A THIRD-ORDER LOW-REGULARITY TRIGONOMETRIC INTEGRATOR FOR THE SEMILINEAR KLEIN-GORDON EQUATION*

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Abstract. In this paper, we propose and analyze a novel third-order low-regularity trigonometric integrator for the semilinear Klein-Gordon equation in the d-dimensional space with $d = 1, 2, 3$. The integrator is constructed based on the full use of Duhamel's formula and the technique of twisted function to the trigonometric integrals. Rigorous error estimates are presented and the proposed method is shown to have third-order accuracy in the energy space under a weak regularity requirement in $H^2 \times H^1$. A numerical experiment shows that the proposed third-order low-regularity integrator is much more accurate than the well-known exponential integrators of order three for approximating the Klein-Gordon equation with nonsmooth solutions.

Key words. Third order scheme, Low-regularity integrator, Error estimate, Klein–Gordon equation

AMS subject classifications. 35L70, 65M12, 65M15, 65M70

1. Introduction. This article concerns the numerical solution of the following semilinear Klein–Gordon equation (SKGE):

$$\begin{cases} \partial_{tt} u(t,x) - \Delta u(t,x) + \rho u(t,x) = f\big(u(t,x)\big), & 0 < t \le T,\ x \in \mathbb{T}^d \subset \mathbb{R}^d, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = v_0(x),\ x \in \mathbb{T}^d, \end{cases} \tag{1.1}$$

under the homogeneous Dirichlet boundary condition, where $u(t,x)$ represents the wave displacement at time t and position x, $d = 1, 2, 3$ is the dimension of x, $\rho \ge 0$ is a given parameter which is interpreted physically as the mass of the particle and $f(u) : \mathbb{R} \to \mathbb{R}$ is a given nonlinear function. Theoretically, the NKGE is globally well-posed for the initial data $(u_0(x), v_0(x)) \in H^\gamma(\mathbb{T}^d) \times H^{\gamma-1}(\mathbb{T}^d)$ for $\gamma \ge 1$ in high dimensions in general. There are many well known examples coming from different choices of $f(u)$. For the very special case $f(u) = 0$, the above Klein-Gordon equation is known as the relativistic version of the Schrödinger equation which correctly describes the spinless relativistic composite particles such as the pion and the Higgs boson [15]. In the case $f(u) = \sin(u)$, the equation (1.1) corresponds to the sine–Gordon equation which arises in various physical applications [4]. If $f(u) = \lambda u^3$ with a given dimensionless parameter λ (positive and negative for defocusing and focusing self-interaction, respectively), the equation covers the cubic wave equation under the choice of $\rho = 0$ and this system widely exists in plasma physics [7].

Classical time discretization methods have been extensively developed and researched in recent decades for solving SKGE, such as splitting methods [1, 9, 16], trigonometric/exponential integrators [3, 8, 17, 18, 35], uniformly accurate methods [2, 5, 6, 11] and structure-preserving methods [10, 12, 14, 25, 26, 27]. For classical methods and their analysis, strong regularity assumptions are unavoidable since every two temporal derivatives in the solution of the wave equation can be converted to two spatial derivatives in the solution. Therefore, in order to have mth-order approximation of the solution $(u, \partial_t u)$ in the space $H^\gamma(\mathbb{T}^d) \times H^{\gamma-1}(\mathbb{T}^d)$ with $\gamma \ge 1$, the initial data of (1.1) is generally required to be in the stronger space $H^{\gamma+m-1}(\mathbb{T}^d) \times H^{\gamma+m-2}(\mathbb{T}^d)$. However, such requirement may not be satisfied since the initial data can be nonsmooth in various real-world applications.

To overcome this barrier, much attention has been paid to the equations with nonsmooth initial data recently. A pioneering work is [28], where a low-regularity exponential-type integrator was designed for nonlinear Schrödinger equation and the proposed scheme could have

*Submitted to the editors DATE.

Funding: This work was supported by NSFC (12371403, 12271426) and Key Research and Development Projects of Shaanxi Province (2023-YBSF-399).

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first-order convergence in H^{γ} for initial data in $H^{\gamma+1}$ (the traditional regularity assumption is $H^{\gamma+2}$ before the work [28]). Since this work [28], many different kinds of low-regularity (LG) integrators have been developed for various equations, including Navier-Stokes equation [22], Dirac equation [33], KdV equation [32, 39], and Schrödinger equations [28, 29, 30, 31].

For the system (1.1) of Klein–Gordon equation, some important integrators with low-regularity property have also been formulated in recently years. A second-order method was presented in [31] and the convergence was shown in the energy space $H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ under the weaker regularity condition $(u_0(x), v_0(x)) \in H^{\frac{7}{4}}(\mathbb{T}^d) \times H^{\frac{3}{4}}(\mathbb{T}^d)$. For a special nonlinear function $f(u) = u^3$ in one dimension $d = 1$, a symmetric low-regularity integrator was derived in [38] which possess second-order accuracy in $H^r(\mathbb{T}^d) \times H^{r-1}(\mathbb{T}^d)$ without loss of regularity of the solution. Recently, for the SKGE (1.1) with a general nonlinear function $f(u)$, second order convergence in $H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ was obtained for a new integrator in [23] under the weaker regularity condition $(u_0(x), v_0(x)) \in H^{1+\frac{d}{4}}(\mathbb{T}^d) \times H^{\frac{d}{4}}(\mathbb{T}^d)$ with $d = 1, 2, 3$. Moreover recently, a symmetric second-order low-regularity integrator was constructed in [36] and it was shown to have the same low regularity as that of [23] but with good long time near conservations.

As far as we know, all the existing low-regularity integrators for the Klein–Gordon equation and/or other equations are devoted to first-order and second-order schemes. It is an interesting question whether higher-order algorithm can achieve low regularity property. The answer is positive but the construction of higher-order algorithms is very challenging. There are two main difficulties in designing higher order low-regularity integrators. The first one is how to control the spatial derivatives in the approximation such that the regularity can be weakened. The second one lies in the complicated structure in the design process which brings great constriction in the formulation of higher-order schemes.

The objective of this article is to construct and analyse a third-order low-regularity integrator (Definition 2.1 given below) for the nonlinear Klein–Gordon equation (1.1) with a general nonlinear function $f(u)$ and the dimension $d = 1, 2, 3$. There are two central ideas to achieve this goal. We first embed the structure of SKGE in the formulation and then apply the technique of twisted function to the trigonometric integrals appeared in the Duhamel's formulation. By carefully selecting the tractable terms from the trigonometric integrals, the spatial derivatives are almost uniformly distributed to the product terms in the remainder and then a new integrator is obtained. With rigorously analysis, the proposed integrator will be shown to have a third order convergence in $H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ under the weaker regularity condition $(u_0(x), v_0(x)) \in H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$ (classical third-order methods require $(u_0(x), v_0(x)) \in H^3(\mathbb{T}^d) \times H^2(\mathbb{T}^d)$). Compared with many existing LG methods, the new integrator presented in this paper has a very simple scheme and it is very convenient to use for the researchers even coming from different disciplines. Moreover, a fully-discrete scheme is also proposed and studied in this article. The error bounds form spatial discretisation and semidiscretisation in time are simultaneously researched for SKGE (1.1) with a nonsmooth initial data.

The rest of this article is organized as follows. In Section 2, we present the formulation of the semi-discrete and fully-discrete methods. The error estimates of the proposed two schemes are rigorously analysed in Section 3. Some numerical results are given in Section 4 to show the superiorities of the new integrators in comparison with some existing methods in the literature. The conclusions are drawn in Section 5.

2. Construction of low-regularity integrator.

In this section we will first introduce the notations and technical tools which are used frequently in this paper. Then the new low-regularity integrator including semi-discrete and fully-discrete schemes will be formulated.

2.1. Notations and technical tools.

For simplicity, we denote by $A \lesssim B$ the statement $A \leq CB$ for a generic constant $C > 0$. The constant C may depend on T and $\|u_0\|_{H^2}$, $\|v_0\|_{H^1}$, and may be different at different occurrences, but is always independent of the time and space discrete number of points and the time/space step sizes.

For the linear differential operator appeared in (1.1), we denote it by a concise notation

$$(\mathcal{A}w)(x) := -\Delta w(x) + \rho w(x) \tag{2.1}$$

with $w(x)$ on $\mathbb{T}^d$. In the construction of the integrator, we will frequently use the following functions

$$\mathrm{sinc}(t) := \frac{\sin(t)}{t}, \ \alpha(t) := \begin{pmatrix} \cos(t\sqrt{\mathcal{A}}) \\ t\mathrm{sinc}(t\sqrt{\mathcal{A}}) \end{pmatrix}, \ \beta(t) := \begin{pmatrix} t\mathrm{sinc}(t\sqrt{\mathcal{A}}) \\ \cos(t\sqrt{\mathcal{A}}) \end{pmatrix}, \ \gamma(t) := \begin{pmatrix} \sqrt{\mathcal{A}}\sin(t\sqrt{\mathcal{A}}) \\ \cos(t\sqrt{\mathcal{A}}) \end{pmatrix}. \tag{2.2}$$

The Sobolev space of functions on the domain $\mathbb{T}^d = [a,b]^d$ is considered in the whole paper, and we shall refer it as $H^\nu(\mathbb{T}^d)$ for any $\nu \geq 0$. The norm on this Sobolev space is denoted by

$$\|f\|^2_{H^\nu} = (b-a)^d \sum_{\xi\in\mathbb{Z}^d} (1+|\xi|^2)^\nu \left|\widehat{f}(\xi)\right|^2,$$

where $\widehat{f}$ is the Fourier transform of $f(x)$ which is defined by $\widehat{f}(\xi) = \frac{1}{(b-a)^d}\int_{\mathbb{T}^d} e^{-ix\cdot\xi} f(x)dx$ with $\xi = (\xi^1, \xi^2, \ldots, \xi^d) \in \mathbb{Z}^d$ and $x = (x^1, x^2, \ldots, x^d) \in \mathbb{Z}^d$. Here $|\cdot|$ is defined as $|\xi| = \sqrt{(\xi^1)^2 + (\xi^2)^2 + \ldots + (\xi^d)^2}$.

For the norm on Sobolev space, some obvious properties are frequently used in this article and we summarize them as follows.

- With the notation $J^\nu := (1-\Delta)^{\frac{\nu}{2}}$ for $\nu \geq 0$, it is clearly that

$$\widehat{J^\nu f}(\xi) = (1+|\xi|^2)^{\frac{\nu}{2}}\widehat{f}(\xi) \quad \text{and} \quad \|J^\nu f\|_{L^2} = \|f\|_{H^\nu}.$$

- For any function $\sigma : \mathbb{Z}^d \to \mathbb{C}^d$ such that $|\sigma(\xi)| \leq C_\sigma(1+|\xi|^2)^m$ with some constants C_σ and m, the operator $\sigma(-i\sqrt{\Delta}) : H^\nu(\mathbb{T}^d) \to H^{\nu-m}(\mathbb{T}^d)$ has the following result

$$\sigma(-i\sqrt{\Delta})f(x) = \sum_{\xi\in\mathbb{Z}^d} \sigma(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}, \quad \left\|\sigma(-i\sqrt{\Delta})f\right\|^2_{H^{\nu-m}} \leq C_\sigma \|f\|^2_{H^\nu}.$$

- For the special operator $\sqrt{\Delta}^{-1}$ which is defined by

$$\widehat{\sqrt{\Delta}^{-1} f}(\xi) = \begin{cases} \dfrac{\widehat{f}(\xi)}{|\xi|}, & \text{when} \quad |\xi| \neq 0, \\ 0, & \text{when} \quad |\xi| = 0, \end{cases}$$

it is straightforward to verify that

$$\left\|\sqrt{\Delta}^{-1} f\right\|_{H^{\nu+1}} \lesssim \|f\|_{H^\nu}.$$

For the operator $\mathcal{A}$ introduced in (2.1), the following estimates are consequences of the properties stated above.

PROPOSITION 2.1. *If $f \in H^\nu$ for any $\nu \geq 0$, then the following results hold*

$$\left\|\cos(t\sqrt{\mathcal{A}})f\right\|_{H^\nu} \leq \|f\|_{H^\nu}, \ \left\|\sin(t\sqrt{\mathcal{A}})f\right\|_{H^\nu} \leq \|f\|_{H^\nu}, \ \left\|\mathrm{sinc}(t\sqrt{\mathcal{A}})f\right\|_{H^\nu} \leq \|f\|_{H^\nu},$$

and

$$\left\|\sqrt{\mathcal{A}}^{-1} f\right\|_{H^{\nu+1}} \lesssim \|f\|_{H^\nu}.$$

Proof. The first statement is obvious. For the second one, it is easy to check that $\left\|\frac{1}{\sqrt{\mathcal{A}}}f\right\|_{H^{\nu+1}} = \|f\|_{H^\nu}$ if $\rho = 1$, $\left\|\frac{1}{\sqrt{\mathcal{A}}}f\right\|_{H^{\nu+1}} < \|f\|_{H^\nu}$ if $\rho > 1$ and $\left\|\frac{1}{\sqrt{\mathcal{A}}}f\right\|_{H^{\nu+1}} < 2\|f\|_{H^\nu}$ if $\rho < 1$. Therefore, the proof is complete. ∎

The following version of the Kato–Ponce inequalities will also be needed in this paper, which was originally given in [21] and developed in [24].

LEMMA 2.1. *(The Kato–Ponce inequalities) If $\nu > 1/2$ and $f, g \in H^\nu$, then we have*

$$\|fg\|_{H^\nu} \lesssim \|f\|_{H^\nu} \|g\|_{H^\nu}, \quad \left\|J^{-1}(Jfg)\right\|_{H^\nu} \lesssim \|f\|_{H^\nu} \|g\|_{H^\nu}.$$

If $\nu \geq 0, \nu_1 > 1/2$ and $f \in H^{\nu+\nu_1}$, $g \in H^\nu$, the inequality

$$\|fg\|_{H^\nu} \lesssim \|f\|_{H^{\nu+\nu_1}} \|g\|_{H^\nu}$$

holds. If $f, g \in H^1$, then

$$\left\|J^{-1}(Jfg)\right\|_{L^2} \lesssim \min\{\|f\|_{L^2} \|g\|_{H^1}, \|g\|_{L^2} \|f\|_{H^1}\}.$$

2.2. Semi-discrete scheme. In this subsection, we construct the semi-discrete numerical method based on twisted functions and Duhamel's formula. For readers' convenience, the x-dependence of the unknown functions is omitted and some technical estimates of remainders are deferred to Section 3, where a rigorous error analysis will be given.

Let $t_n = nh$ with $n = 0, 1, \ldots, N$ be a uniform partition of the time interval $[0, T]$ with stepsize $h = T/N$, where N is any given positive integer. For the nonlinear Klein–Gordon equation (1.1), the Duhamel's formula at $t = t_n + s$ with $s \in \mathbb{R}$ reads

$$\begin{aligned} u(t_n+s) =& \cos(s\sqrt{\mathcal{A}})u(t_n) + s\mathrm{sinc}(s\sqrt{\mathcal{A}})v(t_n) + \int_0^s (s-\theta)\mathrm{sinc}((s-\theta)\sqrt{\mathcal{A}})f\big(u(t_n+\theta)\big)d\theta, \\ v(t_n+s) =& -s\mathcal{A}\mathrm{sinc}(s\sqrt{\mathcal{A}})u(t_n) + \cos(s\sqrt{\mathcal{A}})v(t_n) + \int_0^s \cos((s-\theta)\sqrt{\mathcal{A}})f\big(u(t_n+\theta)\big)d\theta, \end{aligned} \tag{2.3}$$

which plays a crucial role in the method's formulation. Using this formula, we insert the expression of $u(t_n + \theta)$ into the trigonometric integrals on the right hand side of (2.3). Then by carefully selecting the tractable terms from the obtained formulae and dropping some parts which do not affect accuracy and regularity, the semi-discrete scheme is formulated. The detailed procedure is presented blew.

Firstly, we get the expression of $u(t_n + s)$ by the first equation of (2.3) and then insert this into $f\big(u(t_n + s)\big)$, which gives

$$\begin{aligned} f\big(u(t_n+s)\big) =& f\Big(\alpha^\intercal(s)U(t_n)\Big) + f'\Big(\alpha^\intercal(s)U(t_n)\Big)\Big(u(t_n+s) - \alpha^\intercal(s)U(t_n)\Big) \\ &+ R_{f''}(t_n,s)\Big(u(t_n+s) - \alpha^\intercal(s)U(t_n)\Big)^2, \end{aligned} \tag{2.4}$$

where $U(t_n) := (u(t_n), v(t_n))^\intercal$ and

$$R_{f''}(t_n,s) := \int_0^1 \int_0^1 \theta f''\Big((1-\xi)\alpha^\intercal(s)U(t_n) + \xi(1-\theta)\alpha^\intercal(s)U(t_n) + \theta u(t_n+s)\Big)d\xi d\theta. \tag{2.5}$$

Using the expression of $u(t_n + s)$ again in the right hand side of (2.4) yields the following expression:

$$\begin{aligned} f\big(u(t_n+s)\big) =& f\Big(\alpha^\intercal(s)U(t_n)\Big) + f'\Big(\alpha^\intercal(s)U(t_n)\Big)\int_0^{+s} \frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta \\ &+ R_{f''}(t_n,s)\Big(\int_0^{+s} \frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta\Big)^2. \end{aligned} \tag{2.6}$$

Now we obtained a desired form of $f\big(u(t_n+s)\big)$ and based on which, the two trigonometric integrals on the right hand side of (2.3) can be reformulated as

$$
\begin{aligned}
&\int_0^h (h-s)\mathrm{sinc}((h-s)\sqrt{\mathcal{A}})f\big(u(t_n+s)\big)ds = \underbrace{\int_0^h \frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\Big(\alpha^{\intercal}(s)U(t_n)\Big)ds}_{=:\,\mathrm{I}^u}\\
&+\underbrace{\int_0^h \frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f'\Big(\alpha^{\intercal}(s)U(t_n)\Big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta ds}_{=:\,\mathrm{II}^u}\\
&+\underbrace{\int_0^h \frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} R_{f''}(t_n,s)\Big(\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta\Big)^2 ds}_{=:\,\mathrm{III}^u},
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&\int_0^h \cos((h-\theta)\sqrt{\mathcal{A}})f\big(u(t_n+\theta)\big)d\theta = \underbrace{\int_0^h \cos((h-s)\sqrt{\mathcal{A}}) f\Big(\alpha^{\intercal}(s)U(t_n)\Big)ds}_{=:\,\mathrm{I}^v}\\
&+\underbrace{\int_0^h \cos((h-s)\sqrt{\mathcal{A}}) f'\Big(\alpha^{\intercal}(s)U(t_n)\Big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta ds}_{=:\,\mathrm{II}^v}\\
&+\underbrace{\int_0^h \cos((h-s)\sqrt{\mathcal{A}}) R_{f''}(t_n,s)\Big(\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta\Big)^2 ds}_{=:\,\mathrm{III}^v}.
\end{aligned}
\tag{2.8}
$$

The main objective is to find some computable third-order approximations of I^u&I^v, II^u&II^v, and III^u&III^v, which will be derived one by one in the follows.

• **Approximation to I^u and I^v.** We first deal with the integral I^u. To this end, we consider the technology of twisted variable which was widely used in the development of low-regularity time discretizations [22, 23, 28, 29, 30, 31, 33, 38]. In this paper, we introduce a new twisted function

$$F(t_n+s) := \beta(-s)f\big(\alpha^{\intercal}(s)U(t_n)\big).$$

After splitting the term $\frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}$ in I^u into $\alpha^{\intercal}(h)\beta(-s)$, I^u can expressed by the twisted function $F(t_n+s)$:

$$\mathrm{I}^u = \int_0^h \alpha^{\intercal}(h)\beta(-s)f\big(\alpha^{\intercal}(s)U(t_n)\big)ds = \int_0^h \alpha^{\intercal}(h)F(t_n+s)ds.$$

Considering the Newton–Leibniz formula for the twisted function F:

$$F(t_n+s) = F(t_n) + \int_0^s F'(t_n+\zeta)d\zeta,$$

we get

$$
\begin{aligned}
\mathrm{I}^u &= \int_0^h \alpha^{\intercal}(h)F(t_n)ds + \int_0^h \alpha^{\intercal}(h)\int_0^s F'(t_n+\zeta)d\zeta ds\\
&= \int_0^h \alpha^{\intercal}(h)F(t_n)ds + \int_0^h \alpha^{\intercal}(h)F'(t_n+\zeta)(h-\zeta)d\zeta.
\end{aligned}
\tag{2.9}
$$

According to the scheme of $\alpha(h)$ given in (2.2), direct calculation yields the following expression

$$\alpha^{\intercal}(h) = \alpha^{\intercal}(h-s)M(s) \text{ with } M(s) := \begin{pmatrix} \cos(s\sqrt{\mathcal{A}}) & s\mathrm{sinc}(s\sqrt{\mathcal{A}}) \\ -\sqrt{\mathcal{A}}\sin(s\sqrt{\mathcal{A}}) & \cos(s\sqrt{\mathcal{A}}) \end{pmatrix} \text{ for any } s\in\mathbb{R}.$$

Then, (2.9) can be expressed as

$$\begin{aligned}
\mathrm{I}^u =& \int_0^h \alpha^\intercal(h)F(t_n)ds + \int_0^h (h-s)\alpha^\intercal(h-2s)M(2s)F'(t_n+s)ds \\
=& \int_0^h \alpha^\intercal(h)F(t_n)ds + \int_0^h (h-s)\alpha^\intercal(h-2s)M(0)F'(t_n)ds \\
&+ \int_0^h (h-s)\alpha^\intercal(h-2s)\int_0^s \frac{d\big(M(2\zeta)F'(t_n+\zeta)\big)}{d\zeta}d\zeta ds.
\end{aligned}$$

For the term $F'(t_n+\zeta)$ appeared above, it can be computed as:

$$\begin{aligned}
F'(t_n+\zeta) =& \big(\frac{d}{d\zeta}\beta(-\zeta)\big)f\big(\alpha^\intercal(\zeta)U(t_n)\big) + \beta(-\zeta)\frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\
=& \begin{pmatrix} -\cos(\zeta\sqrt{\mathcal{A}})f\big(\alpha^\intercal(\zeta)U(t_n)\big) - \zeta\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})\frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ -\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}})f\big(\alpha^\intercal(\zeta)U(t_n)\big) + \cos(\zeta\sqrt{\mathcal{A}})\frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} \\
=& M(-\zeta)\big(-f\big(\alpha^\intercal(\zeta)U(t_n)\big), \frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big)\big)^\intercal.
\end{aligned} \tag{2.10}$$

This result further leads to

$$\begin{aligned}
&\frac{d}{d\zeta}M(2\zeta)F'(t_n+\zeta) = \frac{d}{d\zeta}M(\zeta)\begin{pmatrix} -f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} \\
=& \Big(\frac{d}{d\zeta}M(\zeta)\Big)\begin{pmatrix} -f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} + M(\zeta)\frac{d}{d\zeta}\begin{pmatrix} -f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} \\
=& \begin{pmatrix} -\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}}) & \cos(\zeta\sqrt{\mathcal{A}}) \\ -\mathcal{A}\cos(\zeta\sqrt{\mathcal{A}}) & -\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}}) \end{pmatrix}\begin{pmatrix} -f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} \\
&+ M(\zeta)\begin{pmatrix} -\frac{d}{d\zeta}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \frac{d^2}{d\zeta^2}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} \\
=& \begin{pmatrix} \sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}})f\big(\alpha^\intercal(\zeta)U(t_n)\big) + \zeta\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})\frac{d^2}{d\zeta^2}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\ \mathcal{A}\cos(\zeta\sqrt{\mathcal{A}})f\big(\alpha^\intercal(\zeta)U(t_n)\big) + \cos(\zeta\sqrt{\mathcal{A}})\frac{d^2}{d\zeta^2}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \end{pmatrix} = \beta(\zeta)\Upsilon(t_n,\zeta),
\end{aligned}$$

with the notation

$$\begin{aligned}
\Upsilon(t_n,\zeta) :=& \mathcal{A}f\big(\alpha^\intercal(\zeta)U(t_n)\big) + \frac{d^2}{d\zeta^2}f\big(\alpha^\intercal(\zeta)U(t_n)\big) \\
=& \mathcal{A}f\big(\alpha^\intercal(\zeta)U(t_n)\big) + f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\gamma^\intercal(\zeta)U(t_n), \gamma^\intercal(\zeta)U(t_n)\big) \\
&- f'\big(\alpha^\intercal(\zeta)U(t_n)\big)\mathcal{A}\alpha^\intercal(\zeta)U(t_n) \\
=& f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\gamma^\intercal(\zeta)U(t_n), \gamma^\intercal(\zeta)U(t_n)\big) \\
&- f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\nabla\alpha^\intercal(\zeta)U(t_n), \nabla\alpha^\intercal(\zeta)U(t_n)\big) \\
&+ \rho f\big(\alpha^\intercal(\zeta)U(t_n)\big) - \rho f'\big(\alpha^\intercal(\zeta)U(t_n)\big)\alpha^\intercal(\zeta)U(t_n).
\end{aligned} \tag{2.11}$$

Here in the last equation, we use the definition of $\mathcal{A}$ (2.1) and split it into $-\Delta$ and ρ. It is noted that the result of $\Upsilon(t_n,\zeta)$ is the key point for reducing the regularity since it changes $\mathcal{A}$ into ∇ in the expression. This good aspect comes from the careful treatment of the twisted function F and the splitting of $\alpha^\intercal(h)$.

With the above results and the splitting of $\Upsilon(t_n,\zeta)$, we derive that

$$\begin{aligned}
\mathrm{I}^u =& \int_0^h \alpha^\intercal(h) ds F(t_n) + \int_0^h (h-s)\alpha^\intercal(h-2s)M(0)dsF'(t_n)\\
&+ \int_0^h (h-s)\alpha^\intercal(h-2s)\int_0^s \beta(\zeta)\Upsilon(t_n,\zeta)d\zeta ds\\
=&h^2\mathrm{sinc}(h\sqrt{\mathcal{A}})f\big(u(t_n)\big) + \begin{pmatrix} h^2/2\mathrm{sinc}(h\sqrt{\mathcal{A}}) \\ \frac{\sin(h\sqrt{\mathcal{A}})-h\sqrt{\mathcal{A}}\cos(h\sqrt{\mathcal{A}})}{2\sqrt{\mathcal{A}}^3} \end{pmatrix}^\intercal \begin{pmatrix} -f\big(u(t_n)\big) \\ f'\big(u(t_n)\big)v(t_n) \end{pmatrix}\\
&+ \int_0^h (h-s)\alpha^\intercal(h-2s)\int_0^s \beta(\zeta)(\Upsilon(t_n,0)+\Upsilon(t_n,\zeta)-\Upsilon(t_n,0))d\zeta ds\\
=&h^2/2\mathrm{sinc}(h\sqrt{\mathcal{A}})f\big(u(t_n)\big) + \frac{\sin(h\sqrt{\mathcal{A}})-h\sqrt{\mathcal{A}}\cos(h\sqrt{\mathcal{A}})}{2\sqrt{\mathcal{A}}^3}f'\big(u(t_n)\big)v(t_n)\\
&+ \frac{2-2\cos(h\sqrt{\mathcal{A}})-h\sqrt{\mathcal{A}}\sin(h\sqrt{\mathcal{A}})}{2\mathcal{A}^2}\Upsilon(t_n,0)+R_1(t_n),
\end{aligned}$$

where the remainder R_1 is defined as

$$R_1(t_n) = \int_0^h (h-s)\alpha^\intercal(h-2s)\int_0^s \beta(\zeta)(\Upsilon(t_n,\zeta)-\Upsilon(t_n,0))d\zeta ds, \tag{2.12}$$

and its boundedness will be shown in Lemma 3.1.

In a very similar way, one deduces that

$$\begin{aligned}
\mathrm{I}^v =& \int_0^h \gamma^\intercal(-h)\beta(-s)f\big(\alpha^\intercal(s)U(t_n)\big)ds\\
=& \int_0^h \gamma^\intercal(-h)F(t_n)ds + \int_0^h \gamma^\intercal(-h)F'(t_n+\zeta)(h-\zeta)d\zeta\\
=& \int_0^h \gamma^\intercal(-h)dsF(t_n) + \int_0^h (h-s)\gamma^\intercal(2s-h)M(0)dsF'(t_n)\\
&+ \int_0^h (h-s)\gamma^\intercal(2s-h)\int_0^s \beta(\zeta)\Upsilon(t_n,\zeta)d\zeta ds\\
=&h(\cos(h\sqrt{\mathcal{A}})+\mathrm{sinc}(h\sqrt{\mathcal{A}}))/2f\big(u(t_n)\big)+h^2/2\mathrm{sinc}(h\sqrt{\mathcal{A}})f'\big(u(t_n)\big)v(t_n)\\
&+h^3\frac{\mathrm{sinc}(h\sqrt{\mathcal{A}})-\cos(h\sqrt{\mathcal{A}})}{2(h\sqrt{\mathcal{A}})^2}\Upsilon(t_n,0)+R_2(t_n)
\end{aligned} \tag{2.13}$$

with the remainder

$$R_2(t_n) = \int_0^h (h-s)\gamma^\intercal(2s-h)\int_0^s \beta(\zeta)(\Upsilon(t_n,\zeta)-\Upsilon(t_n,0))d\zeta ds. \tag{2.14}$$

The estimate of this remainder will be given in Lemma 3.2.

• **Approximation to II^u and II^v.** Now we turn to II^u and II^v. For the first one, fourth-order local error will be derived as follows. From Proposition 2.1, it follows that

$$\begin{aligned}
\|\mathrm{II}^u\|_{H^1} \lesssim& \left\| \int_0^h \frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f'\Big(\alpha^\intercal(s)U(t_n)\Big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}f\big(u(t_n+\theta)\big)d\theta ds\right\|_{H^1}\\
\lesssim& \int_0^h |h-s|\int_0^s \left\|(s-\theta)\mathrm{sinc}((s-\theta)\sqrt{\mathcal{A}})f'\Big(\alpha^\intercal(s)U(t_n)\Big)f\big(u(t_n+\theta)\big)\right\|_{H^1} d\theta ds\\
\lesssim& h^4 \max_{s\in[0,h]}\left\|f'\Big(\alpha^\intercal(s)U(t_n)\Big)\right\|_{H^1}\max_{\zeta\in[0,h]}\left\|f\big(u(t_n+\zeta)\big)\right\|_{H^1}\lesssim h^4.
\end{aligned} \tag{2.15}$$

However, for the part II^v, we only get the following estimate

$$\begin{aligned}\mathrm{II}^v =&\int_0^h \cos((h-s)\sqrt{\mathcal{A}})f'\Big(\alpha^{\intercal}(s)U(t_n)\Big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}f\big(u(t_n+\theta)\big)d\theta ds\\ =&\int_0^h \cos((h-s)\sqrt{\mathcal{A}})f'(u(t_n))\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}d\theta ds f\big(u(t_n)\big)+R_3(t_n)\\ =&h^3\frac{\mathrm{sinc}(h\sqrt{\mathcal{A}})-\cos(h\sqrt{\mathcal{A}})}{2(h\sqrt{\mathcal{A}})^2}f'(u(t_n))f\big(u(t_n)\big)+R_3(t_n),\end{aligned}\tag{2.16}$$

where R_3 is a remainder defined by

$$\begin{aligned}R_3(t_n)=&\int_0^h \cos((h-s)\sqrt{\mathcal{A}})f'\Big(\alpha^{\intercal}(s)U(t_n)\Big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}f\big(u(t_n+\theta)\big)d\theta ds\\ &-\int_0^h \cos((h-s)\sqrt{\mathcal{A}})f'(u(t_n))\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}d\theta ds f\big(u(t_n)\big).\end{aligned}\tag{2.17}$$

The bound of $R_3(t_n)$ will also be studied in Lemma 3.2.

• **Approximation to III^u and III^v.** Finally, we pay attention to the bounds of III^u and III^v which are respectively presented in (2.7) and (2.8). Using Proposition 2.1 again, the following estimate holds

$$\begin{aligned}\|\,\mathrm{III}^u\|_{H^1}\lesssim&\left\|\int_0^h \frac{\sin((h-s)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}R_{f''}(t_n,s)\Big(\int_0^s\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}f\big(u(t_n+\theta)\big)d\theta\Big)^2ds\right\|_{H^1}\\ \lesssim&\int_0^h(h-s)\left\|R_{f''}(t_n,s)\Big(\int_0^s(s-\theta)\mathrm{sinc}((s-\theta)\sqrt{\mathcal{A}})f\big(u(t_n+\theta)\big)d\theta\Big)^2\right\|_{H^1}ds.\end{aligned}$$

In view of (2.5), we obtain

$$\begin{aligned}\|\,\mathrm{III}^u\|_{H^1}\lesssim&\int_0^h(h-s)\left\|R_{f''}(t_n,s)\right\|_{H^1}\int_0^s(s-\theta)^2\left\|f\big(u(t_n+\theta)\big)\right\|_{H^1}^2d\theta ds\\ \lesssim&h^5\max_{s\in[0,h]}\left\|R_{f''}(t_n,s)\right\|_{H^1}\max_{\zeta\in[0,h]}\left\|f\big(u(t_n+\zeta)\big)\right\|_{H^1}^2\lesssim h^5.\end{aligned}\tag{2.18}$$

By the same arguments, it is arrived that

$$\|\,\mathrm{III}^v\|_{L^2}\lesssim h^4.\tag{2.19}$$

As a result, these two parts can be dropped in the numerical scheme without bringing any impact on the accuracy and low-regularity requirement.

Based on these results, we define the following method for the Klein–Gordon equation (1.1) by dropping the remainders $R_1(t_n), R_2(t_n), R_3(t_n)$ and II^u, III^u, III^v.

DEFINITION 2.1. ***(Semi-discrete integrator.)*** *The low-regularity integrator proposed above for the Klein–Gordon equation (1.1) can be written as*

$$\begin{aligned}u_{n+1}=&\cos(h\sqrt{\mathcal{A}})u_n+h\mathrm{sinc}(h\sqrt{\mathcal{A}})v_n+h^2\Phi_1(h\sqrt{\mathcal{A}})f(u_n)+h^3\Psi_1(h\sqrt{\mathcal{A}})f'(u_n)v_n\\ &+h^4\Psi_2(h\sqrt{\mathcal{A}})\Big(f''(u_n)v_n^2-f''(u_n)(\sqrt{\mathcal{A}}u_n)^2+\rho f(u_n)-\rho f'(u_n)u_n\Big),\\ v_{n+1}=&-h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}})u_n+\cos(h\sqrt{\mathcal{A}})v_n+h\Phi_2(h\sqrt{\mathcal{A}})f(u_n)+h^2\Phi_1(h\sqrt{\mathcal{A}})f'(u_n)v_n\\ &+h^3\Psi_1(h\sqrt{\mathcal{A}})\Big(f''(u_n)(v_n^2-(\sqrt{\mathcal{A}}u_n)^2)+\rho f(u_n)-f'(u_n)(\rho u_n-f(u_n))\Big),\end{aligned}\tag{2.20}$$

where $0<h<1$ is a time stepsize, $n=0,1,\dots,T/h-1$ and the coefficient functions of the *integrator are defined by*

$$\begin{aligned}
&\Phi_1(h\sqrt{\mathcal{A}}) = \frac{\operatorname{sinc}(h\sqrt{\mathcal{A}})}{2}, \qquad \Psi_2(h\sqrt{\mathcal{A}}) = \frac{1-\cos(h\sqrt{\mathcal{A}}) - \frac{h}{2}\sqrt{\mathcal{A}}\sin(h\sqrt{\mathcal{A}})}{(h\sqrt{\mathcal{A}})^4},\\
&\Psi_1(h\sqrt{\mathcal{A}}) = \frac{\operatorname{sinc}(h\sqrt{\mathcal{A}}) - \cos(h\sqrt{\mathcal{A}})}{2(h\sqrt{\mathcal{A}})^2},\ \Phi_2(h\sqrt{\mathcal{A}}) = \frac{\cos(h\sqrt{\mathcal{A}}) + \operatorname{sinc}(h\sqrt{\mathcal{A}})}{2}.
\end{aligned} \tag{2.21}$$

Here the prime on f indicates the derivative of $f(u)$ w.r.t. u.

Remark 2.2. From (2.20), it can be observed clearly that this method is a kind of trigonometric integrators. The scheme (2.20) is not complicated even in comparison with some lower-order low-regularity integrators. Therefore, it is convenient to implement the method in practical computations.

2.3. Fully-discrete scheme. This subsection concerns the spatial discretization of (2.20) which can be handled by using trigonometric interpolation [34]. To make the presentation be more concise, it is assumed that $\mathbb{T}^d := [0,1]^d$. Then any function $W \in H_0^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ can be expanded into the Fourier sine series, i.e.,

$$W = \sum_{n_1,n_2,\ldots,n_d=1}^{\infty} W_{n_1,n_2,\ldots,n_d} \sin(n_1\pi x_1)\sin(n_2\pi x_2)\cdots\sin(n_d\pi x_d).$$

Choose a positive integer $N_x > 0$ and denote by $I_{N_x}$and Π_{N_x} the trigonometric interpolation and L^2-orthogonal projection operators onto S_{N_x}, respectively, where the set S_{N_x} is

$$S_{N_x} = \{\sum_{n_1,n_2,\ldots,n_d=1}^{N_x} W_{n_1,n_2,\ldots,n_d} \sin(n_1\pi x_1)\sin(n_2\pi x_2)\cdots\sin(n_d\pi x_d) :\ W_{n_1,n_2,\ldots,n_d} \in \mathbb{R}^2\}.$$

With these notations and based on the semi-discrete scheme given in Definition 2.1, the fully discrete low-regularity integrator is defined as follows:

Definition 2.2. *(**Fully discrete integrator.**) For the semi-discrete integrator (2.20), the fully discrete low-regularity integrator is given by*

$$\begin{aligned}
U_{n+1} =& \cos(h\sqrt{\mathcal{A}})U_n + h\operatorname{sinc}(h\sqrt{\mathcal{A}})V_n + h^2\Phi_1(h\sqrt{\mathcal{A}})I_{N_x}f(U_n) + h^3\Psi_1(h\sqrt{\mathcal{A}})I_{N_x}\big(f'(U_n)V_n\big)\\
&+ h^4\Psi_2(h\sqrt{\mathcal{A}})I_{N_x}\Big(f''(U_n)V_n^2 - f''(U_n)(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U_n) - \rho f'(U_n)U_n\Big),\\
V_{n+1} =& -h\mathcal{A}\operatorname{sinc}(h\sqrt{\mathcal{A}})U_n + \cos(h\sqrt{\mathcal{A}})V_n + h\Phi_2(h\sqrt{\mathcal{A}})I_{N_x}f(U_n) + h^2\Phi_1(h\sqrt{\mathcal{A}})I_{N_x}\big(f'(U_n)V_n\big)\\
&+ h^3\Psi_1(h\sqrt{\mathcal{A}})I_{N_x}\Big(f''(U_n)(V_n^2 - (\sqrt{\mathcal{A}}U_n)^2) + \rho f(U_n) - f'(U_n)(\rho U_n - f(U_n))\Big),
\end{aligned} \tag{2.22}$$

where $0 < h < 1$ is the time stepsize, $n = 0, 1, \ldots, T/h-1$ and the initial values are chosen as $U_0 = \Pi_{N_x}u(x)$ and $V_0 = \Pi_{N_x}v(x)$ for $x \in \{\frac{2n}{2N_x+1} : n = 1,2,\ldots,N_x\}^d$. In practical computation, the trigonometric interpolation I_{N_x} can be implemented with Fast Fourier Transform (FFT).

3. Convergence. In this section, we shall derive the convergence of the proposed semi-discrete and fully-discrete integrators. For each scheme, we will first present the main result and then prove the error estimates for $H^2 \times H^1$ initial data.

3.1. Convergence of semi-discrete scheme.

Theorem 3.1. *Let the nonlinear function f of the Klein–Gordon equation (1.1) satisfy the Lipschitz continuity conditions $\left|f^{(k)}(w)\right| \leq C_0$ for $w \in \mathbb{R}$ and $k = 1,2,3$. Under the regularity condition*

$$(u(0,x), \partial_t u(0,x)) \in [H^2(\mathbb{T}) \bigcap H_0^2(\mathbb{T})] \times H^1(\mathbb{T}), \tag{3.1}$$

there exist positive constants C and h_0 such that for any $h \in (0, h_0]$ the numerical result u_n, v_n produced in Definition 2.1 has the global error:

$$\max_{0\le n\le T/h} \|u_n - u(t_n)\|_{H^1} \le Ch^3, \qquad \max_{0\le n\le T/h} \|v_n - v(t_n)\|_{L^2} \le Ch^3, \tag{3.2}$$

where C depends only on C_0 and T.

The proof is given in the rest part of this section. We begin with the bounds on the remainders $R_1(t_n)$ (2.12), $R_2(t_n)$ (2.14), $R_3(t_n)$ (2.17), which will be derived one by one as follows.

LEMMA 3.1. *Under the conditions of Theorem 3.1, the remainder $R_1(t_n)$ given in (2.12) is bounded by*

$$\|R_1(t_n)\|_{H^1} \lesssim h^4. \tag{3.3}$$

Proof. By using the expression (2.12), $R_1(t_n)$ can be estimated as

$$\begin{aligned}\|R_1(t_n)\|_{H^1} =& \left\| \int_0^h \int_0^s (h-s) \frac{\sin((h-2s+\zeta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} \big(\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)\big) d\zeta ds \right\|_{H^1} \\ \lesssim& \int_0^h (h-s) \int_0^s \left\| \sin((h-2s+\zeta)\sqrt{\mathcal{A}}) \frac{\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)}{\sqrt{\mathcal{A}}} \right\|_{H^1} d\zeta ds \\ \lesssim& \int_0^h (h-s) \int_0^s \left\| \frac{\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)}{\sqrt{\mathcal{A}}} \right\|_{H^1} d\zeta ds \\ \lesssim& h^3 \max_{\zeta\in[0,h]} \left\| \frac{\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)}{\sqrt{\mathcal{A}}} \right\|_{H^1} \lesssim h^3 \max_{\zeta\in[0,h]} \|\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)\|_{L^2}.\end{aligned} \tag{3.4}$$

Then we are devoted to the bound of $\Upsilon(t_n,\zeta) - \Upsilon(t_n,0)$, which can be decomposed into four parts

$$\begin{aligned}\Upsilon(t_n,\zeta) - \Upsilon(t_n,0) =& \underbrace{f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\gamma^\intercal(\zeta)U(t_n)\big)^2 - f''\big(\alpha^\intercal(0)U(t_n)\big)\big(\gamma^\intercal(0)U(t_n)\big)^2}_{=:\Delta_1 f} \\ &+ \underbrace{f''\big(\alpha^\intercal(0)U(t_n)\big)\big(\nabla\alpha^\intercal(0)U(t_n)\big)^2 - f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2}_{=:\Delta_2 f} \\ &+ \underbrace{\rho f\big(\alpha^\intercal(\zeta)U(t_n)\big) - \rho f\big(\alpha^\intercal(0)U(t_n)\big)}_{=:\Delta_3 f} \\ &+ \underbrace{\rho f'\big(\alpha^\intercal(0)U(t_n)\big)\alpha^\intercal(0)U(t_n) - \rho f'\big(\alpha^\intercal(\zeta)U(t_n)\big)\alpha^\intercal(\zeta)U(t_n)}_{=:\Delta_4 f}.\end{aligned} \tag{3.5}$$

In what follows, we will deduce the results for these four terms one by one.

• **Bound on $\|\Delta_1 f\|_{L^2}$.** We first recall the Kato–Ponce inequalities given in Lemma 2.1 and as a result

$$\begin{aligned}\|\Delta_1 f\|_{L^2} \lesssim& \big\|f''\big(\alpha^\intercal(\zeta)U(t_n)\big) - f''\big(\alpha^\intercal(0)U(t_n)\big)\big\|_{H^\theta} \left\|\big(\gamma^\intercal(\zeta)U(t_n)\big)^2\right\|_{L^2} \\ &+ \big\|f''\big(\alpha^\intercal(0)U(t_n)\big)\big\|_{H^\theta} \left\|\big(\gamma^\intercal(\zeta)U(t_n)\big)^2 - \big(\gamma^\intercal(0)U(t_n)\big)^2\right\|_{L^2} \\ \lesssim& \big\|f''\big(\alpha^\intercal(\zeta)U(t_n)\big) - f''\big(\alpha^\intercal(0)U(t_n)\big)\big\|_{H^\theta} \left\|\big(\gamma^\intercal(\zeta)U(t_n)\big)^2\right\|_{L^2} \\ &+ \big\|f''\big(\alpha^\intercal(0)U(t_n)\big)\big\|_{H^\theta} \|\gamma^\intercal(\zeta)U(t_n) + \gamma^\intercal(0)U(t_n)\|_{H^\theta} \|\gamma^\intercal(\zeta)U(t_n) - \gamma^\intercal(0)U(t_n)\|_{L^2}\end{aligned}$$

with $\frac{1}{2} < \theta < 1$. Based on this result, it is needed to estimate

$$\begin{aligned}
&\left\|f''\big(\alpha^\intercal(\zeta)U(t_n)\big) - f''\big(\alpha^\intercal(0)U(t_n)\big)\right\|_{H^\theta} \lesssim \|\alpha^\intercal(\zeta)U(t_n) - \alpha^\intercal(0)U(t_n)\|_{H^\theta} \\
\lesssim &\left\|\cos(\zeta\sqrt{\mathcal{A}})u(t_n) - u(t_n) + \zeta\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})v(t_n)\right\|_{H^\theta} \\
\lesssim &\left\|-2\sin(\zeta\sqrt{\mathcal{A}}/2)\mathrm{sinc}(\zeta\sqrt{\mathcal{A}}/2)\zeta\sqrt{\mathcal{A}}u(t_n)\right\|_{H^\theta} + \zeta\left\|v(t_n)\right\|_{H^\theta} \\
\lesssim &\zeta\big(\left\|u(t_n)\right\|_{H^{1+\theta}} + \left\|v(t_n)\right\|_{H^\theta}\big)
\end{aligned}$$

by using Proposition 2.1. Then the Sobolev embedding theorem shows that

$$\begin{aligned}
\left\|(\gamma^\intercal(\zeta)U(t_n))^2\right\|_{L^2} &\lesssim \|\gamma^\intercal(\zeta)U(t_n)\|_{L^4}^2 \lesssim \|\gamma^\intercal(\zeta)U(t_n)\|_{W^{1,p}}^2 \quad (W^{1,p}\hookrightarrow L^4) \\
&\lesssim \|\gamma^\intercal(\zeta)U(t_n)\|_{W^{\frac{d}{4},2}}^2 \quad (W^{\frac{d}{4},2}\hookrightarrow W^{1,p}) \\
&\lesssim \|\gamma^\intercal(\zeta)U(t_n)\|_{H^{\frac{d}{4}}}^2 \lesssim \left\|\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}})u(t_n) + \cos(\zeta\sqrt{\mathcal{A}})v(t_n)\right\|_{H^{\frac{d}{4}}}^2 \\
&\lesssim \|u(t_n)\|_{H^{1+\frac{d}{4}}}^2 + \|v(t_n)\|_{H^{\frac{d}{4}}}^2 + 2\|u(t_n)\|_{H^{1+\frac{d}{4}}}\|v(t_n)\|_{H^{\frac{d}{4}}}.
\end{aligned}$$

Meanwhile, it follows from (2.2) that

$$\begin{aligned}
\|\gamma^\intercal(\zeta)U(t_n) + \gamma^\intercal(0)U(t_n)\|_{H^\theta} &\lesssim \left\|\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}})u(t_n) + \cos(\zeta\sqrt{\mathcal{A}})v(t_n) + v(t_n)\right\|_{H^\theta} \\
&\lesssim \|u(t_n)\|_{H^{1+\theta}} + \|v(t_n)\|_{H^\theta}.
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
&\|\gamma^\intercal(\zeta)U(t_n) - \gamma^\intercal(0)U(t_n)\|_{L^2} \lesssim \left\|\sqrt{\mathcal{A}}\sin(\zeta\sqrt{\mathcal{A}})u(t_n) + \cos(\zeta\sqrt{\mathcal{A}})v(t_n) - v(t_n)\right\|_{L^2} \\
\lesssim &\left\|\zeta\sqrt{\mathcal{A}}\sqrt{\mathcal{A}}\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})u(t_n)\right\|_{L^2} + \left\|-2\sin(\zeta\sqrt{\mathcal{A}}/2)\mathrm{sinc}(\zeta\sqrt{\mathcal{A}}/2)\zeta\sqrt{\mathcal{A}}v(t_n)\right\|_{L^2} \\
\lesssim &\zeta\|u(t_n)\|_{H^2} + \zeta\|v(t_n)\|_{H^1}.
\end{aligned}$$

Overall, collecting all the estimates together yields

$$\begin{aligned}
\|\Delta_1 f\|_{L^2} \lesssim &\zeta\big(\|u(t_n)\|_{H^{1+\theta}} + \|v(t_n)\|_{H^\theta}\big)\big(\|u(t_n)\|_{H^{1+\frac{d}{4}}}^2 + \|v(t_n)\|_{H^{\frac{d}{4}}}^2 \\
&+ 2\|u(t_n)\|_{H^{1+\frac{d}{4}}}\|v(t_n)\|_{H^{\frac{d}{4}}} + \|u(t_n)\|_{H^2} + \|v(t_n)\|_{H^1}\big) \lesssim \zeta.
\end{aligned}$$

• **Bound on $\|\Delta_2 f\|_{L^2}$.** By the similar argument as $\Delta_1 f$, we decompose $\Delta_2 f$ into four terms:

$$\begin{aligned}
\|\Delta_2 f\|_{L^2} &= \left\|f''\big(\alpha^\intercal(\zeta)U(t_n)\big)\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2 - f''\big(\alpha^\intercal(0)U(t_n)\big)\big(\nabla\alpha^\intercal(0)U(t_n)\big)^2\right\|_{L^2} \\
&\lesssim \left\|f''\big(\alpha^\intercal(\zeta)U(t_n)\big) - f''\big(\alpha^\intercal(0)U(t_n)\big)\right\|_{H^\theta}\left\|\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2\right\|_{L^2} \\
&\quad + \left\|f''\big(\alpha^\intercal(0)U(t_n)\big)\right\|_{H^\theta}\left\|\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2 - \big(\nabla\alpha^\intercal(0)U(t_n)\big)^2\right\|_{L^2}.
\end{aligned}$$

It is noted that this formula contains the same expression as $\Delta_1 f$ and thus we only need to estimate the different two terms: $\left\|\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2\right\|_{L^2}$ and $\left\|\big(\nabla\alpha^\intercal(\zeta)U(t_n)\big)^2 - \big(\nabla\alpha^\intercal(0)U(t_n)\big)^2\right\|_{L^2}$. For the first one, it follows from the Sobolev embedding theorem that

$$\begin{aligned}
\left\|(\nabla\alpha^\intercal(\zeta)U(t_n))^2\right\|_{L^2} &\lesssim \|\nabla\alpha^\intercal(\zeta)U(t_n)\|_{L^4}^2 \lesssim \|\nabla\alpha^\intercal(\zeta)U(t_n)\|_{W^{1,p}}^2 \quad (W^{1,p}\hookrightarrow L^4) \\
&\lesssim \|\nabla\alpha^\intercal(\zeta)U(t_n)\|_{W^{\frac{d}{4},2}}^2 \quad (W^{\frac{d}{4},2}\hookrightarrow W^{1,p}) \\
&\lesssim \|\nabla\alpha^\intercal(\zeta)U(t_n)\|_{H^{\frac{d}{4}}}^2 \lesssim \left\|\cos(\zeta\sqrt{\mathcal{A}})u(t_n) + \sin(\zeta\sqrt{\mathcal{A}})\frac{v(t_n)}{\sqrt{\mathcal{A}}}\right\|_{H^{1+\frac{d}{4}}}^2 \\
&\lesssim \|u(t_n)\|_{H^{1+\frac{d}{4}}}^2 + \|v(t_n)\|_{H^{\frac{d}{4}}}^2 + 2\|u(t_n)\|_{H^{1+\frac{d}{4}}}\|v(t_n)\|_{H^{\frac{d}{4}}}.
\end{aligned}$$

For the second one, we first use the Kato–Ponce inequality to get

$$
\begin{aligned}
&\left\|\big(\nabla\alpha^{\intercal}(\zeta)U(t_n)\big)^2-\big(\nabla\alpha^{\intercal}(0)U(t_n)\big)^2\right\|_{L^2}\\
\lesssim&\left\|\nabla\alpha^{\intercal}(\zeta)U(t_n)+\nabla\alpha^{\intercal}(0)U(t_n)\right\|_{H^\theta}\left\|\nabla\alpha^{\intercal}(\zeta)U(t_n)-\nabla\alpha^{\intercal}(0)U(t_n)\right\|_{L^2}\\
\lesssim&\left\|\cos(\zeta\sqrt{\mathcal{A}})u(t_n)+\sin(\zeta\sqrt{\mathcal{A}})\frac{v(t_n)}{\sqrt{\mathcal{A}}}+u(t_n)\right\|_{H^{1+\theta}}\left\|(\cos(\zeta\sqrt{\mathcal{A}})-1)u(t_n)+\zeta\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})v(t_n)\right\|_{H^1}\\
\lesssim&\big(\left\|u(t_n)\right\|_{H^{1+\theta}}+\left\|v(t_n)\right\|_{H^\theta}\big)\big(\left\|-2\sin(\zeta\sqrt{\mathcal{A}}/2)\mathrm{sinc}(\zeta\sqrt{\mathcal{A}}/2)\zeta\sqrt{\mathcal{A}}u(t_n)\right\|_{H^1}+\left\|\zeta\mathrm{sinc}(\zeta\sqrt{\mathcal{A}})v(t_n)\right\|_{H^1}\big)\\
\lesssim&\zeta\big(\left\|u(t_n)\right\|_{H^{1+\theta}}+\left\|v(t_n)\right\|_{H^\theta}\big)\big(\left\|u(t_n)\right\|_{H^2}+\zeta\left\|v(t_n)\right\|_{H^1}\big).
\end{aligned}
$$

To summarize, we have

$$\|\Delta_2 f\|_{L^2}\lesssim\zeta.$$

- **Bound on $\|\Delta_3 f\|_{L^2}$.** The following estimate can be proved in the same way as above

$$
\begin{aligned}
\|\Delta_3 f\|_{L^2}=&\left\|\rho f\big(\alpha^{\intercal}(\zeta)U(t_n)\big)-\rho f\big(\alpha^{\intercal}(0)U(t_n)\big)\right\|_{L^2}\\
\lesssim&\left\|\alpha^{\intercal}(\zeta)U(t_n)-\alpha^{\intercal}(0)U(t_n)\right\|_{L^2}\lesssim\zeta\big(\left\|u(t_n)\right\|_{H^1}+\left\|v(t_n)\right\|_{L^2}\big).
\end{aligned}
$$

- **Bound on $\|\Delta_4 f\|_{L^2}$.** For the last part, we analogously obtain

$$
\begin{aligned}
\|\Delta_4 f\|_{L^2}=&\left\|\rho f'\big(\alpha^{\intercal}(0)U(t_n)\big)\alpha^{\intercal}(0)U(t_n)-\rho f'\big(\alpha^{\intercal}(\zeta)U(t_n)\big)\alpha^{\intercal}(\zeta)U(t_n)\right\|_{L^2}\\
\lesssim&\left\|f'\big(\alpha^{\intercal}(\zeta)U(t_n)\big)-f'\big(\alpha^{\intercal}(0)U(t_n)\big)\right\|_{H^\theta}\left\|\alpha^{\intercal}(\zeta)U(t_n)\right\|_{L^2}\\
&+\left\|f'\big(\alpha^{\intercal}(0)U(t_n)\big)\right\|_{H^\theta}\left\|\alpha^{\intercal}(\zeta)U(t_n)-\alpha^{\intercal}(0)U(t_n)\right\|_{L^2}\\
\lesssim&\zeta\big(\left\|u(t_n)\right\|_{H^{1+\theta}}+\left\|v(t_n)\right\|_{H^\theta}\big)\big(\left\|u(t_n)\right\|_{L^2}+\zeta\left\|v(t_n)\right\|_{L^2}\big).
\end{aligned}
$$

Consequently, based on the above results, we arrive at

$$\max_{\zeta\in[0,h]}\|\Upsilon(t_n,\zeta)-\Upsilon(t_n,0)\|_{L^2}\lesssim\max_{\zeta\in[0,h]}\zeta\lesssim h. \tag{3.6}$$

Combining this with (3.4) immediately gives the statement of this lemma and the proof is complete. ∎

LEMMA 3.2. *Suppose that the conditions of Theorem 3.1 hold. For the remainders $R_2(t_n)$ and $R_3(t_n)$ respectively defined in (2.14) and (2.17), they are bounded by*

$$\|R_2(t_n)\|_{L^2}\lesssim h^4,\quad \|R_3(t_n)\|_{L^2}\lesssim h^4. \tag{3.7}$$

Proof. Keeping (3.6) in mind, the following estimate can be proved in the same way as Lemma 3.1

$$
\begin{aligned}
\|R_2(t_n)\|_{L^2}=&\left\|\int_0^h\int_0^s(h-s)\cos((h-2s+\zeta)\sqrt{\mathcal{A}})\big(\Upsilon(t_n,\zeta)-\Upsilon(t_n,0)\big)d\zeta ds\right\|_{L^2}\\
\lesssim&\int_0^h\int_0^s|h-s|\left\|\big(\Upsilon(t_n,\zeta)-\Upsilon(t_n,0)\big)\right\|_{L^2}d\zeta ds\\
\lesssim&h^3\max_{\zeta\in[0,h]}\|\Upsilon(t_n,\zeta)-\Upsilon(t_n,0)\|_{L^2}\lesssim h^4.
\end{aligned}
$$

Then we represent the scheme of $R_3(t_n)$

$$
\begin{aligned}
R_3(t_n)=&\int_0^h\cos((h-s)\sqrt{\mathcal{A}})f'\big(\alpha^{\intercal}(s)U(t_n)\big)\int_0^{+s}\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}f\big(u(t_n+\theta)\big)d\theta ds\\
&-\int_0^h\cos((h-s)\sqrt{\mathcal{A}})f'(u(t_n))\int_0^s\frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}d\theta ds f\big(u(t_n)\big)
\end{aligned}
$$

and split it into

$$R_3(t_n) = \int_0^h \cos((h-s)\sqrt{\mathcal{A}})\big(f'\big(\alpha^\intercal(s)U(t_n)\big) - f'(u(t_n))\big)\int_0^{+s} \frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}} f\big(u(t_n+\theta)\big)d\theta ds$$
$$+ \int_0^h \cos((h-s)\sqrt{\mathcal{A}})f'(u(t_n))\int_0^s \frac{\sin((s-\theta)\sqrt{\mathcal{A}})}{\sqrt{\mathcal{A}}}\big(f\big(u(t_n+\theta)\big) - f\big(u(t_n)\big)\big)d\theta ds. \tag{3.8}$$

Some derivations lead to

$$\begin{aligned}\|R_3(t_n)\|_{L^2} \lesssim & \int_0^h \int_0^s \left\|f'\big(\alpha^\intercal(s)U(t_n)\big) - f'(u(t_n))\right\|_{H^\theta} |s-\theta| \left\|f\big(u(t_n+\theta)\big)\right\|_{L^2} d\theta ds \\ & + \int_0^h \int_0^s \|f'(u(t_n))\|_{H^\theta} |s-\theta| \left\|f\big(u(t_n+\theta)\big) - f\big(u(t_n)\big)\right\|_{L^2} d\theta ds \\ \lesssim & h^3 \max_{s\in[0,h]} \left\|f'\big(\alpha^\intercal(s)U(t_n)\big) - f'(u(t_n))\right\|_{H^\theta} + h^3 \max_{\theta\in[0,h]} \left\|f\big(u(t_n+\theta)\big) - f\big(u(t_n)\big)\right\|_{L^2} \\ \lesssim & h^3 \max_{s\in[0,h]} \|\alpha^\intercal(s)U(t_n) - u(t_n)\|_{H^\theta} + h^3 \max_{\theta\in[0,h]} \|u(t_n+\theta) - u(t_n)\|_{L^2} \\ \lesssim & h^3 \max_{s\in[0,h]} s\big(\|u(t_n)\|_{H^{1+\theta}} + \|v(t_n)\|_{H^\theta}\big) + h^3 \max_{\theta\in[0,h]} \theta \left\|v(t_n^\theta)\right\|_{L^2} \lesssim h^4.\end{aligned}$$

The proof is complete. ∎

Sofar we have derived the bounds for the remainders which are dropped in the numerical scheme. Based on them, we can present the error analysis (the proof of Theorem 3.1) in what follows.

Proof of Theorem 3.1.

Proof. To estimate the error of the scheme (2.20)

$$e_n^u := u(t_n) - u_n, \quad e_n^v := v(t_n) - v_n, \quad 0 \le n \le T/h,$$

we shall first consider the local truncation errors which are defined by inserting the solution of (1.1) into (2.20):

$$\begin{aligned}\zeta_n^u :=& u(t_{n+1}) - \cos(h\sqrt{\mathcal{A}})u(t_n) - h\mathrm{sinc}(h\sqrt{\mathcal{A}})v(t_n) - h^2\Phi_1(h\sqrt{\mathcal{A}})f(u(t_n)) \\ & - h^3\Psi_1(h\sqrt{\mathcal{A}})f'(u(t_n))v(t_n) - h^4\Psi_2(h\sqrt{\mathcal{A}})\Big(f''(u(t_n))v(t_n)^2 \\ & - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2 + \rho f(u(t_n)) - \rho f'(u(t_n))u(t_n)\Big), \\ \zeta_n^v :=& v(t_{n+1}) + h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}})u(t_n) - \cos(h\sqrt{\mathcal{A}})v(t_n) - h\Phi_2(h\sqrt{\mathcal{A}})f(u(t_n)) \\ & - h^2\Phi_1(h\sqrt{\mathcal{A}})f'(u(t_n))v(t_n) - h^3\Psi_1(h\sqrt{\mathcal{A}})\Big(f''(u(t_n))(v(t_n)^2 \\ & - (\sqrt{\mathcal{A}}u(t_n))^2) + \rho f(u(t_n)) - f'(u(t_n))(\rho u(t_n) - f(u(t_n)))\Big),\end{aligned} \tag{3.9}$$

where $n = 0, 1, \ldots, T/h$. For the coefficient functions appeared above, it follows from (2.21) that the functions

$$\Phi_1(m), \Phi_2(m), m\Psi_1(m), m^2\Psi_2(m)$$

are uniformly bounded for any $m \in \mathbb{R}$ which immediately leads to

$$\begin{aligned}&\left\|\Phi_1(h\sqrt{\mathcal{A}})y\right\|_{H^s} \lesssim \|y\|_{H^s}, && \left\|\Phi_2(h\sqrt{\mathcal{A}})y\right\|_{H^s} \lesssim \|y\|_{H^s}, \\ &\left\|h\sqrt{\mathcal{A}}\Psi_1(h\sqrt{\mathcal{A}})y\right\|_{H^s} \lesssim \|y\|_{H^s}, && \left\|(h\sqrt{\mathcal{A}})^2\Psi_2(h\sqrt{\mathcal{A}})y\right\|_{H^s} \lesssim \|y\|_{H^s},\end{aligned} \tag{3.10}$$

where we assume that $y \in H^s$ with any $s \geq 0$.

Subtracting the corresponding local error terms (3.9) from the scheme (2.20), we get the recurrence relation for the errors

$$
\begin{aligned}
&e^u_{n+1} - \cos(h\sqrt{\mathcal{A}})e^u_n - h\mathrm{sinc}(h\sqrt{\mathcal{A}})e^v_n = \zeta^u_n + h^2\eta^u_n,\\
&e^v_{n+1} + h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}})e^u_n - \cos(h\sqrt{\mathcal{A}})e^v_n = \zeta^v_n + h\eta^v_n, \qquad n = 0, 1, \dots, T/h,
\end{aligned} \tag{3.11}
$$

where we denote

$$
\begin{aligned}
\eta^u_n :=& \Phi_1(h\sqrt{\mathcal{A}})\big(f(u_n) - f(u(t_n))\big) + h\Psi_1(h\sqrt{\mathcal{A}})\big(f'(u_n)v_n - f'(u(t_n))v(t_n)\big)\\
&+ h^2\Psi_2(h\sqrt{\mathcal{A}})\Big(\big(f''(u_n)v_n^2 - f''(u(t_n))v(t_n)^2\big)\\
&- \big(f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2\big)\\
&+ \rho\big(f(u_n) - f(u(t_n))\big) - \rho\big(f'(u_n)u_n - f'(u(t_n))u(t_n)\big)\Big),\\
\eta^v_n :=& \Phi_2(h\sqrt{\mathcal{A}})\big(f(u_n) - f(u(t_n))\big) + h\Phi_1(h\sqrt{\mathcal{A}})\big(f'(u_n)v_n - f'(u(t_n))v(t_n)\big)\\
&+ h^2\Psi_1(h\sqrt{\mathcal{A}})\Big(\big(f''(u_n)v_n^2 - f''(u(t_n))v(t_n)^2\big)\\
&- \big(f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2\big)\\
&+ \rho\big(f(u_n) - f(u(t_n))\big) - \rho\big(f'(u_n)u_n - f'(u(t_n))u(t_n)\big)\\
&- \big(f'(u_n)f(u_n) - f'(u(t_n))f(u(t_n))\big)\Big).
\end{aligned} \tag{3.12}
$$

The starting value of (3.11) is $e^u_0 = e^v_0 = 0$.

In what follows, the proof is divided into three parts. The first one is about the estimate of local errors ζ^u_n, ζ^v_n, the second is devoted to η^u_n, η^v_n and the last one concerns global errors e^u_n, e^v_n.

• By the bounds (2.15), (2.18)-(2.19) and Lemmas 3.1–3.2, the local errors ζ^u_n, ζ^v_n are estimated as

$$
\begin{aligned}
&\|\zeta^u_n\|_{H^1} \lesssim \|R_1(t_n)\|_{H^1} + \|\text{Part II}^u\|_{H^1} + \|\text{Part III}^u\|_{H^1} \lesssim h^4,\\
&\|\zeta^v_n\|_{L^2} \lesssim \|R_2(t_n)\|_{L^2} + \|R_3(t_n)\|_{L^2} + \|\text{Part III}^v\|_{L^2} \lesssim h^4.
\end{aligned}
$$

With these bounds, the recurrence relation (3.11) and the standard convergent analysis, a coarse error estimate is derived

$$
\|e^u_n\|_{H^1} + \|e^v_n\|_{L^2} \lesssim 1.
$$

Considering further the regularity condition of exact solution (3.1) leads to the boundedness of numerical solution:

$$
\|u_n\|_{H^1} + \|v_n\|_{L^2} \lesssim 1, \quad n = 0, 1, \dots, T/h. \tag{3.13}
$$

• To establish the estimate for the error terms η^u_n and η^v_n defined in (3.12), we consider the terms of (3.12) separately. For the first and last terms, it is easy to see that

$$
\begin{aligned}
&h^2\left\|\Psi_2(h\sqrt{\mathcal{A}})\Big(\rho\big(f(u_n) - f(u(t_n))\big) - \rho\big(f'(u_n)u_n - f'(u(t_n))u(t_n)\big)\Big)\right\|_{H^1}\\
&+ \left\|\Phi_1(h\sqrt{\mathcal{A}})\big(f(u_n) - f(u(t_n))\big)\right\|_{H^1} \lesssim \|e^u_n\|_{H^1},
\end{aligned}
$$

Based on the Lipschitz continuity conditions, Kato–Ponce inequalities and (3.10), it is derived that

$$\begin{aligned}
&\left\|h\Psi_1(h\sqrt{\mathcal{A}})\big(f'(u_n)v_n - f'(u(t_n))v(t_n)\big)\right\|_{H^1}\\
=&\left\|h\sqrt{\mathcal{A}}\Psi_1(h\sqrt{\mathcal{A}})\frac{f'(u_n)v_n - f'(u(t_n))v(t_n)}{\sqrt{\mathcal{A}}}\right\|_{H^1}\\
\lesssim&\left\|f'(u_n)v_n - f'(u(t_n))v(t_n)\right\|_{L^2}\\
\lesssim&\left\|f'(u_n)\right\|_{H^\theta}\left\|e_n^v\right\|_{L^2} + \left\|f'(u_n) - f'(u(t_n))\right\|_{L^2}\left\|v(t_n)\right\|_{H^\theta}\\
\lesssim&\left\|e_n^v\right\|_{L^2} + \left\|e_n^u\right\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|h^2\Psi_2(h\sqrt{\mathcal{A}})\big(f''(u_n)v_n^2 - f''(u(t_n))v(t_n)^2\big)\right\|_{H^1}\\
=&h\left\|h\sqrt{\mathcal{A}}\Psi_2(h\sqrt{\mathcal{A}})\frac{f''(u_n)v_n^2 - f''(u(t_n))v(t_n)^2}{\sqrt{\mathcal{A}}}\right\|_{H^1}\\
\lesssim&h\left\|f''(u_n)v_n^2 - f''(u(t_n))v(t_n)^2\right\|_{L^2}\\
\lesssim&h(\left\|e_n^v\right\|_{L^2} + \left\|e_n^u\right\|_{L^2}).
\end{aligned}$$

With the same arguments, we get

$$\begin{aligned}
&\left\|h^2\Psi_2(h\sqrt{\mathcal{A}})\big(f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2\big)\right\|_{H^1}\\
=&\left\|h^2\mathcal{A}\Psi_2(h\sqrt{\mathcal{A}})\frac{f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2}{\mathcal{A}}\right\|_{H^1}\\
\lesssim&\left\|\frac{f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2}{\mathcal{A}}\right\|_{H^1}\\
\lesssim&\left\|\frac{\big(f''(u_n) - f''(u(t_n))\big)(\sqrt{\mathcal{A}}u_n)^2}{\sqrt{\mathcal{A}}}\right\|_{L^2} + \left\|\frac{f''(u(t_n))\big((\sqrt{\mathcal{A}}u_n)^2 - (\sqrt{\mathcal{A}}u(t_n))^2\big)}{\sqrt{\mathcal{A}}}\right\|_{L^2}.
\end{aligned}$$

According to the results of Lemma 2.1 and the boundedness (3.13), we find

$$\begin{aligned}
&\left\|\frac{\big(f''(u_n) - f''(u(t_n))\big)(\sqrt{\mathcal{A}}u_n)^2}{\sqrt{\mathcal{A}}}\right\|_{L^2}\\
\lesssim&\min\{\left\|f''(u_n) - f''(u(t_n))\right\|_{L^2}\left\|u_n\sqrt{\mathcal{A}}u_n\right\|_{H^1}, \left\|u_n\sqrt{\mathcal{A}}u_n\right\|_{L^2}\left\|f''(u_n) - f''(u(t_n))\right\|_{H^1}\}\\
\lesssim&\left\|u_n^2\right\|_{H^1}\left\|f''(u_n) - f''(u(t_n))\right\|_{H^1} \lesssim \left\|e_n^u\right\|_{H^1}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\frac{f''(u(t_n))\big((\sqrt{\mathcal{A}}u_n)^2 - (\sqrt{\mathcal{A}}u(t_n))^2\big)}{\sqrt{\mathcal{A}}}\right\|_{L^2}\\
\lesssim&\min\{\left\|f''(u(t_n))\right\|_{L^2}\left\|\sqrt{\mathcal{A}}u_n^2 - \sqrt{\mathcal{A}}u^2(t_n)\right\|_{H^1}, \left\|\sqrt{\mathcal{A}}u_n^2 - \sqrt{\mathcal{A}}u^2(t_n)\right\|_{L^2}\left\|f''(u(t_n))\right\|_{H^1}\}\\
\lesssim&\left\|u_n^2 - u^2(t_n)\right\|_{H^1}\left\|f''(u(t_n))\right\|_{H^1} \lesssim \left\|e_n^u\right\|_{H^1}
\end{aligned}$$

Therefore, it is obtained that

$$\left\|h^2\Psi_2(h\sqrt{\mathcal{A}})\big(f''(u_n)(\sqrt{\mathcal{A}}u_n)^2 - f''(u(t_n))(\sqrt{\mathcal{A}}u(t_n))^2\big)\right\|_{H^1} \lesssim \left\|e_n^u\right\|_{H^1}.$$

Combining these results with the first formula of (3.12) gives

$$\left\|\eta_n^u\right\|_{H^1} \lesssim \left\|e_n^u\right\|_{H^1} + \left\|e_n^v\right\|_{L^2}.$$

In a similar way, we can get the same result for η_n^v:

$$\|\eta_n^v\|_{L^2} \lesssim \|e_n^u\|_{H^1} + \|e_n^v\|_{L^2}.$$

- Now we turn back to recurrence relation (3.11) and rewrite it in the following form

$$\begin{pmatrix} e_{n+1}^u \\ e_{n+1}^v \end{pmatrix} = \begin{pmatrix} \cos(h\sqrt{\mathcal{A}}) & h\mathrm{sinc}(h\sqrt{\mathcal{A}}) \\ -h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}}) & \cos(h\sqrt{\mathcal{A}}) \end{pmatrix} \begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix} + \begin{pmatrix} \zeta_n^u + h^2\eta_n^u \\ \zeta_n^v + h\eta_n^v \end{pmatrix}.$$

Using the notation $\left\|\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\| := \sqrt{\|w_1\|_{H^1}^2 + \|w_2\|_{L^2}^2}$ gives

$$\left\|\begin{pmatrix} \cos(h\sqrt{\mathcal{A}}) & h\mathrm{sinc}(h\sqrt{\mathcal{A}}) \\ -h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}}) & \cos(h\sqrt{\mathcal{A}}) \end{pmatrix} \begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\| = \left\|\begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\|.$$

Thus, it is obtained that

$$\begin{aligned} &\left\|\begin{pmatrix} e_{n+1}^u \\ e_{n+1}^v \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\| + \left\|\begin{pmatrix} \zeta_n^u + h^2\eta_n^u \\ \zeta_n^v + h\eta_n^v \end{pmatrix}\right\| \\ \leq &\left\|\begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\| + \sqrt{(\|\zeta_n^u\|_{H^1} + h^2\|\eta_n^u\|_{H^1})^2 + (\|\zeta_n^v\|_{L^2} + h^2\|\eta_n^v\|_{L^2})^2} \\ \leq &\left\|\begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\| + C\sqrt{h^8 + h^6(\|e_n^u\|_{H^1} + \|e_n^v\|_{L^2}) + h^4(\|e_n^u\|_{H^1}^2 + \|e_n^v\|_{L^2}^2)} \\ \leq &\left\|\begin{pmatrix} e_n^u \\ e_n^v \end{pmatrix}\right\| + Ch^4 + Ch^3\sqrt{\|e_n^u\|_{H^1} + \|e_n^v\|_{L^2}} + Ch^2\sqrt{\|e_n^u\|_{H^1}^2 + \|e_n^v\|_{L^2}^2} \end{aligned}$$

Applying the Gronwall inequality yields

$$\sqrt{\|e_{n+1}^u\|_{H^1}^2 + \|e_{n+1}^v\|_{L^2}^2} \leq Ch^3.$$

This shows (3.2) exactly. Therefore, the theorem is confirmed. ∎

3.2. Convergence of fully-discrete scheme.

THEOREM 3.2. *Under the conditions of Theorem 3.1 and the regularity condition*

$$(u(0,x), \partial_t u(0,x)) \in [H^2(\mathbb{T}) \bigcap H_0^2(\mathbb{T})] \times H^1(\mathbb{T}),$$

the numerical solution produced by the fully-discrete scheme (2.22) has the following error bound:

$$\max_{0\leq n\leq T/h-1} \big(\|\Pi_{N_x}U(t_{n+1}) - U_{n+1}\|_{H^1} + \|\Pi_{N_x}V(t_{n+1}) - V_{n+1}\|_{L^2}\big) \leq C\big(h^3 + N_x^{-2}\big), \tag{3.14}$$

where C is the error constant which only depends on T and C_0 given in Theorem 3.1.

Remark 3.3. From this result, it follows that by passing to the limit N_x, the convergence of semi-discretization given in Theorem 3.1 is obtained. In practical applications, a large N_x can be chosen and then the main error of fully-discrete scheme comes from the time discretization.

Proof. Denote the errors of the fully-discrete solution (2.22) by $E_n^U = \Pi_{N_x}U(t_n) - U_n$, $E_n^V = \Pi_{N_x}V(t_n) - V_n$. By the construction of the semi-discrete scheme presented in

Section 2.2, it is obtained that the exact solution satisfies

(3.15)

$$\begin{aligned}\Pi_{N_x}U(t_{n+1}) =& \cos(h\sqrt{\mathcal{A}})\Pi_{N_x}U(t_n) + h\mathrm{sinc}(h\sqrt{\mathcal{A}})\Pi_{N_x}V(t_n)\\ &+ h^2\Phi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}f(U(t_n)) + h^3\Psi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f'(U(t_n))V_n\big)\\ &+ h^4\Psi_2(h\sqrt{\mathcal{A}})\Pi_{N_x}\Big(f''(U(t_n))V(t_n)^2 - f''(U(t_n))(\sqrt{\mathcal{A}}U_n)^2\\ &+ \rho f(U(t_n)) - \rho f'(U(t_n))U(t_n)\Big) + \Pi_{N_x}\Big(R_1(t_n) + \text{Part II}^u + \text{Part III}^u\Big),\\ \Pi_{N_x}V(t_{n+1}) =& - h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}})\Pi_{N_x}U(t_n) + \cos(h\sqrt{\mathcal{A}})\Pi_{N_x}V(t_n)\\ &+ h\Phi_2(h\sqrt{\mathcal{A}})\Pi_{N_x}f(U(t_n)) + h^2\Phi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f'(U(t_n))V(t_n)\big)\\ &+ h^3\Psi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\Big(f''(U(t_n))(V(t_n)^2 - (\sqrt{\mathcal{A}}U(t_n))^2)\\ &+ \rho f(U(t_n)) - f'(U(t_n))(\rho U(t_n) - f(U(t_n)))\Big)\\ &+ \Pi_{N_x}\Big(R_2(t_n) + R_3(t_n) + \text{Part III}^v\Big).\end{aligned}$$

Then considering the difference between (2.22) and (3.15), the following error equation is obtained:

(3.16)

$$\begin{aligned}E^U_{n+1} =& \cos(h\sqrt{\mathcal{A}})E^U_n + h\mathrm{sinc}(h\sqrt{\mathcal{A}})E^V_n + h^2\Phi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f(U(t_n)) - f(U_n)\big)\\ &+ h^3\Psi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f'(U(t_n))V(t_n) - f'(U_n)V_n\big)\\ &+ h^4\Psi_2(h\sqrt{\mathcal{A}})\Pi_{N_x}\Big(f''(U(t_n))V(t_n)^2 - f''(U_n)V_n^2 + f''(U_n)(\sqrt{\mathcal{A}}U_n)^2\\ &- f''(U(t_n))(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U(t_n)) - \rho f(U_n) + \rho f'(U_n)U_n - \rho f'(U(t_n))U(t_n)\Big)\\ &+ \Pi_{N_x}\Big(R_1(t_n) + \text{Part II}^u + \text{Part III}^u\Big) + \widetilde{R_1^U}(t_n) + \widetilde{R_2^U}(t_n) + \widetilde{R_3^U}(t_n),\\ E^V_{n+1} =& - h\mathcal{A}\mathrm{sinc}(h\sqrt{\mathcal{A}})E^U_n + \cos(h\sqrt{\mathcal{A}})E^V_n + h\Phi_2(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f(U(t_n)) - f(U_n)\big)\\ &+ h^2\Phi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\big(f'(U(t_n))V(t_n) - f'(U_n)V_n\big)\\ &+ h^3\Psi_1(h\sqrt{\mathcal{A}})\Pi_{N_x}\Big(f''(U(t_n))V(t_n)^2 - f''(U_n)V_n^2 + f''(U_n)(\sqrt{\mathcal{A}}U_n)^2\\ &- f''(U(t_n))(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U(t_n)) - \rho f(U_n) + \rho f'(U_n)U_n - \rho f'(U(t_n))U(t_n)\\ &+ f'(U(t_n))f(U(t_n)) - f'(U_n)f(U_n)\Big) + \Pi_{N_x}\Big(R_2(t_n) + R_3(t_n) + \text{Part III}^v\Big)\\ &+ \widetilde{R_1^V}(t_n) + \widetilde{R_2^V}(t_n) + \widetilde{R_3^V}(t_n),\end{aligned}$$

where we introduce the following notations to denote remainders

(3.17)

$$\begin{aligned}\widetilde{R_1^U}(t_n) =& h^2\Phi_1(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})f(U_n),\ \widetilde{R_2^U}(t_n) = h^3\Psi_1(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})\big(f'(U_n)V_n\big),\\ \widetilde{R_3^U}(t_n) =& h^4\Psi_2(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})\Big(f''(U_n)V_n^2 - f''(U_n)(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U_n) - \rho f'(U_n)U_n\Big),\end{aligned}$$

and

(3.18)

$$\begin{aligned}\widetilde{R_1^V}(t_n) =& h\Phi_2(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})f(U_n),\ \widetilde{R_2^V}(t_n) = h^2\Phi_1(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})\big(f'(U_n)V_n\big),\\ \widetilde{R_3^V}(t_n) =& h^3\Psi_1(h\sqrt{\mathcal{A}})(\Pi_{N_x} - I_{N_x})\Big(f''(U_n)\big(V_n^2 - (\sqrt{\mathcal{A}}U_n)^2\big) + \rho f(U_n) - f'(U_n)\big(\rho U_n - f(U_n)\big)\Big).\end{aligned}$$

With the same analysis as the bounds (2.15), (2.18)-(2.19) and Lemmas 3.1–3.2, we have

$$\Big\|\Pi_{N_x}\Big(R_2(t_n) + R_3(t_n) + \text{Part III}^v\Big)\Big\|_{L^2} + \Big\|\Pi_{N_x}\Big(R_1(t_n) + \text{Part II}^u + \text{Part III}^u\Big)\Big\|_{H^1} \lesssim h^4.$$

In what follows, we derive the bounds for the terms (3.17)-(3.18) by using mathematical induction on n: assuming that

$$\|U_n\|_{H^2} \le \|\Pi_{N_x} U(t_n)\|_{H^2} + 1, \quad \|V_n\|_{H^1} \le \|\Pi_{N_x} V(t_n)\|_{H^1} + 1, \tag{3.19}$$

we shall prove the following results:

$$\|U_{n+1}\|_{H^2} \le \|\Pi_{N_x} U(t_{n+1})\|_{H^2} + 1, \quad \|V_{n+1}\|_{H^1} \le \|\Pi_{N_x} V(t_{n+1})\|_{H^1} + 1. \tag{3.20}$$

For the first two terms of (3.17), it follows from [23] and (3.19) that

$$\begin{aligned}
\left\|\widetilde{R_1^U}(t_n)\right\|_{H^1} \lesssim & hN_x^{-2}\big(\|U_n\|_{H^2} + \|U_n\|^2_{H^{1+\frac{d}{4}}}\big) \lesssim hN_x^{-1-\frac{d}{4}}\big(\|U_n\|_{H^{1+\frac{d}{4}}} + \|U_n\|^2_{H^1}\big),\\
\left\|\widetilde{R_2^U}(t_n)\right\|_{H^1} \lesssim & hN_x^{-2}\big(\|U_n\|_{H^2} + \|V_n\|_{H^1} + \|U_n\|^2_{H^{1+\frac{d}{4}}} + \|V_n\|^2_{H^{\frac{d}{4}}}\big)\\
\lesssim & hN_x^{-1-\frac{d}{4}}\big(\|U_n\|_{H^{1+\frac{d}{4}}} + \|V_n\|_{H^{\frac{d}{4}}} + \|U_n\|^2_{H^{1+\frac{d}{4}}} + \|V_n\|^2_{H^{\frac{d}{4}}}\big).
\end{aligned}$$

For the third one in (3.17), by noticing $\Psi_2(h\sqrt{\mathcal{A}}) = \frac{1}{4h^2\mathcal{A}}\mathrm{sinc}^2(h\sqrt{\mathcal{A}}/2) - \frac{1}{2h^2\mathcal{A}}\mathrm{sinc}(h\sqrt{\mathcal{A}})$, we deduce that

$$\begin{aligned}
\left\|\widetilde{R_3^U}(t_n)\right\|_{H^1} \lesssim & h\left\|(\Pi_{N_x} - I_N)\frac{1}{\sqrt{\mathcal{A}}^3}\Big(f''(U_n)V_n^2 - f''(U_n)(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U_n) - \rho f'(U_n)U_n\Big)\right\|_{H^1}\\
\lesssim & hN_x^{-2}\left\|\frac{1}{\sqrt{\mathcal{A}}^3}\Big(f''(U_n)V_n^2 - f''(U_n)(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U_n) - \rho f'(U_n)U_n\Big)\right\|_{H^3}\\
\lesssim & hN_x^{-2}\left\|f''(U_n)V_n^2 - f''(U_n)(\sqrt{\mathcal{A}}U_n)^2 + \rho f(U_n) - \rho f'(U_n)U_n\right\|_{L^2}\\
\lesssim & hN_x^{-2}\big(\|V_n\|^2_{L^2} + \|U_n\|^2_{H^2} + \|f(U_n)\|_{L^2} + \|U_n\|_{L^2}\big).
\end{aligned}$$

Similarly, we get

$$\left\|\widetilde{R_1^V}(t_n)\right\|_{L^2} \lesssim hN_x^{-2}, \quad \left\|\widetilde{R_2^V}(t_n)\right\|_{L^2} \lesssim hN_x^{-2}, \quad \left\|\widetilde{R_3^V}(t_n)\right\|_{L^2} \lesssim hN_x^{-2}.$$

By using these estimates and taking the energy norm on both sides of (3.16), we obtain (3.14) and (3.20) with the same arguments given in the proof of Theorem 3.1. The proof is complete. $\square$

4. Numerical test.

In this section, we show the numerical performance of the proposed third-order low-regularity integrator (LGI) by comparing it with the well-known exponential integrators (EIs) [20]. We choose two third-order EIs from [19] (denoted by EI1 and EI2) and an exponential fitting TI from [37] (denoted by EI3).

We shall present the results with an one-dimensional example of (1.1) for simplicity: the nonlinear Klein–Gordon equation (1.1) with $d = 1, \rho = 0, \mathbb{T} = (-\pi, \pi), f(u) = \sin(u)$. We choose the initial values $\psi_1(x)$ and $\psi_2(x)$ in the same way as described in Section 5.1 of [28] and Section 4 of [38]. More precisely, the initial values $\psi_1(x)$ and $\psi_2(x)$ are in the space $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ and has the bound $\|\psi_1(x)\|_{H^1} = \|\psi_2(x)\|_{L^2} = 1$. In the fourier spectral collocation method, we choose $N_x = 1024$ such that the error brought by the spatial discretization can be neglected.

In this test, we display the global errors $err = \frac{\|u_n - u(t_n)\|_{H^1}}{\|u(t_n)\|_{H^1}} + \frac{\|v_n - v(t_n)\|_{L^2}}{\|v(t_n)\|_{L^2}}$ at $T = 1$. The results for different initial value $(\psi_1, \psi_2) \in H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ are given in Figure 1. From the results, it can be observed that for the initial data in $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ with $\theta = 4, 3$, all the methods performance third order convergence. However, for the initial data in a low-regularity space $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ with $\theta = 2$, only the new integrator SLG proposed in this article shows the correct third-order convergence in $H^1(\mathbb{T}) \times L^2(\mathbb{T})$. If the initial data is in a space $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ with $\theta = 1.5, 1, 0.5$ which is lower that the requirement $H^2(\mathbb{T}) \times H^1(\mathbb{T})$

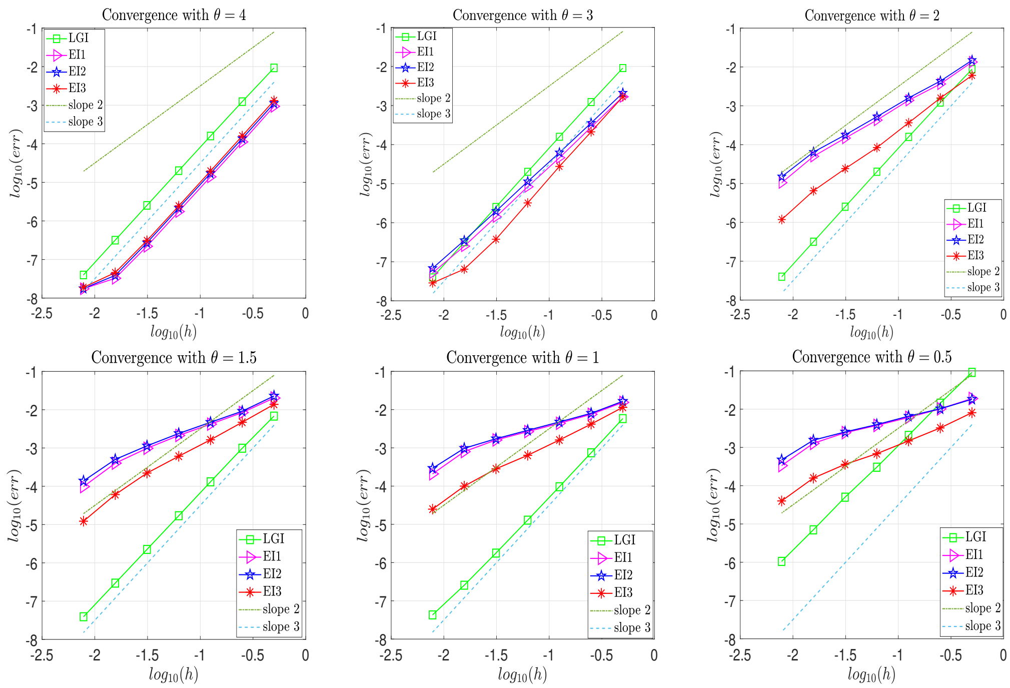

FIG. 1. *Temporal error* $err = \frac{\|u_n - u(t_n)\|_{H^1}}{\|u(t_n)\|_{H^1}} + \frac{\|v_n - v(t_n)\|_{L^2}}{\|v(t_n)\|_{L^2}}$ *at* $T = 1$ *with initial data* $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ *against* h *with* $h = 1/2^k$, *where* $k = 1, 2, \ldots, 7$.

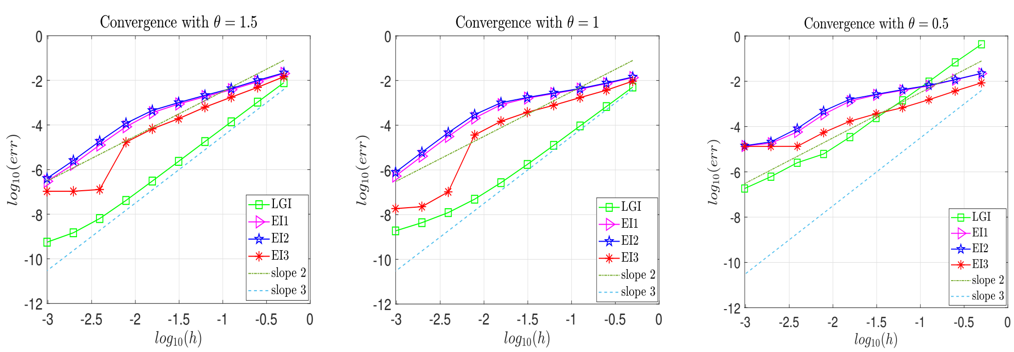

FIG. 2. *Temporal error* $err = \frac{\|u_n - u(t_n)\|_{H^1}}{\|u(t_n)\|_{H^1}} + \frac{\|v_n - v(t_n)\|_{L^2}}{\|v(t_n)\|_{L^2}}$ *at* $T = 1$ *with initial data* $H^\theta(\mathbb{T}) \times H^{\theta-1}(\mathbb{T})$ *against* h *with* $h = 1/2^k$, *where* $k = 1, 2, \ldots, 10$.

given in Theorem 3.1, the expected bad behaviour does not show up in Figure 1. To clarify this issue, we further display the result with more timesteps in Figure 2. The result indicates that if the problem has lower regularity than $H^2(\mathbb{T}) \times H^1(\mathbb{T})$, the new integrator LGI also does not show the correct order accuracy but it still performs much better than the exponential integrators.

To investigate the practical gain from our proposed integrator, we study the efficiency of all the methods. Figure 3 displays the error at $T = 5$ against the CPU time. It can be seen from the results that our proposed method LGI can reach the same error level with remarkably less CPU time, and this clearly demonstrates the more efficiency of LGI than the exponential integrators.

5. Conclusions. In this paper, a low-regularity trigonometric integrator was formulated and analysed for solving the nonlinear Klein-Gordon equation in the d-dimensional space with

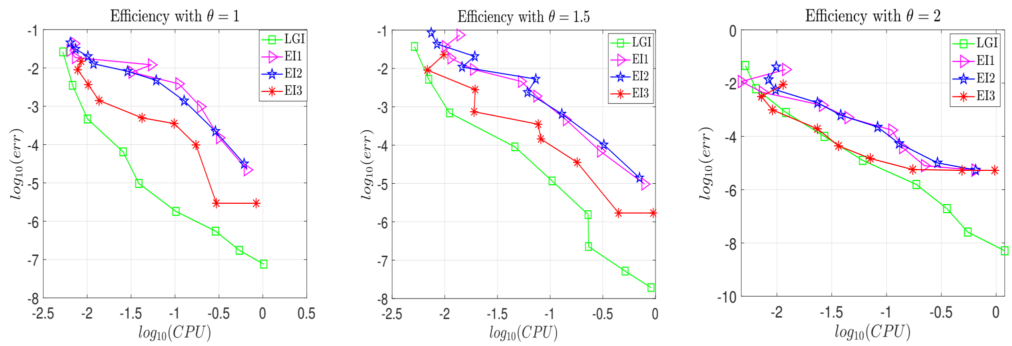

FIG. 3. *Efficiency comparison:* $err = \frac{\|u_n - u(t_n)\|_{H^1}}{\|u(t_n)\|_{H^1}} + \frac{\|v_n - v(t_n)\|_{L^2}}{\|v(t_n)\|_{L^2}}$ *at* $T = 5$ *against different CPU time produced by different* $h = 1/2^k$*, where* $k = 1, 2, \ldots, 7$.

$d = 1, 2, 3$. Rigorous error estimates were given and the proposed integrator was shown to have third-order time accuracy in the energy space under a weak regularity condition. A numerical experiment was carried out and the corresponding numerical results were presented to demonstrate the superiorities of the new integrator in comparison with some well known exponential integrators.

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