POISSONIAN ACTIONS OF POLISH GROUPS

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ABSTRACT. We define and study Poissonian actions of Polish groups as a framework to Poisson suspensions, characterize them spectrally, and provide a complete characterization of their ergodicity. We further construct *spatial* Poissonian actions, answering partially a question of Glasner, Tsirelson & Weiss about Lévy groups. We also construct for every diffeomorphism group an ergodic free spatial probability preserving actions. This constitutes a new class of Polish groups admitting non-essentially countable orbit equivalence relations, obtaining progress on a problem of Kechris.

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1. Introduction

A well-known construction in probability theory is the Poisson point process, in which every standard (typically infinite) measure space $(X, \mathcal{B}, \mu)$ is associated a standard probability space $(X^*, \mathcal{B}^*, \mu^*)$, whose points are configurations of points from X and their distribution is governed by Poisson distribution with intensity μ. In ergodic theory, one naturally associates with every measure preserving transformation T of $(X, \mathcal{B}, \mu)$ a probability preserving transformation T^* of $(X^*, \mathcal{B}^*, \mu^*)$, namely the **Poisson suspension**. In the first part of this work we aim to put the constructions of Poisson point process and the Poisson suspension in a general, more axiomatic framework, thus defining the notion of measure preserving **Poissonian action** of Polish groups.

For a parameter $0 \leqslant \alpha \leqslant \infty$, denote by $\operatorname{Poiss}(\alpha)$ the Poisson distribution with mean α, with the convention that for $\alpha \in \{0, \infty\}$ it is the distribution of the constant α.

Definition 1.1 (Poisson point process)**.** *Let $(X, \mathcal{B}, \mu)$ be a standard measure space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space. A collection*

$$\mathcal{P} = \{P_A : A \in \mathcal{B}\}$$

of random variables that are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Poisson point process** *with the* **base space** $(X, \mathcal{B}, \mu)$*, if the following properties hold:*

(1) P_A has distribution $\operatorname{Poiss}(\mu(A))$ *for every $A \in \mathcal{B}$.*
(2) $P_{A \cup B} = P_A + P_B$ $\mathbb{P}$-a.s. whenever $A, B \in \mathcal{B}$ are disjoint.

Such a Poisson point process $\mathcal{P}$ will be called **generative** *if, in addition,*

(3) The members of $\mathcal{P}$ generate $\mathcal{F}$ modulo $\mathbb{P}$.

Remark 1.2. By the famous Rényi Theorem, which is valid in our general setting, a Poisson point process $\mathcal{P}$ as in Definition 1.1 automatically satisfies that $P_{A_1}, \ldots, P_{A_n}$ are independent whenever $A_1, \ldots, A_n \in \mathcal{B}$ are disjoint.

The measure μ is sometimes referred to as the **intensity** of $\mathcal{P}$. The classical (generative) Poisson point process with an arbitrary base space $(X, \mathcal{B}, \mu)$, is usually constructed on a standard probability space $(X^*, \mathcal{B}^*, \mu^*)$, where X^* consists of Borel simple counting measures on X, and it is the collection

$$\mathcal{N} = \{N_A : A \in \mathcal{B}\} \text{ given by } N_A(\omega) = \omega(A).$$

As we shall see in Proposition 3.1, this construction of Poisson point process amounts to a choice of topology which is not always canonical, and $\mathcal{N}$ in its A-variable becomes a *random measure* on X, a property that is not assumed a priori for general Poisson point process as in Definition 1.1. Nevertheless, as we shall see in Proposition 3.3, all Poisson point processes are essentially unique, and in particular all form random measures in a precise sense. Despite this universality of the Poisson point process, the ability to deviate oneself from the aforementioned concrete construction will be of great importance to us as we shall see in Theorems 3 and 4.

Given a Poisson point process $\mathcal{P}$ as in Definition 1.1, let $\mathcal{B}_\mu$ be the Borel sets in $\mathcal{B}$ with finite measure, and look at the real Hilbert space

$$H(\mathcal{P}) := \overline{\operatorname{span}}\{P_A : A \in \mathcal{B}_\mu\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}).$$

Definition 1.3 (Poissonian action)**.** *Let $\mathcal{P}$ be a generative Poisson point process as in Definition 1.1. A probability preserving action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ of a Polish group G is said to be a* **Poissonian action** *with respect to $\mathcal{P}$, if its Koopman representation preserves $H(\mathcal{P})$ within $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$.*

In the next theorem we provide a characterization of Poissonian action. We will start by introducing the natural source for Poissonian actions, namely the **Poisson suspension** construction, omitting essential technical details that will be treated with a great care in Proposition 3.1. Observe that if T is a measure preserving transformation of $(X, \mathcal{B}, \mu)$, one may define a probability preserving transformation T^* of $(X^*, \mathcal{B}^*, \mu^*)$ by the property that for every ω in an appropriate μ^*-conull set,

$$T^*(\omega) = \omega \circ T^{-1}.$$

Evidently, we have the property

$$N_A(T^*(\omega)) = N_{T^{-1}(A)}(\omega) \text{ for } A \in \mathcal{B} \text{ and for } \mu^*\text{-a.e. } \omega \in X^*.$$

This readily implies that T^* is a Poissonian transformation with respect to the Poisson point process $\mathcal{N}$. As we shall see later on, this source of Poissonian actions is not limited to a single transformation but for every measure preserving action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ of an arbitrary Polish group G we obtain a Poissonian action $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$ with respect to the Poisson point process $\mathcal{N}$. In the Poisson suspension construction, the action $\mathbf{T}$ is referred to as a **base action** of $\mathbf{T}^*$, and for a general Poissonian action this is put as follows.

Definition 1.4 (Base of a Poissonian action)**.** *Let $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ be a Poissonian action of a Polish group G with respect to a generative Poisson point process $\mathcal{P}$ as in Definition 1.3. An action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is called a* **base action** *for $\mathbf{S}$ if*

$$P_A \circ S_g = P_{T_g^{-1}(A)} \quad \mathbb{P}\text{-}a.e. \text{ for every } g \in G \text{ and } A \in \mathcal{B}.$$

The following theorem is our main result about Poissonian actions. We put it in a principled form so to make things clear and the precise formulations can be found in Theorems 4.4 and 4.5 and Corollary 4.6.

Theorem 1. *Suppose $\mathcal{P}$ is a generative Poisson point process as in Definition 1.1, and that a Polish group – the acting group – is given. Then:*

1. *Every measure preserving action on the base space of $\mathcal{P}$ is a base action of an essentially unique Poissonian action with respect to $\mathcal{P}$.*
2. *Every Poissonian action with respect to $\mathcal{P}$ admits an essentially unique base action on the base space of $\mathcal{P}$.*

In the next we completely characterize the ergodicity of Poissonian actions in terms of their base actions, to the generality of measure preserving actions of Polish groups. For $G = \mathbb{Z}$ it was proved by Marchat [27] and other proofs were given later by Grabinsky [16, Theorem 1] and Roy [36, §4.5].

Theorem 2. *Suppose $\mathbf{S}$ is a Poissonian action with a base action $\mathbf{T}$. The following are equivalent.*

1. *$\mathbf{S}$ is ergodic.*
2. *$\mathbf{S}$ is weakly mixing.*
3. *$\mathbf{T}$ admits no invariant set of a positive finite measure.*

The continuation of our study of Poissonian actions is in the more restrictive framework of **spatial actions** of Polish groups. As opposed to the general notion of measure preserving actions, in which every group element corresponds to a transformation that is defined almost everywhere, in spatial actions one requires an actual Borel action which happens to admit an invariant (or quasi-invariant) measure. In many common cases, such as locally compact Polish groups, the *Mackey property* ensures that there is no essential difference between the two notions. However, this completely fails for general Polish groups, as was demonstrated by Becker and by Glasner, Tsirelson & Weiss in *Lévy groups*. We discuss this in Section 5.1.

Observe that the usual construction of the Poisson suspension $\mathcal{N}$ fails in the spatial category. Indeed, the standard Borel space $(X^*, \mathcal{B}^*)$ is classically defined as the space of simple counting Radon measures, namely Radon measures that taking nonnegative integer values and are finite on bounded sets, with respect to an appropriate choice of metric topology. Then for a general Borel transformation T of $(X, \mathcal{B})$, there is no apparent way to identify it as a Borel transformation of $(X^*, \mathcal{B}^*)$, not even when T is a homeomorphism, as it does not ensure that T preserves the class of bounded sets, hence the map $\omega \mapsto \omega \circ T^{-1}$ is not well-defined as transformation of X^*. In order to solve this we introduce a construction of Poisson point process as a random closed set, calling it **Poisson random set**. This provides a construction of Poisson point processes in Polish topologies that may not be locally compact, and manifests the advantage of treating Poisson point processes abstractly.

Recall that if $(X,\mathcal{B})$ is a standard Borel space, then with every Polish topology τ on X that generates $\mathcal{B}$ is associated the **Effros Borel space**,

$$\mathbf{F}_{\tau}(X) := \{F \subset X : X \backslash F \in \tau\},$$

whose points are the τ-closed sets in X. It is known that $\mathbf{F}_{\tau}(X)$ has a structure of a standard Borel space, that will be referred to as the *Effros σ-algebra* and denote it $\mathcal{E}_{\tau}(X)$. We define this in Section 5.2. Thus, a **random closed set** is a probability measure on $(\mathbf{F}_{\tau}(X), \mathcal{E}_{\tau}(X))$.

Theorem 3 (Poisson random set)**.** *Let $(X,\mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$, there are random variables*

$$\{\Xi_A : A \in \mathcal{B}\} \text{ of the form } \Xi_A : \mathbf{F}_{\tau}(X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

with the following property:

> *For every Borel non-atomic measure μ on $(X,\mathcal{B})$ that is τ-locally finite, there exists a unique random closed set μ_{τ} on $(\mathbf{F}_{\tau}(X), \mathcal{E}_{\tau}(X))$, with respect to which $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process with base space $(X,\mathcal{B},\mu)$.*

Our first main result about spatial Poissonian actions allows one to construct spatial probability preserving actions out of spatial infinite measure preserving actions. This will be established for actions with an appropriate Polish topology in the following sense.

Definition 1.5. *A* **locally finite Polish action** *is a measure preserving action $\mathbf{T} : G \curvearrowright (X,\mathcal{B},\mu)$ with a Polish topology for which, simultaneously, $\mathbf{T}$ is Polish and μ is locally finite.*

Theorem 4. *Every locally finite Polish action of a Polish group is a base action of a* spatial *Poissonian action.*

As simple it may look, the construction of spatial Poissonian actions in Theorem 4 provides a valuable tool to construct probability preserving spatial actions for Polish groups without appealing to the Mackey property. Our following results demonstrate the strength of this construction.

In their work on probability preserving spatial actions of Lévy groups, Glasner, Tsirelson & Weiss showed that all such actions are necessarily trivial [13, Theorem 1.1], and they asked whether Lévy groups admit nontrivial nonsingular spatial actions [13, Question 1.2]. Using Theorem 4 we obtain the following partial answer:

Theorem 5. *A Polish group admits a nontrivial probability preserving spatial action if it admits any of the following nontrivial actions:*

(1) A locally finite Polish (measure preserving) action.

(2) A locally finite Polish nonsingular action with continuous Radon-Nikodym cocycle.

In particular, Lévy groups admit no such nontrivial actions.

An immediate strengthening of Theorem 5 can be obtained using a recent result of Kechris, Malicki, Panagiotopoulos & Zielinski [22, Theorem 2.3]. Recall that an action is **faithful** if each group element, except for the identity, acts nontrivially on a positive measure set.

Corollary 1.6. *Every Polish group G admitting one of the actions (1) or (2) as in Theorem 5 which is also faithful, admits a free probability preserving spatial action.*

We now move to use the construction of spatial Poissonian action in Theorem 4 to construct nontrivial spatial actions in the class of diffeomorphism groups. Let M be a Hausdorff connected compact finite dimensional smooth manifold. We call by a **diffeomorphism group** of M, for some $1 \leqslant r \leqslant \infty$, the group

$$\mathrm{Diff}^{r}(M)$$

of all C^{r}-diffeomorphisms of M to itself, considered as a (non-locally compact) Polish group with the compact-open C^{r}-topology.

The aforementioned Mackey property, established by Mackey for locally compact groups, was generalized by Kwiatkowska & Solecki to groups of isometries of locally compact metric spaces and, to the best of our knowledge, currently this is the largest class of Polish groups for which the Mackey property is known to hold. In the following theorem we show that diffeomorphism groups are not in this class, and it is the consequence of two highly nontrivial results: one is a theorem of Thurston, generalizing a theorem by Herman and following a theorem by Epstein, about the simplicity of the identity component in certain diffeomorphism groups, and another by Kwiatkowska & Solecki, following Gao & Kechris, about the topological structure of isometry groups of locally compact metric spaces.

Theorem 6. *Diffeomorphism groups are never groups of isometries of a locally compact metric space.*

Although the Mackey property for diffeomorphism group is unknown, we use the construction of spatial Poissonian actions to construct spatial actions of such groups. In this context it is worth mentioning that by a result of Megrelishvili [28, Theorem 3.1] (see also [13, Remark 1.7]), the homeomorphism group of the unit interval admits no action whatsoever. The picture in diffeomorphism groups turns out to be very different.

Theorem 7. *Every diffeomorphism group admits an ergodic free probability preserving spatial action. Hence, diffeomorphism groups are never Lévy.*

Theorem 7 has a consequence in the theory of equivalence relations on standard Borel spaces. An open problem of Kechris asks whether every non-locally compact Polish group admits a non-essentially countable equivalence relation. Recently, this question was answered affirmatively for groups of isometries of a locally compact metric space by Kechris, Malicki, Panagiotopoulos & Zielinski, and to the best of our knowledge this is the largest class of Polish groups for which the answer is known. Since diffeomorphism groups do not belong to this class according to Theorem 6, the following corollary, which is the result of Theorem 7 together with a theorem by Feldman & Ramsay, shows that the class of diffeomorphism groups constitutes a completely new class with positive answer to Kechris' problem:

Corollary 1.7. *Every diffeomorphism group admits a non-essentially countable orbit equivalence relation.*

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2. General preliminaries

Let $(X,\mathcal{B})$ be a standard Borel space. Thus, X is equipped with a σ-algebra $\mathcal{B}$ that is the Borel σ-algebra of some unspecified Polish topology τ on X. A **transformation** of $(X,\mathcal{B})$ is an invertible Borel mapping $T:X\to X$. A **Borel** (τ**-Polish**) **action** of a Polish group G on X is a jointly Borel (jointly τ-continuous, resp.) map $\mathbf{T}:G\times X\to X$, $\mathbf{T}:(g,x)\mapsto T_g(x)$, such that $T_e=\mathrm{Id}_X$, where $e\in G$ is the group identity, and $T_g\circ T_h=T_{gh}$ for every $g,h\in G$. By a **standard measure space** we refer to $(X,\mathcal{B},\mu)$, where $(X,\mathcal{B})$ is a standard Borel space and μ is a measure that belongs to one of the following of classes:

$\mathcal{M}_1(X,\mathcal{B})$: non-atomic Borel probability measures on X.
$\mathcal{M}_\sigma(X,\mathcal{B})$: non-atomic infinite σ-finite Borel measures on X.
$\mathcal{M}_\sigma^\tau(X,\mathcal{B})$: those measures in $\mathcal{M}_\sigma(X,\mathcal{B})$ that are τ-locally finite.

Being τ-locally finite, by definition, means that every point in X has a τ-neighborhood of finite measure. It is equivalent to the existence of a countable base, or to the existence of a countable open cover, that consists of finite measure sets. The general theory of point processes is usually developed for Radon measure, which are nothing but locally finite measure with respect to a locally compact topology (see e.g. [10, Theorem 7.8]). Here we deal with general Polish topologies.

If $(X,\mathcal{B},\mu)$ is a standard measure space, we denote by

$$\mathcal{B}_\mu$$

the ideal of sets $A \in \mathcal{B}$ for which $\mu(A) < \infty$. We use the common terminology of μ**-a.e.** or μ**-conull**, applied to a specified property of the elements of X, to indicate that the property holds true for all the members of some set in $\mathcal{B}$ whose complement is μ-null, namely is assigned zero by μ. By a **transformation** of $(X, \mathcal{B}, \mu)$ we refer to a bi-measurable bijective map between two μ-conull sets of X. A transformation T of $(X, \mathcal{B})$ is said to be **measure preserving** if $\mu \circ T^{-1} = \mu$, and **nonsingular** if $\mu \circ T^{-1}$ and μ are in the same measure class, namely they are mutually absolutely continuous. We denote by

$$\mathrm{Aut}(X, \mathcal{B}, \mu) \text{ and } \mathrm{Aut}(X, \mathcal{B}, [\mu])$$

the groups of equivalence classes, up to equality on a μ-conull set, of measure preserving and nonsingular transformations, respectively. The latter group clearly depends only on the measure class $[\mu]$ of μ rather than μ itself, and it becomes a Polish group with the weak topology, in which $S_n \xrightarrow[n\to\infty]{} \mathrm{Id}$ if $\mu(S_n A \triangle A) \xrightarrow[n\to\infty]{}$ for every $A \in \mathcal{B}$ with $\mu(A) < \infty$ and $\frac{d\mu\circ S_n}{d\mu} \xrightarrow[n\to\infty]{} 1$ in measure. The former group then becomes a closed, hence Polish, subgroup of the latter (see e.g. [1, end of §1.0], [18, Exercise (17.46)]).

3. Foundations of Poisson point processes

Recall Definition 1.1 for the general notion of Poisson point process. In the following we introduce the usual concrete construction of the Poisson point process that should be regarded as a folklore. The details of this construction will be important to us for the general context and later uses.

Let $(X, \mathcal{B})$ be a standard Borel space. By the Isomorphism Theorem of standard Borel spaces (see e.g. [18, §15.B]) there can be found a complete metric on X that induces a locally compact Polish topology τ on X, and thus we may relate to bounded Borel sets in X with respect to a fixed choice of such a metric. We may further assume that τ admits a countable base that consists of bounded sets and, in fact, we may assume that this topology has all properties of the usual topology on $\mathbb{R}$. Denote by X_τ^* the space of simple counting Radon measures on X. That is, a Borel measure ω on X is in X_τ^* if it satisfies the following properties:

1. (Radon) $\omega(A) < \infty$ for every bounded set $A \in \mathcal{B}$.
2. (counting) $\omega(A) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ for every $A \in \mathcal{B}$.
3. (simple) $\omega(\{x\}) \in \{0, 1\}$ for every $x \in X$.

The space X_τ^* becomes a standard Borel space with the σ-algebra $\mathcal{B}_\tau^*$ that is generated by the canonical mappings

$$\mathcal{N} = \{N_A : A \in \mathcal{B}\}, \quad N_A : X_\tau^* \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \quad N_A : \omega \mapsto \omega(A). \tag{3.0.1}$$

For details on the standard Borel structure of X_τ^* see [6, §9.1].

Suppose now that $(X,\mathcal{B},\mu)$ is a standard infinite measure space, that is we are given a measure $\mu\in\mathcal{M}_{\sigma}(X,\mathcal{B})$. By the Isomorphism Theorem for standard measure spaces (see e.g. [18, §17.F]) there can be found a complete metric that induces a Polish topology τ that satisfies all of the above and, at the same time, turns μ into a Radon measure, i.e. $\mu\in\mathcal{M}_{\sigma}^{\tau}(X,\mathcal{B})$. In fact, up to a Borel isomorphism, we may assume that $(X,\mathcal{B},\mu)$ is $\mathbb{R}$ with its usual Borel structure and the Lebesgue measure. By the classical construction of the Poisson point process, there exists a unique probability measure $\mu_{\tau}^{*}\in\mathcal{M}_{1}(X_{\tau}^{*},\mathcal{B}_{\tau}^{*})$ with respect to which the random variables $\mathcal{N}$ as in (3.0.1) form a generative Poisson point process with base space $(X,\mathcal{B},\mu)$. For details about this classical construction we refer to [5, §2.4], [24, § 3], [39, Proposition 19.4]. In our context we put this as follows:

Proposition 3.1. *For every standard measure space $(X,\mathcal{B},\mu)$ there exists a Polish topology τ and a standard probability space $(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*})$ that is defined uniquely by the property that the collection of random variables $\mathcal{N}$ as in* (3.0.1) *forms a generative Poisson point process with base space $(X,\mathcal{B},\mu)$. Moreover, there is a continuous embedding of Polish groups*

$$\mathrm{Aut}(X,\mathcal{B},\mu)\hookrightarrow\mathrm{Aut}(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}),\quad T\mapsto T^{*},$$

such that for every $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ there is a μ_{τ}^{*}-conull set on which

$$N_{A}\circ T^{*}=N_{T^{-1}(A)}\text{ for every }A\in\mathcal{B}. \tag{3.1.1}$$

Proof. Thanks to the Isomorphism Theorem for standard measure spaces, we may assume that $(X,\mathcal{B},\mu)$ is nothing but the real line with its Lebesgue measure, for which the aforementioned classical construction of the Poisson point process with respect to the usual topology is well known. Let us show the second part. Pick a countable base $\mathcal{O}$ for τ consisting of μ-finite measure sets. An arbitrary element $[T]_{\mu}\in\mathrm{Aut}(X,\mathcal{B},\mu)$ will be mapped to $[T^{*}]_{\mu_{\tau}^{*}}\in\mathrm{Aut}(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*})$ as follows. Pick a representative $T\in[T]_{\mu}$ and consider the measurable set

$$X_{\tau}^{*}(T):=\bigcap_{O\in\mathcal{O}}\bigcap_{n\in\mathbb{Z}}\left\{N_{T^{n}(O)}<\infty\right\}.$$

By the construction of μ_{τ}^{*}, as T preserves μ we see that $\mu_{\tau}^{*}(X_{\tau}^{*}\triangle X_{\tau}^{*}(T))=0$. Let T^{*} be the automorphism of $\left(X_{\tau}^{*}(T),\mathcal{B}_{\tau}^{*}(T),\mu_{\tau}^{*}|_{X_{\tau}^{*}(T)}\right)$ that is given by

$$T^{*}(\omega)=\omega\circ T^{-1},\quad\omega\in X_{T}^{*}.$$

As $\mu_{\tau}^{*}(X_{\tau}^{*}\triangle X_{\tau}^{*}(T))=0$, the element $[T^{*}]_{\mu_{\tau}^{*}}\in\mathrm{Aut}(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*})$ is well-defined. This defines the desired mapping, that from now on we abbreviate without the equivalence class notations, i.e. $\mathrm{Aut}(X,\mathcal{B},\mu)\to\mathrm{Aut}(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*})$, $T\mapsto T^{*}$. It is clearly a homomorphism. In order to see that it is injective, note that if $T\neq\mathrm{Id}_{X}\in\mathrm{Aut}(X,\mathcal{B},\mu)$ there is a Borel set of the form $A = T^{-1}(B) \setminus B$ with $\mu(A) > 0$. Since $\mu\left(A \cap T^{-1}(A)\right) = 0$ we have $\mu_\tau^*\left(N_A \circ T^* > 0, N_A = 0\right) > 0$, hence $T^* \neq \mathrm{Id}_{X_\tau^*} \in \mathrm{Aut}\left(X_\tau^*, \mathcal{B}_\tau^*, \mu_\tau^*\right)$. The equivariance property (3.1.1) is verified by noting that for every $A \in \mathcal{B}_\mu$, for ω in an appropriate μ^*-conull set,

$$N_A \circ T_g^*(\omega) = N_A\left(\omega \circ T_g^{-1}\right) = \omega\left(T_g^{-1}(A)\right) = N_{T_g^{-1}(A)}(\omega).$$

The continuity of this embedding can be verified using elementary considerations, but it is also an immediate consequence of the automatic continuity property of $\mathrm{Aut}(X, \mathcal{B}, \mu)$ by Le Maître [25, Theorem 1.2]. $\square$

Definition 3.2 (Classical Poisson point process)**.** *The construction in Proposition 3.1, while depending on the highly non-canonical choice of τ, serves as a concrete Poisson point process with an arbitrary base space $(X, \mathcal{B}, \mu)$. Ignoring τ, we will refer to it as the* **classical Poisson point process** *and denote it by*

$$(X^*, \mathcal{B}^*, \mu^*) \text{ and } \mathcal{N} = \{N_A : A \in \mathcal{B}\}.$$

While the choice of τ affects directly $X^* = X_\tau^*$ as a subspace of the τ-Radon measures on X, the following proposition shows that the Poisson point process is a universal object to which the choice of τ is irrelevant up to a Borel isomorphism. In fact, we will show that the Poisson point process is universal in the widest sense of Definition 1.1 up to a Borel isomorphism.

Proposition 3.3. *All generative Poisson point processes on the same base space are isomorphic. More explicitly, let $(X, \mathcal{B}, \mu)$ be a standard measure space and $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ be a generative Poisson point process that is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. There is an isomorphism of measure spaces*

$$\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (X^*, \mathcal{B}^*, \mu^*),$$

such that on a $\mathbb{P}$-conull set,

$$N_A \circ \varphi = P_A \text{ for all } A \in \mathcal{B}.$$

Thus, $\mathcal{P}$ is a random measure on X in that for every ω in an appropriate $\mathbb{P}$-conull set, the map $A \mapsto P_A(\omega)$ defines a measure on $(X, \mathcal{B})$.

Proof. For $\omega \in \Omega$ define $\varphi(\omega) : \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ by

$$\varphi(\omega)(A) = P_A(\omega).$$

First we prove that for ω in a $\mathbb{P}$-conull set it holds that $\varphi(\omega) \in X^*$, namely that $\varphi(\omega)$ extends to a genuine measure on $\mathcal{B}$. To this end we verify the conditions appears in [6, p. 17]. The finite additivity of $\varphi(\omega)$ for every $\omega \in \Omega$ is immediate from the definition of $\mathcal{P}$ as a Poisson point process. As for the continuity, we note that the finite additivity implies that $P_A \leqslant P_B$ whenever $A \subset B$, hence if $\mathcal{B}_\mu \ni A_n \searrow \emptyset$ as $n \to \infty$ then the pointwise limit of the monotone descending sequence P_{A_n} as $n \to \infty$ is a nonnegative random variable and, by the dominated convergence theorem, it has zero mean, hence $P_{A_n} \searrow 0$ as $n \to \infty$, establishing the continuity. It follows from [6, Lemma 9.1.XIV] that $\varphi(\omega) \in X^*$ for ω on a $\mathbb{P}$-conull set. Thus we obtain a map $\varphi : \Omega \to X^*$ that is defined on an appropriate $\mathbb{P}$-conull set. In order to see that $\mathbb{P} \circ \varphi^{-1} = \mu^*$, note that for every ω in a $\mathbb{P}$-conull set,

$$N_A(\varphi(\omega)) = \varphi(\omega)(A) = P_A(\omega), \quad A \in \mathcal{B},$$

from which it readily follows that $\mathcal{N}$ forms a Poisson point process on the same base space with respect to $\mathbb{P} \circ \varphi^{-1}$ as well. The uniqueness of the classical Poisson point process implies that $\mathbb{P} \circ \varphi^{-1} = \mu^*$. $\square$

4. Foundations of Poissonian actions

4.1. Preliminaries: actions and representations of Polish groups.

The very definition of 'measure preserving action' of a Polish group has more than one possible meaning, essentially two meanings, which is the source of a substantial part of our study. The first and more general definition can be put in two ways, namely *near actions*, as was put by Zimmer, and *Boolean actions*, a classical object that admit a convenient formulation due to Glasner, Tsirelson & Weiss (see [13, Introduction] and the references therein). The other, more restrictive notion of spatial actions will be presented in the second part of this work, starting in Section 5.

Definition 4.1 (Zimmer). *A* **near action** *of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$ is a jointly measurable map $\mathbf{T} : G \times X \to X$, $\mathbf{T} : (g, x) \mapsto T_g(x)$, such that:*

(1) $T_e = \mathrm{Id}_X$ on a μ-conull set, where $e \in G$ is the identity element.
(2) $T_g \circ T_h = T_{gh}$ on a μ-conull set for every $g, h \in G$.[1]
(3) $\mu \circ T_g^{-1} = \mu$ for every $g \in G$.

There is a natural way to view $\mathrm{Aut}(X, \mathcal{B}, \mu)$ as the group of Boolean isometries of the measure algebra associated with $(X, \mathcal{B}, \mu)$ (i.e. the Boolean algebra of Borel sets in X modulo μ-null sets, with its natural complete metric). With this point of view, Glasner, Tsirelson & Weiss put the notion of Boolean action as follows.

Definition 4.2 (Glasner, Tsirelson & Weiss). *A* **Boolean action** *of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$ is a continuous (or equivalently, measurable)*[2] *homomorphism $\mathbf{T} : G \to \mathrm{Aut}(X, \mathcal{B}, \mu)$.*

[1] It is crucial here that the μ-conull set may depend on g, h.

[2] The equivalence of measurability and continuity for homomorphisms between Polish group is by Pettis' automatic continuity (see e.g. [18, §9.C]).

The reader may recall Proposition 3.1 that suggests why the formulation of Boolean action is the one that is more convenient for our purposes. However, as it was observed by Glasner, Tsirelson & Weiss [13, Introduction], both definitions are essentially the same. Thus, we relate to near actions and Boolean actions simply as **actions**, and denote this unified object by

$$\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu) .$$

We may refer to an action as a *finite action* or an *infinite action*, to indicate whether the underlying measure is a probability measure or an infinite one.

A pair of actions $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ and $\mathbf{T}' : G \curvearrowright (X', \mathcal{B}', \mu')$ are considered to be isomorphic, if there exists a bi-measurable bijection $\varphi : X \to X'$, that is possibly defined only on corresponding conull sets, such that $\mu \circ \varphi^{-1} = \mu'$ and $T'_g \circ \varphi = \varphi \circ T_g$ for each $g \in G$ on a μ-conull set.

Recall that a unitary representation $\mathbf{U}$ of a Polish group G on a Hilbert space $\mathcal{H}$ is a group homomorphism $\mathbf{U} : G \to \mathrm{U}(\mathcal{H})$, $\mathbf{U} : g \mapsto U_g$, where $\mathrm{U}(\mathcal{H})$ denotes the unitary group of $\mathcal{H}$, such that the mapping $(g, f) \mapsto U_g f$ is jointly continuous.[3] When $\mathcal{H}$ is a subspace of some $L^2_{\mathbb{R}}$-space, we say that $\mathbf{U}$ is **unital** if $U_g 1 = 1$ for every $g \in G$ (when 1 is integrable), and that $\mathbf{U}$ is **positivity preserving** if $f \geqslant 0$ implies $U_g f \geqslant 0$ for every $g \in G$ (see [39, p. 207]). The (real) **Koopman representation** of an action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is the unitary representation

$$\mathbf{U} : G \to \mathrm{U}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right), \quad U_g : f \mapsto f \circ T_g^{-1}, \quad g \in G.$$

4.2. **Poissonian actions.** Recall Definition 1.3 for the general notion of Poissonian action. Let us now introduce the Poisson suspension construction in a precise way as a natural source for Poissonian actions. Suppose $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is an action of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$. Consider the classical Poisson point process $\mathcal{N} = \{N_A : A \in \mathcal{B}\}$ that is defined on $(X^*, \mathcal{B}^*, \mu^*)$, and using the continuous embedding introduced in Proposition 3.1, we obtain an action $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$ by composing

$$\mathbf{T}^* : G \to \mathrm{Aut}(X, \mathcal{B}, \mu) \hookrightarrow \mathrm{Aut}(X^*, \mathcal{B}^*, \mu^*), \quad g \mapsto T_g \mapsto T_g^*.$$

The equivariance property (3.1.1) readily implies that the Koopman representation of $\mathbf{T}^*$ preserves $H(\mathcal{P})$, thus $\mathbf{T}^*$ is a Poissonian action with respect to $\mathcal{N}$, and its base action is nothing but $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$.

Remark 4.3. Note that in the setting of Definition 1.4, if $\mathcal{P}$ is not generative we may move to the sub-σ-algebra on Ω that is generated by $\mathcal{P}$. Then if

[3] In fact, from [41, Theorem 4.8.6] or [18, Exercise (9.16) i)] it follows that if the mapping $(g, f) \mapsto U_g f$ is jointly measurable then it is automatically jointly continuous.

the equivariance relations of $\mathbf{S}$ and $\mathbf{T}$ hold, this sub-σ-algebra is $\mathbf{S}$-invariant and we obtain a factor of $\mathbf{S}$ which is a Poissonian action.

We now formulate and prove each of the statements in Theorem 1.

Theorem 4.4. *Let $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ be an action and $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ be a generative Poisson point process that is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. There exists a Poissonian action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ with respect to $\mathcal{P}$ whose base action is $\mathbf{T}$. Moreover, $\mathbf{S}$ is essentially unique in that Poissonian actions associated with isomorphic actions are isomorphic.*

Proof. By Proposition 3.3 we may assume without loss of generality that $(\Omega, \mathcal{F}, \mathbb{P}) = (X^*, \mathcal{B}^*, \mu^*)$ and that $\mathcal{P} = \mathcal{N}$. Then the aforementioned construction of the Poisson suspension, using the embedding described in Proposition 3.1, is a Poissonian action with respect to $\mathcal{N}$ whose base action is $\mathbf{T}$.

In order to show the uniqueness, we start by showing that all Poissonian actions with base action $\mathbf{T}$ are isomorphic to the Poisson suspension $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$. Let $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ be such a Poissonian action with respect to a generative Poisson point process $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ that is defined on $(\Omega, \mathcal{F}, \mathbb{P})$. By Proposition 3.3 there is an isomorphism of probability spaces $\varphi : \Omega \to X^*$ such that $\mathbb{P} \circ \varphi^{-1} = \mu^*$, that interchanges $\mathcal{P}$ and $\mathcal{N}$ in that $N_A \circ \varphi = P_A$ for every $A \in \mathcal{B}$. In order to verify that indeed $\varphi \circ S_g = T_g^* \circ \varphi$ on a $\mathbb{P}$-conull set for every $g \in G$, note that

$$N_A \circ \varphi \circ S_g = P_A \circ S_g = P_{T_g^{-1}(A)} = N_A \circ T_g^* \circ \varphi \quad \text{for every } A \in \mathcal{B},$$

and since $\mathcal{N}$ is generative the desired property follows. Thus, φ is an isomorphism of actions.

Now for the general case, suppose that $\mathbf{T}' : G \curvearrowright (X', \mu')$ is an infinite action that is isomorphic to $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ through $\psi : X \to X'$. From the classical Poisson point process $\mathcal{N}' = \{N'_A : A \in \mathcal{B}'\}$ associated with (X', μ') we obtain the Poisson point process $\mathcal{P} := \left\{N'_A \circ \psi^{-1} : A \in \mathcal{B}\right\}$ associated with $(X, \mathcal{B}, \mu)$. It is then evident that the Poisson suspension $\mathbf{T}'^* : G \curvearrowright (X'^*, \mu'^*)$ is a Poissonian action with base action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ via the Poisson point process $\mathcal{P}$. Hence, by the previous part of the proof it is isomorphic to the Poisson suspension $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$. □

The following theorem generalizes the second statement of Theorem 1, which can be seen as a statement on Koopman representations, to a larger family of unitary representations.

Theorem 4.5. *Suppose $\mathcal{P}$ is a generative Poisson point process that is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. Every unital positivity preserving unitary representation $\mathbf{U} : G \to \mathrm{U}(H(\mathcal{P}))$, admits a Poissonian action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ with respect to $\mathcal{P}$, whose Koopman representation is $\mathbf{U}$. Moreover, $\mathbf{S}$ admits an essentially unique base action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$.*

Noting that Koopman representations are unital and positivity preserving, we obtain from Theorem 4.5 for Koopman representations:

Corollary 4.6. *Every Poissonian action arises from an essentially unique base action.*

For the proof of Theorem 4.5 we introduce two lemmas. First, we will use the following substitution for the notion of unital Koopman operators when dealing with infinite measure spaces. A unitary operator U of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ for some standard measure space $(X,\mathcal{B},\mu)$ will be called **quasi-unital** if

$$\int_X Uf d\mu = \int_X f d\mu \text{ for every } f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu).$$

In particular, U preserves $L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. The group of quasi-unital positivity preserving unitary operators of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ is a closed subgroup of the unitary group of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, hence it is a Polish group.

In the following we formulate in our terminology a well-known fact which is a version of the Banach-Lamperti Theorem for L^2. As it was observed in [42, footnote 3], while the general Banach-Lamperti Theorem is formulated for unitary operators of L^p-spaces for $p \neq 2$, for positivity preserving unitary operators the proof of Lamperti applies for L^2-spaces as well. A byproduct of the following lemma is that when μ is a probability measure, for a positivity preserving unitary operator of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ being unital and quasi-unital is the same.

Lemma 4.7. *For every standard measure space $(X,\mathcal{B},\mu)$, the Koopman embedding*

$$T \mapsto U_T, \quad U_T f = f \circ T^{-1},$$

forms an isomorphism of Polish groups between $\mathrm{Aut}(X,\mathcal{B},\mu)$ and the group of quasi-unital positivity preserving unitary operators of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$.

Proof. By the Banach-Lamperti Theorem in L^2 (see the aforementioned [42, footnote 3]), there is a bijective correspondence between the group of nonsingular transformations of $(X,\mathcal{B},\mu)$ and the group of positivity preserving unitary operators of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, that is given by

$$T \mapsto U_T, \quad U_T f = \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} f \circ T^{-1}.$$

This is a homomorphism of groups and, since T is necessarily measure preserving when U_T is quasi-unital, the measure preserving transformations correspond to the quasi-unital positivity preserving unitary operators. Thus, the restriction of this homomorphism to $\mathrm{Aut}(X,\mathcal{B},\mu)$, which is the usual Koopman embedding, forms a bijective homomorphism from $\mathrm{Aut}(X,\mathcal{B},\mu)$

onto the closed subgroup of quasi-unital positivity preserving unitary operators of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. Finally, the Polish topology of $\mathrm{Aut}(X,\mathcal{B},\mu)$ is, by definition, induced from this correspondance, so this is a homeomorphism. □

For the next step toward proving Theorem 4.5 we will take a further look into unitary operators of $H(\mathcal{P})$. The following objects and their basic properties are presented in more details in Appendix A. Fix a Poisson point process $\mathcal{P}$ as in Definition 1.1. The **first chaos** of $\mathcal{P}$ is the space

$$H_1(\mathcal{P}) := \overline{\mathrm{span}}\left\{P_A - \mu(A) : A \in \mathcal{B}_\mu\right\} \subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P}).$$

As part of the Fock space structure associated with $\mathcal{P}$, there is an isometric isomorphism of Hilbert spaces

$$I_\mu : L^2_{\mathbb{R}}(X,\mathcal{B},\mu) \to H_1(\mathcal{P}), \quad I_\mu : f \mapsto I_\mu(f),$$

that is given by a stochastic integral against $\mathcal{P}$ in an appropriate sense. Recalling the space

$$H(\mathcal{P}) = \overline{\mathrm{span}}\left\{P_A : A \in \mathcal{B}_\mu\right\} \subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P}),$$

the first chaos is its direct summand,

$$H(\mathcal{P}) = H_1(\mathcal{P}) \oplus \mathbb{R}.$$

For every unital positivity preserving unitary operator U of $H(\mathcal{P})$ we have

$$\langle U(I_\mu(f)), 1\rangle = \left\langle I_\mu(f), U^{-1}(1)\right\rangle = \langle I_\mu(f), 1\rangle, \quad f \in L^2_{\mathbb{R}}(X,\mathcal{B},\mu).$$

Thus, U preserves $H_1(\mathcal{P})$ as a direct summand of $H(\mathcal{P})$. Since I_μ is an isometric isomorphism of Hilbert spaces, the conjugation by I_μ forms a map between the unitary groups,

$$\mathrm{U}(H(\mathcal{P})) \to \mathrm{U}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right), \quad U \mapsto U^\mu := I_\mu^{-1} \circ U \circ I_\mu. \tag{4.7.1}$$

This map will be important to the proof of Theorem 4.5. The following lemma deals with its fundamental properties, with its proof following a similar approach to the proof of [37, Proposition 4.4].

Lemma 4.8. *In the map $U \mapsto U^\mu$ as in (4.7.1), unital positivity preserving unitary operators of $H(\mathcal{P})$ are mapped to quasi-unital positivity preserving unitary operator of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$.*

The proof makes essential use of the properties of the Poisson stochastic integral I_μ as well as the fundamentals of infinitely divisible Poissonian (henceforth IDp) random variables. We present this in Appendix A.

Proof. Since I_μ is an isometric isomorphism of Hilbert spaces, U^μ is a unitary operator. Let us fix an arbitrary $f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. As described in Proposition A.1, we have the IDp random variable

$$W_f := I_\mu(U^\mu f) + \int_X f d\mu = U(I_\mu(f)) + \int_X f d\mu = U\left(\int_X f d\mathcal{P}\right),$$

whose Lévy measure is given by

$$\ell_{U^\mu f} = \mu\mid_{\{U^\mu f \neq 0\}} \circ (U^\mu f)^{-1}.$$

Assume further that $f \geqslant 0$ so that also $\int_X f d\mathcal{P} \geqslant 0$ and, since U is positivity preserving, also $W_f \geqslant 0$. From Proposition A.1(3) it follows that

$$\mu(U^\mu f < 0) = \ell_{U^\mu f}(\mathbb{R}_{<0}) = 0 \text{ and } \int_X U^\mu f d\mu = \int_{\mathbb{R}_{\geqslant 0}} t d\ell_{U^\mu f}(t) < \infty.$$

The first property shows that U^μ preserves positivity for every f in a dense subspace, hence it is positivity preserving. The second property shows that $U^\mu f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, so that by Proposition A.1(1),

$$I_\mu(U^\mu f) = \int_X U^\mu f d\mathcal{P} - \int_X U^\mu f d\mu.$$

Plugging this into the definition of W_f and using that $W_f \geqslant 0$, we obtain

$$\int_X U^\mu f d\mu - \int_X f d\mu = \int_X U^\mu f d\mathcal{P} - W_f \leqslant \int_X U^\mu f d\mathcal{P}.$$

With f being fixed, the left hand side is a constant, while the right hand side is a nonnegative IDp random variable that is obtained as a stochastic integral of an integrable function. It follows from [39, Corollary 24.8] that the infimum of the right hand side (as a random variable) is zero, and we conclude that

$$\int_X U^\mu f d\mu \leqslant \int_X f d\mu.$$

Since the map (4.7.1) respects inverses, the same proof shows that the same inequality holds when U^μ is replaced by $(U^\mu)^{-1}$. Both inequalities readily imply that

$$\int_X U^\mu f d\mu = \int_X f d\mu,$$

hence U^μ is quasi-unital. □

Proof of Theorem 4.5. Suppose $\mathbf{U} : G \to \mathrm{U}(H(\mathcal{P}))$ is a unital positivity preserving unitary representation of a Polish group G as in the proposition. We construct a continuous homomorphism along the following arrows

$$G \xrightarrow{\mathbf{U}} \mathrm{U}(H(\mathcal{P})) \xrightarrow{(4.7.1)} \mathrm{U}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right) \xrightarrow{\text{Banach-Lamperti}} \mathrm{Aut}(X,\mathcal{B},\mu)$$

as follows. The first arrow is the given representation $g \mapsto U_g$. The second arrow is the map $U_g \mapsto U_g^\mu$ as in (4.7.1), whose image lies in the closed subgroup of quasi-unital positivity preserving unitary operators by Lemma 4.8. Then on the image of the second arrow, the third arrow $U_g^\mu \mapsto T_g$ is given by Lemma 4.7. Since all the arrows are continuous, the map $g \mapsto T_g$ constitutes an action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$. We now use Theorem 4.4, and take a Poissonian action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ whose base action is $\mathbf{T}$. The equivariance property that relates $\mathbf{T}$ and $\mathbf{S}$ as in Definition 1.4 implies that

$$U_g \left(I_\mu \left(f\right)\right) = I_\mu \left(U_g^\mu f\right) = I_\mu \left(f \circ T_g^{-1}\right) = I_\mu \left(f\right) \circ S_g, \quad g \in G,$$

for every $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$. Thus, $\mathbf{U}$ is the Koopman representation of $\mathbf{S}$, which is a Poissonian action with base action $\mathbf{T}$ as required in the theorem.

It is clear that all actions whose Koopman representation is $\mathbf{U}$ are isomorphic, hence $\mathbf{S}$ is essentially unique. In order to see the essential uniqueness of $\mathbf{T}$, we note that if $\mathbf{T}'$ is another base action for $\mathbf{S}$ then

$$P_{T_g^{-1}(A)} = P_A \circ S_g = P_{T_g'^{-1}(A)} \quad \mathbb{P}\text{-a.e. for every } g \in G \text{ and } A \in \mathcal{B}.$$

However, just as in the uniqueness argument in the proof of Proposition 3.1, if for some $g \in G$ we have $T_g \neq T'_g$, then for $A \in \mathcal{B}_\mu$ for which $\mu\left(T_g^{-1}(A) \cap T_g'^{-1}(A)\right) = 0$ it holds that

$$\mathbb{P}\left(P_{T_g^{-1}(A)} > 0, P_{T_g'^{-1}(A)} = 0\right) > 0,$$

which is a contradiction. This completes the proof. □

4.3. **Ergodicity of Poissonian actions.** Here we prove Theorem 2. Let us first recall some basic definitions. For an action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ we denote the invariant σ-algebra

$$\mathcal{I}_{\mathbf{T}} := \left\{A \in \mathcal{B} : \mu\left(A \triangle T_g^{-1}(A)\right) = 0 \text{ for every } g \in G\right\}.$$

We say that $\mathbf{T}$ is:

- **Null:** for every $A \in \mathcal{I}_{\mathbf{T}}$ either $\mu(A) = 0$ or $\mu(A) = \infty$;
- **Ergodic:** for every $A \in \mathcal{I}_{\mathbf{T}}$ either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$;
- **Doubly Ergodic:** the diagonal action $\mathbf{T} \otimes \mathbf{T} : G \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ is ergodic;
- **Weakly mixing:** for every ergodic action $\mathbf{S} : G \curvearrowright (Y, \mathcal{C}, \nu)$ of G, the diagonal action $\mathbf{T} \otimes \mathbf{S} : G \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ is ergodic.

Remark 4.9. A few general remarks about those properties:

(1) Being null is equivalent to the non-existence of a $\mathbf{T}$-invariant probability measure that is absolutely continuous with respect to μ.

(2) Double ergodicity and weak mixing are equivalent in probability preserving actions of general groups. For locally compact groups this is a classical fact, and it was pointed out to us by Benjy Weiss in a personal communication that this is true in full generality. Indeed, one implication is obvious, and the other was proved by Glasner & Weiss in [15, Theorem 2.1], that while is formulated for locally compact groups the proof holds in full generality. With the terminology of [15, Definition 1.1], this can be seen by looking at the proofs of the implications $DE \implies EIC \implies EUC \implies WM$.

(3) As a consequence of the Lévy 0-1 Law, weak mixing is equivalent to that the infinite diagonal action $\mathbf{T}^{\otimes\mathbb{N}} : G \curvearrowright \left(X^{\mathbb{N}}, \mathcal{B}^{\otimes\mathbb{N}}, \mu^{\otimes\mathbb{N}}\right)$ is ergodic. This will be useful in the proof of Theorem 7.

In proving Theorem 2 we will need the following key result that was established in [31, Theorem 1]. We will refer to the chaos structure of $L^2_{\mathbb{R}}\left(X^*, \mathcal{B}^*, \mu^*\right)$ that arises from its structure as the Fock space of $L^2_{\mathbb{R}}\left(X, \mathcal{B}, \mu\right)$ as we introduce in Appendix A.0.1.

Theorem 4.10 (Parreau & Roy)**.** *Let $(X, \mathcal{B}, \mu)$ be a standard infinite measure space. Every orthogonal projection of $L^2\left(X^*, \mathcal{B}^*, \mu^*\right)$ that preserves its chaos structure and vanishes on the first chaos $H_1(\mathcal{N})$, necessarily vanishes on all higher chaoses, i.e. it is the projection to the constants.*

The following technical lemma was proved for a single transformation in [38, § 3.2], and the following extension to general groups is straightforward.

Lemma 4.11. *Let $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ be an action of a Polish group G, and consider the projection*

$$\pi_{\mathbf{T}} : L^2_{\mathbb{R}}\left(X, \mathcal{B}, \mu\right) \to L^2_{\mathbb{R}}\left(X, \mathcal{I}_{\mathbf{T}}, \mu\right).$$

Then the projection

$$\pi_{\mathbf{T}^*} : L^2_{\mathbb{R}}\left(X^*, \mathcal{B}^*, \mu^*\right) \to L^2_{\mathbb{R}}\left(X^*, \mathcal{I}_{\mathbf{T}^*}, \mu^*\right).$$

preserves the chaos structure of $L^2_{\mathbb{R}}\left(X^*, \mathcal{B}^*, \mu^*\right)$.

Proof. We start with a single transformation $T \in \mathrm{Aut}\left(X^*, \mathcal{B}^*, \mu^*\right)$. Let U_T be the Koopman operator of T and set $\Psi_T = U_T - \mathrm{Id}$, noting that $\mathrm{Im}\pi_T = \ker \Psi_T$. Since U_T preserves the chaos structure of $L^2_{\mathbb{R}}\left(X^*, \mathcal{B}^*, \mu^*\right)$ then so is Ψ_T, so that for every $f = \sum_{n=0}^{\infty} f_n \in L^2_{\mathbb{R}}\left(X^*, \mathcal{B}^*, \mu^*\right)$, where $f_n \in H_n\left(X, \mathcal{B}, \mu\right)$ for every n, we have

$$\Psi_T = \textstyle\sum_{n=0}^{\infty} \Psi_T f_n \text{ and } \Psi_T f_n \in H_n\left(X, \mathcal{B}, \mu\right) \text{ for every } n.$$

It follows that $f \in \ker \Psi_T$ if and only if $\Psi_T f_n = 0$ for every n, that is to say

$$\ker \Psi_T = \textstyle\bigoplus_{n=0}^{\infty} \left(\ker \Psi_T \cap H_n\left(X, \mathcal{B}, \mu\right)\right).$$

However, we have that $\mathrm{Im}\pi_T = \ker\Psi_T$ on each $H_n(X,\mathcal{B},\mu)$, so we obtain

$$\mathrm{Im}\pi_T = \bigoplus_{n=0}^{\infty}\left(\mathrm{Im}\pi_T \cap H_n(X,\mathcal{B},\mu)\right). \tag{4.11.1}$$

Since $\ker\pi_T$ is the orthogonal complement of $\mathrm{Im}\pi_T$ on each $H_n(X,\mathcal{B},\mu)$, we deduce that also

$$\ker\pi_T = \bigoplus_{n=0}^{\infty}\left(\ker\pi_T \cap H_n(X,\mathcal{B},\mu)\right). \tag{4.11.2}$$

Then (4.11.1) and (4.11.2) imply that π_T preserves the chaos structure of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$, proving the lemma for a single transformation.

Now for an action $\mathbf{T}$, if we let $\Psi_g = \Psi_{T_g^*} = U_{T_g^*} - \mathrm{Id}$ for each $g \in G$, then by the same reasoning

$$\bigcap_{g\in G}\ker\Psi_g = \bigoplus_{n=0}^{\infty}\bigcap_{g\in G}\left(\ker\Psi_g \cap H_n(X,\mathcal{B},\mu)\right).$$

Since $\mathrm{Im}\pi_{\mathbf{T}} = \bigcap_{g\in G}\ker\Psi_g$ on each $H_n(X,\mathcal{B},\mu)$, it similarly follows that

$$\mathrm{Im}\pi_{\mathbf{T}} = \bigoplus_{n=0}^{\infty}\left(\mathrm{Im}\pi_{\mathbf{T}} \cap H_n(X,\mathcal{B},\mu)\right)$$

and then that

$$\ker\pi_{\mathbf{T}} = \bigoplus_{n=0}^{\infty}\left(\ker\pi_{\mathbf{T}} \cap H_n(X,\mathcal{B},\mu)\right).$$

This completes the proof of the lemma also for actions. □

Proof of Theorem 2. By Proposition 3.3 and Theorem 1 it is sufficient to prove the theorem for Poisson suspensions $\mathbf{T}^* : G \curvearrowright (X^*,\mathcal{B}^*,\mu^*)$. If the action is null then the projection $\pi_{\mathbf{T}}$ vanishes on the first chaos $H_1(\mathcal{N})$. From Lemma 4.11 and Theorem 4.10 we obtain that $\mathrm{Im}\pi_{\mathbf{T}}$ consists of constant functions, namely the Poisson suspension is ergodic. Conversely, if the action is not null so that $\mathcal{I}_{\mathbf{T}}$ contains a set A with $0 < \mu(A) < \infty$, then N_A is a non-constant, $\mathbf{T}^*$-invariant function in $L^2(X^*,\mathcal{B}^*,\mu^*)$, so the Poisson suspension is not ergodic.

Let us now show that for the Poisson suspension ergodicity implies weak mixing, and in doing so we will use the previous part of the proof twice. The Poisson suspension being ergodic implies that $\mathbf{T} : G \curvearrowright (X,\mathcal{B},\mu)$ is null, and it is clear that in this case also the diagonal action

$$\mathbf{T}\otimes\mathrm{Id} : G \curvearrowright \left(X\times\{0,1\}, \mu\otimes\tfrac{1}{2}(\delta_0+\delta_1)\right)$$

is null, so that the Poisson suspension associated with this latter action is ergodic as well. However, this Poisson suspension, $(\mathbf{T}\otimes\mathrm{Id})^*$, when is taken with respect to the product of the topology τ for which X^* was defined with the discrete topology of $\{0,1\}$, is isomorphic to $\mathbf{T}^*\otimes\mathbf{T}^* : G \curvearrowright (X^*\times X^*,\mathcal{B}^*,\mathcal{B}^*,\mu^*\otimes\mu^*)$, hence $\mathbf{T}^*$ is weakly mixing. □

5. Spatial Poissonian actions

Our main object of study in this part is a more restrictive notion of measure preserving actions, namely spatial actions.

5.1. **Preliminaries: spatial actions and the point-realization problem.**

Definition 5.1. *A (measure preserving)* **spatial action** *of a Polish group G on a standard measure space $(X,\mathcal{B},\mu)$ is a Borel action $\mathbf{T}:G\curvearrowright(X,\mathcal{B})$, $\mathbf{T}:(g,x)\mapsto T_g(x)$, such that $\mu\circ T_g^{-1}=\mu$ for every $g\in G$. We denote spatial actions by*

$$\mathbf{T}:G\overset{\mathrm{sp}}{\curvearrowright}(X,\mathcal{B},\mu)\,.$$

Every spatial action induces a (near/Boolean) action in an obvious way, and in this case it can be thought of as a *point-realization* of the resulting (near/Boolean) action. However, in general, not every action admits a point-realization, and this leads to the point-realization problem in ergodic theory, which revolves around whether a given action admits a point-realization. As it turns out, for important classes of Polish groups this problem has a striking solution.

A Polish group G is said to possess the **Mackey property** (following [19]) if every finite action of G admits a point-realization. The following classes of groups are known to possess the Mackey property:

- **Locally compact Polish groups:** This is the celebrated Mackey-Ramsay Point-Realization Theorem [33, Theorem 3.3].
- **Non-Archimedean groups:** Polish groups with a base of clopen subgroups at the identity. This is Glasner & Weiss' [14, Theorem 4.3].
- **Groups of isometries of a locally compact metric space:** closed subgroups of the group of isometries of a locally compact metric space, with the Polish topology of pointwise convergence. This is Kwiatkowska & Solecki's [23, Theorem 1.1].

The class of groups of isometries of a locally compact metric space includes both, locally compact Polish groups and non-Archimedean groups, and to the best of our knowledge this is the largest class of Polish groups on which the Mackey property is known to hold (beyond Polish groups see [7]).

The fact that there are Polish groups without the Mackey property was demonstrated by Becker for the Abelian group $L^0\left([0,1]\,,S^1\right)$ of measurable functions $[0,1]\to S^1$, identified up to equality on a Lebesgue-conull set, with the topology of convergence in measure (see [13, Appendix A], [32, §7.1] and the references therein).

This was vastly generalized by Glasner, Tsirelson & Weiss [13, Theorem 1.1] in showing that the the Mackey property fails miserably for the class of

Lévy groups, a class of groups that was studied by many following Gromov & Milman (see [32, § 4] and the references therein). This class includes, among others, the group $\mathrm{Aut}(X, \mathcal{B}, \mu)$ itself with the weak topology; the unitary group $\mathrm{U}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$ with the strong operator topology; and, the aforementioned $L^0([0,1], S^1)$. Thus, it was shown by Glasner, Tsirelson & Weiss that Lévy groups admit no finite spatial actions whatsoever except for trivial ones, and a fortiori do not possess the Mackey property. There are also non-Lévy Polish groups that do not possess the Mackey property [14, §6], [30].

Remark 5.2. A spatial action is considered to be trivial if the set of fixed points of the action has full measure. Note that the set of fixed points of a Borel action of a Polish group is a Borel set. This is clearly true for Polish actions, and for Borel action this follows from the theorem of Becker & Kechris [4, § 5.2] by which every Borel action is a Polish action with respect to some Polish topology that induces the given Borel σ-algebra.

Every Poissonian action of a group that possesses the Mackey property, admits a point-realization that serves as a spatial Poissonian action. By a *spatial Poissonian action* we refer to a Poissonian action (as in Definition 1.3) which is also a spatial action. Our goal here is to show that the point-realization problem in Poissonian actions can be solved without appealing to the Mackey property.

5.2. **Poisson random set.** For a standard Borel space $(X, \mathcal{B})$, with every Polish topology τ that generates $\mathcal{B}$ is associated the **Effros Borel space**

$$\mathbf{F}_\tau(X) = \{F \subset X : X \backslash F \in \tau\}.$$

This is a standard Borel space with the **Effros σ-algebra** $\mathcal{E}_\tau(X)$, that is generated by the sets

$$B_O := \{F \in \mathbf{F}_\tau(X) : F \cap O \neq \emptyset\}, \quad O \in \tau.$$

See e.g. [18, Section (12.C)]. A **random closed set** is nothing but a probability measure on $(\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$, namely an element of $\mathcal{M}_1(\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$. A common way to construct random closed sets for locally compact topologies is the Choquet Capacity Theorem (see e.g. [29, §1.1.3]), but here we shall construct the Poisson random closed set also for Polish topologies that may not be locally compact.

Let us denote by

$$(\mathbf{F}^*_\tau(X), \mathcal{E}^*_\tau(X)) \subset (\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$$

the subspace of infinite sets in $\mathbf{F}_\tau(X)$ with its induced Effros σ-algebra. Note that $\mathbf{F}^*_\tau(X)$ is indeed an Effros-measurable set: fixing a countable base $\mathcal{O}$ for τ, for every $n \in \mathbb{Z}_{\geqslant 0}$ the property $\#F \geqslant n$ of $F \in \mathbf{F}_\tau(X)$ is witnessed by the existence of pairwise disjoint $O_1, \ldots, O_n \in \mathcal{O}$ such that $F \in B_{O_1} \cap \cdots \cap B_{O_n}$.

Theorem 5.3 (Poisson random set)**.** *Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$, there are random variables*

$$\{\Xi_A : A \in \mathcal{B}\} \text{ of the form } \Xi_A : \mathbf{F}_\tau^* (X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

and a one-to-one correspondence

$$\mathcal{M}_\sigma^\tau (X, \mathcal{B}) \to \mathcal{M}_1 (\mathbf{F}_\tau^* (X), \mathcal{E}_\tau^* (X)), \quad \mu \mapsto \mu_\tau^*,$$

such that $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process with base space $(X, \mathcal{B}, \mu)$. Furthermore, for every μ the following properties hold.

(1) μ_τ^* is supported on the class of τ-discrete sets.[4]
(2) On the support of μ_τ^* it holds that $\Xi_A (\cdot) = \# (A \cap \cdot)$ for every $A \in \mathcal{B}$.

In order to spot the inherent difficulty in defining the Poisson point process as a random closed set in $\mathbf{F}_\tau (X)$, it should be noted that as long as τ is not locally compact, then for a general $A \in \mathcal{B}$, the function

$$\mathbf{F}_\tau (X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \quad F \mapsto \# (A \cap F),$$

may not be Effros-measurable unless $A \in \tau$. In fact, by a theorem of Christensen this function may fail to be Effros-measurable even if $A \in \mathbf{F}_\tau (X)$ [18, Theorem (27.6)]. In order to resolve this issue we use the following simple modification on the Kuratowski & Ryll-Nardzewski's Selection Theorem.

Proposition 5.4. *Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$ there is a countable collection of mappings*

$$\{\xi_n : n \in \mathbb{N}\} \text{ of the form } \xi_n : \mathbf{F}_\tau^* (X) \to X,$$

such that the following properties hold.

(1) (Measurability) Each ξ_n is Effros-measurable.
(2) (Injectivity) For each $F \in \mathbf{F}_\tau^* (X)$, the mapping $\mathbb{N} \to X$ given by $n \mapsto \xi_n (F)$ is injective.
(3) (Selectivity) For each $F \in \mathbf{F}_\tau^* (X)$, the set $\{\xi_n (F) : n \in \mathbb{N}\}$ is a (countable) dense subset of F.

We will refer to such $\{\xi_n : n \in \mathbb{N}\}$ as a ***measurable injective selection****.*

Proof. By the Kuratowski & Ryll-Nardzewski's Selection Theorem (see [18, Theorem (12.13)]) there is a measurable selection: mappings $\theta_n : \mathbf{F}_\tau (X) \setminus \{\emptyset\} \to X$, $n \in \mathbb{N}$, satisfying the first and the third properties. For each $n \in \mathbb{N}$,

[4] While the class of τ-discrete sets is generally not Effros-measurable, by a theorem of Hurewicz it is co-analytic (see [18, Theorem (27.5), Exercise (27.8)]) hence universally measurable.

restrict the mapping θ_n to $\mathbf{F}^*_\tau(X)$ and modify it into the mapping $\xi_n : \mathbf{F}^*_\tau(X) \to X$ by letting

$$\xi_n(F) = \theta_{\pi_n(F)}(F), \quad F \in \mathbf{F}^*_\tau(X),$$

where $\pi_n : \mathbf{F}^*_\tau(X) \to \mathbb{N}$ is given by

$$\pi_n(F) = \inf\{k \in \mathbb{N} : \#\{\theta_1(F), \ldots, \theta_k(F)\} = n\}.$$

Clearly, $\{\xi_n : n \in \mathbb{N}\}$ forms an injective selection, and an elementary proof by induction on π_n and ξ_n shows that each ξ_n is Effros-measurable. $\square$

Proposition 5.5. *Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$ there are mappings*

$$\{\Xi_A : A \in \mathcal{B}\} \text{ of the form } \Xi_A : \mathbf{F}^*_\tau(X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

with the following properties:

(1) *$\Xi_{A \cup B} = \Xi_A + \Xi_B$ for every disjoint $A, B \in \mathcal{B}$.*
(2) *Each Ξ_A is Effros-measurable, and together $\{\Xi_A : A \in \mathcal{B}\}$ generate the Effros σ-algebra of $\mathbf{F}^*_\tau(X)$.*
(3) *The identity $\Xi_A(F) = \#(A \cap F)$ holds in the following cases:*
  (i) *$A \in \tau$ (and every F).*
  (ii) *There is $O \in \tau$ such that $A \subset O \in \tau$ and $\#(F \cap O) < \infty$.*
  (iii) *$A \in \mathcal{B}$ and F is τ-discrete.*

Proof. Pick a measurable injective selection $\{\xi_n : n \in \mathbb{N}\}$ of $\mathbf{F}^*_\tau(X)$ as in Proposition 5.4, and for $A \in \mathcal{B}$ put

$$\Xi_A(F) = \#\{n \in \mathbb{N} : \xi_n(F) \in A\} = \sum_{n \in \mathbb{N}} 1_{\{\xi_n \in A\}}(F).$$

Property (1) follows from the injectivity of the measurable injective selection. As for Property (2), the Effros-measurability of each Ξ_A follows from the measurability of the measurable injective selection. We now make the observation that if $A \in \tau$ and $C \subset A$ is any countable set, then $\#\left(A \cap \overline{C}\right) = \#(A \cap C)$, where $\overline{C}$ denotes the closure of C, in the sense that they are either infinite together and otherwise both are finite and of the same cardinality. The same is true when $A \subset O \in \tau$ and $C \subset O$ is a finite set, for some $O \in \tau$. Thus, using the properties of measurable injective selection, this establishes Property (3)(i) and (ii), and in a straightforward way also Property (3)(iii). Finally, we note that by Property (3)(i) we have

$$B_O \cap \mathbf{F}^*_\tau(X) = \{\Xi_O > 0\}, \quad O \in \tau,$$

and this completes the proof of the second part of Property (2). $\square$

We can now prove Theorem 5.3. To this end we exploit the existence of the classical Poisson point process for locally compact Polish topologies as follows. Thus, let $(X,\mathcal{B},\mu)$ be a standard infinite measure space. Consider the classical construction of the Poisson point process: Following Proposition 3.1, pick a locally compact Polish topology ϑ for which μ is Radon, and let $(X_{\vartheta}^{*},\mathcal{B}_{\vartheta}^{*},\mu_{\vartheta}^{*})$ with the random variables $N_A:\omega\mapsto\omega(A)$ for $A\in\mathcal{B}$. Consider the map

$$\Phi:X_{\vartheta}^{*}\to\mathbf{F}_{\vartheta}^{*}(X),\quad \Phi(\omega)=\mathrm{Supp}(\omega).$$

Since the Poisson point process consists of simple counting Radon measures, Φ is well-defined. It is injective and, since $\Phi^{-1}(B_O)=\{N_O>0\}$ for every $O\in\tau$, it is measurable. Define now

$$\Xi_A^{\vartheta}:\Phi(X_{\vartheta}^{*})\subset\mathbf{F}_{\vartheta}^{*}(X)\to\mathbb{Z}_{\geqslant0}\cup\{\infty\},\quad \Xi_A^{\vartheta}=N_A\circ\Phi^{-1},\quad A\in\mathcal{B}.$$

Evidently, the random variables $\left\{\Xi_A^{\vartheta}:A\in\mathcal{B}\right\}$ satisfies the condition of Proposition 5.5. Thus, the classical Poisson point process

$$(X_{\vartheta}^{*},\mathcal{B}_{\vartheta}^{*},\mu_{\vartheta}^{*})\ \text{ with }\ \{N_A:A\in\mathcal{B}\}$$

can be naturally identified via Φ with the Poisson random set

$$(\mathbf{F}_{\vartheta}^{*}(X),\mathcal{E}_{\vartheta}^{*}(X),\mu_{\vartheta}^{*})\ \text{ with }\ \left\{\Xi_A^{\vartheta}:A\in\mathcal{B}\right\}. \tag{5.5.1}$$

Proof of Theorem 3. Let $(X,\mathcal{B},\mu)$ be a standard infinite measure space and by the assumption there is a Polish topology τ on X for which $\mu\in\mathcal{M}_{\sigma}^{\tau}(X,\mathcal{B})$. The proof will be divided into three part. In the first part we construct a generating algebra of sets $\mathbf{A}$ on $\mathbf{F}_{\tau}^{*}(X)$. In the second part we define μ_{τ}^{*} by defining it as a pre-measure on $\mathbf{A}$ and extending using the Hahn-Kolmogorov Extension Theorem. In the third part we show the desired properties of μ_{τ}^{*}.

Part 1. By the τ-local-finiteness of μ, there exists a countable base $\mathcal{O}=\{O_1,O_2,\dots\}$ for τ such that $\mu(O_n)<\infty$ for every n. Since τ-closed sets are G_{δ}-sets in τ, by removing the largest τ-open set which is μ-null and restricting τ to the remaining τ-closed set, we may assume that $\mu(O_n)>0$ for every n. For every n, let ρ_n be the finest partition of $O_1\cup\cdots\cup O_n$ that is generated by $\{O_1,\dots,O_n\}$, hence every atom of ρ_n is a τ-open set with finite positive measure. Since $\mathcal{O}$ is a base for τ, the ascending sequence of partitions $(\rho_n)_{n=1}^{\infty}$ converges to the partition into points of X, and the ascending sequence of σ-algebras $(\sigma(\rho_n))_{n=1}^{\infty}$ generates $\mathcal{B}$ modulo μ.

Fix $\{\Xi_A:A\in\mathcal{B}\}$ as in Proposition 5.5. For every $A\in\mathcal{B}$ denote by Π_A the partition of the measurable set $\{\Xi_A<\infty\}$, so that its atoms are $\{\Xi_A=k\}$, $k\in\mathbb{Z}_{\geqslant0}$. More generally, for every finite partition ρ of a set $A\in\mathcal{B}$, denote by Π_ρ the finest partition of $\{\Xi_A < \infty\}$ that is generated by the partitions Π_B, $B \in \rho$. For every n, recalling the partition ρ_n, we put

$$\Pi_n = \Pi_{\rho_n}.$$

Thus, ρ_n is a finite partition of $O_1 \cup \cdots \cup O_n \subset X$ and Π_n is an infinite partition of $\{\Xi_{O_1 \cup \cdots \cup O_n} < \infty\} \subset \mathbf{F}_\tau^*(X)$. Since the σ-algebra of $\mathbf{F}_\tau^*(X)$ is generated by $\{\Xi_O : O \in \tau\}$, it follows that the ascending sequence of partitions $(\Pi_n)_{n=1}^\infty$ converges to the partition into points of $\mathbf{F}_\tau^*(X)$, and the ascending sequence of σ-algebras $(\sigma(\Pi_n))_{n=1}^\infty$ generates the Effros σ-algebra of $\mathbf{F}_\tau^*(X)$. Thus, we obtain that

$$\mathbf{A} := \sigma(\Pi_1) \cup \sigma(\Pi_2) \cup \cdots$$

is an algebra of sets that generates its Effros σ-algebra of $\mathbf{F}_\tau^*(X)$.

Part 2. Observe that the atoms of each Π_n are in one-to-one correspondence with all functions $\kappa : \rho_n \to \mathbb{Z}_{\geqslant 0}$, $B \mapsto \kappa_B$, in that every such function corresponds to the nonempty atom

$$\bigcap_{B \in \rho_n} \{\Xi_B = \kappa_B\} \in \Pi_n.$$

To see why this is true, note that every atom $B \in \rho_n \subset \tau$ is a nonempty open set with $0 < \mu(B) < \infty$, and thus $\mu(X \backslash B) = \infty$. Then pick a (closed) set $F' \subset B$ with $\#F' = \kappa_B$, and look at the closed set $F := F' \cup (X \backslash B)$. By the properties of Ξ_B we see that $\Xi_B(F) = \#(B \cap F) = \kappa_B$, namely $F \in \{\Xi_B = \kappa_B\}$. Since the atoms ρ_n are disjoint, this shows that $\bigcap_{B \in \rho_n} \{\Xi_B = \kappa_B\}$ is nonempty for whatever choice of κ.

Define μ_0^* on $\mathbf{A}$ as follows. For each n, we specify μ_n^* on Π_n by letting $\{\Xi_B, B \in \rho_n\}$ be independent and Poisson distributed with respective means $\mu(B)$, $B \in \rho_n$. Thus, we obtain a set function μ_0^* on $\mathbf{A}$ be letting $\mu_0^* |_{\sigma(\Pi_n)} = \mu_n^*$ for each n. It is not hard to show directly that μ_0^* is a consistent premeasure on $\mathbf{A}$, but it would be shorter to utilize the existence of the classical Poisson point process with respect to some other Polish topology ϑ which is chosen to be locally compact and for which μ is a Radon, as is presented in (5.5.1). First, for every n and $\kappa : \rho_n \to \mathbb{Z}_{\geqslant 0}$ we have the obvious identity

$$\mu_0^* \left(\bigcap_{B \in \rho_n} \{\Xi_B = \kappa_B\} \right) = \mu_\vartheta^* \left(\bigcap_{B \in \rho_n} \{\Xi_B^\vartheta = \kappa_B\} \right) \tag{5.5.2}$$

We now observe that for each n, since Π_n is a countable partition, $\sigma(\Pi_n)$ is nothing but all countable disjoint unions of the atoms of Π_n. Consequently, as the sequence of σ-algebras $(\sigma(\Pi_n))_{n=1}^\infty$ is ascending, the algebra $\mathbf{A}$ consists of countable disjoint unions of atoms of the partitions $(\Pi_n)_{n=1}^\infty$. Then the family of identities (5.5.2) together with the fact that μ_ϑ^* is a probability measure and in particular σ-additive, readily imply that μ_0^* is a consistent pre-measure on $\mathbf{A}$. Finally, by the Hahn-Kolmogorov Extension Theorem, μ_0^* extends to a genuine probability measure μ_τ^* on $\mathbf{F}_\tau^*(X)$.

Part 3. We now show that $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process on the base space $(X, \mathcal{B}, \mu)$. Let $A \in \mathcal{B}_\mu$ be arbitrary. For every n denote

$$A_n = A \cap (O_1 \cup \cdots \cup O_n),$$

and define ρ_n^A to be the finest partition of $O_1 \cup \cdots \cup O_n$ that is generated by $A_n, O_1, \ldots, O_n$. While the sets A_n may not be in τ, since they are contained in $O_1 \cup \cdots \cup O_n$ we may apply Proposition 5.5(3)(ii). Thus, repeating the same construction as above when the partitions $(\rho_n)_{n=1}^\infty$ are replaced by the partitions $\left(\rho_n^A\right)_{n=1}^\infty$, we obtain a probability measure μ_τ^A on $\mathbf{F}_\tau^*(X)$ with respect to which $\Xi_{A\cap O_n}$ is Poisson distributed with mean $\mu(A \cap O_n)$ for every n. Then by standard arguments it follows that Ξ_A is Poisson distributed with mean $\mu(A)$ with respect to μ_τ^A. However, we evidently have that

$$\mu_\tau^A(B_{O_n}) = 1 - e^{-\mu(O_n)} = \mu_\tau^*(B_{O_n}), \quad n = 1, 2, \ldots,$$

and the distribution of a random closed set in $\mathbf{F}_\tau(X)$ is determined by the probabilities of B_O, $O \in \mathcal{O}$ (see e.g. [29, Theorem 1.3.20], whose proof is valid also for a base for the topology), and it follows that $\mu_\tau^A = \mu_\tau^*$. This shows that $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process with respect to μ_τ^* and, moreover, μ_τ^* is unique with respect to this property, establishing that the correspondence $\mathcal{M}_\sigma^\tau(X, \mathcal{B}) \to \mathcal{M}_1(\mathbf{F}_\tau^*, \mathcal{E}_\tau^*(X))$, $\mu \mapsto \mu_\tau^*$, is one-to-one.

Finally, in order to see that μ_τ^* is supported on the class of τ-discrete sets, note that $\bigcap_{n\geqslant 1}\{\Xi_{O_n} < \infty\}$ is an Effros-measurable subset of $\mathbf{F}_\tau^*(X)$ that consists only of τ-discrete sets and, since $\mu_\tau^*(\Xi_{O_n} < \infty) = 1$ for each $n \geqslant 1$, we deduce that it is a μ_τ^*-conull set. ☐

5.3. **Spatial Poisson suspensions.** In this section we prove Theorem 4.

Let $(X, \mathcal{B})$ be a standard Borel space, and τ a Polish topology on X that generates $\mathcal{B}$. For every τ-homeomorphism T of X we define

$$T^* : \mathbf{F}_\tau^*(X) \to \mathbf{F}_\tau^*(X), \quad T^*(F) := T(F) = \{T(x) \in X : x \in F\}.$$

Lemma 5.6. *Let $\mu \in \mathcal{M}_\sigma(X, \mathcal{B})$. If T is a τ-homeomorphism that preserves μ, then T^* is a transformation that preserves μ_τ^* (as in Theorem 5.3).*

Proof. For every $O \in \tau$ we have

$$\begin{aligned}\mu_\tau^*\left(T^{*-1}(B_O)\right) &= \mu^*\left(B_{T^{-1}(O)}\right) = \mu^*\left(\Xi_{T^{-1}(O)} > 0\right) = 1 - e^{-\mu\left(T^{-1}(O)\right)} \\ &= 1 - e^{-\mu(O)} = \mu^*(\Xi_O > 0) = \mu_\tau^*(B_O),\end{aligned}$$

thus $\mu^* \circ T^{*-1}$ and μ^* coincide on B_O, $O \in \tau$. Since the values on B_O, $O \in \tau$, determines random closed sets uniquely (see [29, Theorem 1.3.20]) it follows that T^* preserves μ^* □

Note that $(T \circ S)^* = S^* \circ T^*$ whenever T, S are τ-homeomorphism, and in particular T^* is invertible by $(T^*)^{-1} = \left(T^{-1}\right)^*$. Obviously, $\mathbf{F}^*_\tau(X)$ is T^*-invariant for whatever τ-homeomorphism T of X. Then from a τ-Polish action $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ we obtain the action

$$\mathbf{T}^* : G \curvearrowright (\mathbf{F}^*_\tau(X), \mathcal{E}^*_\tau(X)), \quad \mathbf{T}^* : (g, F) \mapsto T^*_g(F). \tag{5.6.1}$$

Lemma 5.7. *If $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ is a τ-Polish action then $\mathbf{T}^* : G \curvearrowright (\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$, and in particular $\mathbf{T}^* : G \curvearrowright (\mathbf{F}^*_\tau(X), \mathcal{E}^*_\tau(X))$ as in* (5.6.1), *is a Borel action.*

Remark 5.8. In the literature this fact is viewed as elementary (see e.g. [4, § 2.4, Example (ii)], [11, § 3.3]). It is worth mentioning that when τ is locally compact (and only then), the *Fell topology* on $\mathbf{F}_\tau(X)$ is Polish and generates $\mathcal{E}_\tau(X)$ (see [18, Exercise (12.7)]). It can be verified, with the assistance of [41, Theorem 4.8.6], [18, Exercise (9.16) i)], that in this case $\mathbf{T}^*$ is further a Polish action in the Fell topology.

Proof. Evidently $\mathbf{T}^*$ is an action and we verify that $\mathbf{T}^*$ is a Borel map by showing that $\mathbf{T}^{*-1}(B_O)$ is measurable in $G \times \mathbf{F}^*_\tau(X)$ for each $O \in \tau$. To this end we use the Kuratowski & Ryll-Nardzewski's Selection Theorem (see [18, Theorem (12.13)]), to fix a measurable selection $\{\theta_n : n \in \mathbb{N}\}$ for $\mathbf{F}_\tau(X)$, that is a countable collection of Effros-measurable mappings $\theta_n : \mathbf{F}_\tau(X) \to X$ such that $\{\theta_n(F) : n \in \mathbb{N}\}$ is a (countable) dense subset of F for each $F \in \mathbf{F}^*_\tau(X) \setminus \{\emptyset\}$. Fix also a countable base $\mathcal{U}$ for τ.

Let $O \in \tau$ be arbitrary. For each $g \in G$ and $F \in \mathbf{F}_\tau(X)$, using that $T_g^{-1}(O)$ is τ-open,

$$\begin{aligned} T_g(F) \cap O \neq \emptyset &\iff F \cap T_g^{-1}(O) \neq \emptyset \\ &\iff \exists_{n\in\mathbb{N}} \left[\theta_n(F) \in T_g^{-1}(O)\right] \\ &\iff \exists_{n\in\mathbb{N}} \exists_{U\in\mathcal{U}} \left[\left(U \subset T_g^{-1}(O)\right) \wedge (\theta_n(F) \in U)\right], \end{aligned}$$

so we may write

$$\mathbf{T}^{*-1}(B_O) = \bigcup_{n\in\mathbb{N}} \bigcup_{U\in\mathcal{U}} \left\{g \in G : U \subset T_g^{-1}(O)\right\} \times \{F \in \mathbf{F}_\tau(X) : \theta_n(F) \in U\}.$$

Each set $\{F \in \mathbf{F}_{\tau}(X) : \theta_n(F) \in U\}$ is clearly Effros-measurable. In order to see the measurability of each $\left\{g \in G : U \subset T_g^{-1}(O)\right\}$, write

$$\begin{aligned}\left\{g \in G : U \not\subset T_g^{-1}(O)\right\} &= \{g \in G : T_g(U) \cap X \backslash O \neq \emptyset\} \\ &= \bigcup_{x \in X \backslash O} \{g \in G : x \in T_g(U)\}.\end{aligned}$$

Since for each $x \in X$ the map $G \to X$, $g \mapsto T_g^{-1}(x)$, is continuous, the set $\{g \in G : x \in T_g(U)\}$ is open. This readily implies that $\{g \in G : U \subset T_g(O)\}$ is closed hence measurable. □

Proof of Theorem 4. Given a locally finite Polish action $\mathbf{T} : G \overset{\text{sp}}{\curvearrowright} (X, \mathcal{B}, \mu)$ with respect to a Polish topology τ, using the construction of Theorem 5.3, the construction of the action as in (5.6.1), together with Lemmas 5.6 and 5.7, we obtain the spatial action

$$\mathbf{T}^* : G \overset{\text{sp}}{\curvearrowright} \left(\mathbf{F}_{\tau}^*(X), \mathcal{E}_{\tau}^*(X), \mu_{\tau}^*\right).$$

We complete the proof by showing that this is a Poissonian action, whose base action is $\mathbf{T} : G \overset{\text{sp}}{\curvearrowright} (X, \mathcal{B}, \mu)$, with respect to the Poisson point process $\{\Xi_A : A \in \mathcal{B}\}$ that is defined on $\left(\mathbf{F}_{\tau}^*(X), \mathcal{E}_{\tau}^*(X), \mu_{\tau}^*\right)$ as in Theorem 5.3.

Recall that by Proposition 5.5, for every $A \in \mathcal{B}$, we have that $\Xi_A(F) = \#(A \cap F)$ whenever $F \in \mathbf{F}_{\tau}^*(X)$ is τ-discrete. Since μ_{τ}^* is supported on the class of τ-discrete sets, it follows that for F in a μ_{τ}^*-conull set,

$$\Xi_A(T^*(F)) = \#(A \cap T^*(F)) = \#\left(T^{-1}(A) \cap F\right) = \Xi_{T^{-1}(A)}(F).$$

This readily shows that $\mathbf{T}$ is a base action for $\mathbf{T}^*$ as in Definition 1.4, completing the proof of Theorem 4. □

6. Constructing spatial actions from nonsingular spatial actions

Here we aim to establish Theorem 5. Recall the Polish group

$$\mathrm{Aut}(X, \mathcal{B}, [\mu])$$

of the (equivalence classes of) nonsingular transformations of a standard measure space $(X, \mathcal{B}, \mu)$. It is worth mentioning that similarly to $\mathrm{Aut}(X, \mathcal{B}, \mu)$, also $\mathrm{Aut}(X, \mathcal{B}, [\mu])$ is a Lévy group [12, Theorem 6.1], [32, §4.5]. Let us start with the important construction of the Maharam Extension, which allows one to realize the nonsingular transformations of one space as measure preserving transformations of another space.

Proposition 6.1 (Maharam Extension)**.** *For every standard (finite or infinite) measure space* $(X,\mathcal{B},\mu)$ *there exists a standard infinite measure space* $\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$, *with a continuous embedding of Polish groups*

$$\mathrm{Aut}\,(X,\mathcal{B},[\mu])\hookrightarrow \mathrm{Aut}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right),\quad T\mapsto \widetilde{T}.$$

Proof. For a standard measure space $(X,\mathcal{B},\mu)$ define $\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$ by

$$\widetilde{X}=X\times\mathbb{R},\quad \widetilde{\mathcal{B}}=\mathcal{B}\otimes\mathcal{B}\left(\mathbb{R}\right),\quad d\widetilde{\mu}\left(x,t\right)=d\mu\left(x\right)e^{t}dt.$$

Obviously, this is a standard infinite measure space. In order to define the desired embedding, let us denote the Radon-Nikodym cocycle by

$$\nabla:\mathrm{Aut}\,(X,\mathcal{B},[\mu])\times X\to(0,\infty)\,,\quad \nabla_{T}\left(\cdot\right)=\frac{d\mu\circ T^{-1}}{d\mu}\left(\cdot\right)\in L^{0}\left(X,\mathcal{B},\mu\right).$$

Note that this is only an *almost cocycle* in the sense that for every $T,S\in\mathrm{Aut}\,(X,\mathcal{B},[\mu])$ it holds that

$$\nabla_{T\circ S}=\nabla_{T}\circ S\cdot\nabla_{S}\text{ in }L^{0}\left(X,\mathcal{B},\mu\right).$$

We now define the embedding $\mathrm{Aut}\,(X,\mathcal{B},[\mu])\hookrightarrow\mathrm{Aut}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$ by

$$T\mapsto\widetilde{T},\quad \widetilde{T}\left(x,t\right)=\left(T\left(x\right),t-\log\nabla_{T}\left(x\right)\right). \tag{6.1.1}$$

It is straightforward to verify that this is a well-defined, continuous embedding of Polish groups. □

Suppose G is a Polish group. A **nonsingular** (Boolean) **action** of G on a standard measure space $(X,\mathcal{B},\mu)$ is a continuous (equivalently, measurable) homomorphism $\mathbf{T}:G\to\mathrm{Aut}\,(X,\mathcal{B},[\mu])$. We denote such an action by

$$\mathbf{T}:G\curvearrowright(X,\mathcal{B},[\mu])\,.$$

A **nonsingular spatial action** of G on a standard measure space $(X,\mathcal{B},\mu)$ is a Borel action $\mathbf{T}:G\curvearrowright(X,\mathcal{B})$, $\mathbf{T}:(g,x)\mapsto T_{g}\left(x\right)$, such that for every $g\in G$, T_{g} is a nonsingular transformation of $(X,\mathcal{B},\mu)$. We denote such an action by

$$\mathbf{T}:G\overset{\mathrm{sp}}{\curvearrowright}(X,\mathcal{B},[\mu])\,.$$

Using the Maharam Extension construction, from every nonsingular action we obtain a (measure preserving) action, however this does not work in the spatial category since the Radon-Nikodym cocycle is merely an almost cocycle. When G is locally compact, by the Mackey Cocycle Theorem the Radon-Nikodym cocycle admits a pointwise version (*strict version* in the terminology of [3]) and then the Maharam Extension does admit a point-realization. However, as it was shown by Becker [3, Section E], in general a strict version of the Radon-Nikodym cocycle may not exist.

Proof of Theorem 5. If G admits a locally finite Polish action, then by Theorem 4 the spatial Poisson suspension of this locally finite Polish action, with an appropriate Polish topology, is a finite spatial action. If G admits a τ-locally finite Polish nonsingular action $\mathbf{T}:G\overset{\mathrm{sp}}{\curvearrowright}(X,\mathcal{B},[\mu])$, $\mathbf{T}:(g,x)\mapsto T_g(x)$, such that for each $g\in G$ the Radon-Nikodym derivatives $x\mapsto\nabla_g(x):=\nabla_{T_g}(x)$ is τ-continuous, then the Maharam Extension

$$\widetilde{\mathbf{T}}:G\overset{\mathrm{sp}}{\curvearrowright}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right),\quad \widetilde{\mathbf{T}}:(g,(x,t))\mapsto\left(T_g(x),t-\log\nabla_g(x)\right),$$

is a spatial action. In the coming discussion we omit the precise Polish topologies with respect to which the continuity properties are defined, as these are given by products of the obvious topologies on G, X and $\mathbb{R}$. We further claim that this Maharam Extension is a locally finite Polish action. First, clearly the measure $\widetilde{\mu}$ is locally finite. In order to see that the action is Polish, namely that $\widetilde{\mathbf{T}}$ is a continuous map, we note that by the assumption it follows that $\widetilde{\mathbf{T}}$ is separately continuous, and by [18, Theorem (9.14)] it automatically follows that it is indeed jointly continuous. Having that $\widetilde{\mathbf{T}}$ is a locally finite Polish action with respect to $\widetilde{\tau}:=\tau\otimes\tau_{\mathbb{R}}$, where τ is the given Polish topology on X and $\tau_{\mathbb{R}}$ is the usual topology of $\mathbb{R}$, again by Theorem 4 we obtain that the spatial Poisson suspension

$$\widetilde{\mathbf{T}}^{*}:G\overset{\mathrm{sp}}{\curvearrowright}\left(\mathbf{F}^{*}_{\widetilde{\tau}}\left(\widetilde{X}\right),\mathcal{E}^{*}_{\widetilde{\tau}}\left(\widetilde{X}\right),\widetilde{\mu}^{*}_{\widetilde{\tau}}\right)$$

is a finite spatial action of G. □

Proof of Corollary 1.6. By [22, Theorem 2.3], if $\mathbf{S}:G\overset{\mathrm{sp}}{\curvearrowright}(Y,\nu)$ is any finite spatial action that is faithful, the diagonal action on the infinite product $\mathbf{S}^{\mathbb{N}}:G\overset{\mathrm{sp}}{\curvearrowright}\left(Y^{\mathbb{N}},\nu^{\mathbb{N}}\right)$ is essentially free in the sense that there is a ν-conull invariant subset of $Y^{\mathbb{N}}$ (where the invariance is pointwise) on which the action $\mathbf{S}$ is free. Evidently, this diagonal action is also a finite spatial action, so this completes the proof. □

7. Diffeomorphism groups: classification and spatial actions

Let M be a compact smooth manifold. We will always assume, further, that M is d-dimensional Hausdorff connected and without boundary. Let a parameter $1\leqslant r\leqslant\infty$ be fixed, and denote by

$$\mathrm{Diff}^{r}(M)$$

the group of all C^r-diffeomorphisms from M onto itself. It becomes a Polish group with the compact-open C^r-topology. Denote the connected component of the identity, as a normal subgroup, by

$$\mathrm{Diff}^{r}_{o}(M)\lhd\mathrm{Diff}^{r}(M).$$

Diffeomorphism groups are locally connected (see [2, Proposition 1.2.1]), so $\mathrm{Diff}_o^r(M)$ is a clopen subgroup, hence a Polish subgroup, of $\mathrm{Diff}^r(M)$. Of course, $\mathrm{Diff}_o^r(M)$ is also not locally compact. We also note that for every $1 \leqslant r \leqslant \infty$ there is an embedding of Polish groups

$$\mathrm{Diff}_o^\infty(M) \leqslant \mathrm{Diff}_o^r(M).$$

This is because $\mathrm{Diff}^\infty(M)$ is a closed subgroup of $\mathrm{Diff}^r(M)$, together with that the identity component of $\mathrm{Diff}^r(M)$ is precisely the C^r-diffeomorphisms which are C^r-isotopic to the identity (see [2, Corollary 1.2.2]).

Observe that the local-connectedness of $\mathrm{Diff}^r(M)$ implies that it is a (not non-)Archimedean group. Indeed, non-Archimedean groups, admitting a base of clopen subgroups, are totally disconnected. Here we aim to show the stronger statement of Theorem 6. To this end we exploit the following celebrated result that was established by Herman in the case when M is a torus and by Thurston generally. See [2, §2] and the references therein.

Theorem 7.1 (Thurston)**.** *For all M as above, $\mathrm{Diff}_o^\infty(M)$ is a simple group.*

The second tool we need is the following characterization of groups of isometries of a locally compact metric space [23, Theorem 1.2].

Theorem 7.2 (Kwiatkowska & Solecki)**.** *A Polish group G is a group of isometries of a locally compact metric space if and only if it possesses the following property:*

> *Every identity neighborhood contains a closed subgroup H such that G/H is a locally compact space and the normalizer*
> $$N_G(H) := \left\{g \in G : gHg^{-1} = H\right\} \text{ is clopen.}$$

We now formulate and prove Theorem 6.

Theorem 7.3. *For every compact smooth manifold M as above and $1 \leqslant r \leqslant \infty$, the Polish group $\mathrm{Diff}^r(M)$ is not a group of isometries of a locally compact metric space.*

Proof. As we mentioned before, we have the embeddings of Polish groups

$$\mathrm{Diff}_o^\infty(M) \hookrightarrow \mathrm{Diff}_o^r(M) \hookrightarrow \mathrm{Diff}^r(M).$$

Thus, it suffices to prove that $\mathrm{Diff}_o^\infty(M)$ is not a group of isometries of a locally compact metric space. Suppose otherwise toward a contradiction, then by Theorem 7.2 there exists a closed subgroup $H \leqslant \mathrm{Diff}_o^\infty(M)$ for which $\mathrm{Diff}_o^\infty(M)/H$ is a locally compact space and the normalizer $N_{\mathrm{Diff}_o^\infty(M)}(H)$ is clopen. Since $\mathrm{Diff}_o^\infty(M)/H$ is a locally compact space while $\mathrm{Diff}_o^\infty(M)$ is not, it follows that H, hence also $N_{\mathrm{Diff}_o^\infty(M)}(H)$, cannot be the trivial group. Since $N_{\mathrm{Diff}_o^\infty(M)}(H)$ is clopen and $\mathrm{Diff}_o^\infty(M)$ is connected, it follows that

$N_{\mathrm{Diff}_o^\infty(M)}(H) = \mathrm{Diff}_o^\infty(M)$, namely H is a normal subgroup of $\mathrm{Diff}_o^\infty(M)$. This contradicts Theorem 7.1. □

We move now to construct spatial actions of diffeomorphism groups.

Proof of Theorem 7. Let M be a compact smooth manifold and pick a volume form Vol on M. In fact, the following construction works also when M is merely paracompact and accordingly has an infinite total volume in Vol; indeed, the compact-open C^r-topology on $\mathrm{Diff}^r(M)$ is still Polish and the Maharam Extension can be taken with respect to an infinite measure as well. Whether M is compact or merely paracompact, look at the tautological action of $\mathrm{Diff}^r(M)$ on M which is obviously nonsingular with respect to Vol. Thus, we have a nonsingular locally finite Polish action

$$\mathbf{D} : \mathrm{Diff}^r(M) \overset{\mathrm{sp}}{\curvearrowright} (M, \mathrm{Vol}), \quad f.x = f(x).$$

The Radon-Nikodym cocycle of this action is the corresponding Jacobian, which is jointly continuous as a map $\mathrm{Diff}^r(M) \times M \to (0,\infty)$, hence by Theorem 5 we conclude that $\mathrm{Diff}^r(M)$ admits a nontrivial probability preserving spatial action. Using Corollary 1.6 we immediately obtain that there exists also such a free action.

In order to obtain also the ergodicity we look back at the steps in the construction as in the proof of Theorem 5: starting with the locally finite Polish nonsingular action $\mathbf{D}$, we constructed the Maharam Extension $\widetilde{\mathbf{D}}$, from which we constructed the spatial Poisson suspension $\widetilde{\mathbf{D}}^*$ (with respect to the obvious Polish topology on $\widetilde{M}$). Then, in order to obtain a free action, we took the infinite diagonal product action $\left(\widetilde{\mathbf{D}}^*\right)^{\mathbb{N}}$. We then argue that the action $\left(\widetilde{\mathbf{D}}^*\right)^{\mathbb{N}}$ is ergodic. To this end we exploit the fact that the Maharam Extension $\widetilde{\mathbf{D}}$ is ergodic, which will be proved in Appendix B, and then, as every infinite measure preserving action, the ergodicity of $\widetilde{\mathbf{D}}$ implies that it is null. From Theorem 2 it follows its Poisson suspension $\widetilde{\mathbf{D}}^*$ is weakly mixing. Recalling Remark 4.9(3), this readily implies that $\left(\widetilde{\mathbf{D}}^*\right)^{\mathbb{N}}$ is ergodic. This completes the proof of Theorem 7. □

7.1. **Non-essentially countable orbit equivalence relations.** We introduce the necessary background to Corollary 1.7. A general reference to the subject, with many references therein, is [21].

An equivalence relation E on a standard Borel space $(X, \mathcal{B})$ is a set $E \subset X \times X$ such that the condition $x \sim x' \iff (x, x') \in E$ defines an equivalence relation on X. Such an equivalence relation is said to be

Borel if it is a Borel subset of $X \times X$, and **countable** if each of its equivalence classes is countable. The Borel complexity of one equivalence relation relative to another can be tested by the possibility of producing a Borel reduction of the former into the latter in the following sense. A **Borel reduction** of an equivalence relation E on a standard Borel space $(X, \mathcal{B})$ into an equivalence relation F on a standard Borel space $(Y, \mathcal{C})$, is a Borel function $f : X \to Y$ such that

$$\left(x, x'\right) \in E \iff \left(f(x), f\left(x'\right)\right) \in F \text{ for all } x, x' \in X.$$

An equivalence relation is said to be **essentially countable** if it admits a Borel reduction into a countable Borel equivalence relation.

An important class of equivalence relations are **orbit equivalence relations**. If $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ is a Borel action of a Polish group G on a standard Borel space $(X, \mathcal{B})$, the associated orbit equivalence relation is

$$E_{\mathbf{T}}^{X} = \{(x, T_g(x)) : x \in X, g \in G\} \subset X \times X.$$

It was shown by Kechris [20] that if G is locally compact then $E_{\mathbf{T}}^{X}$ is always essentially countable, and he left as open question whether this property characterizes locally compact groups among the Polish groups (see [22, Problem 1.2], [21, Problem 4.16]). Recently, this was answered affirmatively for groups of isometries of a locally compact metric space by Kechris, Malicki, Panagiotopoulos & Zielinski [22]. In their proof they used the aforementioned Mackey property of these groups due to Kwiatkowska & Solecki [23, Theorem 1.2], which is unavailable for diffeomorphism groups in light of Theorem 6.

Proof of Corollary 1.7. By Theorem 7, every diffeomorphism group admits a free spatial action on a standard probability space. If the orbit equivalence relation of this action would be essentially countable, then by a theorem of Feldman & Ramsay [9, Theorem A] (c.f. [22, Theorem 2.1]) it would follow that the acting group is locally compact, which is false, hence this orbit equivalence relation is non-essentially countable. □

8. Open problems

We revisit the question of Glasner, Tsirelson & Weiss [13, Question 1.2] of whether Lévy groups admits nonsingular spatial action. In their proof of the non-existence of probability preserving actions of Lévy groups, it was sufficient to show the non-existence of such *Polish* actions. This reduction was possible thanks to a theorem of Becker & Kechris [4, § 5.2], by which every Borel action of a Polish group admits a Polish topology with respect to which it becomes a Polish action. In locally finite Polish actions, by definition, there exists such a topology that, simultaneously, makes the action

Polish and the measure locally finite. It is then natural to ask about the following refinements of Becker & Kechris' theorem:

Question 8.1. *Let* $\mathbf{T}: G \curvearrowright (X, \mathcal{B})$ *be a Borel action of a Polish group* G*, preserving an infinite measure* μ*. When does there exist a Polish topology on* X *with respect to which, simultaneously,* $\mathbf{T}$ *is Polish and* μ *is locally finite?*

Diffeomorphism groups were shown, in Theorems 6 and 7, to belong neither to the class of groups of isometries of a locally compact metric space nor the class of Lévy groups. Thus it is natural to ask:

Question 8.2. *Do diffeomorphism groups possess the Mackey property?*

The following related question was posed by Moore & Solecki [30, §4]. For a compact smooth manifold M, let $C^{\infty}\left(M, S^{1}\right)$ be the group of C^{∞}-functions from M to the unit circle S^{1} with pointwise multiplication and the compact-open C^{∞}-topology. It was shown by Moore & Solecki [30, Theorem 1.1] that the Mackey property fails for homeomorphisms from M to S^{1}, and they ask about the Mackey property for $C^{\infty}\left(M, S^{1}\right)$, describing it as 'tempting to conjecture'.

The Mackey property of a Polish group refers to all of its actions at once. However, it is possible for a Polish group without the Mackey property to admit spatial actions. Indeed, the homeomorphism group of the circle S^{1} does not possess the Mackey property by the aforementioned result of Moore & Solecki, and yet it acts spatially and ergodically on S^{1} with its Lebesgue measure by $f.z = f(1)\,z$. This suggests to deviate from the general Mackey property and raise the following problem that seems to be widely open:

Problem 8.1. *Consider the following actions a Polish group may posses:*[5]

- **Type** II_1**:** *Spatial ergodic probability preserving actions.*
- **Type** II_∞**:** *Spatial ergodic infinite measure preserving actions.*
- **Type** III**:** *Spatial ergodic nonsingular actions without an absolutely continuous invariant measure.*

Beyond the locally compact case, little is known about whether a given Polish group admits an action of Type II_1, II_∞ *or* III. *A particular important case is Glasner, Tsirelson & Weiss' question: Lévy groups do not admit actions of Type* II_1*, but do they admit actions of Type* III*? What about Type* II_∞*? There are also classification aspects of this problem: does the admission of an action of a certain type imply the admission of an action of some other type? On the contrary, is there a Polish group that admits an action of a certain type but not an action of some other type?*

[5] The notations were borrowed from the Krieger-type of countable groups actions, which in turn were borrowed from the classification of factors in von Neumann algebras.

Remark 8.1. We comment on known results in the above problem

(1) Groups of isometries of locally compact metric spaces admit actions of Types II_∞ (there is a Haar measure) and II_1 (the Poisson suspension of the Haar measure, which is spatial by [23, Theorem 1.1] or by Theorem 4). See [8] with regard to Type III.
(2) Diffeomorphism groups of compact smooth manifolds admit actions of Types III (tautologically), II_∞ (the Maharam Extension) and II_1 (the spatial Poisson suspension as in Theorem 7).
(3) The group $\mathrm{Homeo}_+([0,1])$ of orientation preserving homeomorphisms admits not even a Boolean action whatsoever by a theorem of Megrelishvili [28, Theorem 3.1] (see also [13, Remark 1.7]).
(4) The Maharam Extension construction demonstrates that in certain cases, the admission of a Type III action implies the admission of a Type II_∞ action. The Poisson suspension construction demonstrates that in certain cases, the admission of a Type II_∞ action implies the admission of a Type II_1 action. By Theorem 5, these implications are valid when the actions possess appropriate continuity properties.

Appendix A. The First Chaos of Poisson point processes

Fix a generative Poisson point process $\mathcal{P}=\{P_A : A\in\mathcal{B}\}$ that is defined on $(\Omega,\mathcal{F},\mathbb{P})$ with base space $(X,\mathcal{B},\mu)$ as in Definition 1.1. In the following we describe some fundamental properties of the **first chaos** of $\mathcal{P}$, namely the real Hilbert space

$$H_1(\mathcal{P}):=\overline{\mathrm{span}}\{P_A-\mu(A):A\in\mathcal{B}_\mu\}\subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P}).$$

A.0.1. *Fock space and Chaos decomposition.* The (real, symmetric) **Fock space** associated with a Hilbert space $\mathcal{H}$ is, by definition, the Hilbert space

$$\mathrm{F}(\mathcal{H})=\bigoplus_{n=0}^{\infty}\mathcal{H}^{\odot n},$$

where $\mathcal{H}^{\odot 0}=\mathbb{R}$ and $\mathcal{H}^{\odot n}$ for $n\geqslant 1$ is the symmetric n^{th} tensor product of $\mathcal{H}$, i.e. its elements are the vectors in the usual tensor product $\mathcal{H}^{\otimes n}$ that are invariant to permutations of their coordinates, which are generated by elements of the form

$$u_1\odot\cdots\odot u_n:=\frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)}u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)},$$

and with the inner product that is given by

$$\langle u_1\odot\cdots\odot u_n,v_1\odot\cdots\odot v_n\rangle_{\mathcal{H}^{\odot n}}:=\frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}(n)}\prod_{j=1}^{n}\left\langle u_j,v_{\sigma(j)}\right\rangle_{\mathcal{H}}.$$

Here the operation $\bigoplus$ denotes the operation of taking the Hilbert space obtained as the completion of the direct sum.

For the classical Poisson point process $\mathcal{N}$ that is defined on $(X^*,\mathcal{B}^*,\mu^*)$ with the base space $(X,\mathcal{B},\mu)$, the Fock space decomposition refers to the isometric isomorphism between $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$ and $\mathrm{F}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right)$. See [26] (see also [24, §18]). Suppose $\mathcal{P}$ is a Poisson point process that is defined on $(\Omega,\mathcal{F},\mathbb{P})$ with a base space $(X,\mathcal{B},\mu)$ as in Definition 1.1. Using Proposition 3.3 we obtain an isometric isomorphism between $L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$ and $\mathrm{F}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right)$. Thus, the Hilbert space $L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$ has the **chaos decomposition** into

$$L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})=\bigoplus_{n=0}^{\infty}H_n(\mathcal{P}), \tag{A.0.1}$$

each $H_n(\mathcal{P})$ is called the n^{th} *chaos* with respect to $\mathcal{P}$. The following description of the chaos structure for an abstract $\mathcal{P}$ requires performing stochastic integration against $\mathcal{P}$ as a random measure, which is justified in Proposition 3.3.

- $H_0(\mathcal{P})=\mathbb{R}$.
- $H_1(\mathcal{P}):=\overline{\operatorname{span}}\{P_A-\mu(A):A\in\mathcal{B}_\mu\}\subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$.
- $\cdots$
- $H_n(\mathcal{P})=\overline{\operatorname{span}}\left\{\int_{X^{\odot n}}1_A^{\odot n}d(\mathcal{P}-\mu)^{\odot n}:A\in\mathcal{B}_\mu\right\}\subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$, where $X^{\odot n}$ is the set of sequences in X^n consisting of n disjoint elements.

With this chaos decomposition of $L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$, the isometric isomorphism

$$I_\mu:L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})\to\mathrm{F}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right),$$

is given by stochastic integration against $\mathcal{P}$ as follows.

- $I_\mu:\mathbb{R}f_0\to H_0(\mathcal{P})$ is given by $I_\mu:cf_0\mapsto c$, where $\mathbb{R}f_0$ is a one-dimensional space that is spanned by a distinguish norm-one vector $f_0\in L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, called *vacuum vector*, whose specification will be unimportant for us.
- $I_\mu:L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\to H_1(\mathcal{P})$ is given by the stochastic integral
$$I_\mu:1_A\mapsto\int_X 1_A d(\mathcal{P}-\mu)=P_A-\mu(A),\quad A\in\mathcal{B}_\mu.$$
- $\cdots$
- $I_\mu:L^2_{\mathbb{R}}(X,\mathcal{B},\mu)^{\odot n}\to H_n(\mathcal{P})$ is given by the stochastic integral
$$I_\mu:1_A^{\odot n}\mapsto\int_{X^{\odot n}}1_A^{\odot n}d(\mathcal{P}-\mu)^{\odot n},\quad A\in\mathcal{B}_\mu.$$

In general, every unitary operator U of a Hilbert space $\mathcal{H}$ induces a unitary operator $\mathrm{F}(U)$ of the Fock space $\mathrm{F}(\mathcal{H})$, that is defined by letting $\mathrm{F}(U)$ acts on $\mathcal{H}^{\odot n}$ as the n^{th} tensor product $U^{\otimes n}$. Thus, for every

$T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, by looking at the Koopman operator U_T we obtain the operator $\mathrm{F}(U_T)$ of $\mathrm{F}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right) \cong L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. This operator can also be described without reference to the Fock space structure: if $S \in \mathrm{Aut}(\Omega, \mathcal{F}, \mathbb{P})$ is a Poissonian transformation with base transformation T (for instance, $S = T^*$ as in Proposition 3.1), we obtain the Koopman operator U_S of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. The equivariance property that relates T and S as in Definition 1.4 implies that

$$\mathrm{F}(U_T) = U_S.$$

A.0.2. *Infinitely Divisible distributions and the First Chaos.* We start by recalling some of the basics of infinitely divisible distributions. A general reference to the subject is [39].

A distribution is **infinitely divisible** if for every positive integer n it can be presented as the n^{th}-power convolution of some other distribution. By the fundamental Lévy-Khintchine Representation (see [39, Chapter 2, §8]), every infinitely divisible distribution is completely determined by a triplet

$$(\sigma, \ell, b), \text{ where } \sigma \geqslant 0,\ b \in \mathbb{R} \text{ and } \ell \text{ is a } \textbf{Lévy measure}.$$

By definition, a Lévy measure is a σ-finite Borel (possibly with atoms) measure ℓ on $\mathbb{R}$ with

$$\ell(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} t^2 \wedge 1 d\ell(t) < \infty.$$

According to the Lévy-Khintchine Representation, a random variable W has infinitely divisible distribution associated with a triplet (σ, ℓ, b) if its characteristic function takes the form

$$\mathbb{E}(\exp(itW)) = \exp\left(-\frac{t^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx \cdot 1_{[-1,+1]}(x)\right) d\ell(x) + itb\right).$$

Thus, every infinitely divisible distribution is the convolution of a centred Gaussian distribution with variance σ^2, and another infinitely divisible distribution that is determined by a Lévy measure ℓ and a constant b via the Lévy-Khintchine Representation. An infinitely divisible distribution for which the Gaussian part vanishes, namely $\sigma = 0$, is referred to as **infinitely divisible Poissonian** distribution, henceforth IDp. The fundamental examples of IDp distributions are Poisson distributions and, more generally, compound Poisson distributions. We refer to a random variable as IDp if its distribution is IDp.

Given a generative Poisson point process $\mathcal{P}$, the restriction of the Poisson stochastic integral in the aforementioned chaos decomposition to the first chaos $H_1(\mathcal{P})$, forms an isometric isomorphism of the Hilbert spaces

$$I_\mu : L^2_{\mathbb{R}}(X, \mathcal{B}, \mu) \xrightarrow{\sim} H_1(\mathcal{P}).$$

We introduce some useful properties of this stochastic integral.

Proposition A.1. *In the above setting the following hold.*

(1) *For every $f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$,*
$$I_\mu(f) = \int_X f d\mathcal{P} - \int_X f d\mu.$$
(2) *For every $f \in L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, $I_\mu(f)$ is an* IDp *random variable whose Lévy measure is given by*
$$\ell_f := \mu\mid_{\{f\neq 0\}} \circ f^{-1}.$$
(3) *For every $f \in L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ for which $I_\mu(f)$ is bounded from below,*
$$\ell_f(\mathbb{R}_{<0}) = 0 \text{ and } \int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t) < \infty.$$

Proof. The stochastic integral I_μ is generally defined on the dense subspace $L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ as in part (2), and extends to $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ by continuity. It is a classical fact that if $f \in L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ then $I_\mu(f)$ is an IDp random variable and its Lévy measure is ℓ_f as in part (1) (see e.g. [39, Lemma 20.6], [34, Proposition 2.10] and note that on the dense subspace $L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$ the stochastic integral I_μ differs from the stochastic integrals in these references only by a constant, whence they have the same Lévy measure). This establishes parts (1) and (2).

In order to establish part (3) we exploit the general characterization of IDp random variables that are bounded from below as in [39, Theorem 24.7], by which if $I_\mu(f)$ is bounded from below then its Lévy measure ℓ_f satisfies the following two properties. First, $\ell_f(\mathbb{R}_{<0}) = 0$ as in the first property in part (3). Second, one of the following alternatives occurs (*type A* or *type B* in the terminology of [39, Definition 11.9]):

(1) $\ell(\mathbb{R}_{\geqslant 0}) < \infty$, in which case by the Cauchy-Schwartz inequality
$$\int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t) = \int_{\{f>0\}} f d\mu \leqslant \ell_f(\mathbb{R}_{\geqslant 0}) \left(\int_{\{f>0\}} f^2 d\mu\right) < \infty.$$
(2) $\ell_f(\mathbb{R}_{\geqslant 0}) = \infty$ and $\int_{\{0\leqslant t\leqslant 1\}} t d\ell_f(t) < \infty$, in which case we have
$$\int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t) = \int_{\{0\leqslant t\leqslant 1\}} t d\ell_f(t) + \int_{\{t>1\}} t d\ell_f(t) < \infty,$$
where the finiteness of the second term follows from
$$\int_{\{t>1\}} t d\ell_f(t) = \int_{\{f>1\}} f d\mu \leqslant \int_{\{f>1\}} f^2 d\mu < \infty.$$

This completes the proof of part (3). ∎

Appendix B. Ergodicity of Maharam Extensions

A useful criteria for the ergodicity of the Maharam Extension is based on Krieger's theory of orbit equivalence classification of nonsingular transformations (i.e. nonsingular actions of $\mathbb{Z}$). While this theory, by its nature, applies to countable amenable groups, by reviewing its details one may derive a general criterion to the ergodicity of Maharam Extensions for actions of general groups. In the following discussion, an action of a general group G is a homomorphism from G into $\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)$, and the measurability or continuity of this homomorphism are irrelevant.

Let G be a group. Suppose we are given a nonsingular action of G on a standard measure space $(X,\mathcal{B},\mu)$, that is a homomorphism

$$\mathbf{T}:G\to\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right),\quad \mathbf{T}:(g,x)\mapsto T_g\left(x\right).$$

Denote its Radon-Nikodym cocycle by

$$\nabla_g\left(\cdot\right)=\frac{d\mu\circ T_g}{d\mu}\left(\cdot\right)\in L^1\left(X,\mathcal{B},\mu\right),\quad g\in G.$$

Let η be the measure on $\mathbb{R}$ that is defined by $d\eta\left(t\right)=e^t dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$, and put

$$\widetilde{X}=X\times\mathbb{R},\ \widetilde{\mathcal{B}}=\mathcal{B}\otimes\mathcal{B}\left(\mathbb{R}\right)\ \text{and}\ \widetilde{\mu}=\mu\otimes\eta.$$

The Maharam Extension of $\mathbf{T}$ is the infinite measure preserving action

$$\widetilde{\mathbf{T}}:G\to\mathrm{Aut}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right),\quad \widetilde{T}_g:(x,t)\mapsto\left(T_g\left(x\right),t-\log\nabla_g\left(x\right)\right).$$

A number $s\in\mathbb{R}$ is said to be an **essential value** of $\mathbf{T}$ if:

> Given any $A\in\mathcal{B}$ with $\mu\left(A\right)>0$ and any $\epsilon>0$, there exist $A\supset B\in\mathcal{B}$ with $\mu\left(B\right)>0$ and some $g\in G$, such that $T_g\left(B\right)\subset A$ and $\log\nabla_g\left(B\right)\subset\left(s-\epsilon,s+\epsilon\right)$.

The set of all essential values of $\mathbf{T}$ is called the **ratio set** and is denoted by

$$\mathrm{r}\left(\mathbf{T},\mu\right),\ \text{or also}\ \mathrm{r}\left(\mathbf{T}\right)\ \text{when}\ \mu\ \text{is understood.}$$

The ratio set is a closed subgroup of $\mathbb{R}$ depending only on the measure class of μ, and as such it is a principle invariant in Krieger's theory [40, § 3].

Proposition B.1 (Schmidt)**.** *Suppose* $\mathbf{T}$ *is an ergodic nonsingular action. If* $\mathrm{r}\left(\mathbf{T}\right)=\mathbb{R}$ *then the Maharam Extension* $\widetilde{\mathbf{T}}$ *is ergodic.*

Proof. Let $A\in\widetilde{\mathcal{B}}$ be a $\widetilde{\mathbf{T}}$-invariant set. Denote by $\left(S_s\right)_{s\in\mathbb{R}}$ the flow on $\widetilde{X}$ given by $S_s\left(x,t\right)=\left(x,t+s\right)$. Since $\mathrm{r}\left(\mathbf{T}\right)=\mathbb{R}$, by [40, Theorem 5.2][6] it

[6]Schmidt's theorem is formulated when G is countable and η is the Lebesgue measure. The proof of the part that is being used here remains valid for every group. Also, since η

follows that

$$\widetilde{\mu}\left(A\triangle S_s\left(A\right)\right)=0\text{ for every }s\in\mathbb{R},$$

so that A is $\left(S_s\right)_{s\in\mathbb{R}}$-invariant. By the Fubini Theorem there are $A_1\in\mathcal{B}$ and $A_2\in\mathcal{B}\left(\mathbb{R}\right)$ such that $\widetilde{\mu}\left(A\triangle\left(A_1\times A_2\right)\right)=0$ and A_2 is invariant to all the translations of $\mathbb{R}$, hence either $\eta\left(A_2\right)=0$ or $\eta\left(\mathbb{R}\backslash A_2\right)=0$. Since A is $\widetilde{\mathbf{T}}$-invariant it follows that A_1 is $\mathbf{T}$-invariant, and from the ergodicity of $\mathbf{T}$ it follows that either $\mu\left(A_1\right)=0$ or $\mu\left(X\backslash A_1\right)=0$. Thus, either $\widetilde{\mu}\left(A\right)=0$ or $\widetilde{\mu}\left(\left(X\times\mathbb{R}\right)\backslash A\right)=0$, completing the proof. $\square$

Proposition B.2. *Let M be a paracompact smooth manifold with a volume form* Vol*, let $1\leqslant r\leqslant\infty$, and denote by $\mathbf{D}:\mathrm{Diff}^r\left(M\right)\to\mathrm{Aut}\left(M,\left[\mathrm{Vol}\right]\right)$ the tautological nonsingular action. Then the Maharam Extension $\widetilde{\mathbf{D}}:\mathrm{Diff}^r\left(M\right)\to\mathrm{Aut}\left(\widetilde{M},\widetilde{\mathrm{Vol}}\right)$ is ergodic.*

Remark B.3. While this proposition requires a proof which turns to be somehow technical, it should be regarded as easy. In fact, in common cases much more is known: when M is compact and the dimension is either $d=1$ (Katznelson) or $d\geqslant3$ (Herman), there exists a single diffeomorphism whose Maharam Extension is ergodic. See the introduction of [17, §9] and the references therein.

Proof. Note that $\mathbf{D}$ acts transitively on M (for $d=1$ it is trivial, and for $d\geqslant2$ see [2, Lemma 2.1.10]). Then every function $\phi\in L^1\left(M,\mathrm{Vol}\right)$ that satisfies $\phi\circ f=\phi$ for every $f\in\mathrm{Diff}^r\left(M\right)$ is constant, thus $\mathbf{D}$ is ergodic. Then applying Proposition B.1, it suffices to show that $\mathrm{r}\left(\mathbf{D},\mathrm{Vol}\right)=\mathbb{R}$. Since the defining property of essential values is local in nature, it is enough to verify that on every neighborhood in M, every number is an essential value by diffeomorphisms that are supported on this neighborhood. Let $\varphi:U\to\varphi\left(U\right)\subset\mathbb{R}^d$ be any local chart for some relatively compact neighborhood $U\subset M$, and denote by $\mathrm{Diff}^r\left(U\right)$ those diffeomorphisms $f\in\mathrm{Diff}^r\left(M\right)$ for which $\mathrm{Supp}\left(f\right)\subset U$. Denote by $\varphi_*\mathrm{Vol}$ the push-forward of Vol via φ from U to $\varphi\left(U\right)$. Then with every $f\in\mathrm{Diff}^r\left(\varphi\left(U\right)\right)$, letting $f^{\varphi}:=\varphi^{-1}\circ f\circ\varphi\in\mathrm{Diff}^r\left(U\right)$ we have the relation

$$\frac{d\varphi_*\mathrm{Vol}\circ f}{d\varphi_*\mathrm{Vol}}\circ\varphi=\frac{d\mathrm{Vol}\circ f^{\varphi}}{d\mathrm{Vol}}.$$

Since $\varphi_*\mathrm{Vol}$ is mutually absolutely continuous with the Lebesgue measure on $\varphi\left(U\right)$, which we will abbreviate generally by λ, it follows that

$$\mathrm{r}\left(\mathrm{Diff}^r\left(U\right),\mathrm{Vol}\right)=\mathrm{r}\left(\mathrm{Diff}^r\left(\varphi\left(U\right)\right),\varphi_*\mathrm{Vol}\right)=\mathrm{r}\left(\mathrm{Diff}^r\left(\varphi\left(U\right)\right),\lambda\right),$$

is mutually absolutely continuous with the Lebesgue measure, the choice between those measures does not affect the ergodicity of the Maharam extension.

where the former ratio set is the one of the tautological action of $\mathrm{Diff}^r(U) \subset \mathrm{Diff}^r(M)$ on U, and the latter ratio set is the one of the tautological action of $\mathrm{Diff}^r(\varphi(U)) \subset \mathrm{Diff}^r(\mathbb{R}^d)$ on $\varphi(U)$. Thus, it suffices to show that $\mathrm{r}(\mathrm{Diff}^r(\Omega), \lambda) = \mathbb{R}$ for every open domain $\Omega \subset \mathbb{R}^d$ equipped with the Lebesgue measure λ and the nonsingular tautological action of $\mathrm{Diff}^r(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be an open domain and let $s \in \mathbb{R}$ be arbitrary. It is typically hard to compute essential values on a general Borel set, so it is a common practice to compute essential values on Borel sets in a certain generating collection and then using approximation (see e.g. [8, Fact 2.7]). In our case of the Euclidean space $\mathbb{R}^d$ we work with the collection of open cubes. Thus, to show that $s \in \mathrm{r}(\mathrm{Diff}^r(\Omega), \lambda)$ it suffices to verify the following property.

> For every open cube $C \subset \Omega$ and every $\epsilon > 0$, there is a Borel set $C_0 \subset C$ and a diffeomorphism $f_0 \in \mathrm{Diff}^r(\Omega)$, such that

$$\lambda(C_0) \geqslant e^{-|s|}\lambda(C), \ f_0(C_0) \subset C \text{ and } \log \nabla_{f_0}(C_0) \subset (s-\epsilon, s+\epsilon).$$

(The cost of considering only cubes is that not only C_0 is of positive measure, but further the ratio of the volume of C_0 and of C stays away from zero). Indeed, given an open cube $C = C_p(u_0)$ centred at $u_0 \in \Omega$ with side length $p > 0$ (the ϵ is irrelevant as we will see), let $C_0 = C_{p/\left(e^{|s|/d}\right)}(u_0)$ and let $f_0 \in \mathrm{Diff}^r(\Omega)$ be any diffeomorphism such that

$$f_0(u) = e^{s/d}(u - u_0) + u_0 \text{ for every } u \in C_0.$$

It is then evident that

$$\lambda(C_0) = \left(p/e^{|s|/d}\right)^d = e^{-|s|}\lambda(C), \ f_0(C_0) \subset C \text{ and } \log \nabla_{f_0}|_{C_0} \equiv s. \quad \square$$

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