EXISTENCE OF A GLOBAL FUNDAMENTAL SOLUTION FOR HÖRMANDER OPERATORS

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Abstract. We prove that for a simply connected manifold M – and a vast class of non-simply connected manifolds – the existence of a fundamental solution for a differential operator $\mathcal{L} = \sum_{\alpha\in\mathbb{N}^q} r_\alpha \cdot X^\alpha$ of finite degree over M, follows via a saturation method from the existence of a fundamental solution for the associated "lifted" operator over a group G. Given some smooth complete vector fields $X_1, \ldots, X_q$ on M, suppose that they generate a finite dimensional Lie algebra $\mathfrak{g}$ satisfying the Hörmander's condition. The simply connected Lie group G is the unique such that $\mathrm{Lie}(G) \cong \mathfrak{g}$. It has the property that a right G-action exists over M, faithful and transitive, inducing a natural projection $E\colon G \to M$.

We generalize an approach developed by Biagi and Bonfiglioli. In particular we represent the group G as a direct product $M \times G^z$ where the model fiber G^z has a group structure. Our approach doesn't need any nilpotency hypothesis on the group G, therefore it broadens the spectrum of cases where techniques of the kind "lifting and approximation", as those introduce by the works of Rothschild, Stein, Goodman, can be applied.

Keywords. Partial Differential Equation's; Hörmander operators; Lifting and approximation theorem; Lie algebra of vector fields; Lie group action; Integration of vector fields.

1. Intro

Consider a family of smooth vector fields $X_1, \ldots, X_q$ over an orientable manifold M, such that $\mathrm{Lie}(X_1, \ldots, X_q)$ verifies the Hörmander's condition, *i.e.* it spans at every point x the tangent plane T_xM. Our main objects of study are differential operators $\mathcal{L}$ such that

$$\mathcal{L}(x) = \sum_{|\alpha|\leq k} r_\alpha(x) \cdot X^\alpha(x), \quad \forall x \in M, \tag{1.1}$$

where any $\alpha \in \mathbb{N}^q$ is a multi-index of length bounded by $k > 0$ and the $r_\alpha\colon M \to \mathbb{R}$ are smooth coefficients. The problem of finding solutions for operators of this kind has been largely studied, in particular starting from the degree 2 case $\mathcal{L} = \sum X_j^2$ over $M = \mathbb{R}^m$. In this case the operator is hypoelliptic, as proved in the paramount article [12]. Moreover, if a homogeneous group structure exists over $\mathbb{R}^m$ such that $\mathcal{L}$ is left invariant and homogeneous of degree 2, then Folland built in [8] a homogeneous fundamental solution for $\mathcal{L}$.

In their crucial work [15], Rothschild and Stein introduced a "lifting and approximation" technique that allows to locally approximate the operator $\mathcal{L}$ with a homogeneous left invariant operator, even if we don't have a group structure on M. They recovered the group structure by building a higher dimensional nilpotent group G_U for a covering family of bounded open subsets $U \subset M$. The Lie algebra $\mathrm{Lie}(G_U)$ is generated by vector fields that are free up to some finite step r, and a natural projection $E\colon G_U \to U$ exists. The idea is that we can lift the

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vector fields X_i to G_U, and this nilpotent group has a natural projection to the tangent bundle over U (see also [14]), therefore it allows a local polynomial approximation of the lifted vector fields. These tools allow estimates on the Sobolev norm of u and $\mathcal{L}u$ for any distribution u over U, and the study of the hypoellipticity of $\mathcal{L}$. For a focus on the Sobolev estimates, see [6].

In a similar fashion, Goodman develop a slightly different lifting in [11], such that the group G_U is generated by vector fields $\widetilde{X}_i$ that are E-related to the X_i, meaning that

$$\widetilde{X}(f \circ E) = Xf \quad \forall f \in C^\infty(M).$$

Rothschild-Stein and Goodman are both local approaches, meaning that any fundamental solution for $\mathcal{L}$ is built over any neighborhood U, and then glued together. The works [8, 9] by Folland are instead focused on a *global* approach on $\mathbb{R}^m$ where the vector fields X_i are homogeneous with respect to a set of dilations δ_λ and therefore the nilpotency of the higher dimensional group is preserved. In this case a global lifting $E\colon \mathbb{G} \to \mathbb{R}^m$ exists, where $\mathbb{G}$ is a Carnot group "generated" by the vector fields X_i as in the Goodman's lifting. As a consequence, the Carnot sub-Laplacian $\mathcal{L}_\mathbb{G}$ and the operator $\mathcal{L} = \sum X_i^2$ are E-related. For a wider study of this operator see also the monograph [5]. The advantage of the Folland's approach is that it allows a global representation and therefore it improves the knowledge of the associated Sobolev spaces.

As Folland and Stein treated in [10, 8], over a Carnot group the sub-Laplacian $\mathcal{L}_\mathbb{G}$ has always a fundamental solution $\Gamma_\mathbb{G}$. The question then arises, about the existence of a fundamental solution Γ for $\mathcal{L}$ which is in some sense E-related to $\Gamma_\mathbb{G}$. Biagi and Bonfiglioli answer positively to this question in [1], by showing that it is possible to obtain Γ from $\Gamma_\mathbb{G}$ with a "saturation" process. In their work they consider any differential operator in the form (1.1) over $\mathbb{R}^m$. The Carnot group $\mathbb{G}$ is then represented through a change of coordinates as a product $\mathbb{R}^m \times \mathbb{R}^p$, and the projection $E\colon \mathbb{G} \to \mathbb{R}^m$ becomes the first coordinate projection. If $\widetilde{\mathcal{L}}$ is the lifting of $\mathcal{L}$ and $\widetilde{\Gamma}$ the fundamental solution for $\widetilde{\mathcal{L}}$, then

$$\Gamma(x;y) = \int_{\mathbb{R}^p} \widetilde{\Gamma}(x,0;y,t)\mathrm{d}t \quad \forall x,y \in \mathbb{R}^m,\ x \neq y \tag{1.2}$$

is in fact the fundamental solution for $\mathcal{L}$ over $\mathbb{R}^m$. Their construction also allows for estimates on the original operator $\mathcal{L}$, developed for example in [3, 4].

In the present work we continue the program initiated by Biagi and Bonfiglioli, by considering a large class of differential operators over a manifold M and with no homogeneity conditions. In particular the differential operators considered are in the form (1.1) with the Lie algebra $\mathrm{Lie}(X_1, \dots, X_q)$ that is complete and finite dimensional but has no nilpotency conditions. Therefore we are expanding the cases where we are able to use the "lifted" vector fields, overcoming the polynomial-kind approximations that are typical in the "lifting and approximation" classic framework. This allows for global representations of the fundamental solutions even without a dilation structure on M.

The central tool of our analysis is the Fundamental Theorem of Lie Algebra Actions (see Theorem 2.2), whose usefulness in this setting has been emphasized by the same Biagi and Bonfiglioli in [2]. If G is any finite dimensional Lie group with a right action $\mu\colon M \times G \to M$, then its differential induces a Lie algebra morphism $T_\mu\colon\ \mathrm{Lie}(G) \to \mathfrak{X}(M)$ called infinitesimal generator of μ. The Fundamental Theorem states the converse, if G is a finite dimensional Lie group, M a smooth manifold and $T\colon\ \mathrm{Lie}(G) \to \mathfrak{X}(M)$ a complete Lie algebra morphism, then there exists a unique right G-action $\mu\colon M\times G\to M$ whose infinitesimal generator is T. If G is the unique connected and simply connected Lie group G such that $\mathrm{Lie}(G)=\mathrm{Lie}(X_1,\dots,X_q)$, then the Hörmander's condition corresponds to the transitivity of the induced right action $\mu\colon M\times G\to M$. By fixing a starting point $z\in M$, we obtain the Folland-style morphism $E:=\mu(z,-)\colon G\to M$.

We are now able to show a product representation

$$G\cong M\times G^z,$$

where $G^z=E^{-1}(z)$ is the central G fiber and has a group structure. This construction allows for a general saturation procedure and therefore introduce the main result of our work (see Theorem 4.5 for the details).

Theorem. *If $\mathcal{L}$ is a differential operator of the kind* (1.1), *$\widetilde{\mathcal{L}}$ its lifting through E and $\widetilde{\Gamma}$ is a fundamental solution for $\widetilde{\mathcal{L}}$, then*

$$\Gamma(x;y)=\overline{\rho}(x)\cdot\int_{G^z}\widetilde{\Gamma}(x,s;y,t)\mathrm{d}t,\ \forall x,y\in M\ x\neq y, \tag{1.3}$$

is a fundamental solution for $\mathcal{L}$, where $\overline{\rho}\colon M\to\mathbb{R}$ is a smooth everywhere non-null function and s any element of the group G^z.

It has to be noted that, by exploiting the action of the central fiber G^z on any other E-fiber, we can prove that

$$\widetilde{\mathcal{L}}^*=\mathcal{L}^*+\sum_{|\gamma|\geq 1} r^*_{\beta,\gamma}(x)\cdot X^\beta Y^\gamma,$$

where the vectors Y are defined along the G^z coordinate, and the coefficients depend only on the M coordinate. The evaluation of the differential operator is therefore invariant along the fibers of the E map. This "vertical invariance" of the lifting is a key result, allowing the saturation procedure even without a dilation structure on M or G.

In Section 2 we introduce the notation and the Fundamental theorem of Lie algebra actions. In 3 the group G generated by the given vector fields on M, is represented as a direct product between M and a group fiber. In Section 4 our main theorem is stated and proved for simply connected manifolds, while in Section 5 we consider a class of non-simply connected manifolds where the same result is still valid.

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2. Basics

2.1. **Basic notations.** We consider a smooth simply connected manifold M and a bunch of vector fields over it. We denote by $\mathfrak{X}(M)$ the space of vector fields over M. Given $X_1,X_2,\dots,X_q\in\mathfrak{X}(M)$, if $\alpha=(\alpha_1,\dots,\alpha_q)$ is a multi-index in $\mathbb{N}^q$, we use X^α as a short notation for

$$X^\alpha=X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_q^{\alpha_q}.$$

Let $\mathfrak{g}$ be the Lie algebra generated by the X_is, that is the algebra generated by the X_is and their commutators of any order

$$[X_{i_1},[X_{i_2}[\dots,X_{i_k}]]],$$

then we suppose that $\mathfrak{g}$ respects three main conditions stated here.

Remark 2.1. These conditions are in fact a completeness condition and a re-writing of the well known Hörmander's condition:

(1) $\mathfrak{g}$ is finite dimensional, if necessary we will complete the sequence above to a basis $\{X_1, \dots, X_n\}$;
(2) for any $X \in \mathfrak{g}$, X is a complete vector field;
(3) at every $x \in M$, $\mathfrak{g}_x := \{X_x \mid X \in \mathfrak{g}\}$ is the whole tangent space $T_x M$.

If the three conditions are respected and we denote by m the M dimension, then by definition $m \leq n$.

We use the following notation for a differential operator $\mathcal{L}$ of order k over M,

$$\mathcal{L} = \sum_{|\alpha| \leq k} r_\alpha \cdot X^\alpha,$$

where $|\alpha|$ is the length of the multi-index α, and the r_α are real coefficients.

By Lie's Third Theorem (see [16]) we know that there exists unique a connected and simply connected Lie group G such that $\mathrm{Lie}(G) \cong \mathfrak{g} = \mathrm{Lie}(X_1, \dots, X_q)$. We denote by $L\colon \mathrm{Lie}(G) \to \mathfrak{g}$ the isomorphism, and sometimes we will use the notation $\widetilde{X}$ for $L^{-1}X$ with X any vector field in $\mathfrak{g}$. Therefore $L\widetilde{X} = X$.

2.2. **Lie algebra maps and right group actions.** The key ingredient of our work is the Fundamental Theorem of Lie Algebra Actions. We recall it here. Consider a simply connected Lie group G, a smooth manifold M and a right G-action

$$\mu\colon M \times G \to M.$$

When it is clear from the context, we will denote by $z \cdot \xi$ the action $\mu(z, \xi)$. We denote by $\mu^{(z)}$ the map $\mu(z, -)\colon G \to M$.

This action induces a map from the Lie algebra associated to G to the vector space $T_z M$ for any point z of M. Indeed, to $\widetilde{X} \in \mathrm{Lie}(G)$ we can associate the vector

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu(z, \mathrm{Exp}_G(t\widetilde{X})) = \mathrm{d}_e \mu^{(z)}(\widetilde{X}).$$

Moreover we observe that, if we denote by $L_\xi\colon G \to G$ the left ξ multiplication inside G, then

$$\mu^{(z \cdot \xi)} = \mu^{(z)} \circ L_\xi,$$

this implies that

$$\mathrm{d}_e \mu^{(z \cdot \xi)} = \mathrm{d}_\xi \mu^{(z)} \circ \mathrm{d}_e L_\xi\colon \mathrm{Lie}(G) \to T_{z \cdot \xi} M.$$

As any vector $\widetilde{X}$ in $\mathrm{Lie}(G)$ is left-invariant, the formula above shows that

$$\mathrm{d}_e \mu^{(z \cdot \xi)} \widetilde{X} = \mathrm{d}_\xi \mu^{(z)} \widetilde{X}. \tag{2.1}$$

This allows to define a map

$$T_\mu\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$$

which is called the *infinitesimal generator* of the μ action. The important result about the infinitesimal generator T_μ is the fact that it is a Lie algebra morphism ([13, Theorem 20.15]). Given any Lie algebra $\mathfrak{h}$ and a Lie algebra morphism $T\colon \mathfrak{h} \to \mathfrak{X}(M)$, this morphism is called complete if for any $\widetilde{X} \in \mathfrak{h}$, the vector field $T(\widetilde{X})$ is complete.

Theorem 2.2 (Fundamental Theorem of Lie Algebra Actions, [13, Theorem 20.16])**.** *Let M be a smooth manifold, G a simply connected Lie group and $T\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$ a complete Lie algebra morphism. Then, there exists a unique right G-action $\mu\colon M \times G \to M$ whose infinitesimal generator is T.*

Thanks to the above theorem, if we consider the aforementioned isomorphism

$$L\colon \mathrm{Lie}(G) \to \mathfrak{g} \subset \mathfrak{X}(M),$$

this is the infinitesimal generator of a right G-action

$$\mu_L\colon M \times G \to M.$$

As already said, we denote by $E\colon G \to M$ the map $\mu_L^{(z)}$. As a consequence of (2.1) and the definition of the L isomorphism, for any $X \in \mathfrak{g}$, $\widetilde{X} = L^{-1}X$ and X are E-related, meaning that

$$\mathrm{d}_\xi E(\widetilde{X}) = X_{E(\xi)}. \tag{2.2}$$

This is also proved by Biagi and Bonfiglioli, [2, Formula (3.10)].

Observe that, by construction, the E map respects the equality

$$E(\xi) = \Psi_1^{L \log_G(\xi)}(z) \tag{2.3}$$

for ξ in an opportune neighborhood of the G identity element e, where Ψ_t^X is the usual flow operator along the vector field X.

We denote by $G^x := E^{-1}(x)$ the fiber over any point $x \in M$ and observe that in the case of $z = E(e)$, the fiber G^z is a subgroup of G. Moreover, it is possibile to define a G^z-action on any fiber G^x by left multiplication. Indeed, if $\xi \in G^z$ and $\eta \in G^x$, $\xi\eta \in G^x$ for any $x \in M$. This action is faithful and transitive.

Remark 2.3. If M is simply connected, then the E-fibers are connected too. Indeed, if a path γ relying two connected components of G^x exists for some $x \in M$, then its image $E(\gamma)$ would be an un-contractible loop.

Example 2.4. Consider the case of the Grushin operator $\mathcal{L} = \partial_{x_1}^2 + (x_1\partial_{x_2})^2$ defined over $\mathbb{R}^2$. We consider the smooth and complete vector fields $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$. Even if they don't generate linearly the tangent space at $(0,0)$, their Lie algebra respects the Hörmander's condition everywhere. Indeed, $\mathfrak{g} = \mathrm{Lie}(X_1, X_2) = \langle X_1, X_2, X_3\rangle$ with $X_3 = [X_1, X_2] = \partial_{x_2}$, which span $\mathbb{R}^2$ at every $(x_1, x_2) \in \mathbb{R}^2$.

The connected and simply connected group whose Lie algebra is $\mathfrak{g}$ is the first Heisenberg group $\mathbb{H}^1$. If we start at $z = (0,0)$ setting ξ_1, ξ_2, ξ_3 as exponential coordinates of the first type on $\mathbb{H}^1$ with respect to the base X_1, X_2, X_3, then

$$E(\xi_1, \xi_2, \xi_3) = \exp_{(0,0)}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = \left(\xi_1, \xi_3 + \frac{\xi_1\xi_2}{2}\right),$$

where exp is the usual exponential map on $\mathbb{R}^2$.

3. Coordinate system on the group

3.1. **The group as a fiber bundle.** Given the isomorphism $L\colon \mathrm{Lie}(G) \to \mathfrak{g}$ as above, a simply connected smooth manifold M and a point $z \in M$, we consider the map $E = \mu_L^{(z)}$ and the z stabilizer G^z.

Remark 3.1. By the hypothesis in Remark 2.1, for any $z \in M$ there exists a vector subspace $\mathfrak{g}' \subset \mathfrak{g}$ of dimension m and a neighborhood U of z such that $\mathfrak{g}'_x = T_x M$ for any $x \in U$. Therefore it is an exponential-kind diffeomorphic (in a neighborhood of the origin) map

$$\begin{aligned} &\mathfrak{g}' \to M \\ &X \mapsto \Psi_1^X(z). \end{aligned}$$

We denote by $\overline{\log}\colon U \to \mathfrak{g}'$ its inverse. This allows to define a trivialization of the map $E\colon G \to M$. Indeed, if $G_U := E^{-1}(U)$, consider the map $G_U \to U \times G^z$,

$$\xi \mapsto \left(E(\xi),\ \xi \cdot \mathrm{Exp}_G(-L^{-1}\overline{\log}(E(\xi)))\right).$$

This is well defined as a consequence of (2.3). Equivalently, $\xi = s \cdot \mathrm{Exp}_G(\widetilde{X})$ where $s \in G^z$ and $\widetilde{X}$ is in $\mathfrak{g}'$.

We built a local trivialization of the bundle $G \to M$, proving it is a fiber bundle.

Remark 3.2. The *vertical* bundle associated to E is the subbundle of $TG \to M$ defined as $V(E) := \mathrm{Ker}(\mathrm{d}E) \subset TM$. An Ehresmann connection on a fiber bundle is the data of a *horizontal bundle*, meaning a sub-bundle $H(E) \subset TM$ such that $V(E) \oplus H(E) = TM$. Observe that this implies $H_\xi(E) \cong T_{E(\xi)}M$ for any $\xi \in G$.

The differential $\mathrm{d}E$ is surjective at any $\xi \in G$, therefore if we have an Ehresmann connection $H(E)$ and if $x = E(\xi)$, there exists a neighborhood $U \subset M$ of x and a smooth subvariety $\widetilde{U} \subset G$ containing ξ such that $E|_{\widetilde{U}}$ is a diffeomorphism between $\widetilde{U}$ and U, and $T\widetilde{U} = H(E)|_{\widetilde{U}}$. Over U we can therefore define $\ell\colon U \to G$ as the E inverse. By simple connectedness of M, we can impose $\ell(z) = e$ the identity element of G, and extend ℓ to a section $\ell\colon M \to G$ on the whole manifold.

Example 3.3. We consider again the case of the Grushin operator and the induced group $\mathbb{H}^1$ acting on $\mathbb{R}^2$, as in Example 2.4. Biagi and Bonfiglioli in their work [1] consider the coordinate change

$$(\xi_1, \xi_2, \xi_3) \mapsto \begin{cases} x_1 &= \xi_1 \\ x_2 &= \xi_3 + \frac{\xi_1\xi_2}{2} \\ x_3 &= \xi_2 \end{cases}$$

In these coordinates the map E is the coordinate projection $\mathbb{H}^1 \cong \mathbb{R}^3 \to \mathbb{R}^2$ and the group assumes the usual matrix representation.

In this case $V(E) = \langle x_1\widetilde{X}_3 - \widetilde{X}_2\rangle$ by construction, and the product induces naturally an Ehresmann connection $H(E) = \langle \partial_{x_1}, \partial_{x_3}\rangle = \langle \widetilde{X}_1, \widetilde{X}_3\rangle$. Observe that the liftings of X_1, X_2 are

$$\begin{aligned} L^{-1}X_1 &= \partial_{x_1} \\ L^{-1}X_2 &= x_1\partial_{x_2} + \partial_{x_3}, \end{aligned}$$

therefore the two vector fields are everywhere linearly independent.

The section ℓ above allows to build a trivialization of G. Observe that $G^x = G^z\ell(x)$ by construction, therefore we have

$$\begin{aligned} \psi\colon G &\xrightarrow{E \times R^{-1}_{\ell\circ E}} M \times G^z \\ \xi &\mapsto (E(\xi), \xi \cdot \ell(E(\xi))^{-1}) \end{aligned}$$

as a smooth isomorphism. Therefore the inverse diffeomorphism defines a different coordinate system on G,

$$(x, s) \mapsto s \cdot \ell(x) \in G. \tag{3.1}$$

In this setting, the fiber G^x is identified with $\{x\} \times G^z \subset M \times G^z$. Moreover, if $E(\xi) = x$, then the following equalities hold for the tangent space to the G^x-fiber,

$$T_\xi G^x = \mathrm{d}R_{\ell(x)}\left(T_{\xi\ell(x)^{-1}} G^z\right) = \mathrm{Ker}(\mathrm{d}_\xi E). \tag{3.2}$$

We thus represented G as a smooth fiber bundle over M such that every fiber is identified with the G^z group.

Let ω_G be the left invariant volume form over G (unique up to a multiplicative constant). If $\widetilde{X}_1, \dots, \widetilde{X}_n$ is a basis of $\mathrm{Lie}(G)$ and $\xi_1, \dots, \xi_n$ the exponential coordinates of the first kind induced on G by the exponential map with respect to this basis, then we can set

$$\omega_G = \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n.$$

Moreover, we consider the scalar product that makes $\widetilde{X}_1, \dots, \widetilde{X}_n$ in an orthonormal basis. At every point $\xi \in G$, this scalar product induces a split

$$\mathrm{Lie}(G) = \mathrm{Ker}(\mathrm{d}_\xi E)^\perp \oplus \mathrm{Ker}(\mathrm{d}_\xi E). \tag{3.3}$$

Then, considering the definition of ψ and the identification (3.2), there is a natural identification

$$T_\xi G = \mathrm{Ker}(\mathrm{d}_\xi E)^\perp \oplus T_\xi G^x \xrightarrow{\mathrm{d}\psi = \mathrm{d}E \oplus \mathrm{d}R^{-1}_{\ell(x)}} T_x M \oplus T_{\xi\ell(x)^{-1}} G^z = T_{(x, \xi\ell(x)^{-1})}(M \times G^z). \tag{3.4}$$

If for every $\xi \in G$, $\omega_{\mathrm{Ker}}(\xi)$ and $\omega_\perp(\xi)$ are the naturally induced volume forms on $\mathrm{Ker}(\mathrm{d}_\xi E)$ and $\mathrm{Ker}(\mathrm{d}_\xi E)^\perp$ respectively, then the following wedge formula is also true,

$$\omega_G = \omega_\perp \wedge \omega_{\mathrm{Ker}}.$$

Remark 3.4. Observe that once we fix the orientation of $\widetilde{X}_1, \dots, \widetilde{X}_n$, ω_G is univocally determined while $\omega_\perp$ and ω_{Ker} are determined together up to orientation

In the following sections we are going to precise how the coordinate change to $M \times G^z$ acts on the natural volume form over this last space, $\mathrm{Vol}_M \wedge \omega_{\mathrm{Ker}}$. In particular, we are going to prove the following important coordinate change formula.

Theorem 3.5. *There exists a smooth everywhere non-null function $\overline{\rho} \colon M \to \mathbb{R}$, such that at any point $\xi \in G$ with $x = E(\xi)$,*

$$\omega_G(\xi) = \overline{\rho}(x) \cdot \mathrm{d}_\xi \psi^* \left(\mathrm{Vol}_M \wedge \omega_{\mathrm{Ker}}\right).$$

3.2. **Coordinate change on the base manifold.** We revisit the coordinate change rule on volume forms in order to adapt it to our situation.

If V is a (real) vector space of finite dimension m, for any basis $\boldsymbol{v} = v_1, \dots, v_m$ of V, we denote by $\omega_{\boldsymbol{v}}$ the m-form such that $\omega_{\boldsymbol{v}}(v_1, \dots, v_m) = 1$. Consider a scalar product $\langle , \rangle$ over V, if $u \in V$ we denote by u^* the dual linear map $\langle u, - \rangle \in V^*$. For example if $\boldsymbol{e} = e_1, \dots, e_m$ is an orthonormal basis of V, then

$$\omega_{\boldsymbol{e}} = e_1^* \wedge \cdots \wedge e_m^*.$$

This form is called volume form of V and sometimes also denoted by $\omega_V = \omega_{\boldsymbol{e}}$. By fixing the sign of the volume form, we chose the orientation of any orthonormal basis. If V is the tangent space to a Riemannian oriented manifold M and the scalar product on it is the associated Riemannian metric, then we denote the volume form by Vol_M.

If $B\colon V \to V$ is a linear isomorphism, we use the notation $B^*\omega$ or $\omega(B-)$ for the form

$$B^*\omega(w_1, \dots, w_m) = \omega(B(w_1, \dots, w_m)) = \omega(Bw_1, \dots, Bw_m).$$

As $\omega(B-) = \det(B) \cdot \omega$, if $B\boldsymbol{v}$ is the basis $Bv_1, \dots, Bv_m$, we have $B^*\omega_{B\boldsymbol{v}} = \omega_{B\boldsymbol{v}}(B-) = \omega_{\boldsymbol{v}}$ and therefore

$$\omega_{B\boldsymbol{v}} = \frac{1}{\det B} \cdot \omega_{\boldsymbol{v}}, \tag{3.5}$$

which is the classic coordinate change formula.

Remark 3.6. In this setting we recall a well known volume formula. Indeed, given a basis $\boldsymbol{v} = v_1, \dots, v_m$ of V, if B is the map sending e_i to v_i for any $i = 1, \dots, m$, then $\omega_{\boldsymbol{v}} = \omega_V(B^{-1}-)$, which gives $\omega_V = \det(B) \cdot \omega_{\boldsymbol{v}}$. We observe that $\det(B) = \sqrt{\det(BB^T)}$ and BB^T is precisely the matrix of the scalar products $g_{ii'} = (\langle v_i, v_{i'} \rangle)_{ii'}$, thus

$$\omega_V = \sqrt{\det(g_{ii'})} \cdot \omega_{\boldsymbol{v}},$$

where we supposed that $v_1, \dots, v_m$ is positively oriented and therefore $\det(B) = \det(B^T) > 0$.

We show a slightly more general result. If V' is another (real, finite dimensional) vector space of dimension m' equipped with a scalar product and $B\colon V' \to V$ a linear surjective map, let $\omega_\perp$ be the volume form on $\mathrm{Ker}(B)^\perp$.

Lemma 3.7. *If ω_V is the volume form on V, then*

$$B^*\omega_V = \pm\sqrt{\det(BB^T)} \cdot \omega_\perp.$$

Remark 3.8. Here the $\pm$ sign means that the orientation determined by B is not canonical.

Proof. Consider an orthonormal positively oriented basis $\boldsymbol{v}$ of $\mathrm{Ker}(B)^\perp$, then by construction $B\boldsymbol{v}$ is a basis of L, and $\omega_{B\boldsymbol{v}}(B-) = \omega_{\boldsymbol{v}} = \omega_\perp$. Moreover, consider the map $C\colon V \to \mathrm{Ker}(B)^\perp$ sending the orthonormal basis $\boldsymbol{e}$ of L in $\boldsymbol{v}$, then $\omega_{\boldsymbol{e}} = \omega_V$ and

$$\omega_{B\boldsymbol{v}} = \omega_{BC\boldsymbol{e}} = \frac{1}{\det(BC)} \omega_V.$$

By construction the matrix C is orthogonal and $\det(BB^T) = \det(BCC^TB^T) = \det(BC)^2$, which proves the lemma. $\square$

We are going to apply Lemma 3.7 in our case, with the linear map given by the restriction of the differential $\mathrm{d}_\xi E\colon \mathrm{Ker}(\mathrm{d}_\xi E)^\perp \to T_xM$, where $x = E(\xi)$ as usual.

Remark 3.9. Observe that by the Formula (2.2) the image of $\mathrm{d}_\xi E$ only depends on the chosen vector on $\mathrm{Lie}(G)$ and the point $E(\xi)$, *i.e.* it is a function of $E(\xi)$ and not ξ. In particular,

$$\mathrm{d}_\xi E = \mathrm{ev}_{E(\xi)} \circ L. \tag{3.6}$$

In the following we use the notation $\mathcal{X}(x) := \mathrm{d}_\xi E\colon \mathrm{Lie}(G) \to T_xM$ where $x = E(\xi)$.

Remark 3.10. This point is crucial for the following of our work, in particular because it states that the subspaces $\mathrm{Ker}(\mathrm{d}_\xi E) \subset \mathrm{Lie}(G)$ is invariant along the G^x fibers. Moreover, for any $x \in M$ we have the isomorphism

$$\mathrm{Ker}(\mathcal{X}(x)) = \mathrm{Ker}(\mathrm{d}_\xi E) \cong \mathrm{Lie}(G^z) \subset \mathrm{Lie}(G), \tag{3.7}$$

but for different $x \in M$ it corresponds to different inclusions of $\mathrm{Lie}(G^z)$ inside $\mathrm{Lie}(G)$.

If $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$, and $\widetilde{X}_1, \dots, \widetilde{X}_n$ the "lifted" basis on $\mathrm{Lie}(G)$, then the map $\mathcal{X}$ is represented by the $m \times n$ matrix

$$\mathcal{X}(x) = (X_1(x)| \dots |X_n(x)).$$

Moreover, $\mathcal{X}^T(x)$ is the adjoint operator, and therefore

$$\mathrm{Ker}(\mathrm{d}_\xi E)^\perp = \mathrm{Ker}(\mathcal{X}(x))^\perp = \mathrm{Im}(\mathcal{X}(x)^T).$$

We can conclude by Lemma 3.7, obtaining the coordinate change formula for the M component in the isomorphism $\psi \colon G \xrightarrow{\sim} M \times G^z$.

$$\left(\mathrm{d}_\xi E|_{\mathrm{Ker}(\mathrm{d}E)^\perp}\right)^* \mathrm{Vol}_M = \pm\sqrt{\det(\mathcal{X}(x)\mathcal{X}^T(x))} \cdot \omega_\perp. \tag{3.8}$$

As already said, the $\pm$ sign means that the orientation induced by $\mathcal{X}(x)$ is not canonical. Anyway, up to re-ordering the basis $X_1, \dots, X_n$ of $\mathfrak{g}$, we can assume that the sign above is $+$ in an opportune neighborhood of M, and therefore by orientability of M the scalar factor is positive for any point of M.

Remark 3.11. The determinant above is everywhere non-null because of the Hörmander-like hypothesis on the vector fields $X_1, \dots, X_n$.

3.3. **Coordinate change on the fibers.** As seen in (3.2), the right multiplication $R_{\ell(x)}$ induces an isomorphism between the tangent spaces of the fibers G^z and G^x. Therefore, if we consider the action induced on the volume form of this tangent spaces, that we called ω_{Ker}, we obtain

$$\mathrm{d}R^*_{\ell(x)}\left(\omega_{\mathrm{Ker}}(\xi)\right) = c(\xi) \cdot \left(\omega_{\mathrm{Ker}}(\xi\ell(x)^{-1})\right),$$

where $x = E(\xi)$ and c is an everywhere non-null smooth function.

Lemma 3.12. *The scaling function c is constant on any fiber G^x of $G \xrightarrow{E} M$.*

Proof. By construction the scalar product on G, and therefore the volume form, is invariant by left multiplication. As pointed out in Remark 3.10 the subspace $\mathrm{Ker}(\mathrm{d}E) \subset \mathrm{Lie}(G)$ is invariant along the fibers G^x, meaning it is invariant under left multiplication by elements in the subgroup $G^z \subset G$.

All this implies that the volume form ω_{Ker} is invariant by left multiplication by G^z too. That is, for any $s \in G^z$,

$$\omega_{\mathrm{Ker}} = \mathrm{d}L_s^*\omega_{\mathrm{Ker}}.$$

Moreover, left and right multiplication are commutative, therefore by developing the calculations we obtain

$$\begin{aligned} c(\xi) \cdot \omega_{\mathrm{Ker}}\left(s^{-1}\xi\ell(x)^{-1}\right) &= \mathrm{d}L_s^*\mathrm{d}R^*_{\ell(x)}\omega_{\mathrm{Ker}}(\xi) \\ &= \mathrm{d}R^*_{\ell(x)}\mathrm{d}L_s^*\omega_{\mathrm{Ker}}(\xi) \\ &= c(s^{-1}\xi) \cdot \omega_{\mathrm{Ker}}\left(s^{-1}\xi\ell(x)^{-1}\right). \end{aligned}$$

Thus, $c(\xi) = c(s^{-1}\xi)$ for any $s \in G^z$ which is equivalent to the thesis of the lemma. □

With a slight abuse of notation we denote by $c(x)$ the value of $c(\xi)$ on the fiber G^x, obtaining a smooth and everywhere non-null function $c\colon M \to \mathbb{R}$. We can resume the result above with the formula

$$\mathrm{d}R^*_{\ell(x)}\omega_{\mathrm{Ker}} = c(x) \cdot \omega_{\mathrm{Ker}}. \tag{3.9}$$

Formulas (3.8) and (3.9) allow to prove Theorem 3.5. Indeed, by what we showed, if $E(\xi) = x$, then

$$\begin{aligned}\omega_G(\xi) &= (\omega_\perp \wedge \omega_{\mathrm{Ker}})(\xi)\\ &= (\rho(x) \cdot \mathrm{d}_\xi E^* \,\mathrm{Vol}_M) \wedge \Big(c(x) \cdot (\mathrm{d}R^{-1}_{\ell(x)})^* \omega_{\mathrm{Ker}}\Big)\\ &= \overline{\rho}(x) \cdot \mathrm{d}_\xi \psi^* \left(\mathrm{Vol}_M \wedge \omega_{\mathrm{Ker}}\right),\end{aligned}$$

where we used the notation

$$\rho(x) = \frac{1}{\sqrt{\det\left(\mathcal{X}(x)\mathcal{X}(x)^T\right)}} \tag{3.10}$$

$$\overline{\rho}(x) = \rho(x) \cdot c(x) \tag{3.11}$$

and the fact that

$$\mathrm{d}_\xi \psi = \Big(\mathrm{d}_\xi E|_{\mathrm{Ker}(\mathrm{d}E)^\perp}\Big) \oplus \mathrm{d}R^{-1}_{\ell(x)}.$$

4. Saturating fundamental solutions

4.1. **Lifting differential operators.** Consider a differential operator $\mathcal{L}$ over M

$$\mathcal{L} = \sum_{|\alpha| \le k} r_\alpha(x) X^\alpha,$$

where the sequence $X_1, \dots, X_n$ of complete vector fields is, as before, a basis of the Lie algebra $\mathfrak{g} \subset \mathfrak{X}(M)$ and the coefficients r_α are smooth functions over M. The map $E\colon G \to M$ induces a canonical lifting of $\mathcal{L}$ on G, that is

$$\widetilde{\mathcal{L}} := \sum_{|\alpha| \le k} r_\alpha(E(\xi)) \widetilde{X}^\alpha.$$

For any vector field $\widetilde{X} \in \mathrm{Lie}(G)$, we define the vector fields $\widetilde{X}^K(\xi)$ and $\widetilde{X}^\perp(\xi)$, its projections to $\mathrm{Ker}(\mathrm{d}_\xi E)$ and $\mathrm{Ker}(\mathrm{d}_\xi E)^\perp$ respectively. If $X \in \mathfrak{g}$, $\widetilde{X} = L^{-1}X$ and $E(\xi) = x$, then we have

$$\mathrm{d}_\xi E(\widetilde{X}^\perp) = X(x) \tag{4.1}$$

as $\mathrm{d}E$ is the first coordinate projection in $TG \cong TM \times TG^z$.

Regarding the vector component along the G^x fiber, first we observe that the vectors $\widetilde{X}^K$ are invariant with respect to left multiplication by elements of G^z. Indeed, by Remark 3.9 the subspace $\mathrm{Ker}(\mathrm{d}_\xi E) \subset \mathrm{Lie}(G) = T_\xi G$ is invariant with respect to left multiplication L_s if $s \in G^z$. At the same time, the vectors in $\mathrm{Lie}(G)$ and the scalar product that we introduced on it, are invariant with respect to left multiplication by any G element. This proves that the projection $\widetilde{X}^K$ of $\widetilde{X}$ on $\mathrm{Ker}(\mathrm{d}E)$ only depends on the fiber G^x where it is taken.

As a consequence of all this, if $\widetilde{X}^K(\xi)$ is the vertical component of a vector $\widetilde{X} \in \mathrm{Lie}(G) = T_\xi G$ at the point ξ such that $E(\xi) = x$, then the coordinate change described in (3.4) identifies it with

$$\mathrm{d}R^{-1}_{\ell(x)}\widetilde{X}^K(\xi) \in \mathrm{Ker}(\mathrm{d}_{\xi\ell(x)^{-1}}E) = T_{\xi\ell(x)^{-1}}G^z \cong \mathrm{Lie}(G^z) \subset \mathrm{Lie}(G).$$

Observe that the map $\widetilde{X} \mapsto \mathrm{d}R^{-1}_{\ell(x)}\widetilde{X}^K$ is linear for any $\xi \in G$, therefore we can write it as a linear (degenerate) morphism

$$\mathcal{M}(\xi)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G^z),$$

varying smoothly with respect to ξ.

Lemma 4.1. *The map $\mathcal{M}$ introduced above is invariant along any G^x fiber.*

Proof. For any $\widetilde{X} \in \mathrm{Lie}(G)$, $s \in G^z$ and $x \in M$,

$$\begin{aligned}
\mathcal{M}(\xi)\widetilde{X}(s\xi\ell(x)^{-1}) &= \mathrm{d}L_s \mathrm{d}R^{-1}_{\ell(x)}\widetilde{X}^K(\xi)\\
&= \mathrm{d}R^{-1}_{\ell(x)}\mathrm{d}L_s\widetilde{X}^K(\xi)\\
&= \mathrm{d}R^{-1}_{\ell(x)}\widetilde{X}^K(s\xi)\\
&= \mathcal{M}(s\xi)\widetilde{X}(s\xi\ell(x)^{-1}),
\end{aligned}$$

where we used the commutativity of left and right multiplication. Since this is true for any $\widetilde{X} \in \mathrm{Lie}(G)$ and $s \in G^z$, then $\mathcal{M}$ is constant on any fiber G^x. □

With another abuse of notation we write $\mathcal{M}(x)$ for the map $\mathcal{M}$ on the fiber G^x, obtaining a smooth and everywhere maximum rank morphism $\mathcal{M}\colon M \to \mathrm{Lin}(\mathrm{Lie}(G), \mathrm{Lie}(G^z))$ such that

$$\mathrm{d}R^{-1}_{\ell(x)}\widetilde{X}^K = \mathcal{M}(x)\widetilde{X}. \tag{4.2}$$

Thanks to (4.1) and (4.2) we write down the lifting of any vector field $X \in \mathfrak{g}$ in the coordinate system of $M \times G^z$. In particular, X is lifted to $\widetilde{X}$ on TG and to

$$(\mathrm{d}E\widetilde{X}^\perp, \mathrm{d}R^{-1}_{\ell\circ E}\widetilde{X}^K) = (X, (\mathcal{M}\circ E)\widetilde{X})$$

on $TM \times TG^z$. In particular the lifted operator $\widetilde{\mathcal{L}}$ can be represented as

$$\begin{aligned}
\widetilde{\mathcal{L}} &= \sum_{|\alpha|\le k} r_\alpha(x)(X + \mathcal{M}(x)\widetilde{X})^\alpha\\
&= \sum_{|\alpha|\le k} r_\alpha(x)X^\alpha + \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|\ge 1}} r_\alpha(x)X^\beta(\mathcal{M}(x)\widetilde{X})^\gamma.
\end{aligned}$$

If we consider a basis $Y_1, \dots, Y_p$ of $\mathrm{Lie}(G^z)$ and consider the representation in this basis of $\mathcal{M}$, we get in particular $\mathcal{M}(x)\widetilde{X}_i = \sum_{j=1}^p \mathcal{M}_{ij}(x)Y_j$ for any $i = 1, \dots m$. After rearranging the terms, we get

$$\begin{aligned}
\widetilde{\mathcal{L}} &= \sum_{|\alpha|\le k} r_\alpha(x)X^\alpha + \sum_{\substack{|\beta+\gamma|\le k\\ |\gamma|\ge 1}} r_{\beta,\gamma}(x)X^\beta Y^\gamma\\
&= \mathcal{L} + R.
\end{aligned}$$

Remark 4.2. Observe that with this representation of the lifted operator, if we consider the coordinates (x, s) on $M \times G^z$, the X vector fields only act on the x coordinates while the Y vector fields only act on the s coordinates. This separation of the coordinates will be crucial in generalizing the Biagi-Bonfiglioli technique.

As a consequence, if we define in the usual way the dual operator $\mathcal{L}^*$ over $M \times G^z$, it has a similar form

$$\mathcal{L}^* = \sum_{|\alpha| \le k} r^*_\alpha(x) X^\alpha + \sum_{\substack{|\beta+\gamma| \le k \\ |\gamma| \ge 1}} r^*_{\beta,\gamma}(x) X^\beta Y^\gamma. \tag{4.3}$$

Example 4.3. Consider the operator $\mathcal{L} = \partial^2_{x_1} + (\sin(x_1)\partial_{x_2})^2$. As in the case of the Grushin operator the Lie algebra generated by $X_1 = \partial_{x_1}$ and $X_2 = \sin(x_1)\partial_{x_2}$ verifies the Hörmander's condition everywhere, but in this case $\mathfrak{g} = \mathrm{Lie}(X_1, X_2) = \langle X_1, X_2, X_3 \rangle$ is not nilpotent. Indeed $[X_1, X_2] = X_3 = \cos(x_1)\partial_{x_2}$ while $[X_1, X_3] = -X_2$.

If G is the simply connected Lie group whose Lie algebra is $\mathfrak{g}$, and we put on $\mathrm{Lie}(G)$ the metric induced by the orthonormal basis $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3$, then the map $E\colon G \to \mathbb{R}^2$ respects

$$\begin{aligned} \mathrm{Ker}(\mathrm{d}E) &= \langle X_3' = \cos(x_1)\widetilde{X}_2 - \sin(x_1)\widetilde{X}_3 \rangle \\ \mathrm{Ker}(\mathrm{d}E)^\perp Y &= \langle \widetilde{X}_1,\ X_2' = \sin(x_1)\widetilde{X}_2 + \cos(x_1)\widetilde{X}_3 \rangle. \end{aligned}$$

If ξ_1, ξ_2, ξ_3 are the exponential coordinates of the first type associated to $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3$, then the coordinates x_1, x_2, x_3 associated to the vector fields $\widetilde{X}_1, X_2', X_3'$ in $\mathrm{Lie}(G)$ are

$$\begin{aligned} x_1 &= \xi_1 \\ x_2 &= \frac{\xi_2}{\xi_1}(1 - \cos(\xi_1)) + \frac{\xi_3}{\xi_1}\sin(\xi_1) \\ x_3 &= \frac{\xi_2}{\xi_1}\sin(\xi_1) - \frac{\xi_3}{\xi_1}(1 - \cos(\xi_1)). \end{aligned}$$

We can now observe, by recovering the group operation in these coordinates, that $G \cong \mathbb{R}^1 \rtimes \mathbb{R}^2$, the universal cover of the roto-translation group $SE(2)$. Moreover, we can write down the liftings in the new coordinates,

$$\begin{aligned} L^{-1}X_1 &= \partial_{x_1} \\ L^{-1}X_2 &= \sin(x_1)\partial_{x_2} + \cos(x_1)\partial_{x_3}. \end{aligned}$$

4.2. **Main result.** We are going to introduce the notion of a fundamental solution for the lifted operator $\widetilde{\mathcal{L}}$ and state that if such a solution exists, then it is possible to saturate it and obtain a fundamental solution for $\mathcal{L}$ on the base manifold M.

Remark 4.4. A fundamental solution of a differential operator $\widetilde{\mathcal{L}}$ over the group G is a smooth function

$$\widetilde{\Gamma}_G\colon (G \times G) \backslash \Delta(G) \to \mathbb{R},$$

where $\Delta(G) \subset G \times G$ is the diagonal, $\widetilde{\Gamma}_G$ is in $L^1_{\mathrm{loc}}(G \times G)$ and for any $\widetilde{\varphi} \in C^\infty_0(G)$

$$\int_G \widetilde{\Gamma}(\xi; \eta) \widetilde{\mathcal{L}}^* \widetilde{\varphi}(\eta) \omega_G(\eta) = -\widetilde{\varphi}(\xi), \tag{4.4}$$

where ω_G is the already introduced left invariant Haar volume form. Moreover $\widetilde{\Gamma}$ respects the following properties,

(1) $\widetilde{\Gamma}(\xi;\eta) > 0$ for any $\xi \neq \eta$ in G;
(2) $\widetilde{\Gamma}(\xi;-)\colon G\backslash\{\xi\} \to \mathbb{R}$ is smooth and $\widetilde{\mathcal{L}}$-harmonic, meaning that $\widetilde{\mathcal{L}}\widetilde{\Gamma}(\xi;-) = 0$ over $G\backslash\{\xi\}$;
(3) if we consider the restricted function

$$\widetilde{\Gamma}(\xi;-)\Big|_{G^y}$$

over the fiber G^y for some $y \in M$ and $y \neq E(\xi)$, then $\widetilde{\Gamma}(\xi;-)|_{G^y}$ is in $L^1(G^y)$;
(4) for any $K \subset M$ compact subset and $E^{-1}(K) \subset G$ its preimage, then $\widetilde{\Gamma}(\xi;-)$ is in $L^1(E^{-1}(K))$.

Theorem 4.5. *Consider a simply connected Riemannian manifold M with a differential operator $\mathcal{L} = \sum r_\alpha \cdot X^\alpha$ on it, and such that the Lie algebra $\mathfrak{g} \subset \mathfrak{X}(M)$ generated by $X_1, X_2, \ldots, X_q$ satisfies the hypothesis of Remark 2.1. Moreover, let G be the unique connected and simply connected Lie group such that* $\mathrm{Lie}(G) \cong \mathfrak{g}$.

If a fundamental solution $\widetilde{\Gamma}(\xi;\eta)$ exists for the lifted operator $\widetilde{\mathcal{L}}$ over G with the properties listed above, then a fundamental solution $\Gamma(x;y)$ for the differential operator $\mathcal{L}$ over M exists too, and in particular the latter is obtained as an integral

$$\Gamma(x;y) := \rho(y) \cdot \int_{G^y} \widetilde{\Gamma}(\xi;\eta)\omega_{\mathrm{Ker}}(\eta), \quad \forall x, y \in M,\ x \neq y, \tag{4.5}$$

where $\rho\colon M \to \mathbb{R}$ is the smooth and everywhere non-null function introduced in (3.10)*, while ξ is any element of G^x. Moreover, the function Γ verifies the following properties,*

(1) $\Gamma(x;y) > 0$ for any $x \neq y$ in M;
(2) $\Gamma(x;-)\colon M\backslash\{x\} \to \mathbb{R}$ is smooth and $\mathcal{L}\Gamma(x;-) = 0$ over the same domain;
(3) $\Gamma(x;-)$ is in $L^1_{\mathrm{loc}}(M)$.

Remark 4.6. As already said, ω_{Ker} is the restriction of the volume form ω_G along the vertical bundle TG^x for any $x \in M$. Therefore in the formula (4.5) the fundamental solution $\widetilde{\Gamma}$ is integrated along one the E fibers.

Remark 4.7. Preliminary to the proof, we observe that the integral in (4.5) depends on the chosen point $\xi \in G$. Anyway, in many cases its value is independent on this choice. For example, if the fundamental solution is in the form $\widetilde{\Gamma}(\xi,\eta) = \widetilde{\Gamma}_G(\xi^{-1}\eta)$ with $\widetilde{\Gamma}_G$ a locally integrable function on G, smooth outside a pole at the identity e. This is the case for the solution of the sub-Laplacian on a Carnot group.

In general, if we have some additional condition that gives the unicity of $\widetilde{\Gamma}$, for example some decay property at ∞ when $G \cong \mathbb{R}^n$, then we can prove that Γ is the same for any ξ. In this case, for any $g \in G$ we have

$$\widetilde{\Gamma}(g\xi;g\eta) = \widetilde{\Gamma}(\xi;\eta) \quad \forall \xi,\eta \in G,\ \xi \neq \eta. \tag{4.6}$$

In order to prove this equation, denote by $L_g\widetilde{\varphi}(\xi)$ the function $\widetilde{\varphi}(g\xi)$ for any compactly supported smooth function $\widetilde{\varphi}\colon G \to \mathbb{R}$ and $g \in G$, then

$$\begin{aligned}\int_G \widetilde{\Gamma}(g\xi; g\eta)\widetilde{\mathcal{L}}^*\widetilde{\varphi}(\eta)\omega_G &= \int_G \widetilde{\Gamma}(g\xi; g\eta)\widetilde{\mathcal{L}}^*(L_{g^{-1}}\widetilde{\varphi}(g\eta))\omega_G \\ &= \int_G \widetilde{\Gamma}(g\xi; \eta')\widetilde{\mathcal{L}}^*(L_{g^{-1}}\widetilde{\varphi}(\eta'))\omega_G \\ &= -L_{g^{-1}}\widetilde{\varphi}(g\xi) \\ &= -\widetilde{\varphi}(\xi),\end{aligned}$$

where we used the fact that if $\eta' = g\eta$, then $\omega_G(\eta') = \omega_G(\eta)$ by left invariance. By the unicity of $\widetilde{\Gamma}$, this proves Equation (4.6).

This allows to conclude that Equation (4.5) is independent of the ξ choice. Indeed, if we consider another element ξ' of G^x, this means that $\xi' = s\xi$ with $s \in G^z$. We observe that $\eta' = s\eta$ is again in G^y if $\eta \in G^y$, therefore we use this coordinate change and obtain

$$\int_{G^y} \widetilde{\Gamma}(\xi'; \eta')\omega_{\mathrm{Ker}}(\eta') = \int_{G^y} \widetilde{\Gamma}(\xi; \eta)\omega_{\mathrm{Ker}}(s\eta) = \int_{G^y} \widetilde{\Gamma}(\xi; \eta)\omega_{\mathrm{Ker}}(\eta).$$

Here we used that $\omega_{\mathrm{Ker}}(\eta') = \omega_{\mathrm{Ker}}(\eta)$ by left invariance with respect to G^z multiplication (see proof of Lemma 3.12).

4.3. **The saturation method.** In this section and the following we are going to prove Theorem 4.5 via a saturation method that follows the idea introduced by Biagi and Bonfiglioli in [1].

We will use the notations (x, s) or (y, t) for the coordinates on $M \times G^z$. In particular we choose over G^z the exponential coordinates (s or t) with respect a basis of $\mathrm{Lie}(G^z)$. Therefore, for the volume form over any fiber G^x we can write $\omega_{\mathrm{Ker}} = \mathrm{d}s$ or $\mathrm{d}t$ in order to make the calculations clearer.

With another minor abuse of notation we denote by $\widetilde{\Gamma}(x, s; y, t)$ the "read" of the fundamental solution $\widetilde{\Gamma}$ in the latter coordinate system. Observe that the L^1 hypothesis in Theorem 4.5 translates as the fact that $\widetilde{\Gamma}(x, s; y, -)\colon G^z \to \mathbb{R}$ is of class $L^1(G^z)$ for any (x, s) and $y \neq x$, while $\widetilde{\Gamma}(x, s; -, -)$ is of class $L^1(K \times G^z)$ for any compact subset $K \subset M$. Through the coordinate change formula (3.9) we have

$$\begin{aligned}\int_{G^y} \widetilde{\Gamma}(\xi; \eta)\omega_{\mathrm{Ker}}(\eta) &= \int_{G^z} \widetilde{\Gamma}(x, s; y, t)\omega_{\mathrm{Ker}}(R_{\ell(y)}(t)) \\ &= c(y) \cdot \int_{G^z} \widetilde{\Gamma}(x, s; y, t)\mathrm{d}t,\end{aligned}$$

and this allows to rewrite Equation (4.5),

$$\Gamma(x; y) = \overline{\rho}(y) \cdot \int_{G^z} \widetilde{\Gamma}(x, s; y, t)\mathrm{d}t, \quad \forall x, y \in M,\ x \neq y. \tag{4.7}$$

where we used the definition (3.11).

We write also the defining equation of the fundamental solution $\widetilde{\Gamma}$ in the new coordinate setting. For any function $\widetilde{\varphi} \in C_0^\infty(G)$ we use the same notation by writing $\widetilde{\varphi}(x, s) = \widetilde{\varphi}(\xi)$ if

$(x,s)=\psi(\xi)$. Therefore,

$$\begin{aligned}-\widetilde{\varphi}(x,s)=-\widetilde{\varphi}(\xi)&=\int_G \widetilde{\Gamma}(\xi;\eta)\widetilde{\mathcal{L}}^*\widetilde{\varphi}(\eta)\omega_G\\&=\int_G \widetilde{\Gamma}(\xi;\eta)\widetilde{\mathcal{L}}^*\widetilde{\varphi}(\eta)\overline{\rho}(E(\eta))\mathrm{d}\psi^*(\mathrm{Vol}_M\wedge\omega_{\mathrm{Ker}})\\&=\int_{M\times G^z}\widetilde{\Gamma}(x,s;y,t)\widetilde{\mathcal{L}}^*\widetilde{\varphi}(y,t)\overline{\rho}(y)(\mathrm{Vol}_M\wedge\mathrm{d}t),\end{aligned}$$

where we used the result of Theorem 3.5.

We can now prove Theorem 4.5. In order to have a simpler notation, in the following we impose $s=e$ in the above formula, which means imposing $\psi(\xi)=(x,e)$, while we will use the notation $\psi(\eta)=(y,t)$. The proof is the same for any other choice of s, and in many cases also the value of the integral, as we exposed in Remark 4.7.

Consider two compactly supported functions $\varphi\in C_0^\infty(M)$ and $\theta\in C_0^\infty(G^z)$ such that $\theta(e)=1$, and define the product function

$$\widetilde{\varphi}:=\varphi\cdot\theta.$$

Therefore, by definition of fundamental solution and by Equation (4.3),

$$\begin{aligned}\int_G \widetilde{\Gamma}(\xi;\eta)\cdot\widetilde{\mathcal{L}}^*(\widetilde{\varphi}(\eta))\omega_G=-\widetilde{\varphi}(\xi)=-\varphi(x)\cdot\theta(e)=-\varphi(x)=\\=\int_{M\times G^z}\widetilde{\Gamma}\cdot\theta\cdot(\mathcal{L}^*\varphi)\cdot\overline{\rho}\cdot(\mathrm{Vol}_M\wedge\mathrm{d}t)+\int_{M\times G^z}\widetilde{\Gamma}\cdot R^*(\varphi\theta)\cdot\overline{\rho}\cdot(\mathrm{Vol}_M\wedge\mathrm{d}t),\end{aligned}$$

We denote by I and II the two integrals in the right side, and we want to prove that for an opportune sequence of functions θ_j with $j\to+\infty$, we have two convergence results

$$\begin{aligned}I&\to\int_M\Gamma(x;y)\mathcal{L}^*\varphi(y)\,\mathrm{Vol}_M\\II&\to 0.\end{aligned}$$

4.3.1. *The convergence of I.* We suppose that the $\theta_j\colon G^z\to\mathbb{R}$ are compactly supported cut-off functions such that $\{\theta_j=1\}\uparrow_j G^z$ and $\theta_j(e)=1$ for each j. Observe that if $\mathrm{supp}(\varphi)\subset K$ a compact subset of M, then we have the inequality

$$\left|\mathcal{L}^*\varphi\cdot\widetilde{\Gamma}\cdot\theta_j\cdot\overline{\rho}\right|\le C\widetilde{\Gamma},$$

where C is a constant depending on K. As $\widetilde{\Gamma}(x,e;-,-)$ is integrable over $K\times G^z$, by dominated convergence we have

$$I_j\xrightarrow{j\to+\infty}\int_{M\times G^z}\widetilde{\Gamma}(x,e;y,t)\cdot\mathcal{L}^*\varphi(y)\cdot\overline{\rho}(y)(\mathrm{Vol}_M\wedge\mathrm{d}t).$$

By the integrability of $\widetilde{\Gamma}(x,e;y,-)$ over G^z and the Fubini's Theorem, we get an equivalent formulation of the same result,

$$I_j\to\int_M\mathcal{L}^*\varphi(y)\cdot\left(\int_{G^z}\widetilde{\Gamma}(x,e;y,t)\mathrm{d}t\right)\overline{\rho}(y)\,\mathrm{Vol}_M=\int_M\Gamma(x;y)\cdot\mathcal{L}^*\varphi(y)\,\mathrm{Vol}_M$$

as we intended to prove.

4.3.2. *The convergence of II.* As showed in (4.3),

$$R^*(\varphi(y)\cdot\theta_j(t)) = \sum_{|\gamma|\geq 1} r^*_{\beta,\gamma}(y)\cdot X^\beta\varphi(y)\cdot Y^\gamma\theta_j(t).$$

If the $\theta_j\colon G^z\to\mathbb{R}$ are compactly supported cut-off functions invading G^z, then $R^*(\varphi\theta_j)$ tends pointwise to 0 over G^z because the term $Y^\gamma\theta_j$ tends pointwise to 0. Therefore, if we can use again the dominated convergence argument, we can conclude. For this reason we want to prove that $|\overline{\rho}(y)\cdot r^*_{\beta,\gamma}(y)\cdot X^\beta\varphi\cdot Y^\gamma\theta_j|$ is bounded over $K\times G^z$. In fact, it suffices to bound the terms $|Y^\gamma\theta_j|$ because the other terms depend only on the coordinate y and must be bounded over the compact K.

Therefore, if we build a sequence θ_j such that $|Y^\gamma\theta_j|\leq C$ for any j and for any γ of bounded length (C depending on the compact set K), then

$$\left|\widetilde{\Gamma}\cdot\overline{\rho}\cdot R^*(\varphi\theta_j)\right| \leq \widetilde{\Gamma}\cdot\overline{\rho}(y)\cdot\sum_{|\gamma|\geq 1}\left|r^*_{\beta,\gamma}(y)X^\beta\varphi(y)\right|\cdot|Y^\gamma\theta_j(t)|\leq C'\cdot\widetilde{\Gamma}$$

for some constant C'. As $\widetilde{\Gamma}(x,e;-,-)$ is in $L^1(K\times G^z)$ by hypothesis, we can conclude

$$II_j\to 0.$$

4.4. **The θ sequence.** In this section we introduce a "good" sequence of functions θ_j in order to complete the previous proof. We recall that we want a sequence $\theta_j\colon G^z\to\mathbb{R}$ for $j=1,2,\dots$ such that

$$\theta_j(e)=1\ \forall j\ \text{ and }\ \{\theta_j=1\}\uparrow_j G^z.$$

Moreover we want the terms $Y^\gamma\theta_j$ to be bounded, uniformly in j, for γ of bounded length.

Let's introduce a cut-off function $\theta_0\colon G^z\to\mathbb{R}$ such that $\operatorname{supp}\theta_0\subset K$ where K is a compact G subset, and there exists an open neighborhood $U\subset K$ of e such that $\theta_0|_U\equiv 1$. For any $g\in G^z$ we define $\theta_g := g_*\theta_0=\theta_0(g^{-1}\cdot -)$ and we denote by U_g, K_g the pushforward of the U and K set respectively,

$$U_g=g_*U=\{x|\ g^{-1}x\in U\},\quad K_g=g_*K.$$

By construction the K_g are all compact.

Consider a vector field $Y\in\operatorname{Lie}(G^z)$ seen as a left invariant vector field in TG^z, we observe that for any $g,h\in G^z$,

$$Y_h\theta_g=(\mathrm{d}L_hY_e)\theta_0(g^{-1}\cdot -)=Y_e\theta_0(g^{-1}h\cdot -)=Y_{g^{-1}h}\theta_0. \tag{4.8}$$

Remark 4.8. This implies that the bounds of the function $Y^\gamma\theta_g\colon G^z\to\mathbb{R}$ for some γ multi-index, are the same of the function $Y^\gamma\theta_0\colon G^z\to\mathbb{R}$ for any $g\in G^z$, and both are clearly bounded because their support are included in K_g and K. In this work, we will only consider the case of $\gamma=0$, but this result gives a tool for developing estimates in the $M\times G^z$ coordinate system.

Consider a basis $Y_1, \dots, Y_p$ of $\mathrm{Lie}(G^z)$, and the metric that makes these vectors orthonormal. For ε sufficiently small we identify a small ball of radius ε around $0 \in \mathrm{Lie}(G^z)$, with a neighborhood of any $g \in G^z$. We denote by $B(g, \varepsilon) \subset G^z$ this ball-neighborhood. We define,

$$U^{(\varepsilon)} := \{g \in U \mid B(g, \varepsilon) \subset U\}.$$

We choose ε sufficiently small that $U^{(\varepsilon)}$ is an e neighborhood. Observe that as a consequence for any $h \in G^z$, $U_h^{(\varepsilon)} = h_* U^{(\varepsilon)}$ is a h^{-1} neighborhood and

$$U_h^{(\varepsilon)} = \{g \in U_h \mid B(g, \varepsilon) \subset U_h\}.$$

Consider a numerable sequence $h_1, h_2, \dots$ such that $\left(U_{h_i}^{(\varepsilon)}\right)_i$ is a covering of G^z. Such a sequence exists because G^z is second countable, and therefore by a theorem of Lindelöf (see [7, Theorem VIII.6.3]) every open covering has a countable subcovering.

We introduce

$$\overline{\theta}_j := \max_{i \leq j} \theta_{h_i}.$$

As a consequence of Remark 4.8, the functions $Y^\gamma \overline{\theta}_j$, for γ of bounded length, have the same bounds. Observe however that they are not defined in every point of G^z as the max might not be derivable everywhere.

In order to obtain some smooth functions θ_j such that all the $Y^\gamma \theta_j$ have the same bounds, we consider a mollifier τ_ε defined in a neighborhood of $0 \in \mathrm{Lie}(G^z)$ and therefore also in a neighborhood of $e \in G^z$. As a mollifier, τ_ε is a smooth function verifying the following two properties

- $\operatorname{supp} \tau_\varepsilon \subset B(e, \varepsilon)$;
- $\int_{G^z} \tau_\varepsilon = 1$.

We introduce the sequence θ_j via a mollification, that is a convolution operation with the mollifier τ_ε,

$$\theta_j := \overline{\theta}_j \star \tau_\varepsilon \quad \forall j = 1, 2, \dots \tag{4.9}$$

The new θ_j functions are smooth because obtained via a convolution with a smooth function. Observe that by construction $B(e, \varepsilon)$ has finite measure and the functions $Y^\gamma \tau_\varepsilon$ are all bounded (for γ of bounded length) because the support of τ_ε is included in a compact subset. We write

$$|Y^\gamma \tau_\varepsilon| \leq C, \quad |\overline{\theta}_j| \leq \sup |\theta_0| = C'.$$

Therefore the $Y^\gamma \theta_j$ are bounded. Indeed,

$$\begin{aligned} |Y^\gamma \theta_j(h)| &= \left| \int_{G^z} Y^\gamma \tau_\varepsilon(g^{-1}h) \cdot \overline{\theta}_j(g) \mathrm{d}g \right| \\ &\leq \int_{G^z} \left| Y^\gamma \tau_\varepsilon(g^{-1}h) \right| \cdot \left| \overline{\theta}_j(g) \right| \mathrm{d}g \\ &\leq |B(e, \varepsilon)| \cdot CC', \end{aligned}$$

for any γ of length $\leq k$ and for any j.

We also observe that for any j, we have

$$\{\theta_j = 1\} \supset \bigcup_{i \leq j} U_{h_i}^{(\varepsilon)}.$$

By definition of the sequence h_i, this implies that the sets $\{\theta_j = 1\}$ progressively invade G^z. Therefore we have completed the verification of the hypothesis on the θ_j sequence detailed in Section 4.3.2, and this in turn completes the proof of the equality

$$\int_M \Gamma(x;y)\mathcal{L}^*\varphi(y)\,\mathrm{Vol}_M = -\varphi(x) \quad \forall\varphi \in C_0^\infty(M).$$

It remains to prove that $\Gamma(x;-)$ is locally integrable on M for any $x \in M$, and the same function is smooth and $\mathcal{L}$-harmonic on $M\backslash\{x\}$ for any $x \in M$.

The first property is a direct consequence of $\widetilde{\Gamma}(\xi;-)$ being in $L^1(K\times G)$ for any compact subset $K \subset M$.

For the other properties, observe that $\mathcal{L}\Gamma(x;-) = 0$ on $M\backslash\{0\}$ as distributions, therefore as $\mathcal{L}$ is an Hörmander's operator it is C^∞-hypoelliptic and $\Gamma(x;-) \in C^\infty(M\backslash\{x\})$. This also implies that the equality $\mathcal{L}\Gamma(x;-) = 0$ is true on $M\backslash\{x\}$ as functions.

5. The case of some non-simply connected varieties and groups

In this section we show an analogous result for a class of non-simply connected (but orientable) smooth manifolds. Given a smooth manifold N consider the Lie algebra generated by smooth vector fields $\mathrm{Lie}(X_1,\dots,X_q) = \mathfrak{g} \subset \mathfrak{X}(N)$, respecting the hypothesis of Remark 2.1. If G is the unique connected and simply connected Lie group such that $\mathrm{Lie}(G) \cong \mathfrak{g}$, we denote by L this isomorphism and moreover we consider the universal cover M of N, therefore a regular surjection $M \to N$ with discrete fibers exists, and M is simply connected.

The vector fields X_i extend naturally to smooth vector fields over M, so we can see $\mathfrak{g}$ as a finite dimensional sub-algebra of $\mathfrak{X}(M)$. If we denote by μ_L the right G-action induced on M by the infinitesimal generator $L\colon \mathrm{Lie}(G) \to \mathfrak{g}$, then we make the hypothesis that a discrete subgroup $H \subset G$ exists, such that

$$N = M/H. \tag{5.1}$$

Moreover, if we denote by $R_h\colon M \to M$ the right action by $h \in H$ on M, then by the previous hypothesis,

$$\mathrm{d}R_h X = X, \quad \forall h \in H,\ \forall X \in \mathfrak{g}. \tag{5.2}$$

Theorem 5.1. *If G is the group described above and H a discrete subgroup respecting the property (5.2), then H lies in the center of G and therefore the right G-action μ_L on M induces a right $\frac{G}{H}$-action on the smooth variety $N = M/H$.*

In order to prove it, we need two lemmas.

Lemma 5.2. *There is no element of the group G acting trivially on M. Equivalently, there exists no $g \in G$ such that $\mu_L(x,g) = x$ for every $x \in M$.*

Proof. If $x_1,\dots,x_k$ are M points, we denote by $\mathfrak{h}(x_1,\dots,x_k)$ the following subspace of $\mathfrak{g}$

$$\mathfrak{h}(x_1,\dots,x_k) = \{X \in \mathfrak{g} \mid X(x_i) = 0\ \forall i = 1,\dots,k\}.$$

If n is the $\mathfrak{g}$ dimension as a (real) vector space, which is the same dimension of G as a (real) manifold, and m the M dimension, then there exists $n-m+1$ points $x_1,\dots,x_{n-m+1} \in M$ such that

$$\mathfrak{h}(x_1,\dots x_{n-m+1}) = \{0\}. \tag{5.3}$$

Indeed, $\dim \mathfrak{h}(x_1) = n - m$ because of the hypothesis on $\mathfrak{g}$, then for any $i \geq 2$ we consider $X \in \mathfrak{h}(x_1, \dots, x_{i-1}) \backslash \{0\}$ and chose a point x_i such that $X(x_i) \neq 0$. It exists because $X \neq 0$ as a vector field. Therefore $\dim \mathfrak{h}(x_1, \dots, x_i) \leq \dim \mathfrak{h}(x_1, \dots, x_{i-1}) - 1$ and by induction we prove (5.3).

We consider the product manifold $\widetilde{M} = M^{\times(n-m+1)}$ of $n-m+1$ copies of M. There exists a natural right G-action on $\widetilde{M}$. We denote by $\widetilde{G} \subset G$ the subgroup of elements acting trivially on the point $(x_1, \dots, x_{n-m+1}) \in \widetilde{M}$. As $\widetilde{M}$ is simply connected, the group $\widetilde{G}$ must be connected. At the same time, the tangent space to $\widetilde{G}$ at the origin is $\mathrm{Lie}(\widetilde{G}) = \mathfrak{h}(x_1, \dots, x_{n-m+1}) = \{0\}$. Therefore $\widetilde{G}$ is trivial and the proof is concluded. □

Lemma 5.3. *If h is an H element and g any G element, then the right actions of hg and gh coincide, meaning that*

$$\mu_L(x, gh) = \mu_L(x, hg) \quad \forall x \in M.$$

Proof. We fix $x \in M$ and $h \in H$ and we prove that $\mu_L(x, g) = \mu_L(x, hgh^{-1})$. Observe that $\nu(x, g) = \mu_L(x, hgh^{-1})$ is in fact a right G-action and

$$\mathrm{d}\nu^{(x)} \widetilde{X} = \mathrm{d}R_h \mathrm{d}\mu_L^{(x)} \mathrm{d}L_{h^{-1}} \widetilde{X},$$

for any $\widetilde{X} \in \mathrm{Lie}(G)$. Because of property (5.2) and the left invariance of any $\widetilde{X}$ in $\mathrm{Lie}(G)$, we obtain

$$\mathrm{d}\nu^{(x)} = \mathrm{d}\mu_L^{(x)},$$

and because of the Fundamental Theorem 2.2, this implies that the two actions are the same. □

As a consequence, for any $h \in H$ and $g \in G$, the commutator $ghg^{-1}h^{-1}$ acts trivially on any $x \in M$, and therefore by Lemma 5.2, $ghg^{-1}h^{-1} = e$. This proves that H is a subgroup of the G center. Moreover, we have this further result.

Lemma 5.4. *If an element $h \in H$ acts trivially on some $x \in M$, then $h = e$. More generally, the intersection between the G-center and the x-stabilizer, is the trivial group $\{e\}$ for any x.*

Proof. Observe that if h is in the G-center (which is the case for any element in H) and $x \cdot h = x$, then $x \cdot g \cdot h = x \cdot h \cdot g = x \cdot g$. Therefore, as the G-action is transitive, any point on M is in the form $x \cdot g$ and h acts trivially on M. By Lemma 5.2 this implies $h = e$. □

Consider any differential operator $\mathcal{L}_N$ over the smooth variety $N = M/H$ in the form $\mathcal{L}_N = \sum_{|\alpha| \leq k} r_\alpha \cdot X^\alpha$ where the r_α are $C^\infty(N)$ coefficients. We can canonically extend the coefficients over M, and therefore the whole differential operator. We denote the extended operator by $\mathcal{L}_M$. As in the first part of our work, this lifts to G, as the differential operator $\widetilde{\mathcal{L}}_G = \sum (r_\alpha \circ E) \widetilde{X}^\alpha$.

From now on, with a little abuse of notation we denote by $E \colon G/H \to N = M/H$ the induced map on the quotients, and by $(G/H)^x = E^{-1}(x)$ the preimage of any $x \in N$. By our construction, the operatore $\widetilde{\mathcal{L}}_G$ induces naturally a differential operator $\widetilde{\mathcal{L}}_{G/H}$ over G/H which is E related to $\mathcal{L}_N$.

Example 5.5. We consider the analogous operator to Example 4.3 defined over $S^1 \times \mathbb{R}$, meaning the operator $\mathcal{L} = \partial_\theta^2 + (\sin(\theta)\partial_x)^2$. In this case $X_1 = \partial_\theta$ and $X_2 = \sin(\theta)\partial_x$ are naturally associated to the vector fields defined on the previous example with the same name.

We have the same Lie algebra $\mathfrak{g} = \mathrm{Lie}(X_1, X_2) = \langle X_1, X_2, X_3\rangle$ and the associated Lie group $G = \mathbb{R} \rtimes \mathbb{R}^2$.

The universal cover of $N = S^1 \times \mathbb{R}$ is $M = \mathbb{R}^2$, and we have $M = N/H$ where H is the discrete group of rotations $\{(2\pi k, 0, 0),\ k \in \mathbb{Z}\} \subset G$.

All our construction pass through the quotient by H on G, and we obtain the associated vector fields on the group $G/H = S^1 \rtimes \mathbb{R}^2$, that is the roto-traslation group $SE(2)$,

$$L^{-1}X_1 = \partial_\theta$$
$$L^{-1}X_2 = \sin(\theta)\partial_x + \cos(\theta)\partial_y.$$

We suppose that there exists a fundamental solution $\widetilde{\Gamma}_{G/H}$ for $\widetilde{\mathcal{L}}_{G/H}$ defined over the variety $((G/H) \times (G/H))\backslash\Delta$ and such that

(1) $\widetilde{\Gamma}_{G/H}(\xi;\eta) > 0$ for any $\xi \neq \eta$ in G/H;
(2) $\widetilde{\Gamma}_{G/H}(\xi;-)\colon (G/H)\backslash\{\xi\} \to \mathbb{R}$ is $\widetilde{\mathcal{L}}_{G/H}$-harmonic;
(3) the restricted function $\widetilde{\Gamma}_{G/H}(\xi;-)\Big|_{(G/H)^y}$ for some $y \in N$ such that $y \neq E(\xi)$ is in $L^1((G/H)^y)$;
(4) for any $K \subset N$ compact subset and $E^{-1}(K) \subset G/H$ its preimage, $\widetilde{\Gamma}_{G/H}(\xi;-)$ is in $L^1(E^{-1}(K))$.

Theorem 5.6. *Consider a simply connected Riemannian manifold M. Suppose that the Lie algebra $\mathfrak{g} \subset \mathfrak{X}(M)$ generated by the vector fields $X_1, X_2, \ldots, X_q$ satisfies the hypothesis of Remark 2.1. Moreover, let G be the unique connected and simply connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$. By construction a right G-action on M exists. If $H \subset G$ is a discrete subgroup such that $\mathrm{d}R_h X = X$ for any $h \in H$ and $X \in \mathfrak{g}$, then by Theorem 5.1 there exists right $\frac{G}{H}$-action on the smooth variety $N = M/H$.*

Consider a differential operator $\mathcal{L}_N = \sum_{|\alpha|\leq k} r_a \cdot X^\alpha$ on N. If a fundamental solution $\widetilde{\Gamma}_{G/H}(\xi;\eta)$ exists for the lifted operator $\widetilde{\mathcal{L}}_{G/H}$ over G/H with the properties listed above, then a fundamental solution $\Gamma_N(x;y)$ for the differential operator $\mathcal{L}_N$ exists too, and in particular the latter is obtained as an integral

$$\Gamma_N(x;y) := \rho(x) \cdot \int_{(G/H)^y} \widetilde{\Gamma}_{G/H}(\xi;\eta)\omega_{\mathrm{Ker}}(\eta), \quad \forall x, y \in N,\ x \neq y, \tag{5.4}$$

where $\rho\colon N \to \mathbb{R}$ is a smooth and everywhere non-null function, while ξ is any element of the fiber $(G/H)^x$. Moreover, the function Γ_N verifies the following properties,

(1) $\Gamma_N(x;y) > 0$ for any $x \neq y$ in N;
(2) $\Gamma_N(x;-)\colon N\backslash\{x\} \to \mathbb{R}$ is smooth and $\mathcal{L}_N\Gamma_N(x;-) = 0$ over the same domain;
(3) $\Gamma_N(x;-)$ is in $L^1_{\mathrm{loc}}(N)$.

Proof. Given the fundamental solution $\widetilde{\Gamma}_{G/H}$ for $\widetilde{\mathcal{L}}_{G/H}$, we can naturally define $\widetilde{\Gamma}_G(\xi;\eta)$ over $(G \times G)\backslash\overline{\Delta}$ where $\overline{\Delta}$ is the preimage by $G \times G \to (G/H) \times (G/H)$ of the diagonal $\Delta(G/H)$. Observe that $\widetilde{\Gamma}_G$ is a fundamental solution for $\widetilde{\mathcal{L}}_G$ that satisfies the hypothesis of Theorem 4.5 with the slight adaptation that $\widetilde{\Gamma}_G(\xi;-) \in L^1(G^y)$ for any $y \in M$ outside the H-orbit of $x = E(\xi)$. We denote by Γ_M the fundamental solution to $\mathcal{L}_M$ induced by $\widetilde{\Gamma}_G$ and Formula (4.5).

Our construction implies that

$$\Gamma_M(x;y) = \Gamma_M(xh_1;yh_2) \quad \forall h_1, h_2 \in H,\ x, y \in M\ xH \neq yH.$$

Therefore it is possible to define $\Gamma_N\colon (N \times N)\backslash\Delta \to \mathbb{R}$. Observe that Γ_N is a fundamental solution for $\mathcal{L}_N$ respecting the properties in the theorem.

Observe that by Lemma 5.4, the action of H is trivial on any E-fiber. Equivalently, G^y is in bijection with $(G/H)^y$ via the map induced by the quotient, for any $y \in N = M/H$. Observe moreover that the construction of the map $\mathcal{X}(x)$ (see Remark 3.9) is H-invariant, and therefore the function $\rho\colon M \to \mathbb{R}$ is H-invariant too, therefore by an abuse of notation we can define a smooth and everywhere non-null function with the same name $\rho\colon N \to \mathbb{R}$. We can conclude that Formula (5.4) gives the result value of Γ_N because the quotient pass through the integral. □

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