-Rao metric* is a Riemannian metric on the positive orthant given by

$$g_{\mu}^{\text{FR}}(v, w) := \sum_{x \in \mathbb{X}} \frac{v_x w_x}{\mu_x} \quad \text{for all } v, w \in \mathbb{R}^{\mathbb{X}}, \mu \in \mathbb{R}_{> 0}^{\mathbb{X}}, \quad (2.1)$$

where we denote the induced norm by  $\|v\|_{g_{\mu}^{\text{FR}}} := g_{\mu}^{\text{FR}}(v, v)^{\frac{1}{2}}$ . The Fisher-Rao metric was introduced in the seminal works of C. R. Rao [45, 46] to provide lower bounds on the statistical error in parameter estimation known as the Cramer-Rao bound. This geometric approach to statistical estimation has subsequently led to the development of the field of information geometry, where N. N. Čencov characterized the Fisher-Rao metric as the unique Riemannian metric (up to scaling) that is invariant under sufficient statistics [17, 7, 8]. Despite its central role in statistics, our main

motivation for studying the Fisher-Rao metric is for its use in reinforcement learning, where it has been used to design natural gradient algorithms as well as trust region methods [6, 35, 42]. Further, it is very closely related to entropic regularization in linear programming, which enjoys immense popularity particularly in computational optimal transport [44, 51], see also [55] for a detailed discussion of entropy regularized linear programming.

The Fisher-Rao metric is closely connected to the negative Shannon entropy

$$\phi(\mu) = -H(\mu) := \sum_{x \in \mathbb{X}} \mu_x \log \mu_x \quad \text{for all } \mu \in \mathbb{R}_{>0}^{\mathbb{X}} \quad (2.2)$$

as it is induced by the Hessian of the (negative) entropy, i.e., we have

$$g_{\mu}^{\text{FR}}(v, w) = v^{\top} \nabla^2 \phi(\mu) w \quad \text{for all } v, w \in \mathbb{R}^{\mathbb{X}}, \mu \in \mathbb{R}_{>0}^{\mathbb{X}}. \quad (2.3)$$

As such, the Fisher-Rao metric falls into the class of *Hessian metrics* that have been studied in convex optimization; we refer to [5, 37] for general well-posedness and convergence results. An important concept in the analysis of Hessian gradient flows is the *Bregman divergence* induced by  $\phi$ , which in the case of the negative entropy is given by the *KL-divergence*

$$D_{\text{KL}}(\mu, \nu) := \phi(\mu) - \phi(\nu) - \nabla \phi(\nu)(\mu - \nu) = \sum_{x \in \mathbb{X}} \mu_x \log \frac{\mu_x}{\nu_x} - \sum_{x \in \mathbb{X}} \mu_x + \sum_{x \in \mathbb{X}} \nu_x \quad (2.4)$$

for  $\mu, \nu \in \mathbb{R}_{>0}^{\mathbb{X}}$  with  $\mu \ll \nu$ , where we use the common convention  $0 \log \frac{0}{0} := 0$ .

Consider now a continuously differentiable function  $f: \mathbb{R}_{>0}^{\mathbb{X}} \rightarrow \mathbb{R}$  that we assume to be differentiable on  $\mathbb{R}_{>0}^{\mathbb{X}}$  that we want to optimize over a polytope  $P = \mathbb{R}_{>0}^{\mathbb{X}} \cap \mathcal{L}$ , where  $\mathcal{L}$  is a linear space. We denote the gradient of  $f: \mathbb{R}_{>0}^{\mathbb{X}} \rightarrow \mathbb{R}$  at  $\mu \in \mathbb{R}_{>0}^{\mathbb{X}}$  with respect to the Fisher-Rao metric by  $\nabla^{\text{FR}} f(\mu)$  and call it the *Fisher-Rao gradient*. Further, we denote the Fisher-Rao gradient of  $f: \text{int}(P) \rightarrow \mathbb{R}$  by  $\nabla_P^{\text{FR}} f(\mu) \in TP$ , which is uniquely determined by

$$g_{\mu}^{\text{FR}}(\nabla_P^{\text{FR}} f(\mu), v) = df(\mu)v \quad \text{for all } v \in TP. \quad (2.5)$$

Note that  $\nabla_P^{\text{FR}} f(\mu)$  is the projection of  $\nabla^{\text{FR}} f(\mu)$  with respect to the Fisher-Rao metric onto  $TP$ . By examining the definition of the Fisher-Rao metric we see that this is equivalent to

$$\langle \nabla^2 \phi(\mu) \nabla_P^{\text{FR}} f(\mu), v \rangle = \langle \nabla f(\mu), v \rangle \quad \text{for all } v \in TP. \quad (2.6)$$

We say that  $(\mu_t)_{t \in [0, T]} \subseteq \text{int}(P)$  solves the *Fisher-Rao gradient flow* if it solves the gradient flow with respect to the Fisher-Rao metric, i.e., if

$$\partial_t \mu_t = \nabla_P^{\text{FR}} f(\mu_t) \quad \text{for all } t \in [0, T]. \quad (2.7)$$

By using the characterization (2.6) of  $\nabla_P^{\text{FR}} f(\mu_t)$ , we see that  $(\mu_t)_{t \in [0, T]} \subseteq \text{int}(P)$  solves the Fisher-Rao gradient flow (2.7) if and only if we have

$$\langle \nabla^2 \phi(\mu_t) \partial_t \mu_t, v \rangle = \langle \nabla f(\mu_t), v \rangle \quad \text{for all } v \in TP, t \in [0, T]. \quad (2.8)$$

In the remainder, we study linear programs and work in the following setting.

**Setting 2.1.** We consider a finite set  $\mathbb{X}$  and a linear program

$$\max c^{\top} \mu \quad \text{subject to } \mu \in P, \quad (2.9)$$

with cost  $c \in \mathbb{R}^{\mathbb{X}}$  and feasible region  $P = \mathbb{R}_{>0}^{\mathbb{X}} \cap \mathcal{L}$  with  $P \cap \mathbb{R}_{>0}^{\mathbb{X}} \neq \emptyset$ , where  $\mathcal{L} \subseteq \mathbb{R}^{\mathbb{X}}$  is an affine space. By  $(\mu_t)_{t \in [0, T]} \subseteq \text{int}(P)$  we denote a solution of the Fisher-Rao gradient flow (2.7) with initial condition  $\mu_0 \in P \cap \mathbb{R}_{>0}^{\mathbb{X}}$  and potential  $f(\mu) = c^{\top} \mu$ , where  $T \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ .

Fisher-Rao gradient flows are closely connected to the solutions of KL-regularized linear programs,  $c^{\top} \mu - \lambda D_{\text{KL}}(\mu, \mu_0)$ . The family of solutions of the regularized problems parametrized by the regularization strength  $\lambda$  is referred to as the (entropic) *central path* in optimization [14].

**Proposition 2.2** (Central path property). *Consider Setting 2.1. Then  $\mu_t$  is uniquely characterized by*

$$\mu_t = \arg \max \left\{ c^\top \mu - t^{-1} D_{\text{KL}}(\mu, \mu_0) : \mu \in P \right\} \quad \text{for all } t \in (0, T). \quad (2.10)$$

*Proof.* Let  $\hat{\mu}_t \in P$  denote the unique maximizer of  $g(\mu) := c^\top \mu - t^{-1} D_{\text{KL}}(\mu, \mu_0)$  over  $P$  for  $t > 0$ , then surely  $\hat{\mu}_t \in \text{int}(P)$ . Thus,  $\hat{\mu}_t$  is uniquely determined by  $\langle \nabla g(\hat{\mu}_t), v \rangle = 0$  for all  $v \in TP$ . Direct computation yields  $\nabla g(\mu) = c - t^{-1}(\nabla \phi(\mu) - \nabla \phi(\mu_0))$  and hence  $\hat{\mu}_t$  is uniquely determined by

$$t \langle c, v \rangle = \langle \nabla \phi(\hat{\mu}_t) - \nabla \phi(\mu_0), v \rangle \quad \text{for all } v \in TP.$$

On the other hand, for the gradient flow we can use (2.8) and compute for  $v \in TP$

$$\begin{aligned} \langle \nabla \phi(\mu_t) - \nabla \phi(\mu_0), v \rangle &= \int_0^t \partial_s \langle \nabla \phi(\mu_s), v \rangle ds \\ &= \int_0^t \langle \nabla^2 \phi(\mu_s) \partial_s \mu_s, v \rangle ds \\ &= \int_0^t \langle \nabla f(\mu_s) \mu_s, v \rangle ds \\ &= \int_0^t \langle c, v \rangle ds = t \langle c, v \rangle. \end{aligned}$$

This shows  $\mu_t = \hat{\mu}_t$  as claimed.  $\square$

We can use the central path property to show  $O(t^{-1})$  convergence. The following corollary can be generalized to arbitrary convex objectives [5].

**Corollary 2.3** (Sublinear convergence rate). *Consider Setting 2.1 and assume that the linear program (2.9) admits a solution  $\mu^\star \in P$ . Then it holds that*

$$c^\top \mu^\star - c^\top \mu_t \leq \frac{D_{\text{KL}}(\mu^\star, \mu) - D_{\text{KL}}(\mu_t, \mu)}{t} \leq \frac{D_{\text{KL}}(\mu^\star, \mu_0)}{t} \quad \text{for all } t \in [0, T]. \quad (2.11)$$

*Proof.* By the central path property, we have

$$c^\top \mu_t - t^{-1} D_{\text{KL}}(\mu_t, \mu) \geq c^\top \mu^\star - t^{-1} D_{\text{KL}}(\mu^\star, \mu).$$

Rearranging yields the result.  $\square$

One can use the central path property to show the long-time existence of Fisher-Rao gradient flows. Again, the following result can be generalized to a large class of Hessian geometries and potentials  $f$ , see [5, 37], albeit with more delicate proofs.

**Theorem 2.4** (Well-posedness of FR GFs). *Consider Setting 2.1. Then there exists a unique global solution  $(\mu_t)_{t \geq 0} \subseteq \text{int}(P)$  of the Fisher-Rao gradient flow (2.7).*

*Proof.* The local existence and uniqueness follows from the Picard-Lindelöf theorem. Hence, it suffices to show that the Fisher-Rao gradient flow does not hit the boundary  $\partial P$  in finite time. By the central path property, this is equivalent to the statement that the solutions of all KL-regularized problems (2.10) lie in the interior  $\text{int}(P)$  of the polyhedron, which can be easily checked.  $\square$

### 3. CONVERGENCE OF FISHER-RAO GRADIENT FLOWS

We have seen that Fisher-Rao gradient flows converge globally at a sublinear rate  $O(t^{-1})$ . We now build on this analysis and show that once the gradient flow enters a vicinity of the optimizer, it converges at a quasi-exponential rate  $O(t^\kappa e^{-\Delta t})$ , where  $\Delta > 0$  depends on the geometry of the linear program and  $\kappa > 0$  depends on the initial condition  $\mu_0$ . We consider linear programs of the following form.

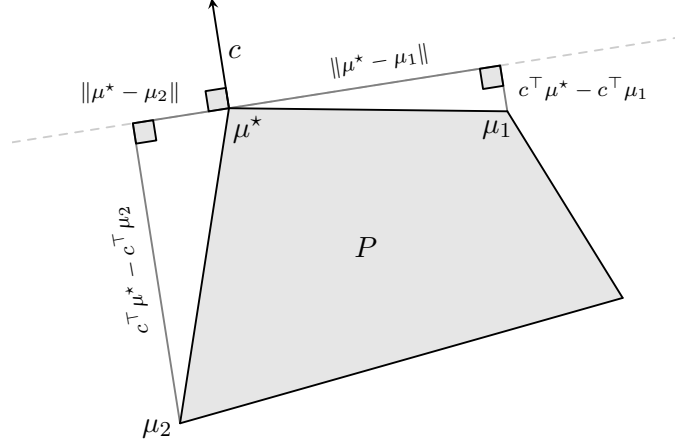


FIGURE 1. Visualization of the suboptimality gap  $\Delta$  appearing in Theorem 3.2 associated to the linear program (3.1); note that  $\Delta$  deteriorates when  $c$  is almost orthogonal to a face of  $P$ .

**Setting 3.1.** We consider a finite set  $\mathbb{X}$  and a linear program

$$\max c^\top \mu \quad \text{subject to } \mu \in P, \quad (3.1)$$

with cost  $c \in \mathbb{R}^{\mathbb{X}}$  and feasible region  $P = \Delta_{\mathbb{X}} \cap \mathcal{L}$  with  $P \cap \mathbb{R}_{>0}^{\mathbb{X}} \neq \emptyset$ , where  $\mathcal{L} \subseteq \mathbb{R}^{\mathbb{X}}$  is an affine space. By  $(\mu_t)_{t \geq 0} \subseteq \text{int}(P)$  we denote the solution of the Fisher-Rao gradient flow (2.7) with initial condition  $\mu_0 \in P \cap \mathbb{R}_{>0}^{\mathbb{X}}$  and the potential  $f(\mu) = c^\top \mu$ .

The following result is the main contribution of this article, where we defer the proof to Section 3.1. We first establish it under the assumption that the linear program (3.1) admits a unique solution, and provide a generalization in Theorem 3.13.

**Theorem 3.2** (Linear convergence of Fisher-Rao GFs of LPs). *Consider Setting 3.1 and assume that the linear program (3.1) admits a unique solution  $\mu^* \in P$ . Let*

$$\Delta := \min \left\{ \frac{c^\top \mu^* - c^\top \mu}{\|\mu^* - \mu\|_{\text{TV}}} : \mu \in N(\mu^*) \right\}, \quad (3.2)$$

where  $N(\mu^*)$  denotes the set of neighboring vertices of  $\mu^*$  and set

$$t_0 := \frac{2D_{\text{KL}}(\mu^*, \mu_0)}{\Delta \cdot \min\{\mu_x^* : \mu_x^* > 0\}}. \quad (3.3)$$

Then for any  $t \geq t_0$  we have

$$D_{\text{KL}}(\mu^*, \mu_t) \leq D_{\text{KL}}(\mu^*, \mu_0) \exp \left( -\Delta(t - t_0) + 2t_0\Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.4)$$

as well as

$$c^\top \mu^* - c^\top \mu_t \leq \Delta D_{\text{KL}}(\mu^*, \mu_0) \exp \left( -\Delta(t - t_0) + 2t_0\Delta \log \left( \frac{t + t_0}{2t_0} \right) \right). \quad (3.5)$$

The constant  $\Delta$  depends on the geometry of the linear program, see Figure 1. Indeed, the quotient  $\frac{c^\top \mu^* - c^\top \mu}{\|\mu^* - \mu\|_{\text{TV}}}$  is the slope of the objective along the edge  $\mu^* - \mu$ . Consequently,  $\Delta$  decreases when the cost  $c$  is closer to orthogonal to a face of  $P$ .

Using the central path property of Fisher-Rao gradient flows and initializing at the maximum entropy distribution in  $P$  yields the following result.



**Corollary 3.3** (Entropic regularization error). *Consider Setting 3.1 and assume that the linear program (3.1) admits a unique solution  $\mu^\star \in P$ . For  $t > 0$  denote by  $\mu_t^\star$  the unique solution of the entropy-regularized linear program*

$$\max c^\top \mu + t^{-1} H(\mu) \quad \text{subject to } \mu \in P, \quad (3.6)$$

where  $H$  denotes the Shannon entropy. Then for any  $t \geq t_0$  we have

$$D_{\text{KL}}(\mu^\star, \mu_t^\star) \leq R_H \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.7)$$

as well as

$$c^\top \mu^\star - c^\top \mu_t^\star \leq \Delta R_H \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.8)$$

where  $\Delta > 0$  and  $t_0 \geq 0$  are defined in (3.2) and (3.3), respectively, and

$$R_H := \max_{\mu \in P} H(\mu) - \min_{\mu \in P} H(\mu) \leq \log |\mathbb{X}| \quad (3.9)$$

denotes the entropic radius of  $P$ .

Similar to the convergence result, here too one can remove the uniqueness assumption, see Remark 3.14.

**Remark 3.4** (Comparison with existing results). In [18] it was shown that the regularization error for entropy-regularized linear programs decays exponentially fast, without quantifying the convergence rate. The convergence rate of the error, as well as that of Fisher-Rao gradient flows, was subsequently studied in [55, 51], establishing a rate  $O(e^{-\delta t})$  with

$$\delta := \frac{\min \{c^\top \mu^\star - c^\top \mu : \mu \in \text{Vert}(P) \setminus \{\mu^\star\}\}}{\max \{\|\mu\|_1 : \mu \in P\}}. \quad (3.10)$$

For polytopes  $P \subseteq \Delta_{\mathbb{X}}$  that we consider here, we have

$$\delta = \min \{c^\top \mu^\star - c^\top \mu : \mu \in \text{Vert}(P)\} = \min \{c^\top \mu^\star - c^\top \mu : \mu \in N(\mu^\star)\} \leq \Delta,$$

showing that Theorem 3.2 offers an improvement of these previous results. For the special case  $P = \Delta_{\mathbb{X}}$ , for which a matching lower bound was constructed in [55], the two constants agree. Indeed, it is easily checked that  $\delta = \Delta$  if and only if there is a neighboring vertex  $\mu \in N(\mu^\star)$  which has minimal optimality gap  $c^\top \mu^\star - c^\top \mu$  and has disjoint support from  $\mu^\star$ . For  $P = \Delta_{\mathbb{X}}$  this is always true, but for Markov decision processes the feasible region of the (dual) linear program is a strict subset  $\mathcal{D} \subsetneq \Delta_{\mathbb{S} \times \mathbb{A}}$  and under the standard exploratory Assumption 5.2 we have  $\delta < \Delta$ . In Section 5.2 we provide an explicit example where  $\delta < \Delta$ .

Further, for gradient flows with respect to a Riemannian metric of the form

$$g_\mu^\sigma(v, w) := \sum_{x \in \mathbb{X}} \frac{v_x w_x}{\mu_x^\sigma} \quad (3.11)$$

one can show  $O(t^{-\frac{1}{\sigma-1}})$  convergence for  $\sigma \in (1, 2)$ , see [39]. Note that this can be extended to the case  $\sigma = 2$ , which corresponds to logarithmic barriers for which it is well known that the central path converges at a  $O(t^{-1})$  rate [14, Section 11.2].

**Remark 3.5** (Tightness). For  $P = \Delta_{\mathbb{X}}$  we have  $\mu_t(x) \sim e^{-tc_x}$  as can be seen from the first order stationarity conditions; hence, in this case, the bound is tight. For general  $P$ , in Section 5.2 we provide empirical evidence that our bound on the exponent is sometimes but not always tight depending on the specific  $c$ .

We now present the proofs of the results.

**3.1. Convergence of Fisher-Rao gradient flows.** At the heart of the proof lies the following result, which can easily be extended to general Hessian geometries. For this, one can follow the reasoning in [5, Proposition 4.9], which treats general Hessian geometries, but does not allow for time-dependent constants  $\kappa_t$  and assumes the lower bound (3.12) in a neighborhood of  $\mu^\star$  and not only along the trajectory.

**Lemma 3.6.** *Consider Setting 2.1 and assume that there is an optimizer  $\mu^\star \in P$  and  $\kappa_t > 0$  for  $t > t_0 \geq 0$  such that*

$$c^\top \mu^\star - c^\top \mu_t \geq \kappa_t D_{\text{KL}}(\mu^\star, \mu_t) \quad \text{for all } t > t_0. \quad (3.12)$$

*Then we have*

$$D_{\text{KL}}(\mu^\star, \mu_t) \leq D_{\text{KL}}(\mu^\star, \mu_0) \exp \left( - \int_{t_0}^t \kappa_s ds \right) \quad \text{for all } t \geq t_0, \quad (3.13)$$

*as well as*

$$c^\top \mu^\star - c^\top \mu_t \leq \kappa_t D_{\text{KL}}(\mu^\star, \mu_0) \exp \left( - \int_{t_0}^t \kappa_s ds \right) \quad \text{for all } t \geq t_0. \quad (3.14)$$

For the proof of this result, we require the following identity.

**Lemma 3.7.** *Consider Setting 2.1, whereby we allow  $f: \mathbb{R}_{>0}^\mathbb{X} \rightarrow \mathbb{R}$  to be an arbitrary differentiable function, and fix  $\mu \in P$ . Then for any  $t \geq 0$  it holds that*

$$\partial_t D_{\text{KL}}(\mu, \mu_t) = \langle \nabla f(\mu_t), \mu_t - \mu \rangle. \quad (3.15)$$

*Proof.* Denoting the negative Shannon entropy by  $\phi$ , we compute

$$\partial_t D_{\text{KL}}(\mu, \mu_t) = -\partial_t \phi(\mu_t) - \partial_t \langle \nabla \phi(\mu_t), \mu - \mu_t \rangle = \langle \nabla^2 \phi(\mu_t) \partial_t \mu_t, \mu_t - \mu \rangle.$$

Now (2.8) yields the claim.  $\square$

*Proof of Lemma 3.6.* Using (3.15) and (3.12) we find that for  $t \geq T$  it holds that

$$\partial_t D_{\text{KL}}(\mu^\star, \mu_t) = c^\top \mu_t - c^\top \mu^\star \leq -\kappa_t D_{\text{KL}}(\mu^\star, \mu_t).$$

Now Gronwall's inequality yields (3.13). By Corollary 2.3 we have for any  $h > 0$  that

$$c^\top \mu^\star - c^\top \mu_t \leq \frac{D_{\text{KL}}(\mu^\star, \mu_{t-h})}{h} \leq D_{\text{KL}}(\mu^\star, \mu_{t_0}) \cdot \frac{\exp \left( - \int_{t_0}^{t-h} \kappa_s ds \right)}{h}.$$

Taking the limit  $h \rightarrow 0$  yields (3.14).  $\square$

The lower bound (3.12) can be interpreted as a form of strong convexity under which the objective value controls the Bregman divergence, see also [31, 10] for a discussion of gradient domination and strong convexity conditions in Bregman divergence. To show that such a lower bound holds in the case of the linear program (3.1), we first lower bound the sub-optimality gap  $c^\top \mu^\star - c^\top \mu_t$  in terms of an arbitrary norm, where we will later use the total variation distance.

**Lemma 3.8.** *Consider a polytope  $P \subseteq \mathbb{R}^\mathbb{X}$  and let  $c \in \mathbb{R}^\mathbb{X}$  be such that the linear function  $\mu \mapsto c^\top \mu$  has a unique maximizer over  $P$  at vertex  $\mu^\star \in P$  and consider an arbitrary semi-norm  $\|\cdot\|: \mathbb{R}^\mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$ . Setting  $\frac{c}{0} := +\infty$  for  $c > 0$ , we have*

$$\Delta := \min \left\{ \frac{c^\top \mu^\star - c^\top \mu}{\|\mu^\star - \mu\|} : \mu \in N(\mu^\star) \right\} > 0, \quad (3.16)$$

*where  $N(\mu^\star)$  denotes the set of neighboring vertices of  $\mu^\star$  and it holds that*

$$c^\top \mu^\star - c^\top \mu \geq \Delta \cdot \|\mu^\star - \mu\| \quad \text{for all } \mu \in P. \quad (3.17)$$



*Proof.* Note that since  $\mu^\star$  is the unique maximizer of  $\mu \mapsto c^\top \mu$  over  $P$  it holds that  $c^\top \mu^\star - c^\top \mu > 0$  for every neighboring vertex  $\mu \in N(\mu^\star)$ , which implies  $\Delta > 0$ . If  $\Delta = +\infty$ , then  $\|\mu^\star - \mu\| = 0$  for all  $\mu \in P$  and hence (3.17) holds trivially, so let  $\Delta < +\infty$ . The polytope  $P$  is contained in the cone

$$C = \left\{ \mu^\star + \sum_{\nu \in N(\mu^\star)} \alpha_\nu (\nu - \mu^\star) : \alpha_\nu \geq 0 \text{ for all } \nu \in N(\mu^\star) \right\}$$

generated by the edges adjacent to  $\mu^\star$ , see [59, Lemma 3.6]. Hence, for any  $\mu \in P$  there are non-negative weights  $\alpha_\nu \geq 0$  for  $\nu \in N(\mu^\star)$  such that

$$\mu = \mu^\star + \sum_{\nu \in N(\mu^\star)} \alpha_\nu (\nu - \mu^\star).$$

Now we compute

$$c^\top \mu^\star - c^\top \mu = \sum_{\nu \in N(\mu^\star)} \alpha_\nu (c^\top \mu^\star - c^\top \nu) \geq \Delta \cdot \sum_{\nu \in N(\mu^\star)} \alpha_\nu \|\mu^\star - \nu\|. \quad (3.18)$$

Further, by the triangle inequality we have

$$\|\mu^\star - \mu\| = \left\| \sum_{\nu \in N(\mu^\star)} \alpha_\nu (\nu - \mu^\star) \right\| \leq \sum_{\nu \in N(\mu^\star)} \alpha_\nu \|\nu - \mu^\star\|,$$

which completes the proof.  $\square$

**Lemma 3.9.** *Consider a finite set  $\mathbb{X}$  and a probability distribution  $\mu \in \Delta_{\mathbb{X}}$ . Let  $c > 1$  and set*

$$\delta := \frac{c-1}{c+1} \cdot \min\{\mu_x : \mu_x > 0\} > 0. \quad (3.19)$$

*Then for all  $\nu \in \Delta_{\mathbb{X}}$  satisfying  $\|\mu - \nu\|_\infty \leq \delta$  it holds that*

$$D_{\text{KL}}(\mu, \nu) \leq c \cdot \|\mu - \nu\|_{\text{TV}}. \quad (3.20)$$

*Proof.* We bound the individual summands in the KL-divergence

$$D_{\text{KL}}(\mu, \nu) = \sum_{x \in \mathbb{X}} \mu_x \log \left( \frac{\mu_x}{\nu_x} \right) = \sum_{x \in X} \mu_x \log \left( \frac{\mu_x}{\nu_x} \right),$$

where  $X := \{x \in \mathbb{X} : \mu_x > 0\}$ . If  $\mu_x, \nu_x > 0$  then

$$\begin{aligned} \mu_x \log \left( \frac{\mu_x}{\nu_x} \right) &= \mu_x (\log(\nu_x + (\mu_x - \nu_x)) - \log(\nu_x)) \\ &\leq \mu_x \left( \log(\nu_x) + \frac{\mu_x - \nu_x}{\nu_x} - \log(\nu_x) \right) \\ &= (\mu_x - \nu_x) \cdot \frac{\mu_x}{\nu_x}, \end{aligned} \quad (3.21)$$

where we used the convexity  $\log(t+h) \leq \log(t) + h/t$  for  $t > 0, t+h > 0$ . We set  $\varepsilon := \frac{c-1}{2} \in (0, 1)$ , such that

$$\delta = \frac{\varepsilon}{1+\varepsilon} \cdot \min\{\mu_x : \mu_x > 0\}.$$

If  $\|\mu - \nu\|_\infty \leq \delta$  then

$$\nu_x \geq \mu_x - \delta \geq \mu_x \left( 1 - \frac{\varepsilon}{1+\varepsilon} \right) = \frac{\mu_x}{1+\varepsilon}$$

as well as

$$\nu_x \leq \mu_x + \delta \leq \mu_x \left( 1 + \frac{\varepsilon}{1+\varepsilon} \right) \leq \mu_x \left( 1 + \frac{\varepsilon}{1-\varepsilon} \right) = \frac{\mu_x}{1-\varepsilon}$$

and therefore  $1 - \varepsilon \leq \frac{\mu_x}{\nu_x} \leq 1 + \varepsilon$ . If  $\mu_x \geq \nu_x$  then

$$(\mu_x - \nu_x) \cdot \frac{\mu_x}{\nu_x} \leq (1 + \varepsilon)(\mu_x - \nu_x) = \mu_x - \nu_x + \varepsilon|\mu_x - \nu_x|,$$

and if  $\mu_x < \nu_x$  then

$$(\mu_x - \nu_x) \cdot \frac{\mu_x}{\nu_x} \leq (1 - \varepsilon)(\mu_x - \nu_x) = \mu_x - \nu_x + \varepsilon|\mu_x - \nu_x|. \quad (3.22)$$

Together with (3.21) summing over  $x$  yields

$$D_{\text{KL}}(\mu, \nu) \leq \sum_{x \in X} (\mu_x - \nu_x) + \varepsilon \sum_{x \in X} |\mu_x - \nu_x| \leq \sum_{x \in X} (\mu_x - \nu_x) + 2\varepsilon \|\mu - \nu\|_{\text{TV}}. \quad (3.23)$$

It remains to estimate the first part. Setting  $X^c := \mathbb{X} \setminus X$ , we have

$$\sum_{x \in X} (\mu_x - \nu_x) = \sum_{x \in \mathbb{X}} (\mu_x - \nu_x) - \sum_{x \in X^c} (\mu_x - \nu_x) = - \sum_{x \in X^c} (\mu_x - \nu_x) = \sum_{x \in X^c} |\mu_x - \nu_x|$$

since  $\mu_x = 0$  for  $x \in X^c$ . Now we can estimate

$$2 \sum_{x \in X} (\mu_x - \nu_x) = \sum_{x \in X} (\mu_x - \nu_x) + \sum_{x \in X^c} |\mu_x - \nu_x| \leq \|\mu - \nu\|_1 = 2\|\mu - \nu\|_{\text{TV}}. \quad (3.24)$$

Combining (3.23) and (3.24) yields

$$D_{\text{KL}}(\mu, \nu) \leq (1 + 2\varepsilon) \|\mu - \nu\|_{\text{TV}} = c \cdot \|\mu - \nu\|_{\text{TV}}.$$

□

**Corollary 3.10** (Local KL-TV estimate). *Consider a finite set  $\mathbb{X}$  and a probability distribution  $\mu \in \Delta_{\mathbb{X}}$ . Then for all  $\nu \in \Delta_{\mathbb{X}}$  satisfying*

$$\|\mu - \nu\|_{\infty} < \min\{\mu_x : \mu_x > 0\} \quad (3.25)$$

*it holds that*

$$D_{\text{KL}}(\mu, \nu) \leq \frac{\min\{\mu_x : \mu_x > 0\} + \|\mu - \nu\|_{\infty}}{\min\{\mu_x : \mu_x > 0\} - \|\mu - \nu\|_{\infty}} \cdot \|\mu - \nu\|_{\text{TV}}. \quad (3.26)$$

*Proof.* This is a direct consequence of Lemma 3.9. Indeed, for  $\varepsilon > 0$  small enough we have

$$\|\mu - \nu\|_{\infty} \leq \frac{2 - \varepsilon}{2 + \varepsilon} \cdot \min\{\mu_x : \mu_x > 0\}$$

and thus by Lemma 3.9 with  $c = 1 + \varepsilon$  we have

$$D_{\text{KL}}(\mu, \nu) \leq (1 + \varepsilon) \|\mu - \nu\|_{\text{TV}}.$$

As this holds for all  $\varepsilon > 0$  small enough, the claim follows. □

Before we prove the convergence of Fisher-Rao gradient flows we restate the result.

**Theorem 3.2** (Linear convergence of Fisher-Rao GFs of LPs). *Consider Setting 3.1 and assume that the linear program (3.1) admits a unique solution  $\mu^* \in P$ . Let*

$$\Delta := \min \left\{ \frac{c^\top \mu^* - c^\top \mu}{\|\mu^* - \mu\|_{\text{TV}}} : \mu \in N(\mu^*) \right\}, \quad (3.2)$$

*where  $N(\mu^*)$  denotes the set of neighboring vertices of  $\mu^*$  and set*

$$t_0 := \frac{2D_{\text{KL}}(\mu^*, \mu_0)}{\Delta \cdot \min\{\mu_x^* : \mu_x^* > 0\}}. \quad (3.3)$$

*Then for any  $t \geq t_0$  we have*

$$D_{\text{KL}}(\mu^*, \mu_t) \leq D_{\text{KL}}(\mu^*, \mu_0) \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.4)$$

as well as

$$c^\top \mu^\star - c^\top \mu_t \leq \Delta D_{\text{KL}}(\mu^\star, \mu_0) \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right). \quad (3.5)$$

*Proof.* Setting  $\delta := \min\{\mu_x^\star : \mu_x^\star > 0\}$  and using Lemma 3.8 with  $\|\cdot\|_{\text{TV}}$  and Corollary 3.10 we have

$$c^\top \mu^\star - c^\top \mu_t \geq \Delta \|\mu^\star - \mu_t\|_{\text{TV}} \geq \Delta \cdot \frac{\delta - \|\mu^\star - \mu_t\|_\infty}{\delta + \|\mu^\star - \mu_t\|_\infty} \cdot D_{\text{KL}}(\mu^\star, \mu_t),$$

if  $\|\mu^\star - \mu_t\|_\infty < \delta$ . By Corollary 2.3 we have

$$\|\mu^\star - \mu_t\|_\infty \leq 2\|\mu^\star - \mu_t\|_{\text{TV}} \leq \frac{2c^\top(\mu^\star - \mu_t)}{\Delta} \leq \frac{2D_{\text{KL}}(\mu^\star, \mu_0)}{\Delta \cdot t}.$$

Hence, for  $t > t_0$  we have  $\|\mu^\star - \mu_t\|_\infty < \delta$ . In this case, we can estimate

$$\frac{\delta - \|\mu^\star - \mu_t\|_\infty}{\delta + \|\mu^\star - \mu_t\|_\infty} \leq \frac{\delta - 2D_{\text{KL}}(\mu^\star, \mu_0)\Delta^{-1}t^{-1}}{\delta + 2D_{\text{KL}}(\mu^\star, \mu_0)\Delta^{-1}t^{-1}} = \frac{t - t_0}{t + t_0} =: \kappa_t.$$

Thus for  $t > t_0$  we have  $c^\top \mu^\star - c^\top \mu_t \geq \Delta \kappa_t D_{\text{KL}}(\mu^\star, \mu_t)$ , and Lemma 3.6 together with

$$\int_{t_0}^t \frac{s - t_0}{s + t_0} ds = s - 2t_0 \log(s + t_0) \Big|_{s=t_0}^{s=t} = (t - t_0) - 2t_0 \log \left( \frac{t + t_0}{2t_0} \right)$$

yield the result.  $\square$

**3.2. Estimating the regularization error.** Using the central path property we can deduce an estimate on the regularization error from the convergence results for the Fisher-Rao gradient flow. If the uniform distribution is contained in  $P$ ,  $\mu_{\text{Unif}} \in P$ , then the claim follows simply by setting  $\mu_0 := \mu_{\text{Unif}}$  as

$$D_{\text{KL}}(\mu, \mu_{\text{Unif}}) = -H(\mu) + \log|\mathbb{X}|. \quad (3.27)$$

If the uniform distribution is not contained in  $P$ , we can choose its information projection as an initial distribution  $\mu_0$  to the same effect. Indeed, recall that for

$$\mu_0 = \arg \min_{\mu \in P} D_{\text{KL}}(\mu, \mu_{\text{Unif}}) = \arg \max_{\mu \in P} H(\mu) \quad (3.28)$$

we have by the Pythagorean theorem that

$$D_{\text{KL}}(\mu, \mu_{\text{Unif}}) = D_{\text{KL}}(\mu, \mu_0) + D_{\text{KL}}(\mu_0, \mu_{\text{Unif}}) \quad (3.29)$$

for all  $\mu \in P$ , see [8, Theorem 2.8]. Now we can estimate the regularization error.

**Corollary 3.3** (Entropic regularization error). *Consider Setting 3.1 and assume that the linear program (3.1) admits a unique solution  $\mu^\star \in P$ . For  $t > 0$  denote by  $\mu_t^\star$  the unique solution of the entropy-regularized linear program*

$$\max c^\top \mu + t^{-1} H(\mu) \quad \text{subject to } \mu \in P, \quad (3.6)$$

where  $H$  denotes the Shannon entropy. Then for any  $t \geq t_0$  we have

$$D_{\text{KL}}(\mu^\star, \mu_t^\star) \leq R_H \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.7)$$

as well as

$$c^\top \mu^\star - c^\top \mu_t^\star \leq \Delta R_H \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right), \quad (3.8)$$

where  $\Delta > 0$  and  $t_0 \geq 0$  are defined in (3.2) and (3.3), respectively, and

$$R_H := \max_{\mu \in P} H(\mu) - \min_{\mu \in P} H(\mu) \leq \log|\mathbb{X}| \quad (3.9)$$

denotes the entropic radius of  $P$ .

*Proof.* Recall that by the central path property the Fisher-Rao gradient flow  $(\mu_t)_{t \geq 0}$  satisfies

$$\mu_t = \arg \max \left\{ c^\top \mu - t^{-1} D_{\text{KL}}(\mu, \mu_0) : \mu \in P \right\}.$$

If we choose  $\mu_0$  as the information projection according to (3.28) we obtain by the Pythagorean theorem that

$$D_{\text{KL}}(\mu, \mu_0) = D_{\text{KL}}(\mu, \mu_{\text{Unif}}) + H(\mu_0) - \log|\mathbb{X}| = H(\mu_0) - H(\mu).$$

This shows that

$$\mu_t = \arg \max \left\{ c^\top \mu + t^{-1} H(\mu) : \mu \in P \right\},$$

i.e., that  $\mu_t$  is the solution of the entropy regularized linear program (3.6). Now the claim follows from Theorem 3.2 and  $D_{\text{KL}}(\mu^*, \mu_0) = H(\mu_0) - H(\mu^*) \leq R_H$ .  $\square$

**3.3. Non-unique maximizers.** Both Theorem 3.2 and Corollary 3.3 are formulated under the assumption that the linear program (3.1) admits a unique solution. This is satisfied for almost all cost vectors  $c \in \mathbb{R}^{\mathbb{X}}$ , however, it can be generalized to all costs. To this end, we use the following generalization of Lemma 3.8 where we defer the proof to the appendix as it follows similar arguments.

**Lemma 3.11.** *Consider a polytope  $P \subseteq \mathbb{R}^{\mathbb{X}}$  and denote by  $F^*$  the face of maximizers of the linear function  $\mu \mapsto c^\top \mu$  over  $P$ . Denote the set of neighboring vertices of a vertex  $\mu$  by  $N(\mu)$  and let  $\|\cdot\|: \mathbb{R}^{\mathbb{X}} \rightarrow \mathbb{R}_{\geq 0}$  be an arbitrary semi-norm. Then either  $F^* = P$  or with  $\frac{c}{0} := +\infty$  for  $c > 0$ , we have*

$$\Delta := \min \left\{ \frac{c^\top \mu^* - c^\top \mu}{\|\mu^* - \mu\|} : \mu^* \in \text{vert}(F^*), \mu \in N(\mu^*) \setminus F^* \right\} > 0, \quad (3.30)$$

and further

$$c^\top \mu^* - c^\top \mu \geq \Delta \cdot \inf_{\mu^* \in F^*} \|\mu^* - \mu\| \quad \text{for all } \mu \in P. \quad (3.31)$$

To proceed like in the proof with a unique maximizer, we need to identify the limit of  $\mu_t$  in  $F^*$ . For linear objectives function the limit  $\mu^*$  is the information projection of  $\mu_0$  to  $F^*$ , see [5, Corollary 4.8]. We include a proof here for the sake of completeness.

**Corollary 3.12** (Implicit bias of Fisher-Rao GF). *Consider Setting 3.1 and denote the face of maximizers of the linear program (3.1) by  $F^*$ . Then it holds that*

$$\lim_{t \rightarrow +\infty} \mu_t = \mu^* = \arg \min_{\mu \in F^*} D_{\text{KL}}(\mu, \mu_0). \quad (3.32)$$

*In words, the Fisher-Rao gradient flow converges to the information projection of  $\mu_0$  to  $F^*$ , i.e., it selects the optimizer which has the minimum KL-divergence from  $\mu_0$ .*

*Proof.* By compactness of  $P$ , the sequence  $(\mu_{t_n})_{n \in \mathbb{N}}$  has at least one accumulation point for any  $t_n \rightarrow +\infty$ . Hence, we can assume without loss of generality that  $\mu_{t_n} \rightarrow \hat{\mu}$  and it remains to identify  $\hat{\mu}$  as the information projection  $\mu^* \in F^*$ .

Surely, we have  $\hat{\mu} \in F^*$  as  $c^\top \hat{\mu} = \lim_{n \rightarrow \infty} c^\top \mu_{t_n} = \max_{\mu \in P} c^\top \mu$  by Corollary 2.3. Further, by the central path property we have for any optimizer  $\mu' \in F^*$  that

$$c^\top \mu_t - t^{-1} D_{\text{KL}}(\mu_t, \mu_0) \geq c^\top \hat{\mu} - t^{-1} D_{\text{KL}}(\mu', \mu_0)$$

and therefore

$$D_{\text{KL}}(\mu', \mu_0) - D_{\text{KL}}(\mu_t, \mu_0) \geq t c^\top (\mu' - \mu_t) \geq 0.$$

Hence, we have

$$D_{\text{KL}}(\hat{\mu}, \mu_0) = \lim_{n \rightarrow \infty} D_{\text{KL}}(\mu_{t_n}, \mu_0) \leq D_{\text{KL}}(\mu', \mu_0)$$

and can conclude by minimizing over  $\mu' \in F^*$ .  $\square$

**Theorem 3.13.** *Consider Setting 3.1, assume that the linear program is non trivial, i.e., that  $F^\star \neq P$ , where  $F^\star$  denotes the face of optimizers, and denote the information projection of  $\mu_0$  to  $F^\star$  by  $\mu^\star \in F^\star$  and set*

$$\Delta := \min \left\{ \frac{c^\top \mu^\star - c^\top \mu}{\|\mu^\star - \mu\|_{\text{TV}}} : \mu^\star \in \text{vert}(F^\star), \mu \in N(\mu^\star) \setminus F^\star \right\} > 0. \quad (3.33)$$

*Then for any  $\kappa \in (0, \Delta)$  there is  $t_\kappa \in \mathbb{R}_{\geq 0}$  such that for any  $t \geq t_\kappa$  we have*

$$D_{\text{KL}}(\mu^\star, \mu_t) \leq D_{\text{KL}}(\mu^\star, \mu_0) e^{-\kappa(t-t_\kappa)} \quad (3.34)$$

*and*

$$c^\top \mu^\star - c^\top \mu_t \leq \Delta D_{\text{KL}}(\mu^\star, \mu_0) e^{-\kappa(t-t_\kappa)}. \quad (3.35)$$

*Proof.* Corollary 3.12 shows that  $\mu_t \rightarrow \mu^\star$ . Let  $\mu_t^\star \in F^\star$  denote the  $\|\cdot\|_{\text{TV}}$ -projection of  $\mu_t$  onto  $F^\star$ , i.e., be such that

$$\|\mu_t - \mu_t^\star\|_{\text{TV}} = \min_{\mu' \in F^\star} \|\mu' - \mu_t\|_{\text{TV}} \rightarrow 0 \quad \text{for } t \rightarrow +\infty$$

as  $\mu_t \rightarrow \mu^\star \in F^\star$ . Now we have

$$\|\mu_t^\star - \mu^\star\|_{\text{TV}} \leq \|\mu_t^\star - \mu_t\|_{\text{TV}} + \|\mu_t - \mu^\star\|_{\text{TV}} \rightarrow 0 \quad \text{for } t \rightarrow +\infty$$

and hence  $\mu_t^\star \rightarrow \mu^\star$ . Note that  $\mu^\star \in \text{int}(F^\star)$ , i.e., has maximal support in  $F^\star$  and hence  $\mu_t^\star \ll \mu^\star$ , see Lemma A.2. Together with  $\mu_t^\star \rightarrow \mu^\star$  this yields

$$\delta_t := \min\{\mu_t^\star(x) : \mu_t^\star(x) > 0\} \rightarrow \min\{\mu^\star(x) : \mu^\star(x) > 0\} > 0.$$

Combining Corollary 3.10 and Lemma 3.11 yields

$$D_{\text{KL}}(\mu_t^\star, \mu_t) \leq \frac{\delta_t + \|\mu_t^\star - \mu_t\|_{\text{TV}}}{\delta_t - \|\mu_t^\star - \mu_t\|_{\text{TV}}} \cdot \Delta^{-1} (c^\top \mu^\star - c^\top \mu_t),$$

where the right hand side converges to  $\Delta^{-1} (c^\top \mu^\star - c^\top \mu)$  for  $t \rightarrow +\infty$ . Hence, for  $\kappa < \Delta$  and  $t$  large enough, we have

$$\kappa D_{\text{KL}}(\mu^\star, \mu_t) \leq \kappa D_{\text{KL}}(\mu_t^\star, \mu_t) \leq c^\top \mu^\star - c^\top \mu_t,$$

where we used that  $\mu^\star$  is the information projection of  $\mu_t$  to  $F^\star$  and  $\mu_t^\star \in F^\star$ , therefore establishing (3.12). Now we can conclude utilizing Lemma 3.6.  $\square$

A bound on the time  $t_\kappa$  could be obtained through a refinement of Lemma 3.8 showing  $c^\top \mu^\star - c^\top \mu \geq \Delta \cdot \|\mu^\star - \mu\|_{\text{TV}}$  for the information projection  $\mu^\star$  of  $\mu \in P$  to  $F^\star$ . Another approach to control  $t_\kappa$  is to quantify the convergence of  $\mu_t^\star \rightarrow \mu^\star$ .

**Remark 3.14** (Estimating the regularization error). *Just like before, we can estimate the regularization error with the same argument as in Corollary 3.3. In this case, the guarantee (3.34) holds with the entropic radius  $R_H$  instead of  $D_{\text{KL}}(\mu^\star, \mu_0)$ .*

#### 4. CONVERGENCE OF NATURAL GRADIENT FLOWS

In practice, it is often not feasible to perform optimization in the space of measures, and therefore one often resorts to parametric models. Natural gradients were introduced by S. Amari [6] and are designed to mimic the Fisher-Rao gradient flow by preconditioning the Euclidean gradient in parameter space with the Fisher information matrix. To study natural gradient methods, we work in the following setting.

**Setting 4.1.** *We consider a finite set  $\mathbb{X}$  and a polytope  $P = \Delta_{\mathbb{X}} \cap \mathcal{L}$  with  $P \cap \mathbb{R}_{>0}^{\mathbb{X}} \neq \emptyset$ , where  $\mathcal{L} \subseteq \mathbb{R}^{\mathbb{X}}$  is an affine space. Further, we consider a differentiable parametrization  $\mathbb{R}^p \rightarrow \text{int}(P); \theta \mapsto \mu_\theta$  and a (possibly nonlinear) differentiable objective function  $f: \mathbb{R}_{>0}^{\mathbb{X}} \rightarrow \mathbb{R}$ , and write  $f(\theta) = f(\mu_\theta)$ .*

We work in continuous time and consider the following evolution of parameters.

**Definition 4.2** (Natural gradient flow). *Consider Setting 4.1. We call*

$$\partial_t \theta_t = F(\theta_t)^+ \nabla f(\theta_t) \quad (4.1)$$

*the natural gradient flow, where  $F(\theta)^+$  denotes the pseudo-inverse of the Fisher information matrix with entries*

$$F(\theta)_{ij} = \sum_{x \in \mathbb{X}} \frac{\partial_i \mu_\theta(x) \partial_j \mu_\theta(x)}{\mu_\theta(x)} = g_{\mu_\theta}^{\text{FR}}(\partial_i \mu_\theta, \partial_j \mu_\theta). \quad (4.2)$$

**4.1. Compatible function approximation.** In this subsection, and more precisely in Proposition 4.4, we describe the natural gradient direction as the minimizer of a linear least squares regression problem with features  $\phi_\theta(x) = \nabla_\theta \log \mu_\theta(x)$ , which is known as *compatible function approximation* in reinforcement learning. This can be used to approximate the natural gradient using samples from the distribution  $\mu_\theta$ .

The measure  $\mu_t = \mu_{\theta_t}$  does not necessarily evolve according to the Fisher-Rao gradient flow on the polytope  $P$  (2.7) even if  $\theta_t$  satisfies the natural gradient flow in the parameter space (4.1). In the next lemma we describe the discrepancy between  $\partial_t \mu_t = \partial_t \theta_t^\top \nabla_\theta \mu_{\theta_t}$  and the Fisher-Rao gradient  $\nabla_P^{\text{FR}} f(\mu_t)$ .

**Lemma 4.3.** *Consider Setting 4.1 and a parameter evolution  $\partial_t \theta_t = v_t$  and write  $\mu_t = \mu_{\theta_t}$ . Then we have*

$$\|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 = L(v_t, \theta_t) - C(\theta_t), \quad (4.3)$$

where

$$L(w, \theta) := \mathbb{E}_{\mu_\theta} \left[ \left( w^\top \nabla_\theta \log \mu_\theta(x) - \nabla f(\mu_\theta)(x) \right)^2 \right] \quad (4.4)$$

is an  $l^2$ -regression error and  $C(\theta_t) := \inf_{\nu \in TP} \|\nabla_P^{\text{FR}} f(\mu_t) - \nu\|_{g_{\mu_{\theta_t}}^{\text{FR}}}^2$  a projection error.

*Proof.* The Fisher-Rao gradient  $\nabla_P^{\text{FR}} f(\mu_t)$  of  $f: P \rightarrow \mathbb{R}$  is the Fisher-Rao projection of the Fisher-Rao gradient  $\nabla^{\text{FR}} f(\mu_t)$  of  $f: \mathbb{R}_{>0}^{\mathbb{X}} \rightarrow \mathbb{R}$  onto  $TP$ . Hence, by the Pythagorean theorem, we have

$$\|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 = \|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 + \|\nabla_P^{\text{FR}} f(\mu_t) - \nabla^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2.$$

Since  $\nabla_P^{\text{FR}} f(\mu_t)$  is the projection of  $\nabla^{\text{FR}} f(\mu_t)$  to  $TP$ , we obtain

$$\|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 = \|\partial_t \mu_t - \nabla^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 - C(\theta_t).$$

Further, by the chain rule, we have

$$\partial_t \mu_t(x) = \partial_t \theta_t^\top \nabla_\theta \mu_{\theta_t}(x) = v_t^\top \nabla_\theta \mu_{\theta_t}(x).$$

Using  $\nabla^{\text{FR}} f(\mu) = \nabla f(\mu) \odot \mu$  we conclude

$$\begin{aligned} \|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}}^2 &= \left\| v_t^\top \nabla_\theta \mu_\theta - \nabla f(\mu_\theta) \odot \mu_\theta \right\|_{g_{\mu_\theta}^{\text{FR}}}^2 \\ &= \mathbb{E}_{\mu_\theta} \left[ \frac{\left( v_t^\top \nabla_\theta \mu_\theta(x) - \nabla f(\mu_\theta)(x) \mu_\theta(x) \right)^2}{\mu_\theta(x)^2} \right] \\ &= \mathbb{E}_{\mu_\theta} \left[ \left( v_t^\top \nabla_\theta \log \mu_\theta(x) - \nabla f(\mu_\theta)(x) \right)^2 \right] \\ &= L(v_t, \theta). \end{aligned}$$

□



**Proposition 4.4** (Compatible function approximation). *Consider Setting 4.1, let  $F(\theta)$  denote the Fisher-information matrix, and let  $L$  be defined as in (4.4). Then  $v \in \mathbb{R}^p$  is a natural gradient at  $\theta \in \mathbb{R}^p$ , i.e., satisfies  $F(\theta)v = \nabla_\theta f(\theta)$ , if and only if*

$$v \in \arg \min_{w \in \mathbb{R}^p} L(w, \theta). \quad (4.5)$$

*Proof.* The objective function  $L(w, \theta)$  is given, up to a constant, by

$$\left\| w^\top \nabla_\theta \mu_\theta \right\|_{g_{\mu_\theta}^{\text{FR}}}^2 - 2g_{\mu_\theta}^{\text{FR}}(w^\top \nabla_\theta \mu_\theta, \nabla^{\text{FR}} f(\mu_\theta)) = w^\top F(\theta)w - 2\nabla f(\theta)^\top w.$$

The global minimizers are characterized by the normal equation  $F(\theta)w = \nabla f(\theta)$ .  $\square$

In the context of compatible function approximation

$$\varepsilon_t^2 = \min_{w \in \mathbb{R}^p} L(w, \theta_t) = \min_{w \in \mathbb{R}^p} \mathbb{E}_{\mu_t} \left[ \left( w^\top \nabla_\theta \log \mu_{\theta_t}(x) - \nabla f(\mu)(x) \right)^2 \right] \quad (4.6)$$

is often called the *approximation error*. Examining the objective  $L(w, \theta)$  and using Lemma 4.3 we see that the natural gradient flow minimizes the discrepancy between  $\partial_t \mu_t$  and the Fisher-Rao gradient  $\nabla_P^{\text{FR}} f(\mu_t)$ . In this case, the evolution  $\partial_t \mu_t$  is given by the orthogonal projection of the Fisher-Rao gradient onto the tangent space of the parametrized model. A similar property holds for any natural gradient defined using a Riemannian metric on the polytope [7, 52, 40].

**Corollary 4.5** (Projection property). *Consider a solution  $(\theta_t)_{t \in [0, T]}$  of the natural gradient flow (4.1). We denote the projection with respect to the Fisher-Rao metric onto the generalized tangent space*

$$T_\theta P := \text{span}\{\partial_{\theta_i} \mu_\theta : i = 1, \dots, p\} = \{w^\top \nabla_\theta \mu_\theta : w \in \mathbb{R}^p\} \subseteq TP$$

*by  $P_\theta^{\text{FR}}$ . Then it holds that*

$$\partial_t \mu_t = P_{\theta_t}^{\text{FR}}(\nabla_P^{\text{FR}} f(\mu_t)). \quad (4.7)$$

*In particular, if  $T_{\theta_t} P = TP$  then  $\partial_t \mu_t = \nabla_P^{\text{FR}} f(\mu_t)$ .*

*Proof.* By Proposition 4.4 the natural gradient direction  $v_t$  is a minimizer of  $L(\cdot, \theta_t)$ . By Lemma 4.3 this yields

$$\begin{aligned} \|\partial_t \mu_t - \nabla_P^{\text{FR}} f(\mu_t)\|_{g_{\mu_t}^{\text{FR}}} &= \min_{w \in \mathbb{R}^p} \left\| w^\top \nabla_\theta d_\theta - \nabla_P^{\text{FR}} f(\mu_t) \right\|_{g_{\mu_t}^{\text{FR}}} \\ &= \min_{\nu \in T_\theta P} \left\| \nu - \nabla_P^{\text{FR}} f(\mu_t) \right\|_{g_{\mu_t}^{\text{FR}}}. \end{aligned}$$

In particular, this shows that  $\partial_t \mu_t$  is the projection of  $\nabla_P^{\text{FR}} f(\mu_t)$  onto  $T_\theta P$ .  $\square$

**4.2. Convergence of natural gradient flows.** We start with a generalization of Corollary 2.3 to cover cases where the evolution of  $\mu_t$  only approximately follows the Fisher-Rao gradient flow.

**Proposition 4.6** (A perturbed convergence result). *Consider Setting 3.1, a differentiable curve  $\mu: [0, \infty) \rightarrow \text{int}(P)$  and a differentiable convex objective  $f: \mathbb{R}_{>0}^X$ . Assume that  $f$  admits a maximizer  $\mu^*$  over  $P$  with value  $f^*$ . It holds that*

$$f^* - f(\mu_t) \leq \frac{D_{\text{KL}}(\mu^*, \mu_0) - D_{\text{KL}}(\mu^*, \mu_t)}{t} + t^{-1} \int_0^t \varepsilon_s \delta_s ds, \quad (4.8)$$

where  $\delta_t^2 := \chi^2(\mu^*, \mu_t)$  and  $\varepsilon_t^2 := \|\nabla_P^{\text{FR}} f(\mu_t) - \partial_t \mu_t\|_{g_{\mu_t}^{\text{FR}}}^2$ .

*Proof.* We compute

$$\begin{aligned}
\partial_t D_{\text{KL}}(\mu^\star, \mu_t) &= -\partial_t \phi(\mu_t) - \partial_t \langle \nabla \phi(\mu_t), \mu^\star - \mu_t \rangle \\
&= \langle \nabla^2 \phi(\mu_t) \partial_t \mu_t, \mu_t - \mu^\star \rangle \\
&= g_{\mu_t}^{\text{FR}}(\partial_t \mu_t, \mu_t - \mu^\star) \\
&= g_{\mu_t}^{\text{FR}}(\nabla_P^{\text{FR}} f(\mu_t), \mu_t - \mu^\star) + g_{\mu_t}^{\text{FR}}(\nabla_P^{\text{FR}} f(\mu_t) - \partial_t \mu_t, \mu_t - \mu^\star) \\
&= \nabla f(\mu_t)^\top (\mu_t - \mu^\star) + g_{\mu_t}^{\text{FR}}(\nabla_P^{\text{FR}} f(\mu_t) - \partial_t \mu_t, \mu_t - \mu^\star) \\
&\leq \nabla f(\mu_t)^\top (\mu_t - \mu^\star) + \varepsilon_t \delta_t \\
&\leq f(\mu_t) - f(\mu^\star) + \varepsilon_t \delta_t,
\end{aligned}$$

where we used Lemma 4.3 and Proposition 4.4 as well as  $\|\mu_t - \mu^\star\|_{g_{\mu_t}^{\text{FR}}}^2 = \chi^2(\mu^\star, \mu_t)$ . Integration and rearranging now yields (4.8).  $\square$

If  $(\mu_t)_{t \geq 0}$  solves the Fisher-Rao gradient flow, we have  $\varepsilon_t = 0$  and recover Corollary 2.3. For natural gradient flows, we obtain the following result.

**Corollary 4.7.** *Consider Setting 4.1 and a solution  $(\theta_t)_{t \geq 0}$  of the natural gradient flow (4.1) for a convex objective  $f$  and set  $\mu_t := \mu_{\theta_t}$ . Then (4.8) holds with*

$$\varepsilon_t^2 \leq \min_{w \in \mathbb{R}^p} \mathbb{E}_{\mu_\theta} \left[ \left( w^\top \nabla_\theta \log \mu_{\theta_t}(x) - \nabla f(\mu_t)(x) \right)^2 \right]. \quad (4.9)$$

*Proof.* Combine Proposition 4.6 with Lemma 4.3 and Proposition 4.4.  $\square$

**Remark 4.8** (Baseline). *In reinforcement learning, baselines are often used when estimating the natural policy gradient from samples to reduce the variance of the estimates [54]. This amounts to projecting the gradient of the objective to the tangent space of the model. In our setting, this corresponds to projecting  $\nabla f(\mu) \odot \mu$  to the tangent space  $TP$  with respect to the Fisher-Rao metric  $g_\mu^{\text{FR}}$ , see also [51, Subsection 4.1.1]. In the special case  $P = \Delta_{\mathbb{X}}$  the Fisher-Rao projection of  $\nabla f(\mu) \odot \mu$  is given by  $\nabla f(\mu) \odot \mu - \kappa \mu$ , where  $\kappa = \sum_x \nabla f(\mu)(x)$  and the corresponding compatible function approximation objective is given by*

$$\tilde{L}(w, \theta) := \mathbb{E}_{\mu_\theta} \left[ \left( w^\top \nabla_\theta \log \mu_\theta(x) - (\nabla f(\mu)(x) - \kappa) \right)^2 \right].$$

**4.3. Global convergence for multi-player games.** For a rich enough para-metrization Corollary 4.5 ensures that  $(\mu_{\theta_t})_{t \geq 0}$  follows the Fisher-Rao gradient flow in which case Theorem 3.2 implies the linear convergence of the natural gradient flow (4.1). A common example is the softmax parametrization  $\mu_\theta(x) \propto e^{\theta(x)}$ . For multi-player games with suitable payoff structure, the dynamics of the individual players decouple [13], which allows us to provide a global convergence result for parametric models with exponentially fewer parameters than the softmax parametrization.

**Theorem 4.9.** *Consider a differentiable parametrization of conditional probabilities  $\{m_\theta : \theta \in \mathbb{R}^p\} = \text{int}(\Delta_{\mathbb{X}}^n)$ , where  $n \in \mathbb{N}$  and  $\mathbb{X}$  is a finite set, and suppose that*

$$\text{span} \{ \partial_{\theta_i} m_\theta : i = 1, \dots, p \} = T \Delta_{\mathbb{X}}^n \quad \text{for all } \theta \in \mathbb{R}^p. \quad (4.10)$$

*Define a corresponding parametric independence model as*

$$\mu_\theta(x) := \prod_{i=1}^n m_\theta(x_i | i) \quad \text{for all } x \in \mathbb{X}^n. \quad (4.11)$$

*Further, consider*

$$c \in \text{span} \left\{ \mathbf{1}_{\mathbb{X}} \otimes \dots \otimes \delta_x \otimes \dots \otimes \mathbf{1}_{\mathbb{X}} : x \in \mathbb{X}, i = 1, \dots, n \right\} \subseteq \mathbb{R}^{\mathbb{X}^n} \quad (4.12)$$

and the linear payoff  $f(\mu) = c^\top \mu$  and the natural gradient flow (4.1). Then  $(\mu_{\theta_t})_{t \geq 0}$  solves the Fisher-Rao gradient flow in  $\Delta_{\mathbb{X}^n}$  and hence, we have

$$\mu_t(x) = \frac{e^{tc(x)}}{\sum_{x'} e^{tc(x')}} \quad \text{for all } x \in \mathbb{X}^n. \quad (4.13)$$

*Proof.* The Segre embedding  $\Delta_{\mathbb{X}}^n \rightarrow \Delta_{\mathbb{X}^n}$ ,  $(\mu_i)_{i=1,\dots,n} \mapsto \otimes_{i=1}^n \mu_i$  is an isometry with respect to the product Fisher-Rao metric, i.e., the sum of the Fisher metrics over the individual factors, and the Fisher-Rao metric [34, 13]. In particular, this implies that  $(\mu_{\theta_t})_{t \geq 0}$  solves the Fisher-Rao gradient flow with respect to  $f$  restricted the independence model

$$\mathcal{I} := \left\{ \bigotimes_{i=1}^n \mu_i : \mu_i \in \Delta_{\mathbb{X}} \text{ for } i = 1, \dots, n \right\} \subseteq \Delta_{\mathbb{X}^n},$$

as  $\partial_t \mu_t = P_{T_{\mu_t} \mathcal{I}} \nabla^{\text{FR}} f(\mu_t) = \nabla^{\text{FR}} f|_{\mathcal{I}}(\mu_t)$ , see [52]. Condition (4.12) implies that  $f$  factorizes along the marginalization map and hence the independence model  $\mathcal{I}$  is invariant under the Fisher-Rao gradient flow [13]. Thus,  $(\mu_{\theta_t})_{t \geq 0}$  solves the Fisher-Rao gradient flow with potential  $f$  in  $\Delta_{\mathbb{X}^n}$ , which can be solved explicitly [55].  $\square$

Note that a model parametrizing  $\Delta_{\mathbb{X}}^n$  only requires  $n(|\mathbb{X}| - 1)$  parameters, whereas a model parametrizing the joint distributions  $\Delta_{\mathbb{X}^n}$  requires  $|\mathbb{X}|^n - 1$  parameters. However, we require the cost vector  $c$  to lie in an  $n|\mathbb{X}|$ -dimensional subspace of  $\mathbb{R}^{\mathbb{X}^n}$ .

## 5. CONVERGENCE OF STATE-ACTION NATURAL POLICY GRADIENTS

Having studied general linear programs we now turn to the reward optimization problem in infinite-horizon discounted Markov decision processes. Reward optimization is well known to be equivalent to a linear program and the state-action natural policy gradient flow corresponds to the Fisher-Rao gradient flow inside the state-action polytope [25, 39]. We give a short overview of the required notions and refer to [23] for a thorough introduction to Markov decision processes.

In Markov decision processes (MDPs), we are concerned with controlling the state  $s \in \mathbb{S}$  of some system through an action  $a \in \mathbb{A}$  in order to achieve an optimal behavior over time. The evolution of the system is described by a Markov kernel  $P \in \Delta_{\mathbb{S} \times \mathbb{A}}^{\mathbb{S} \times \mathbb{A}}$ , where  $P(s'|s, a)$  denotes the probability of transitioning from state  $s$  to  $s'$  under action  $a$ . Here, we work with finite state and action spaces  $\mathbb{S}$  and  $\mathbb{A}$ . A (stochastic) policy is a Markov kernel  $\pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}$ , where  $\pi(a|s)$  denotes the probability of selecting action  $a$  when in state  $s$ . For a fixed policy  $\pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}$  and an initial distribution  $\mu \in \Delta_{\mathbb{S}}$  we obtain a Markov process over  $\mathbb{S} \times \mathbb{A}$  according to  $S_0 \sim \mu$  and

$$A_t \sim \pi(\cdot|S_t), \quad S_{t+1} \sim P(\cdot|S_t, A_t) \quad \text{for } t \in \mathbb{N}, \quad (5.1)$$

and we denote its law by  $\mathbb{P}^{\pi, \mu}$ . We consider a *instantaneous reward vector*  $r \in \mathbb{R}^{\mathbb{S} \times \mathbb{A}}$  indicating how favorable a certain state and action combination is. As a criterion for the performance of a policy  $\pi$  we consider the *infinite horizon discounted reward*

$$R(\pi) := (1 - \gamma) \mathbb{E}_{\mathbb{P}^{\pi, \mu}} \left[ \sum_{t \in \mathbb{N}} \gamma^t r(S_t, A_t) \right], \quad (5.2)$$

where the *discount factor*  $\gamma \in [0, 1)$  is fixed and ensures convergence. The reward optimization problem is given by

$$\max R(\pi) \quad \text{subject to } \pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}. \quad (5.3)$$

An important role in Markov decision processes play the *state-action distributions*  $d^\pi \in \Delta_{\mathbb{S} \times \mathbb{A}}$ , which are given by

$$d^\pi(s, a) := (1 - \gamma) \sum_{t \in \mathbb{N}} \gamma^t \mathbb{P}^{\pi, \mu}(S_t = s, A_t = a). \quad (5.4)$$

They determine the reward as

$$R(\pi) = \sum_{s \in \mathbb{S}, a \in \mathbb{A}} r(s, a) d^\pi(s, a) = r^\top d^\pi. \quad (5.5)$$

The set of state-action distributions has been characterized as a polytope in the seminal works of Cyrus Derman, see [19].

**Proposition 5.1** (State-action polytope). *The set  $\mathcal{D} = \{d^\pi : \pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}\} \subseteq \Delta_{\mathbb{S} \times \mathbb{A}}$  of state-action distributions is a polytope given by*

$$\mathcal{D} = \Delta_{\mathbb{S} \times \mathbb{A}} \cap \left\{ d \in \mathbb{R}^{\mathbb{S} \times \mathbb{A}} : \ell_s(d) = 0 \text{ for all } s \in \mathbb{S} \right\}, \quad (5.6)$$

where the defining linear equations are given by

$$\ell_s(d) = \sum_{a \in \mathbb{A}} d(s, a) - \gamma \sum_{s' \in \mathbb{S}, a' \in \mathbb{A}} P(s|s', a') d(s', a') - (1 - \gamma) \mu(s). \quad (5.7)$$

We refer to  $\mathcal{D}$  as the *state-action polytope*. This leads to the linear programming formulation of Markov decision processes [25], given by<sup>1</sup>

$$\max r^\top d \quad \text{subject to } d \in \mathcal{D}. \quad (5.8)$$

The state-action polytope  $\mathcal{D} = \Delta_{\mathbb{S} \times \mathbb{A}} \cap \mathcal{L}$  falls under the class of polytopes studied in Section 3. Given a state-action distribution  $d \in \mathcal{D}$ , we can compute a corresponding policy  $\pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}$  with  $d = d^\pi$  by conditioning,

$$\pi(a|s) = \frac{d(s, a)}{\sum_{a' \in \mathbb{A}} d(s, a')} \quad \text{for all } a \in \mathbb{A}, s \in \mathbb{S}, \quad (5.9)$$

if this is well-defined, see [38, 29], which leads us to the following assumption.

**Assumption 5.2** (State exploration). *For any policy  $\pi \in \Delta_{\mathbb{A}}^{\mathbb{S}}$  the discounted state distribution is positive, i.e.,  $\sum_{a \in \mathbb{A}} d^\pi(s, a) > 0$  for all  $s \in \mathbb{S}$ .*

This assumption is satisfied if  $\mu(s) > 0$  for all  $s \in \mathbb{S}$  as  $\sum_{a \in \mathbb{A}} d^\pi(s, a) \geq (1 - \gamma) \mu(s)$ . This assumption is standard in linear programming approaches to Markov decision processes; policy gradient methods can fail to converge if it is violated [25, 33].

Policy optimization algorithms parameterize the policy  $\pi_\theta$  and optimize  $\theta$ . As we study gradient-based approaches we work under the following assumption.

**Assumption 5.3** (Differentiable parametrization). *We consider a differentiable policy parametrization  $\mathbb{R}^p \rightarrow \text{int}(\Delta_{\mathbb{A}}^{\mathbb{S}}); \theta \mapsto \pi_\theta$ .*

We consider continuous-time natural policy gradient methods that optimize the parameters  $\theta$  of a parametric policy  $\pi_\theta$  according to

$$\partial_t \theta_t = G(\theta_t)^\top \nabla R(\theta_t), \quad (5.10)$$

where we write  $R(\theta) = R(\pi_\theta)$ . Here  $G(\theta)$  denotes a Gramian matrix with entries  $G(\theta)_{ij} = g_{d_\theta}(\partial_{\theta_i} d_\theta, \partial_{\theta_j} d_\theta)$ , where we write  $d_\theta = d^{\pi_\theta}$  and  $g_d$  denotes a Riemannian metric on the state-action polytope  $\mathcal{D}$ . In this context, the matrix  $G(\theta)$  is referred to as a preconditioner. Various choices have been suggested for  $G(\theta)$ , such as for example

$$G_K(\theta) := \sum_s d_\theta(s) \sum_a \frac{\partial_{\theta_i} \pi_\theta(a|s) \partial_{\theta_j} \pi_\theta(a|s)}{\pi_\theta(a|s)}, \quad (5.11)$$

<sup>1</sup>Sometimes, this is referred to as the dual linear programming formulation of Markov decision processes, where the primal linear program has the optimal value function as its solution.

which is a weighted sum of Fisher-information matrices over the individual states [24, 9, 43] which has been studied extensively in the literature. We focus on the so-called *state-action natural gradient* given by the Fisher information matrix of the state-action distribution [35],

$$G_M(\theta)_{ij} := F(\theta)_{ij} = \sum_{s,a} \frac{\partial_{\theta_i} d_\theta(s,a) \partial_{\theta_j} d_\theta(s,a)}{d_\theta(s,a)} = g_{d_\theta}^{\text{FR}}(\partial_{\theta_i} d_\theta, \partial_{\theta_j} d_\theta). \quad (5.12)$$

This choice was observed to reduce the severity of plateaus and is closely connected to the trust region method known as relative entropy policy search (REPS) [42].

**5.1. Convergence of state-action natural policy gradients.** Now that we have built a convergence theory for general natural gradient flows we elaborate on the consequences for state-action natural policy gradients.

**Corollary 5.4** (Sublinear convergence under function approximation). *Consider a finite discounted Markov decision process, suppose Assumption 5.2 and Assumption 5.3 hold, and consider a solution of the natural policy gradient flow (5.10) for  $G = G_M$ . Then it holds that*

$$R^* - R(\theta_t) \leq \frac{D_{\text{KL}}(d^*, d_0)}{t} + t^{-1} \int_0^t \delta_s \varepsilon_s ds, \quad (5.13)$$

where  $\delta_t^2 := \chi^2(d^*, d_t)$  and  $\varepsilon_t^2 := \min_{w \in \mathbb{R}^p} \|\nabla_{\mathcal{D}}^{\text{FR}} f(\mu_t) - w^\top \nabla_\theta d_{\theta_t}\|_{g_{d_t}^{\text{FR}}}^2$  and

$$\varepsilon_t^2 \leq \min_{w \in \mathbb{R}^p} \mathbb{E}_{d_t} \left[ \left( w^\top \nabla_\theta \log d_{\theta_t}(s,a) - r(s,a) \right)^2 \right]. \quad (5.14)$$

*Proof.* This is Corollary 4.7 for state-action natural policy gradients.  $\square$

**Remark 5.5** (Inexact gradient evaluations). *If the parameters follow the evolution  $\partial_t \theta_t = v_t$ , then we can apply Proposition 4.6 to see that (5.13) remains valid with*

$$\varepsilon_t^2 := \left\| \nabla_{\mathcal{D}}^{\text{FR}} f(\mu_t) - v_t^\top \nabla_\theta d_{\theta_t} \right\|_{g_{d_t}^{\text{FR}}}^2 \leq \mathbb{E}_{d_t} \left[ \left( v_t^\top \nabla_\theta \log d_{\theta_t}(s,a) - r(s,a) \right)^2 \right].$$

**Remark 5.6** (Comparison to Kakade's natural policy gradient). *For Kakade's natural policy gradient in discrete time without entropy regularization in the function approximation regime, the value converges as  $O(\frac{1}{t})$  up to a remainder stemming from function approximation [4]. Compared to (5.13), the  $O(\frac{1}{t})$  involves a conditional KL term corresponding to the Kakade geometry, which is induced by the conditional entropy rather than the entropy. More importantly, however, it comes with a multiplicative distribution mismatch coefficient, where it is unclear whether it remains bounded during optimization. However, it is unclear whether this is inherent to Kakade's natural policy gradient or an artifact of the proof. The remainder term in [4] again depends on the distribution mismatch and on a concentrability coefficient similar to  $\chi^2(d^*, d_t)$ . Another difference between Kakade's and state-action natural policy gradients is that the compatible function approximation regresses the (estimated)  $Q$  or advantage function instead of the reward vector  $r$ , therefore leading to a different approximation error  $\tilde{\varepsilon}_t$ . Further, Kakade's natural policy gradient without entropy regularization in a function approximation setting has been shown to converge linearly when using geometrically increasing step sizes [56, 3, 57].*

Finally, entropy regularization with strength  $\lambda$  leads to  $O(e^{-\lambda t})$  convergence up to a remainder term, where the same  $\chi^2$ -divergence appears in the remainder term albeit with a different approximation error term [15].

We have studied general policy parameterizations and have seen that the corresponding state-action distributions evolve according to the projection of the Fisher-Rao gradient flow. A particularly nice case is given by parameterizations that are rich enough to express all policies as in this

case the state-action distributions exactly evolve according to the Fisher-Rao gradient flow. This is why we consider the following condition for policy parameterizations.

**Definition 5.7** (Regular tabular parametrization). *We say that a differentiable parametrization  $\mathbb{R}^p \rightarrow \text{int}(\Delta_{\mathbb{A}}^{\mathbb{S}}); \theta \mapsto \pi_{\theta}$  is a regular tabular parametrization if it is surjective and satisfies*

$$\text{span}\{\partial_{\theta_i} \pi_{\theta} : i = 1, \dots, p\} = T\Delta_{\mathbb{A}}^{\mathbb{S}} \quad \text{for all } \theta \in \mathbb{R}^p. \quad (5.15)$$

Since  $\pi \mapsto d^{\pi}$  is a diffeomorphism between  $\text{int}(\Delta_{\mathbb{A}}^{\mathbb{S}})$  and  $\text{int}(\mathcal{D})$ , see [39], we have

$$\text{span}\{\partial_{\theta_i} d_{\theta} : i = 1, \dots, p\} = T\mathcal{D} \quad \text{for all } \theta \in \mathbb{R}^p$$

for a regular policy parametrization.

Regular parametrizations include the following common examples:

- *Expressive exponential families:* For a feature map  $\phi: \mathbb{S} \times \mathbb{A} \rightarrow \mathbb{R}^p$  and  $\theta \in \mathbb{R}^p$  we consider the log-linear policy

$$\pi_{\theta}(a|s) := \frac{e^{\theta^{\top} \phi(s,a)}}{\sum_{a'} e^{\theta^{\top} \phi(s,a')}}.$$

If  $\text{rank}\{\phi(s,a) : s \in \mathbb{S}, a \in \mathbb{A}\} = |\mathbb{S}| \cdot |\mathbb{A}|$  this provides a regular tabular parametrization, see [47, Remark 2.4]. In particular, this includes *tabular softmax* policies, where  $\pi_{\theta}(a|s) \propto e^{\theta_{s,a}}$  which is the arguably most commonly studied policy class in policy gradient methods.

- *Escort transform:* To reduce the plateaus of vanilla policy gradients when working with softmax policies the so-called *escort transform* given by

$$\pi_{\theta}(a|s) := \frac{|\theta_{s,a}|^p}{\sum_{a'} |\theta_{s,a'}|^p},$$

for  $p \geq 1$  was introduced in [32].

For regular tabular parameterizations, the state-action distributions  $d_t$  evolve according to the Fisher-Rao gradient flow inside the state-action polytope  $\mathcal{D}$ . Hence, we can apply our general convergence theory to obtain the following result.

**Corollary 5.8** (Linear convergence for tabular parametrizations). *Consider a finite discounted Markov decision process, suppose Assumption 5.2 and Assumption 5.3 hold, and consider a solution of the natural gradient flow (5.10) for a regular tabular parametrization and write  $d_t = d_{\theta_t}$ . Further, assume that there is a unique optimal policy  $\pi^*$  with state-action distribution  $d^*$  and optimal reward  $R^*$ . Then we have*

$$D_{\text{KL}}(d^*, d_t) \leq D_{\text{KL}}(d^*, d_0) \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right) \quad (5.16)$$

and

$$R^* - R(\theta_t) \leq \Delta D_{\text{KL}}(d^*, d_0) \exp \left( -\Delta(t - t_0) + 2t_0 \Delta \log \left( \frac{t + t_0}{2t_0} \right) \right) \quad (5.17)$$

for all  $t \geq t_0$ , where

$$\Delta := \min \left\{ \frac{R^* - R(\pi)}{\|d^* - d^{\pi}\|_{\text{TV}}} : \begin{array}{l} \pi \text{ is deterministic and} \\ \text{agrees with } \pi^* \text{ on all but one state} \end{array} \right\} > 0, \quad (5.18)$$

and  $N(d^*)$  denotes the neighboring vertices of  $d^*$  in the state-action polytope  $\mathcal{D}$  and

$$t_0 := \frac{2D_{\text{KL}}(d^*, d_0)}{\Delta \cdot \min\{d^*(s) : s \in \mathbb{S}\}}. \quad (5.19)$$

*Proof.* Note that we can apply Theorem 3.2 and hence it remains to provide the explicit expressions for  $\Delta$  and  $t_0$ . Note that since  $\pi^*$  is deterministic, we have  $d^*(s, a) = d^*(s)$  if  $a$  is optimal and  $d^*(s, a) = 0$  otherwise. Hence, we have

$$\min\{d^*(s, a) : s \in \mathbb{S}, a \in \mathbb{A}\} = \min\{d^*(s) : s \in \mathbb{S}\}$$



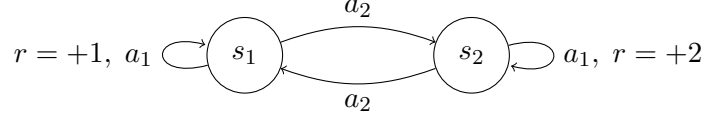


FIGURE 2. Transition graph and reward of the MDP example.

showing (5.19). For the explicit expression of  $\Delta$ , note that the neighboring vertices in  $\mathcal{D}$  and  $\Delta_{\mathbb{A}}^{\mathbb{S}}$  correspond to each other [38]. Hence,  $d^{\pi}$  is a neighbor of  $d^{\star}$  if  $\pi$  is deterministic and agrees with  $\pi^{\star}$  on all but one state.  $\square$

**Remark 5.9** (Comparison to Kakade’s NPG). *Much like the state-action natural policy gradient, Kakade’s natural policy gradient with exact gradient evaluations has been shown to converge linearly without the need of entropy regularized setting [27]. Here, the discrete-time setting is studied and NPG is interpreted as soft policy iteration. This is used to show a convergence rate of  $R^{\star} - R(\pi_k) = O(e^{-ck})$  for any  $c \in (0, \Delta_K)$ , where*

$$\Delta_K := -(1 - \gamma)^{-1} \max \{A^{\star}(s, a) : a \neq a_s^{\star}\} \geq \Delta.$$

Indeed, by the performance difference lemma, we have

$$\Delta = \min_{d \in N(d^{\star})} \frac{d^{\top} A^{\star}}{(1 - \gamma) \cdot \|d^{\star} - d\|_{\text{TV}}} = -(1 - \gamma)^{-1} \max_{d \in N(d^{\star})} \frac{d^{\top} A^{\star}}{\|d^{\star} - d\|_{\text{TV}}}.$$

Note that since  $d \in N(d^{\star})$  and since  $A^{\star}(s, a_s^{\star}) = 0$  we have  $\pi(a_0|s_0) = 1$  with  $a_0 \neq a_{s_0}^{\star}$  for and  $\pi(a_s^{\star}|s) = 1$  for  $s \neq s_0$ . Now we have  $d^{\top} A^{\star} = d(s_0)A^{\star}(s_0, a_0) \leq 0$ . Further, we estimate

$$\begin{aligned} 2\|d^{\star} - d\|_{\text{TV}} &= \sum_{s \neq 0} |d^{\star}(s) - d(s)| + d^{\star}(s_0) + d(s_0) \\ &\geq \sum_{s \neq 0} (d^{\star}(s) - d(s)) + d^{\star}(s_0) + d(s_0) \\ &= (1 - d^{\star}(s_0)) - (1 - d(s_0)) + d^{\star}(s_0) + d(s_0) = 2d(s_0). \end{aligned}$$

Overall, this yields  $\frac{d^{\top} A^{\star}}{\|d^{\star} - d\|_{\text{TV}}} \geq A^{\star}(s_0, a_0)$  and therefore

$$\Delta \leq -(1 - \gamma) \max \{A^{\star}(s, a) : a \neq a_s^{\star}\} = \Delta_K.$$

Hence, the best-known convergence rate of Kakade’s NPG is faster compared to the rate we provide for the state-action natural policy gradient. However, in the general case, no lower bounds are known for either gradient methods. In our computational example, both converge at the same exponential rate  $O(e^{-\Delta_K t})$  even if  $\Delta < \Delta_K$ .

**5.2. A computational example.** We use the following example proposed by Kakade [24] and which was also used in [9, 35]. Computer code for all experiments is made available under <https://github.com/muellerjohannes/fisher-rao-GFs-LPs>. We consider an MDP with two states  $s_1, s_2$  and two actions  $a_1, a_2$ , with the transitions and instantaneous rewards shown in Figure 2.

We adopt the initial distribution  $\mu(s_1) = 0.8, \mu(s_2) = 0.2$  and work with a discount factor of  $\gamma = 0.9$ . We can explicitly compute the rewards of the four deterministic policies to be  $R_1 = 0.98, R_2 = 1.2, R_3 = 1.84$  and  $R_4 = 0$ , and this way determine the optimal policy. Consequently, we can compute the exponent  $\Delta$  given in Corollary 5.8 to be  $\Delta = 0.8$ . In contrast, the exponent  $\delta$  given in [55, 51] is  $\delta = 0.64$ . In Remark 3.4 we observed that  $\delta \leq \Delta$ , and this now provides an explicit example where  $\delta < \Delta$ . Finally, we compute the constant  $\Delta_K = 0.8$  that describes the exponent in the convergence rate of Morimura’s natural policy gradient [27].

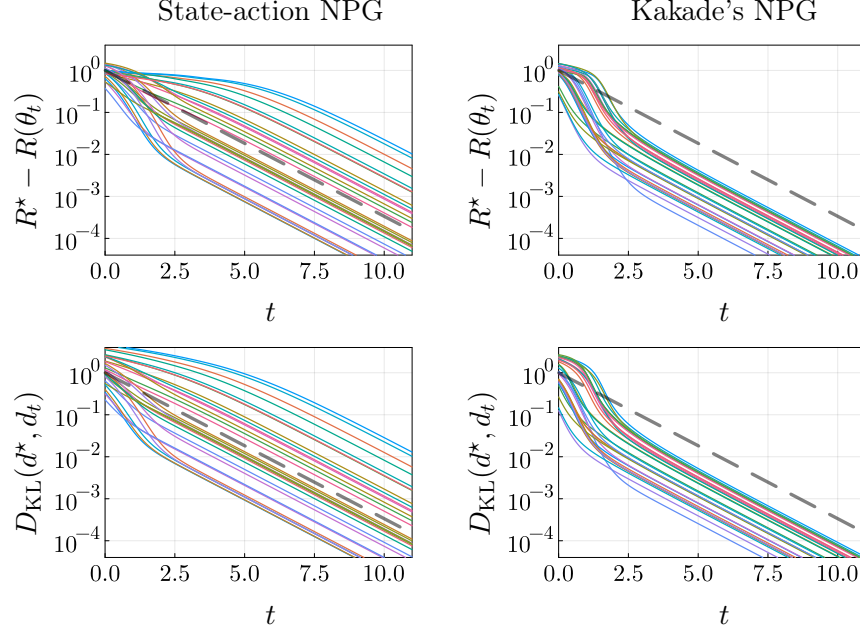


FIGURE 3. Shown are the suboptimality gap  $R^* - R(\theta_t)$  (top row) and the KL-divergence  $D_{\text{KL}}(d^*, d_t)$  (bottom row) for the state-action NPG (left column) and Kakade's NPG (right column) plotted in a logarithmic scale, along with the predicted exponential decay  $e^{-\Delta t} = e^{-\Delta_{\text{K}} t}$  (dashed line), see Corollary 5.8 and [27] for state-action and Kakade's NPG, respectively.

5.2.1. *Experimental setup and findings.* To illustrate our theoretical findings, we run both state-action natural gradients as well as Kakade's natural policy gradient applied to a tabular soft-max parametrization for 30 random initializations. In order to prevent a blow-up of the parameters we use the update rule

$$\theta_{k+1} = \theta_k + \eta \cdot \frac{G(\theta_k)^+ \nabla R(\theta_k)}{\|G(\theta_k)^+ \nabla R(\theta_k)\|}, \quad (5.20)$$

where  $\eta > 0$ . This ensures  $\|\theta_{k+1} - \theta_k\| = \eta$  and corresponds to using an adaptive step size

$$\eta_k = \frac{\eta}{\|G(\theta_k)^+ \nabla R(\theta_k)\|}.$$

Consequently, we consider the *virtual time*  $t_k := \sum_{l=0}^{k-1} \eta_l$  in our experiments as  $\theta_k \approx \tilde{\theta}_{t_k}$  if  $(\tilde{\theta}_t)_{t \geq 0}$  solves the natural policy gradient flow. Figure 3 plots the suboptimality gap  $R^* - R(\theta_t)$  as well as the KL-divergence  $D_{\text{KL}}(d^*, d_t)$  for the two different natural policy gradient methods and the same 30 random initializations. Additionally, the gray dashed line indicates the exponential decay rate  $O(e^{-\Delta t}) = O(e^{-\Delta_{\text{K}} t})$  guaranteed by Corollary 5.8 and by [27], respectively. We see that for all trajectories both the suboptimality gap  $R^* - R(\theta_t)$  as well as the KL-divergence to the optimal state-action distribution  $D_{\text{KL}}(d^*, d_t)$  decay at this guaranteed rate for both the state-action and Kakade's natural policy gradient method.

5.2.2. *A second example and non-tightness.* We complement our computational example by studying the same Markov decision process from Figure 2 but with a modified reward vector given by

$$r = \begin{matrix} & a_1 & a_2 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \end{matrix}. \quad (5.21)$$

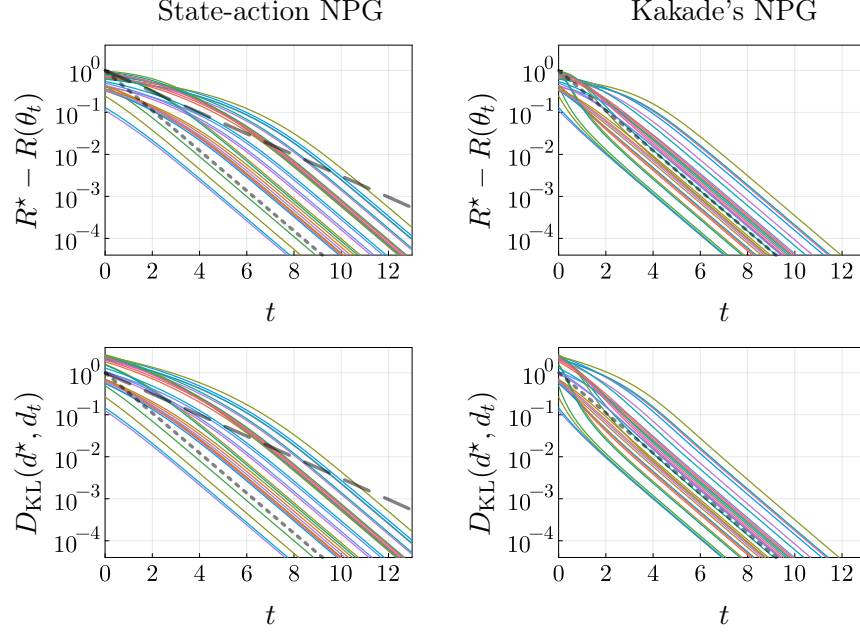


FIGURE 4. Shown are the suboptimality gap  $R^* - R(\theta_t)$  (top) and the KL-divergence  $D_{\text{KL}}(d^*, d_t)$  (bottom) for the state-action NPG (left) and Kakade's NPG (right); shown are also the guaranteed exponential decay rates  $e^{-\Delta t}$  for the state-action NPG (dashed line) and  $e^{-\Delta_K t}$  for Kakade's NPG (dotted line). Although the guarantees are different, both methods exhibit the same fast decay rate.

As above we can compute the three constants  $\delta$ ,  $\Delta$  and  $\Delta_K$ , and obtain  $\delta \approx 0.5326$ ,  $\Delta \approx 0.5789$  and  $\Delta_K = 1.1$ . Here again  $\delta < \Delta$  as it is always guaranteed under the Assumption 5.2, see Remark 3.4. Further, in this example we have  $\Delta < \Delta_K$ . We conduct the same experiment as before and report the findings in Figure 4. In the plots concerning the state-action natural policy gradient, we plot both the guaranteed decay  $O(e^{-\Delta t})$  (gray dashed line) and the decay  $O(e^{-\Delta_K t})$  guaranteed for Kakade's natural policy gradient (gray dotted line). We see that both methods exhibit the convergence rate  $O(e^{-\Delta_K t})$ . In particular, this indicates that our convergence analysis of the Fisher-Rao gradient, although improving on known results, is still not tight for general problems.

## 6. CONCLUSION AND OUTLOOK

We study Fisher-Rao gradient flows of linear programs and show they converge linearly with an exponent that depends on the geometry of the linear program. This yields an estimate on the error introduced by entropic regularization of the linear program, which improves existing guarantees. We extend this analysis to natural gradient flows for general parametrized measure models and show they converge at a sublinear rate  $O(\frac{1}{t})$  up to an approximation error and mismatch of the trajectory to the solution  $\chi^2$ -divergence. In particular, our results yield  $O(\frac{1}{t})$  convergence of state-action natural policy gradients without regularization under function approximation and linear convergence of state-action natural policy gradients for general tabular parametrizations. Finally, we provide computational examples illustrating our results.

Our results improve previous results, but some further improvements may be possible. In particular, we use the best global constant  $\Delta > 0$  for which the estimate (3.17) holds for all  $\mu \in P$ . However, if one can improve this constant along the trajectory  $(\mu_t)_{t \geq 0}$  this would directly imply an improvement of the convergence rate. A natural way to approach this is to characterize the direction from which the flow  $(\mu_t)_{t \geq 0}$  is approaching the global optimizer  $\mu^*$ . Another interesting

direction for future work is to study the statistical complexity of the state-action natural policy gradients. Finally, it could be explored whether our convergence results can be used in order to modify the cost to achieve a faster convergence without changing the optimizer, which is known as reward shaping in the context of reinforcement learning.

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#### APPENDIX A. AUXILIARY RESULTS

For the sake of completeness, we provide the proofs of some auxiliary results.

**Lemma A.1.** *Consider a polytope  $P \subseteq \mathbb{R}^X$ , a face  $F \subseteq P$  and consider the cone*

$$C := \text{cone} \left\{ \nu - \mu : \mu \in \text{vert}(F), \nu \in N(\mu) \setminus F \right\},$$

*which is generated by the edges pointing out of  $F$ . Then we have  $P \subseteq F + C$ .*

*Proof.* This is generalization of [59, Lemma 3.6], which covers the case that  $F$  consists of a single vertex. We will show that

$$F + C \supseteq \tilde{P} := \bigcap \left\{ H \subseteq \mathbb{R}^X : H \text{ is a halfspace, } P \subseteq H, H \cap F \neq \emptyset \right\} \supseteq P,$$

for which we pick an element  $u \in \tilde{P}$ . Consider a hyperplane  $H = \{\mu : a^\top \mu = \alpha\}$  separating  $F$  and  $\text{vert}(P) \setminus F$  and consider the *face figure*  $P/F := P \cap H$ , which is a polytope. Now, we consider a translation  $\tilde{H} = \{\mu : a^\top \mu = \beta\}$  of  $H$ , such that  $u \in \tilde{H}$ . Now we have

$$\tilde{P} = \text{conv} \left\{ \mu + \frac{a^\top \mu - \beta}{a^\top \mu - a^\top \nu} \cdot (\nu - \mu) : \mu \in \text{vert}(F), \nu \in N(\mu) \setminus F \right\},$$

see [60, Proposition 2.30]. Hence, we can choose convex weights  $\lambda_i$  such that

$$u = \sum_i \lambda_i (\mu_i + \alpha_i (\nu_i - \mu_i)) = \sum_i \lambda_i \mu_i + \sum_i \alpha_i \lambda_i (\nu_i - \mu_i) \in F + C,$$

where  $\mu_i \in \text{vert}(F), \nu_i \in N(\mu_i) \setminus F$ . □

**Lemma 3.11.** *Consider a polytope  $P \subseteq \mathbb{R}^X$  and denote by  $F^*$  the face of maximizers of the linear function  $\mu \mapsto c^\top \mu$  over  $P$ . Denote the set of neighboring vertices of a vertex  $\mu$  by  $N(\mu)$  and let  $\|\cdot\| : \mathbb{R}^X \rightarrow \mathbb{R}_{\geq 0}$  be an arbitrary semi-norm. Then either  $F^* = P$  or with  $\frac{c}{0} := +\infty$  for  $c > 0$ , we have*

$$\Delta := \min \left\{ \frac{c^\top \mu^* - c^\top \mu}{\|\mu^* - \mu\|} : \mu^* \in \text{vert}(F^*), \mu \in N(\mu^*) \setminus F^* \right\} > 0, \quad (3.30)$$

and further

$$c^\top \mu^* - c^\top \mu \geq \Delta \cdot \inf_{\mu^* \in F^*} \|\mu^* - \mu\| \quad \text{for all } \mu \in P. \quad (3.31)$$

*Proof.* If  $F^* \neq P$ , then surely  $c^\top \mu^* - c^\top \mu > 0$  for some vertex  $\mu$ , which implies  $\Delta > 0$ . Now we can proceed analogously to the proof of Lemma 3.8. To simplify notation let us define the set  $E := \{\mu - \mu^* : \mu \in N(\mu^*) \setminus F^*, \mu^* \in \text{vert}(F^*)\}$  of edges such that exactly one of the two endpoints is contained in  $F^*$ . Then, the polytope  $P$  is contained in

$$F^* + C = \left\{ \mu^* + \sum_{e \in E} \alpha_e e : \mu^* \in F^*, \alpha_e \geq 0 \text{ for all } e \in E \right\},$$

see Lemma A.1 and hence we can write  $\mu \in P$  as  $\mu = \mu^\star + \sum_e \alpha_e e$  for some  $\mu^\star \in F^\star$ . Just like in the proof of Lemma 3.8 we obtain

$$\Delta \|\mu^\star - \mu\|_1 \leq \Delta \sum_e \alpha_e \|e\|_1 \leq \sum_e \langle v, -e \rangle = \langle v, \mu^\star \rangle - \langle v, \mu \rangle = f^\star - \langle v, \mu \rangle.$$

□

**Lemma A.2** (Information projections have maximal support). *Consider a polytope  $P = \Delta_{\mathbb{X}} \cap L$  for an affine space  $L$  and a face  $F$  of  $P$ . Further, let  $\hat{\mu} \in F$  be the information projection of  $\mu \in \text{int}(P)$  to  $F$ , then  $\hat{\mu} \in \text{int}(F)$ .*

*Proof.* The information projection  $\hat{\mu} \in F$  is characterized by

$$D_{\text{KL}}(\hat{\mu}, \mu) = \min_{\mu' \in F} D_{\text{KL}}(\mu', \mu).$$

Assume that  $\hat{\mu} \in \partial F$ , then  $\mu_{x_0} = 0$  for some  $x_0 \in \mathbb{X}$ . Consider now  $v \in \mathbb{R}^{\mathbb{X}}$  such that  $\hat{\mu} + tv \in \text{int}(F)$  for  $t > 0$  small enough, then surely  $v_{x_0} > 0$ . By convexity of the KL-divergence (or simply of  $s \mapsto s \log s$ ), we have

$$D_{\text{KL}}(\hat{\mu}, \mu) \geq D_{\text{KL}}(\hat{\mu} + tv, \mu) + t \partial_t D_{\text{KL}}(\hat{\mu} + tv, \mu),$$

where  $\partial_t D_{\text{KL}}(\hat{\mu} + tv, \mu) \rightarrow -\infty$  for  $t \rightarrow 0$ . Hence, this shows  $D_{\text{KL}}(\hat{\mu}, \mu) > D_{\text{KL}}(\hat{\mu} + tv, \mu)$  for  $t$  small enough contradicting that  $\hat{\mu}$  is the information projection of  $\mu$ . □

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