THE COMPLEMENT OF TROPICAL CURVES IN MODERATE POSITION ON TROPICAL SURFACES

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Abstract. López de Medrano, Rincón and Shaw defined the Chern classes on tropical manifolds as an extension of their theory of the Chern–Schwartz–MacPherson cycles on matroids. This makes it possible to define the Riemann–Roch number of tropical Cartier divisors on compact tropical manifolds. In this paper, we introduce the notion of a moderate position, and discuss a conjecture that the Riemann–Roch number $\mathrm{RR}(X;D)$ of a tropical submanifold D of codimension 1 in moderate position on a compact tropical manifold X is equal to the topological Euler characteristic of the complement $X \setminus D$. In particular, we prove it and its generalization when $\dim X = 2$ and X admits a Delzant face structure.

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1. Introduction

1.1. **Background.** López de Medrano, Rincón and Shaw defined the Chern classes and the Todd classes on tropical manifolds in [LdMRS23] as an extension of their theory of the Chern–Schwartz–MacPherson cycles on matroids [LdMRS20]. In particular, this makes it possible to define the Riemann–Roch number $\mathrm{RR}(X;D)$ for any (sedentarity-0) tropical Cartier divisor D on a compact (purely) n-dimensional tropical manifold X;

$$\mathrm{RR}(X;D) := \int_X \mathrm{ch}(\mathcal{L}(D))\,\mathrm{td}(X) \tag{1.1}$$

where $\mathcal{L}(D)$ is the tropical line bundle associated with D, $\mathrm{ch}(\mathcal{L}(D)) := \sum_{i=0}^{\infty} \frac{c_1(\mathcal{L}(D))^i}{i!}$ is the Chern character of $\mathcal{L}(D)$ and $\int_X$ is the trace map of X. We can extend the Riemann–Roch number for tropical cycles of codimension 1 by the Poincaré duality of tropical cohomology (Definition 3.9). In algebraic geometry, the Riemann–Roch number $\mathrm{RR}(X;D)$ of a given Cartier divisor D on a smooth projective variety corresponds to the Euler characteristic of the sheaf cohomology of the invertible sheaf associated with D by the Hirzebruch–Riemann–Roch theorem or the Grothendieck–Riemann–Roch theorem.

In tropical geometry, invertible sheaves, particularly the structure sheaf on tropical manifolds (e.g. [MZ14, §1]) do not form sheaves of Abelian groups. Consequently, there is no *direct* analog of the Euler characteristic of the sheaf cohomology of tropical Cartier divisors now. (A different tropical analog of the Euler characteristic of line bundles is pursued in [Tsu23] by using ideas of the Strominger–Yau–Zaslow conjecture and microlocal sheaf theory, but this approach is somewhat special.)

The tropical Riemann–Roch theorem for compact tropical curves was proved by both Gathmann–Kerber [GK08] and Mikhalkin–Zharkov [MZ08] independently as a generalization of the Riemann–Roch theorem for finite graphs established by Baker–Norine [BN07]. The generalization of these results and their derivatives have developed in works such as [AC13, Car21, Sum21, BU22, GUZ22]. However, researchers have not yet established the tropical Riemann–Roch theorem for higher-dimensional compact tropical manifolds as a direct generalization of the tropical Riemann–Roch theorem for compact tropical curves by [GK08, MZ08].

Regardless of the specifics, it seems that understanding the tropical geometric meaning of $\mathrm{RR}(X;D)$ is important. At least, there exists a common interpretation for a geometric meaning of the Riemann–Roch number $\mathrm{RR}(X;0)$ for the trivial divisor. In [LdMRS23, Conjecture 6.13], López de Medrano, Rincón and Shaw conjectured that $\mathrm{RR}(X;0)$ for a compact tropical manifold X is equal to the topological Euler characteristic $\chi(X)$ of X, and they proved it when X is a compact tropical surface admitting a Delzant face structure [LdMRS23, Theorem 6.3]. They also proved a sufficiently broad range of tropical surfaces admitting Delzant face structures [LdMRS23, Corollary 6.11]. (The tropical Noether formula was studied in [Sha15] previously. We also note a study of the Noether formula for tropical complexes of dimension 2 in [Car15, Proposition 1.3]). Many researchers expect that the topological Euler characteristic of compact tropical manifolds should be an analog of that of the sheaf cohomology of the structure sheaves on complete smooth algebraic varieties. In fact, the two are highly related via a good degeneration of algebraic varieties [IKMZ19, Corollary 2]. Besides, the Conjecture 6.13 of [LdMRS23] is true for compact integral affine manifolds from Klingler's proof of Chern's conjecture for special affine manifolds [Kli17] since the Todd class of integral affine manifold is trivial.

In this paper, we discuss a geometrical meaning of $\mathrm{RR}(X;D)$ when D is in several non-trivial cases. We mainly consider it when X is a compact tropical surface, but we expect that there exists a generalization for higher-dimensional compact tropical manifolds.

1.2. **Main results.** Firstly, we recall some elementary properties of divisors on algebraic varieties. To clarify the similarities we wish to investigate, we will recall them under strong conditions.

Let D be a nonsingular divisor on a nonsingular algebraic variety X, and a morphism $\iota\colon D \to X$ the closed embedding. Then, there exists the following exact sequence of sheaves:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \iota_*\mathcal{O}_D \to 0. \tag{1.2}$$

From this, we get the following equation for the Euler characteristic of the sheaf cohomology:

$$\chi(X;\mathcal{O}_X(-D)) = \chi(X;\mathcal{O}_X) - \chi(D;\mathcal{O}_D). \tag{1.3}$$

From (1.3), the conjecture below is very natural and seems to be widely believed among many researchers before [LdMRS23], if we mildly ignore a technical condition of tropical submanifolds [LdMRS23, Definition 2.14] (see also [Sha15, Definition 4.3]) and their definition of the Todd class of tropical manifolds.

Conjecture 1.1. Let X be a compact (purely) n-dimensional tropical manifold and D a tropical submanifold of codimension 1 on X [LdMRS23, Definition 2.14]. Then,

$$\mathrm{RR}(X;-D) = \chi(X) - \chi(D) = \chi(H_c^\bullet(X \setminus D;\mathbb{R})) \tag{1.4}$$

where $H_c^\bullet(X \setminus D;\mathbb{R})$ is the cohomology with compact support of $X \setminus D$.

Example 1.2. If $\dim X = 1$, then Conjecture 1.1 is trivial. Let $\dim X = 2$ and K_X the canonical cycle of X [Mik06, Definition 5.8]. The adjunction formula for tropical submanifold of codimension 1 on X ([Sha15, Theorem 6] or [LdMRS23, Theorem 5.2]) gives

$$\mathrm{RR}(X; -D) = \frac{\deg(D.D + K_X.D)}{2} + \mathrm{RR}(X; 0) = -\chi(D) + \mathrm{RR}(X; 0). \tag{1.5}$$

where $K_X.D$ is the intersection of K_X and D in the meaning of [Sha15]. Therefore, if X admits a Delzant face structure, then Conjecture 1.1 is true by [LdMRS23, Theorem 6.3]. (In Proposition A.2, we will see the compatibility of different definition of intersection numbers which is needed.)

The main conjecture of this paper is the following which gives the geometric meaning of $\mathrm{RR}(X; D)$, i.e., the dual of $\mathrm{RR}(X; -D)$.

Conjecture 1.3. Let D be a tropical submanifold of codimension 1 of an n-dimensional compact tropical manifold X [LdMRS23, Definition 2.14]. If D is in *moderate position* (Definition 2.4) on X, then

$$\mathrm{RR}(X; D) = \sum_{k=0}^{\infty} \chi(|D^k|) = \chi(H^\bullet(X \setminus D; \mathbb{R})) \tag{1.6}$$

where $|D^k|$ is the support of the k-th power of D in X (Notation 3.5).

The first equation of Conjecture 1.3 is a certain tropical analog of [Hir95, §20.6.(14)], and we will explain it in Example 3.13 and Remark 3.14. We believe that other researchers also expect Conjectures 1.1 and 1.3 for the reason that, when $X := \mathbb{T}P^n$, for the tropical hypersurface $D := V(F)$ defined by a d-degree tropical homogeneous smooth polynomial F, Conjectures 1.1 and 1.3 follow as a consequence of facts that are classically known. We will see it in Example 2.9. However, Conjecture 1.3 is more non-trivial than Conjecture 1.1. In fact, many tropical submanifolds are in moderate position, but we can easily construct of tropical submanifolds which are not in moderate position. We also remark that the first equation of Conjecture 1.3 should hold for more general cases.

Example 1.4. Let C be a compact tropical curve and C_{reg} the set of all points whose valency are 2. A point p on C is in moderate position if and only if $p \in C_{\mathrm{reg}}$.

From easy calculation, for any tropical submanifold D of codimension 1, i.e., finite subsets contained in C_{reg}, we get

$$\mathrm{RR}(C; D) = \sharp(D) + \chi(C) = \chi(C \setminus D). \tag{1.7}$$

Therefore, Conjecture 1.3 is true for C. The first equation above also holds when D is not in moderate position.

In Proposition 2.7 and Remark 2.8, we will see other examples for evidence why it seems that Conjecture 1.3 is true. Moreover, we also prove the second equation of Conjecture 1.3 when D is relatively uniform on X in Theorem 2.14. One of interesting points of Conjectures 1.1 and 1.3 is that the complement of a tropical submanifold in a tropical manifold is *not* usually considered an analog of Zariski open subset of algebraic variety, but related with the Riemann–Roch number. For instance, the complement of two points on a tropical elliptic curve $\mathbb{R}/\mathbb{Z}$ is disconnected. This feature is far from that of the analytification of Zariski open subsets of algebraic varieties. (In fact, Mikhalkin defined a tropical analog of the open subset of a given polynomial as the complement of the full graph [Mik06, §3.3] of a tropical polynomial in [Mik06, Remark 3.5 and Example 3.6].) On the other hand, in [NS16, AB19], interesting studies have been conducted on the complements of tropical varieties in $\mathbb{R}^n$ based on motivations different from our paper.

Incidentally, the author arrived at Conjecture 1.3 as a derivation from the study in [Tsu23] in order to construct a concrete method of creating permissible C^∞-divisors on compact tropical manifolds which satisfies [Tsu23, Conjecture 1.2]. We will see relationships between this paper and [Tsu23] in Remark 2.18. The relationships which are remarked in Remark 2.18 suggest that the cohomology of the complement of a tropical submanifold of codimension 1 contains a piece of data of a homological invariant of the line bundle associated with it.

One of main theorem in this paper is the following, and its proof is not difficult:

Theorem 1.5 (Main theorem)**.** Let D be a tropical submanifold of codimension 1 in moderate position on a compact tropical surface X. Then,

$$\chi(X \setminus D) = \frac{\deg((D - K_X).D)}{2} + \chi(X). \tag{1.8}$$

From [LdMRS23, Theorem 6.3] and Theorem 1.5, we get the following corollary:

Corollary 1.6. If X is a compact tropical surface admitting a Delzant face structure, then Conjecture 1.3 is true.

From this point, we will also consider an extension of both Conjectures 1.1 and 1.3. We recall well-known properties in algebraic geometry again. Let X be a nonsingular projective variety, D a nonsingular divisor on X and $\iota\colon D \to X$ the inclusion of D. For any divisor D' on X, we have the short exact sequence:

$$0 \to \mathcal{O}_X(D' - D) \to \mathcal{O}_X(D') \to \mathcal{O}_X(D') \otimes_{\mathcal{O}_X} \iota_*\mathcal{O}_D \to 0. \tag{1.9}$$

From the projection formula for locally free sheaves, we get

$$\chi(X; \mathcal{O}_X(D') \otimes_{\mathcal{O}_X} \iota_*\mathcal{O}_D) = \chi(X; \iota_*(\iota^*\mathcal{O}_X(D') \otimes_{\mathcal{O}_D} \mathcal{O}_D)) = \chi(D; \iota^*\mathcal{O}_X(D')). \tag{1.10}$$

If D' is nonsingular and the intersection $D' \cap D$ is a nonsingular effective divisor on D, then

$$\chi(D; \iota^*\mathcal{O}_X(D')) = \chi(D; \mathcal{O}_D(D' \cap D)). \tag{1.11}$$

It may seem that the assumption for a pair (D, D') above is too strict. However, through the application of the theorem of Bertini, we can select a pair (D, D') of divisors satisfying the condition above and $D_0 \sim D' - D$ for a given divisor D_0 on X. For such a pair, the following equation holds:

$$\chi(X; \mathcal{O}_X(D' - D)) = \chi(X; \mathcal{O}_X(D')) - \chi(D; \mathcal{O}_D(D' \cap D)). \tag{1.12}$$

By combining the observation and Conjecture 1.3, we can also expect the following conjecture:

Conjecture 1.7. Let X be a compact n-dimensional tropical manifold. Let D, D' be tropical submanifolds of codimension 1 on X or empty. Assume D and D' satisfy the following conditions:

1. (1) D' is in moderate position on X.
2. (2) The restrction $D'|_D$ of D' on D is a tropical submanifold of D such that its support is $D' \cap D$.
3. (3) $D' \cap D$ is in moderate position on D.

Then,

$$\mathrm{RR}(X; D' - D) = \sum_{k=0}^{\infty} \chi(|D'^k|) - \sum_{k=0}^{\infty} \chi(|(D'|_D)^k|) = \chi(H^\bullet(X \setminus D', D \setminus D'; \mathbb{R})) \\ (= \chi(X \setminus D') - \chi(D \setminus (D' \cap D))) \tag{1.13}$$

where $H^\bullet(X \setminus D', D \setminus D'; \mathbb{R})$ is the relative cohomology of the pair $(X \setminus D', D \setminus D')$.

Definition 1.8. A pair (D, D') of tropical submanifolds of codimension 1 on a compact tropical manifold X is *in moderate position* if (D, D') satisfies the condition (1)-(3) in Conjecture 1.7.

In this paper, we won't discuss how many pairs of tropical submanifolds of codimension 1 in moderate position exist, but we expect there exists a sufficient number of them. Instead, we will see that we can deduce Conjecture 1.7 from Conjecture 1.3 and a conjecture about the Todd class of tropical manifolds (Conjecture 3.7) in Proposition 3.15. From this observation, we will generalize Theorem 1.5 to Theorem 3.17.

Remark 1.9. If both D and D' is empty, then Conjecture 1.7 is equivalent to [LdMRS23, Conjecture 6.13]. Conjecture 1.7 is equivalent to Conjecture 1.1 when D' is empty and D is not. Conjecture 1.3 is equivalent to Conjecture 1.1 when D is empty and D' is not. Therefore, we can consider Conjecture 1.7 as a generalization of the three conjecture above. Moreover, it is important that the RHS of (1.13) can be considered as a certain homological data. This data is also related with the study in [Tsu23], so we expect Conjecture 1.7 is highly related with homological mirror symmetry.

1.3. **Outline of this paper.** In Section 2, we mainly discuss Conjecture 1.3 and give a proof of Theorem 1.5. In Section 3, we mainly discuss Conjecture 1.7 and Proposition 3.15. We also give a generalization of Theorem 1.5 in Theorem 3.17.

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2. Tropical submanifolds in moderate position

2.1. **Tropical manifolds.** In this subsection, we recall the theory of tropical manifolds from [Sha11, MZ14, MR18, GS23, LdMRS23]. We also recall it from other references if necessary. We mainly follow the sheaf theoretic approach of rational polyhedral spaces in [GS23]. For simplicity, we adopt the definition of tropical manifold in [LdMRS23, Definition 2.3]. Every tropical manifold in the sense of [LdMRS23] induce a structure of a rational polyhedral space naturally, and it is a tropical manifold in the sense of the preprint version [GS19, Definition 6.1] of [GS23]. To distinguish between the subtle differences in the definitions of tropical manifolds according to different papers, similar to [GS23, §6], we will refer to tropical manifolds in the sense of [GS19, Definition 6.1] as *locally matroidal* rational polyhedral spaces.

Notation 2.1. Throughout this paper, $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$. Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space [GS23, Definition 2.2]. The *dimension* $\dim X$ of X is the homological dimension of locally compact Hausdorff spaces (e.g. [Ive86, Chapter III. Definition 9.4]). The *local dimension* of X at $x \in X$ will be denoted by $\dim_x X$ [Ive86, Chapter III. Definition 9.10]. A rational polyhedral space $(X, \mathcal{O}_X^\times)$ is *pure dimensional* if $\dim X$ is finite and $\dim X = \dim_x X$ for any $x \in X$. These definitions are compatible with [MR18, Definition 7.1.1]. We write X_{reg} for the set of points in X such that they have an open neighborhood which is isomorphic to an integral affine manifold [GS23, Definition 2.7] (see [JRS18, §4.1]). See also [KS06, Definition 3] for the sheaf theoretical definition of integral affine manifolds. We write $X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$.

For a given loopless matroid M, let L_M be the *tropical linear space* of M in the sense of [GS23, §2.2]. When we consider L_M as a tropical cycle on a vector space, we say L_M the matroidal tropical cycle associated to M like [LdMRS23]. We write $U_{r,n}$ for the uniform of matroid of rank r over $[n] := \{1, \ldots, n\}$.

In this paper, a rational polyhedral subspace is meant in the following sense (cf. [LdMRS23, Definition 2.14]).

Definition 2.2. Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and Y a subspace of X. A rational polyhedral space $(Y, \mathcal{O}_Y^\times)$ is a *rational polyhedral subspace* of $(X, \mathcal{O}_X^\times)$ if for any $x \in Y$, there exists a chart $\psi \colon U \to V (\subset \mathbb{T}^n)$ of X [GS23, Definition 2.2] such that $x \in U$ and the restriction $\psi|_{U \cap Y} \colon U \cap Y \to \psi(U \cap Y)$ is also a chart of Y.

We note that there exists rational polyhedral spaces $(X, \mathcal{O}_X^\times)$ and $(Y, \mathcal{O}_Y^\times)$ such that $X = Y$ and the identity map of X is a morphism from $(Y, \mathcal{O}_Y^\times)$ to $(X, \mathcal{O}_X^\times)$ but not an isomorphism. (We can construct such an example from a tropical analog of Frobenius morphism.) Every locally polyhedral set of a rational polyhedral space [GS23, Definition 2.4 (d)] has a natural structure of a rational polyhedral subspace.

Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and $\mathrm{LC}_x X$ the local cone of X at x $(\in X)$ [GS23, §2.2]. Follwoing [MR18, Definition 7.1.8] and [LdMRS23, Definition 2.3], an *atlas* of X means a family $\{(U_i, \psi_i)\}_{i \in I}$ of charts $\psi_i \colon U_i \to V_i (\subset \mathbb{T}^{n_i})$ such that $\bigcup_{i \in I} U_i = X$.

Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space which is regular at infinity [GS23, §6.1] (see also [MZ14, Definition 1.2] and [MR18, Definition 7.2.4 and Corollary 7.2.11]). Then, every point x in X has an open neighborhood U_x which is isomorphic to an open subset of $\mathrm{LC}_x X \times \mathbb{T}^{m_x}$ for some $m_x \in \mathbb{Z}_{\geq 0}$. Therefore, every rational polyhedral space which is regular at infinity is paracompact and locally contractible, so the singular cohomology of it is isomorphic to the sheaf cohomology of the constant sheaf on it, and thus we identify the two cohomologies. Moreover, every nonempty rational polyhedral space which is regular at infinity has the sedentarity function $\mathrm{sed}_X \colon X \to \mathbb{Z}$ on X [MR18, Definition 7.2.6] (see also [LdMRS23, Definition 2.4]).

From definition, for any $x \in X$

$$\mathrm{sed}_X(x) + \dim \mathrm{LC}_x X = \dim_x X. \tag{2.1}$$

The sedentarity function sed_X is upper semiconstant [MR18, Definition 7.1.11] so sed_X is upper semicontinuous. In particular, the following subsets are locally polyhedral [MR18, Proposition 7.1.12]:

$$X^{[\geq k]} := \{p \in X \mid \mathrm{sed}_X(p) \geq k\}, \quad X_\infty := X^{[\geq 1]}. \tag{2.2}$$

The polyhedral subspace X_∞ is called the *boundary* of X (e.g. [LdMRS23]), or the *divisor at infinity* of X (e.g. [MR18, Definition 7.2.9]). If X is a tropical toric variety, then X_∞ is a direct analog of toric boundary. A rational polyhedral space $(X, \mathcal{O}_X^\times)$ which is regular at infinity is *locally matroidal* if $\mathrm{LC}_x X \simeq L_M$ for some loopless matroid M [GS23, §6]. When a data $(X, \{\psi_\alpha \colon U_\alpha \to \Omega_\alpha \subset X_\alpha\}_{\alpha \in \mathcal{I}})$ is a tropical manifold (in the sense of [LdMRS23, Definition 2.3]), then the associated rational polyhedral space is locally matroidal.

Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and $(Y, \mathcal{O}_Y^\times)$ a rational polyhedral subspace. For every $x \in Y$, the inclusion map $\iota \colon Y \to X$ induces an injection $\iota_{*,x} \colon \mathrm{LC}_x Y \to \mathrm{LC}_x X$ from the local cone $\mathrm{LC}_x Y$ of Y at x to that of X at x. As long as it does not lead to confusion, we will identify $\mathrm{LC}_x Y$ with $\iota_{*,x}(\mathrm{LC}_x Y)$, and consider $T_x Y$ as a subspace of $T_x X$. We set

$$\operatorname{codim}(Y/X) \coloneqq \dim X - \dim Y, \quad \operatorname{codim}_x(Y/X) \coloneqq \dim_x X - \dim_x Y. \tag{2.3}$$

The rational polyhedral subspace Y of X is a rational polyhedral space of *codimension* d if Y and X are pure dimensional and $\operatorname{codim}(Y/X) = d$.

Let $(X, \mathcal{O}_X^\times)$ and $(Y, \mathcal{O}_Y^\times)$ be rational polyhedral spaces which is regular at infinity and assume that $(Y, \mathcal{O}_Y^\times)$ is a rational polyhedral subspace of $(X, \mathcal{O}_X^\times)$. Then, the following equations and inequalities hold for all $x \in Y$:

$$\mathrm{sed}_X(x) - \mathrm{sed}_Y(x) = \operatorname{codim}_x(Y/X) - \operatorname{codim}_0(\mathrm{LC}_x Y / \mathrm{LC}_x X), \tag{2.4}$$

$$\operatorname{codim}_x(Y/X) \geq \mathrm{sed}_X(x) - \mathrm{sed}_Y(x) \geq 0. \tag{2.5}$$

In particular, if $\operatorname{codim}_x(Y/X) = 1$, then $\mathrm{sed}_X(x) - \mathrm{sed}_Y(x) = 0$ or 1. Following [LdMRS23, §2.5], the rational polyhedral subspace Y of X is *sedentarity*-0 if $\mathrm{sed}_X(x) = \mathrm{sed}_Y(x)$ for all $x \in Y$.

An injective morphism $f \colon Y \to X$ of rational polyhedral spaces is *locally matroidal* if both X and Y are locally matroidal and for any inclusion $\mathrm{LC}_x Y \subset \mathrm{LC}_x X$ of the local cones at x comes from the inclusion $L_M \subset L_N$ induced from some two matroids M, N with the common ground set (see also [FR13, §3] or [Sha13, §2.4]). If X is a tropical manifold and Y is a tropical submanifold of X [LdMRS23, Definition 2.14], then the inclusion map $Y \hookrightarrow X$ is locally matroidal. We note that a codimension 1 tropical submanifold of a given tropical manifold X is essentially same with a locally degree 1 divisor on X [Sha15, Definition 4.3]. (We thank Kris Shaw for answering our question about this). As stressed in [LdMRS23, Example 2.15], there exist examples such that Y is a tropical manifold and a rational polyhedral subspace of another tropical manifold X, but Y is not a tropical submanifold of X. We can see such examples in [Vig10, BS15, Sha15]. The support of every tropical cycle in a given rational polyhedral space is a closed subset of it, and thus every tropical submanifold of a given tropical manifold X is a closed subset of X.

2.2. **Moderate position.** Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space. For $x \in X$, let $\mathrm{lineal}(X, x)$ be the (maximal) lineality space $\mathrm{lineal}(\mathrm{LC}_x X)$ of $\mathrm{LC}_x X (\subset T_x X)$ [GS21, §2.1] (see also [LdMRS23, §3]).

Remark 2.3. The lineality space of the local cone $\mathrm{LC}_x X$ at a point x in a rational polyhedral space $(X, \mathcal{O}_X^\times)$ in the sense of [GS21, §2.1] is equivalent to the maximal lineality space of $\mathrm{LC}_x X$ in [FR13, §5]. We can check about this as follows: Since every conical rational polyhedral set P has an isomorphism $P \simeq P/\mathrm{lineal}(P) \times \mathrm{lineal}(P)$, we may assume $\mathrm{lineal}(P)$ is trivial. Therefore, to see that the two coincide, it is sufficient to observe that conical rational polyhedral sets always possess a fan structure. We can give a proof of it like that of the existence theorem of a triangulation of compact convex polyhedron in $\mathbb{R}^n$ (cf. [RS82, Theorem 2.11]). From definition, every conical rational polyhedral set P is a finite union $\bigcup_{i\in I}\sigma_i$ of rational polyhedral cones σ_i. Besides, we may assume every σ_i is strongly convex. For every $i \in I$, there exists a complete fan Σ_i on $\mathbb{R}^n$ such that $\sigma_i \in \Sigma_i$. (We can prove it by an application of Sumihiro's compactification theorem for toric varieties [Sum74, Theorem 3].) The refinement $\bigwedge_{i\in I}\Sigma_i$ of a family $\{\Sigma_i\}_{i\in I}$ of fans gives a fan structure of P.

Definition 2.4. Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and $(Y, \mathcal{O}_Y^\times)$ a rational polyhedral subspace of X. The rational polyhedral subspace Y is in *moderate position* on X if for any $x \in Y$

$$\operatorname{lineal}(Y, x) \subsetneq \operatorname{lineal}(X, x). \tag{2.6}$$

We remark that the complement $X \setminus Y$ of tropical submanifold Y in moderate position on X does not usually satisfy the condition of finite type [MR18, Definition 7.1.14 (c)].

Example 2.5. We retain the notation of Definition 2.4.

(1) If Y is in moderate position on X, then $Y \cap \{x \in X \mid \operatorname{lineal}(X, x) = \{0\}\} = \varnothing$.
(2) The rational polyhedral subspace $Y \cap X_{\mathrm{reg}}$ of X_{reg} is in moderate position on X_{reg}. In particular, Y is always in moderate position when X is an integral affine manifold.

If Y is in moderate position, then the inclusion $\mathrm{LC}_x Y \to \mathrm{LC}_x X$ induces the following exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{lineal}(Y,x) & \longrightarrow & T_x Y & \longrightarrow & T_x Y/\operatorname{lineal}(Y,x) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{lineal}(X,x) & \longrightarrow & T_x X & \longrightarrow & T_x X/\operatorname{lineal}(X,x) & \longrightarrow & 0
\end{array}.$$

Proposition 2.6. Let X be a tropical manifold and Y a tropical submanifold of codimension 1 in moderate position on X. Then, Y is a sedentarity-0 submanifold of X.

Proof. If Y is in moderate position, then $\mathrm{LC}_x Y \not\simeq \mathrm{LC}_x X$ for all $x \in Y$. On the other hand, if $x \in Y$ satisfies $\operatorname{sed}_X(x) - \operatorname{sed}_Y(x) = 1$, then $\dim \mathrm{LC}_x Y = \dim \mathrm{LC}_x X$, and thus $\mathrm{LC}_x Y \simeq \mathrm{LC}_x X$ [FR13, Lemma 2.4]. Therefore, $\operatorname{sed}_X(x) - \operatorname{sed}_Y(x) = 0$ when Y is in moderate position. $\square$

Proposition 2.7. Let X be a purely n-dimensional compact integral affine manifold and D a tropical submanifold of codimension 1. Then, Conjecture 1.1 is equivalent to Conjecture 1.3. In particular, Conjecture 1.3 is true for all compact integral affine manifolds when [LdMRS23, Conjecture 6.13] is true for any compact tropical manifold.

Proof. Since X is an integral affine manifold, D is always in moderate position in X. Since $\operatorname{td}(X) = 1$ (see Definition 3.8), we have

$$\mathrm{RR}(X; -D) = (-1)^{\dim X}\,\mathrm{RR}(X; D). \tag{2.7}$$

Besides, $X \setminus D$ is a topological manifold, and thus we have the following equation from the Poincaré duality:

$$\chi_c(X \setminus D) = (-1)^{\dim X} \chi(X \setminus D) \tag{2.8}$$

where $\chi_c(X \setminus D) := \chi(H_c^\bullet(X \setminus D; \mathbb{R}))$. Therefore, Proposition 2.7 follows from (2.7) and (2.8). □

Remark 2.8 (Integral affine manifolds with singularities)**.** We expect that we can generalize Conjecture 1.3 and Proposition 2.7 for integral affine manifold with singularities (e.g. see [GS06, KS06, Rud21]).

Normally, integral affine manifolds with singularities are analogs of complex manifolds whose canonical bundles are numerically trivial. If M is a compact and connected complex manifold whose canonical bundle K_M is numerically trivial, then $c_1(K_M)$ is a torsion, i.e., $l \cdot c_1(K_M) = 0$ for some $l \in \mathbb{Z}_{>0}$. From the Atiyah–Singer index formula and the Serre duality, for any divisor D on M, we have

$$\chi(M; \mathcal{O}_M(-D)) = (-1)^{\dim M} \chi(M; \mathcal{O}_M(D)). \tag{2.9}$$

The equation (2.9) is similar with Proposition 2.7, and one of essential points of the proof of it is that $X \setminus D$ is a topological manifold. Therefore, we expect that we can generalize Conjectures 1.3 and 1.7 for integral affine manifolds with singularities and this description is compatible with tropical contractions from tropical manifolds to integral affine manifold with singularities [Yam21].

Example 2.9 ([LdMRS23, Example 2.11])**.** Let e_i the i-th coordinate vector of $\mathbb{R}^n$ and Δ_n be the standard n-simplex generated by e_i $(i = 1, \ldots, n)$ and the origin of $\mathbb{R}^n$. The normal fan Σ of Δ_n induces a compactification X_Σ of $\mathbb{R}^n$. The space X_Σ is a typical example of tropical toric varieties [Kaj08, Pay09] and X_Σ is isomorphic to the tropical projective n-space $\mathbb{T}P^n$ [Mik06, Example 3.10]. From a direct calculation of the Čech cohomology of $\mathcal{O}_{X_\Sigma}^\times$, we get $H^1(X_\Sigma; \mathcal{O}_{X_\Sigma}^\times) \simeq H^{1,1}(X_\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$. In general, the Picard group of toric schemes over semifields is studied in [JMT19]. (See also [Mey11] and [MR18, Chapter 3] for more detail about tropical toric varieties.) Let f be a Laurent polynomial on $\mathbb{R}^n$ whose Newton polytope is $d\Delta_n$. The closure $\overline{V_\mathbb{T}(f)}$ of $V_\mathbb{T}(f)$ in X_Σ is can be considered as an analog of a hypersurface of projective n-space of a homogeneous polynomial of degree d (see also [MR18, Definition 3.4.6]). The degree of $\overline{V_\mathbb{T}(f)}$ on $\mathbb{T}P^n$ is equal to that of f and the first Chern class of the line bundle $\mathcal{L}(\overline{V_\mathbb{T}(f)})$. The tropical hypersurface $V_\mathbb{T}(f)$ of f is *smooth* (in the sense of [MS15, §4.5]) if the regular subdivision of f is unimodular. If the regular subdivision of f is unimodular, then $\overline{V_\mathbb{T}(f)}$ is a tropical manifold. Let $d\Delta_n(\mathbb{Z})$ be the set of lattice points in $d\Delta_n$, and $\mathrm{int}(d\Delta_n)(\mathbb{Z})$ the set of lattice points in the (relative) interior of $d\Delta_n$. It is well-known that the complement $\mathbb{T}P^n \setminus \overline{V_\mathbb{T}(f)}$ is homotopy equivalent to $d\Delta_n(\mathbb{Z})$ and $\overline{V_\mathbb{T}(f)}$ is homotopy equivalent to the $\sharp(\mathrm{int}(d\Delta_n)(\mathbb{Z}))$-th bouquet of $(n-1)$-spheres (e.g. [MS15, Proposition 3.1.6] or [MR18, Proposition 3.4.12]). In particular, every connected component of $\mathbb{T}P^n \setminus \overline{V_\mathbb{T}(f)}$ is a locally closed polyhedron in $\mathbb{T}P^n$, so it is contractible. Moreover, the definition of Chern classes of tropical toric manifolds is compatible with that of algebraic toric manifolds (cf. [CLS11, Proposition 13.1.2]). Therefore, the Todd class of $\mathbb{T}P^n$ also has the same representation of that of algebraic toric manifolds (see also [CLS11, Theorem 13.1.6]).

Therefore, we have

$$\mathrm{RR}(\mathbb{T}P^n; \overline{V_{\mathbb{T}}(f)}) = \sharp d\Delta_n(\mathbb{Z}) = \chi(\mathbb{T}P^n \setminus \overline{V_{\mathbb{T}}(f)}). \tag{2.10}$$

We can also get the following equation similarly:

$$\mathrm{RR}(\mathbb{T}P^n; -\overline{V_{\mathbb{T}}(f)}) = (-1)^n \sharp \operatorname{int}(d\Delta_n(\mathbb{Z})) = \chi_c(\mathbb{T}P^n \setminus \overline{V_{\mathbb{T}}(f)}). \tag{2.11}$$

The second equation of (2.11) follows from that $\chi_c(\mathbb{T}P^n \setminus \overline{V_{\mathbb{T}}(f)}) = 1 - \chi(\overline{V_{\mathbb{T}}(f)})$ and $\overline{V_{\mathbb{T}}(f)}$ is homotopy equivalent to the $\sharp \operatorname{int}(d\Delta_n(\mathbb{Z}))$-th bouquet of $(n-1)$-dimensional spheres.

2.3. **Relatively uniform tropical submanifolds.** In this subsection, we define a good class of tropical submanifolds in moderate position and prove the second equation of Conjecture 1.3 for this case in Theorem 2.14.

Definition 2.10. Let X be an n-dimensional tropical manifold X and $\iota\colon D \to X$ is the embedding map of a codimension-1 and sedentarity-0 tropical submanifold D on X. The tropical submanifold D of X is *relatively uniform* on X if, for any $x \in D$, the pushforward morphism $\iota_{*x}\colon \mathrm{LC}_x D \to \mathrm{LC}_x X$ is isomorphic to

$$\mathrm{id}_{L_M} \times i\colon L_M \times L_{U_{r,r+1}} \to L_M \times L_{U_{r+1,r+1}} \tag{2.12}$$

where M is a loopless matroid, r is a positive integer and i is the inclusion map induced from the inclusion $U_{r,r+1} \subset U_{r+1,r+1}$.

We note that (2.12) itself can be represented by a matroidal map induced from the parallel connection of matroids (see [GS21, Lemma 3.1] or [LdMRS23, Proposition 3.7]). We don't know whether there exists a tropical submanifold in moderate position but not relatively uniform, so Definition 2.10 may be equivalent to Definition 2.4.

In the following proposition, we use Notation 3.5 but it is elementary.

Proposition 2.11. Let X be a purely n-dimensional tropical manifold and D a relatively uniform tropical submanifold of codimension 1 on X. Then, for any $x \in |D^j|$ for some $j \in \mathbb{Z}_{>0}$ there exists the following isomorphism:

$$\mathrm{LC}_x\, |D^j| \simeq L_{M_x} \times L_{U_{r-j+1,r+1}} \tag{2.13}$$

where M_x is a loopless matroid which is independent of the choice of j.

Proof. By direct calculation, we have $|(L_{U_{r,r+1}})^j| = L_{U_{r-j+1,r+1}}$ (see [AR10, Example 3.9]). The proposition is local, so we may assume $X = \mathbb{T}^m \times L_M \times L_{U_{r+1,r+1}}$ and $D = \mathbb{T}^m \times L_M \times L_{U_{r,r+1}}$ for some loopless matroid M. Let $\pi_{|D^j|}\colon |D^j| \to L_{U_{r-j+1,r+1}}$ be the projection. Then, $D = \pi_X^* L_{U_{r,r+1}}$ and $D|_D = \pi_D^*(L_{U_{r,r+1}}|_{L_{U_{r,r+1}}}) = \pi_D^* L_{U_{r-1,r+1}} = \mathbb{T}^m \times L_M \times L_{U_{r-1,r+1}}$. By repeating this, we obtain (2.13). □

The following proposition is also elementary, but important.

Proposition 2.12. Let $X := L_{U_{n+1,n+1}} \simeq \mathbb{R}^n$ and $D := L_{U_{n,n+1}}$. For $U_i \in \pi_0(X \setminus D)$, we write $\overline{U_i}$ the closure of U_i in X and $(X \setminus D)^\dagger := \bigsqcup_{\pi_0(X \setminus D)} \overline{U_i}$. Then, the canonical inclusion $i\colon X \setminus D \to (X \setminus D)^\dagger$ is locally aspheric in the sense of [Ogu18, Chapter V. Corollary 1.3.2], and the composition of i with the canonical map $\iota\colon (X \setminus D)^\dagger \to X$ is the inclusion map $j\colon X \setminus D \to X$. In particular,

$$Rj_* \mathbb{R}_{X \setminus D} = \iota_! \mathbb{R}_{(X \setminus D)^\dagger}. \tag{2.14}$$

Remark 2.13. We can generalize the above partial compactification $(X \setminus D)^\dagger$ and Proposition 2.12 when X is a tropical manifold and D is a relatively uniform tropical submanifold of codimension 1 on X naturally. When X and D is compact, then the partial compactification $(X \setminus D)^\dagger$ of $X \setminus D$ is also compact.

Theorem 2.14. *Let X be a purely n-dimensional compact tropical manifold and D a relatively uniform tropical submanifold of codimension 1 on X. Then,*

$$\chi(X \setminus D) = \sum_{k=0}^{\infty} \chi(|D^k|) = \sum_{k=0}^{n} (k+1)\chi_c(|D^k| \setminus |D^{k+1}|). \tag{2.15}$$

Proof. Let $j\colon X \setminus D \to X$ be the inclusion map of $X \setminus D$. Then, we have

$$H^\bullet(X \setminus D; \mathbb{R}_{X\setminus D}) \simeq \mathbb{H}^\bullet(X; Rj_*\mathbb{R}_{X\setminus D}). \tag{2.16}$$

From (2.13), we also get

$$Rj_*\mathbb{R}_{X\setminus D} \simeq j_*\mathbb{R}_{X\setminus D}, \quad \chi((j_*\mathbb{R}_{X\setminus D})_x) = \sum_{k=0}^{\infty} 1_{|D^k|}(x). \tag{2.17}$$

Fix a finite tropical atlas $\mathcal{U}$ for X. From the inclusion-exclusion principle and the Meyer–Vietoris sequence of sheaves (e.g. [Ive86, p.185]), we only need to check

$$\chi(H_c^\bullet(U; (j_*\mathbb{R}_{X\setminus D})|_U)) = \sum_{k=0}^{\infty} \chi_c(|D^k| \cap U). \tag{2.18}$$

for any $(U, \phi) \in \mathcal{U}$. Since $\mathbb{T}^m$ is homeomorphic to $\mathbb{R}^m_{\geq 0}$ via the extended exponential map $\exp\colon \mathbb{T}^m \to \mathbb{R}^m_{\geq 0}$, we may consider every rational polyhedral set in some $\mathbb{T}^m$ as a subanalytic set in $\mathbb{R}^m$. From definition, we can choose an open subset V of $\mathbb{R}^m$ such that $\exp \circ \phi(U)$ is a closed subset of V. The restriction of every rational polyhedral set in $\mathbb{T}^n$ on V is also subanalytic [KS94, Proposition 8.2.2.(iii)] on V. Let $\iota := \exp \circ \phi$. Then, $\iota_*(j_*\mathbb{R}_{X\setminus D})|_U$ is an $\mathbb{R}$-constructible sheaf on V [KS94, Definition 8.4.3] from Proposition 2.12. From [KS94, Theorem 9.7.1] and (2.17), we obtain (2.18). □

Remark 2.15. In the proof of Theorem 2.14, we considered $Rj_*\mathbb{R}_{X\setminus D}$ in order to calculate the Euler characteristic of $X \setminus D$. This method comes from *Euler calculus* [Vir88, Sch91]. Euler calculus is an effective way to calculate the Euler characteristic of the sheaf cohomology of constructible sheaves. We can consider the Euler characteristic of compact supports of locally closed subsets of X as an analog of signed measure, and the RHS of (2.15) as an integration of the following simple function over it [Sch91, (3.4)]:

$$\chi(Rj_*\mathbb{R}_{X\setminus D})(x) := \chi((Rj_*\mathbb{R}_{X\setminus D})_x) \simeq \chi(\varinjlim_{U \ni x} H^\bullet((X \setminus D) \cap U; \mathbb{R})), \quad (x \in X). \tag{2.19}$$

The value $\chi(Rj_*\mathbb{R}_{X\setminus D})(x)$ is called the *local Euler–Poincaré index* of $Rj_*\mathbb{R}_{X\setminus D}$ at x. Euler calculus is also a useful way to investigate other constructible sheaves on tropical manifolds. As stressed in [Rau23, Remark 4.8], we can interpret the first equation of the tropical Poincaré–Hopf theorem [Rau23, Theorem 4.7] as the Euler integration of the total complex $\Omega^\bullet_{\mathbb{Z},X}$ of the sheaves of tropical p-forms.

2.4. **For tropical surfaces.** For the proof of Theorem 1.5, we use the following well-known proposition for Bergman fans: the support of the Bergman fan of a loopless matroid of rank 2 is isomorphic to the support of the Bergman fan of a uniform matroid $U_{2,n}$ on $[n] := \{1, \ldots, n\}$. In fact, we can deduce it from that the supports of the Bergman fans of loopless matroids are isomorphic when the simplification of them are isomorphic. Shaw classified local cones which comes from for some tropical surface in [Sha15, Corollary 2.4].

Proposition 2.16. Let S be a compact tropical surface and C is a tropical submanifold of codimension 1 in moderate position on S. Then,

$$|C^2| = C_{\text{sing}} \cap S_{\text{reg}}. \tag{2.20}$$

Proof. First, we will see the classification of the embeddings $\operatorname{LC}_x C \subset \operatorname{LC}_x S$ of local cones. If C is in moderate position on S, then $\dim \operatorname{lineal}(S, x) = 1, 2$ for any $x \in C$.

(1) Suppose $\dim \operatorname{lineal}(S, x) = 2$. Then, $\operatorname{LC}_x S \simeq \mathbb{R}^2$ and $\operatorname{LC}_x C$ should be isomorphic to $L_{U_{2,3}}$ or $L_{U_{2,2}}$. In particular, $\operatorname{val}_C(x) = 2, 3$ in this case.
(2) We assume $\dim \operatorname{lineal}(S, x) = 1$ and $x \in C \setminus C_\infty$. Then, $\dim \operatorname{lineal}(C, x) = 0$. In particular, $\operatorname{val}_C(x) \geq 3$. From [Sha15, Corollary 2.4], we may assume $\operatorname{LC}_x S \simeq L_{U_{2,m}} \times \mathbb{R} \simeq L_{U_{2,m} \oplus U_{1,1}}$ for some $m (\geq 3)$. From this description, we can check $\operatorname{val}_C(x) \leq m$. In particular, we can consider $T_x C$ as a proper vector subspace of $T_x S$.

 Besides, we can check $\operatorname{LC}_x C \cap \operatorname{lineal}(S, x) = \{0\}$ by indirect proof. In fact, if $\operatorname{LC}_x C \cap \operatorname{lineal}(S, x) \neq \{0\}$, then $L := \operatorname{Im}(T_x C \to T_x S / \operatorname{lineal}(S, x))$ is a proper subspace of $T_x S / \operatorname{lineal}(S, x)$. On the other hand, the convex hull of $L \cap (\operatorname{LC}_x S / \operatorname{lineal}(S, x))$ does not contain any nontrivial vector subspace. This contradicts with $\dim T_x S \geq \dim T_0(\operatorname{LC}_x C) \geq 2$.

 Since $\operatorname{LC}_x C \cap \operatorname{lineal}(S, x) = \{0\}$, $T_x C \cap \operatorname{lineal}(S, x) = \{0\}$. Then, $T_x C \cap \operatorname{LC}_x S \simeq L_{U_{2,m}} \simeq \operatorname{LC}_x C$ from the Balancing condition of $\operatorname{LC}_x C$. In particular, $\operatorname{LC}_x C$ is the intersection cycle of $T_x C$ and $\operatorname{LC}_x S$ in $T_x S$, so the self-intersection cycle of $\operatorname{LC}_x C$ in $\operatorname{LC}_x S$ is trivial.

From the above classification of the embeddings of local cones, we obtain the proof. □

Remark 2.17. We can also prove from [Sha13, Theorem 4.2] or [Sha15, Theorem 4.11] since the intersection pairing in the sense of [GS23] is compatible with that of [Sha15, LdMRS23] (see Proposition A.1).

From now on, we will prove Theorem 1.5. Most of the proof is the same as Theorem 2.14, but for simplicity, we will show it independently of Theorem 2.14.

Proof of Theorem 1.5. Let $j \colon S \setminus C \to S$ be the inclusion map of $S \setminus C$. Then, the following equation holds:

$$\chi(S \setminus C) = \chi(Rj_* \mathbb{R}_{S \setminus C}) := \chi(\mathbb{H}^\bullet(S; Rj_* \mathbb{R}_{S \setminus C})). \tag{2.21}$$

If C is in moderate position on S, then the classification of the local cone of C gives

$$(Rj_* \mathbb{R}_{S \setminus C})_x \simeq \begin{cases} \mathbb{R}[0], & \text{if } x \in S \setminus C, \\ \mathbb{R}^3[0], & \text{if } x \in C_{\text{sing}} \cap S_{\text{reg}}, \\ \mathbb{R}^2[0], & \text{otherwise.} \end{cases} \tag{2.22}$$

In particular, $Rj_*\mathbb{R}_{S\setminus C} \simeq j_*\mathbb{R}_{S\setminus C}$. Since $j^{-1}j_* = \mathrm{id}_{S\setminus C}$, from [KS94, Proposition 2.3.6 (v)] there exists the following short exact sequence:

$$0 \to j_!\mathbb{R}_{S\setminus C} \to j_*\mathbb{R}_{S\setminus C} \to (j_*\mathbb{R}_{S\setminus C})_C \to 0. \tag{2.23}$$

In particular, we have

$$\chi(S \setminus C) = \chi_c(S \setminus C) + \chi((j_*\mathbb{R}_{S\setminus C})|_C). \tag{2.24}$$

From definition of the pushforward of sheaves, $(j_*\mathbb{R}_{S\setminus C})|_C$ is locally constant on C_{reg} and the stalk of it is $\mathbb{R}^2$. Therefore, we also have

$$\chi((j_*\mathbb{R}_{S\setminus C})|_C) = 2\chi_c(C_{\mathrm{reg}}) + 3\chi_c(C_{\mathrm{sing}} \cap S_{\mathrm{reg}}) + 2\chi_c(C_{\mathrm{sing}} \cap S_{\mathrm{sing}}). \tag{2.25}$$

Therefore,

$$\begin{aligned}
\chi(S \setminus C) &= \chi_c(S \setminus C) + 3\chi_c(C_{\mathrm{sing}} \cap S_{\mathrm{reg}}) + 2\chi_c(C \setminus (C_{\mathrm{sing}} \cap S_{\mathrm{reg}})) && (2.26)\\
&= \chi_c(S \setminus C) + 2\chi_c(C) + \chi_c(C_{\mathrm{sing}} \cap S_{\mathrm{reg}})\\
&= \chi(S) + \chi(C) + \chi(|C^2|) && (2.27)\\
&= \frac{\deg(C.(C - K_S))}{2} + \chi(S).
\end{aligned}$$

In the last equation, we use the adjunction formula of tropical curves [Sha15, Theorem 4.11]. □

Proof of Corollary 1.6. It follows from Theorem 1.5 and Proposition A.2. □

Remark 2.18. In this remark, we discuss relationships between Conjecture 1.3 and [Tsu23] in some cases. This remark highly depends on [Tsu23], and you can skip this remark to understand the main theorem of this paper. As explained in the introduction of this paper, the Euler characteristic $\chi_c(X \setminus D)$ of the cohomology of compact support of the complement of a tropical submanifold D on a compact tropical manifold X is an analog of (1.3), so it is natural to expect that Conjecture 1.1 holds. We will explain about algebraic geometrical backgrounds of the first equation of Conjecture 1.3 in Example 3.13 and Remark 3.14. However, the author does not know some theorem in algebraic geometry or Berkovich geometry which suggests the *second* equation of Conjecture 1.3 directly as like Conjecture 1.1, except toric varieties or special cases like them. On the other hand, based on the study in [Tsu23], we can expect that $H^\bullet(X \setminus D)$ is highly related with the theory of Floer cohomology of Lagrangian submanifolds, the homological mirror symmetry and the Strominger–Yau–Zaslow conjecture. Below, we will see about it by using examples. For simplicity, let X be $\mathbb{R}/\mathbb{Z}$ and D a finite subset of X. Then, the following isomorphisms of graded modules hold:

$$H^\bullet(X \setminus D; \mathbb{Z}) \simeq \bigoplus_{E \in \pi_0(X\setminus D)} H^\bullet(E; \mathbb{Z}) \simeq \bigoplus_{E \in \pi_0(X\setminus D)} \mathbb{Z}[0], \tag{2.28}$$

$$H_c^\bullet(X \setminus D; \mathbb{Z}) \simeq \bigoplus_{E \in \pi_0(X\setminus D)} H_c^\bullet(E; \mathbb{Z}) \simeq \bigoplus_{E \in \pi_0(X\setminus D)} \mathbb{Z}[-1]. \tag{2.29}$$

We can also define similar graded modules for C^∞-divisors on X. In [Tsu23], the author proposed the following approach inspired from microlocal sheaf theory and the Strominger–Yau–Zaslow conjecture.

Let $\mathcal{A}_X^{0,0}$ be the sheaf of $(0,0)$-superforms on X [JSS19, Definition 2.24]. Then, the following exact sequence exists:

$$0 \to \mathcal{O}_X^\times \to \mathcal{A}_X^{0,0} \to \mathcal{A}_X^{0,0}/\mathcal{O}_X^\times \to 0. \tag{2.30}$$

Since $\mathcal{A}_X^{0,0}$ is acyclic, the connecting homomorphism

$$H^0(X; \mathcal{A}_X^{0,0}/\mathcal{O}_X^\times) \to H^1(X; \mathcal{O}_X^\times); s \mapsto \mathcal{L}(s). \tag{2.31}$$

is surjective. The group $\mathrm{Div}^\infty(X) := H^0(X; \mathcal{A}_X^{0,0}/\mathcal{O}_X^\times)$ is an analog of the group of Cartier divisors when X is a boundaryless tropical manifold. In [Tsu23], elements in $\mathrm{Div}^\infty(X)$ are called *C^∞-divisors* on X. Let $\mathcal{C}_X^0$ the sheaf of continuous functions on X. Then, the group of tropical Cartier divisors on X and $\mathrm{Div}^\infty(X)$ is in $\Gamma(X; \mathcal{C}_X^0/\mathcal{O}_X^\times)$. We note that the author defined the group of C^∞-divisor for *any* tropical manifold in [Tsu23]. If B is an integral affine manifold, then every C^∞-divisor on B defines a Lagrangian section of the standard Lagrangian torus fibration $\check{f}_B \colon \check{X}(B) \to B$ on B, we can consider C^∞-divisors as derivative objects of Lagrangian sections of Lagrangian torus fibrations.

For a given finite set D on $X = \mathbb{R}/\mathbb{Z}$, we can find a C^∞-divisor s on X such that $\mathcal{L}(D) = \mathcal{L}(s)$ as follows. Let $\pi \colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the canonical universal covering. The pullback $\pi^* D$ is a principal divisor of a convex function f on $\mathbb{R}$ such that the set of strictly convex points of f is $\pi^{-1}(D)$ and satisfies a quasi-periodicity. By smoothing of f, we can find a quasi-periodic convex C^∞-function g such that $f(x) = g(x)$ on the complement of a sufficiently small neighborhood of $\pi^{-1}(D)$. This g gives a C^∞-divisor s_α on X such that the associated line bundle on X is equal to $\mathcal{L}(D)$. Moreover, we can deform s_α to another linearly equivalent C^∞-divisor s_1 on X such that the intersection $s_0 \cap s_1 := L_0 \cap L_{s_1}$ of the associated Lagrangian section L_{s_1} of s_1 and the zero section L_0 is contained in $X \setminus D$ and $\sharp(E \cap L_0 \cap L_{s_1}) = 1$ for any $E \in \pi_0(X \setminus D)$. Then, s_1 is a *permissible* C^∞-divisor and $\pi^* s_1$ is the principal divisor of a strictly convex C^∞-function. More generally, we can construct a map $s \colon [0,1] \to \Gamma(X; \mathcal{C}_X^0/\mathcal{O}_X^\times)$ satisfies

(1) $s(0) = D$, $s(1) = s_1$ and $s(t)$ is a prepermissible C^∞-divisor for all $t \in (0,1]$,
(2) for all $t \in [0,1]$, the line bundle $\mathcal{L}(s(t)) \in \mathrm{Pic}(X)$ associated with $s(t)$ is equal,
(3) $X \setminus D \supset s_0 \cap s(t) \supset s_0 \cap s(u)$ for all $0 < t \leq u \leq 1$,
(4) $s_0 \cap s(t)$ is homotopy equivalent to $X \setminus D$ for all $t \in (0,1]$,
(5) for any $x \in X \setminus D$, there exists $t \in (0,1]$ such that $x \in s_0 \cap s(t)$.

In [Tsu23], the author defined the graded module $\mathrm{LMD}^\bullet(X; s)$ for a permissible C^∞-divisor s on a compact tropical manifold X. By using the theory of [LdMRS23], we can refine the Conjecture 1.2 in [Tsu23] like this: the Euler characteristic of $\mathrm{LMD}^\bullet(X; s)$ is equal to the Riemann–Roch number of $\mathcal{L}(s)$. ([Tsu23, Conjecture 1.2] does not give an explicit definition of the Riemann–Roch number since this conjecture appeared before [LdMRS23].)

If $X = \mathbb{R}/\mathbb{Z}$, then X is an integral affine manifold and the graded module $\mathrm{LMD}^\bullet(X; s_1)$ for s_1 is isomorphic to the Floer complex associated with the Lagrangian section L_{s_1} of s_1 [KS01, Remark 13] as a graded module (see also [Tsu23,

§4.4] for more detail about it). In our case, we have the following isomorphisms:

$$\mathrm{LMD}^\bullet(X; s_1) \simeq \bigoplus_{p \in s_0 \cap s_1} \mathbb{Z}[0] \simeq H^\bullet(X \setminus D; \mathbb{Z}), \tag{2.32}$$

$$\mathrm{LMD}^\bullet(X; -s_1) \simeq \bigoplus_{p \in s_0 \cap s_1} \mathbb{Z}[-1] \simeq H^\bullet_c(X \setminus D; \mathbb{Z}). \tag{2.33}$$

The method presented here is indeed not canonical, but the isomorphisms are not coincidences. In fact, a similar logic works for tropical toric varieties.

Let P be a top dimensional Delzant lattice polytope in $\mathbb{R}^n$ and X_P the tropical toric variety of P. Recall that, if f is a tropical Laurent polynomial on $\mathbb{R}^n$ whose Newton polynomial is P and the regular subdivision of f is unimodular, then the closure $\overline{V(f)}$ of the hypersurface of f in X_P is a tropical submanifold of X_P and the set of connected component of $X_P \setminus \overline{V(f)}$ is bijective with the set $P(\mathbb{Z})$ of lattice points in P. By using the Maslov dequantization of a family of nonnegative valued Laurent polynomial which converges to f, we get a C^∞-divisor s_f on X_P whose line bundle is isomorphic to that of $\overline{V(f)}$, $s_0 \cap s_f \subset X_P \setminus \overline{V(f)}$, and $\pi_0(s_0 \cap s_f)$ is bijective with $\pi_0(X_P \setminus \overline{V(f)})$. Then, we have

$$\begin{aligned} \mathrm{LMD}^\bullet(X_P; s_f) &\simeq \bigoplus_{p \in P \cap \mathbb{Z}^n} \mathbb{Z}[0] \simeq \bigoplus_{W \in \pi_0(X_P \setminus \overline{V(f)})} H^\bullet(W; \mathbb{Z}) \\ &\simeq H^\bullet(X_P \setminus \overline{V(f)}; \mathbb{Z}), \end{aligned} \tag{2.34}$$

$$\begin{aligned} \mathrm{LMD}^\bullet(X_P; -s_f) &\simeq \bigoplus_{p \in \mathrm{int}(P)(\mathbb{Z})} \mathbb{Z}[-\dim X_P] \simeq \bigoplus_{W \in \pi_0(X_P \setminus \overline{V(f)})} H^\bullet_c(W; \mathbb{Z}) \\ &\simeq H^\bullet_c(X_P \setminus \overline{V(f)}; \mathbb{Z}). \end{aligned} \tag{2.35}$$

See also [Tsu23, Appendix D] for more detail about C^∞-divisors on tropical toric manifolds associated with lattice polytopes.

From these examples, it seems that the cohomology of the complement of tropical submanifolds D of codimension 1 on a compact tropical manifold X has a piece of data of homological invariant of $\mathcal{L}(D)$. From analogy of Floer cohomology, the author expects that we can define the cohomology of $\mathcal{L}(D)$ if we can define a good "Floer differential" for the graded module $H^\bullet(X \setminus D)$ (over some Novikov field). One of reason why we need some "Floer differential" for $H^\bullet(X \setminus D)$ is explained in Remark 2.19. We also note that Demazure's theorem [Dem70] is also related with the current remark (see also [CLS11, §9.1]).

Remark 2.19. Let C be a compact tropical curve and D a finite set of $C \setminus C_{\mathrm{sing}}$. Then, there exist the following relationships between the rank of cohomology of compact support of $C \setminus D$ and the Baker–Norine rank $r(D)$ of D [GK08, Definition 1.12]:

$$\operatorname{rank} H^0_c(C \setminus D) = r(D) + 1 = 0, \quad \operatorname{rank} H^1_c(C \setminus D) = r(K_C - D) + 1. \tag{2.36}$$

The Baker–Norine rank $r(D)$ is an analog of the rank of linear systems of divisors on algebraic varieties. There is an explanation that the Baker–Norine rank is truly a good analog of the rank of linear systems in [Bak08]. From (2.36), we can say the rank of $H^\bullet_c(C \setminus D)$ also behave like the cohomology of a line bundle on an algebraic variety. On the other hand, the cohomology $H^\bullet(C \setminus D)$ does not necessarily behave as an analog of the cohomology of a line bundle on an algebraic variety.

From now on, let C be a compact tropical curve of genus 2 like [MZ08, Figure 1]. Let E be a connected component of $C \setminus C_{\mathrm{sing}}$, and D a nonempty finite subset of E. Then,

$$\operatorname{rank} H^0(C \setminus D) = \sharp(D), \quad \operatorname{rank} H^1(C \setminus D) = 1. \tag{2.37}$$

On the other hand, $\deg(K_C - D) = 2 - \sharp(D)$. Therefore, when $\sharp(D) > 2$, we get

$$r(D) + 1 = \sharp(D) - 1, \quad r(K_C - D) + 1 = 0. \tag{2.38}$$

In particular, $\operatorname{rank} H^0(C \setminus D) > r(D)+1$ in this case. The equation $\operatorname{rank} H^1(C \setminus D) = 1$ is independent of the degree of D. It is a very different point from behavior of the cohomology of invertible sheaves on algebraic varieties. This is a main reason why we stress that we need to define a "Floer differential" for $H^\bullet(C \setminus D)$ in order to capture a homological data of tropical line bundles in Remark 2.18.

We expect that we can define such an appropriate differential of $H^\bullet(C \setminus D)$ by a combinatorial way as an analog of the Floer complex of Lagrangian sections on trivalent graphs [AEK22]. We also expect the homological mirror symmetry gives a new perspective on the specialization inequality of the Baker–Norine rank [Bak08, Lemma 2.8].

3. The Riemann–Roch number of pairs of tropical submanifolds

In this section, we discuss Conjecture 1.7. In order to discuss Conjecture 1.7, we need preparation from the theory of tropical homology and Chern classes of tropical manifolds. First, we note the theory of Chern classes of tropical manifold [LdMRS23] is new and the Chern(–Schwartz–MacPherson) class of tropical manifolds is not defined from a direct analog of the theory of vector bundles on algebraic varieties. Therefore, there is much that is not sufficiently understood about Chern classes of tropical manifolds. (The theory of tropical vector bundle is studied in [All12, JMT24] or [AP20, Theorem 1.8].) On the other hand, we can use facts which arise from elementary properties of multiplicative sequences (or m-sequences) [Hir95, §1] (see also [MS74, §19]). From these properties, we can investigate the Todd classes of tropical manifolds to some extent.

In this section, we mainly discuss properties which the Todd classes should have if a natural conjecture for tropical Chern classes (Conjecture 3.7) holds. We also prove a generalization of Theorem 1.5 in Theorem 3.17.

3.1. **Tropical homology.** We recall the theory of tropical homology from [MZ14, JRS18, GS23]. In this subsection, let Q be a subring of $\mathbb{R}$ and $p, q \in \mathbb{Z}$. Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and $\Omega^p_{\mathbb{Z},X}$ the sheaf of tropical p-forms on X [GS23, Definition 2.7] (cf. [MZ14, §2.4]). Let $\omega^\bullet_X$ be the dualizing complex of X (e.g. [KS94, Definition 3.1.16]). The (p,q)-tropical cohomology $H^{p,q}(X; Q)$ and the (p,q)-tropical Borel–Moore homology $H^{\mathrm{BM}}_{p,q}(X; Q)$ of $(X, \mathcal{O}_X^\times)$ are the following Q-modules [GS23, Definition 4.1 and 4.3]:

$$H^{p,q}(X; Q) \coloneqq \operatorname{Hom}_{D^b(\mathbb{Z}_X)}(\mathbb{Z}_X, \Omega^p_{\mathbb{Z},X}[q]) \otimes_{\mathbb{Z}} Q, \tag{3.1}$$

$$H^{\mathrm{BM}}_{p,q}(X; Q) \coloneqq \operatorname{Hom}_{D^b(\mathbb{Z}_X)}(\Omega^p_{\mathbb{Z},X}[q], \omega^\bullet_X) \otimes_{\mathbb{Z}} Q. \tag{3.2}$$

For comparison with the original tropical homology [MZ14, IKMZ19], see also [GS23, Remark 2.8 and Theorem 4.20]. Let $\Omega^\bullet_{\mathbb{Z},X} \coloneqq \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \Omega^j_{\mathbb{Z},X}[-j]$ be the total complex of $\Omega^p_{\mathbb{Z},X}$ (see [Sma17, Proposition 3.1]). Since $\Omega^\bullet_{\mathbb{Z},X}$ is a sheaf of trivial and graded-commutative dga, so the hypercohomology $\mathbb{H}^\bullet(X;\Omega^\bullet_{\mathbb{Z},X})$ of $\Omega^\bullet_{\mathbb{Z},X}$ has a natural graded-commutative ring structure (see [GW23, Remark 21.130]). Besides, $\Omega^\bullet_{\mathbb{Z},X}$ is trivial, so there exists an isomorphism $\mathbb{H}^\bullet(X;\Omega^\bullet_{\mathbb{Z},X}) \simeq \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}(X;\mathbb{Z})$ as Abelian groups. If $f\colon X \to Y$ is a morphism of rational polyhedral spaces, then f induces the pullback $f^*\colon \mathbb{H}^\bullet(Y;\Omega^\bullet_{\mathbb{Z},Y}) \to \mathbb{H}^\bullet(X;\Omega^\bullet_{\mathbb{Z},X})$ [GS23, Proposition 4.18] and f^* is a graded ring homomorphism. If $f\colon X \to Y$ is a proper morphism, then f induces the pushforward $f_*\colon H^{\mathrm{BM}}_{p,q}(X;Q) \to H^{\mathrm{BM}}_{p,q}(Y;Q)$ [GS23, Definition 4.9].

Let $Z_k(X)$ be the group of tropical k-cycles on X [GS23, Definition 3.5]. The presheaf $U \to Z_k(U)$ on X is a sheaf, and we write $\mathscr{Z}^X_k$ for it [GS23, p.591]. For any $k \in \mathbb{Z}_{\geq 0}$, there exists the *cycle map* $\mathrm{cyc}_X\colon Z_k(X) \to H^{\mathrm{BM}}_{k,k}(X;\mathbb{Z})$ and $f_* \circ \mathrm{cyc}_X = \mathrm{cyc}_Y \circ f_*$ for any proper morphism $f\colon X \to Y$ of rational polyhedral spaces [GS23, Definition 5.4 and Corollary 5.8] (see also [JRS18, Definition 4.13]). A purely n-dimensional rational polyhedral space $(X, \mathcal{O}^\times_X)$ *admits a fundamental class* if the constant function on X with value 1 forms an n-dimensional tropical cycle $1_{X_{\mathrm{reg}}}$ [GS23, §6.1], and the image $[X] \coloneqq \mathrm{cyc}_X(1_{X_{\mathrm{reg}}}) \in H^{\mathrm{BM}}_{n,n}(X;\mathbb{Z})$ is called the *fundamental class* of X (see also [JRS18, Definition 4.8]). For simplicity, we sometimes write X as $1_{X_{\mathrm{reg}}}$. From definition, $H^{\mathrm{BM}}_{n,n}(X;\mathbb{Z}) \coloneqq \mathrm{Hom}_{D^b(\mathbb{Z}_X)}(\Omega^n_X[n], \boldsymbol{\omega}^\bullet_X)$, so the multiplicative structure of $\Omega^\bullet_X$ and the fundamental class of X induces the following morphism:

$$\eta^X_p\colon \Omega^{n-p}_X[n] \otimes^L_{\mathbb{Z}_X} \Omega^p_X \to \Omega^n_X[n] \xrightarrow{[X]} \boldsymbol{\omega}^\bullet_X. \tag{3.3}$$

The tensor-hom adjunction of η^X_p is the following [GS23, p.627]:

$$\delta^X_p\colon \Omega^{n-p}_X[n] \to \mathcal{D}_{\mathbb{Z}_X}(\Omega^p_X) \coloneqq R\mathcal{H}om_{\mathbb{Z}_X}(\Omega^p_X, \boldsymbol{\omega}^\bullet_X). \tag{3.4}$$

Of course, $\delta^X_0 = [X]$. Besides, the natural isomorphism from $\mathrm{Hom}_{D^b(\mathbb{Z}_X)}(-\otimes^L_{\mathbb{Z}_X}\Omega^p_X[q], \boldsymbol{\omega}^\bullet_X)$ to $\mathrm{Hom}_{D^b(\mathbb{Z}_X)}(-, \mathcal{D}_{\mathbb{Z}_X}(\Omega^p_X[q]))$ gives the following commutative diagram for any $\alpha\colon \mathbb{Z}_X \to \Omega^{n-p}_X[n-q]$:

$$\begin{tikzcd}
\mathrm{Hom}_{D^b(\mathbb{Z}_X)}(\Omega^{n-p}_X[n-q] \otimes^L_{\mathbb{Z}_X} \Omega^p_X[q], \boldsymbol{\omega}^\bullet_X) \arrow[r] \arrow[d] & \mathrm{Hom}_{D^b(\mathbb{Z}_X)}(\Omega^{n-p}_X[n-q], \mathcal{D}_{\mathbb{Z}_X}(\Omega^p_X[q])) \arrow[d] \\
\mathrm{Hom}_{D^b(\mathbb{Z}_X)}(\Omega^p[q], \boldsymbol{\omega}^\bullet_X) \arrow[r] & \mathrm{Hom}_{D^b(\mathbb{Z}_X)}(\mathbb{Z}_X, \mathcal{D}_{\mathbb{Z}_X}(\Omega^p[q])).
\end{tikzcd}$$

The morphism of each row of this commutative diagram is an isomorphism. In particular, the image of η^X_p by the homomorphism on the left column of the commutative diagram is the cap product $\alpha \frown [X]$ of α and $[X]$ [GS23, §4.6]. (For the sake of visibility, we have reversed the action as in [GS23, §4.6].) Therefore, we can identify the homomorphism

$$\mathbb{H}^{-q}(\delta^X_p)\colon H^{n-p,n-q}(X;\mathbb{Z}) \to \mathbb{H}^{-q}(X;\mathcal{D}_{\mathbb{Z}_X}(\Omega^p_X))$$

with the cap product homomorphism

$$\cdot \frown [X]\colon H^{n-p,n-q}(X;\mathbb{Z}) \to H^{\mathrm{BM}}_{p,q}(X;\mathbb{Z}). \tag{3.5}$$

In particular, if X is compact, then the unique morphism $a_X\colon X \to \{\mathrm{pt}\}$ to the one-point space $\{\mathrm{pt}\}$ defines the trace map $\int_X c \coloneqq a_{X*}(c \frown [X]) \in H^{\mathrm{BM}}_{0,0}(\{\mathrm{pt}\};\mathbb{Z}) \simeq \mathbb{Z}$ for $c \in H^{\bullet,\bullet}(X;\mathbb{Z})$. An n-dimensional rational polyhedral space $(X, \mathcal{O}^\times_X)$ admitting with a fundamental class satisfies *Poincaré–Verdier duality* if δ^X_k is an isomorphism for all $k \in \mathbb{Z}_{\geq 0}$ [GS23, Definition 6.4]. If $(X, \mathcal{O}^\times_X)$ satisfies Poincaré–Verdier duality, then (3.5) is an isomorphism [GS23, Coroolary 6.9]. We can see every tropical manifold satisfies the Poincaré–Verdier duality from [JRS18, Proposition 5.5] and [GS23, Theorem 6.7]. The integer valued Poincaré duality for tropical manifolds admitting a global face structure was firstly proved in [JRS18, Theorem 5.3]. In recent work, considerations have also been made using the Poincaré–Verdier duality and the local Poincaré duality for rational polyhedral spaces as indicators of smoothness (e.g. [Aks23, AP21, GS23]).

For simplicity, we will use the following notation.

Notation 3.1. Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space admitting a fundamental class $[X]$ and satisfying the Poincaré–Verdier duality. We write PD_X for the inverse of the cap product homomorphism $\cdot \frown [X]$, and $[Z]_{\mathrm{PD}} \coloneqq \mathrm{PD}_X(Z)$ for all $Z \in H_{p,q}^{\mathrm{BM}}(X;\mathbb{Z})$.

3.2. **Tropical Cartier divisors.** Let $(X, \mathcal{O}_X^\times)$ be a rational polyhedral space and $\mathrm{PAff}_{\mathbb{Z},X}$ the sheaf of piecewise integral affine linear functions on X (see [JRS18, Definition 4.1] or [GS23, Definition 3.8 and Remark 3.9]). From definition, there exist the following exact sequence of sheaves:

$$0 \to \mathcal{O}_X^\times \to \mathrm{PAff}_{\mathbb{Z},X} \to \mathrm{PAff}_{\mathbb{Z},X} / \mathcal{O}_X^\times \to 0. \tag{3.6}$$

Besides, we use the notations below:

$$\mathrm{Div}(X)^{[0]} \coloneqq H^0(X; \mathrm{PAff}_{\mathbb{Z},X} / \mathcal{O}_X^\times), \quad \mathcal{D}iv_X^{[0]} \coloneqq \mathrm{PAff}_{\mathbb{Z},X} / \mathcal{O}_X^\times. \tag{3.7}$$

The connecting homomorphism $\delta\colon \mathrm{Div}(X)^{[0]} \to H^1(X; \mathcal{O}_X^\times)$ induced from (3.6) is surjective [JRS18, Proposition 4.6]. From now on, for a given $D \in \mathrm{Div}(X)^{[0]}$, let $\mathcal{L}(D)$ be the associated line bundle of D (see [GS23, §3.5] and [MZ08, §4.3]).

Remark 3.2. The definition of rational functions on tropical manifolds varies depending on authors. Therefore, the definition of tropical Cartier divisors on tropical manifolds also varies. In [JRS18, GS23], the previously mentioned group $\mathrm{Div}(X)^{[0]}$ is called the group of tropical Cartier divisors on X, but this is different from the meaning in [LdMRS23]. By [Sha15, Proposition 3.27], every tropical cycle of codimension 1 on a tropical manifold is a tropical Cartier divisor in the sense of [Sha15, LdMRS23]. In contrast, there exist codimension 1 tropical cycle which does not come from any elements in $\mathrm{Div}(X)^{[0]}$. For example, the point $\{-\infty\}$ in $\mathbb{T}$ does not come from $\mathrm{Div}(\mathbb{T})^{[0]}$.

On the other hand, there exist advantages of the definition of tropical Cartier divisors in the sense of [JRS18, GS23]. One of them is that every morphism $f\colon X \to Y$ of rational polyhedral spaces always induces the pullback $f^*\colon \mathrm{Div}(Y)^{[0]} \to \mathrm{Div}(X)^{[0]}$, and the pullback is compatible with that of Picard groups $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ [GS23, Propoisition 3.15].

Moreover, there exists a natural pairing [GS23, §3.4] (cf. [AR10, Definition 6.5]):

$$\mathrm{Div}(X)^{[0]} \times Z_k(X) \to Z_{k-1}(X); (D, A) \mapsto D \cdot A. \tag{3.8}$$

In particular, the following equation holds [GS23, Proposition 5.12]:

$$\mathrm{cyc}_X(D \cdot A) = c_1(\mathcal{L}(D)) \frown \mathrm{cyc}_X(A). \tag{3.9}$$

Notation 3.3. Let A be a tropical k-cycle on a rational polyhedral space $(X, \mathcal{O}_X^\times)$. We write $[A] \coloneqq \mathrm{cyc}_X(A)$. We also write $[A]_{\mathrm{PD}} \coloneqq \mathrm{PD}_X(\mathrm{cyc}_X(A))$ when $(X, \mathcal{O}_X^\times)$ satisfies the Poincaré duality.

When $(X, \mathcal{O}_X^\times)$ is a purely n-dimensional rational polyhedral space which admits a fundamental class, there exists the following homomorphism [JRS18, Definition 4.14]:

$$\operatorname{div}_X \colon \operatorname{Div}(X)^{[0]} \to Z_{n-1}(X); D \mapsto D \cdot X. \tag{3.10}$$

See also [JRS18, Theorem 4.15].

Let A be a tropical k-cycle on X , $|A|$ the support of A [GS23, Definition 3.5], and $i \colon |A| \to X$ the inclusion of $|A|$. We note that we can consider A as an element of $Z_k(|A|)$ and the trivial projection formula holds from definition:

$$i_*(D|_{|A|} \cdot A) = D \cdot A. \tag{3.11}$$

Proposition 3.4. If X is a purely n-dimensional tropical manifold, then the homomorphism (3.10) is injective.

Proof. The divisor map for each open subset of X naturally induces a sheaf homomorphism

$$\operatorname{div}_X \colon \mathcal{D}iv_X^{[0]} \to \mathscr{Z}_{n-1}^X. \tag{3.12}$$

Therefore, it is enough to check the homomorphism of sheaves is injective. Let $\mathcal{B}$ be an open basis of X. Then, the category of $\mathcal{B}$-sheaves ([GW10, p.49-50]) is equivalent to the category of sheaves on X, so we only need to check $\operatorname{div}_U \colon \mathcal{D}iv_X^{[0]}(U) \to Z_{n-1}(U)$ for any $U \in \mathcal{B}$. Since X is a tropical manifold, we may assume U is isomorphic to an open subset of $\mathrm{LC}_x X \times \mathbb{T}^n$ for some $x \in X$. Moreover, we may assume every piecewise integer affine linear function on U is the pullback of a piecewise integer affine linear function on $\mathrm{LC}_x X$ by the projection $\mathrm{LC}_x X \times \mathbb{T}^n \to \mathrm{LC}_x X$ locally. If X is boundaryless, then a generalization of Proposition 3.4 holds [GS21, Theorem 4.5]. Therefore, it has been proved. $\square$

Notation 3.5. From now on, we identify $\operatorname{Div}(X)^{[0]}$ with $\operatorname{div}_X(\operatorname{Div}(X)^{[0]})$ when X is a tropical manifold. Under this identification, we can consider every sedentarity-0 tropical submanifold as an element of $\operatorname{Div}(X)^{[0]}$. In particular, from (3.8), we can define the *k-th power* D^k of $D \in \operatorname{Div}(X)^{[0]}$ as an element of $Z_{n-k}(X)$ and satisfies

$$[D^k]_{\mathrm{PD}} = [D]_{\mathrm{PD}}^k = c_1(\mathcal{L}(D))^k. \tag{3.13}$$

For simplicity, $D^0 := X$. Additionally, we suppose D is a tropical submanifold on X and the pullback $D|_D$ of D is a tropical submanifold on D. For the embedding morphism $\iota_D \colon D \to X$, we obtain the following equation:

$$\iota_{|D|*}(D|_{|D|}) = D \cdot D. \tag{3.14}$$

We can identify the support $|D|_{|D|}|$ with $|D^2|$ within the scope where confusion does not arise. Let $D^{(1)} := D$ and $D^{(i+1)} := D^{(i)}|_{|D^{(i)}|} = D|_{|D^{(i)}|}$ for all $i \in \mathbb{Z}_{>0}$. Moreover, if $D^{(k)}$ $(k = 1, \ldots, m)$ is tropical submanifolds of $|D^{k-1}|$ and $|D^k| = |D^{(k)}|$, then

$$\iota_{|D^{(k)}|*}(D^{(k+1)}) = \iota_{|D^{(k)}|*}(D^{(k)}|_{|D^{(k)}|}) = D \cdot D^k = D^{k+1}. \tag{3.15}$$

Thus, we can identify $|D^{k+1}|$ with $|D^{(k+1)}|$ within the scope where confusion does not arise.

3.3. **Chern classes and Todd classes of tropical manifolds.** In this subsection, we start to discuss Conjecture 1.7 and give a generalization of Theorem 1.5.

Notation 3.6. Let X be a purely n-dimensional tropical manifold and $\mathrm{csm}_k(X)$ the k-th Chern–Schwartz–MacPherson cycle of X [LdMRS23, Definition 3.4]. We note that we can define the k-th Chern–Schwartz–MacPherson cycle of $\mathrm{csm}_k(X)$ by a sheaf theoretical approach from [LdMRS23, Proposition 3.11 and Lemma 3.12] and [GS23, Lemma 4.13] (see also [SW22, Lemma 5.4]). Then, we write the k-th Chern class of X as

$$c_k^{\mathrm{sm}}(X) := \mathrm{PD}_X \circ \mathrm{cyc}_{n-k}(\mathrm{csm}_{n-k}(X)) \in H^{k,k}(X;\mathbb{Z}). \tag{3.16}$$

This notation $c_k^{\mathrm{sm}}(X)$ is used to avoid confusion with the Chern classes of divisors. As like this, we write the total Chern class of X as $c^{\mathrm{sm}}(X) := \sum_{k=0}^{n} c_k^{\mathrm{sm}}(X)$. For a given $L \in H^{1,1}(X;\mathbb{Z})$, we write the *Chern character* of L as follows:

$$\mathrm{ch}(L) := \exp(L) := \sum_{i=0}^{\infty} \frac{L^i}{i!} \in H^{\bullet,\bullet}(X;\mathbb{Q}). \tag{3.17}$$

In order to discuss the Todd classes of tropical manifolds, we recall fundamental properties of Chern classes of complex manifolds from [Hir95, Ful98, CLS11]. Let M be a complex manifold and D a nonsingular divisor on D. Let $\mathcal{T}_M$ be the tangent bundle of M. For an embedding $\iota\colon D \to M$ of D, we have the following exact sequence:

$$0 \to \mathcal{T}_D \to \iota^*\mathcal{T}_M \to \iota^*\mathcal{O}_M(D) \to 0. \tag{3.18}$$

From the axioms of Chern classes, (3.18) gives the adjunction formula for total Chern classes:

$$\iota^* c(\mathcal{T}_M) = c(\mathcal{T}_D) c(\iota^*\mathcal{O}_M(D)). \tag{3.19}$$

Since $c(\mathcal{O}_M(D)) = 1 + c_1(\mathcal{O}_M(D))$, the k-th part of (3.19) is

$$\iota^* c_k(\mathcal{T}_M) = c_k(\mathcal{T}_D) + c_{k-1}(\mathcal{T}_D) c_1(\iota^*\mathcal{O}_M(D)). \tag{3.20}$$

We can discuss an analog of (3.19), and expect the following conjecture holds:

Conjecture 3.7. Let X be a pure dimensional compact tropical manifold and D a tropical submanifold of X of codimension 1. Then, the following equations hold for $k \in \mathbb{Z}_{\geq 0}$ and the inclusion $\iota\colon D \to X$:

$$\iota^* c^{\mathrm{sm}}(X) = c^{\mathrm{sm}}(D)(1 + \iota^*[D]_{\mathrm{PD}}), \tag{3.21}$$

$$\iota^* c_k^{\mathrm{sm}}(X) = c_k^{\mathrm{sm}}(D) + c_{k-1}^{\mathrm{sm}}(D)\iota^*[D]_{\mathrm{PD}} \; (\in H^{k,k}(D;\mathbb{Z})). \tag{3.22}$$

We also expect that a version of Conjecture 3.7 for tropical Chow groups [Sha15, Definition 3.30] also holds. When $\dim X = 1$, Conjecture 3.7 holds trivially. When $\dim X = 2$, Conjecture 3.7 is just another representation of the adjunction formula of locally degree 1 tropical curves on tropical surfaces ([Sha15, Theorem 6], [LdMRS23, Theorem 5.2]). We will see later what corollaries appear when Conjecture 3.7 is true.

From now on, we discuss the Todd classes of tropical manifolds. At first, we recall m-sequences. The notion of m-sequence is introduced by Hirzebruch in [Hir95, §1], but some part of explanation in [MS74, §19] by Milnor–Stasheff are more comprehensible, and thus we also follow from [MS74, §19]. Let $\mathrm{Todd} := (\mathrm{Todd}_j)_{j\in\mathbb{Z}_{\geq 0}}$ be the Todd m-sequence [Hir95, §1.7]. In complex geometry, the k-th Chern class $c_k(M) := c_k(\mathcal{T}_M)$ of a given complex manifold M is defined as an element of

$H^{2k}(M;\mathbb{Z})$. Let $c(M) := \sum_{i=0}^{\infty} c_i(M)$ be the total Chern class of M in the real valued even cohomology ring $H^{\mathrm{even}}(X;\mathbb{R})$. Since $H^{\mathrm{even}}(M;\mathbb{R})$ is commutative and a graded $\mathbb{R}$-algebra and $c_0(M) = 1$, so the total Chern class of M defines the Todd class $\mathrm{td}(M) := \sum_{j=0}^{\infty} \mathrm{Todd}_j(c_1(M), \dots, c_j(M))$ of M [Hir95, §10].

We can define the Todd class $\mathrm{td}(X)$ of a given tropical manifold X by replacing $H^{\mathrm{even}}(M;\mathbb{R})$ with $\bigoplus_{i=0}^{\infty} H^{i,i}(X;\mathbb{R})$ (Definition 3.8). In [LdMRS23, Conjecture 6.13], the Todd class of tropical manifold is defined as a tropical cycle. In this paper, we use the Poincaré dual of the image of the cycle map of it in order to get closer to the notation in algebraic geometry.

Definition 3.8 ([LdMRS23, §6])**.** Let X be a purely n-dimensional tropical manifold. The j-th *Todd class* of X is a cycle in $H^{j,j}(X;\mathbb{R})$ defined from the Todd m-sequence $\mathrm{Todd} := (\mathrm{Todd}_j)_{j\in\mathbb{Z}_{\geq 0}}$ with respect to $\bigoplus_{i=0}^{\infty} H^{i,i}(X;\mathbb{R})$ as follows:

$$\mathrm{td}_j(X) := \mathrm{Todd}_j(c_1^{\mathrm{sm}}(X), \dots, c_j^{\mathrm{sm}}(X)), \tag{3.23}$$

$$\mathrm{td}(X) := \mathrm{Todd}(c^{\mathrm{sm}}(X)) := \sum_{j=0}^{\infty} \mathrm{td}_j(X). \tag{3.24}$$

Definition 3.9 ([Hir95, §12.1.(2)])**.** Let X be a purely n-dimensional compact tropical manifold. The *Riemann–Roch number* of $D(\in Z_{n-1}(X))$ is the following number:

$$\mathrm{RR}(X;D) := \int_X \mathrm{ch}([D]_{\mathrm{PD}})\,\mathrm{td}(X). \tag{3.25}$$

If $D \in \mathrm{Div}(X)^{[0]}$, then we can write (3.25) as (1.1) from [GS23, Proposition 5.12].

Following the theory of algebraic geometry, we define the *generalized Gysin map* associated with proper morphism between tropical manifolds (see [CLS11, Chapter 13. Appendix] for classical cases). We note that the generalized Gysin map for tropical manifolds have already appeared in [AP20].

Definition 3.10 (cf. [AP20])**.** Let X and Y be pure dimensional tropical manifolds and Q a subring of $\mathbb{R}$. Let $f\colon X \to Y$ be a proper morphism. The *generalized Gysin map* of f is the morphism $f_!\colon H^{\bullet,\bullet}(X;Q) \to H^{\bullet,\bullet}(Y;Q)$ which commutes with the following diagram:

$$\begin{tikzcd} H^{\bullet,\bullet}(X;Q) \arrow[r, "f_!"] \arrow[d, "\cdot\frown[X]"'] & H^{\bullet,\bullet}(Y;Q) \arrow[d, "\cdot\frown[Y]"] \\ H^{\mathrm{BM}}_{\bullet,\bullet}(X;Q) \arrow[r, "f_*"'] & H^{\mathrm{BM}}_{\bullet,\bullet}(Y;Q) \end{tikzcd} . \tag{3.26}$$

From definition, there exists the following projection formula.

Proposition 3.11. Let $f\colon X \to Y$ be a proper morphism between pure dimensional tropical manifolds X and Y. Then, for any $c \in H^{\bullet,\bullet}(Y;Q)$ and $d \in H^{\bullet,\bullet}(X;Q)$ the following equation holds:

$$f_!(f^*(c)\cdot d) = c \cdot f_!(d). \tag{3.27}$$

In particular, $f_!(f^*(c)) = c \cdot f_!(1) = c \cdot [X]_{\mathrm{PD}}$ when X is a rational polyhedral subspace of Y and f is the inclusion map of X.

Proof. Since $\cdot \frown [Y]$ is an isomorphism, we only need to prove the following:

$$f_!(f^*(c) \cdot d) \frown [Y] = (c \cdot f_!(d)) \frown [Y]. \tag{3.28}$$

We can get the projection formula for generalized Gysin map from that for the pushforward of tropical Borel–Moore homologies as follows:

$$\begin{aligned}(c \cdot f_!(d)) \frown [Y] &= c \frown (f_!(d) \frown [Y]) = c \frown f_*(d \frown [X]) && (3.29)\\ &= f_*(f^*(c) \frown (d \frown [X])) = f_*((f^*(c) \cdot d) \frown [X]) && (3.30)\\ &= f_!(f^*(c) \cdot d) \frown [Y]. && (3.31)\end{aligned}$$

□

Example 3.12. Let X be a purely n-dimensional tropical manifold and D a tropical submanifold of codimension 1 on X. If Conjecture 3.7 is true, then the generalized Gysin map of the inclusion map $\iota\colon D \to X$ induces

$$c^{\mathrm{sm}}(X) \cdot [D]_{\mathrm{PD}} = \iota_! c^{\mathrm{sm}}(D)(1 + [D]_{\mathrm{PD}}). \tag{3.32}$$

The first degree part of the equation above is

$$c_1^{\mathrm{sm}}(X) \cdot [D]_{\mathrm{PD}} = \iota_! c_1^{\mathrm{sm}}(D) + [D]_{\mathrm{PD}} \cdot [D]_{\mathrm{PD}}. \tag{3.33}$$

We may consider (3.33) as the Poincaré dual of the adjunction formula proved in [LdMRS23, Theorem 5.2]. By using the canonical divisor of tropical manifolds, (3.33) can be rewritten as follows:

$$\iota_! [K_D]_{\mathrm{PD}} = ([K_X]_{\mathrm{PD}} + [D]_{\mathrm{PD}})[D]_{\mathrm{PD}}. \tag{3.34}$$

If $\dim X = 2$, then this equation also gives the adjunction formula of tropical surfaces which is proved in [Sha15, Theorem 4.11] since $H^{2,2}(X;\mathbb{Z}) \simeq \mathbb{Z}$.

Example 3.13 (cf. [CLS11, Chapter 13 Appendix])**.** We retain the notation in Example 3.12. We also assume Conjecture 3.7 is true again. From (3.21) and the property of multiplicative sequences,

$$\iota^* \operatorname{td}(X) = \operatorname{td}(D) \frac{\iota^*[D]_{\mathrm{PD}}}{1 - \exp(-\iota^*[D]_{\mathrm{PD}})}. \tag{3.35}$$

From (3.35) we also get

$$\operatorname{td}(D) = \iota^* \left(\frac{1 - \exp(-[D]_{\mathrm{PD}})}{[D]_{\mathrm{PD}}} \operatorname{td}(X) \right). \tag{3.36}$$

Moreover, the projection formula for ι gives

$$\iota_!(\operatorname{td}(D)) = (1 - \operatorname{ch}(-[D]_{\mathrm{PD}})) \operatorname{td}(X). \tag{3.37}$$

The equation (3.37) is an analog of the Grothendieck–Riemann–Roch theorem for a special case.

Let D' be a tropical $(n-1)$-cycle on X. Then, the projection formula for the embedding $\iota\colon D \to X$ gives

$$\begin{aligned}\iota_!(\operatorname{ch}(\iota^*([D']_{\mathrm{PD}})) \operatorname{td}(D)) &= \operatorname{ch}([D']_{\mathrm{PD}}) \iota_!(\operatorname{td}(D))\\ &= (\operatorname{ch}([D']_{\mathrm{PD}}) - \operatorname{ch}([D' - D]_{\mathrm{PD}})) \operatorname{td}(X). && (3.38)\end{aligned}$$

If D' is in $\mathrm{Div}^{[0]}(X)$, then the pullback $\iota^* D' = D'|_D$ of D' satisfies the following:

$$\mathrm{RR}(X; D' - D) = \mathrm{RR}(X; D') - \mathrm{RR}(D; D'|_D). \tag{3.39}$$

Similarity to the case of algebraic varieties, we can define the *virtual T-genus* of $D_1, \dots, D_m \in Z_{n-1}(X)$ as follows [Hir95, §11.2]:

$$T(X) \coloneqq \mathrm{RR}(X;0), \quad T(X; D_1, \dots, D_m) \coloneqq \int_X \prod_{j=1}^{m} (1 - \exp(-[D_j]_{\mathrm{PD}}))\,\mathrm{td}(X). \tag{3.40}$$

We can see $T(X; D) = \mathrm{RR}(D; 0)$ when Conjecture 3.7 is true.

As explained in [Hir95, §20.6], the following equation holds

$$\exp([D_1]_{\mathrm{PD}}) = (1 - (1 - \exp(-[D_1]_{\mathrm{PD}})))^{-1} = \sum_{j=0}^{n} (1 - \exp(-[D_1]_{\mathrm{PD}}))^j. \tag{3.41}$$

From this, we obtain the following equation [Hir95, §20.6.(14)]:

$$\mathrm{RR}(X; D_1) = \sum_{j=0}^{n} T(X; \overbrace{D_1, \dots, D_1}^{j}). \tag{3.42}$$

Additionally, we assume every D_i $(i = 1, \dots, m)$ is in $\mathrm{Div}^{[0]}(X)$ and $D_1 = D$. Then, from (3.39) and Proposition 3.11, we also obtain the following equations [Hir95, Theorem 11.2.1]:

$$T(X; D_1, \dots, D_m) = T(D_1; D_2|_{D_1}, \dots, D_m|_{D_1}), \tag{3.43}$$

$$\sum_{j=1}^{n} T(X; \overbrace{D, \dots, D}^{j}) = \mathrm{RR}(D; 0) + \sum_{j=1}^{n-1} T(D; \overbrace{D|_D, \dots, D|_D}^{j}) = \mathrm{RR}(X; D|_D). \tag{3.44}$$

If $D|_D$ is also a tropical submanifold, then we can repeat this process.

Remark 3.14. In this remark, we see another explanation of an algebraic geometrical meaning of the first equation of Conjecture 1.3. Let $a_X \colon X \to \operatorname{Spec} k$ be a complete nonsingular algebraic variety over k and $K_0(X)$ the Grothendieck ring of X (see, e.g., [Ful98, §15.1]). For a coherent sheaf $\mathcal{F}$ on X, let $[\mathcal{F}]$ be the isomorphism class of $\mathcal{F}$ in $K_0(X)$. The addition of $K_0(X)$ is given from the direct sum of coherent sheaves and the multiplication of $K_0(X)$ is given from the derived tensor product of them. The unit of $K_0(X)$ is $[\mathcal{O}_X]$ where $\mathcal{O}_X$ is the structure sheaf of X. Since a_X is a proper morphism, so a_X induces the pushforward group homomorphism $a_{X*} \colon K_0(X) \to K_0(\operatorname{Spec} k)$, and $a_{X*}([\mathcal{F}]) = \chi(X; \mathcal{F})$. For simplicity, we write $\chi \coloneqq a_{X*}$. Let D be a nonsingular divisor on X and $\iota \colon D \to X$ the embedding morphism. Then, the following equations hold:

$$[\mathcal{O}_X(-D)] = [\mathcal{O}_X] - [\iota_* \mathcal{O}_D], \quad [\mathcal{O}_X(D)] = ([\mathcal{O}_X] - [\iota_* \mathcal{O}_D])^{-1} = \sum_{k=0}^{\infty} [\iota_* \mathcal{O}_D]^k, \tag{3.45}$$

$$\chi(X; \mathcal{O}_X(-D)) = \chi([\mathcal{O}_X] - [\iota_* \mathcal{O}_D]), \quad \chi(X; \mathcal{O}_X(D)) = \sum_{k=0}^{\infty} \chi([\iota_* \mathcal{O}_D]^k). \tag{3.46}$$

This means that (3.46) is the algebraic geometric counterpart of the first equation in Conjectures 1.1 and 1.3.

The following proposition demonstrates the relationships among the various conjectures presented in this paper.

Proposition 3.15. If [LdMRS23, Conjecture 6.13], Conjectures 1.3 and 3.7 are true, then Conjecture 1.7 is also true.

Proof. We suppose Conjecture 3.7 is true. From the assumption, D' is sedentarity-0.

If Conjecture 1.3 and [LdMRS23, Conjecture 6.13] is true, then

$$\begin{aligned}\mathrm{RR}(X; D' - D) =& \mathrm{RR}(X; D') - \mathrm{RR}(D; D'|_D) \\ =& \chi(X \setminus D') - \chi(D \setminus (D' \cap D)).\end{aligned}$$

Therefore, the proposition has been proved. □

Remark 3.16. The above proposition suggests that the most difficult part for proving Conjecture 1.7 is obtaining a proof of [LdMRS23, Conjecture 6.13].

Based on the above, we will generalize Theorem 1.5.

Theorem 3.17. Let X be a compact tropical surface and (D, D') a pair of tropical submanifolds of codimension 1 in moderate position on X. Then,

$$\chi(X \setminus D') - \chi(D \setminus (D' \cap D)) = \frac{\deg((D' - D).(D' - D - K_X))}{2} + \chi(X). \quad (3.47)$$

In particular, Conjecture 1.7 is true when X admits a Delzant face structure.

Proof. Since $\dim X = 2$, Conjecture 3.7 is true from [LdMRS23, Theorem 5.2]. From Theorem 1.5 and (3.39), we have

$$\begin{aligned}\frac{\deg((D' - D).(D' - D - K_X))}{2} =& \mathrm{RR}(X; D') - \mathrm{RR}(X; 0) - \mathrm{RR}(D; D' \cap D) \quad (3.48) \\ =& \chi(X \setminus D') - \chi(X) - \chi(D \setminus (D' \cap D)). \quad (3.49)\end{aligned}$$

□

Remark 3.18. Let X be an n-dimensional compact tropical manifold and (D, D') a pair of codimension 1 tropical submanifold in moderate position on X. In Remark 2.18, we stressed the cohomology $H^\bullet(X \setminus D)$ of the complement $X \setminus D$ is related with the graded modules $\mathrm{LMD}^\bullet(X; s)$ associated with a permissible C^∞-divisor s on X. In this remark, we explain $\chi(X \setminus D') - \chi(D \setminus (D' \cap D))$ is also the Euler characteristic of the relative cohomology $H^\bullet(X \setminus D, D \setminus D'; \mathbb{R})$ of the pair $(X \setminus D', D \setminus D')$ [Ive86, Chapter IV. Definition 8.1] (see also [Bre97, Chapter II. Proposition 12.3]).

Let $\mathbb{R}_X$ is the constant sheaf of $\mathbb{R}$ on X and $\mathbf{Mod}(\mathbb{R}_X)$ be the category of sheaves of $\mathbb{R}_X$-modules. Let Z be a locally closed subset of X and $j\colon Z \to X$ the inclusion map. For a given $\mathcal{F} \in \mathbf{Mod}(\mathbb{R}_X)$, let $(\mathcal{F})_Z := j_! j^{-1}\mathcal{F}$ [KS94, Proposition 2.3.6]. The functor $(\cdot)_Z\colon \mathbf{Mod}(\mathbb{R}_X) \to \mathbf{Mod}(\mathbb{R}_X); \mathcal{F} \to (\mathcal{F})_Z$ is exact and has the right adjoint left exact functor $\Gamma_Z\colon \mathbf{Mod}(\mathbb{R}_X) \to \mathbf{Mod}(\mathbb{R}_X)$ [KS94, Definition 2.3.8]. If Z is open, then $\Gamma_Z \mathcal{F} \simeq j_* j^{-1}\mathcal{F}$ [KS94, Proposition 2.3.9 (iii)]. Therefore, the following exact sequence exists

$$0 \to (\mathbb{R}_X)_{X\setminus D} \to \mathbb{R}_X \to (\mathbb{R}_X)_D \to 0. \quad (3.50)$$

The derived functor $R\Gamma_{X\setminus D'}$ also gives the following exact triangle:

$$R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D}) \to R\Gamma_{X\setminus D'}(\mathbb{R}_X) \to R\Gamma_{X\setminus D'}((\mathbb{R}_X)_D) \to R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D})[1]. \quad (3.51)$$

Let $i\colon D\to X$ be the inclusion map of D. From [KS94, (2.3.20)], we get

$$R\Gamma_{X\setminus D'}((\mathbb{R}_X)_D)\simeq i_*R\Gamma_{D\setminus D'}\mathbb{R}_D, \tag{3.52}$$

$$\mathbb{H}^\bullet(X;R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D}))\simeq H^\bullet(X\setminus D';(\mathbb{R}_{X\setminus D'})_{X\setminus(D\cup D')}). \tag{3.53}$$

The cohomology $H^\bullet(X\setminus D';(\mathbb{R}_{X\setminus D'})_{X\setminus(D\cup D')})$ is just the relative cohomology $H^\bullet(X\setminus D, D\setminus D';\mathbb{R})$ of the pair $(X\setminus D', D\setminus D')$. Hence, the following equation holds

$$\chi(\mathbb{H}^\bullet(X;R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D})))=\chi(X\setminus D')-\chi(D\setminus(D'\cap D)). \tag{3.54}$$

Besides, from definition we can check

$$\mathbb{H}^\bullet(X;R\Gamma_X((\mathbb{R}_X)_{X\setminus D}))\simeq H_c^\bullet(X\setminus D),\quad \mathbb{H}^\bullet(X;R\Gamma_{X\setminus D'}((\mathbb{R}_X)_X))\simeq H^\bullet(X\setminus D'). \tag{3.55}$$

Therefore, the graded module $\mathbb{H}^\bullet(X;R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D}))$ is a generalization of both $H_c^\bullet(X\setminus D)$ and $H^\bullet(X\setminus D)$, and $\mathbb{H}^\bullet(X;R\Gamma_{X\setminus D'}((\mathbb{R}_X)_{X\setminus D}))$ is also related with graded modules associated with permissible C^∞-divisors that are studied in [Tsu23].

We expect that Conjecture 1.7 is also true for a pair (D_1,D_2) of rational polyhedral subspaces in moderate position on X such that each D_i is a finite union of codimension 1 tropical submanifolds whose intersections satisfy a good condition.

Appendix A. Compatibility of intersections of tropical cycles

In this appendix, we note the compatibility of intersections of tropical cycles in [GS23, LdMRS23, Sha15].

Proposition A.1. Let $(X,\mathcal{O}_X^\times)$ be a tropical manifold and $D_1,D_2\in \mathrm{Div}^{[0]}(X)$. Then, the intersection D_1*D_2 in the sense of [LdMRS23, §2.4] is equal to $D_1\cdot D_2$ [GS23, §3.4].

Proof. The definition of the intersection of Cartier divisors of both are local. Therefore, we only need to find an atlas $\mathcal{U}:=\{(\psi_i:U_i\to V_i)\}_{i\in I}$ of X such that U_i satisfies Proposition A.1. For any $x\in X$, there exists a standard chart [MR18, Definition 7.2.10], i.e., a chart $\psi_x\colon U_x\to \mathrm{LC}_x X\times\mathbb{T}^{\mathrm{sed}_X(x)}$ such that $\psi_x(x)=(0,-\infty)\in \mathrm{LC}_x X\times\mathbb{T}^{\mathrm{sed}_X(x)}$. Furthermore, U_x can be replaced with the product $V_x\times W_x$, where V_x is a contractible open neighborhood of 0 in $\mathrm{LC}_x X$, and W_x is a contractible open neighborhood of $-\infty$ in $\mathbb{T}^{\mathrm{sed}_X(x)}$. Moreover, we also may assume $H^1(V_x\times W_x;\mathcal{O}^\times_{V_x\times W_x})=0$, and there exists an open subset B_x of T_xX and a piecewise integer affine linear function f on B_x such that $\mathrm{div}_{V_x\times W_x}(D_1|_{V_x\times W_x})=\mathrm{div}_{V_x}(f|_{V_x})\times W_x$. Therefore, in the context of [GS23], as well as in the context of [LdMRS23], the intersection of sedentarity-0 Cartier divisors at x can be regarded as the direct product of the intersection of principal divisors on V_x and W_x, and thus it is enough to prove Proposition A.1 when $\mathrm{sed}_X(x)=0$. From now on, we assume $\mathrm{sed}_X(x)=0$ and V_x is a closed subset of B_x. Let $i\colon V_x\to B_x$ be the closed inclusion map of V_x. Then, for any $D_0\in\mathrm{Div}(V_x)$

$$i_*((f|_{V_x})\cdot D_0)=(f)\cdot i_*D_0\in Z_{n-2}(B_x). \tag{A.1}$$

Since B_x is an open subset of a finite dimensional real vector space, the pairing in the sense of [GS23, §3.4] is determined by [AR10, Definition 3.4]. By the same logic, the pairing in the sense of [LdMRS23, §2.4] is determined by the stable intersection of tropical Cartier divisors and tropical cycles in $\mathbb{R}^n$ [Mik06, Definition 4.4] from [Sha11, Proposition 2.1.9].

As mentioned in [Sha13, §2], it has been shown in [Rau16] and [Kat12] that the intersection of tropical cycles on $\mathbb{R}^n$ in the sense of [AR10] coincides with the intersection in the sense of [RGST05, Mik06]. $\square$

Next, we will see the compatibility of the intersection number of tropical 1-cycles in compact surfaces (admitting a global face structure) in the sense of [Sha15, LdMRS23] and in our sense. Before proving it, we recall eigenwave homomorphism from [IKMZ19, JRS18]. Every rational polyhedral space $(X, \mathcal{O}_X^\times)$ has the following exact sequence:

$$0 \to \mathbb{R}_X \to \mathcal{O}_X^\times \to \Omega^1_{\mathbb{Z},X} \to 0. \tag{A.2}$$

The exact sequence (A.2) defines the connecting homomorphism $\phi\colon H^{1,1}(X;\mathbb{Z}) \to H^{0,2}(X;\mathbb{R})$. The connecting homomorphism ϕ is the dual of the eigenwave homomorphism $\hat{\phi}\colon H^{\mathrm{BM}}_{n-1,n-1}(X;\mathbb{Z}) \to H^{\mathrm{BM}}_{n,n-2}(X;\mathbb{R})$ [MZ14, (5.2)] (see also [JRS18, Definition 2.9]) when X is a tropical manifold admitting a global face structure. In fact, both of them are compatible with the Poincaré duality for tropical manifolds admitting a global face structure [JRS18, Lemma 5.13].

Proposition A.2. Let X be a compact tropical surface admitting a global face structure and D_1, D_2 tropical 1-cycles on X. Then,

$$\deg(D_1.D_2) = \int_X [D_1]_{\mathrm{PD}} \cdot [D_2]_{\mathrm{PD}}. \tag{A.3}$$

where $D_1.D_2$ is the intersection of D_1 and D_2 in the sense of [Sha15].

Proof. Let $i = 1, 2$. From [MZ14, Theorem 5.4] (or more generally [JRS18, Theorem 1.1]), $[D_i] := \mathrm{cyc}_X(D_i)$ is in the kernel of the eigenwave homomorphism $\hat{\phi}\colon H^{\mathrm{BM}}_{1,1}(X;\mathbb{Z}) \to H^{\mathrm{BM}}_{2,0}(X;\mathbb{R})$ (see also [GS23, Theorem 5.13] to verify that the cycle map in [JRS18, Definition 4.13] is equivalent to the cycle map in [GS23, Definition 5.4]). Since $\hat{\phi}$ is compatible with the connecting homomorphism $\phi\colon H^{1,1}(X;\mathbb{Z}) \to H^{0,2}(X;\mathbb{R})$ [JRS18, Lemma 5.13], we can find $D_i' \in \mathrm{Div}^{[0]}(X)$ such that $[D_i'] = c_1(\mathcal{L}(D_i')) \frown [X] = [D_i]$ from [GS23, Proposition 5.12]. Then, from [Sha15, Proposition 3.34] we get

$$D_1.D_2 = D_1'.D_2', \quad \int_X [D_1]_{\mathrm{PD}} \cdot [D_2]_{\mathrm{PD}} = \int_X [D_1']_{\mathrm{PD}} \cdot [D_2']_{\mathrm{PD}}. \tag{A.4}$$

Therefore, we may assume D_1 and D_2 are in $\mathrm{Div}(X)^{[0]}$. Moreover, the definition of trace map, [GS23, Proposition 5.12], and the projection formula deduce

$$\int_X [D_1]_{\mathrm{PD}} \cdot [D_2]_{\mathrm{PD}} = a_{X*}(c_1(\mathcal{L}(D_1)) \frown [D_2]) = a_{X*}(\mathrm{cyc}_X(D_1 \cdot D_2)). \tag{A.5}$$

By [Sha11, Theorem 3.1.7], the intersection number of tropical 1-cycles in [Sha15] is equivalent to that in [Sha11]. Furthermore, the intersection of $D_1, D_2 \in \mathrm{Div}^{[0]}(X)$ defined in [LdMRS23, §2.4], is based on [Sha11], so the intersection in [LdMRS23] and that in [Sha15] are compatible with each other. From Proposition A.1, the intersection of D_1 and D_2 in [LdMRS23] equivalent to [GS23], so we obtain (A.3). $\square$

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