ON 5-CYCLES AND STRONG 5-SUBTOURNAMENTS IN A TOURNAMENT OF ODD ORDER N

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Dedicated to Richard Brualdi on the occasion of his 80th birthday

Abstract. Let T be a tournament of odd order $n \geq 5$, $c_m(T)$ be the number of its m-cycles, and $s_m(T)$ be the number of its strongly connected m-subtournaments. Due to work of L.W. Beineke and F. Harary, it is well known that $s_m(T) \leq s_m(RLT_n)$, where RLT_n is the regular locally transitive tournament of order n. For $m = 3$ and $m = 4$, $c_m(T)$ equals $s_m(T)$, but it is not so for $m \geq 5$. As J.W. Moon pointed out in his note in 1966, the problem of determining the maximum of $c_m(T)$ seems very difficult in general (i.e. for $m \geq 5$). In the present paper, based on the Komarov-Mackey formula for $c_5(T)$ obtained recently, we prove that $c_5(T) \leq (n+1)n(n-1)(n-2)(n-3)/160$ with equality holding iff T is doubly regular. A formula for $s_5(T)$ is also deduced. With the use of it, we show that $s_5(T) \leq (n+1)n(n-1)(n-3)(11n-47)/1920$ with equality holding iff $T = RLT_n$ or $n = 7$ and T is regular or $n = 5$ and T is strong. It is also proved that for a regular tournament T of (odd) order $n \geq 9$, a lower bound $(n+1)n(n-1)(n-3)(17n-59)/3840 \leq s_5(T)$ holds with equality iff T is doubly regular. These results are compared with the ones recently obtained by the author for $c_5(T)$.

Keywords: tournament; transitive tournament; locally transitive tournament; regular tournament; doubly regular tournament; 5-cycle, strong 5-subtournament; the Komarov-Mackey formula

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§1. Introduction

A *tournament* T of order n (or, merely, n-tournament T) is an orientation of the complete graph K_n. So, there exists exactly one arc between any two vertices of T. If a pair (i, j) is an arc in T, we say that the vertex i *dominates* the vertex j and write $i \to j$. For two vertex-disjoint sets (tournaments) I and J, the notation $I \Rightarrow J$ means that each vertex of I dominates every vertex of J. We say that vertices of T are replaced by some tournaments $S^{(1)}, ..., S^{(n)}$ if the latter are taken instead of the former and the binary relation $\to$ between the vertices is replaced by the binary relation $\Rightarrow$ between the tournaments. For a tournament S, we write $T \cdot S$ if each vertex of T is replaced by a copy of S. The tournament obtained in result is called the *composition* of T and S.

Let $N^+(i)$ be the *out-set* of i (i.e. the set of vertices dominated by i) and $N^-(i)$ be the *in-set* of i (i.e. the set of vertices dominating i). The subtournaments induced

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by these vertex-sets will be denoted by the same symbols. The quantities $\delta_i^+ = |N^+(i)|$ and $\delta_i^- = |N^-(i)|$ are called the *out-degree* and *in-degree* of i, respectively. They are related as follows: $\delta_i^+ = n-1-\delta_i^-$. However, we will consider both the out-degrees and the in-degrees of the vertices, which are also called the *scores* and *co-scores* of the vertices. The *score-list* (*co-score-list*) of a tournament is the list of the out-degrees (in-degrees) of the vertices, usually arranged in non-decreasing order. A score-list is *balanced* if it coincides with the co-score-list. We also say that a non-decreasing sequence of non-negative integers $\delta_1 \leq ... \leq \delta_k \leq ... \leq \delta_n$ is a (co)-score-list if there is a tournament of order n whose (co)-scores are $\delta_1, ..., \delta_k, ..., \delta_n$. According to the Landau criterion (see [21], [26], and [9]), this holds if and only if $\delta_1 + ... + \delta_k \geq \binom{k}{2}$ for each $k = 1, ..., n-1$ and $\delta_1 + ... + \delta_n = \binom{n}{2}$.

A vertex in T is a *source* if its out-degree is $n-1$. In turn, a vertex in T is a *sink* if its in-degree is $n-1$. Any n-tournament admits at most one source and at most one sink. Moreover, there exists precisely one n-tournament each of whose subtournaments has exactly one sink and exactly one source, namely, the *transitive* tournament TT_n of order n. It is called so because if $i \to k \to j$, then $i \to j$ and this (transitive) condition uniquely determines TT_n: if $1, ..., n$ is its (unique) Hamiltonian path, then $j \to i$ if and only if $i > j$.

A tournament is *strongly connected* (or, merely, *strong*) if for any two distinct vertices i and j, there is a path from i to j. We assume that the tournament $\circ$ of order 1 (consisting of a unique vertex) is strong. Denote by $\mathcal{ST}_n$ the set of all strong tournaments of order n. If T is not contained in $\mathcal{ST}_n$, then it is obtained from some TT_k, where $2 \leq k \leq n$, by replacing its vertices $1, ..., k$ with strong tournaments $T^{(1)}, ..., T^{(k)}$, respectively. In this case, we write $T = T^{(1)} \Rightarrow ... \Rightarrow T^{(k)}$.

For a family $\mathcal{F}_m$ of digraphs of order m, let $n_{\mathcal{F}_m}(T)$ be the number of copies of elements of $\mathcal{F}_m$ in T (i.e. the number of subdigraphs of T that are isomorphic to elements of $\mathcal{F}_m$). If $\mathcal{F}_m$ consists of the unique element D_m, then we simply write $n_{D_m}(T)$. A problem of determining the maximum (minimum) of $n_{\mathcal{F}_m}(T)$ or $n_{D_m}(T)$ in the class $\mathcal{T}_n$ of all tournaments of order n or some subclass of $\mathcal{T}_n$ can be posed. In the present paper, we consider the cases of $\mathcal{F}_m = \mathcal{ST}_m$ and $D_m = \vec{C}_m$, where $\vec{C}_m$ is the *directed cycle of length m*. For simplicity, denote $n_{\mathcal{ST}_m}(T)$ and $n_{\vec{C}_m}(T)$ by $s_m(T)$ and $c_m(T)$, respectively.

We first consider the quantity $s_m(T)$. As for $m \geq 3$, we have $\mathcal{ST}_m \cap \mathcal{T}_{m-1} \Rightarrow \circ = \emptyset$, the inequality

$$s_m(T) \leq \binom{n}{m} - n_{\mathcal{T}_{m-1} \Rightarrow \circ}(T) \tag{1}$$

holds with equality iff every subtournament of order m in T is either strong or contains a sink. In particular, this condition is satisfied if T is *locally$^+$ transitive*, i.e. the out-set of each vertex induces a transitive subtournament. Note that in (1), the term $n_{\mathcal{T}_{m-1} \Rightarrow \circ}(T)$ can be replaced by $n_{\circ \Rightarrow \mathcal{T}_{m-1}}(T)$. The new upper bound will be attained when T is *locally$^-$ transitive*, i.e. the in-set of each vertex induces a transitive subtournament. As

$$\sum_{i=1}^{n} n_{TT_3}\Big(N^+(i)\Big) = n_{\circ \Rightarrow TT_3}(T) = n_{TT_4}(T) = n_{TT_3 \Rightarrow \circ}(T) = \sum_{i=1}^{n} n_{TT_3}\Big(N^-(i)\Big),$$

T with balanced (co)-score-list is locally$^+$ transitive if and only if it is locally$^-$ transitive. A tournament having both of these properties is *locally transitive*. According to Theorem 4 [1], for a given balanced (co)-score-list $\delta_1 \leq ... \leq \delta_n$, there exists a locally transitive tournament with this (co)-score-list if and only if any integer between δ_1 and δ_n is contained in $\{\delta_1, ..., \delta_n\}$.

Note that

$$n_{\mathcal{T}_p \Rightarrow \circ}(T) = \sum_{i=1}^{n} n_{\mathcal{T}_p}\Big(N^-(i)\Big) = \sum_{i=1}^{n} \binom{\delta_i^-}{p}. \tag{2}$$

A well-known combinatorial result (presented, for instance, in [4]) states that for a given integer $p \geq 2$, the sum is a minimum if δ_i^- are as nearly equal as possible (the same also holds if one replaces δ_i^- by δ_i^+ in the sum). Moreover, by our Lemma 1, for $n \geq 2p-1$, "if" can be replaced by "if and only if" in this statement. The condition that δ_i^- are as nearly equal as possible means that if n is odd, each δ_i^- equals $\frac{n-1}{2}$ and hence, T is *regular* (for this case, each δ_i^+ also equals $\frac{n-1}{2}$; so, this quantity can be called the *semi-degree* and denoted by δ); if n is even, half the in-degrees are $\frac{n}{2}$ and the others are $\frac{n}{2}-1$, i.e. T is *near regular* (as one can see, it is a one-vertex-deleted subtournament of some regular tournament of order $n+1$).

It is not difficult to check that for each odd n, there exists exactly one n-tournament that is both regular and locally transitive (see [2]). It can be defined on the ring $\mathbb{Z}_n$ of residues modulo n by the rule $i \to j$ iff the difference $j-i$ (as a residue) is contained in the subset $\{1, ..., \frac{n-1}{2}\}$ of $\mathbb{Z}_n$. We call it the *regular locally transitive* tournament and denote it by RLT_n. The above arguments imply that for odd n, the maximum of $s_m(T)$ in the class $\mathcal{T}_n$ is attained at RLT_n (see [4] and [10]). However, to determine all maximizers of $s_m(T)$ in the class $\mathcal{T}_n$, one needs to get an exact expression for $s_m(T)$. It is possible to do when $m = 3, 4$, and 5 because for these values of m, any non-strong tournament of order m admits either a sink or a source. So, in these cases, we have

$$s_m(T) = \binom{n}{m} - n_{\mathcal{T}_{m-1} \Rightarrow \circ}(T) - n_{\circ \Rightarrow \mathcal{T}_{m-1}}(T) + n_{\circ \Rightarrow \mathcal{T}_{m-2} \Rightarrow \circ}(T) =$$

$$\binom{n}{m} - \sum_{i=1}^{n} \binom{\delta_i^-}{m-1} - \sum_{i=1}^{n} \binom{\delta_i^+}{m-1} + \sum_{i=1}^{n} n_{\mathcal{T}_{m-2} \Rightarrow \circ}\Big(N^+(i)\Big). \tag{3}$$

Note that $\circ \Rightarrow \circ = TT_2$. As each 2-subtournament of any tournament is TT_2, for $m = 3$, the last two terms in (3) are cancelled in pairs. So, the maximum of $s_3(T)$ in the class $\mathcal{T}_n$ is attained only at the class $\mathcal{R}_n$ of regular tournaments of order n or the class $\mathcal{NR}_n$ of near regular tournaments of order n if n is odd or even, respectively [15]. In turn, as $\mathcal{T}_2 \Rightarrow \circ = \{TT_2 \Rightarrow \circ\} = \{TT_3\}$ and a tournament admits only transitive 3-subtournaments if and only if it is transitive itself, for each odd $n \geq 5$, the maximum of $s_4(T)$ in the class $\mathcal{T}_n$ is attained if and only if T is regular and locally transitive, i.e. $T = RLT_n$. Finally, one can show that a tournament of order $p \geq 4$ admits only 4-subtournaments belonging to $\mathcal{T}_3 \Rightarrow \circ$ if and only if it is contained in $\mathcal{T}_3 \Rightarrow TT_{p-3}$. Based on this fact and (3), we prove in Section 2 that for each odd $n \geq 9$, the maximum of $s_5(T)$ in the class $\mathcal{T}_n$ is attained only at RLT_n, while for $n = 7$, it is achieved if and only if T is regular.

For $m = 4$ and $m = 5$, the quantity $s_m(T)$ can take different values on elements of $\mathcal{R}_n$ or $\mathcal{NR}_n$. A lower bound on $s_m(T)$ in the class of tournaments of order n with given (co)-score-list can be obtained from (3) if for each $i = 1, ..., n$, one replaces $n_{\mathcal{T}_{m-2} \Rightarrow \circ}\Big(N^+(i)\Big)$ by its minimum possible value. (For $m = 4$, this was first done in [18].) So, equality (2) and the above lemma on the combinatorial sum imply that the lower bound on $s_m(T)$ is attained if for each $i = 1, ..., n$, the out-set $N^+(i)$ induces a regular or near regular tournament when $\left|N^+(i)\right|$ is odd or even, respectively. A tournament having this property is called *locally$^+$ regular*.

In the last sum in (3), the term $n_{\mathcal{T}_{m-2}\Rightarrow\circ}\Big(N^+(i)\Big)$ can be replaced by the term $n_{\circ\Rightarrow\mathcal{T}_{m-2}}\Big(N^-(i)\Big)$. So, a new lower bound on $s_m(T)$ can be obtained if for each $i = 1, ..., n$, one replaces $n_{\circ\Rightarrow\mathcal{T}_{m-2}}\Big(N^-(i)\Big)$ by its minimum possible value. It is attained if for each $i = 1, ..., n$, the in-set $N^-(i)$ induces a regular or near regular tournament when $\left|N^-(i)\right|$ is odd or even, respectively. Such a tournament is called *locally$^-$ regular*. One can show that a tournament with balanced (co)-score-list is locally$^+$ regular if and only if it is locally$^-$ regular. In this case, it is *locally regular*.

A regular locally regular tournament of order n is called *doubly-regular* or *nearly-doubly-regular* when $\frac{n-1}{2}$ is odd and hence, $n \equiv 3 \mod 4$ or $\frac{n-1}{2}$ is even and hence, $n \equiv 1 \mod 4$, respectively. Denote by $\mathcal{DR}_n$ and $\mathcal{NDR}_n$ the corresponding subfamilies of $\mathcal{R}_n$. If the value of $s_m(T)$ does not depend on a particular choice of $T \in \mathcal{DR}_n$ or $T \in \mathcal{NDR}_n$, then we will simply write $s_m(\mathcal{DR}_n)$ or $s_m(\mathcal{NDR}_n)$. The same rule also concerns the other quantities introduced below.

Under this convention, the lower bound on $s_m(T)$ in the class $\mathcal{R}_n$ suggested above for the case of $m = 4$ and $m = 5$ can be written as $s_m\big((\mathcal{N})\mathcal{DR}_n\big) \le s_m(T)$, where the choice of $\mathcal{NDR}_n$ or $\mathcal{DR}_n$ depends on the residue (1 or 3) of n modulo 4. Our Lemma 1 implies that for $m = 4$ and $n \ge 5$ or $m = 5$ and $n \ge 11$, the equality holds if and only if $T \in \mathcal{DR}_n$ (when $n \equiv 3 \mod 4$) or $T \in \mathcal{NDR}_n$ (when $n \equiv 1 \mod 4$). However, it is not so for $m = 5$ and $n = 9$. For this case, the bound is attained if and only if the out-set of each vertex of T includes no sink. In Section 2, we show that besides two elements of $\mathcal{NDR}_9$, there exist three more such regular tournaments of order 9.

The problem of non-emptiness of $\mathcal{DR}_n$ for each $n \equiv 3 \mod 4$ is open up to now, while one can present infinitely many DR's, there are methods for constructing DR's from those of smaller orders and according to a common opinion, at least one element of $\mathcal{DR}_n$ exists for any possible order n (see [25]). The same can be also said about $\mathcal{NDR}_n$, where $n \equiv 1 \mod 4$, (see [22], [23], and [33]), while much less is known about NDR's than about DR's. We say that for odd n, the $\exists$-property holds if the set $\mathcal{RLR}_n$ of regular locally regular tournaments of order n is not empty, i.e. either $\mathcal{DR}_n \neq \emptyset$ (when $n \equiv 3 \mod 4$) or $\mathcal{NDR}_n \neq \emptyset$ (when $n \equiv 1 \mod 4$).

Note that if $m = 3$ or 4, then there exists exactly one strong tournament of order m. For $m = 3$, it is called the *cyclic triple* and is denoted by Δ. In turn, the unique strong tournament ST_4 of order 4 can be obtained from Δ by replacing one of its vertices with TT_2. It is not difficult to check that either of the strong tournaments contains precisely one Hamiltonian cycle. So, for $m = 3$ and $m = 4$, the equality $c_m(T) = s_m(T)$ holds and hence, we can apply the above-mentioned (classical) results on $s_m(T)$ to $c_m(T)$.

As it was first pointed out in [20] (see also [21]), for the case of $m = 5$, the problem of determining the maximum of $c_m(T)$ in the class $\mathcal{T}_n$ is much more difficult. (Note that $|\mathcal{ST}_5| = 6$ and the number of 5-cycles varies from 1 to 3 on $\mathcal{ST}_5$.) One can show that in contrast with $s_5(T)$ and $c_4(T)$, the collection of the out-sets of vertices in T does not determine $c_5(T)$. However, there also exists a formula for $c_5(T)$ expressed in terms of the co-scores of vertices in the out-sets of vertices in T. It has been obtained only recently by Komarov and Mackey (see [16]). With the use of it, they present an upper bound on $c_5(T)$ in the class $\mathcal{T}_n$ (which, however, is not sharp for any $n \geq 5$) and prove that the maximum of $c_5(T)$ in the class $\mathcal{T}_n$ is asymptotically the same as the expected number $E_n c_5$ of 5-cycles in a random tournament of order n. In Section 3, based on the Komarov-Mackey formula, we show that for each $T \in \mathcal{T}_n$, where n is odd and $n \geq 5$, the inequality $c_5(T) \leq E_{n+1}c_5$ holds with equality iff $T \in \mathcal{DR}_n$. (In particular, this means that the upper bound is sharp for infinitely many n.) So, if the (very plausible) conjecture on the existence of at least one element in $\mathcal{DR}_n$ for each possible n is true, then the problem of determining the maximum of $c_5(T)$ in the class $\mathcal{T}_n$ is settled for $n \equiv 3 \mod 4$. The other cases are still open. At the end of the section, the reader will find a conjecture on possible maximizers of $c_5(T)$ in $\mathcal{T}_n$ for $n \equiv 1 \mod 4$. Finally, in Section 4, we give concluding comments on $s_m(T)$ and $c_m(T)$ in the case of arbitrary m.

§2. All maximizers of $\mathbf{s_5(T)}$ in the class $\mathcal{T}_\mathbf{n}$, where n is odd, and a lower bound on $\mathbf{s_5(T)}$ in the class $\mathcal{R}_\mathbf{n}$

We have already mentioned and used the well-known fact that the minimum of the sum $\sum\limits_{i=1}^{n} \binom{\delta_i^-}{p}$ is achieved when δ_i^- are as nearly equal as possible. For our purposes, we need to show that this condition is also necessary if $p \geq 2$ and n is large enough.

Lemma 1. *Let $p \geq 2$. Then for $n \geq 2p-1$, the sum $\sum\limits_{i=1}^{n} \binom{\delta_i^-}{p}$ attains its minimum in the class $\mathcal{T}_n$ if and only if the corresponding tournament is regular (when n is odd) or near regular (when n is even).*

Proof. We first consider the minimization problem in the class Ω_n of ordered sequences ω_n of non-negative integers $\delta_1^- \leq ... \leq \delta_n^-$ whose sum is equal to $\frac{n(n-1)}{2}$ for the case of $n \geq 2p$. If $\delta_1^- \geq p-1$ and $\delta_n^- > \delta_1^- + 1$, then

$$\binom{\delta_1^- + 1}{p} - \binom{\delta_1^-}{p} = \frac{\delta_1^- ...(\delta_1^- - p + 2)}{(p-1)!} <$$

$$\frac{(\delta_n^- - 1)...(\delta_n^- - p + 1)}{(p-1)!} = \binom{\delta_n^-}{p} - \binom{\delta_n^- - 1}{p}.$$

In turn, if $\delta_1^- < p-1$, then this inequality also holds because the LHS equals 0, while the RHS is positive as $\delta_n^- \geq \lceil \frac{n-1}{2} \rceil \geq p$. So, in both cases, we have

$$\binom{\delta_1^-}{p} + \binom{\delta_n^-}{p} > \binom{\delta_1^- + 1}{p} + \binom{\delta_n^- - 1}{p}$$

and hence, the sum $\sum\limits_{i=1}^{n} \binom{\delta_i^-}{p}$ will strictly decrease if we replace δ_n^- and δ_1^- with $\delta_n^- - 1$ and $\delta_1^- + 1$, respectively. (Obviously, after ordering, the new sequence will belong to Ω_n.)

We see that the minimum is achieved if $\delta_n^- - \delta_1^- \leq 1$. Let us show that this condition uniquely determines ω_n in the class Ω_n. If $\delta_n^- - \delta_1^- = 0$, then $\delta_1^- = ... = \delta_n^- = \frac{n-1}{2}$, where n must be odd. For any other ω_n, we have $\delta_1^- < \frac{n-1}{2} < \delta_n^-$ and hence, if n is odd, then $\delta_n^- - \delta_1^- \geq 2$. So, if $\delta_n^- - \delta_1^- = 1$, then n must be even,

$\delta_1^- = \frac{n}{2} - 1$ and $\delta_n^- = \frac{n}{2}$. Let p be the integer such that $\frac{n}{2} - 1 = \delta_p^- < \delta_{p+1}^- = \frac{n}{2}$. Then the condition $\delta_1^- + ... + \delta_n^- = p\big(\frac{n}{2} - 1\big) + (n-p)\frac{n}{2} = \frac{n(n-1)}{2}$ implies that $p = \frac{n}{2}$. In both cases, denote the sequence obtained in result by $\hat{\omega}_n$.

The arguments presented above imply that for $n \geq 2p$, the minimum of the sum $\sum\limits_{i=1}^{n} \binom{\delta_i^-}{p}$, where $p \geq 2$, in Ω_n is attained only at $\hat{\omega}_n$. In the case where $n = 2p - 1$, the sequence $\hat{\omega}_n$ is a unique element of Ω_n for which the sum is equal to zero. (For any other sequence of Ω_n, where $n = 2p - 1$, we have $\delta_n^- \geq p$ and hence, the sum is greater than or equal to 1.) So, we have determined all minimizers of the sum in Ω_n for any $n \geq 2p - 1$. The same result also holds for the minimum in the class $\mathcal{T}_n$ because $\hat{\omega}_n$ is the co-score-list of a regular n-tournament or a near regular n-tournament when n is odd or even, respectively. The lemma is proved.

Denote by $\Delta \cdot TT_2$ the (near regular) tournament of order 6 obtained from Δ by replacing each vertex with TT_2. It is well known that there exist exactly three regular tournaments of order 7, namely, RLT_7, the unique element DR_7 of $\mathcal{DR}_7$, and the Kotzig tournament Kz_7 which is uniquely determined by the condition that $\Delta \cdot TT_2$ is its one-vertex-deleted subtournament. As we have already seen, for each odd n, the maximum of $s_5(T)$ in the class $\mathcal{T}_n$ is equal to

$$s_5(RLT_n) = \binom{n}{5} - n\binom{\frac{n-1}{2}}{4} = \frac{(n+1)n(n-1)(n-3)(11n-47)}{1920}. \qquad (4)$$

The description of $\mathcal{R}_7$ and Lemma 1 taken together allow us to determine all maximizers of $s_5(T)$ in the class $\mathcal{T}_n$.

Proposition 1. *For each odd $n \geq 9$, the maximum of $s_5(T)$ in the class $\mathcal{T}_n$ is attained only at RLT_n. For $n = 7$, it is also achieved for DR_7 and Kz_7.*

Proof. Recall that $\mathcal{T}_4$ consists of ST_4, $\circ \Rightarrow \Delta$, $\Delta \Rightarrow \circ$, and TT_4. As $\{\Delta \Rightarrow \circ\} \cup \{TT_4\} = \mathcal{T}_3 \Rightarrow \circ$, for $m = 5$, we can rewrite (3) as

$$s_5(T) = \binom{n}{5} - \sum_{i=1}^{n} \binom{\delta_i^-}{4} - \sum_{i=1}^{n} n_{ST_4}\Big(N^+(i)\Big) - \sum_{i=1}^{n} n_{\circ\Rightarrow\Delta}\Big(N^+(i)\Big). \qquad (5)$$

For the case of RLT_n, each of the three sums in (5) attains its minimum. The first one equals $n\binom{\frac{n-1}{2}}{4}$ and the others equal 0 (see (4)). The same must also hold for any maximizer T of order $n \geq 7 = 2\cdot 4 - 1$. Equality (5) and Lemma 1 imply that T is a regular tournament with semi-degree $\delta \geq 3$ the out-set of each of whose vertices contains no ST_4 or $\circ \Rightarrow \Delta$ as a subtournament. Obviously, this condition always holds if $\delta = 3$. So, we can assume that $\delta \geq 4$.

Any tournament of order δ is obtained from some transitive tournament by replacing its vertices with strong tournaments. The latter are strong components of the tournament. If the order of at least one of them is not less than 4, then by the Moon vertex-pancyclic theorem, one can always find a copy of ST_4 in it. So, if the tournament contains no ST_4, then each of its strong components is either $\circ$ or Δ. In turn, if it does not admit $\circ \Rightarrow \Delta$ at that, then only its first strong component can be Δ. Hence, it is either TT_δ or $\Delta \Rightarrow TT_{\delta-3}$. This implies that the out-set of each vertex of T belongs to $\mathcal{T}_3 \Rightarrow TT_{\delta-3}$.

Note that the converse T^- of T (obtained from T by reversing all of its arcs) is also a maximizer and hence, the out-set of each of its vertices is contained in

$\mathcal{T}_3 \Rightarrow TT_{\delta-3}$. The out-set of a vertex in T^- is the converse of the in-set of the same vertex in T. So, the in-set of each vertex of T belongs to $TT_{\delta-3} \Rightarrow \mathcal{T}_3$. In particular, for each i, the in-set $N^-(i)$ contains a source. Denote it by j. Obviously, $N^-(j)$ is a subset of $N^+(i)$. Since $|N^+(i)| = |N^-(j)| = \delta$, we have $N^+(i) = N^-(j)$. As for $\delta \geq 4$, $\mathcal{T}_3 \Rightarrow TT_{\delta-3} \bigcap TT_{\delta-3} \Rightarrow \mathcal{T}_3 = \{TT_\delta\}$, the structure of $N^+(i)$ and $N^-(j)$ described above implies that $N^+(i) \cong TT_\delta$. Hence, $T \cong RLT_n$. The proposition is proved.

For comparison with (4), note that

$$c_5(RLT_n) = \frac{(n+1)n(n-1)(n-3)(3n-11)}{480}.$$

This expression for $c_5(RLT_n)$ was first obtained in [5], while it is also presented in [28]. Let $\Delta(\circ, TT_3, \circ)$ be the 5-tournament obtained from Δ by replacing one of its vertices with TT_3 and $\Delta(TT_2, \circ, TT_2)$ be the 5-tournament obtained from Δ by replacing two of its vertices with TT_2. If we add RLT_5 to them, then we obtain a list of all strong locally transitive 5-tournaments (only they can be strong 5-subtournaments of RLT_n!). Either of the first two 5-tournaments admits exactly one hamiltonian cycle, while RLT_5 contains two 5-cycles. So,

$$c_5(RLT_n) = n_{\Delta(\circ,TT_3,\circ)}(RLT_n) + n_{\Delta(TT_2,\circ,TT_2)}(RLT_n) + 2n_{RLT_5}(RLT_n),$$

while

$$s_5(RLT_n) = n_{\Delta(\circ,TT_3,\circ)}(RLT_n) + n_{\Delta(TT_2,\circ,TT_2)}(RLT_n) + n_{RLT_5}(RLT_n).$$

The known expressions for $s_5(RLT_n)$ and $c_5(RLT_n)$ imply that

$$n_{RLT_5}(RLT_n) = \frac{(n+3)(n+1)n(n-1)(n-3)}{1920}.$$

Theorem 1.2 [13] allows us to suggest that the maximum of $n_{RLT_5}(T)$ in the class $\mathcal{T}_n$ is attained at RLT_n. Based on this theorem, one can also conjecture that

$$n_{\Delta(\circ,TT_3,\circ)}(T) \leq n_{\Delta(\circ,TT_3,\circ)}(RLT_n) = \frac{(n+1)n(n-1)(n-3)(n-5)}{384}.$$

However, Theorem 1.3 [13] means that for large odd n, the maximum of the quantity $n_{\Delta(TT_2,\circ,TT_2)}(T)$ in the class $\mathcal{T}_n$ cannot be achieved for RLT_n, while RLT_n has the same number of copies of $\Delta(\circ, TT_3, \circ)$ and $\Delta(TT_2, \circ, TT_2)$. The results of [13] show that the problem of determining the maximum number of copies of a given 5-tournament in the class $\mathcal{T}_n$ is trivial only for TT_5. For other 5-tournaments, it seems to be very difficult.

Denote by $\Delta \cdot \Delta$ the tournament of order 9 obtained from the cyclic triple Δ by replacing each of its vertices with a copy of Δ. Lemma 1 and formula (3) allow us not only to determine all maximizers of $s_5(T)$ in the class $\mathcal{T}_n$ but also to obtain a sharp lower bound on $s_5(T)$ in the class $\mathcal{R}_n$.

Proposition 2. *Let T be a regular tournament of (odd) order n. If $n \equiv 3 \mod 4$, then the inequality*

$$\frac{n(n+1)(n-1)(n-3)(17n-59)}{3840} = s_5(\mathcal{DR}_n) \leq s_5(T)$$

holds with equality iff T is doubly-regular (i.e. $T \in \mathcal{DR}_n$) or T is an arbitrary regular 7-tournament. In turn, if $n \equiv 1 \mod 4$, then we have

$$\frac{n(n-1)(17n^3-93n^2+127n-243)}{3840} = s_5(\mathcal{NDR}_n) \leq s_5(T)$$

with equality holding iff T is nearly-doubly-regular (i.e. $T \in \mathcal{NDR}_n$) or T is isomorphic to $\Delta \cdot \Delta$ or one of the following two regular tournaments of order 9 with adjacency matrices

$$\begin{pmatrix} 0&1&1&1&1&0&0&0&0\\ 0&0&1&1&1&1&0&0&0\\ 0&0&0&1&0&1&0&1&1\\ 0&0&0&0&1&0&1&1&1\\ 0&0&1&0&0&0&1&1&1\\ 1&0&0&1&1&0&1&0&0\\ 1&1&1&0&0&0&0&1&0\\ 1&1&0&0&0&1&0&0&1\\ 1&1&0&0&0&1&1&0&0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0&1&1&1&1&0&0&0&0\\ 0&0&1&0&1&0&1&1&0\\ 0&0&0&1&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&1\\ 0&0&0&1&0&1&0&1&1\\ 1&1&1&0&0&0&0&0&1\\ 1&0&0&1&1&1&0&0&0\\ 1&0&1&0&0&1&1&0&0\\ 1&1&0&0&0&0&1&1&0 \end{pmatrix}.$$

Proof. For a regular tournament T with semi-degree δ (and hence, of order $n = 2\delta+1$) and $m = 5$, formula (3) takes the following form

$$s_5(T) = \binom{n}{5} - 2n\binom{\delta}{4} + \sum_{i=1}^{n} n_{\mathcal{T}_3 \Rightarrow \circ}\Big(N^+(i)\Big). \tag{6}$$

This formula, equality (2), and Lemma 1 imply that for odd $\delta \geq 1$, we have

$$s_5(T) \geq \binom{n}{5} - 2n\binom{\delta}{4} + n\delta\binom{\frac{\delta-1}{2}}{3} = \frac{\delta(\delta-1)(\delta+1)(2\delta+1)(17\delta-21)}{240}.$$

For odd $\delta \geq 5$, the equality holds if and only if for each $i = 1, ..., n$, the out-set $N^+(i)$ is a regular tournament of order δ. In turn, if $\delta = 3$, then any $N^+(i)$ satisfies $n_{\mathcal{T}_3 \Rightarrow \circ}\Big(N^+(i)\Big) = 0$. So, for any regular tournament T of order 7, the equality holds. The case $\delta = 1$ is trivial as Δ is the unique regular tournament of order 3 and it is also the unique element of $\mathcal{DR}_3$.

Formula (6), equality (2), and Lemma 1 imply that for even $\delta \geq 2$, we have

$$s_5(T) \geq \binom{n}{5} - 2n\binom{\delta}{4} + n\frac{\delta}{2}\left(\binom{\frac{\delta}{2}}{3} + \binom{\frac{\delta}{2}-1}{3}\right) = \frac{\delta(2\delta+1)(17\delta^3-21\delta^2-2\delta-24)}{240}.$$

For even $\delta \geq 6$, the equality holds if and only if for each $i = 1, ..., n$, the out-set $N^+(i)$ is a near regular tournament of order δ. For $\delta = 4$, this lower bound is attained if and only if for each i, the out-set $N^+(i)$ is not contained in $\mathcal{T}_3 \Rightarrow \circ$, i.e. it is either ST_4 or $\circ \Rightarrow \Delta$. It is not difficult to check that the out-set of each vertex in the composition $\Delta \cdot \Delta$ induces $\circ \Rightarrow \Delta$. It is also shown in [33] that there are exactly two nearly-doubly-regular tournaments of order 9 (the out-set of each of their vertices induces ST_4). Can one present other examples of $T \in \mathcal{R}_9$ the out-sets of whose vertices induce ST_4 or $\circ \Rightarrow \Delta$?

Note that the score-lists of ST_4 and $\circ \Rightarrow \Delta$ are $(1,1,2,2)$ and $(1,1,1,3)$, respectively. The score-lists of the remaining tournaments of order 4 contain 0. Non-diagonal entries of the matrix $AA^\top$, where A is the adjacency matrix and $A^\top$ is its transpose, form the score-lists of the out-sets of the vertices. So, we have to find regular tournaments T of order 9 for which $AA^\top > 0$, i.e. the matrix $AA^\top$ has only positive entries. According to [3], there exist exactly 15 (non-isomorphic) regular tournaments of order 9. Our exhaustive manual consideration of the list of all elements of $\mathcal{R}_9$ presented therein shows that there are another two regular tournaments of order 9 with $AA^\top > 0$. Their adjacency matrices were given in the statement of the proposition. Note that for either of them, there exists a vertex whose out-set induces ST_4 and there exists a vertex whose out-set induces $\circ \Rightarrow \Delta$. So, in some sense, they are placed between two elements of $\mathcal{NDR}_9$ and $\Delta \cdot \Delta$. Finally, the case $\delta = 2$ is trivial because the unique regular tournament of order 5 is nearly-doubly-regular (it is isomorphic to RLT_5) and hence, the statement also holds for $n = 5$. The proposition is proved.

As we have seen in the introduction, formula (3) and Lemma 1 imply that for $T \in \mathcal{R}_n$, the inequality $s_4\big((\mathcal{N})\mathcal{DR}_n\big) \le s_4(T)$ or, the same, $n_{ST_4}\big((\mathcal{N})\mathcal{DR}_n\big) \le n_{ST_4}(T)$ holds with equality if and only if $T \in (\mathcal{N})\mathcal{DR}_n$. The same also takes place for the quantity $n_{TT_4}(T)$. For $\circ \Rightarrow \Delta$ or $\circ \Leftarrow \Delta$, the minimum number of copies of either of them is attained at RLT_n and equals 0. The same also holds for all tournaments of order 5 with the exception of TT_5, RLT_5, $\Delta(\circ, TT_3, \circ)$, and $\Delta(TT_2, \circ, TT_2)$ because only they are locally transitive and hence, only they can be 5-subtournaments of RLT_n. For each of them, the problem of determining the minimum number of copies in the class $\mathcal{R}_n$ is difficult. It is so even for TT_5. Indeed, the identity

$$n_{TT_5}(T) = \sum_{i=1}^{n} n_{TT_4}\Big(N^+(i)\Big) = \sum_{i=1}^{n} n_{TT_4}\Big(N^-(i)\Big)$$

and the lower bound $n_{TT_4}\big((\mathcal{N})\mathcal{DR}_n\big) \le n_{TT_4}(T)$ allow us to get a lower bound on $n_{TT_5}(T)$. (For $n \equiv 7 \mod 8$, the corresponding expression is given in the last section of [20].) For $n \ge 11$, it is attained only at a *regular locally doubly-regular tournament* (when $n \equiv 7 \mod 8$), i.e. the out-set (in-set) of each vertex is doubly regular, or a *regular locally nearly-doubly-regular tournament* (when $n \equiv 3 \mod 8$), i.e. the out-set (in-set) of each vertex is nearly doubly regular. Denote by $\mathcal{RLDR}_n$ and $\mathcal{RLNDR}_n$ the corresponding subfamilies of $\mathcal{R}_n$. Are they non-empty?

For $n = p^k$, where p is a prime that is congruent to 3 modulo 4 and k is an odd positive integer, define the *quadratic residue tournament* QR_n as the tournament such that its vertex-set is the Galois field $GF(n)$ and $i \to j$ if and only if $j - i$ is a non-zero square in $GF(n)$. For each such n, the tournament QR_n has high regularity properties. In particular, it is doubly regular. One can check that $\mathcal{RLDR}_7 = \{QR_7\}$, while $\mathcal{RLDR}_{15} = \emptyset$ and for $n = 23, 31$, and 47, the tournament QR_n does not belong to the set $\mathcal{RLDR}_n$. Moreover, $\mathcal{RLNDR}_{11} = \{QR_{11}\}$, $\mathcal{RLNDR}_{19} = \{QR_{19}\}$, and $\mathcal{RLNDR}_{27}$ contains QR_{27}, while QR_{43} is not contained in $\mathcal{RLNDR}_{43}$. We conjecture that a regular locally doubly-regular tournament of order n does not exist for any $n > 7$ (and so, the Moon lower bound on $n_{TT_5}(T)$ is not sharp for $n > 7$) and for sufficiently large n, the set $\mathcal{RLNDR}_n$ is also empty. For these values of n, we have no good candidates for minimizers of $n_{TT_5}(T)$. At the moment, we can only state that any sequence of minimizers of growing order n

must be quasi-random (see, for instance, [11]) or, the same, asymptotically doubly regular (the corresponding definition will be given in Section 4 below).

Proposition 2 and the main result of [4] imply that for $m = 4$ or 5 and $T \in \mathcal{R}_n$, we have $s_m\big((\mathcal{N})\mathcal{DR}_n\big) \le s_m(T) \le s_m(RLT_n)$. On the other hand, it is shown in [28] (see also Proposition A.1 in Appendix A) that

$$c_5(T) + 2c_4(T) = \frac{n(n-1)(n+1)(n-3)(n+3)}{160} \tag{7}$$

and hence, $c_5(RLT_n) \le c_5(T) \le c_5\big((\mathcal{N})\mathcal{DR}_n\big)$. So, while $c_m(T)$ and $s_m(T)$ coincide for $m = 3$ and $m = 4$, they demonstrate quite different behaviour for the next value $m = 5$. In the next section, we prove that the inequality $c_5(T) \le c_5(\mathcal{DR}_n)$ holds for each $T \in \mathcal{T}_n$, where n is odd.

§3. An upper bound on $\mathbf{c_5(T)}$ in the class $\mathcal{T}_\mathbf{n}$, where n is odd

For $m = 4$, formula (3) can be rewritten as

$$c_4(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{\delta_i^-}{3} - \sum_{i=1}^{n} c_3\Big(N^+(i)\Big)$$

or

$$c_4(T) = \binom{n}{4} - \sum_{i=1}^{n} \binom{\delta_i^+}{3} - \sum_{i=1}^{n} c_3\Big(N^-(i)\Big).$$

Replacing all δ_i^+ and δ_i^- by $\frac{n-1}{2}$, summing both identities and then dividing the sum obtained by 2 yield a formula for the number of 4-cycles in $T \in \mathcal{R}_n$:

$$c_4(T) = \frac{(n+1)n(n-1)(n-3)}{48} - \frac{1}{2}\sum_{i=1}^{n} c_3\Big(N^+(i)\Big) - \frac{1}{2}\sum_{i=1}^{n} c_3\Big(N^-(i)\Big).$$

In turn, this formula and identity (7) imply that for a regular tournament T of order n, we have

$$c_5(T) = \frac{n(n-1)(n+1)(n-3)(3n-11)}{480} + \sum_{i=1}^{n} c_3\big(N^+(i)\big) + \sum_{i=1}^{n} c_3\big(N^-(i)\big).$$

We see that two regular n-tournaments with the same collection of the out-sets and in-sets of vertices have the same number of 5-cycles. However, it is not so even for near regular locally transitive tournaments of given even order n. In particular, $\Delta \cdot TT_2$ has 6 cycles of length 5, while $RLT_7 - \circ$ (it is obtained by replacing one vertex of RLT_5 with TT_2) admits 8 cycles of length 5. This is a direct consequence of the following three facts: (a) any near regular tournament of order 6 is 2-strong; (b) $c_5(RLT_5) = 2$; (c) any other strong locally transitive tournament of order 5 admits exactly one hamiltonian cycle.

As one can see, generally, the quantity $c_5(T)$ is not uniquely determined by the out-sets of the vertices. In [16], a formula for $c_5(T)$ was obtained in terms of the *intersection numbers* (or, the *edge-scores*).[1] For any two vertices i and j, they are defined as follows

$$\delta_{ij}^{++} = |N^+(i) \cap N^+(j)|, \quad \delta_{ij}^{+-} = |N^+(i) \cap N^-(j)|,$$

and

$$\delta_{ij}^{--} = |N^-(i) \cap N^-(j)|, \quad \delta_{ij}^{-+} = |N^-(i) \cap N^+(j)|.$$

Formulas (2) and (3) imply that

$$s_5(T) = \binom{n}{5} - \sum_{i=1}^{n} \binom{\delta_i^-}{4} - \sum_{i=1}^{n} \binom{\delta_i^+}{4} + \sum_{(i,j)\in\mathcal{A}(T)} \binom{\delta_{ij}^{+-}}{3},$$

where $\mathcal{A}(T)$ is the arc-set of T. The formula for $c_5(T)$ can be written in our notation as follows:

$$8c_5(T) = 6\binom{n}{5} + \sum_{(i,j)\in\mathcal{A}(T)} f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-}),$$

where

$$f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-}) = -(\delta_{ij}^{+-} + \delta_{ij}^{-+})(\delta_{ij}^{++} - \delta_{ij}^{--})^2 - (\delta_{ij}^{++} + \delta_{ij}^{--})(\delta_{ij}^{+-} - \delta_{ij}^{-+})^2 +$$
$$2(\delta_{ij}^{++} + \delta_{ij}^{--})(\delta_{ij}^{+-} + \delta_{ij}^{-+}). \tag{8}$$

Note that

$$\delta_{ij}^{++} + \delta_{ij}^{--} + \delta_{ij}^{+-} + \delta_{ij}^{-+} = n - 2. \tag{9}$$

Based on the formula for $c_5(T)$ and (9), the authors of [16] showed that

$$c_5(T) \le \frac{3}{4}\binom{n}{5} + \frac{1}{4}\binom{n}{2}\Big(\frac{n-2}{2}\Big)^2 = \frac{n(n-1)(n-2)(n^2-2n+2)}{160}.$$

A more detailed analysis of (8) allows us to obtain the following proposition.

Theorem 1. *For a tournament T of odd order $n \ge 5$, the inequality*

$$c_5(T) \le \frac{(n+1)n(n-1)(n-2)(n-3)}{160} \tag{10}$$

holds with equality iff T is doubly-regular (i.e. $T \in \mathcal{DR}_n$).

Proof. Let us first give an upper bound on the value of the function $f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-})$ in (8) for a given arc (i,j) of T. Since $n-2$ is odd, equality (9) implies that at least one of inequalities $\delta_{ij}^{-+} \ne \delta_{ij}^{+-}$ and $\delta_{ij}^{++} \ne \delta_{ij}^{--}$ holds.

Assume first that $\delta_{ij}^{-+} \ne \delta_{ij}^{+-}$ (and hence, $\delta_{ij}^{+-} + \delta_{ij}^{-+} \ge 1$). Then (8) implies that

$$f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-}) \le -\big(\delta_{ij}^{++} + \delta_{ij}^{--}\big) + 2\big(\delta_{ij}^{++} + \delta_{ij}^{--}\big)\big(\delta_{ij}^{+-} + \delta_{ij}^{-+}\big)$$

[1] As the authors of [16] remark, the combinatorial sense of their formula is not well understood. We would also like to note that while the known expressions for $c_5(T)$ in the classes $\mathcal{R}_n$ and $\mathcal{NR}_n$ contain only the numbers of 3-cycles in some subtournaments, at the moment, we are unable to present such a formula for $c_5(T)$ in the general case (recall that it exists for $c_4(T)$).

with equality holding iff $\delta_{ij}^{++} = \delta_{ij}^{--} \geq 1$ and $\delta_{ij}^{+-} = \delta_{ij}^{-+} \pm 1$ or $\delta_{ij}^{++} = \delta_{ij}^{--} = 0$. Let $\Delta_{ij} = \delta_{ij}^{++} + \delta_{ij}^{--}$. By (9), we have $\delta_{ij}^{+-} + \delta_{ij}^{-+} = n - 2 - \Delta_{ij}$. So, the above inequality can be rewritten as

$$f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-}) \leq -\Delta_{ij} + 2\Delta_{ij}\big(n - 2 - \Delta_{ij}\big).$$

Note that

$$-\Delta_{ij} + 2\Delta_{ij}\big(n - 2 - \Delta_{ij}\big) = \frac{(n-3)(n-2)}{2} - 2\Big(\Delta_{ij} - \frac{n-3}{2}\Big)\Big(\Delta_{ij} - \frac{n-2}{2}\Big).$$

For integers Δ_{ij} and n, the subtrahend in the right-hand side of the equality is always non-negative. It is equal to zero iff $\Delta_{ij} = \frac{n-3}{2}$ or $\Delta_{ij} = \frac{n-2}{2}$. The first number is strictly greater than 0 if $n \geq 5$ and the last number is not an integer if n is odd. Hence, for the case $\delta_{ij}^{-+} \neq \delta_{ij}^{+-}$, we have

$$f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-}) \leq \frac{(n-3)(n-2)}{2} \tag{11}$$

with equality holding iff either

$$\delta_{ij}^{++} = \delta_{ij}^{--} = \delta_{ij}^{+-} = \frac{n-3}{4} \quad \text{and} \quad \delta_{ij}^{-+} = \frac{n+1}{4} \quad \big(\text{case } \delta_{ij}^{-+} = \delta_{ij}^{+-} + 1\big) \tag{a}$$

or

$$\delta_{ij}^{++} = \delta_{ij}^{--} = \delta_{ij}^{-+} = \frac{n-3}{4} \quad \text{and} \quad \delta_{ij}^{+-} = \frac{n+1}{4} \quad \big(\text{case } \delta_{ij}^{-+} = \delta_{ij}^{+-} - 1\big). \tag{b}$$

Note that the value of $f(\delta_{ij}^{++}, \delta_{ij}^{--}, \delta_{ij}^{-+}, \delta_{ij}^{+-})$ remains the same if one interchanges the pairs $(\delta_{ij}^{++}, \delta_{ij}^{--})$ and $(\delta_{ij}^{+-}, \delta_{ij}^{-+})$. Thus, the arguments given above imply that for the case $\delta_{ij}^{++} \neq \delta_{ij}^{--}$, inequality (11) also holds with equality iff either

$$\delta_{ij}^{+-} = \delta_{ij}^{-+} = \delta_{ij}^{--} = \frac{n-3}{4} \quad \text{and} \quad \delta_{ij}^{++} = \frac{n+1}{4} \quad \big(\text{case } \delta_{ij}^{++} = \delta_{ij}^{--} + 1\big) \tag{c}$$

or

$$\delta_{ij}^{+-} = \delta_{ij}^{-+} = \delta_{ij}^{++} = \frac{n-3}{4} \quad \text{and} \quad \delta_{ij}^{--} = \frac{n+1}{4} \quad \big(\text{case } \delta_{ij}^{++} = \delta_{ij}^{--} - 1\big). \tag{d}$$

Inequality (11) taken together with (8) implies that

$$c_5(T) \leq \frac{3}{4}\binom{n}{5} + \frac{1}{8}\sum_{(i,j)\in\mathcal{A}(T)} \frac{(n-3)(n-2)}{2} =$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)}{160} + \frac{n(n-1)(n-2)(n-3)}{32} =$$

$$= \frac{n(n-1)(n-2)(n-3)(n+1)}{160}.$$

Assume that this upper bound is attained. Then for each arc $(i,j) \in \mathcal{A}(T)$, the equality in (11) holds and hence, exactly one of cases $(a)-(d)$ is realized. In all cases, we have $\delta_{ij}^{++} \geq \frac{n-3}{4}$. Note that δ_{ij}^{++} is the out-degree of j in the subtournament induced by $N^+(i)$. For a tournament with minimum out-degree δ_{min}^+ (maximum out-degree δ_{max}^+), its order is at least $2\delta_{min}^+ + 1$ (at most $2\delta_{max}^+ + 1$) with equality holding iff it is regular. Hence, for each vertex i of T, the inequality $\delta_i^+ \geq \frac{n-1}{2}$ holds. As the order of T is n, in fact, we have $\delta_i^+ = \frac{n-1}{2}$ for each i and hence, $\delta_{ij}^{++} = \frac{n-3}{4}$ for each $(i,j) \in \mathcal{A}(T)$, i.e. T is doubly regular.

Note that if $i \to j$, then the intersection numbers are related as follows:

$$\delta_i^+ = 1 + \delta_{ij}^{++} + \delta_{ij}^{+-}, \quad \delta_i^- = \delta_{ij}^{--} + \delta_{ij}^{-+}$$

and

$$\delta_j^+ = \delta_{ij}^{++} + \delta_{ij}^{-+}, \quad \delta_j^- = 1 + \delta_{ij}^{--} + \delta_{ij}^{+-}.$$

From this it follows that if

$$\delta_i^+ = \delta_i^- = \delta_j^+ = \delta_j^- = \frac{n-1}{2} \quad \text{and} \quad \delta_{ij}^{++} = \frac{n-3}{4},$$

then

$$\delta_{ij}^{++} = \delta_{ij}^{--} = \delta_{ij}^{+-} = \frac{n-3}{4} \quad \text{and} \quad \delta_{ij}^{-+} = \frac{n+1}{4}.$$

This means that for each arc (i,j) of $T \in \mathcal{DR}_n$, case (a) always holds. Thus, the upper bound is achieved if and only if $T \in \mathcal{DR}_n$. The theorem is proved.

Unfortunately, for $n \equiv 1 \mod 4$, upper bound (10) is not sharp. It was shown in [28] that for this case, in the class $\mathcal{R}_n$, an upper bound

$$c_5(T) \leq \frac{n(n-1)(n^3 - 4n^2 + n - 14)}{160} = c_5(\mathcal{NDR}_n) \tag{12}$$

holds with equality if and only if $T \in \mathcal{NDR}_n$. However, inequality (12) does not hold in the class $\mathcal{T}_5$ because the number of hamiltonian cycles in the tournament $\Delta(\circ, \Delta, \circ)$ obtained from the cyclic triple Δ by replacing one of its vertices with a copy of it is equal to the number of hamiltonian paths in Δ and so, $c_5\big(\Delta(\circ, \Delta, \circ)\big) = 3$, while according to (12), we have $c_5(\mathcal{NDR}_5) = 2$.

Nevertheless, our computer search shows that the maximum numbers of 5-cycles in $\mathcal{T}_n$ and $\mathcal{R}_n$ coincide for $n = 9$. We believe that the Komarov-Mackey formula for $c_5(T)$ and arguments which are similar to those used in the proof of Theorem 1 will allow us to prove that for $n \equiv 1 \mod 4$ and $n > 5$, upper bound (12) also holds in the class $\mathcal{T}_n$, but we will not try to do this here because without any doubt, a possible proof for this case contains much more routine calculations and hence, should be considered in a separate paper.

We see that for $m = 5$, the problem of determining the maximum number of m-cycles in the class $\mathcal{T}_n$ is not so simple as that in the case of $m = 3$ or $m = 4$. Note that the length $m = 5$ is critical for many combinatorial problems involving m-cycles. In particular, the question of decomposition of complete tripartite graphs into cycles of odd length m also becomes non-trivial for $m = 5$ (see [19]).

§4. Concluding remarks on $\mathbf{s_m(T)}$ and $\mathbf{c_m(T)}$ in the case of arbitrary $\mathbf{m}$

For arbitrary $m \geq 3$, the right-hand side of (3) is equal to the number $w_m(T)$ of subtournaments of order m containing neither sink nor source in T. By Lemma 1, for each $m \geq 3$ and odd n with $\exists$-property, the minimum of $w_m(T)$ in the class $\mathcal{R}_n$ is attained at $(\mathcal{N})\mathcal{DR}_n$. For $m = 3, 4$, and 5, we have $w_m(T) = s_m(T)$. However, it is not so for $m = 6$. For getting an expression for $s_m(T)$ in this case, we have to subtract $n_{\Delta \Rightarrow \Delta}(T)$ from the right-hand side of (3). To obtain an explicit formula for $n_{\Delta \Rightarrow \Delta}(T)$, we need to consider the intersection numbers of higher order, namely, $\delta_{ikj}^{\pm\pm\pm} = |N^{\pm}(i) \cap N^{\pm}(k) \cap N^{\pm}(j)|$. As a consequence, a possible expression for $s_m(T)$ becomes more complicated for $m \geq 6$. However, we think that the necessary corrections to the right-hand side of (3) do not essentially change the situation and conjecture that the minimum of $s_m(T)$ in the class $\mathcal{R}_n$ is also achieved for some element of $(\mathcal{N})\mathcal{DR}_n$.

Note that the connectivity number of a regular tournament of order n is at least $\lceil \frac{n}{3} \rceil$ (see Lemma 4.1 [34]). Hence, for any n that is sufficiently close to m, any two regular n-tournaments have the same number of strong m-subtournaments, namely, $\binom{n}{m}$, and so, in this case, we can write $s_m(\mathcal{R}_n) = \binom{n}{m}$ (when $n \approx m$). However, it is not so if n is large enough. We suggest that for arbitrary $m \geq 4$ (not only for $m = 4$ and $m = 5$) and sufficiently large odd n with $\exists$-property, any minimizer of $s_m(T)$ in the class $\mathcal{R}_n$ is contained in $(\mathcal{N})\mathcal{DR}_n$, while the maximum of $s_m(T)$ in the class $\mathcal{T}_n$ is attained only at RLT_n. The latter can be also conjectured for $n_{RLT_m}(T)$ if m is odd (see Conjecture 5.5 in [12] and Conjecture 3 in [6]). The known expressions for this quantity in the case of $m = 3$ and $m = 5$ allow us to suggest that

$$n_{RLT_m}(RLT_n) =$$

$$\frac{(n+m-2)(n+m-4)...(n+1)n(n-1)...(n-m+4)(n-m+2)}{2^{m-1}m!}.$$

In our further papers, we will try to confirm this conjecture.

Note that the trace $tr_m(T)$ of the mth power of the adjacency matrix of T is equal to the number of closed m-walks on T and hence, $mc_m(T) \leq tr_m(T)$. For $m = 3, 4$, and 5, any such walk can be obtained by a shift along some m-cycle. Hence, for these values of m, we have $mc_m(T) = tr_m(T)$ for each T. However, it is not so for $m = 6$ because repeating a closed 3-walk (3-cycle) provides a closed 6-walk. Nevertheless, for arbitrary m, there exists $C_m > 0$ not depending on n such that

$$tr_m(T) - mc_m(T) < C_m n^{m-1}. \tag{13}$$

Inequality (13) allows us to study the asymptotical properties of $c_m(T)$ in different subfamilies of $\mathcal{T}_n$ as $n \to \infty$ with the use of the known spectral properties of tournament matrices.

Let $R_n^{(m)}$ be a regular tournament of order n which maximizes the number of cycles of length $m \geq 4$ in the class $\mathcal{R}_n$. It is shown in [32] that $R_n^{(m)}$ is *asymptotically doubly regular* (i.e. $n_{\circ \Rightarrow \Delta}(R_n^{(m)}) = n_{\circ \Rightarrow \Delta}(\mathcal{DR}_n) + o(n^4)$; here, $\circ \Rightarrow \Delta$ can be replaced by any element of $\mathcal{T}_4$) or, the same, *quasi-random* if and only if m is not a multiple of 4. Our Theorem 1 means that $c_5(T) \leq c_5(\mathcal{DR}_n)$ for each $T \in \mathcal{T}_n$, where n is odd. In turn, Theorem 4 [28] shows that $c_6(T) \leq c_6(\mathcal{DR}_n)$ for each $T \in \mathcal{R}_n$, where $n \geq 7$. Finally, in [29], many serious arguments are given for supporting the conjecture that $c_7(T) \leq c_7(\mathcal{DR}_n)$ for each $T \in \mathcal{R}_n$, where $n \geq 7$. Note that at the moment, $(5, 5)$ is the only known pair (m, n) for which the maxima of $c_m(T)$ in $\mathcal{T}_n$ and $\mathcal{R}_n$ are distinct. All these facts and also the results of [14] allow us to suggest that for each $m \equiv 1, 2, 3 \mod 4$ and sufficiently large odd n with $\exists$-property, we have

$$\max\{c_m(T) : T \in \mathcal{T}_n\} = \max\{c_m(T) : T \in (\mathcal{N})\mathcal{DR}_n\}. \tag{14}$$

For these values of m, the maximum of $c_m(T)$ in the class $\mathcal{R}_n$ (and even in the class $\mathcal{T}_n$ as one of the main theorems of [14] states) is asymptotically the same as the expected number $E_n c_m$ of m-cycles in a random n-tournament. According to G. Korvin (see [17]), for $m \geq 3$, we have $E_n c_m = \big(n\big)_m / \big(m2^m\big)$, where $\big(n\big)_m = n(n-1)...(n-m+1)$. It is shown in [24] and [27] that for $m = 5$, the quantity $c_m(\mathcal{DR}_n)$ is equal to $E_{n+1}c_m$ (see also Theorem 1). The same also holds for $m = 3$ (see [15]). The known expressions for $c_m(\mathcal{DR}_n)$ in the case of $m = 6, 7, 8$, and 9 obtained in [28], [29], [30], and [31], respectively, imply that for $6 \leq m \leq 9$ and $n \geq m$, we have

$$\big(n\big)_m / \big(m2^m\big) = E_n c_m < c_m(\mathcal{DR}_n) < E_{n+1}c_m = \big(n+1\big)_m / \big(m2^m\big).$$

Based on this fact, it is natural to suggest that for $m > 5$ and odd $n \geq m$, the strict inequality

$$\max\{c_m(T) : T \in (\mathcal{N})\mathcal{DR}_n\} < E_{n+1}c_m = \big(n+1\big)_m / \big(m2^m\big) \tag{15}$$

always holds. So, if both of our conjectures (14) and (15) are true, then in the case of $m \equiv 1, 2, 3 \mod 4$ and odd $n \geq m$, for an n-tournament T, the inequality $c_m(T) \leq \big(n+1\big)_m / \big(m2^m\big)$ holds with equality iff $m = 3$ and T is regular or $m = 5$ and T is doubly regular. This cannot be true for $m \equiv 0 \mod 4$ because according to [28] and [29], $c_m\big(RLT_n\big) = \frac{1+(-1)^{\frac{m}{2}}\beta(m)}{m2^m} n^m + O(n^{m-1})$, where $\beta(m)$ is the (positive) coefficient of z^{m-1} in the Maclaurin expansion of the trigonometric function $\tan z$. According to our conjecture, for such m and sufficiently large odd n, the maximum of $c_m(T)$ in the class $\mathcal{T}_n$ is attained at RLT_n as in the case of $s_m(T)$. The results of our paper [30] show that the condition "n should be large enough" is essential even for $m = 8$. For this case, n should be strictly greater than 37. (Recall that for $m = 4$, the inequality $c_m(T) \leq c_m(RLT_n)$ holds for each $T \in \mathcal{T}_n$, where n is an arbitrary odd number.)

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Appendix A. Identity relating $\mathbf{c_4(T)}$ and $\mathbf{c_5(T)}$ in the case of a regular n-tournament T

We have already used an identity including $c_4(T)$, $c_5(T)$, and n for $T \in \mathcal{R}_n$ above several times. It was (first) obtained in [28]. In this paper, it is proved with the use of purely matrix methods. More precisely, it is deduced from a matrix identity whose proof essentially uses the fact that for a regular tournament, its adjacency matrix A commutes with the all ones matrix J. One can also prove it with the use of the Komarov-Mackey formula for $c_5(T)$. In the present paper, we give a proof based on the spectral properties of T. This proof is straightforward and uses no preliminary results. By this reason, we present it here.

Proposition A.1 [28]. *For a regular tournament T of (odd) order n, we have*

$$c_5(T) + 2c_4(T) = \frac{n(n-1)(n+1)(n-3)(n+3)}{160}.$$

Proof. Let $tr_m(T)$ be the trace of the mth power of the adjacency matrix A of T. Since for $m = 3, 4$, and 5, we have $mc_m(T) = tr_m(T)$, it suffices to relate $tr_5(T)$, $tr_4(T)$, and n. The Perron root (spectral radius) of A equals $\frac{n-1}{2}$ and its algebraic multiplicity is equal to 1. According to [7], all the other (non-Perron) eigenvalues $\lambda_1, ..., \lambda_{n-1}$ lie on the line $\Re(\lambda) = -\frac{1}{2}$. Denote by ρ_j the imaginary part of λ_j, where $j = 1, ..., n-1$. The binomial formula for exponent 4 implies that

$$\lambda_j^4 = \Big(-\frac{1}{2} + i\rho_j\Big)^4 = \frac{1}{16} - \frac{1}{2}i\rho_j - \frac{3}{2}\rho_j^2 + 2i\rho_j^3 + \rho_j^4.$$

In turn, the binomial formula for exponent 5 means that

$$\lambda_j^5 = \Big(-\frac{1}{2} + i\rho_j\Big)^5 = -\frac{1}{32} + \frac{5}{16}i\rho_j + \frac{5}{4}\rho_j^2 - \frac{5}{2}i\rho_j^3 - \frac{5}{2}\rho_j^4 + i\rho_j^5.$$

So,

$$\lambda_j^5 + \frac{5}{2}\lambda_j^4 = \frac{1}{8} - \frac{5}{2}\rho_j^2 + i\Big(-\frac{15}{16}\rho_j + \frac{5}{2}\rho_j^3 + \rho_j^5\Big). \qquad (A.1)$$

As A is a matrix with real entries, for each odd m, the mth moment $\sum_{j=1}^{n-1}\rho_j^m$ is equal to zero. Moreover, according to [8], we have[2]

$$\sum_{j=1}^{n-1}\rho_j^2 = \frac{n(n-1)}{4}.$$

Thus, summing over j from 1 to $n-1$ in $(A.1)$ and adding the terms associated with the Perron root yield

$$tr_5(T) + \frac{5}{2}tr_4(T) = \frac{(n-1)^5}{32} + 5\frac{(n-1)^4}{32} + \frac{n-1}{8} - \frac{5}{8}n(n-1) =$$

$$\frac{(n-1)}{32}\Big\{(n-1)^4 + 5(n-1)^3 + 4 - 20n\Big\} = \frac{(n-1)}{32}\Big\{n^4 + n^3 - 9n^2 - 9n\Big\} =$$

$$\frac{(n-1)(n^2+n)(n^2-9)}{32} = \frac{n(n-1)(n+1)(n-3)(n+3)}{32}.$$

Recalling that $tr_5(T) = 5c_5(T)$ and $tr_4(T) = 4c_4(T)$ completes the proof.

[2]The equality presented below is a simple consequence of the binomial formula for exponent 2 and the evident equality $tr_2(T) = 0$.

References

[1] B. Alspach and C. Tabib, A note on the number of 4-circuits in a tournament, Ann. Discrete Math. 12 (1982), 13-19.

[2] A. Astié-Vidal, V. Dugat, Autonomous parts and decomposition of regular tournaments, Discrete Math. 111 (1993), 27-36.

[3] A. Astié-Vidal, V. Dugat, Z. Tuza, Construction of non-isomorphic regular tournaments, Ann. Discrete Math. 52 (1992), 11-23.

[4] L.W. Beineke, F. Harary, The maximum number of strongly connected subtournaments, Canad. Math. Bull. 8 (1965), 491-498.

[5] D.M. Berman, On the number of 5-cycles in a tournament, in: Proc. Sixth Southeastern Conf. Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975), pp. 101-108. Congress Num. XIV, Utilitas Math., Winnipeg, Man., 1975.

[6] D. Burke, B. Lidický, F. Pfender, M. Phillips, Inducibility of 4-vertex tournaments, Preprint arXiv:2103.07047v1 (2021).

[7] A. Brauer and I.C. Gentry, On the characteristic roots of tournament matrices, Bull. Amer. Math. Soc. 74 (1968), 1133-1135.

[8] A. Brauer and I.C. Gentry, Some remarks on tournament matrices, Linear Algebra Appl. 5 (1972), 311-318.

[9] R.A. Brualdi, J. Shen, Landau's inequalities for tournament scores and a short proof of a theorem on transitive sub-tournaments, J. Graph Theory 38 (2001), 244-254.

[10] U. Colombo, Sui circuiti nei grafi completi, Boll. Unione Mat. Ital. 19 (1964), 153-170.

[11] L.N. Coregliano and A. Razborov, On the density of transitive tournaments, J. Graph Theory 85 (2017), 12-21.

[12] L.N. Coregliano, Quasi-carousel tournaments, J. Graph Theory 88 (2018), 192-210.

[13] L.N. Coregliano, R.F. Parente, C.M. Sato, On the maximum density of fixed strongly connected tournaments, Electronic J. Comb. 26 (1), # P.1.44 (2019).

[14] A. Grzesik, D. Král', L.M. Lovász, J. Volec, Cycles of a given length in tournaments, J. Combin. Theory Ser. B 158 (2023), 117-145.

[15] M.G. Kendall and B. Babington Smith, On the method of paired comparisons, Biometrika 33 (1940), 239-251.

[16] N. Komarov, J. Mackey, On the number of 5-cycles in a tournament, J. Graph Theory 86 (2017), 341-356.

[17] G. Korvin, Some combinatorial problems on complete directed graphs, in: Theory of Graphs (P. Rosenstiehl, Ed.), Gordon and Breach, New York, and Dunod, Paris, 1967, pp. 197-203.

[18] A. Kotzig, Sur le nombre des 4-cycles dans un tournoi, Mat. Casopis Sloven. Akad. Vied. 18 (1968), 247-254.

[19] E.S. Mahmoodian, M. Mirzakhani, Decomposition of complete tripartite graphs into 5-cycles, in: Combinatorics advances (Tehran, 1994). Mathematics and its applications (C.J. Colbourn, E.S. Mahmoodian, Eds.), Springer, Boston,

MA, vol. 329 (1995), pp. 235-241; Kluwer Acad. Publ., Dordrecht, 1995, pp. 75-81.

[20] J.W. Moon, On subtournaments of a tournament, Canad. Math. Bull. 9 (1966), 297-301.

[21] J.W. Moon, *Topics on tournaments*, Holt, Rinehart and Winston, New York, 1968.

[22] A. Moukouelle, Construction of a new class of near-homogeneous tournaments, C.R. Acad. Sci. Paris Ser. I 327 (1998), 913-916.

[23] A. Moukouelle, Morphology of tournaments and distribution of 3-cycles, PhD. Thesis, University Aix-Marseille III (1998).

[24] J. Plesnik, On homogeneous tournaments, Acta Fac. Rerum Natur. Univ. Comenian Math. Publ. 21 (1968), 25-34.

[25] K.B. Reid and E. Brown, Doubly regular tournaments are equivalent to skew-Hadamard matrices, J. Combin. Theory Ser. A 12 (1972), 332-338.

[26] K.B. Reid and L.W. Beineke, Tournaments, in: Selected topics in Graph Theory, Vol. 2 (L.W. Beineke and R.J. Wilson, Eds.), Academic Press, New York, 1979, pp. 169-204.

[27] P. Rowlison, On 4-cycles and 5-cycles in regular tournaments, Bull. London Math. Soc. 18 (1986), 135-139.

[28] S.V. Savchenko, On 5-cycles and 6-cycles in regular n-tournaments, J. Graph Theory 83 (2016), 44-77.

[29] S.V. Savchenko, On the number of 7-cycles in regular n-tournaments, Discrete Math. 340 (2017), 264-285.

[30] S.V. Savchenko, On the number of 8-cycles for two particular regular tournaments of order n with diametrically opposite local properties, arXiv:2403.07629 (2024).

[31] S.V. Savchenko, On the number of 9-cycles in a doubly-regular tournament, (in preparation).

[32] S.V. Savchenko, Bernoulli numbers and m-cycles in regular n-tournaments, Proc. Amer. Math. Soc. (will be submitted for publication).

[33] C. Tabib, Caractérisation des tournois presqu'homogènes, Ann. Discrete Math. 8 (1980), 77-82.

[34] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory Series B 28 (1980), 142-163.