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HOLOMORPHICALLY CONJUGATE POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$ ARE POLYNOMIALLY CONJUGATE

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ABSTRACT. We confirm a conjecture of Friedland and Milnor: if two polynomial automorphisms f and $g \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$ with dynamical degree > 1 are conjugate by some holomorphic diffeomorphism $\varphi\colon \mathbf{C}^2 \to \mathbf{C}^2$, then φ is a polynomial automorphism; thus, f and g are conjugate inside $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$. We also discuss a number of variations on this result.

1. INTRODUCTION

1.1. **Conjugacy classes in** $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{K}})$**.** In their very influential paper [10], Friedland and Milnor study the dynamical and group-theoretic properties of polynomial automorphisms of the affine plane defined over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. In particular, they provide a dichotomy for conjugacy classes in $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$, which we describe below.

For any field $\mathbf{K}$, Jung's theorem says that the group $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{K}})$ is the amalgamated product of the group of *affine automorphisms* $\mathsf{GL}_2(\mathbf{K}) \ltimes \mathbf{K}^2$ and the group of *elementary automorphisms*

$$(x, y) \mapsto (\alpha x + p(y), \beta y + \gamma),$$

with amalgamation along their intersection. If f is an element of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{K}})$, we denote by $\deg(f)$ the degree of the formulas defining f, and set

$$\lambda_1(f) = \lim_{n\to+\infty} \deg(f^n)^{1/n}.$$

Friedland and Milnor prove the following:

- either $\lambda_1(f) = 1$, and then f is conjugate to an elementary or an affine automorphism; moreover, if $\mathbf{K}$ is algebraically closed, every affine automorphism is conjugate to an elementary automorphism by a linear change of variable;
- or $\lambda_1(f)$ is an integer larger than 1, f is conjugate to a *generalized Hénon map*, that is a finite composition of Hénon mappings $(x, y) \mapsto (ay + p_i(x), x)$ with $\deg(p_i) \geqslant 2$. In that case, $\lambda_1(f)$ is the product of the degrees of the p_i.

As explained in [10], it follows that the dynamics of f is interesting only when f is conjugate to a generalized Hénon map; for instance, for $f \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$, the topological entropy of the map $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ is equal to $\log(\lambda_1(f))$ (see [13]).

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The Bass-Serre theory shows that $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{K}})$ acts on a simplicial tree, with stabilizers of vertices corresponding to conjugates of the affine group and the elementary group. Then, the classification of Friedland and Milnor corresponds to the following dichotomy: $\lambda_1(f) = 1$ if and only if the induced action of f on the Bass-Serre tree fixes some vertex, while $\lambda(f) > 1$ when f acts as a non-trivial translation on some invariant geodesic axis. In accordance with this description, we will refer to these two cases as *elliptic* and *loxodromic*, respectively.

1.2. **Conjugation under biholomorphisms and rigidity.** Let us now suppose that $\mathbf{K} = \mathbf{C}$. Friedland and Milnor also study the following question: *if two polynomial automorphisms are conjugate by some holomorphic diffeomorphism of* $\mathbf{C}^2$*, are they also conjugate by an element of* $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$? Automorphisms with distinct dynamical degrees cannot be conjugate by a homeomorphism $\mathbf{C}^2 \to \mathbf{C}^2$, because their topological entropies are distinct. In particular, elliptic and loxodromic classes should be treated separately. A complete answer is given in [10] for elementary automorphisms: the answer is *yes* if such an automorphism admits a periodic point, and *no* otherwise [10, Theorem 6.10 and Lemma 6.12]. For generalized Hénon maps, Friedland and Milnor also prove that the answer is yes for maps of degree 2 and 3, and conjecture that the same result holds in the general case [10, Theorem 7.1]. In this note we confirm this conjecture.

Theorem A. *Let f and g be loxodromic polynomial automorphisms of the affine plane, defined over some subfield $\mathbf{K}$ of $\mathbf{C}$. If $\varphi\colon \mathbf{C}^2 \to \mathbf{C}^2$ is a biholomorphism that conjugates f to g, i.e. $\varphi \circ f \circ \varphi^{-1} = g$, then φ is a polynomial automorphism, and it is defined over a finite extension of $\mathbf{K}$. Moreover, one can find $\psi \in \mathsf{Aut}(\mathbb{A}^2_{\overline{\mathbf{K}}})$ such that $\psi \circ f = g \circ \psi$ and $\deg(\psi) \leqslant 2^{57}(\deg(f)\deg(g))^{29}$.*

A natural approach to this problem is to use eigenvalues at periodic orbits and to show that loxodromic automorphisms with the same eigenvalues are conjugate in $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$. This is precisely what is done in [10] in degree $\leqslant 3$. However, for arbitrary degree, such a multiplier rigidity result is not available, so we use a different method. In a nutshell, the main ingredient will be the existence and rigidity of the canonical invariant currents of f and g; these currents $T^\pm_f$ and $T^\pm_g$ must be respected by the conjugacy, i.e. $\varphi_\star T^\pm_f = cT^\pm_g$ for some $c > 0$, and their rigidity properties will be the key to our proof.

When $f = g$, the corresponding result was proven in [6, Proposition 8.1] and [5]; we shall use this particular case below. Related results were also obtained by Bera, Pal and Verma in [4], with an approach similar to ours.

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2. Proof of Theorem A

2.1. **Preliminaries on Hénon maps (see** [10] **and** [12]**).** Let $f \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$ be a generalized Hénon map of degree d (so $d = \lambda_1(f)$). Its birational extension to the projective plane $\mathbb{P}^2(\mathbf{C})$ has an indeterminacy point at $I^+_f := [0:1:0]$, and it contracts the line at infinity $\{z = 0\}$ onto the indeterminacy point $I^-_f := [1:0:0]$ of f^{-1}. Let K^+_f (resp. K^-_f) be the subset of $\mathbf{C}^2$ of points with a bounded forward (resp. backward) orbit; its closure in $\mathbb{P}^2(\mathbf{C})$ intersects the line at infinity at I_f^+ (resp. at I_f^-). Let $\|\cdot\|$ denote the usual euclidean norm on $\mathbf{C}^2$. Then, the function $G_f^+ : \mathbf{C}^2 \to \mathbf{R}$ defined by

$$G_f^+(p) = \lim_{n\to+\infty} \frac{1}{d^n} \log^+ \|f^n(p)\|$$

is continuous, plurisubharmonic, and non-negative; it vanishes exactly along K_f^+ and satisfies $G_f^+ \circ f = dG_f^+$. The difference $p \mapsto G_f^+(p) - \log^+ \|p\|$ extends to a continuous function on $\mathbb{P}^2(\mathbf{C})\setminus\{I_f^+\}$, as follows for instance from [12, Thm 1.7.1]. The closed, positive current

$$T_f^+ = \frac{\mathrm{i}}{\pi}\partial\overline{\partial} G_f^+$$

satisfies $f^\star T_f^+ = dT_f^+$; it is supported on the boundary of K_f^+ and up to a positive multiplicative constant, *it is the unique closed positive current supported by* K_f^+ (see [9, Theorem 7.12]). Changing f into f^{-1}, one constructs similar objects K_f^-, G_f^- and T_f^-.

The intersection $K_f = K_f^+ \cap K_f^-$ is a compact f-invariant subset of $\mathbf{C}^2$, it is maximal for this property, and the topological entropy of f on K_f is equal to $\log(d)$. The automorphism f has infinitely many saddle periodic points, all of them contained in K_f (see [1]). If $q \in \mathbf{C}^2$ is such a saddle periodic point, its stable manifold $W^s(q)$ is the image of a holomorphic, injective, immersion $\mathbf{C} \to \mathbf{C}^2$, *the image of which is dense in* ∂K_f^+; the current T_f^+ can be recovered asymptotically as a current of integration on $W^s(p)$ (see [3]) The function $G_f := \max(G_f^+, G_f^-)$ is continuous and plurisubharmonic and vanishes exactly on K_f. Moreover

$$G_f^+(p) \leqslant \log^+ \|p\| + O(1) \text{ and } G_f(p) = \log^+ \|p\| + O(1)$$

as $\|p\|$ goes to $+\infty$; see Corollary 2.6 and Proposition 3.8 in [2]. The product $T_f^+ \wedge T_f^-$ is an invariant probability measure, supported in K_f; it coincides with $(\frac{\mathrm{i}}{\pi}\partial\overline{\partial} G_f)^2$.

2.2. **The proof.** Conjugating f and g by elements of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$, we may assume that both of them are generalized Hénon maps. Let $\varphi \colon \mathbf{C}^2 \to \mathbf{C}^2$ be a biholomorphism such that $\varphi \circ f \circ \varphi^{-1} = g$. Then $\varphi \circ f^n = g^n \circ \varphi$ for all $n \in \mathbf{Z}$, $\varphi(K_f^\pm) = K_g^\pm$, $\varphi(K_f) = K_g$, and f and g have the same topological entropy, hence the same degree d.

Lemma. *There exists a positive real number c such that* $\varphi^\star T_g^+ = cT_f^+$ *and* $G_g^+ \circ \varphi = cG_f^+$.

Proof. The first statement follows from the fact, recalled above, that the only positive closed $(1,1)$-currents supported on K_g^+ are of the form cT_g^+, $c > 0$.

To prove the second fact, we observe that $H = cG_f^+ - G_g^+ \circ \varphi$ is a pluriharmonic function on $\mathbf{C}^2$ with $K_f^+ \subset \{H = 0\}$. Assume that $H \not\equiv 0$ and fix a holomorphic function h on $\mathbf{C}^2$ such that H is the real part $\Re(h)$. Then $\{H = 0\}$ admits a unique singular foliation by Riemann surfaces: its leaves are the level sets $\{h = i\alpha\}$, $\alpha \in \mathbf{R}$. On the other hand, for any saddle periodic point q of f, the stable manifold $W^s(q) \subset \mathbf{C}^2$ is a connected, immersed, Riemann surface contained in K_f^+; thus $W^s(q)$ is an irreducible component of $\{h = i\alpha\}$ for some α. This is a contradiction since $W^s(q)$ is dense in the boundary of K_f^+, so it is not embedded at any point. $\square$

Similarly, there exists a constant $c' > 0$ such that $G_g^- \circ \varphi = c' G_f^-$. In fact, from

$$\int \varphi^\star(T_g^+ \wedge T_g^-) = \int T_f^+ \wedge T_f^- = 1$$

we deduce that $c' = 1/c$, but this will not be used in what follows. From this we infer that

$$0 \leqslant G_g \circ \varphi = \max(G_g^+ \circ \varphi, G_g^- \circ \varphi) = \max\left(cG_f^+, c'G_f^-\right) \leqslant \max(c, c')G_f.$$

On the other hand, $G_f(p) = \log\|p\| + O(1)$, and $G_g(p) = \log\|p\| + O(1)$ as $\|p\| \to \infty$. Since φ is proper, we deduce that

$$\log\|\varphi(p)\| + O(1) = G_g(\varphi(p)) \leqslant \max(c, c')G_f(p) \leqslant \max(c, c')\log\|p\| + O(1),$$

that is $\|\varphi(p)\| \leqslant C\|p\|^e$ for some constant $C > 0$ and for $e = \max(c, c')$. This implies that φ is defined by polynomial formulas. Similarly, φ^{-1} is also defined by polynomial formulas, and φ is a polynomial automorphism.

If φ and ψ are two automorphisms conjugating f to g, then $\varphi \circ \psi^{-1}$ commutes with f. In [11, 5], it is proven that the centralizer of f in $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$ contains $f^{\mathbf{Z}}$ as a finite index subgroup. Thus, given any integer $D \geqslant 1$, the set $\left\{\psi \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})\,;\ \psi \circ f \circ \psi^{-1} = g,\ \deg(\psi) \leqslant D\right\}$ is finite, and it is non-empty for $D \geqslant \deg(\varphi)$. Write $\psi = (\psi_1, \psi_2)$, with ψ_1 and ψ_2 in $\mathbf{C}[x, y]$ of degree $\leqslant D$. The constraints $\psi \circ f \circ \psi^{-1} = g$ and $\psi \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$ correspond to polynomial equations in the coefficients of ψ_1 and ψ_2; the coefficients of these equations are in the field of definition $\mathbf{K}$ of f and g. Thus, the solutions lie in a finite extension of $\mathbf{K}$.

It remains to show that one can find an automorphism ψ that conjugates f to g and has degree at most $2^{57}(\deg(f)\deg(g))^{29}$. From [5, Theorem 4.10], one can find such a conjugacy in the group $\mathsf{Bir}(\mathbb{P}^2_{\overline{\mathbf{K}}})$. Thus, to conclude one only needs to apply the following lemma of independent interest. □

Lemma. *Let f and $g \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$ be two Hénon automorphisms. If ψ is a birational transformation of the affine plane that f to g, then ψ is an automorphism.*

Proof. Let us consider f, g, and ψ as birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{C}}$. Let E be the union of the irreducible curves $E_i \subset \mathbb{A}^2_{\mathbf{C}}$ which are contracted by ψ (there strict transform is a point of $\mathbb{P}^2_{\mathbf{C}}$). From $\psi \circ f = g \circ \psi$ we deduce that E is f-invariant, and since a Hénon map does not preserve any algebraic curve, we deduve that E is empty. Similarly, ψ^{-1} does not contract any curve $F \subset \mathbb{A}^2_{\mathbf{C}}$.

It remains to show that ψ does not have any indeterminacy point in $\mathbb{A}^2(\mathbf{C})$. To see this, we resolve the indeterminacies of ψ by a finite sequence of blow-ups: this provides a smooth projective surface X together with two birational morphisms ε, $\eta\colon X \to \mathbb{P}^2_{\mathbf{C}}$ such that $\psi = \eta \circ \varepsilon^{-1}$; we assume that X is minimal for this property. Suppose that $q \in \mathbb{A}^2(\mathbf{C})$ is an indeterminacy point of ψ and denote by $D \subset X$ the tree of rational curves which is mapped to q by ε. The curve D must intersect an irreducible curve $C \subset X$ which is contracted by η and is not contained in D; indeed, otherwise, X would not be a minimal resolution of the indeterminacies. But then, C is the strict transform by ε of an irreducible plane curve C' that contains q. This curve C' is contracted by ψ and contains q, in contradiction with $E = \varnothing$. □

Example. Fix an integer $m \geqslant 2$, and a positive integer D which is not the m-th power of an integer. One easily sees that the Hénon maps $f(x,y) = (y, x + y^{m+1})$ and $g(x,y) = (y, x + Dy^{m+1})$ are conjugate over $\mathbf{Q}(D^{1/m})$ but not over $\mathbf{Q}$. Indeed, $h(x,y) = (\alpha x, \alpha y)$ conjugates f to g if $\alpha^m = 1/D$, and from [11] we know that the centralizer of f is the semi-direct product of $f^{\mathbf{Z}}$ and the cyclic group of order m generated by $(x,y) \mapsto (e^{2i\pi/m}x, e^{2i\pi/m}y)$.

3. Discussion and complements

(a) – Theorem A is still true when φ is a proper holomorphic semiconjugacy:

Theorem B. *If f and g are loxodromic automorphisms of $\mathbf{C}^2$ and $\varphi : \mathbf{C}^2 \to \mathbf{C}^2$ is a proper holomorphic map such that $\varphi \circ f = g \circ \varphi$, then φ is a polynomial automorphism.*

Proof. Note that we also have $\varphi \circ f^{-1} = g^{-1} \circ \varphi$. The properness of φ implies successively that

(1) $\varphi(\mathbf{C}^2) = \mathbf{C}^2$ by Remmert's proper mapping theorem;
(2) $\varphi^{-1}(K_g^{\pm}) = K_f^{\pm}$;
(3) $\varphi^\star(T_g^+)$ (resp. $\varphi^\star(T_g^-)$) is a positive closed current supported on K_f^+ (resp. K_f^-).

Indeed, $\varphi^\star(T_g^+)$ (resp. $\varphi^\star(T_g^-)$) is non-zero by the first item. Hence $\varphi^\star T_g^{\pm} = c^{\pm} T_f^{\pm}$, with $c^{\pm} > 0$, and exactly as in Theorem A we conclude that φ is polynomial. Finally, the Jacobian of φ does not vanish, because otherwise, by the relation $\varphi \circ f = g \circ \varphi$, $\{\mathrm{Jac}(\varphi) = 0\}$ would be an f-invariant algebraic curve, and such a curve does not exist by [2, Proposition 4.2]. Thus, since φ is proper, it is a covering, hence an automorphism, and we are done. □

(b) – Take $\mathbf{K} = \mathbf{R}$. If $f \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{R}})$ has no fixed point in $\mathbf{R}^2$ and f preserves the orientation of $\mathbf{R}^2$, then f is conjugate to a translation by some real analytic diffeomorphism of $\mathbf{R}^2$ (see [7]). Thus, *Theorem A fails for polynomial automorphisms of* $\mathbf{R}^2$. On the other hand one may expect some rigidity properties when $h_{\mathrm{top}}(f|_{\mathbf{R}^2}) > 0$, for instance when $K_f \subset \mathbf{R}^2$.

(c) – If U is a small ball in the complement of $K_f^+ \cup K_f^-$, then its orbit under the action of f is wandering. So, we can find a C^∞ diffeomorphism of $\mathbf{C}^2$ that commutes to f, is the identity on the complement of the f-orbit of U, and is not the identity on U. This shows that, as it is stated, *Theorem A fails in the C^∞ setting.*

On the other hand, the following lemma shows that the canonical currents are preserved under C^1 conjugacy. Thus, the rigidity of positive closed currents with support in K^+ is not really needed in the proof of Theorem A, we could replace it by this lemma.

Lemma. *Let f and g be Hénon automorphisms of the plane $\mathbb{A}^2_{\mathbf{C}}$. If φ is a C^1 diffeomorphism of $\mathbf{C}^2$ conjugating f to g, then $\varphi^\star T_g^+ = cT_f^+$ for some real number $c \neq 0$.*

Sketch of proof. For both f and g, T^+ is a laminar current whose laminar structure is entirely defined in terms of the smooth dynamics [1]. More precisely, the structure theory of strongly laminar currents [8, §5] shows that these currents can be viewed in a precise way as foliation cycles on a weak lamination, whose leaves are Pesin stable manifolds and whose transverse measure is induced by the unstable conditionals of the unique measure of maximal entropy. It follows that if φ is a C^1 diffeomorphism conjugating f and g, $\varphi^\star T_g^+$ is a closed laminar current subordinated to the same weak lamination as T_f^+; more precisely, $\varphi^\star T_g^+$ is

- a closed current;
- a laminar current with respect a transverse signed measures (the sign takes care of the orientation of the local leaves, since φ may reverse this orientation);
- absolutely continuous with respect to T_f^+; in particular it is supported by K_f^+.

On the other hand, the fact that T_f^+ is an extremal point in the cone of positive closed currents (see e.g. [12, §2.2]) entails that the transverse invariant measure of T_f^+ is ergodic (see [8, Cor. 5.8]) and this implies that $\varphi^\star T_g^+$ is a constant multiple of T_f^+. $\square$

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