Well-Posedness of the generalised Dean-Kawasaki Equation with correlated noise on bounded domains

Shyam Popat*

March 29, 2024

ABSTRACT. In this paper, we extend the notion of stochastic kinetic solutions introduced in [FG24] to establish the well-posedness of stochastic kinetic solutions of generalized Dean-Kawasaki equations with correlated noise on bounded, C^2-domains with Dirichlet boundary conditions. The results apply to a wide class of non-negative boundary data, which is based on certain a priori estimates for the solutions, that encompasses all non-negative constant functions including zero and all smooth functions bounded away from zero.

1 Introduction

We will consider the well-posedness of the generalised Dean-Kawasaki initial boundary value problem on a C^2 and bounded domain $U \subset \mathbb{R}^d$,

$$\begin{cases} \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ \dot{\xi}^F + \nu(\rho)), & \text{on } U \times (0,T], \\ \Phi(\rho) = \bar{f}, & \text{on } \partial U \times [0,T], \\ \rho(\cdot, t=0) = \rho_0, & \text{on } U \times \{t=0\}. \end{cases} \tag{1}$$

The main uniqueness and existence assumptions for the non-linear functions Φ, σ and ν are given in Assumptions 3.1 and 4.2 respectively. The assumptions allow us to consider a wide range of relevant stochastic PDE such as the full range of fast diffusion and porous medium equations, i.e. $\Phi(\rho) = \rho^m$ for every $m \in (0,\infty)$ and the degenerate square root $\sigma(\rho) = \sqrt{\rho}$. Subsequently we will refer to this choice of Φ and σ, and the alternative choice $\sigma(\rho) = \Phi^{1/2}(\rho)$ as the model case. The Stratonovich noise $\circ\dot{\xi}^F$ is white in time and sufficiently regular in space, see Definition 2.1.

We will prove the existence and uniqueness of a stochastic kinetic solutions of (1), in the sense of Definition 2.8 below. In the definition we do not insist that ρ is continuous all the way to the boundary and so we can't define the boundary condition in a pointwise sense. We only require the solution ρ to be locally in the Sobolev space $W^{1,2}(U)$, i.e. when cutoff away from zero and infinity, and otherwise just to be $L^1(U)$ regular. Hence the boundary condition in (1) is expressed indirectly via $\Phi(\rho)$ and is done using the trace theorem, see Section 5.5 of [Eva22]. Furthermore, the boundary condition does not depend on time, $\bar{f} = \bar{f}(x^*)$ for $x^* \in \partial U$. The restriction on the boundary data $\bar{f}$ that we are able to handle comes from assuming finiteness of the various boundary terms that appear in the a priori energy estimates.

A detailed summary outlining various reasons why one would be interested in studying stochastic PDE such as (1) is given in Section 1.3 of [FG24]. To briefly summarise, they arise as fluctuating continuum models for interacting particle systems, see [GLP98], and they can also be used to describe the hydrodynamic limit of simple particle processes, for instance the simple exclusion process, see [QRV99], and the zero range process, see Section 4.3 of [DSZ16].

In [FG24], Fehrman and Gess prove the well-posedness of equations such as (1) on the torus. Motivated by the particle system application, we will follow and extend the techniques introduced there in order to extend the well-posedness to a bounded domain. In this context the Dirichlet boundary condition enables one to model sources and sinks.

*Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK (popat@maths.ox.ac.uk)

1.1 Background and relevant literature

The Dean-Kawasaki equation was derived independently in the physics literature by Dean [Dea96] and Kawasaki [Kaw94]. In [Dea96], Dean considered an N particle system $\{X^i\}_{i=1}^N$ following Langevin dynamics with pairwise interaction potential V^1,

$$\dot{X}_t^i = \frac{1}{N}\sum_{j=1}^{N} V^1(X_t^i - X_t^j) + \sqrt{2}\dot{\beta}_t^i.$$

Via a formal argument replacing space time white noise and Brownian motion, the empirical measure $\rho_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is shown to satisfy the equation

$$\partial_t \rho^N = \Delta\rho^N - \nabla\cdot\left(\rho^N \nabla V^1 * \rho^N\right) - \frac{\sqrt{2}}{\sqrt{N}}\nabla\cdot\left(\sqrt{\rho^N}\xi\right), \tag{2}$$

where ξ is space time white noise. The irregularity of space time white noise implies that equations like (2) and (1) with space time white noise are not renormalisable using Hairer's regularity structures [Hai14] or Gubinelli, Imkeller and Perkowski's paracontrolled distributions [GIP15], even in dimension $d = 1$. Since the derivative hits space time white noise in (2), one can only consider solutions ρ^N in the space of distributions rather than functions, in which case the square root $\sqrt{\rho^N}$ as well as the product $\sqrt{\rho^N}\xi$ have no meaning. In fact, Konarovskyi, Lehmann and von Renesse show in [KLvR19, KLvR20] that the only martingale solutions to (2) or (1) with space time white noise are the "trivial" empirical measures ρ^N defined above. Furthermore, they show that one at least needs to add a drift correction term into the equations to obtain non-trivial solutions. It is for this reason that the Dean-Kawasaki equation is described as a "rigid mathematical object" in [CFIR23]. However, this rigidity can be overcome when suitable regularisations are considered, for instance smoothing or truncating the noise. We now discuss previous work on regularised versions of the Dean-Kawasaki equation based on relevance with the present work.

As mentioned above, this work primarily follows the techniques presented in Fehrman and Gess, [FG24]. There the authors consider a white in time, spatially correlated noise, see Definition 2.1 below. Even though the noise is sufficiently regular in space the well-posedness of equation (1) is tricky due to the square root singularity $\sigma(\rho) = \sqrt{\rho}$. Indeed, the Itô to Stratonovich conversion of (1) introduces a term with factor $(\sigma'(\rho))^2\nabla\rho$ that creates a singularity at zero, see derivation of equation (3) and Remark 2.4 below. The lack of local integrability of $log(\rho)$ means that even a weak solution to (1) can not be considered. Instead, the authors consider stochastic kinetic solutions, see Definition 2.8 below, where the compact support in the velocity variable restricts the solution away from it's zero set, see Remark 2.11. The kinetic formulation and the notion of kinetic solutions was first introduced in the PDE setting by Lions, Perthame and Tadmor [LPT94] where the authors considered the kinetic formulation for multidimensional scalar conservation laws. They chose the term 'kinetic equation' due to its analogy with the classical kinetic models such as Boltzmann or Vlasov models, see [Cer88]. Later, Chen and Perthame [CP03] were able to treat non-isotropic degenerate parabolic-hyperbolic equations. A new chain rule type condition introduced there allowed kinetic equations to be well defined, even when the macroscopic fluxes are not locally integrable.

Before proceeding, it is worth mentioning the relationship between stochastic kinetic solutions and weak solutions in the context of the Dean-Kawasaki equation. When the diffusion coefficient σ is sufficiently smooth so that we can make sense of weak solutions to (1), a weak solution is a stochastic kinetic solution, see Proposition 5.21 of [FG24]. Conversely, under additional assumptions on σ, stochastic kinetic solutions are also weak solutions, see Corollary 5.31 of [FG24].

Other related works are that of Fehrman and Gess [FG19, FG21] where they prove the path by path well-posedness of equations like (1) via a kinetic approach, motivated by the theory of stochastic viscosity solutions, see [LS98b, LS98a, LS00], and stochastic conservation laws, see [LPS13, LPS14, GS14, GS17]. The pathwise well-posedness is proven via rough path techniques, which imposes extra regularity on the diffusion coefficient σ that is needed to overcome the roughness of the noise ξ^F. In the recent work of Clini [Cli23] the result was extended to a smooth bounded domain with zero Dirichlet boundary conditions in the case of the porous media and fast diffusion equations, i.e.

$\Phi(\rho) = \rho^m, \nu(\rho) = 0$ in (1).
Another extension of [FG24] was that of Wang, Wu and Zhang [WWZ22]. The authors are able to show the existence of renormalised kinetic solutions to non-local equations such as (2) where we have a convolution, rather than the local interaction $\nu(\rho)$ in (1) considered here and in [FG24]. There the interaction kernel (V^1 in (2)) is assumed to satisfy the Ladyzhenskaya-Prodi-Serrin (LPS) condition, a regularity condition first studied in the context of Navier-Stokes equations in [Pro59, Ser61, Lad67] and applied to SDEs and distributional dependent SDEs in [KR05, RZ21] respectively. The main difficulty lies in the proof of existence, where the lack of uniform $L^p(\mathbb{T}^d)$ estimates of weak solutions of the regularised equation implies that the kinetic measures of the regularised equation are not uniformly bounded over $[0, T] \times \mathbb{T}^d \times \mathbb{R}$. One needs to instead use entropy estimates, similar to Proposition 4.24 below, to show the tightness of the kinetic measures of the approximate equations.
Another approach was by Dareiotis and Gess [DG20] where they constructed probabilistic solutions to (1) via an entropy formulation. The advantage of the kinetic approach considered here and in [FG24] over the entropy approach is that, by the precise identification of the kinetic defect measure, the kinetic approach can handle $L^1(U)$ integrable initial data, whereas the entropy approach requires $L^m(U)$ integrable initial data in the model case $\Phi(\xi) = \xi^m$. Furthermore, the approach in [DG20] required $C^{1,\alpha}$-regularity from the noise coefficient σ, and therefore remained far from the critical square root case $\sigma(\rho) = \sqrt{\rho}$.
Djurdjevac, Kremp and Perkowski in [DKP22] were able to prove the existence of strong solutions to equations like (1) in the case of truncated noise and mollified square root. The proof follows by a variational approach for a transformed equation and energy estimates for the approximate (Galerkin projected) system. However, again it is worth mentioning that the approach can only handle smooth coefficients and so also can not handle the square root singularity.
Also relevant is the work of Martini and Mayorcas [MM22], where local well-posedness of equations like (1) with space time white noise is proven when the square root $\sigma(\rho) = \sqrt{\rho}$ is replaced by $\sigma = \sqrt{\rho_{det}}$ where ρ_{det} solves a related deterministic PDE. The proof is via paracontrolled distribution theory.
Finally, we mention that there is an extensive literature surrounding the well-posedness of stochastic nonlinear diffusion equations on a smooth, bounded domains in $\mathbb{R}^d$ with zero Dirichlet boundary conditions. In [BBDPR06], Barbu, Bogachev, Da Prato and Röckner are able to show the existence of weak solutions of additive equations of the form

$$\partial_t \rho = \Delta\Phi(\rho) + \sigma(\rho)\dot{W}_t,$$

where W_t is cylindrical Brownian motion, $\sigma = \sqrt{Q}$ in the case that Q is linear, non-negative and of finite trace and Φ' satisfies a polynomial growth condition. See [BDPR09] for a proof of strong solutions in the porus medium case with lipschitz σ. Well-posedness of similar equations with multiplicative noise for dimension $1 \leq d \leq 3$ was shown by Barbu, Da Prato and Röckner [BDPR08a, BDPR08b].

1.2 Organisation and main contributions

In Section 2 we setup the problem. Firstly, in Section 2.1 we give the definition and assumptions on the nose ξ^F and subsequently in Section 2.2 we define a stochastic kinetic solution. The setup is analogous to Chapters 2 and 3 of [FG24], the main difference being that we incorporate the boundary condition in point two of the definition of stochastic kinetic solution, Definition 2.8.
In Section 3.1 we state the required assumptions for uniqueness as well as the definitions of convolution kernels and cutoff functions. Given that we are working on a bounded domain, in order to avoid boundary terms that we don't have control over appearing when integrating by parts, our solution concept Definition 2.8 is modified so that our test functions are also compactly supported in space. When formalising this in the uniqueness proof, this amounts to multiplying the test functions in [FG24] by a new spacial cutoff ι_γ, see Definition 3.2. It is worthwhile to mention that the restriction of working on a C^2 domain U arises so that we are able to differentiate the distance function that forms part of the definition of the spacial cutoff, as well as to apply Sobolev embedding theorems later on. In Section 3.2 the uniqueness is proven. The main novelty in the proof is in the techniques

used to bound the new terms arising when the gradient hits the spacial cutoff.
In Section 4.1 we begin by proving $L^2(U)$ a priori energy estimates of an appropriately regularised version of (1) in Proposition 4.14. We use this to prove further space-time regularity results in the remainder of the section. In Section 4.2 we prove an entropy estimate for the equation, similar to Proposition 5.18 of [FG24]. A localised version of the entropy estimate is used to prove a bound for the decay of the kinetic measure at zero, a statement needed in the uniqueness proof but proved in Section 4.3 for convenience. All of the estimates follow from applying Itô's formula. An important novelty is that we introduce harmonic PDEs (see for example Definitions 4.8 and 4.22) with carefully chosen boundary data that ensures we obtain functions which vanish along the boundary when applying Itô's formula, and so we can integrate by parts without worrying about additional boundary terms. The boundary regularity assumptions needed for our energy estimates will impose restrictions the boundary data $\bar{f}$ that we can consider.
The rest of the existence arguments, including tightness and compactness arguments of Section 4.4 follow directly from arguments from Chapter 5 of [FG24] so we are brief and simply include the main ideas for completeness.

2 Setup and Kinetic formulation

2.1 Definition of the noise

We briefly introduce and state the assumptions needed for the noise term ξ^F. Both the definition and assumptions are identical to those introduced in Chapter 2 of [FG24].

Definition 2.1 (The noise ξ^F)**.** *Let $F := \{f_k : U \to \mathbb{R}\}_{k\in\mathbb{N}}$ be a sequence of continuously differentiable functions and $\{B^k : [0,T] \to \mathbb{R}^d\}_{k\in\mathbb{N}}$ a sequence of independent, d-dimensional Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The noise ξ^F, superscripted by F to denote dependence on $\{f_k\}_k$, is defined by*

$$\xi^F : U \times [0,T] \to \mathbb{R}^d, \qquad \xi^F(x,t) := \sum_{k=1}^{\infty} f_k(x)B_t^k.$$

For ease of notation let's define three quantities related to the spacial component of the noise,

$$F_1 : U \to \mathbb{R} \quad \text{defined by} \quad F_1(x) := \sum_{k=1}^{\infty} f_k^2(x);$$

$$F_2 : U \to \mathbb{R}^d \quad \text{defined by} \quad F_2(x) := \sum_{k=1}^{\infty} f_k(x)\nabla f_k(x) = \frac{1}{2}\sum_{k=1}^{\infty} \nabla f_k^2(x);$$

$$F_3 : U \to \mathbb{R} \quad \text{defined by} \quad F_3(x) := \sum_{k=1}^{\infty} |\nabla f_k(x)|^2.$$

We need to make the following assumptions on the noise.

Assumption 2.2 (Assumption on noise)**.** *Suppose that F_i, $i = 1, 2, 3$ are continuous on U. Furthermore assume $\nabla \cdot F_2$ is bounded on U.*

Note that by Hölder's inequality, the boundedness of F_1 and F_3 imply the partial sums of F_2 are absolutely convergent.

2.2 Definition of stochastic kinetic solution

We begin by re-writing equation (1) using Itô noise. By Definition 2.1 of the noise we have

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ \dot{\xi}^F + \nu(\rho)) = \Delta\Phi(\rho) - \nabla \cdot \nu(\rho) - \sum_{k=1}^{\infty} \nabla \cdot (\sigma(\rho) f_k \circ dB_t^k).$$

Denoting $F_{\sigma,k}(\xi,x) := \sigma(\xi)f_k(x)$ for fixed $x \in U$, the Itô to Stratonovich conversion formula (see Section 3.3 of [Oks13]), the chain rule and product rule give formally that

$$\begin{aligned}
\partial_t\rho &= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot\left(\frac{\partial F_{\sigma,k}(\rho,x)}{\partial B^k}\right) \\
&= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot\left(f_k\sigma'(\rho)\frac{\partial\rho}{\partial B^k}\right) \\
&= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot(f_k\sigma'(\rho)\nabla(f_k\sigma(\rho))) \\
&= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho)]^2\nabla\rho + \sigma'(\rho)\sigma(\rho)F_2),
\end{aligned} \tag{3}$$

which we will equivalently sometimes write in the formal SDE notation as

$$d\rho_t = \Delta\Phi(\rho)\,dt - \nabla\cdot(\sigma(\rho)\,d\xi^F + \nu(\rho)\,dt) + \frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho)]^2\nabla\rho + \sigma'(\rho)\sigma(\rho)F_2)\,dt.$$

The below remark illustrates how to interpret integrals involving the divergence of the Itô noise in (3). We use it when interpreting the kinetic equation (5) below.

Remark 2.3. *For $\mathcal{F}_t$-adapted processes $g_t \in L^2(\Omega\times[0,T];L^2(U))$ and $h_t \in L^2(\Omega\times[0,T];H^1(U))$ and any $t\in[0,T]$ we define*

$$\int_0^t\int_U g_s\nabla\cdot(h_s\,d\xi^F) = \sum_{k=1}^{\infty}\left(\int_0^t\int_U g_sf_k\nabla h_s\cdot dB_s^k + \int_0^t\int_U g_sh_s\nabla f_k\cdot dB_s^k\right).$$

Remark 2.4. *In the model case $\sigma(\rho)=\rho^{1/2}$, the first correction term arising in the Itô equation* (3) *is $\frac{1}{8}\nabla\cdot(F_1\rho^{-1}\nabla\rho) = \frac{1}{8}\nabla\cdot(F_1\nabla log(\rho))$. If the solution ρ approaches 0 at any time the above term is a singular. In fact it is not even clear how we can define the notion of a weak solution since we don't know if $log(\rho)$ is locally integrable.*

We now turn our attention to providing the kinetic formulation for the generalised Dean-Kawasaki equation (3). By now the kinetic formulation is well understood, and for the Dean-Kawasaki equation the full derivation can be found in Chapter 3 of [FG24]. We briefly describe the motivation below. Suppose that for a convex function $S\in C^2(\mathbb{R};\mathbb{R})$ we were interested in properties of functions of the solution $S(\rho)$, where ρ solves equation (3). To derive the equation satisfied by $S(\rho)$ one applies Itô's formula

$$dS(\rho) = S'(\rho)d\rho + \frac{1}{2}S''(\rho)d\langle\rho\rangle.$$

However, one can not do this directly since ρ is not regular enough to apply Itô's formula, and instead we work on the level of the regularised equation.

Definition 2.5 (Regularised equation)**.** *The regularised generalised Dean-Kawasaki Equation ρ^α is defined for every $\alpha\in(0,1)$ by*

$$\partial_t\rho^\alpha = \Delta\Phi(\rho^\alpha) + \alpha\Delta\rho^\alpha - \nabla\cdot(\sigma(\rho^\alpha)\dot{\xi}^F + \nu(\rho^\alpha)) + \frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho^\alpha)]^2\nabla\rho^\alpha + \sigma'(\rho^\alpha)\sigma(\rho^\alpha)F_2). \tag{4}$$

After obtaining an equation for $S(\rho^\alpha)$, one aims to re-write the equation in terms of the kinetic function.

Definition 2.6 (Kinetic function)**.** *Given a non-negative solution ρ of equation* (3)*, define the kinetic function $\chi : U\times[0,T]\times\mathbb{R}\to\{0,1\}$ by*

$$\chi(x,t,\xi) = \mathbb{1}_{\{0\le\xi\le\rho(x,t)\}}.$$

The kinetic function can equivalently be viewed as $\chi:\mathbb{R}\times\mathbb{R}\to\{0,1\}$, $\chi=\chi(\rho,\xi)$.

In Lions, Perthame and Tadmor [LPT94] the kinetic function is called the velocity distribution or velocity profile since there they view ξ as a velocity variable. Here we will adopt the same nomenclature. By analogy with the theory of gasses, χ can be called a pseudo-Maxwellian. It is not then difficult to obtain a distributional equation for the kinetic function by using identities such as, if $S(0)=0$,

$$S(\rho(x,t)) = \int_{\mathbb{R}} S'(\xi)\chi(x,\xi,t)\,d\xi.$$

Finally, in taking the regularisation limit ($\alpha \to 0$), one needs to control a term containing $\alpha|\nabla\rho^\alpha|^2$ where we have the competing decay of α and divergence of $|\nabla\rho^\alpha|$ in the limit. On the level of the kinetic equation, in the theory of entropy solutions, see for instance Chapter 2 of [CP03], this is precisely quantified by the kinetic measure q.

Definition 2.7 (Kinetic measure)**.** *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty)},\mathbb{P})$ be a filtered probability space. A kinetic measure q is a map from Ω to the set of non-negative, locally finite measures on $U\times(0,\infty)\times[0,T]$ such that, for every $\psi\in C_c^\infty(U\times(0,\infty))$ we have*

$$(\omega,t)\in(\Omega,[0,T]) \to \int_0^t\int_{\mathbb{R}}\int_U \psi(x,\xi)\,dq(\omega)$$

is $\mathcal{F}_t$ predictable.

The resulting equation forms the basis of the definition of stochastic kinetic solution. We encapsulate some of the properties of the kinetic measure in points three and four of the definition below.

Definition 2.8 (Stochastic kinetic solution of (3))**.** *Let $\rho_0\in L^1(\Omega;L^1(U))$ be a non-negative $\mathcal{F}_0$ measurable initial condition. A stochastic kinetic solution of (3) is a non-negative, almost surely continuous $L^1(U)$ valued $\mathcal{F}_t$-predictable function $\rho\in L^1(\Omega\times[0,T];L^1(U))$ that satisfies*

1. *Integrability of flux: we have*

$$\sigma(\rho)\in L^2(\Omega;L^2(U\times[0,T])) \quad \text{and} \quad \nu(\rho)\in L^1(\Omega;L^1(U\times[0,T];\mathbb{R}^d)).$$

2. *Boundary condition, local regularity of solution: for each $k\in\mathbb{N}$*

$$[(\Phi(\rho)\wedge k)\vee 1/k]-[(\bar f\wedge k)\vee 1/k]\in L^2(\Omega;L^2([0,T];H_0^1(U))).$$

Furthermore, there exists a kinetic measure q that satisfies:

3. *Regularity: almost surely, in the sense of non-negative measures,*

$$\delta_0(\xi-\rho)\Phi'(\xi)|\nabla\rho|^2\le q \text{ on } U\times(0,\infty)\times[0,T].$$

4. *Vanishing at infinity: we have*

$$\lim_{M\to\infty}\mathbb{E}\left(q(U\times[M,M+1]\times[0,T])\right)=0.$$

5. *The kinetic equation: for every $\psi\in C_c^\infty(U\times(0,\infty))$ and every $t\in(0,T]$, almost surely,*

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi(x,\xi,t)\psi(x,\xi)\,dx\,d\xi = \int_{\mathbb{R}}\int_U \chi(x,\xi,t=0)\psi(x,\xi)\,dx\,d\xi \\
&-\int_0^t\int_U \left(\Phi'(\rho)\nabla\rho+\frac12F_1[\sigma'(\rho)]^2\nabla\rho+\frac12\sigma'(\rho)\sigma(\rho)F_2\right)\cdot\nabla\psi(x,\xi)|_{\xi=\rho}\,dx\,ds \\
&-\int_0^t\int_{\mathbb{R}}\int_U \partial_\xi\psi(x,\xi)\,dq+\frac12\int_0^t\int_U\left(\sigma'(\rho)\sigma(\rho)\nabla\rho\cdot F_2+\sigma(\rho)^2F_3\right)\partial_\xi\psi(x,\rho)\,dx\,ds \\
&-\int_0^t\int_U\psi(x,\rho)\nabla\cdot(\sigma(\rho)\,d\xi^F)\,dx-\int_0^t\int_U\psi(x,\rho)\nabla\cdot\nu(\rho)\,dx\,ds. \qquad (5)
\end{aligned}$$

We conclude the chapter with a few remarks on the above definition.

Remark 2.9. *In the kinetic equation* (5) *we write* $\nabla\psi(x,\xi)|_{\xi=\rho}$ *to emphasize that we take gradient in the first component of* ψ *rather than the full gradient of* $\psi(x,\rho)$*.*
We abuse notation and write $\partial_\xi\psi(x,\rho)$ *to mean* $\partial_\xi\psi(x,\xi)|_{\xi=\rho}$*.*

Remark 2.10. *Since we will assume* Φ *is strictly increasing (Assumption 3.1), the second point implies that locally* $\rho \in H^1(U)$*.*

Remark 2.11. *In the kinetic equation it's essential that we integrate against test functions* ψ *that are compactly supported in* $U \times (0,\infty)$*. Firstly, noting Remark 2.4, the compact support in the velocity variable* $\xi \in (0,\infty)$ *implies that equation* (5) *needs to hold only away from the zero set of the solution. Secondly, the compact support in space implies that we can integrate by parts and without worrying about boundary terms. The velocity and spacial compactness requirement motivates the introduction cutoff functions in Definition 3.2 that will be present in many of the choices of test function* ψ *we make.*

3 Uniqueness

In Section 3.1 we begin with some assumptions on the coefficients Φ, ν, σ of equation (3) needed for uniqueness. The assumptions are the same as in [FG24] and allow us to consider the model cases. We then introduce smoothing kernels and cutoff functions in Definition 3.2.
The uniqueness is proved in Theorem 3.5. By taking limits of the various convolution kernels and cutoffs in the correct order, the only difference to the torus, Theorem 4.7 of [FG24], is the need to bound new terms arising when the gradient hits the spacial cutoff. In the proof we will use a result bounding the decay of the kinetic measure at zero, Proposition 4.6 of [FG24], which, for convenience, we prove in the next chapter in Section 4.3.

3.1 Assumptions and definition of convolution kernels and cutoff functions

We begin with the assumptions needed for uniqueness, identical to Assumption 4.1 of [FG24].

Assumption 3.1 (Uniqueness assumptions)**.** *Suppose* $\Phi, \sigma \in C([0,\infty))$ *and* $\nu \in C([0,\infty);\mathbb{R}^d)$ *satisfy the five assumptions:*

1. *We have* $\Phi, \sigma \in C^{1,1}_{loc}([0,\infty))$ *and* $\nu \in C^1_{loc}([0,\infty);\mathbb{R}^d)$*.*
2. *The function* Φ *is strictly increasing and starts at* 0*:* $\Phi(0) = 0$ *with* $\Phi' > 0$ *on* $(0,\infty)$*.*
3. *At least linear decay of* σ^2 *at* 0*: There exists a constant* $c \in (0,\infty)$ *such that*
$$\limsup_{\xi\to 0^+} \frac{\sigma^2(\xi)}{\xi} \le c.$$
In particular this implies that $\sigma(0) = 0$*.*
4. *Regularity of oscillations of of* σ^2 *at infinity: There is a* $c \in [1,\infty)$ *such that*
$$\sup_{\xi'\in[0,\xi]} \sigma^2(\xi') \le c(1+\xi+\sigma^2(\xi)) \quad \textit{for every } \xi \in [0,\infty).$$
5. *Regularity of oscillations of of* ν *at infinity: There is a constant* $c \in [1,\infty)$ *such that*
$$\sup_{\xi'\in[0,\xi]} |\nu(\xi')| \le c(1+\xi+|\nu(\xi)|) \quad \textit{for every } \xi \in [0,\infty).$$

We refer to Remark 4.2 of [FG24] for a comprehensive discussion on the final two assumptions. The assumptions are satisfied in the case that the functions σ^2, ν are increasing, or are uniformly continuous, or grow linearly at infinity. The assumption is more general than any of the above three examples and essentially amounts to a restriction on the growth of the magnitude of oscillations, rather than frequency of oscillations at infinity.
We now define the convolution kernels and cutoff functions required in the uniqueness proof.

Definition 3.2 (Convolution kernels and cutoff functions).

- *Convolution kernel in space and velocity: for every $\epsilon, \delta \in (0,1)$ let $\kappa_d^\epsilon : U \to [0,\infty)$ and $\kappa_1^\delta : \mathbb{R} \to [0,\infty)$ be standard convolution kernels/ mollifiers of scale ϵ and δ on U and $\mathbb{R}$ respectively. That is to say, let $\kappa_d \in C_c^\infty(\mathbb{R}^d), \kappa_1 \in C_c^\infty(\mathbb{R})$ be non-negative and integrate to one. For $\epsilon, \delta \in (0,1)$ define*

$$\kappa_d^\epsilon(x) = \frac{1}{\epsilon^d}\kappa_d\left(\frac{x}{\epsilon}\right), \quad \kappa_1^\delta(\xi) = \frac{1}{\delta}\kappa_1\left(\frac{\xi}{\delta}\right).$$

Let $\kappa^{\epsilon,\delta}$ be defined by the product

$$\kappa^{\epsilon,\delta}(x,y,\xi,\eta) := \kappa_d^\epsilon(x-y)\kappa_1^\delta(\xi-\eta), \qquad (x-y,\xi,\eta) \in U \times \mathbb{R}^2.$$

- *Cutoff of small velocity ξ: for every $\beta \in (0,1)$ let $\phi_\beta : \mathbb{R} \to [0,1]$ be the unique non-decreasing piecewise linear function that satisfies*

$$\phi_\beta(\xi) = 1 \text{ if } \xi \geq \beta, \quad \phi_\beta(\xi) = 0 \text{ if } \xi \leq \beta/2, \quad \phi'_\beta = \frac{2}{\beta}\mathbb{1}_{\beta/2\leq\xi\leq\beta}.$$

- *Cutoff of large velocity ξ: for every $M \in \mathbb{N}$ let $\zeta_M : \mathbb{R} \to [0,1]$ be the unique non-increasing piecewise linear function that satisfies*

$$\zeta_M(\xi) = 1 \text{ if } \xi \leq M, \quad \zeta_M(\xi) = 0 \text{ if } \xi \geq M+1, \quad \zeta'_M = -\mathbb{1}_{M\leq\xi\leq M+1}.$$

- *Spacial cutoff around boundary: for γ sufficiently small (depending on geometry of domain U), start by introducing the interior regions*

$$U_\gamma := \{x \in U : d(x,\partial U) \geq \gamma\} \subset U, \qquad \partial U_\gamma := \{x \in U : d(x,\partial U) = \gamma\},$$

where d is the usual Euclidean distance in $\mathbb{R}^d$.
The cutoff function is such that it takes the value 1 in the interior of the domain, 0 along the boundary, and linearly interpolates between the two. Explicitly we consider the function

$$\iota_\gamma(x) := \frac{d(x,\partial U) \wedge \gamma}{\gamma} = \begin{cases} 1, & \text{if } d(x,\partial U) > \gamma, \\ \gamma^{-1} d(x,\partial U), & \text{if } 0 \leq d(x,\partial U) \leq \gamma. \end{cases} \tag{6}$$

We will repeatedly use below that for any fixed γ, one can approximate the cutoff function above by a sequence of compactly supported functions. In this way we may abuse notation and describe ι_γ itself as being compactly supported. Let us now discuss how to define the gradient of the spacial cutoff.

Remark 3.3 (Derivative of spacial cutoff). *To define the spacial derivative of the function ι_γ, we will differentiate the distance function. We know that the distance function is differentiable if and only if for every x we can find a unique closest point $x^* := \Pi_{\partial U}(x)$ on the boundary to x. Looking at the definition of the cutoff (6), we only want to differentiate the distance function for $x \in U \setminus U_\gamma$, so it follows that we only need to assume this property for points x sufficiently close to the boundary. This can be done in any C^2 domain, see [Foo84].*
In this case, letting v_x denote the inward pointing unit normal at the boundary to point $x \in U$, with x^* as above, the first derivative is given by

$$\nabla\iota_\gamma(x) = \gamma^{-1}\frac{x-x^*}{|x-x^*|}\mathbb{1}_{U\setminus U_\gamma}(x) := \gamma^{-1}v_x\mathbb{1}_{U\setminus U_\gamma}(x),$$

which in particular implies that the size of the first derivative is of the order γ^{-1},

$$|\nabla\iota_\gamma(x)| = \gamma^{-1}\mathbb{1}_{U\setminus U_\gamma}(x).$$

3.2 Uniqueness proof

Before proving the uniqueness theorem, we need to prove an integration by parts lemma against the kinetic function. Since we will only deal with test functions that are compactly supported in space, the statement reads the same as the torus, see Lemma 4.4 of [FG24]. We will use the lemma with test function being the convolution kernel $\psi = \kappa^{\epsilon,\delta}$.

Lemma 3.4 (Integration by parts against kinetic function)**.** *Let $\psi \in C_c^\infty(U \times (0,\infty))$ be a compactly supported test function (in both arguments) and χ the kinetic function as defined in Definition 2.7. Then*

$$\int_{\mathbb{R}} \int_U \nabla_x \psi(x,\xi)\chi(x,\xi,r)\,dx\,d\xi = -\int_U \psi(x,\rho(x,r))\nabla\rho(x,r)\,dx.$$

Proof. Let $\Psi : U \times (0,\infty) \to \mathbb{R}$ be a function satisfying $\partial_\xi \Psi(x,\xi) = \psi(x,\xi)$, $\Psi(x,0) = 0$. By the definition of kinetic function, for any $r \in [0,T]$ we have

$$\begin{aligned}
\int_U \int_{\mathbb{R}} \nabla_x \psi(x,\xi)\chi(x,\xi,r)\,d\xi\,dx &= \int_U \int_0^{\rho(x,r)} \nabla_x \psi(x,\xi)\,d\xi\,dx \\
&= \int_U \int_0^{\rho(x,r)} \partial_\xi \nabla_x \Psi(x,\xi)\,d\xi\,dx \\
&= \int_U \nabla_x \Psi(x,\rho(x,r)) - \nabla_x \Psi(x,0)\,dx \\
&= \int_U \nabla \Psi(x,\rho(x,r)) - \partial_\xi \Psi(x,\rho(x,r))\nabla\rho(x,r)\,dx \\
&= \int_{\partial U} \Psi(x,\rho(x,r)) \cdot \hat{\eta}\,dx - \int_U \psi(x,\rho(x,r))\nabla\rho(x,r)\,dx,
\end{aligned}$$

where the final equality is due to the divergence theorem. Note that the above is an equality of vectors, the first term on the right hand side in the final line denotes a vector where the $i'th$ component is the function Ψ dotted with the $i'th$ direction outward pointing unit normal η_i, and it vanishes due to the compact support of ψ. In the remainder of the paper we write $\hat{\eta} = (\hat{\eta}_i)_{i=1}^d$ to denote the outward pointing unit normal at the boundary ∂U. ☐

We are now in a position to prove the uniqueness of stochastic kinetic solutions of (3). For the proof we will assume the following decay of the kinetic measure at zero, which is proved in Section 4.3.

$$\liminf_{\beta \to 0} \left(\beta^{-1} q(U \times [\beta/2, \beta] \times [0,T]) \right) = 0.$$

Theorem 3.5. *Suppose that the coefficients Φ, σ, ν of equation* (3) *and the coefficients of noise ξ^F satisfy Assumptions 2.2, 3.1, and further suppose we have the above decay of the kinetic measure at zero. Suppose ρ^1 and ρ^2 are two stochastic kinetic solutions of* (3) *in the sense of Definition 2.8 with $\mathcal{F}_0-$measurable initial data $\rho_0^1, \rho_0^2 \in L^1(\Omega; Ent(U))$ respectively. Then almost surely*

$$\sup_{t \in [0,T]} \|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{L^1(U)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(U)}.$$

Remark 3.6.

- *Note that the pathwise contraction property in the equation above implies the pathwise continuity of solutions with respect to the initial condition. This is a stronger result than the uniqueness of equation* (3).
- *The proof follows along the same lines as on the torus, see Theorem 4.7 of [FG24]. Hence in the proof below we omit the majority of the bounds that just follow from there, instead focusing our attention on the new terms arising as a result of having to include the spacial cutoff ι_γ as part of the test function.*
- *In the below proof and throughout the paper we use c to denote a running constant and not specify what it depends on unless it is important.*

Proof. Let χ^1, χ^2 be the kinetic functions of ρ^1, ρ^2 respectively. For every $\epsilon, \delta \in (0,1)$, $i \in \{1,2\}$ and $\kappa^{\epsilon,\delta}$ as in Definition 3.2 define the smoothed kinetic functions

$$\chi^{\epsilon,\delta}_{t,i}(y,\eta) := (\chi^i(\cdot,\cdot,t) * \kappa^{\epsilon,\delta})(y,\eta), \;\; t \in [0,T], y \in U, \eta \in \mathbb{R}.$$

We have by definition of convolution kernels that for $x, y \in U$ and for $\xi, \eta \in \mathbb{R}$

$$\nabla_x \kappa^\epsilon_d(y-x) = -\nabla_y \kappa^\epsilon_d(y-x), \quad \partial_\xi \kappa^\delta_1(\eta-\xi) = -\partial_\eta \kappa^\delta_1(\eta-\xi).$$

This implies, as a result of the kinetic equation (5), that for every $\epsilon, \delta \in (0,1)$ there is a subset of full probability such that we have for every $i \in \{1,2\}$, $t \in [0,T]$ and $(y,\eta) \in U_{2\epsilon} \times (2\delta, \infty)$ such that the convolution kernel is compactly supported

$$\begin{aligned}
&\chi^{\epsilon,\delta}_{s,i}(y,\eta)\Big|_{s=0}^t := (\chi^i(\cdot,\cdot,s) * \kappa^{\epsilon,\delta})(y,\eta)\Big|_{s=0}^t := \int_{\mathbb{R}}\int_U \chi^i(x,\xi,s)\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dx\,d\xi\Big|_{s=0}^t \\
&= \nabla_y \cdot \left(\int_0^t \int_U \left(\Phi'(\rho^i)\nabla(\rho^i) + \frac{1}{2}F_1[\sigma'(\rho^i)]^2\nabla\rho^i + \frac{1}{2}\sigma'(\rho^i)\sigma(\rho^i)F_2\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\,dx\,ds\right) \\
&+ d_\eta\left(\int_0^t\int_{\mathbb{R}}\int_U \kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dq^i\right) - \frac{1}{2}\partial_\eta\left(\int_0^t\int_U \left(\sigma'(\rho^i)\sigma(\rho^i)\nabla\rho^i \cdot F_2 + \sigma(\rho^i)^2F_3\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\,dx\,ds\right) \\
&- \int_0^t\int_U \kappa^{\epsilon,\delta}(x,y,\rho,\eta)\nabla\cdot(\sigma(\rho^i)\,d\xi^F)\,dx - \int_0^t\int_U \kappa^{\epsilon,\delta}(x,y,\rho,\eta)\nabla\cdot\nu(\rho^i)\,dx\,ds. \qquad (7)
\end{aligned}$$

To find an expression of the difference in solutions, we want to deal with a regularised version of

$$\int_U |\rho^1(x,s)-\rho^2(x,s)|\,dx\Big|_{s=0}^t = \int_U\int_{\mathbb{R}} \chi^1(\xi,\rho(x,s)) + \chi^2(\xi,\rho(x,s)) - 2\chi^1(\xi,\rho(x,s))\chi^2(\xi,\rho(x,s))\,d\xi\,dx\Big|_{s=0}^t. \qquad (8)$$

We begin by treating the regularised version of the first two terms on the right hand side of equation (8). Testing equation (7) against smooth approximations of the product of cutoff functions $\zeta_M\phi_\beta\iota_\gamma$, which are smooth and compactly supported, and subsequently taking the limit of the approximations yields

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,i}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\Big|_{s=0}^t = \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\Phi'(\rho^i)\nabla\rho^i + \frac{1}{2}F_1[\sigma'(\rho^i)]^2\nabla\rho^i + \frac{1}{2}\sigma'(\rho^i)\sigma(\rho^i)F_2\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\zeta_M(\eta)\phi_\beta(\eta)\cdot\nabla\iota_\gamma\,dy\,dx\,ds\,d\eta \\
&-\int_0^t\int_{\mathbb{R}^2}\int_{U^2}\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dq^i\,dy\,d\eta \\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\sigma'(\rho^i)\sigma(\rho^i)\nabla\rho^i\cdot F_2 + \sigma(\rho^i)^2F_3\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dy\,dx\,ds\,d\eta \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\nabla\cdot(\sigma(\rho^i)\,d\xi^F)\,dy\,dx\,ds\,d\eta \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\nabla\cdot\nu(\rho^i)\,dy\,dx\,ds\,d\eta.
\end{aligned}$$

For convenience we split this up into three parts,

$$\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,i}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\Big|_{s=0}^t = I^{i,cut}_t + I^{i,mart}_t + I^{i,cons}_t,$$

with the cutoff term being the first three lines on the right hand side, the martingale term being the noise term on the fourth line and the conservative term being the term in the final line. Note in particular that the first terms on the right hand side containing the derivative of the spacial cutoff $\nabla\iota_\gamma$ is a new term compared to the torus case which a priori diverges like γ^{-1}.
To obtain an expression for the final term (the mixed term) in (8), we introduce the notation

$(x,\xi) \in U \times \mathbb{R}$ for arguments in χ^1 and related quantities and $(x',\xi') \in U \times \mathbb{R}$ for arguments of χ^2 and related quantities. For brevity we also introduce the notation

$$\bar{k}^{\epsilon,\delta}_{s,1}(x,y,\eta) := \kappa^{\epsilon,\delta}(x,y,\eta,\rho^1(x,s)), \qquad \bar{k}^{\epsilon,\delta}_{s,2}(x',y,\eta) := \kappa^{\epsilon,\delta}(x',y,\eta,\rho^2(x',s)).$$

In the below computations, since we smoothed the kinetic function, we are allowed to use it as part of an admissible test function. The stochastic product rules tells us that almost surely we have, for $\beta \in (0,1)$, $M \in \mathbb{N}$, γ sufficiently small depending on the domain U, $\delta \in (0,\beta/4)$, $\epsilon \in (0,\gamma/4)$,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\chi^{\epsilon,\delta}_{s,2}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\Big|_{s=0}^t \\
&= \int_0^t\int_{\mathbb{R}}\int_U \left(\chi^{\epsilon,\delta}_{s,1}(y,\eta)d\chi^{\epsilon,\delta}_{s,2}(y,\eta) + \chi^{\epsilon,\delta}_{s,2}(y,\eta)d\chi^{\epsilon,\delta}_{s,1}(y,\eta) + d\langle\chi^{\epsilon,\delta}_1,\chi^{\epsilon,\delta}_1\rangle_s(y,\eta)\right)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta \\
&= \int_{\mathbb{R}}\int_U \Bigg(\chi^{\epsilon,\delta}_{s,1}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,2}(y,\eta)\Big|_{s=0}^t\right] + \chi^{\epsilon,\delta}_{s,2}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,1}(y,\eta)\Big|_{s=0}^t\right] \\
&\qquad\qquad + \left[\langle\chi^{\epsilon,\delta}_1,\chi^{\epsilon,\delta}_2\rangle_s(y,\eta)\Big|_{s=0}^t\right]\Bigg)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta. \qquad (9)
\end{aligned}$$

Using equation (7) we can write the first term on the final line of (9) as

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,2}(y,\eta)\right]\Big|_{s=0}^t\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta \\
&= \int_{\mathbb{R}}\int_U \zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\Bigg[\nabla_y\cdot\left(\int_0^t\int_U\left(\Phi'(\rho^2)\nabla\rho^2\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\right) \\
&+ \nabla_y\cdot\left(\int_0^t\int_U\left(\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2 + \frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\right) \\
&+ \partial_\eta\left(\int_0^t\int_{\mathbb{R}}\int_U \kappa^{\epsilon,\delta}(x',y,\xi,\eta)\,dq^2\right) - \frac{1}{2}\partial_\eta\left(\int_0^t\int_U\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2 + \sigma(\rho^2)^2F_3\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\right)\,ds \\
&- \int_0^t\int_U \bar{k}^{\epsilon,\delta}_{s,2}\nabla\cdot(\sigma(\rho^2)\,d\xi^F)\,dx' - \int_0^t\int_U \bar{k}^{\epsilon,\delta}_{s,2}\nabla\cdot\nu(\rho^2)\,dx'\,ds\Bigg]\,dy\,d\eta.
\end{aligned}$$

We integrate by parts and move derivatives onto (smooth approximations of) the product $\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)$, which are smooth, compactly supported so can be done using classical integration by parts. We use the product rule when integrating in y then the integration by parts lemma, Lemma 3.4 noting the convolution kernel is compactly supported since $y,\eta \in U_{2\epsilon}\times(2\delta,\infty)$:

$$\begin{aligned}
\nabla_y\chi^{\epsilon,\delta}_{s,1}(y,\eta) &:= \int_{\mathbb{R}}\int_U \chi^i(x,\xi,s)\nabla_y\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dx\,d\xi \\
&= -\int_{\mathbb{R}}\int_U \chi^i(x,\xi,s)\nabla_x\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dx\,d\xi \\
&= -\int_U \bar{k}^{\epsilon,\delta}_{s,i}\nabla\rho^i(x,r)\,dx,
\end{aligned}$$

to obtain the decomposition

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}}\int_U &\chi^{\epsilon,\delta}_{s,1}(y,\eta)d\chi^{\epsilon,\delta}_{s,2}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta \\
&= I^{1,2,err}_t + I^{1,2,meas}_t + I^{1,2,cut}_t + I^{1,2,mart}_t + I^{1,2,cons}_t.
\end{aligned}$$

Adding the term

$$\int_0^t\int_{\mathbb{R}}\int_{U^3}[\Phi'(\rho^1)]^{1/2}[\Phi'(\rho^2)]^{1/2}\nabla\rho^1\cdot\nabla\rho^2\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dx\,dx'\,dy\,d\eta\,ds$$

to the error term and taking it away from the measure term gives for each term separately (note below that we get an extra spacial and an extra real integral due to the definition of convolution in

$\chi^{\epsilon,\delta}$):

$$
\begin{aligned}
&I_t^{1,2,err} = \\
&-\int_0^t\int_{\mathbb{R}}\int_{U^3}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\Phi'(\rho^2)\nabla\rho^2\cdot\nabla\rho^1\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\,dx\,dx'\,ds\,dy\,d\eta \\
&-\int_0^t\int_{\mathbb{R}}\int_{U^3}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\left(\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2\cdot\nabla\rho^1+\frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\cdot\nabla\rho^1\right)\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\,dx\,dx'\,ds\,dy\,d\eta \\
&-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_{U^3}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2+\sigma(\rho^2)^2F_3\right)\bar{k}^{\epsilon,\delta}_{s,2}\bar{k}^{\epsilon,\delta}_{s,1}\,dx'\,ds\,dy\,d\eta \\
&+\int_0^t\int_{\mathbb{R}}\int_{U^3}[\Phi'(\rho^1)]^{1/2}[\Phi'(\rho^2)]^{1/2}\nabla\rho^1\cdot\nabla\rho^2\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dx\,dx\,dx'\,dy\,d\eta\,ds,
\end{aligned}
$$

measure term

$$
\begin{aligned}
I_t^{1,2,meas} &= \int_0^t\int_{\mathbb{R}^2}\int_{U^3}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(x',y,\xi,\eta)\bar{k}^{\epsilon,\delta}_{s,1}\,dq^2(x',\xi,s)\,dx\,dy\,d\eta \\
&-\int_0^t\int_{\mathbb{R}}\int_{U^3}[\Phi'(\rho^1)]^{1/2}[\Phi'(\rho^2)]^{1/2}\nabla\rho^1\cdot\nabla\rho^2\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dx\,dx'\,dy\,d\eta\,ds,
\end{aligned}
$$

cutoff term defined by

$$
\begin{aligned}
I_t^{1,2,cut} &= -\int_0^t\int_{\mathbb{R}^2}\int_{U^2}\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\chi^{\epsilon,\delta}_{s,1}(y,\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(x',y,\xi,\eta)\,dq^2(x',\xi,s)\,dy\,d\eta \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_{U^2}\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\chi^{\epsilon,\delta}_{s,1}(y,\eta)\iota_\gamma(y)\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2+\sigma(\rho^2)^2F_3\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\,dy\,d\eta \\
&-\int_0^t\int_{\mathbb{R}}\int_{U^2}\zeta_M(\eta)\phi_\beta(\eta)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla_y\iota_\gamma(y)\cdot\left(\Phi'(\rho^2)\nabla\rho^2+\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2+\frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right) \\
&\qquad\qquad\times\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\,dy\,d\eta,
\end{aligned}
$$

martingale term

$$
I_t^{1,2,mart} = -\int_0^t\int_{\mathbb{R}}\int_{U^2}\bar{k}^{\epsilon,\delta}_{s,2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla\cdot(\sigma(\rho^2)\,d\xi^F)\,dx'\,dy\,d\eta,
$$

and conservative term

$$
I_t^{1,2,cons} = -\int_0^t\int_{\mathbb{R}}\int_{U^2}\bar{k}^{\epsilon,\delta}_{s,2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla\cdot\nu(\rho^2)\,dx'\,ds\,dy\,d\eta.
$$

Note again that the challenge arises from the final line of the cutoff term involving the derivative $\nabla\iota_\gamma$. An analogous decomposition holds for the second term on the right hand side of (9). We denote error, measure, cutoff, martingale and conservative terms of the second term up to time $t\in[0,T]$ by $I_t^{2,1,\cdot}$, where we again artificially add an error term and subtract it from the measure term. Finally we deal with the quadratic variation term in equation (9). Let's begin by noticing, using Definition

2.1 of the noise ξ^F, that formally

$$
\begin{aligned}
&d\langle \chi^{\epsilon,\delta}_{\cdot,1}, \chi^{\epsilon,\delta}_{\cdot,2}\rangle_s(y,\eta) := d\langle (\chi^1 * \kappa^{\epsilon,\delta}), (\chi^2 * \kappa^{\epsilon,\delta})\rangle_s(y,\eta)\\
&= \int_{U^2}\int_{\mathbb{R}^2} d\langle \chi^1, \chi^2\rangle_s \kappa^{\epsilon,\delta}_{s,1}(y,x,\eta,\xi)\kappa^{\epsilon,\delta}_{s,2}(y,x',\eta,\xi')\, d\xi\, d\xi'\, dx\, dx'\\
&= \int_{U^2}\int_{\mathbb{R}^2} \delta_0(\xi-\rho^1)\delta_0(\xi'-\rho^2)\nabla\cdot\left(\sigma(\rho^1)\sum_{k=1}^{\infty} f_k(x)dB^k_s\right)\nabla\cdot\left(\sigma(\rho^2)\sum_{j=1}^{\infty} f_j(x')dB^j_s\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times \kappa^{\epsilon,\delta}_{s,1}(x,y,\xi,\eta)\kappa^{\epsilon,\delta}_{s,2}(x',y,\xi',\eta)\, d\xi\, d\xi'\, dx\, dx'\\
&= \sum_{j,k=1}^{\infty}\int_{U^2} \left(f_k\sigma'(\rho^1)\nabla\rho^1 + \sigma(\rho^1)\nabla f_k\right)\left(f_j\sigma'(\rho^2)\nabla\rho^2 + \sigma(\rho^2)\nabla f_j\right) d\langle B^k_\cdot, B^j_\cdot\rangle_s \bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2} dx dx'\\
&= \sum_{k=1}^{\infty}\int_{U^2} \left(f_k\sigma'(\rho^1)\nabla\rho^1 + \sigma(\rho^1)\nabla f_k\right)\left(f_j\sigma'(\rho^2)\nabla\rho^2 + \sigma(\rho^2)\nabla f_j\right) \bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\, dx\, dx'\, ds.
\end{aligned}
$$

The above can be made rigorous by integrating against smooth approximations of the product $\phi_\beta\zeta_M\iota_\gamma$ and rather than the multiplication of delta functions, and using the integration by parts lemma, Lemma 3.4. One obtains

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\int_U d\langle \chi^{\epsilon,\delta}_{\cdot,1}, \chi^{\epsilon,\delta}_{\cdot,2}\rangle_s(y,\eta)\, \phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\, dy\, d\eta\\
&= \sum_{k=1}^{\infty}\int_0^t\int_{\mathbb{R}}\int_{U^3} \left(f_k\sigma'(\rho^1)\nabla\rho^1 + \sigma(\rho^1)\nabla f_k\right)\cdot\left(f_j\sigma'(\rho^2)\nabla\rho^2 + \sigma(\rho^2)\nabla f_j\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times \bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\, dx\, dx'\, dy\, d\eta\, ds.
\end{aligned}
$$

Putting this together, it follows from equation (9) and the subsequent above computations that we have the decomposition:

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\chi^{\epsilon,\delta}_{s,2}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\, dy\, d\eta\Big|_{s=0}^{t}\\
&\qquad\qquad\qquad\qquad = I^{err}_t + I^{meas}_t + I^{mix,cut}_t + I^{mix,mart}_t + I^{mix,cons}_t. \qquad (10)
\end{aligned}
$$

We put all four terms from the quadratic variation term into the error term and regroup the terms. Note that the addition of the artificial term in terms $I^{1,2,err}$ and $I^{2,1,err}$ factorises with a term from the quadratic variation and allows it to be controlled, for more detail see equation (4.24) and subsequent computations in [FG24]. Similarly, the measure term just arises from the first two components of (9),

$$
I^{meas}_t = I^{1,2,meas}_t + I^{2,1,meas}_t.
$$

The contribution from the mixed term (third term of (8)) in the cutoff, martingale and conservative terms just comes from the sum of the first two terms of equation (9) and are denoted as mixed terms above in (10). Finally, we return back to the equation of interest that governs the L^1 difference of two solutions, equation (8). One has the decomposition

$$
\int_{\mathbb{R}}\int_U \left(\chi^{\epsilon,\delta}_{s,1} + \chi^{\epsilon,\delta}_{s,2} - 2\chi^{\epsilon,\delta}_{s,1}\chi^{\epsilon,\delta}_{s,2}\right)\phi_\beta\zeta_M\iota_\gamma\Big|_{s=0}^{t} = -2I^{err}_t - 2I^{meas}_t + I^{mart}_t + I^{cut}_t + I^{cons}_t. \qquad (11)
$$

The error and measure term were defined above and arise solely from the mixed term (10), the final term on the left hand side of (11). The martingale, cutoff and conservative terms arise from all three terms in the left hand side of equation (11),

$$
I^{mart,cut,cons}_t = I^{1,mart,cut,cons}_t + I^{2,mart,cut,cons}_t - 2I^{mix,mart,cut,cons}_t.
$$

Let us deal with each term on the right hand side of (11) separately.

Measure term.

Firstly, by Hölder's inequality and the regularity property (point three) in Definition 2.8 of stochastic kinetic solution, we have

$$I_t^{meas} \geq 0.$$

Error term.

Following the computations from equation (4.24) to (4.26) of [FG24], we have

$$\limsup_{\delta\to 0}\left(\limsup_{\epsilon\to 0}|I_t^{err}|\right) = 0.$$

Cutoff term.

We have for every $\beta \in (0,1), M \in \mathbb{N}, \gamma > 0$ sufficiently small, $\delta \in (0,\beta/4), \epsilon \in (0,\gamma/4)$ that for every $t \in [0,T]$,

$$\begin{aligned}
&I_t^{cut} := I_t^{1,cut} + I_t^{2,cut} - 2I_t^{mix,cut}\\
&= \int_0^t\int_{\mathbb{R}^2}\int_{U^2}\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)(-1+2\chi_{s,2}^{\epsilon,\delta})\,dq^1(x,\xi,s)\,dy\,d\eta\\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1\cdot F_2+\sigma(\rho^1)^2F_3\right)\bar{k}_{s,1}^{\epsilon,\delta}(1-2\chi_{s,2}^{\epsilon,\delta})\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dy\,dx\,ds\,d\eta\\
&+\int_0^t\int_{\mathbb{R}^2}\int_{U^2}\kappa^{\epsilon,\delta}(y,x',\eta,\xi)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)(-1+2\chi_{s,1}^{\epsilon,\delta})\,dq^2(x',\xi,s)\,dy\,d\eta\\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2+\sigma(\rho^2)^2F_3\right)\bar{k}_{s,2}^{\epsilon,\delta}(1-2\chi_{s,1}^{\epsilon,\delta})\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dy\,dx\,ds\,d\eta\\
&+\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\Phi'(\rho^1)\nabla\rho^1+\frac{1}{2}F_1[\sigma'(\rho^1)]^2\nabla\rho^1+\frac{1}{2}\sigma'(\rho^1)\sigma(\rho^1)F_2\right)\\
&\qquad\qquad\times\bar{k}_{s,1}^{\epsilon,\delta}\zeta_M(\eta)\phi_\beta(\eta)\cdot\nabla\iota_\gamma(y)(-1+2\chi_{s,2}^{\epsilon,\delta})\,dy\,dx\,ds\,d\eta\\
&+\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\Phi'(\rho^2)\nabla\rho^2+\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2+\frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\\
&\qquad\qquad\times\bar{k}_{s,2}^{\epsilon,\delta}\zeta_M(\eta)\phi_\beta(\eta)\cdot\nabla\iota_\gamma(y)(-1+2\chi_{s,1}^{\epsilon,\delta})\,dy\,dx'\,ds\,d\eta.
\end{aligned}\tag{12}$$

Let us begin by bounding the final two lines of I_t^{cut} above, comprising of the new terms involving gradients of the spacial cutoff. We take the $\epsilon,\delta\to 0$ limits first and use the distributional inequality for $i,j\in\{1,2\}$,

$$\begin{aligned}
\lim_{\epsilon,\delta\to 0}\bar{k}_{s,i}^{\epsilon,\delta}(x,y,\eta,\rho)(-1+2\chi_{s,j}^{\epsilon,\delta}(y,\eta)) &\to \delta_0(x-y)\delta_0(\eta-\rho^i)sgn(\rho^j-\eta)\\
&= \delta_0(x-y)\delta_0(\eta-\rho^i)sgn(\rho^j-\rho^i).
\end{aligned}\tag{13}$$

This means that the final two lines of the cutoff (12) can be realised in the $\epsilon,\delta\to 0$ as

$$\begin{aligned}
&\int_0^t\int_U\left(\Phi'(\rho^1)\nabla\rho^1+\frac{1}{2}F_1[\sigma'(\rho^1)]^2\nabla\rho^1+\frac{1}{2}\sigma'(\rho^1)\sigma(\rho^1)F_2\right)\\
&\qquad\qquad\times\zeta_M(\rho^1)\phi_\beta(\rho^1)\cdot\nabla\iota_\gamma(y)sgn(\rho^2-\rho^1)\,dy\,ds\\
&+\int_0^t\int_{U^2}\left(\Phi'(\rho^2)\nabla\rho^2+\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2+\frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\\
&\qquad\qquad\times\zeta_M(\rho^2)\phi_\beta(\rho^2)\cdot\nabla\iota_\gamma(y)sgn(\rho^1-\rho^2)\,dy\,ds.
\end{aligned}\tag{14}$$

The terms of the two lines are combined using the fact that $sgn(\rho^1-\rho^2) = -sgn(\rho^2-\rho^1)$.

We will deal with terms that have a factor of $\nabla\rho$ and terms that do not separately. Let's consider the first terms in both the lines of (14). Start by defining the function $\Phi_{M,\beta}$ to be the unique function such that $\Phi_{M,\beta}(0) = 0$ and

$$\Phi'_{M,\beta}(\xi) = \zeta_M(\xi)\phi_\beta(\xi)\Phi'(\xi) \geq 0.$$

This says that the function $\Phi_{M,\beta}$ is non-decreasing. Hence, with the convention that $sgn(0) = 1$, using Remark 3.3 to define the spacial derivative of the cutoff, the notation $v_y := \frac{y-y*}{|y-y*|}$ for the inward pointing unit normal, and the fundamental theorem of calculus, we can show that the difference of the first terms of (14) is non-negative

$$\begin{aligned}
&-\int_0^t \int_U \left(\nabla\Phi_{M,\beta}(\rho^2) - \nabla\Phi_{M,\beta}(\rho^1)\right) \cdot \nabla\iota_\gamma(y)\, sgn(\rho^2 - \rho^1)\, dy\, ds \\
&= -\int_0^t \int_U \left(\nabla\Phi_{M,\beta}(\rho^2) - \nabla\Phi_{M,\beta}(\rho^1)\right) \cdot \nabla\iota_\gamma(y)\, sgn(\Phi_{M,\beta}(\rho^2) - \Phi_{M,\beta}(\rho^1))\, dy\, ds \\
&= -\int_0^t \int_U \nabla|\Phi_{M,\beta}(\rho^2) - \Phi_{M,\beta}(\rho^1)| \cdot \nabla\iota_\gamma(y)\, dy\, ds \\
&= -\gamma^{-1}\int_0^t \int_{U\setminus U_\gamma} \nabla|\Phi_{M,\beta}(\rho^2) - \Phi_{M,\beta}(\rho^1)| \cdot v_y\, dy\, ds \\
&= -\gamma^{-1}\int_0^t \int_0^\gamma \int_{\partial U_z} \nabla|\Phi_{M,\beta}(\rho^2(y^* + zv_y, s)) - \Phi_{M,\beta}(\rho^1(y^* + zv_y, s))| \cdot v_y\, dy\, dz\, ds \\
&= -\gamma^{-1}\int_0^t \int_0^\gamma \int_{\partial U_z} \frac{\partial}{\partial z}|\Phi_{M,\beta}(\rho^2(y^* + zv_y, s)) - \Phi_{M,\beta}(\rho^1(y^* + zv_y, s))|\, dy\, dz\, ds \\
&= \gamma^{-1}\int_0^t \int_{\partial U} |\Phi_{M,\beta}(\rho^2) - \Phi_{M,\beta}(\rho^1)| - \gamma^{-1}\int_0^t \int_{\partial U_\gamma} |\Phi_{M,\beta}(\rho^2) - \Phi_{M,\beta}(\rho^1)| \leq 0. \qquad (15)
\end{aligned}$$

The first term on the final line vanishes since the solutions coincide on the boundary, and the second term is non-positive because the integrand is clearly non-negative for every fixed $\gamma > 0$. By repeating the same arguments, noting that $\frac{1}{2}F_1(\sigma'(\rho^2))^2 \geq 0$, one can conclude that the combination of second terms of (14) are non-positive for every $\gamma > 0$. For the final term of (14), we have by Remark 3.3 as well as by the boundedness of F_2 and sgn

$$\begin{aligned}
&\frac{1}{2}\int_0^t \int_U \left(\sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2) - \sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)\right) sgn(\rho^2 - \rho^1)F_2 \cdot \nabla\iota_\gamma(y)\, dy\, ds \\
&\leq c\gamma^{-1}\int_0^t \int_U \left|\sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2) - \sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)\right| \mathbb{1}_{U\setminus U_\gamma}(y)\, dy\, ds.
\end{aligned}$$

For ease of notation, for every M, β define the function $G_{M,\beta}$ by $G_{M,\beta}(\xi) = \sigma'(\xi)\sigma(\xi)\zeta_M(\xi)\phi_\beta(\xi)$, which is bounded and Lipschitz due to the fact that σ and σ' are locally lipschitz, and also it is clearly zero outside $[\beta/2, M+1]$. For every $y \in U$ sufficiently close to the boundary (i.e. γ sufficiently small above) let $y^* = y^*(y)$ denote the unique closest point on the boundary to y. For $i = 1, 2$ we denote $\rho^i(y^*) := \Phi^{-1}\left(\bar{f}(y^*)\right)$ and $\rho^i_{M,\beta}(y^*) := (\rho^i(y^*) \vee \beta/2) \wedge (M+1)$ for $y^* \in \partial U$. By adding and subtracting this boundary data, the triangle inequality and Lipschitz property of $G_{M,\beta}$, we have

$$\begin{aligned}
&c\gamma^{-1}\int_0^t \int_{U\setminus U_\gamma} \left|\sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2) - \sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)\right|\, dy\, ds \\
&\leq c\gamma^{-1}\int_0^t \int_{U\setminus U_\gamma} \left(|G_{M,\beta}(\rho^2) - G_{M,\beta}(\rho^2(y^*))| + |G_{M,\beta}(\rho^1(y^*)) - G_{M,\beta}(\rho^1)|\right)\, dy\, ds \\
&\leq c\gamma^{-1}\int_0^t \int_{U\setminus U_\gamma} \left(|\rho^2_{M,\beta}(y,s) - \rho^2_{M,\beta}(y^*,s)| + |\rho^1_{M,\beta}(y^*,s) - \rho^1_{M,\beta}(y,s)|\right)\, dy\, ds. \qquad (16)
\end{aligned}$$

Both of the above terms can be handled in the same way. Write for $i = 1, 2$, fixed distance from the boundary $\gamma' \in (0,\gamma)$, fixed time $s \in [0,t]$, and for running constant $c \in (0,\infty)$,

$$\begin{aligned}
\int_{\partial U_{\gamma'}} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*,s)| &= \int_{\partial U_{\gamma'}} \left| \int_0^{\gamma'} \nabla \rho^i_{M,\beta}\left(y^* + z v_y, s\right)\, dz \right| \\
&\le \int_{U\setminus U_{\gamma'}} |\nabla \rho^i_{M,\beta}(y,s)|\, dy \\
&\le |U \setminus U_{\gamma'}|^{1/2}\, \|\nabla \rho^i_{M,\beta}\|_{L^2(U\setminus U_{\gamma'})} \\
&\le c(\gamma')^{1/2}\, \|\nabla \rho^i_{M,\beta}\|_{L^2(U\setminus U_{\gamma'})} \\
&\le c(\gamma')^{1/2}\, \|\nabla \rho^i_{M,\beta}\|_{L^2(U)},
\end{aligned}$$

where in the final line we made the norm independent of γ'. We therefore bound the terms of (16) by

$$\begin{aligned}
c\,\gamma^{-1} \int_0^t \int_{U\setminus U_\gamma} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*)|\, dy\, ds &= c\,\gamma^{-1} \int_0^t \int_0^\gamma \int_{\partial U_{\gamma'}} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*)|\, dy\, d\gamma'\, ds \\
&\le c\,\gamma^{-1} \int_0^t \|\nabla \rho^i_{M,\beta}\|_{L^2(U)} \left(\int_0^\gamma \sqrt{\gamma'}\, d\gamma' \right) ds \\
&\le c\,\gamma^{1/2} \|\nabla \rho^i_{M,\beta}\|_{L^1([0,t];L^2(U))},
\end{aligned}$$

which converges to 0 as $\gamma \to 0$ for fixed M, β. Therefore we managed to show that the final two lines of the cutoff (12) consisting of the new terms are non-positive in the $\gamma \to 0$ limit. The below is an important point about the remaining cutoff terms, as well as martingale and conservative terms that are yet to be analysed.

Following the computations in the uniqueness proof in [FG24], the terms involving F_2 in the second and fourth lines of the cutoff term (12), as well as martingale and conservative terms are handled by integration by parts. More precisely the ϵ, δ limits are taken first, then integration by parts is performed before taking M, β limits. On the bounded domain, the presence of a spacial cutoff ι_γ leads to additional terms with factors $\nabla \iota_\gamma$ when integrating by parts. We emphasize that these new terms can be bounded in a similar way to the third term in (14). We will explain how to bound these terms, which will allow us to conclude.

Firstly, to bound the terms involving F_2 in the second and fourth line of (12), analogously to above, in the $\epsilon, \delta \to 0$ limit we use the distributional equality (13), followed by the equality $sgn(\rho^1 - \rho^2) = -sgn(\rho^2 - \rho^1)$ and finally the product rule to evaluate the derivative of the cutoffs to get

$$\begin{aligned}
&\lim_{\epsilon,\delta\to 0} \Bigg(\frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{U^2} \sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1 \cdot F_2 \bar{k}^{\epsilon,\delta}_{s,1}(1 - 2\chi^{\epsilon,\delta}_{s,2}) \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\, \iota_\gamma(y)\, dy\, dx\, ds\, d\eta \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{U^2} \sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2 \cdot F_2 \bar{k}^{\epsilon,\delta}_{s,2}(1 - 2\chi^{\epsilon,\delta}_{s,1}) \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\, \iota_\gamma(y)\, dy\, dx\, ds\, d\eta \Bigg) \\
&= \frac{1}{2} \int_0^t \int_U \left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2 - \sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1 \right) \cdot F_2 \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta)) sgn(\rho^2 - \rho^1)\, \iota_\gamma\, dx\, ds \\
&= \frac{1}{4} \int_0^t \int_U \left(\nabla\sigma^2(\rho^2) - \nabla\sigma^2(\rho^1) \right) \cdot F_2 \left(\mathbb{1}_{M<\rho<M+1} + \beta^{-1} \mathbb{1}_{\beta/2<\rho<\beta} \right) sgn(\rho^2 - \rho^1)\, \iota_\gamma\, dx\, ds \\
&= \frac{1}{4\beta} \int_0^t \int_U \nabla \left(\sigma^2\left((\rho^2 \vee \beta) \wedge \beta/2 \right) - \sigma^2\left((\rho^1 \vee \beta) \wedge \beta/2 \right) \right) \cdot F_2 sgn(\rho^2 - \rho^1)\, \iota_\gamma\, dx\, ds \\
&\quad + \frac{1}{4} \int_0^t \int_U \nabla \left(\sigma^2\left((\rho^2 \vee (M+1)) \wedge M \right) - \sigma^2\left((\rho^1 \vee (M+1)) \wedge M \right) \right) \cdot F_2 sgn(\rho^2 - \rho^1)\, \iota_\gamma\, dx\, ds.
\end{aligned}$$

Note that σ^2 is not necessarily increasing, so we can't use that $sgn(\rho^2 - \rho^1) = sgn(\sigma^2(\rho^2) - \sigma^2(\rho^1))$ as in (15). Instead we smooth out the sign function by writing $sgn^\delta := sgn * \kappa_1^\delta$ for $\delta \in (0,1)$ before integrating by parts. The terms involving M are handled in the same way as the terms involving

β, so we illustrate how to handle the β term. For convenience introduce the shorthand notation $\rho^i_\beta(x,t) := \big(\rho^i(x,t)\vee\beta\big)\wedge\beta/2$ for $i=1,2$, $(x,t)\in U\times[0,T]$. The terms involving β can be written as

$$\begin{aligned}
&\frac{1}{4\beta}\int_0^t\int_U \nabla\left(\sigma^2(\rho^2_\beta)-\sigma^2(\rho^1_\beta)\right)\cdot F_2 sgn(\rho^2-\rho^1)\iota_\gamma\,dx\,ds\\
&=\lim_{\delta\to0}\frac{1}{4\beta}\int_0^t\int_U \nabla\left(\sigma^2(\rho^2_\beta)-\sigma^2(\rho^1_\beta)\right)\cdot F_2 sgn^\delta(\rho^2-\rho^1)\iota_\gamma\,dx\,ds\\
&=-\lim_{\delta\to0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta)-\sigma^2(\rho^1_\beta)\right)\nabla\cdot F_2 sgn^\delta(\rho^2-\rho^1)\iota_\gamma\,dx\,ds\\
&\quad-\lim_{\delta\to0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta)-\sigma^2(\rho^1_\beta)\right)\nabla(\rho^2-\rho^1)\cdot F_2(sgn^\delta)'(\rho^2-\rho^1)\iota_\gamma\,dx\,ds\\
&\quad-\lim_{\delta\to0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta)-\sigma^2(\rho^1_\beta)\right)\nabla\iota_\gamma\cdot F_2\, sgn^\delta(\rho^2-\rho^1)\,dx\,ds.
\end{aligned}$$

For the first two terms we can directly take the $\gamma\to0$ limit since the cutoff converges point-wise to 1 and so they can be handled analogously as on the torus, see the computation leading from equation (4.28) to equation (4.31) in [FG24]. We only need to consider the final term involving the gradient of spacial cutoff, which we can bound in an analogous way to the final term in equation (14) after realising that $\sigma^2(\cdot\vee\beta\wedge\beta/2)$ is Lipschitz for every fixed $\beta>0$, F_2 and sgn^δ are bounded, and noting we take $\gamma\to0$ limit before M and β limits.
To show that the first and third lines of the cutoff term (12) vanish we use precisely the decays of the kinetic measure at zero and infinity. Putting (12) and subsequent computations together, we conclude

$$\lim_{M\to\infty,\beta\to0}\lim_{\gamma\to0}\lim_{\epsilon,\delta\to0} I^{cut}_t\le 0.$$

Martingale term.
Following the analysis from equation (4.27) to (4.36) presented in [FG24], we have that, for the unique function $\Theta_{M,\beta}:[0,\infty)\to[0,\infty)$ defined by $\Theta_{M,\beta}(0)=0$, $\Theta'_{M,\beta}(\xi)=\phi_\beta(\xi)\zeta_M(\xi)\sigma'(\xi)$

$$\begin{aligned}
\lim_{\epsilon,\delta\to0} I^{mart}_t &= \int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\nabla\cdot\left(\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)d\xi^F\right)\\
&+\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\left(\phi_\beta(\rho^1)\zeta_M(\rho^1)\sigma(\rho^1)-\Theta_{M,\beta}(\rho^1)\right)\nabla\cdot d\xi^F\\
&-\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\left(\phi_\beta(\rho^2)\zeta_M(\rho^2)\sigma(\rho^2)-\Theta_{M,\beta}(\rho^2)\right)\nabla\cdot d\xi^F.
\end{aligned}$$

The final two terms can be handled directly as in [FG24] by first directly taking the $\gamma\to0$ limit. For the first term, using again the regularisation of the sign function and subsequently integrating by parts gives two terms. When the derivative hits the regularised sign, the term can be handled in the same way as [FG24] after immediately taking the $\gamma\to0$ limit, and we get a new term when the derivative hits the spacial cutoff,

$$\int_0^t\int_U sgn^\delta(\rho^2-\rho^1)\left(\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\right)\nabla\iota_\gamma\cdot d\xi^F.$$

By using the Burkholder-Davis-Gundy inequality (see for example Theorem 4.1 of [RY13]), the boundedness of the sgn^δ as well as the bound on the derivative of the spacial cutoff given in Remark

3.3, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\int_U sgn^\delta(\rho^2-\rho^1)\left((\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\nabla\iota_\gamma\cdot d\xi^F\right|\right)\\
&\leq c\mathbb{E}\left(\int_0^T\left(\int_U sgn^\delta(\rho^2-\rho^1)\left((\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)|\nabla\iota_\gamma|\right)^2 ds\right)^{1/2}\\
&\leq c\gamma^{-1}\,\mathbb{E}\left(\int_0^T\left(\int_U\left|(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right|\mathbb{1}_{U\setminus U_\gamma}\right)^2 ds\right)^{1/2}.
\end{aligned}
$$

The inner spacial integral is handled in an analogous way to the final term of (14) after once again noticing $\Theta_{M,\beta}$ is Lipschitz due to the cutoffs. By Young's inequality we can write

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\int_U sgn(\rho^2-\rho^1)\left((\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\nabla\iota_\gamma\cdot d\xi^F\right|\right)\\
&\leq c\gamma^{-1}\,\mathbb{E}\left(\int_0^T\left(\sum_{i=1}^2\|\nabla\rho^i_{M,\beta}\|_{L^2(U)}\gamma^{3/2}\right)^2 ds\right)^{1/2}\\
&\leq c\gamma^{1/2}\sum_{i=1}^2\mathbb{E}\|\nabla\rho^i_{M,\beta}\|_{L^2([0,T];L^2(U))}.
\end{aligned}
$$

The final term converges to zero in the $\gamma\to 0$ limit for fixed M,β. This implies almost sure convergence of this new term along a sub-sequence, that is

$$
\lim_{M\to\infty,\beta\to 0}\left(\lim_{\gamma\to 0}\left(\lim_{\epsilon,\delta\to 0} I_t^{mart}\right)\right)=0.
$$

Conservative term.
The same arguments as the martingale term, in particular equation (4.31) to (4.35) of [FG24], also apply here. After again taking the ϵ,δ limit we have

$$
\lim_{\epsilon,\delta\to 0} I_t^{cons}=\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\left(\nabla\cdot\nu(\rho^1)\zeta_M\phi_\beta(\rho^1)-\nabla\cdot\nu(\rho^2)\zeta_M\phi_\beta(\rho^2)\right).
$$

First define the Lipschitz vector valued function $\Psi_{M,\beta,\nu}=(\Psi_{M,\beta,\nu,i})_{i=1}^d$ such that for $\nu=(\nu_i)_{i=1}^d$

$$
\Psi_{M,\beta,\nu,i}(0)=0,\quad \frac{\partial\Psi_{M,\beta,\nu,i}}{\partial x_i}(\xi)=\frac{\partial^2\nu_i}{\partial x_i^2}(\xi)\phi_\beta(\xi)\zeta_M(\xi).
$$

Then with a similar re-writing as the martingale term, we have

$$
\begin{aligned}
\lim_{\epsilon,\delta\to 0} I_t^{cons}&=\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\nabla\cdot\left(\Psi_{M,\beta,\nu}(\rho^1)-\Psi_{M,\beta,\nu}(\rho^2)\right)\\
&\quad+\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\left(\nabla\cdot\nu(\rho^1)\zeta_M\phi_\beta(\rho^1)-\nabla\cdot\Psi_{M,\beta,\nu}(\rho^1)\right)\\
&\quad-\int_0^t\int_U sgn(\rho^2-\rho^1)\iota_\gamma\left(\nabla\cdot\nu(\rho^2)\zeta_M\phi_\beta(\rho^2)-\nabla\cdot\Psi_{M,\beta,\nu}(\rho^2)\right).
\end{aligned}
$$

The final two terms can be handled analogously to the final two terms appearing in the martingale term after taking a $\gamma\to 0$ limit. The first term can be handled using integration by parts, analogous to the martingale term as shown above. Here we use the $L^1(U)$ integrability of $\nu(\rho)$ and the final assumption of Assumption 3.1 to apply the dominated convergence theorem, and conclude that along subsequences

$$\lim_{M\to\infty,\beta\to 0}\left(\lim_{\gamma\to 0}\left(\lim_{\epsilon,\delta\to 0} I_t^{cons}\right)\right)=0.$$

Conclusion.

Putting everything together we get from (11) and subsequent handling of each term, that there are random sub-sequences $\epsilon,\delta,\beta,\gamma\to 0$, $M\to\infty$ along which

$$\begin{aligned}\int_{\mathbb{R}}\int_U |\chi_s^1-\chi_s^2|^2\Big|_{s=0}^t &= \lim_{\beta\to 0,M\to\infty}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0}\int_{\mathbb{R}}\int_U |\chi_{s,1}^{\epsilon,\delta}-\chi_{s,2}^{\epsilon,\delta}|^2\phi_\beta\zeta_M\iota_\gamma\Big|_{s=0}^t\\ &= \lim_{\beta\to 0,M\to\infty}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0}\left(-2I_t^{err}-2I_t^{meas}+I_t^{mart}+I_t^{cut}+I_t^{cons}+I_t^{bound}\right)\leq 0.\end{aligned}$$

This implies that

$$\int_U|\rho^1(\cdot,t)-\rho^2(\cdot,t)|=\int_{\mathbb{R}}\int_U|\chi_t^1-\chi_t^2|^2\leq\int_{\mathbb{R}}\int_U|\chi_0^1-\chi_0^2|^2=\int_U|\rho_0^1-\rho_0^2|.$$

□

4 Existence

In this chapter we construct a stochastic kinetic solution of the generalised Dean-Kawasaki Equation (3) in the sense of Definition 2.8. The existence consists of three steps. Firstly, in Section 4.1 we will prove $L^2(U)$ energy estimates for a suitable regularised version of (3). To do this we will use the regularised equation (4) and smooth the non-linearity σ. We then proceed to prove further space-time regularity results for weak solutions of the regularised equation.
As an aside, in 4.2 is dedicated to proving an entropy estimate for the equation and a localised version of this argument helps us to prove a statement about the Kinetic measure at zero in Section 4.3. For all of the energy estimates we will need to introduce harmonic PDE's (for example in Definitions 4.8 and 4.22) that allow certain functions to vanish along the boundary when applying Itô's formula.
The second and third steps are analogous to Chapter 5 of [FG24], and so we are brief in their presentation. For the second step, in the first part of Section 4.4 we show that there exists a stochastic kinetic solution to the regularised equation. Since all the coefficients are regular, the proof follows by a projection argument, where the projected system is just a finite dimensional system of stochastic differential equations and so has a unique strong solution. The final step, illustrated in the latter half of Section 4.4, requires us to pass to the limit in the regularisation.

4.1 A priori estimates for the regularised equation

In this section we start with some definitions as well as stating the relevant assumptions needed for uniqueness. The main result of the section is the subsequent $L^2(U)$ energy estimate of the regularised equation in Proposition 4.14. We conclude by proving some higher order spacial regularity of the solution in Corollary 4.15 and higher order space-time regularity of the solution cutoff away from zero in Proposition 4.18 that will be essential in the tightness arguments.
The estimates will be proven with respect to the regularised equation (4), which we recall is given by

$$\partial_t\rho^\alpha=\Delta\Phi(\rho^\alpha)+\alpha\Delta\rho^\alpha-\nabla\cdot(\sigma(\rho^\alpha)\dot{\xi}^F+\nu(\rho^\alpha))+\frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho^\alpha)]^2\nabla\rho^\alpha+\sigma'(\rho^\alpha)\sigma(\rho^\alpha)F_2),$$

defined for $\alpha\in(0,1)$ with boundary condition $\Phi(\rho^\alpha)=\bar{f}$. For ease of notation when proving estimates about the regularised equation we denote the regularised equation by ρ instead of ρ^α. Motivated with trying to write the factor $\Phi'(\rho)|\nabla\rho|^2$ in the regularised kinetic measure as a single gradient term, we introduce the below auxiliary function.

Definition 4.1 (Auxiliary function corresponding to Φ). *Let Φ be any $C([0,\infty)]) \cap C^1_{loc}(0,\infty)$ function that is strictly increasing with $\Phi(0) = 0$. Define $\Theta_\Phi \in C([0,\infty)]) \cap C^1_{loc}(0,\infty)$ to be the unique function satisfying $\Theta_\Phi(0) = 0$ and*

$$\Theta'_\Phi(\xi) = (\Phi'(\xi))^{1/2}.$$

We now state the assumptions needed for existence. Some of the assumptions overlap with the uniqueness assumptions, Assumption 3.1.

Assumption 4.2 (Existence assumptions). *Suppose $\Phi, \sigma \in C([0,\infty)) \cap C^1_{loc}((0,\infty))$ and $\nu \in C([0,\infty);\mathbb{R}^d) \cap C^1_{loc}((0,\infty);\mathbb{R}^d)$ satisfy the six assumptions in the same spirit as Assumption 5.2 of [FG24]:*

1. *We have $\Phi(0) = \sigma(0) = 0$ and $\Phi' > 0$ on $(0,\infty)$.*
2. *There exists constants $m \in (0,\infty), c \in (0,\infty)$ such that for every $\xi \in [0,\infty)$*
$$\Phi(\xi) \leq c(1 + \xi^m).$$
3. *There is a constant $c \in (0,\infty)$ such that, we have for every $\xi \in [0,\infty)$*
$$\Phi'(\xi) \leq c(1 + \xi + \Phi(\xi)).$$
4. *For Θ_Φ defined in Definition 4.1, we have that either for constants $c \in (0,\infty)$ and $\theta \in [0,1/2]$ that for every $\xi \in (0,\infty)$,*
$$(\Theta'_\Phi(\xi))^{-1} := \Phi'(\xi)^{-1/2} \leq c\xi^\theta, \tag{17}$$
or we have constants $c \in (0,\infty)$, $q \in [1,\infty)$ such that for every $\xi, \eta \in [0,\infty)$
$$|\xi - \eta|^q \leq c|\Theta_\Phi(\xi) - \Theta_\Phi(\eta)|^2. \tag{18}$$
5. *For a constant $c \in (0,\infty)$ and every $\xi \in [0,\infty)$ we have*
$$\sigma^2(\xi) \leq c(1 + \xi + \Phi(\xi)).$$
6. *For each $\delta \in (0,1)$ there is a constant $c_\delta \in (0,\infty)$ such that for every $\xi \in (\delta,\infty)$,*
$$\frac{(\sigma'(\xi))^4}{\Phi'(\xi)} \leq c_\delta(1 + \xi + \Phi(\xi)).$$

Furthermore we have the additional new assumptions, used to bound new boundary terms that arise in the estimates:

7. *For a constant $c \in (0,\infty)$ and every $\xi \in (0,\infty)$ we have*
$$\Theta_\Phi(\xi) \geq c(\xi^{\frac{m+1}{2}} - 1).$$
8. *There is a constant $c \in (0,\infty)$ such that for every $\xi \in [0,\infty)$*
$$|\nu(\xi)|^2 + |\sigma(\xi)\sigma'(\xi)|^2 \leq c(1 + \xi + \Phi(\xi)).$$
9. *The anti-derivative of ν, defined element-wise for $i = 1,\ldots,d$, by $\Theta_{\nu,i}(0) = 0$, $\Theta'_{\nu,i}(\xi) = \nu_i(\xi)$ satisfies for $i = 1\ldots,d$ that $\Theta_{\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}_i \in L^1(\partial U)$, where $\bar{f}$ is the boundary condition for the regularised equation.*
10. *We have $\sigma^2(\Phi^{-1}(\bar{f})) \in L^1(\partial U)$.*

11. *Either $\bar{f}$ is constant, or for the unique function Ψ_σ defined by $\Psi_\sigma(1) = 0$, $\Psi'_\sigma(\xi) = F_1[\sigma'(\xi)]^2$, we have*
$$\bar{f} \in L^2(\partial U), \quad \Phi^{-1}(\bar{f}) \in L^2(\partial U), \quad \Psi_\sigma(\Phi^{-1}(\bar{f})) \in L^2(\partial U).$$

Remark 4.3. *The fourth point in the above assumption enables us to consider $\Phi(\xi) = \xi^m$ for every $m \in (0,\infty)$. If $m < 1$ then $\Phi'(\xi)^{-1/2} = m^{-1/2}\xi^{\frac{1-m}{2}}$ so satisfies* (17). *On the other hand, if $m \geq 1$ then by Remark 4.4 we have $c|\Theta_\Phi(\xi) - \Theta_\Phi(\eta)|^2 = cm|\xi^{\frac{m+1}{2}} - \eta^{\frac{m+1}{2}}|^2$ so satisfies* (18) *with $q = m+1$.*

Remark 4.4 (Growth assumption on Θ_Φ)**.** *The lower bound on the growth of Θ_Φ in point seven of the above assumption is essential for obtaining $L^k(U)$ estimates of the solution in Proposition 4.12 below. Formally, one should think that in order to obtain $L^k(U)$ estimates for the solution, one needs to harness the additional regularity in equation* (4) *coming from when the Laplacian acts on $\Phi(\rho) = \rho^m$. In the model case $\Phi'(\xi) = m\xi^{m-1}$ and so the assumption is satisfied since*
$$\Theta_\Phi(\xi) = m^{1/2}\int_0^\xi \eta^{(m-1)/2}\, d\eta = \frac{2m^{1/2}}{m+1}\xi^{(m+1)/2}.$$

Remark 4.5 (Constraint on boundary conditions)**.** *For the final assumption in the final point, we have in the model case that $\Psi_\sigma(\xi) \sim \log(\xi)$. Hence the assumption still enables us to handle boundary data $\bar{f}$ that is constant or uniformly bounded away from zero.*

We once again need to deal with singularities from the Itô to Stratonovich conversion and ensure the integrals below are well defined, so need the below assumption smoothing σ. We state it as a separate assumption since it is not necessary and will subsequently be dispensed of via an approximation argument in Lemma 4.32.

Assumption 4.6. *Let $\sigma \in C([0,\infty)) \cap C^\infty((0,\infty))$ with $\sigma(0) = 0$ and $\sigma' \in C_c^\infty([0,\infty))$.*

For the regularised equation with smoothed σ we can make sense of a weak solution:

Definition 4.7 (Weak solution of regularised equation (4))**.** *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2, 4.2 and 4.6. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable. A weak solution ρ of* (4) *with initial condition ρ_0 is a continuous $L^2(U)$ valued, non-negative $\mathcal{F}_t$-predictable process such that $\rho - g \in L^2([0,T]; H_0^1(U))$ and $\Theta_\Phi(\rho) \in L^2([0,T]; H^1(U))$, and such that for every $\psi \in C_c^\infty(U)$, almost surely for every $t \in [0,T]$,*
$$\begin{aligned}\int_U \rho(x,t)\psi(x)\,dx = &\int_U \rho_0\psi\,dx - \int_0^t\int_U \Phi'(\rho)\nabla\rho\cdot\nabla\psi\,dx\,dt - \alpha\int_0^t\int_U \nabla\rho\cdot\nabla\psi\,dx\,dt \\ &+ \int_0^t\int_U \nu(\rho)\cdot\nabla\psi\,dx\,dt + \int_0^t\int_U \sigma(\rho)\nabla\psi\cdot\,d\xi^F\,dt \\ &- \frac{1}{2}\int_0^t\int_U F_1(\sigma'(\rho))^2\nabla\rho\cdot\nabla\psi\,dx\,dt - \frac{1}{2}\int_0^t\int_U \sigma(\rho)\sigma'(\rho)F_2\cdot\nabla\psi\,dx\,dt.\end{aligned}$$

We introduce two PDEs that will allow us to avoid boundary terms in our energy estimate, Proposition 4.14.

Definition 4.8 (The PDEs g and h_M)**.** *Let $\bar{f}$ be the boundary condition in the regularised equation* (4)*. Define $g : U \to \mathbb{R}$ to be the below harmonic function that captures the regularity of the solution on the boundary*
$$\begin{cases} -\Delta g = 0 & \text{on } U, \\ g = \Phi^{-1}(\bar{f}) & \text{on } \partial U. \end{cases}$$
For the function $S_M : [0,\infty) \to [0,\infty)$ satisfying $S''_M(\xi) = \mathbb{1}_{M_1<\xi<M_2}(\xi)$ with $0 < M_1 < M_2$, define $h_M : U \to \mathbb{R}$ to be the below harmonic function satisfying
$$\begin{cases} -\Delta h_M = 0 & \text{on } U, \\ h_M = S'_M(\Phi^{-1}(\bar{f})) & \text{on } \partial U. \end{cases}$$

Note that S'_M is $[0, M_2 - M_1]-$valued, and therefore by the maximum principle h_M is bounded by $M_2 - M_1$. Before stating the assumption we will need on g, we first state an important remark motivating the harmonic PDEs introduced in this paper.

Remark 4.9 (Choice of harmonic PDEs.)**.** *The boundary data of the PDEs are chosen to ensure certain functions vanish along the boundary when applying Itô's formula, and consequently allows integration by parts without picking up boundary terms. They are all also chosen to be harmonic in U, and this is for several reasons, illustrated using the example of PDE g in equation* (23) *in Proposition 4.14 below.*

1. *Looking at the final line of* (23)*, several terms include integrands with a factor of ∇g multiplied by another gradient term. To bound these terms, integrating by parts and moving the derivative onto the ∇g, since g is harmonic we have that that they turn into boundary terms. That is, for $f : U \times [0,T] \to \mathbb{R}$ such that the below integrals are well defined, we have*

$$\int_U \nabla f(x,t) \cdot \nabla g(x) = -\int_U f(x,t)\Delta g(x) + \int_{\partial U} f(x^*,t)\frac{\partial g}{\partial \hat{\eta}}(x^*) = \int_{\partial U} f(x^*,t)\frac{\partial g}{\partial \hat{\eta}}(x^*).$$

2. *For the remaining terms in the final line of equation* (23)*, to handle integrands where ∇g is multiplied by terms without another gradient we use either Hölder's or Young's inequality. In both of these cases we have to bound the $L^2(U)$ norm of ∇g. Since g is harmonic, testing the PDE against the solution and integrating by parts allows us to estimate this quantity,*

$$\int_U |\nabla g|^2 = \int_{\partial U} g\frac{\partial g}{\partial \hat{\eta}}.$$

3. *It is a fascinating fact that on a C^1 domain, one can bound the $L^p(\partial U)$ norm of the normal derivative of a harmonic PDE simply by a norm of the boundary condition. Here we work over a C^2 domain so can apply the result.*
   To be more specific, consider as in [FJJR78] Laplace's equation $\Delta u = 0$ on U with $u|_{\partial U} = \bar{f}$. We define the $H^1(\partial U)$ norm of the boundary data $\bar{f}$ as the definition of the $L_1^p(\partial D)$ norm on page 176 of [FJJR78](with p = 2). Then we will repeatedly use a special case of Theorem 2.4 (iii) of [FJJR78] that says that there exists a constant $c \in (0,\infty)$ independent of $\bar{f}$ so that

$$\left\|\frac{\partial u}{\partial \hat{\eta}}\right\|_{L^2(\partial U)} \le c\|\bar{f}\|_{H^1(\partial U)}.$$

   It is clear that such a statement would be useful, for instance in bounding terms on the right hand side of the above two points. Importantly, this allows us to express bounds in terms of norms of the boundary data $\bar{f}$ directly rather than in terms of norms the corresponding PDEs.

With the above remark in mind, we state below assumptions.

Assumption 4.10 (Assumption on g)**.** *Either the boundary data $\bar{f}$ is constant, or using the PDE for g in Definition 4.8, we have $\Phi^{-1}(\bar{f}) \in H^1(\partial U)$ where the norm is defined as the $L_1^p(\partial D)$ norm in page 176 of [FJJR78]. Further assume that either the boundary data $\bar{f}$ is bounded, or we have $\Theta_\Phi(g) \in H^1(U)$.*

Remark 4.11 (Distinguishing constant and non-constant boundary conditions)**.** *The analysis in the energy estimates below (as well as the entropy estimate of Proposition 4.24) is far simpler in the case that the boundary condition is constant, say $\bar{f} = a \ge 0$. In this case the PDEs g and h_M as defined in Definition 4.8 are solved by constant functions, for instance*

$$g(x) = \Phi^{-1}(a), \qquad x \in \bar{U}.$$

Hence the gradient ∇g as well as the normal derivative $\frac{\partial g}{\partial \hat{\eta}}$ are zero, and so terms involving either of these factors vanish immediately.

We arrive to the first result of the section, and obtain a bound for powers of the solution by comparing it with the function Θ_Φ defined in Definition 4.1. Such an estimate is required because we are not on the torus so do not have preservation of mass, and nor do we have enough regularity to quantify the flux along the boundary, see Remark 4.26 below.

Proposition 4.12 (Estimate for $L^1_t L^k_x$ norm of the solution)**.** *Suppose that Φ satisfies the polynomial growth condition in point two of Assumption 4.2 and the PDE g as in Definition 4.8 satisfies Assumption 4.10. If ρ is a weak solution of the regularised equation (4) in the sense of Definition 4.7, then one has, for every $\epsilon > 0$ and $k \in (0, m+1)$, a constant $c \in (0,\infty)$ depending only on k, m and U,*

$$\int_0^T \int_U |\rho|^k \leq c \left(\frac{T}{\epsilon} + \epsilon T \|\Theta_\Phi(g)\|_{H^1(U)} + \epsilon \int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2 \right).$$

If the boundary data $\bar{f}$ is bounded by constant K, we obtain the simplified bound with constant again depending only on k, m and U,

$$\int_0^T \int_U |\rho|^k \leq c \left(\frac{T}{\epsilon} + \epsilon T + \epsilon \int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2 \right).$$

Remark 4.13. *Using Jensen's inequality, the above proposition also provides a bound for powers of the $L^1(U)$ norm of the solution, since*

$$\int_0^T \left(\int_U |\rho| \right)^k = |U|^k \int_0^T \left(\frac{1}{|U|} \int_U |\rho| \right)^k \leq |U|^{k-1} \int_0^T \int_U |\rho|^k.$$

The proposition implies that for the full range $m \in (0,\infty)$ we at least have an $L^1_t L^1_x$ estimate for the solution, since $k \in (0, m+1)$. Apart from the bounding the L^1_x norm, the result is useful when $m, k > 1$, since if $k < 1$ we can just use interpolation to obtain

$$\int_0^T \int_U |\rho|^k \leq \int_0^T \int_U (1 + |\rho|).$$

Proof. The bound on Θ_Φ given in point seven of Assumption 4.2 gives, for a running constant c depending on k, U and m,

$$\int_0^T \int_U |\rho|^k \leq cT + c \int_0^T \int_U |\Theta_\Phi(\rho)|^{\frac{2k}{m+1}}.$$

Note that here the exponent in the integrand is strictly less than two. Applying Young's inequality with ϵ and exponent $\frac{m+1}{k} > 1$ and Jensen's inequality then gives

$$\begin{aligned}
\int_0^T \int_U |\Theta_\Phi(\rho)|^{\frac{2k}{m+1}} &\leq \int_0^T \frac{(m+1-k) 1^{\frac{m+1}{m+1-k}}}{\epsilon(m+1)} + \frac{\epsilon k}{m+1} \left(\int_U \Theta_\Phi(\rho)^{\frac{2k}{m+1}} \right)^{\frac{m+1}{k}} \\
&\leq \frac{cT}{\epsilon} + c\epsilon \int_0^T \int_U \Theta_\Phi(\rho)^2.
\end{aligned}$$

Using the trivial inequality $a^2 \leq 2(a-b)^2 + b^2$ with $b = \Theta_\Phi(g)$, where the PDE g is as in Definition 4.8, and subsequently applying Poincaré inequality gives the first claim,

$$\begin{aligned}
\int_0^T \left(\int_U |\rho| \right)^k &\leq \frac{cT}{\epsilon} + c\epsilon \int_0^T \int_U (\Theta_\Phi(\rho) - \Theta_\Phi(g))^2 + c\epsilon \int_0^T \int_U \Theta_\Phi(g)^2 \qquad (19) \\
&\leq \frac{cT}{\epsilon} + c\epsilon \int_0^T \int_U |\nabla (\Theta_\Phi(\rho) - \Theta_\Phi(g))|^2 + c\epsilon T \int_U \Theta_\Phi(g)^2 \\
&\leq \frac{cT}{\epsilon} + c\epsilon \int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2 + c\epsilon T \|\Theta_\Phi(g)\|_{H^1(U)}.
\end{aligned}$$

For the second claim, if the boundary data is bounded by constant K, then we can use the comparison principle which tells us that

$$\int_0^T \left(\int_U |\rho| \right)^k \leq \int_0^T \left(\int_U |\tilde{\rho}| \right)^k ,$$

where $\tilde{\rho}$ solves the same equation as ρ but with boundary condition K. Repeating the steps above to bound the norm on the right hand side, we see that when we arrive to (19) we add and subtract the constant

$$\Theta_\Phi(K) = \frac{2m^{1/2}}{m+1} K^{\frac{m+1}{2}},$$

which can subsequently be absorbed into the running constant. This gives the second claim. $\square$

Proposition 4.14 (Energy estimate for solution of regularised equation with smooth, bounded σ)**.** *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2, 4.2 and 4.6, and let $\alpha \in (0,1)$, $T \in [1,\infty)$. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable, $\Theta_{\nu,i}$ and Ψ_σ be defined as in Assumption 4.2 and g, h_M be the PDEs defined in Definition 4.8 satisfying Assumption 4.10. If ρ is a weak solution of the regularised equation* (4) *in the sense of Definition 4.7 then one has for $c \in (0,\infty)$ independent of α, the estimate*

$$\begin{aligned}
&\frac{1}{2} \sup_{t\in[0,T]} \mathbb{E}\left(\int_U (\rho(x,t) - g(x))^2 \right) + \mathbb{E}\left(\int_U \int_0^T |\nabla \Theta_\Phi(\rho)|^2 \right) + \mathbb{E}\left(\alpha \int_U \int_0^T |\nabla \rho|^2 \right) \\
&\leq \frac{1}{2}\|\rho_0 - g\|^2_{L^2(U)} + T\left(c + c\|\Theta_\Phi(g)\|_{H^1(U)} + \sum_{i=1}^d \int_{\partial U} \Theta_{\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}_i + c\|\sigma^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)} \right) \\
&+ T\left(\|\bar{f}\|^2_{L^2(\partial U)} + \|\Psi_\sigma(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)} + c(1+\alpha)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)} \right). \qquad (20)
\end{aligned}$$

Let the function S_M and the PDE h_M be defined as in Definition 4.8, and for $i = 1, \ldots, d$ define the functions $\Theta_{M,\nu,i} : \mathbb{R} \to \mathbb{R}$ by $\Theta_{M,\nu,i}(0) = 0$ and $\Theta'_{M,\nu,i}(\xi) = \mathbb{1}_{M_1<\xi<M_2}\nu_i(\xi)$. Then we have for every $M_1 < M_2 \in (0,\infty)$ the existence of constant $c \in (0,\infty)$ independent of both M_1, M_2 such that

$$\begin{aligned}
&\mathbb{E}\int_U \int_0^T \mathbb{1}_{M_1<\rho<M_2} \left(\Phi'(\rho)|\nabla\rho|^2 + \alpha|\nabla\rho|^2 \right) \leq \mathbb{E}\int_U (\rho_0(x) - M_1)_+ + \|h_M\|_{L^2(U)}\mathbb{E}\|\rho_T\|_{L^2(U)} \\
&+ c\left\|S'_M(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)} \mathbb{E}\int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2 + c\mathbb{E}\int_U\int_0^T \mathbb{1}_{\rho\geq M_1}\sigma^2(\rho\wedge M_2) \\
&+ T\sum_{i=1}^d \int_{\partial U} \Theta_{M,\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta} + cT\int_{\partial U} \left(\sigma^2((\Phi^{-1}(\bar{f}) \wedge M_2) \vee M_1) - \sigma^2(M_1) \right) \\
&+ cT\left\|S'_M(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)} \left(T + \|\Theta_\Phi(g)\|_{H^1(U)} + \left\|\bar{f}\right\|_{L^2(\partial U)} + \alpha\left\|\Phi^{-1}(\bar{f})\right\|_{L^2(\partial U)} + \left\|\Psi_\sigma(\Phi^{-1}(\bar{f}))\right\|_{L^2(\partial U)} \right). \qquad (21)
\end{aligned}$$

Note that the final two terms on the left hand side of (20) are the approximate kinetic measures q^α, and the bound in (21) is a bound on the approximate kinetic measures when ρ is bounded away from zero and infinity. The right hand side of (21) is written deliberately to emphasise that it converges to zero as $M_1, M_2 \to \infty$ due to the fact that $h_M \equiv 0$ on $\bar{U}$ as $M_1 \to \infty$.

Proof. To prove the first claim, applying Itô's formula to the solution minus the PDE with correct boundary data so that the difference vanishes along the boundary, gives

$$\int_U (\rho(x,s) - g(x))^2\, dx\Big|_{s=0}^t = \int_U \int_0^t d(\rho-g)^2 = \int_U\int_0^t 2(\rho-g)\,d(\rho-g) + \frac{1}{2}2(\rho-g)^0 d\langle\rho-g\rangle_s\, dx. \qquad (22)$$

For the first term on the right hand side, noting g does not depend on time so $d(\rho - g) = d\rho$ and integrating by parts gives

$$\int_U \int_0^t 2(\rho - g)\, d(\rho - g)\, dx$$
$$= -2 \int_U \int_0^t \nabla(\rho - g) \cdot \left(\nabla\Phi(\rho) + \alpha\nabla\rho - \sigma(\rho)\dot{\xi}^F - \nu(\rho) + \frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2 \right) ds\, dx.$$

The Itô correction can be easily evaluated considering $d\langle \rho - g\rangle = d\langle \rho \rangle$ and using the definition of the noise,

$$\int_U \int_0^t d\langle \rho - g\rangle_s\, dx = \int_U \int_0^t F_1(\sigma'(\rho))^2|\nabla\rho|^2 + 2\sigma\sigma'(\rho)F_2 \cdot \nabla\rho + F_3\sigma^2(\rho)\, ds\, dx.$$

Putting these two together, we get from (22) that

$$\frac{1}{2}\int_U (\rho(x,s) - g(x))^2\, dx\Big|_{s=0}^t$$
$$= -\int_U \int_0^t \left(\Phi'(\rho)|\nabla\rho|^2 + \alpha|\nabla\rho|^2 - \sigma(\rho)\nabla\rho \cdot \dot{\xi}^F - \nabla\rho \cdot \nu(\rho) - \frac{1}{2}\sigma'(\rho)\sigma(\rho)\nabla\rho \cdot F_2 - \frac{1}{2}F_3\sigma^2(\rho) \right)$$
$$+ \int_U \int_0^t \nabla g \cdot \left(\nabla\Phi(\rho) + \alpha\nabla\rho - \sigma(\rho)\dot{\xi}^F - \nu(\rho) + \frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2 \right). \tag{23}$$

Let's consider each term of (23) in turn. The first two terms on the right hand side are precisely those in the estimate, so we move them to the left hand side. The noise term in both lines vanish after taking an expectation. The fourth term remains, but we can write it as a boundary integral in the following way. As in Assumption 4.2, for $i = 1, \ldots, d$ let $\Theta_{\nu,i}$ denote the anti-derivative of ν_i. We then obtain

$$\int_U \int_0^t \nabla\rho \cdot \nu(\rho)\, ds\, dx = \sum_{i=1}^d \int_U \int_0^t \partial_i\left(\Theta_{\nu,i}(\rho)\right) ds\, dx = \sum_{i=1}^d \int_{\partial U} \int_0^t \Theta_{\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}_i\, ds\, dx.$$

The fifth term can be bounded by integration by parts, noting either $\nabla \cdot F_2 = 0$ or it is bounded and we can bound σ^2 by $c(1 + \rho + \Phi(\rho))$, and the fact that $F_2 \cdot \hat{\eta}$ is bounded. We have

$$\frac{1}{4}\int_U \int_0^t \nabla\sigma^2(\rho) \cdot F_2\, ds\, dx = -\frac{1}{4}\int_U \int_0^t \sigma^2(\rho)\nabla \cdot F_2\, ds\, dx + \frac{1}{4}\int_{\partial U} \int_0^t \sigma^2(\Phi^{-1}(\bar{f}))F_2 \cdot \hat{\eta}\, ds\, dx$$
$$\leq c\int_U \int_0^t (1 + \rho + \Phi(\rho))\, ds\, dx + c\int_{\partial U} \int_0^t \sigma^2(\Phi^{-1}(\bar{f}))\, ds\, dx.$$

The final term on the second line of (23) can be bounded by the first term in the above inequality after noting that F_3 is bounded and once more bounding $\sigma^2(\rho)$ by $c(1 + \rho + \Phi(\rho))$.
The terms on the final line of (23) involving ∇g would all vanish if the boundary condition was constant. Otherwise they can be bounded precisely as described in points one and two of Remark 4.9. The first, second and fifth terms involving another gradient are reduced to boundary terms using integration by parts and noting g is harmonic, whereas the remaining terms are handled using Young's inequality and the bounds in point eight of Assumption 4.2. Putting everything in (23) together we have

$$\frac{1}{2}\mathbb{E}\left(\int_U (\rho(x,s) - g(x))^2\, dx\Big|_{s=0}^t \right) + \mathbb{E}\left(\int_U \int_0^t |\nabla\Theta_\Phi(\rho)|^2 \right) + \mathbb{E}\left(\alpha\int_U \int_0^t |\nabla\rho|^2 \right)$$
$$\leq c\left(t + \mathbb{E}\int_0^t \int_U |\rho| + \mathbb{E}\int_0^t \int_U \Phi(\rho) \right) + t\sum_{i=1}^d \int_{\partial U} \Theta_{\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}_i + ct\int_{\partial U} \sigma^2(\Phi^{-1}(\bar{f}))$$
$$+ \alpha t\int_{\partial U} \Phi^{-1}(\bar{f})\frac{\partial g}{\partial\hat{\eta}} + t\int_{\partial U} \bar{f}\frac{\partial g}{\partial\hat{\eta}} + t\int_{\partial U} \Psi_\sigma(\Phi^{-1}(\bar{f}))\frac{\partial g}{\partial\hat{\eta}} + t\|\nabla g\|_{L^2(U)}^2,$$

where Ψ_σ was defined in point eleven of Assumption 4.2. We used the fact that the boundary terms in the final two lines are deterministic and do not depend on time, and are all well defined due to the final three assumptions in Assumption 4.2.
The first three terms in the final line can be further bounded using Young's inequality point three of Remark 4.9 and the final term can also be bounded using points two and three of Remark 4.9 as well as Hölder's inequality.
To bound the integral involving $\Phi(\rho)$ in the second line we first note the polynomial growth condition $\Phi(\rho) \le c(1+\rho^m)$. We then use Proposition 4.12 to bound this term and the term involving the $L^1(U)$ norm of the solution,

$$\int_0^t \int_U (|\rho| + |\Phi(\rho)|) \le cT + c\left(\frac{T}{\epsilon} + \epsilon T \|\Theta_\Phi(g)\|_{H^1(U)} + \epsilon \int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2\right).$$

Choosing ϵ so small that the final term can be absorbed into the left hand side of the estimate and taking the supremum over $t \in [0,T]$ we obtain the first estimate (20).
To prove the the second estimate (21), for the function $S_M : [0,\infty) \to [0,\infty)$ defined in Definition 4.8, apply Itô's formula to a regularised version of $\Psi_{S_M} : U \times [0,\infty) \to \mathbb{R}$ defined by

$$\Psi_{S_M}(x,0) = 0, \quad \partial_\xi \Psi_{S_M}(x,\xi) = S_M'(\xi) - h_M(x),$$

where h_M satisfies the PDE in Definition 4.8 and ensures that $\partial_\xi \Psi_{S_M}(x,\rho)$ vanishes along the boundary. Grouping the ∇h_M and $\nabla \rho$ terms, we have

$$\begin{aligned}
&\partial_t \int_U \Psi_{S_M}(x,\rho(x,t))\,dx = \int_U \partial_\xi \Psi_{S_M}(x,\rho(x,t))\partial_t \rho\,dx + \frac{1}{2}\int_U \partial_\xi^2 \Psi_{S_M}(x,\rho(x,t))\partial_t \langle \rho \rangle_t\,dx \\
&= -\int_U \int_0^t S''(\rho)\nabla\rho \cdot \left(\nabla\Phi(\rho) + \alpha\nabla\rho - \sigma(\rho)\,d\xi^F - \nu(\rho) - \sigma(\rho)\sigma'(\rho)F_2\right) + \frac{1}{2}\int_U \int_0^t S''(\rho)F_3\sigma^2(\rho) \\
&+ \int_U \int_0^t \nabla h_M \cdot \left(\nabla\Phi(\rho) + \alpha\nabla\rho - \sigma(\rho)\,d\xi^F - \nu(\rho) + \frac{1}{2}F_1(\sigma'(\rho))^2\nabla\rho + \sigma(\rho)\sigma'(\rho)F_2\right). \qquad (24)
\end{aligned}$$

We deal with the terms in the same way as the first estimate. The first two terms form part of the estimate so are moved to the left hand side, the noise terms vanish in expectation and the fourth term can be re-written as a boundary integral. For the fifth term in the right hand side of (24) we use the distributional equality

$$\mathbb{1}_{M_1<\rho<M_2}\sigma(\rho)\sigma'(\rho)\nabla\rho = \frac{1}{2}\mathbb{1}_{M_1<\rho<M_2}\nabla\sigma^2(\rho) = \frac{1}{2}\nabla\left(\sigma^2((\rho\wedge M_2)\vee M_1) - \sigma^2(M_1)\right).$$

Then using integration by parts gives

$$\begin{aligned}
\frac{1}{2}\int_U \int_0^T S''(\rho)\sigma(\rho)\sigma'(\rho)\nabla\rho \cdot F_2 = &-\frac{1}{4}\int_U \int_0^T \left(\sigma^2((\rho\wedge M_2)\vee M_1) - \sigma^2(M_1)\right)\nabla\cdot F_2 \\
&+ \frac{1}{4}\int_{\partial U}\int_0^T \left(\sigma^2((\Phi^{-1}(\bar f)\wedge M_2)\vee M_1) - \sigma^2(M_1)\right)F_2\cdot\hat\eta.
\end{aligned}$$

This implies by the boundedness of $\nabla \cdot F_2$, F_3 and $F_2 \cdot \hat\eta$ that

$$\begin{aligned}
\frac{1}{2}\int_U \int_0^T S''(\rho)\left(\sigma(\rho)\sigma'(\rho)\nabla\rho\cdot F_2 + F_3\sigma^2(\rho)\right) \le{}& c\int_U \int_0^T \mathbb{1}_{\rho\ge M_1}\sigma^2(\rho\wedge M_2) \\
&+ c\int_{\partial U}\int_0^T \left(\sigma^2((\Phi^{-1}(\bar f)\wedge M_2)\vee M_1) - \sigma^2(M_1)\right).
\end{aligned}$$

Again we mention that the terms in (24) involving ∇h_M would vanish if our boundary condition was constant. Otherwise they are dealt with in a similar way to the first estimate, except that we use Hölder's inequality everywhere rather than Young's inequality. In this way we keep the

M dependence in these terms through h_M. Explicitly, we have for the terms that involve another gradient using points one and three of Remark 4.9,

$$\int_U\int_0^T \nabla h_M\cdot\left(\nabla\Phi(\rho)+\alpha\nabla\rho+\frac{1}{2}F_1(\sigma'(\rho))^2\nabla\rho\right)=T\int_{\partial U}\frac{\partial h_M}{\partial\hat{\eta}}\left(\bar{f}+\alpha\Phi^{-1}(\bar{f})+\Psi_\sigma(\Phi^{-1}(\bar{f}))\right)$$
$$\leq cT\left\|S_M'(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)}\left(\left\|\bar{f}\right\|_{L^2(\partial U)}+\alpha\left\|\Phi^{-1}(\bar{f})\right\|_{L^2(\partial U)}+\left\|\Psi_\sigma(\Phi^{-1}(\bar{f}))\right\|_{L^2(\partial U)}\right).$$

For the remaining two terms, the bound in point eight of Assumption 4.2 alongside Hölder's inequality and point two and three of Remark 4.9 gives

$$\int_U\int_0^T \nabla h_M\cdot(-\nu(\rho)+\sigma(\rho)\sigma'(\rho)F_2)\leq c\|\nabla h_M\|_{L^2(U)}\int_U\int_0^T(1+\rho+\Phi(\rho))$$
$$\leq c\left\|S_M'(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)}\int_U\int_0^T(1+\rho+\Phi(\rho)).$$

The final two terms can be dealt with using Proposition 4.12, just as in the first estimate. Here we can't absorb the gradient term that appears, and it stays on the right hand side of the estimate, and it is bounded as a consequence of the first estimate.
To complete the estimate we move the first term on the left hand involving Ψ_{S_M} in (24) to the right hand side. To handle it, the definition of Ψ_{S_M} implies that $\Psi_{S_M}(x,\rho)=S_M(\rho)-h(x)\rho$. Furthermore the product $h_M\rho_0$ and $\int_U S_M(\rho(x,T))$ are non-negative so can be removed from the estimate. Using the bound $S_M(\rho_0(x))\leq(\rho_0(x)-M_1)_+$ and Holder's inequality to bound the boundary terms and integral of $h\rho_T$, noting that the $L^2(U)$ norm of ρ_T is bounded using the first estimate, completes the estimate. □

Our goal is to use the above estimates to prove fractional Sobolev spacial regularity of the solution. We will need the below lemma that will allow us to prove regularity of (fractional) spacial derivatives of the solution, see Lemma 5.11 of [FG24] for proof.

Lemma 4.15. *Let Φ satisfy Assumption 4.2. Let $z\in H^1(U)$ be non-negative. If Φ satisfies* (17) *(allows us to handle $0<m<1$) then*

$$\|\nabla z\|_{L^1(U;\mathbb{R}^d)}\leq\|z\|^{\theta}_{L^1(U)}\|\nabla\Theta_\Phi\|_{L^2(U;\mathbb{R}^d)}.$$

If Φ satisfies (18) *(allows us to handle $m\geq 1$) then for every $\beta\in(0,1\wedge 2/q)$, for $c\in(0,\infty)$ depending on β,*

$$\|z\|_{W^{\beta,1}(U)}\leq c\left(\|z\|_{L^1(U)}+\|\nabla\Theta_\Phi\|^{\frac{2}{q}}_{L^2(U;\mathbb{R}^d)}\right).$$

We have the following regularity estimate.

Corollary 4.16 (Regularity of solution to regularised equation)**.** *Let ξ^F,Φ,σ and ν satisfy Assumptions 2.2, 4.2 and 4.6, and fix the regularisation $\alpha\in(0,1)$ and terminal time $T\in[1,\infty)$. Let further $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable, $\Theta_{\nu,i}$ and Ψ_σ be defined as in Assumption 4.2 and g be the PDE defined in Definition 4.8 satisfying Assumption 4.10. Let ρ be a weak solution of* (4) *in the sense of Definition 4.7.*

- *If Φ satisfies* (17)*, then for $c\in(0,\infty)$ independent of α and T, we have*

$$\mathbb{E}\left(\|\rho\|_{L^1([0,T];W^{1,1}(U))}\right)$$
$$\leq c\|\rho_0-g\|^2_{L^2(U)}+cT\left(1+\|\Theta_\Phi(g)\|_{H^1(U)}+\sum_{i=1}^d\int_{\partial U}\Theta_{\nu,i}(\Phi^{-1}(\bar{f}))\cdot\hat{\eta}_i\right)$$
$$+cT\left(\|\sigma^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)}+\|\bar{f}\|^2_{L^2(\partial U)}+\|\Psi_\sigma(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)}+(1+\alpha)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)}\right).$$

- *If Φ satisfies* (18)*, then for all $\beta \in (0, 2/q \wedge 1)$ there is a constant $c \in (0,\infty)$ depending on β, but independent of α and T, such that*

$$\begin{aligned}
&\mathbb{E}\left(\|\rho\|_{L^1([0,T];W^{\beta,1}(U))}\right) \\
&\quad \leq c\|\rho_0 - g\|^2_{L^2(U)} + cT\left(1 + \|\Theta_\Phi(g)\|_{H^1(U)} + \sum_{i=1}^d \int_{\partial U} \Theta_{\nu,i}(\Phi^{-1}(\bar{f}))\cdot\hat{\eta}_i\right) \\
&\quad + cT\left(\|\sigma^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)} + \|\bar{f}\|^2_{L^2(\partial U)} + \|\Psi_\sigma(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)} + (1+\alpha)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)}\right).
\end{aligned}$$

Proof. For both estimates we can use Proposition 4.12 to bound the $L^1_tL^1_x$ norms with $\epsilon = 1$. The first estimate then follows from the first item in Lemma 4.15, Young's inequality, the fact that $\theta \in [0, 1/2]$ and Proposition 4.14.
The second estimate follows from the second point in Lemma 4.15, the fact that $q \in [1,\infty)$ and Proposition 4.14. ☐

The final result we want to show is a higher order space-time regularity of solutions cut away from their zero set. The result will be useful when proving the existence of a weak solution of equation (4) with smooth and bounded σ, and will subsequently motivate the introduction of a new metric on $L^1_xL^1_t$, see Definition 4.33.

Definition 4.17 (Cutoff away from zero)**.** *For $\beta \in (0,1)$ let ϕ_β be the piecewise linear cutoff in Definition 3.2. Let $\tilde{\phi}_\beta \in C^\infty([0,\infty))$ be a smooth approximation of ϕ_β. That is to say $0 \leq \tilde{\phi}_\beta \leq 1$ is non-decreasing and satisfies $\tilde{\phi}_\beta(\xi) = 1$ for $\xi \geq \beta$, $\tilde{\phi}_\beta(\xi) = 0$ for $\xi \leq \beta/2$ with $|\tilde{\phi}'_\beta(\xi)| \leq c/\beta$ for constant $c \in (0,\infty)$ independent of β.*
For each $\delta \in (0,1)$ define $\Phi_\beta \in C^\infty([0,\infty))$ by

$$\Phi_\beta(\xi) = \tilde{\phi}_\beta(\xi)\xi.$$

Proposition 4.18. *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2, 4.2 and 4.6, and let $\alpha \in (0,1)$. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable, $\Theta_{\nu,i}$ and Ψ_σ be defined as in Assumption 4.2 and g be the PDE defined in Definition 4.8 satisfying Assumption 4.10. Let ρ be a weak solution of the regularised equation* (4) *in the sense of Definition 4.7.*
For every $\delta \in (0, 1/2)$ and $s > \frac{d}{2} + 1$ there is a constant $c \in (0,\infty)$ depending on β, δ and s but independent of α and T such that

$$\begin{aligned}
&\mathbb{E}\left(\|\Phi_\beta(\rho)\|_{W^{\delta,1}([0,T];H^{-s}(U))}\right) \leq cT\|\rho_0 - g\|^2_{L^2(U)} \\
&\quad + cT^2\left(1 + \|\Theta_\Phi(g)\|_{H^1(U)} + \sum_{i=1}^d \int_{\partial U} \Theta_{\nu,i}(\Phi^{-1}(\bar{f}))\cdot\hat{\eta}_i + \|\sigma^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)}\right) \\
&\quad + cT^2\left(\|\bar{f}\|^2_{L^2(\partial U)} + \|\Psi_\sigma(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)} + (1+\alpha)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)}\right).
\end{aligned}$$

The proof is analogous to Proposition 5.14 [FG24], we just give the main idea below.

Proof. Similarly to the derivation of the kinetic equation, it follows by Itô's formula and then bringing the $\Phi'_\beta(\rho)$ inside the derivative and using the product rule, that

$$\begin{aligned}
d\Phi_\beta(\rho) &= \Phi'_\beta(\rho)d\rho + \frac{1}{2}\Phi''_\beta(\rho)d\langle\rho\rangle \\
&= \nabla\cdot\left(\Phi'_\beta(\rho)\nabla\Phi(\rho) + \alpha\Phi'_\beta(\rho)\nabla\rho - \Phi'_\beta(\rho)\sigma(\rho)d\xi^F - \Phi'_\beta(\rho)\nu(\rho)\right) \\
&+ \nabla\cdot\left(\frac{1}{2}F_1\Phi'_\beta(\rho)(\sigma'(\rho))^2\nabla\rho + \frac{1}{2}\Phi'_\beta(\rho)\sigma(\rho)\sigma'(\rho)F_2\right) \\
&- \Phi''_\beta(\rho)\nabla\rho\cdot\nabla\Phi(\rho) - \alpha\Phi''_\beta(\rho)|\nabla\rho|^2 + \Phi''_\beta(\rho)\sigma(\rho)\nabla\rho\cdot d\xi^F + \Phi''_\beta(\rho)\nabla\rho\cdot\nu(\rho) \\
&+ \frac{1}{2}\Phi''_\beta(\rho)\left(\sigma(\rho)\sigma'(\rho)F_2\cdot\nabla\rho + F_3\sigma^2(\rho)\right)
\end{aligned}$$

Doing so will allow us to compute the the fractional Sobolev norm more easily. Integrating in time, and writing some derivatives in terms of the function Θ_Φ, we have for fixed $x \in U$ the decomposition $\Phi_\beta(\rho(x,t)) = \Phi_\beta(\rho_0(x)) + I_t^{f.v.} + I_t^{mart}$ with

$$I_t^{mart} := -\int_0^t \nabla \cdot \left(\Phi'_\beta(\rho)\sigma(\rho) d\xi^F\right) + \int_0^t \Phi''_\beta(\rho)\sigma(\rho)(\Phi'(\rho))^{-1/2}\nabla\Theta_\Phi(\rho)\cdot d\xi^F,$$

and

$$\begin{aligned} I_t^{f.v.} := & \int_0^t \nabla \cdot \Big(\Phi'_\beta(\rho)(\Phi'(\rho))^{1/2}\nabla\Theta_\Phi(\rho)\Big) - \int_0^t \Phi''_\beta(\rho)|\nabla\Theta_\Phi(\rho)|^2 + \alpha\int_0^t \nabla\cdot(\Phi'_\beta(\rho)\nabla\rho) \\ & - \alpha\int_0^t \Phi''_\beta(\rho)|\nabla\rho|^2 + \frac{1}{2}\int_0^t \nabla\cdot\left(F_1\Phi'_\beta(\rho)\frac{(\sigma'(\rho))^2}{(\Phi'(\rho))^{1/2}}\nabla\Theta_\Phi(\rho)\right) + \frac{1}{2}\int_0^t \nabla\cdot\left(\Phi'_\beta(\rho)\sigma(\rho)\sigma'(\rho)F_2\right) \\ & + \frac{1}{2}\int_0^t \Phi''_\beta(\rho)\frac{\sigma(\rho)\sigma'(\rho)}{\Phi'(\rho)^{1/2}}F_2\cdot\nabla\Theta_\Phi(\rho) + \frac{1}{2}\int_0^t \Phi''_\beta(\rho)F_3\sigma^2(\rho) - \int_0^t \nabla\cdot(\Phi'_\beta(\rho)\nu(\rho)) \\ & - \int_0^t \nabla\cdot(\Phi'_\beta(\rho)\nu(\rho)) + \int_0^t \Phi''_\beta(\rho)\nabla\rho\cdot\nu. \end{aligned}$$

We begin by showing the martingale term is in $W^{\delta,2}([0,T];H^{-s}(U))$, that is

$$\|I_t^{mart}\|^2_{W^{\delta,2}([0,T];H^{-s}(U))} := \int_0^T \|I_t^{mart}\|^2_{H^{-s}(U)}\,dt + \int_0^T\int_0^T \frac{\|I_t^{mart}-I_s^{mart}\|^2_{H^{-s}(U)}}{|t-s|^{1+2\delta}}\,ds\,dt < \infty.$$

Since $s > \frac{d}{2}+1$ the Sobolev embedding theorem tells us that for test functions ϕ that we use in the definition of negative fractional Sobolev norm, there is a constant $c \in (0,\infty)$ such that

$$\|\phi\|_{L^\infty(U)} + \|\nabla\phi\|_{L^\infty(U;\mathbb{R}^d)} \leq c\|\phi\|_{H^s(U)} \tag{25}$$

Using this and an argument similar to Lemma 2.1 of [FG95] alongside the Burkholder-Davis-Gundy inequality, the fact that Φ''_β is supported on $[\beta/2,\beta]$ as well as the bounds in Assumption 4.2, that the second term in the definition of the norm satisfies

$$\begin{aligned} & \mathbb{E}\left(\int_0^T\int_0^T \frac{\|I_t^{mart}-I_s^{mart}\|^2_{H^{-s}(U)}}{|t-s|^{1+2\delta}}\,ds\,dt\right) \\ & \leq c\mathbb{E}\left(\int_0^T \|\Theta_\Phi(\rho)\mathbb{1}_{\beta/2\leq\rho\leq\beta}\|^2_{L^2(U)} + \|\sigma(\rho)\|^2_{L^2(U)}\,ds\right) \\ & \leq c\mathbb{E}\left(|U|T + \int_0^T\int_U \left(|\rho_t| + |\nabla\Theta_\Phi|^2 + |\Phi(\rho)|\right)\right). \end{aligned}$$

Analogously for the first term in the norm we have the same bound

$$\int_0^T \|I_t^{mart}\|^2_{H^{-s}(U)}\,dt \leq c\mathbb{E}\left(|U|T + \int_0^T\int_U \left(|\rho_t| + |\nabla\Theta_\Phi|^2 + |\Phi(\rho)|\right)\right).$$

Putting these together, it follows from Proposition 4.12 that

$$\|I_t^{mart}\|^2_{W^{\delta,2}([0,T];H^{-s}(U))} \leq c\mathbb{E}\left(T + T\|\Theta_\Phi(g)\|_{H^1(U)} + \int_0^T\int_U |\nabla\Theta_\Phi|^2\right).$$

We next show that the finite variation term is in $W^{1,1}([0,T];H^{-1}(U))$, that is

$$\|I^{f.v.}\|_{W^{1,1}([0,T];H^{-s}(U))} := \int_0^T \|I^{f.v.}\|_{H^{-s}(U)} + \int_0^T \left\|\frac{d}{dt}I^{f.v.}\right\|_{H^{-s}(U)} < \infty.$$

It follows from (25), the fact that Φ'_β and Φ''_β are supported on $[\beta/2,\infty)$ and $[\beta/2,\beta]$ respectively and Young's inequality, that we can bound the first term by

$$\begin{aligned}\int_0^T \left\|I^{f.v.}\right\|_{H^{-s}(U)} \leq c\int_0^T\int_U\int_0^t \big(&\Phi'(\rho)\mathbb{1}_{\rho>\beta/2} + |\nabla\Theta_\Phi(\rho)|^2\mathbb{1}_{\rho>\beta/2} + \alpha|\nabla\rho|\mathbb{1}_{\rho>\beta/2}\\ &+\frac{\sigma'(\rho)^4}{\Phi'(\rho)}\mathbb{1}_{\rho>\beta/2} + |\sigma(\rho)\sigma'(\rho)|\mathbb{1}_{\rho>\beta/2} + \frac{(\sigma(\rho)\sigma'(\rho))^2}{\Phi'(\rho)}\mathbb{1}_{\rho>\beta/2} + |\nu(\rho)|\mathbb{1}_{\rho>\beta/2}\\ &+(1-\alpha)|\nabla\rho|^2\mathbb{1}_{\beta/2<\rho<\beta} + \sigma^2\mathbb{1}_{\beta/2<\rho<\beta} + |\nu|^2\mathbb{1}_{\beta/2<\rho<\beta}\big)\, ds\, dx\, dt.\end{aligned}$$

Using the fundamental theorem of calculus, the second finite variation term consists of the same terms but without the inner ds integral. Using that the inner integral is increasing in t, the fact that $\alpha\in(0,1)$ and so $\alpha x\leq 1+\alpha x^2$, the fact that the three terms in the final line are bounded over the indicator set, points three six and eight of Assumption 4.2 to bound the various coefficients and using Proposition 4.12 analogously to the martingale term, we have

$$\mathbb{E}\left\|I^{f.v.}\right\|_{W^{1,1}([0,T];H^{-s}(U))} \leq c(1+T)\mathbb{E}\left(T + T\|\Theta_\Phi(g)\|_{H^1(U)} + \int_0^T\int_U \left(|\nabla\Theta_\Phi|^2 + \alpha|\nabla\rho|^2\right)\right).$$

Using the trivial fact that there is a constant c such that $(1+T)<cT$, the estimate then follows by the first energy estimate in Proposition 4.14 alongside the continuous embeddings $W^{\beta,2}, W^{1,1}\hookrightarrow W^{\beta,1}$ for every $\beta\in(0,1/2)$. □

4.2 Entropy estimate

In this section we prove an entropy estimate for weak solutions of the regularised Dean-Kawasaki equation, following Proposition 5.18 of [FG24]. The estimates will not be used in the remainder of the work, but they are provided because of their connection with the study of large deviation principles, see for instance the introduction of [FG23].
We begin with the definition of entropy space where the initial condition will live, followed by the assumptions needed for the entropy type estimate. These are analogous to Definition 5.16 and Assumptions 5.17 of [FG24], see also Remark 5.15 there.

Definition 4.19 (Entropy space)**.** *The space of non-negative, $L^1(U)$ functions with finite entropy is the space*

$$Ent(U) := \left\{\rho\in L^1(U) : \rho\geq 0 \text{ almost everywhere, with } \int_U \rho\log(\rho)<\infty\right\}.$$

We say that a function $\rho:\Omega\to L^1(U)\cap Ent(U)$ is in the space $L^1(\Omega; Ent(U))$ if ρ is $\mathcal{F}_0$ measurable and

$$\mathbb{E}\left[\|\rho\|_{L^1(U)} + \int_U \rho\log(\rho)\right]<\infty.$$

Assumption 4.20 (Entropy assumptions)**.** *Let $\Phi,\sigma\in C([0,\infty))$ and $\nu\in C([0,\infty);\mathbb{R}^d)$ satisfy the following assumptions from Assumptions 5.17 of [FG24].*

1. *There exists a constant $c\in(0,\infty)$ such that $|\sigma(\xi)|\leq c\Phi^{1/2}(\xi)$ for every $\xi\in[0,\infty)$.*
2. *There exists a constant $c\in(0,\infty)$ such that for every $\xi\in[0,\infty)$*
$$\Phi'(\xi)\leq c(1+\xi+\Phi(\xi)). \tag{26}$$
3. *We have $\nabla\cdot F_2 = 0$.*
4. *We have that $\log(\Phi)$ is locally integrable on $[0,\infty)$.*

Furthermore, we have the two additional new assumptions, analogous to the new assumptions in Assumption 4.2.

5. *Either $F_2 = 0$ or the unique function $\Theta_{\Phi,\sigma}$ defined by*

$$\Theta_{\Phi,\sigma}(1) = 0, \qquad \Theta'_{\Phi,\sigma}(\xi) = \frac{\Phi'(\xi)\sigma'(\xi)\sigma(\xi)}{\Phi(\xi)}$$

satisfies $\Theta_{\Phi,\sigma}(\Phi^{-1}(\bar{f})) \in L^1(\partial U)$.

6. *For $i = 1, \dots, d$, the unique functions $\Theta_{\Phi,\nu,i}$ defined by*

$$\Theta_{\Phi,\nu,i}(0) = 0, \qquad \Theta'_{\Phi,\nu,i}(\xi) = \frac{\Phi'(\xi)\nu_i(\xi)}{\Phi(\xi)}$$

satisfy $\Theta_{\Phi,\nu,i}(\Phi^{-1}(\bar{f})) \cdot \eta_i \in L^1(\partial U)$.

We give remarks about the new assumptions.

Remark 4.21.

- *In the model case the function in point five is given by $\Theta_{\Phi,\sigma}(\xi) = m\log(\xi)$. In the case of zero boundary conditions we therefore require $F_2 = 0$.*
- *We can without loss of generality consider $\nu(0) = 0$, and since ν is lipschitz in the model case we have that the function in point six is $\Theta_{\Phi,\nu,i}(\xi) = c\xi$ for some constant $c \in (0, \infty)$ and the assumption is satisfied.*

We now introduce the family of PDEs $\{v_\delta\}$.

Definition 4.22. *For every $\delta \in (0, 1)$ let v_δ satisfy the PDE*

$$\begin{cases} -\Delta v_\delta = 0, & \text{in } U, \\ v_\delta = \log(\bar{f} + \delta), & \text{on } \partial U, \end{cases} \tag{27}$$

where $\bar{f}$ is the boundary condition in (1).

We will need to bound normal derivatives of v_δ in the $\delta \to 0$ limit, motivating the below assumption.

Assumption 4.23 (Assumption on v_δ). *Either the boundary data $\bar{f}$ is constant or the solution of the PDE v_0 defined by*

$$\begin{cases} -\Delta v_0 = 0, & \text{in } U, \\ v_0 = \log(\bar{f}), & \text{on } \partial U, \end{cases}$$

satisfies $\log(\bar{f}) \in H^1(\partial U)$, where the norm is defined as the $L^p_1(\partial D)$ norm in page 176 of [FJJR78].

We now present the entropy estimate, the proof follows Proposition 5.18 of [FG24]. We once again repeat the comment in Remark 4.11 and emphasise that the below analysis is simplified if the boundary condition is constant.

Proposition 4.24 (Entropy estimate). *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2, 4.2, 4.6 and 4.20, and suppose Assumption 4.23 concerning v_0 is satisfied. Let the functions $\Theta_{\Phi,\sigma}, \{\Theta_{\Phi,\nu,i}\}_{i=1}^d, \Psi_\sigma$ be defined as in Assumption 4.20. Let $\alpha \in (0, 1)$, $T \in [1, \infty)$ and suppose the weak solution of the regularised equation* (4) *has $\mathcal{F}_0$-measurable initial condition satisfying $\rho_0 \in L^1(\Omega; Ent(U))$. For the function $\Psi_{\Phi,0} : U \times [0, \infty) \to \mathbb{R}$ defined by $\Psi_{\Phi,0}(x, 0) = 0$ and $\partial_\xi \Psi_{\Phi,0}(x, \xi) = \log(\Phi(\xi)) - v_0(x)$, we have, for a constant $c \in (0, \infty)$ independent of α and T, the bound*

$$\begin{aligned} &\mathbb{E}\int_U \Psi_{\Phi,0}(x, \rho(x, t))\,dx + 4\mathbb{E}\int_0^T \int_U |\nabla \Phi^{1/2}(\rho)|^2 + \alpha \mathbb{E}\int_0^T \int_U \frac{\Phi'(\rho)}{\Phi(\rho)}|\nabla \rho|^2 \\ &\quad \le \mathbb{E}\int_U \Psi_{\Phi,0}(x, \rho_0(x))\,dx + cT + cT\|\Theta_\Phi(g)\|_{H^1(U)} + c\mathbb{E}\int_0^T \int_U |\nabla \Theta_\Phi(\rho)|^2 \\ &\quad + cT\int_{\partial U} \Theta_{\Phi,\sigma}(\Phi^{-1}(\bar{f})) + T\sum_{i=1}^d \int_{\partial U} \Theta_{\Phi,\nu,i}(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}_i + T\|\bar{f}\|^2_{L^2(\partial U)} \\ &\quad + \alpha T\|\Phi^{-1}(\bar{f})\|^2_{L^2(\partial U)} + T\|\Psi_\sigma(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)} + c(1 + \alpha)T\|\log(\Phi^{-1}(\bar{f}))\|^2_{H^1(\partial U)}. \end{aligned}$$

Proof. To obtain the bound, apply Itô's formula to the regularised function $\Psi_{\Phi,\delta}: U \times [0,\infty) \to \mathbb{R}$ defined by $\Psi_{\Phi,\delta}(x,0) = 0$ and $\partial_\xi \Psi_{\Phi,\delta}(x,\xi) = \log(\Phi(\xi)+\delta) - v_\delta(x)$, where v_δ satisfies the PDE (27) in Definition 4.22 and once again ensures that $\partial_\xi \Psi_{\Phi,\delta}(x,\rho)$ vanishes along the boundary. We get after integrating the first order term by parts, that

$$\begin{aligned}
&\int_U \Psi_{\Phi,\delta}(x,\rho(x,t))\,dx\Big|_{s=0}^{T} \\
&= \int_0^T \int_U -\frac{|\nabla\Phi(\rho)|^2}{\Phi(\rho)+\delta} - \frac{\alpha\Phi'(\rho)|\nabla\rho|^2}{\Phi(\rho)+\delta} + \frac{\sigma(\rho)\Phi'(\rho)\nabla\rho\cdot d\xi^F}{\Phi(\rho)+\delta} + \frac{\Phi'(\rho)\nabla\rho\cdot\nu(\rho)}{\Phi(\rho)+\delta} \\
&\qquad\qquad + \frac{\Phi'(\rho)\sigma'(\rho)\sigma(\rho)\nabla\rho\cdot F_2}{2(\Phi(\rho)+\delta)} + \frac{F_3\Phi'(\rho)\sigma^2(\rho)}{2(\Phi(\rho)+\delta)} \\
&+ \int_0^T \int_U \nabla v_\delta \cdot \left(\nabla\Phi(\rho) + \alpha\nabla\rho - \sigma(\rho)d\xi^F - \nu(\rho) + \frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2\right).
\end{aligned} \tag{28}$$

The terms are handled in an analogous way to energy estimates already seen thus far in Propositions 4.14 and 4.29, so we are brief. We move the first two terms to the left and side of the estimate, noting that the distributional inequality allows us to rewrite the first term as $\nabla\Phi^{1/2}(\rho) = \frac{\Phi'(\rho)}{2\Phi^{1/2}(\rho)}\nabla\rho$

$$\int_U \frac{|\nabla\Phi(\rho)|^2}{\Phi(\rho)+\delta} = \int_U \frac{4\Phi(\rho)}{\Phi(\rho)+\delta}|\nabla\Phi^{1/2}(\rho)|^2.$$

After taking expectation the third term is killed as well as the noise term in the second line. The fourth and fifth terms can be re-written as boundary integrals. For the fourth, we use the functions $\Theta_{\Phi,\nu,\delta,i}$ for $i=1,\ldots,d$ defined by $\Theta_{\Phi,\nu,\delta,i}(0)=0, \Theta'_{\Phi,\nu,\delta,i}(\xi) = \frac{\Phi'(\xi)\nu_i(\xi)}{\Phi(\xi)+\delta}$, and for the fifth we define the unique function $\Theta_{\Phi,\sigma,\delta}$ satisfying $\Theta_{\Phi,\sigma,\delta}(0)=0, \Theta'_{\Phi,\sigma,\delta}(\xi) = \frac{\Phi'(\xi)\sigma'(\xi)\sigma(\xi)}{\Phi(\xi)+\delta}$, and note that either $F_2 = 0$, or use integration by parts alongside the assumptions $\nabla\cdot F_2 = 0$ and the fact that $F_2\cdot\hat{\eta}$ is bounded. For the final term in the first line of (28), by the assumption $\sigma \leq c\Phi^{1/2}$, the fact that $\frac{x}{x+\delta} < 1$ for every $\delta > 0$ and the assumption $\Phi'(\xi) \leq c(1+\xi+\Phi(\xi))$, we obtain

$$\frac{1}{2}\int_0^T\int_U \frac{F_3\Phi'(\rho)\sigma^2(\rho)}{\Phi(\rho)+\delta} \leq c\int_0^T\int_U \Phi'(\rho) \leq c\left(|U|T + \int_0^T\int_U(\rho+\Phi(\rho))\right).$$

The terms involving ∇v_δ in (28) would all vanish if the boundary condition was constant. Otherwise they are handled in the way described in points one and two of Remark 4.9. The first, second and fifth terms in (28) with other gradient terms are handled using integrate by parts and turn into boundary terms. As for the terms without derivatives, using the $L^2(U)$ integrability of ∇v_δ implied by point two of Remark 4.9, Young's inequality, the boundedness of F_2, the equation (27) satisfied by v_δ and Assumption 4.2, we have

$$\int_0^T\int_U \nabla v_\delta\cdot(-\nu(\rho) + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2) \leq cT\|\log(\bar{f}+\delta)\|^2_{H^1(\partial U)} + c\int_0^T\int_U(1+\rho+\Phi(\rho))\,dx\,dt.$$

Putting everything together we get

$$\begin{aligned}
&\mathbb{E}\int_U \Psi_{\Phi,\delta}(x,\rho(x,t))\,dx + 4\mathbb{E}\int_0^T\int_U \frac{\Phi(\rho)}{\Phi(\rho)+\delta}|\nabla\Phi^{1/2}(\rho)|^2 + \alpha\mathbb{E}\int_0^T\int_U \frac{\Phi'(\rho)}{\Phi(\rho)+\delta}|\nabla\rho|^2 \\
&\quad\leq \mathbb{E}\int_U \Psi_{\Phi,\delta}(x,\rho_0(x))\,dx + c\left(|U|T + \mathbb{E}\int_0^T\int_U(|\rho|+|\Phi(\rho)|)\right) \\
&\quad + cT\int_{\partial U}\Theta_{\Phi,\sigma,\delta}(\Phi^{-1}(\bar{f})) + T\sum_{i=1}^d\int_{\partial U}\Theta_{\Phi,\nu,\delta,i}(\Phi^{-1}(\bar{f}))\cdot\hat{\eta}_i \\
&\quad + T\left(\int_{\partial U}\bar{f}\frac{\partial v_\delta}{\partial\hat{\eta}} + \alpha\int_{\partial U}\Phi^{-1}(\bar{f})\frac{\partial v_\delta}{\partial\hat{\eta}} + \int_{\partial U}\Psi_\sigma(\Phi^{-1}(\bar{f}))\frac{\partial v_\delta}{\partial\hat{\eta}} + c\|\log(\bar{f}+\delta)\|^2_{H^1(\partial U)}\right).
\end{aligned} \tag{29}$$

Once again we used the fact that that the boundary terms in the final two lines are deterministic and don't depend on time. To obtain the desired estimate, we wish to take the $\delta \to 0$ limit in equation (29), and therefore we need a handle over the boundary terms which all depend on δ. Alongside Young's inequality, the new assumptions in Assumption 4.20 and Assumption 4.2 precisely allow us to do this. Finally, to bound the integral of ρ and $\Phi(\rho)$ in the second line, we use Proposition 4.12 in the same manor as the energy estimates, noting that we can't absorb the term involving $|\nabla\Theta_\Phi(\rho)|^2$ into the left hand side, but the term is bounded by the first energy estimate (20). The estimate is proven. □

Remark 4.25 (Comparing entropy estimate with proof of kinetic measure at zero)**.** *Note that the entropy estimate could be used to prove a statement about the kinetic measure at zero if we include the assumption that $\frac{\Phi(\xi)}{\Phi'(\xi)} \leq c\xi$. We have*

$$2\beta^{-1}\mathbb{E}\left(q(U \times [\beta/2, \beta] \times [0,T])\right) \leq \liminf_{\alpha \to 0} \mathbb{E}\left(\int_0^T \int_U \int_{\mathbb{R}} \frac{1}{\xi} \mathbb{1}_{\beta/2 \leq \xi \leq \beta} dq^\alpha\right).$$

Subsequently the assumption gives

$$\begin{aligned}\frac{1}{\xi} dq^a &= \frac{1}{\xi}\delta_0(\xi - \rho^\alpha)\left(\frac{4\Phi(\rho^\alpha)}{\Phi'(\rho^\alpha)}|\nabla\Phi^{1/2}(\rho^\alpha)|^2 + \alpha|\nabla\rho^\alpha|^2\right)\\ &\leq c\delta_0(\xi - \rho^\alpha)\left(4|\nabla\Phi^{1/2}(\rho^\alpha)|^2 + \alpha\frac{\Phi'(\rho)}{\Phi(\rho)}|\nabla\rho^\alpha|^2\right).\end{aligned}$$

One sees that this is the precise quantity which we showed was bounded in the entropy estimate above. Consequently by dominated convergence theorem, with the indicator present, the kinetic measures go to zero.
The reason we don't use this estimate is due to the first term on the right hand side of the estimate, which requires $\rho_0 \in L^1(\Omega; Ent(U))$. For the definition of stochastic kinetic solution, Definition 2.8, we only have $\rho_0 \in L^1(\Omega; L^1(U))$. We circumvent this in the sequel by choosing a test function that cuts off the logarithm at 1.

4.3 Decay of kinetic measure at zero

In this section we will prove the decay of the kinetic measure at zero required in the uniqueness proof,

$$\liminf_{\beta \to 0}\left(\beta^{-1}q(U \times [\beta/2, \beta] \times [0,T])\right) = 0.$$

First of all we begin with a remark that illustrates why we can't bound the decay of the kinetic measure using the kinetic equation as in Proposition 4.6 of [FG24].

Remark 4.26. *Adapting the proof of Proposition 4.6 of [FG24], test the kinetic equation* (5) *against smooth approximations of the product $\zeta_M\phi_\beta\iota_\gamma$ and subsequently take the limit in the approximations. We end up with an additional term when the spacial gradient hits the cutoff ι_γ. Taking expectation to kill the noise term and taking the limit as $\gamma \to 0$ (which we need to take before M, β limits due to Remark 2.4) one needs to consider the term*

$$-\lim_{\gamma \to 0}\mathbb{E}\left(\int_0^t \int_U \zeta_M(\rho)\phi_\beta(\rho)\left(\Phi'(\rho)\nabla\rho + \frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2\right)\cdot\nabla\iota_\gamma(x)\,dx\,ds\right).$$

We can not make sense of this limit because we don't have sufficient regularity of the first two terms. For example to take the limit in the first term we would need to use the trace theorem and therefore require $\nabla\Phi(\rho) \in H^1_{loc}(U)$. However, we only have $\Phi(\rho) \in H^1_{loc}(U)$, i.e. $\nabla\Phi(\rho) \in L^2_{loc}(U)$.
Note that similar terms arose in the uniqueness proof, for instance recall equation (14)*. Crucially, there the terms appear as a difference of two solutions and can only be handled because the first two terms above have a sign for every fixed $\gamma > 0$.*

We consider the following PDE, where the boundary condition is the logarithm cutoff at 1.

Definition 4.27 (The PDE v)**.** *Let $\bar{f}$ be the boundary condition of the regularised equation* (4)*. Define the function $S : [0,\infty) \to [0,\infty)$ by $S(0) = 0$ and $S''(\xi) = \frac{1}{\xi}\mathbb{1}_{0\le\xi\le 1}$. Define the harmonic PDE $v : U \to \mathbb{R}$ by*

$$\begin{cases} -\Delta v = 0 & \text{on } U, \\ v = S'(\Phi^{-1}(\bar{f})) & \text{on } \partial U. \end{cases}$$

By integrating we have that $S'(\xi) = \log(\xi \wedge 1)$ and $S(\xi) = (\xi \wedge 1)\log(\xi \wedge 1) - (\xi \wedge 1)$.

Assumption 4.28 (Assumptions for kinetic measure)**.**

1. *Either the boundary data $\bar{f}$ is constant or the solution of PDE v satisfies $S'(\Phi^{-1}(\bar{f})) \in H^1(\partial U)$, where the norm is defined as the $L^p_1(\partial D)$ norm in page 176 of [FJJR78].*
2. *We have that $v \in L^2(U)$.*
3. *Either $F_2 = 0$, or the unique function Θ_σ defined by $\Theta_\sigma(1) = 0$, $\Theta'_\sigma(\xi) = \frac{\sigma(\xi)\sigma'(\xi)}{\xi}$ satisfies $\Theta_\sigma(\Phi^{-1}(\bar{f}) \wedge 1) \in L^1(\partial U)$.*
4. *Either $\bar{f}$ is constant, or for the unique function Ψ_σ defined by $\Psi_\sigma(0) = 0$, $\Psi'_\sigma(\xi) = F_1[\sigma'(\xi)]^2$, we have $\Psi_\sigma(\Phi^{-1}(\bar{f})) \in L^2(\partial U)$.*

For point three, note that the $L^1(\partial U)$ integrability of Θ_σ is guaranteed if, for example, $\sigma(\xi)\sigma'(\xi) \le c\xi^\alpha$, where $\alpha > 0$ and $c \in (0,\infty)$.

Proposition 4.29. *Let ξ^F, Φ, σ and ν satisfy Assumption 2.2, 3.1 and Assumption 4.28. Let further $\rho_0 \in L^1(\Omega; L^1(U))$ be non-negative and $\mathcal{F}_0$ measurable and v be defined as in Definition 4.27. If ρ is a stochastic kinetic solution of* (3) *in the sense of Definition 2.8, then it follows almost surely that*

$$\liminf_{\beta\to 0} \left(\beta^{-1} q(U \times [\beta/2,\beta] \times [0,T])\right) = 0.$$

Proof. Let us begin by noting that, whilst we don't know the precise form of the limiting measure q due to the presence of a parabolic defect measure in the limit (see [CP03]), for the regularised equation we have a precise equation for the kinetic measures (see proof of Proposition 5.21 of [FG24]) given by

$$dq^\alpha = \delta_0(\xi - \rho^\alpha)\left(\Phi'(\rho^\alpha)|\nabla\rho^\alpha|^2 + \alpha|\nabla\rho^\alpha|^2\right).$$

However, by Fatou's lemma for measures, we have

$$\begin{aligned} \liminf_{\beta\to 0} 2\beta^{-1}\mathbb{E}\left(q(U \times [\beta/2,\beta] \times [0,T])\right) &\le \liminf_{\beta\to 0} \mathbb{E}\left(\int_0^T \int_U \int_{\beta/2}^{\beta} \frac{1}{\xi}\, dq\right) \\ &= \liminf_{\beta\to 0} \mathbb{E}\left(\int_0^T \int_U \int_{\mathbb{R}} \frac{1}{\xi}\mathbb{1}_{\beta/2\le\xi\le\beta}\, dq\right) \\ &\le \liminf_{\alpha\to 0}\liminf_{\beta\to 0} \mathbb{E}\left(\int_0^T \int_U \int_{\mathbb{R}} \frac{1}{\xi}\mathbb{1}_{\beta/2\le\xi\le\beta} dq^\alpha\right). \end{aligned}$$

Analogous to the energy estimates of Proposition 4.14 and to the entropy estimate of Proposition 4.24, apply Itô's formula to a regularised version of the function $\Psi : U \times [0,\infty) \to \mathbb{R}$ defined by $\Psi(x,0) = 0, \partial_\xi\Psi(x,\xi) = S'(\xi) - v(x)$. One obtains, for the functions Θ_σ and Ψ_σ as in Assumption

4.28, $\Theta_{\nu,i}$ defined by $\Theta_{\nu,i}(0)=0$, $\Theta_{\nu,i}'(\xi)=\nu_i(\xi)/\xi$, that there is a constant $c\in(0,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}\left(\int_U\int_0^t\int_{\mathbb{R}}\frac{1}{\xi}\mathbb{1}_{0\le\xi\le1}\,dq^\alpha\right)=\mathbb{E}\left(\int_U\int_0^t\frac{1}{\rho}\mathbb{1}_{0\le\rho\le1}\left(\Phi'(\rho)|\nabla\rho|^2+\alpha|\nabla\rho|^2\right)\right)\\
&\le\mathbb{E}\int_U(\Psi(x,\rho_0)-\Psi(x,\rho_t))+cT+T\sum_{i=1}^d\int_{\partial U}\Theta_{\nu,i}(\Phi^{-1}(\bar f)\wedge1)\cdot\hat\eta_i+cT\int_{\partial U}\Theta_\sigma(\Phi^{-1}(\bar f)\wedge1)\\
&+T\|\log(\Phi^{-1}(\bar f)\wedge1)\|_{H^1(\partial U)}\left(\|\bar f\|_{L^2(\partial U)}+\alpha\|\Phi^{-1}(\bar f)\|_{L^2(\partial U)}+\|\Psi_\sigma(\Phi^{-1}(\bar f))\|_{L^2(\partial U)}\right)\\
&+T\|\nabla v\|_{L^2(U)}\left(c\|\sigma(\rho)\sigma'(\rho)\|_{L^2(U)}+\|\nu(\rho)\|_{L^2(U;\mathbb{R}^d)}\right)
\end{aligned}\tag{30}$$

Since $\int_{\partial U}|\nabla v|^2=\int_{\partial U}v\frac{\partial v}{\partial\hat\eta}=\int_{\partial U}\log(\Phi^{-1}(\bar f)\wedge1)\frac{\partial v}{\partial\hat\eta}$, if $\Phi^{-1}(\bar f)>1$ then the terms in the final two lines vanish since $log(1)=0$.
Furthermore, we have for the first term on the right hand side, that

$$\Psi(x,\rho_0)=S(\rho_0)-\rho_0v(x)=(\rho_0\wedge1)\log(\rho_0\wedge1)-(\rho_0\wedge1)-\rho_0v(x).$$

And so by the non-negativity of the solution, the initial condition and v, we have by disposing of the negative terms,

$$\mathbb{E}\int_U(\Psi(x,\rho_0)-\Psi(x,\rho_t))\le\mathbb{E}\int_U((\rho_0\wedge1)\log(\rho_0\wedge1)+(\rho_t\wedge1)-\rho_tv(x)).$$

Using Hölder's inequality, the second assumption of Assumption 4.28 and the $L^2(U)$ energy estimate (20), this term is bounded.
Hence putting everything together, we showed that the right hand side of (30) the final term in the above inequality is bounded. However, by working along the dyadic scale $\beta^{(i)}=2^{-i}$ for $i=0,1,\ldots$, we have

$$\sum_{i=0}^\infty\mathbb{E}\left(\int_U\int_0^t\int_{\mathbb{R}}\frac{1}{\beta^{(i)}}\mathbb{1}_{\beta^{(i)}/2\le\xi\le\beta^{(i)}}\,dq^\alpha\right)\le\mathbb{E}\left(\int_U\int_0^t\int_{\mathbb{R}}\frac{1}{\xi}\mathbb{1}_{0\le\xi\le1}\,dq^\alpha\right)\le c.$$

The infinite sum being bounded by a constant (that is decreasing in α) implies that the individual elements of the sum converge to zero, which proves the claim. $\square$

4.4 Existence of solution to generalised Dean-Kawasaki equation

In what follows the arguments are identical to that on the torus and so follow Chapter 5 of [FG24]. We are therefore brief and just provide the main ideas for completeness.
In this subsection we start in Proposition 4.30 by proving the existence of a weak solution of the regularised Dean-Kawasaki equation with smooth and bounded σ (in the sense of Assumption 4.6). We then show that the constructed weak solution is also a stochastic kinetic solution, in the sense that it satisfies a kinetic equation similar to equation (5).
The goal will then subsequently be in Lemma 4.32 to remove the assumption that σ is smooth and bounded, which will be done with an approximation argument. To take the regularisation α limit, showing the existence of a solution to the generalised Dean-Kawasaki equation (3) is done in Theorem 4.38.

Proposition 4.30 (Existence of weak solution to regularised equation (4) with smooth and bounded σ)**.** *Let ξ^F,Φ,σ and ν satisfy Assumptions 2.2, 4.2 and 4.6, and let $\alpha\in(0,1)$. Let further $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable.*
Then there exists a weak solution of the regularised equation (4) *in the sense of Definition 4.7. Additionally the solution satisfies the energy estimates of Proposition 4.14.*

Proof. The idea is to approximate all coefficients by regular ones, use Galerkin projection argument to show existence and then take limits in the correct order.
Start by considering a sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of smooth bounded non-decreasing functions starting at

zero such that Φ_n and Φ_n' converge locally uniformly to Φ and Φ' as $n \to \infty$.
Next consider sequence $\{\nu_n\}_{n\in\mathbb{N}}$ of smooth approximations of ν that converge to ν locally uniformly as $n \to \infty$.
For $K \in \mathbb{N}$ consider the finite dimensional approximation of the noise $\xi^{F,K} := \sum_{k=1}^K f_k(x)B_t^k$, and also the truncated coefficients $F_1^K := \sum_{k=1}^K f_k^2$, $F_2^K := \sum_{k=1}^K f_k \nabla f_k$.
For $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis in $L^2(U)$ which is orthogonal in $H^1(U)$ and $M \in \mathbb{N}$, define $\Pi_M : L^2(U \times [0,T]) \to L^2(U \times [0,T])$ be the projection onto the first M orthonormal basis vectors. That is to say, for any $g \in L^2(U \times [0,T])$,

$$\Pi_M g(x,t) := \sum_{k=1}^M g_k(t) e_k(x)$$

where $g_k(t) := \int_U g(x,t)e_k(x)\,dx$. For brevity denote the space of projected L^2 function by $L^2_M := \Pi_M(L^2(U \times [0,T]))$. Consider the below projected equation with regularised coefficients posed on the space $L^2(\Omega; L^2_M)$

$$\begin{aligned} d\rho =& \Pi_M \left(\Delta\Phi_n(\rho)\,dt + \alpha\Delta\rho\,dt - \nabla\cdot(\sigma(\rho)\,d\xi^{F,K} + \nu_n(\rho)\,dt)\right) \\ &+ \Pi_M \left(\frac{1}{2}\nabla\cdot(F_1^K[\sigma'(\rho)]^2\nabla\rho + \sigma'(\rho)\sigma(\rho)F_2^K)\,dt\right). \end{aligned}$$

The equation is equivalent to a finite dimensional stochastic differential equation. Since Φ_n, ν_n, σ are all smooth and bounded functions, the system has a unique strong solution.
Then we pass to the various limits, first as $M \to \infty$, followed by $K \to \infty$ and finally the limit as $n \to \infty$. We rely on simpler versions of the energy estimates of the previous section (e.g. equation (4.14) and Proposition 4.18) and the compact embedding given by Aubin-Lions-Simon lemma ([Aub63, Lio69, Sim86]) to do this.
The resulting solution is continuous in $L^2(U)$ as a consequence of Itô's formula. The non-negativity of solution follows by applying Itô's formula to approximations of $min(0,\rho)$, similar to what was done in estimate (21). □

Proposition 4.31 (Stochastic kinetic solution of regularised DK equation (4) with smooth and bounded σ)**.** *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2, 4.2 and 4.6, and let $\alpha \in (0,1)$. Let $\rho_0 \in L^{m+1}(\Omega; L^1(U)) \cap L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable.*
Let ρ be a weak solution of (4) with smooth and bounded σ in the sense of Definition 4.7, and $\chi(x,\xi,t) = \mathbb{1}_{0<\xi<\rho(x,t)}$ be the kinetic function on $U \times (0,\infty) \times [0,T]$. Then ρ is a stochastic kinetic solution in the sense that, almost surely, for every $\psi \in C_c^\infty(U \times \mathbb{R})$, $t \in [0,T]$:

$$\begin{aligned} \int_{\mathbb{R}}\int_U \chi(x,\xi,t)\psi(x,\xi)\,dx\,d\xi =& \int_{\mathbb{R}}\int_U \chi(x,\xi,t=0)\psi(x,\xi)\,dx\,d\xi - \alpha\int_0^t\int_U \nabla\rho\cdot\nabla\psi(x,\xi)|_{\xi=\rho} \\ &- \int_0^t\int_U \left(\Phi'(\rho)\nabla(\rho) + \frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho + \frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2\right)\cdot\nabla\psi(x,\xi)|_{\xi=\rho}\,dx\,dt \\ &- \int_0^t\int_U \partial_\xi\psi(x,\rho)\Phi'(\xi)|\nabla\rho|^2 - \alpha\int_0^t\int_U \partial_\xi\psi(x,\rho)|\nabla\rho|^2 \\ &+ \frac{1}{2}\int_0^t\int_U \left(\sigma'(\rho)\sigma(\rho)\nabla\rho\cdot F_2 + \sigma(\rho)^2F_3\right)\partial_\xi\psi(x,\rho)\,dx\,dt \\ &- \int_0^t\int_U \psi(x,\rho)\nabla\cdot(\sigma(\rho)\,d\xi^F)\,dx - \int_0^t\int_U \psi(x,\rho)\nabla\cdot\nu(\rho). \end{aligned}$$

The derivatives of the test function are again interpreted in the sense of Remark 2.9.

Proof (idea). The proof follows precisely the steps for deriving the stochastic kinetic equation (5). Begin by using Itô's formula to derive an equation for $S(\rho)$ for a smooth and bounded function $S : \mathbb{R} \to \mathbb{R}$. Secondly derive a formula for the integral

$$\int_U S(\rho)\psi(x)$$

for test function $\psi \in C_c^\infty(U)$ using Definition 4.7 of a weak solution. Finally the kinetic equation is derived, noting the density of linear combinations of functions of the form $S'(\xi)\psi(x)$ in $C_c^\infty(U \times \mathbb{R})$. One noteworthy point is that, as mentioned above, the kinetic measure corresponding to the solution ρ constructed above is

$$q = \delta_0(\xi - \rho)\Phi'(\xi)|\nabla\rho|^2 + \alpha\delta_0(\xi - \rho)|\nabla\rho|^2 = \delta_0(\xi - \rho)\left(|\nabla\Theta_\Phi(\rho)|^2 + \alpha|\nabla\rho|^2\right).$$

The measure is finite due to the estimates of Proposition 4.14 and satisfies the other assumptions of a kinetic measure as in Definition 2.8 due to Assumption 4.2.
For further details see proof of Proposition 5.21 of [FG24]. $\square$

Next we wish to extend the well-posedness to the generalised Dean-Kawasaki equation (3). The first step is to dispense of the regularity assumption on σ of Assumption 4.6.

Lemma 4.32 (Approximating σ in C^1_{loc})**.** *Let σ satisfy Assumption 4.2. Then one can find a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that for each n, σ_n satisfies Assumption 4.6. Further, the sequence uniformly satisfy Assumption 4.2 and $\sigma_n \to \sigma$ in $C^1_{loc}((0,\infty))$.*

The proof follows from constructing smooth bounded approximations by convolution which can be done due to the regularity of σ from Assumption 4.2.
The difficulty in extending the well-posedness to (3) is that the weak solution constructed in Proposition 4.30 does not have a stable $W_t^{\beta,1}H^{-s}$ energy estimate. We only have stable $W_t^{\beta,1}H^{-s}$ for the solution bounded away from its zero set, as in Proposition 4.18. We deal with this by defining the below metric on $L^1_tL^1_x$. Tightness of the cutoff solution $\Phi_\delta(\rho)$ as in Definition (4.17) will be proved with respect to this metric.

Definition 4.33 (New metric on $L^1_tL^1_x$)**.** *For $\delta \in (0,1)$ let Ψ_δ be defined as in Definition (4.17). Define $D : L^1([0,T]; L^1(U)) \to [0,\infty)$ by*

$$D(f,g) = \sum_{k=1}^\infty 2^{-k}\left(\frac{\|\Psi_{1/k}(f) - \Psi_{1/k}(g)\|_{L^1([0,T];L^1(U))}}{1 + \|\Psi_{1/k}(f) - \Psi_{1/k}(g)\|_{L^1([0,T];L^1(U))}}\right).$$

Lemma 4.34. *The function D defined above is a metric on $L^1([0,T]; L^1(U))$. The metric topology defined by D coincides with the strong norm topology on $L^1([0,T]; L^1(U))$.*

The proof of the above lemma can be found as Lemma 5.24 in [FG24]. Instead of assuming σ in the regularised equation (4) satisfies Assumption 4.6, we define an approximate equation with σ_n as defined in Lemma 4.32 approximating σ.

Definition 4.35 (Regularised and smoothed σ equation)**.** *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2 and 4.2, and let $T \in [1,\infty)$. Let $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable, and σ_n as in Lemma 4.32.*
For every $n \in \mathbb{N}$ and $\alpha \in (0,1)$, define $\rho^{a,n}$ to be the stochastic kinetic solution of

$$\begin{aligned} d\rho^{\alpha,n} &= \Delta\Phi(\rho^{\alpha,n})\,dt + \alpha\Delta\rho^{\alpha,n}\,dt - \nabla\cdot(\sigma_n(\rho^{\alpha,n})\,d\xi^F + \nu(\rho^{\alpha,n})\,dt) \\ &\quad + \frac{1}{2}\nabla\cdot(F_1[\sigma_n'(\rho^{\alpha,n})]^2\nabla\rho^{\alpha,n} + \sigma_n'(\rho^{\alpha,n})\sigma_n(\rho^{\alpha,n})F_2)\,dt, \end{aligned} \tag{31}$$

in $U \times (0,T)$ with initial data ρ_0 and boundary condition $\Phi(\rho) = \bar{f}$ as constructed in Proposition 4.31.

The below proposition is a key element of the existence proof. The proof can be found in Proposition 5.26 and 5.27 of [FG24].

Proposition 4.36 (Tightness of laws of $\rho^{\alpha,n}$ in $L^1_tL^1_x$ and of martingale term in C^γ_t)**.** *Let ξ^F, Φ, σ and ν satisfy Assumptions 2.2 and 4.2. Let $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$ measurable, σ_n as in Lemma 4.32 and kinetic solutions $\rho^{\alpha,n}$ be as in Definition 4.35.*

1. *The laws of $\{\rho^{\alpha,n}\}_{\alpha\in(0,1),n\in\mathbb{N}}$ are tight on $L^1([0,T];L^1(U))$ with respect to the strong norm topology.*
2. *For each test function $\psi \in C_c^\infty(U\times(0,\infty))$, $\gamma\in(0,1/2)$ the laws of the martingales*

$$M_t^{\alpha,n,\psi} := \int_0^t \int_U \psi(x,\rho^{\alpha,n})\nabla\cdot(\sigma_n(\rho^{\alpha,n})d\xi^F)$$

are tight on $C^\gamma([0,T])$.

One of the main results to prove existence will come from the below technical lemma, see Lemma 1.1 of [GK96] for proof.

Lemma 4.37. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\bar{X}$ be a complete separable metric space. Then a sequence $\{X_n:\Omega\to\bar{X}\}$ of $\bar{X}$ valued random variables converges in probability as $n\to\infty$ if and only if for every pair of sequences $(n_k,m_k)_{k=1}^\infty$ with $n_k,m_k\to\infty$ as $k\to\infty$, there is a further sub-sequence $(n_{k'},m_{k'})_{k=1}^\infty$ such that the joint laws $(X_{n_{k'}},X_{m_{k'}})$ converge weakly as $k'\to\infty$ to a probability measure μ on $\bar{X}\times\bar{X}$ satisfying $\mu(\{(x,y)\in\bar{X}\times\bar{X}: x=y\})=1$.*

We state the main existence result, which is stated as Theorem 5.29 in [FG24]. The full proof can be found there, we will just explain the main idea by putting all the previous results from this section together.

Theorem 4.38 (Existence of solution to (3))**.** *Let ξ^F,Φ,σ and ν satisfy Assumptions 2.2 and 4.2. Let $\rho_0\in L^1(\Omega;L^1(U))$ be non-negative and $\mathcal{F}_0$ measurable.*
Then there exists a stochastic kinetic solution to the generalised Dean-Kawasaki equation (3) in the sense of Definition 2.8. Furthermore, the solution satisfies the estimates of Proposition 4.14.

Proof. We provide the main steps of the proof and omit the technical details.

1. **Tightness.**
   Recall the stochastic kinetic solutions $\{\rho^{\alpha,n}\}_{\alpha\in(0,1),n\in\mathbb{N}}$ as defined in Definition 4.35, martingales $M^{\alpha,n,\psi}$ as introduced in Proposition 4.36 and introduce the measures

$$q^{\alpha,n} := \delta_0(\xi-\rho^{\alpha,n})\left(|\nabla\Theta_\Phi(\rho^{\alpha,n})|^2+\alpha|\rho^{\alpha,n}|^2\right).$$

   Proposition 4.31 gives us existence of the solutions and the energy estimate of Proposition 4.14 allows us to deduce that $\{q^{\alpha,n}\}_{\alpha\in(0,1),n\in\mathbb{N}}$ are finite kinetic measures.
   Using the kinetic equation given in Proposition 4.31, one can write an an equation for the kinetic functions $\chi^{\alpha,n}$ of $\rho^{\alpha,n}$. Fixing a dense sequence of functions $\{\psi_j\}_{j\in\mathbb{N}}$ of $C_c^\infty(U\times(0,\infty))$ in the strong $H^s(U\times(0,\infty))$ topology (for $s>d/2+1$), we consider the random variables coming from the kinetic equation of $\chi^{\alpha,n}$:

$$X^{\alpha,n} := (\rho^{\alpha,n},\nabla\Theta_{\Phi,p}(\rho^{\alpha,n}),\alpha\nabla\rho^{\alpha,n},q^{\alpha,n},(M^{\alpha,n,\psi_j})_{j\in\mathbb{N}})$$

   on the space

$$\bar{X} := L^1(U\times(0,T))\times L^2\left(U\times(0,T);\mathbb{R}^d\right)^2\times\mathcal{M}(U\times(0,\infty)\times[0,T])\times C\left([0,T]\right)^{\mathbb{N}}\right).$$

   Equip $\bar{X}$ with the product topology, with the strong topology on $L^1(U\times(0,T))$, the weak topologies on $L^2(U\times(0,T);\mathbb{R}^d)$ and $\mathcal{M}(U\times(0,\infty)\times[0,T])$ and topology of component wise convergence in the strong norm on $C([0,T])^{\mathbb{N}}$, in particular using the norm constructed before:

$$D_C((f_k),(g_k)) = \sum_{k=1}^\infty 2^{-k}\left(\frac{\|f_k-g_k\|_{C([0,T])}}{1+\|f_k-g_k\|_{C([0,T])}}\right).$$

   To show convergence in probability of the random variables $X^{\alpha,n}$ we try to use Lemma 4.37. To this end, we consider two subsequences (α_k,n_k), (β_k,m_k) such that $\alpha_k,\beta_k\to 0$ and $n_k,m_k\to\infty$ as $k\to\infty$ Consider the laws on $\bar{Y}:=\bar{X}\times\bar{X}\times C([0,T])^{\mathbb{N}}$ of

$$(X^{\alpha_k,n_k},X^{\beta_k,m_k},B),$$

where $B = (B^j)_{j\in\mathbb{N}}$ are the Brownian motions defined in the noise ξ^F in Definition 2.1.
The energy estimate in Proposition 4.14 alongside the two tightness results in Proposition 4.36 show that the laws of $(X^{\alpha,n})$ are tight on $\bar{X}$.

2. **Skorokhod representation theorem.**
By Prokhrov's theorem, passing to a sub-sequence still denoted by $k \to \infty$, there is a probability measure μ on $\bar{Y}$ such that $(X^{\alpha_k,n_k}, X^{\beta_k,m_k}, B) \to \mu$ as $k \to \infty$.
$\bar{X}$ being separable implies that $\bar{Y}$ is separable, so we can apply Skorokhod representation theorem. It tells us that there is an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for every k,
$$(\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k) = (X^{\alpha_k,n_k}, X^{\beta_k,m_k}, B) \qquad \text{in law on } \bar{Y},$$
$$(\tilde{Y}, \tilde{Z}, \tilde{\beta}) = \mu \qquad \text{in law on } \bar{Y},$$
and we have the almost sure convergence as $k \to \infty$:
$$(\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k) \to (\tilde{Y}, \tilde{Z}, \tilde{\beta})$$
in the space $\bar{X}$ and $C([0,T])$. To apply Proposition 4.37 we will show $\tilde{Y} = \tilde{Z}$.

3. **Characterising $\tilde{Y}$.**
It follows from the equality in law of $\tilde{Y}^k$ and X^{α_k,n_k} that there is a $\tilde{\rho}^k \in L^\infty(\Omega \times [0,T]; L^1(U))$ and $\tilde{G}_1^k, \tilde{G}_2^k, \tilde{q}^k, (\tilde{M}^{k,\psi_j})_{j\in\mathbb{N}}$ in the appropriate spaces such that
$$\tilde{Y}^k = (\tilde{\rho}^k, \tilde{G}_1^k, \tilde{G}_2^k, \tilde{q}^k, (\tilde{M}^{k,\psi_j})_{j\in\mathbb{N}}).$$
By converting various expectations $\tilde{\mathbb{E}}$ into expectations $\mathbb{E}$ by using the equalities in law above, and further using that $\alpha_k \nabla\rho^k \rightharpoonup 0$ in $L_t^2 L_x^2$ by energy estimates (Proposition 4.14) tells us that in the limit as $k \to \infty$
$$\tilde{Y} = (\tilde{\rho}, \nabla(\Theta_{\Phi,p}(\tilde{\rho}), 0, \tilde{q}, (\tilde{M}^j)_{j\in\mathbb{N}}),$$
where $\tilde{p} \in L^1(\tilde{\Omega} \times [0,T]; L^1(U))$ and $\tilde{q}$ is the corresponding kinetic measure.
It remains to characterise $\tilde{M}^j$, and to do this we first need to characterise $\tilde{\beta}^k$.

4. **The path $\tilde{\beta}$ is a Brownian Motion.**
Writing for each k, $\tilde{\beta}^k := (\tilde{\beta}^{k,j})_{j\in\mathbb{N}}$, and the limiting process $\tilde{\beta} = (\tilde{\beta}^j)_{j\in\mathbb{N}}$, one obtains using the same trick of interchanging expectations $\tilde{\mathbb{E}}$ and $\mathbb{E}$ using equalities in law that, by proving first for $\tilde{\beta}^{k,j}$ then passing to the limit in k, that for any $F : \bar{Y} \to \mathbb{R}$, $0 \le s \le t \le T$, $j \in \mathbb{N}$
$$\tilde{\mathbb{E}}\left(F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right)\left(\tilde{\beta}_t^j - \tilde{\beta}_s^j\right)\right) = 0.$$
Identically for $i, j \in \mathbb{N}$, $0 \le s \le t \le T$,
$$\tilde{\mathbb{E}}\left(F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right)\left(\tilde{\beta}_t^j\tilde{\beta}_t^i - \tilde{\beta}_s^j\tilde{\beta}_s^i - \delta_{ij}(t-s)\right)\right) = 0,$$
where δ_{ij} is the Kronecker delta. Using these and the fact that $\tilde{\beta}^j$ has almost surely continuous paths,we conclude using Lévy's characterisation that $\tilde{\beta}^j$ are independent one dimensional Brownian motions with respect to the filtration
$$\mathcal{G}_t = \sigma(\tilde{Y}|_{[0,t]}, \tilde{Z}|_{[0,t]}, \tilde{\beta}|_{[0,t]}).$$
By continuity and uniform integrability $\tilde{\beta}$ is a Brownian motion with respect to the augmented filtration $\bar{\mathcal{G}}$ of $\mathcal{G}$.

5. **$(\tilde{M}^j)_{j\in\mathbb{N}}$ are $\bar{\mathcal{G}}_t$ martingales.**
   The statement follows using a similar technique as the previous point. First showing for $j \in \mathbb{N}$, $0 \le s \le t \le T$ and $k \in \mathbb{N}$,
$$\tilde{\mathbb{E}}\left(F\left(\tilde{Y}^k|_{[0,s]}, \tilde{Z}^k|_{[0,s]}, \tilde{\beta}^k|_{[0,s]}\right)\left(\tilde{M}_t^{k,\psi_j} - \tilde{M}_s^{k,\psi_j}\right)\right) = 0.$$
   The result follows by taking the limit as $k \to \infty$ using the uniform integrability of the martingales.

6. **$(\tilde{M}^j)_{j\in\mathbb{N}}$ are stochastic integrals with respect to $\tilde{\beta}$.**
   Again this follows from the same techniques as before. First proving the results for the approximations and then taking a limit as $k \to \infty$, we can prove that
$$\tilde{\mathbb{E}}\left(F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right)\left(\tilde{M}_t^j \tilde{\beta}_t^i - \tilde{M}_s^j \tilde{\beta}_s^i - \int_s^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_i)\right)\right) = 0,$$
   where recall f_i are defined as the spacial components of the noise ξ^F. Hence this shows for each $i \in \mathbb{N}$,
$$\tilde{M}_t^j \tilde{\beta}_t^i - \int_s^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_i) \qquad \text{is a } \mathcal{G}-\text{martingale.}$$
   It is easy to see by uniform integrability and continuity that the process is also a $\bar{\mathcal{G}}_t$ martingale. Identical arguments show that for $j \in \mathbb{N}$
$$(\tilde{M}_t^j)^2 - \int_0^t \sum_{k=1}^{\infty}\left(\int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_k)\right)^2$$
   is a continuous $\bar{\mathcal{G}}_t$ martingale. Putting everything together, due to an explicit calculation using the quadratic variation of Brownian motion, for every $j \in \mathbb{N}$, $t \in [0,T]$,
$$\tilde{E}\left(\left(\tilde{M}_t^j - \int_0^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})\tilde{\xi}^F)\right)^2\right) = 0,$$
   where $\tilde{\xi}^F$ is defined analogously to ξ^F but with Brownian Motion $\tilde{\beta}$ on $\tilde{\Omega}$. It follows that $\tilde{M}_t^j = \int_0^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})\tilde{\xi}^F)$.

7. **Tying up loose ends.**
   One needs to show the following technical steps in order to finish the proof.
   (a) Show the limiting kinetic measure $\tilde{q}$ is almost surely a kinetic measure for $\tilde{\rho}$.
   (b) Show that $\sigma(\tilde{\rho})$ is in L^2.
   (c) Remove the set $\mathcal{A} := \{t \in [0,T] : \tilde{q}(\{t\} \times U \times \mathbb{R}) \neq 0\}$. Outside $\mathcal{A}$ there is no ambiguity when writing the kinetic equation for the kinetic function $\tilde{\chi}$.
   (d) Show that $\tilde{\rho} \in L^1([0,T]; L^1(U))$.
   This is quite a technical step, it involves looking at left and right continuous representations of $\tilde{\rho}$. One needs to study properties of the left and right kinetic functions $\chi^{\pm}$. Conclude by showing that the measure $\tilde{q}$ almost surely has no atoms in time.

8. **Conclusion.**
   We showed the existence of $\tilde{\rho}$ with representative in $L^1(\tilde{\Omega} \times [0,T]; L^1(U))$ $\tilde{\rho}$ is a stochastic kinetic solution in the sense of Definition 2.8 with respect to Brownian Motion $\tilde{\beta}$ and filtration $(\bar{\mathcal{G}}_t)_{t\in[0,T]}$. That is to say, we showed the existence of a probabilistically weak solution. We now explain how to extend this to a probabilistically strong solution.
   Repeating all steps from Step 3 of the above, it follows that we can characterise $\tilde{Z}$ as
$$\tilde{Z} = (\bar{\rho}, \nabla(\Theta_{\Phi,p}(\bar{\rho}), 0, \bar{q}, (\bar{M}^j)_{j\in\mathbb{N}}).$$

Continuing, one shows there is an L^1 continuous representation of $\bar{\rho}$ which is a stochastic kinetic solution in the sense of Definition 2.8 with respect to Brownian Motion $\tilde{\beta}$ and filtration $(\bar{\mathcal{G}}_t)_{t\in[0,T]}$.
The uniqueness theorem, Theorem 3.5 tells us $\tilde{\rho} = \bar{\rho}$ almost surely in $L^1_t L^1_x$.
By Lemma 4.37, it follows that after passing $\{\rho^{a,n}\}$ to a sub-sequence α_k, n_k on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there is a random variable $\rho \in L^1(\Omega \times [0,T]; L^1(U))$ such that ρ^{α_k,n_k} converges to ρ in probability.
Working along a further sub-sequence we have that ρ^{α_k,n_k} converges almost surely to ρ. Repeating the steps again above we can show ρ is a stochastic kinetic solution of the generalised Dean-Kawasaki equation (3) in the sense of Definition 2.8. Noting that $\rho^{\alpha,n}$ are all probabilistically strong solutions, ρ is also a probabilistically strong solution. The energy estimates follow from the estimates for the regularised equations and the weak lower semicontinuity of the Sobolev norm.

□

Acknowledgements

The author is grateful to his PhD advisor Benjamin Fehrman for the constant academic support and guidance. This work would not have been possible without him.
This publication is based on work supported by the EPSRC Centre for Doctoral Training in Mathematics of Random Systems: Analysis, Modelling and Simulation (EP/S023925/1).

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