Discrete Scale Invariance and $U(2)$ Family of Two-Body Contact Interactions in One Dimension

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Abstract

Because of the absence of indistinguishability constraint, interparticle interactions between nonidentical particles have in general much more variety than those between identical particles. In particular, it is known that there exists a $U(2)$ family of two-body contact interactions between nonidentical particles in one spatial dimension. This paper studies breakdown of continuous scale invariance to discrete scale invariance under this $U(2)$ family of two-body contact interactions in two-body problems of nonidentical particles on the half line. We show that, in contrast to the corresponding identical-particle problem, there exist two distinct channels that admit geometric sequences of two-body bound states.

1 Introduction

Discrete scale invariance is an invariance under enlargements and reductions of the system size by constant factors that form a geometric sequence. If a quantum many-body system enjoys this invariance, bound-state energies of many-body bound states, if exist, must form a geometric sequence. The most prominent example that realizes this phenomenon is the three-body problem of identical bosons in three dimensions under two-body short-range interactions. This example, first discovered by Efimov in 1970 [1], has now been experimentally observed and also theoretically generalized to several directions; see, e.g., [2] for a recent review. The purpose of this paper is to present a new discrete scale-invariant model of nonidentical particles in one dimension. The key is the absence of indistinguishability constraint in nonidentical-particle problems.

In general, the indistinguishability of identical particles requires interparticle interactions to be permutation invariant. This permutation invariance places a stringent constraint on possible many-body interactions. For example, for two-body contact interactions in one spatial dimension, identical bosons and fermions (with no internal degrees of freedom) have no other choice but to interact via delta- and epsilon-function potentials, respectively [3]. In contrast, there is no such constraint for distinguishable particles, meaning that there are much more variety of two-body contact interactions for nonidentical particles than those for identical particles. In fact, in one spatial dimension, the most general two-body contact interaction consistent with the probability conservation (unitarity) is known to be described by a 2×2 unitary matrix $U \in U(2)$ [4, 5], which contains $\dim U(2) = 4$ independent real parameters. However, scale-invariance breaking under this $U(2)$ family of two-body interactions has never been studied in the literature.

This paper is aimed at studying breakdown of continuous scale invariance to discrete scale invariance under the $U(2)$ family of two-body contact interactions. To simplify the analysis, we shall focus on two-body problems on the half line, which is exactly solvable and known to exhibit discrete scale invariance for the case of identical particles [6]. We shall show that, in a certain subspace of the parameter space of $U(2)$, there arise two distinct channels that admit geometric sequences of two-body bound states.

The rest of the paper is organized as follows. We first formulate the problem in section 2. We consider two nonidentical particles on the half line with two-body contact interaction. Under the assumption that the interaction potential satisfies a scaling law, we see that the two-body problem is reduced to a set of two differential equations—the radial and angular Schrödinger equations—where the former determines the two-body energy spectrum while the latter determines the scale-invariance breaking. Section 3 studies the whole parameter space of scale-invariant two-body contact interactions in one dimension. We first show that, for nonidentical particles, there exists a $U(2)$ family of scale-invariant two-body contact interactions. We then discuss the quantization condition for eigenvalues of the angular Schrödinger equation. This quantization condition becomes particularly simple if the masses of the particles are the same. Section 4 studies such a mass-degenerate case and shows that, in a certain parameter region, there arise two distinct geometric sequences of two-body bound states. We conclude in section 5. Appendix A presents a derivation of the $U(2)$ family of two-body contact interactions.

2 Scale-invariant two-body problem on the half line

To begin with, let us fix some notation. Let x_1 and x_2 be the coordinates of two nonidentical particles of mass m_1 and m_2 on the half line $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, respectively. Let $V(x_1, x_2)$ be a potential for interparticle interactions between these particles. The Hamiltonian of such a two-body system is then given by

$$H = -\sum_{j=1}^{2} \frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V(x_1, x_2). \tag{1}$$

We wish to understand when and how continuous scale invariance is broken down to discrete scale invariance under a general two-body contact interaction. To study this, we focus on the potential V that fulfills the following properties:

- **Property 1. (Scaling law)**

$$V(\mathrm{e}^t x_1, \mathrm{e}^t x_2) = \mathrm{e}^{-2t} V(x_1, x_2), \quad \forall t \in \mathbb{R}. \tag{2}$$

- **Property 2. (Contactness)**

$$\mathrm{supp}(V) = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 = x_2\}. \tag{3}$$

Here $\mathrm{supp}(V)$ stands for the support of V. The first property (2) is necessary to be scale invariant. The second property (3), on the other hand, describes the situation where the particles interact only at two-body coincidence points. As we will see in the next section, such contact interactions are best described by connection conditions at the two-particle coincidence points. Throughout this section, however, we shall use the potential V as a convenient placeholder. Note that we do not impose the translation invariance $V(x_1 + a, x_2 + a) = V(x_1, x_2)$ ($a \in \mathbb{R}$) on the potential because the translation invariance is already broken by the boundary.

Now, let us first discuss the impact of the scaling law (2) in two-body problems on the half line. To this end, let us introduce the polar coordinate system (r, θ) in the two-body configuration space $\mathbb{R}_+^2$. We write

$$x_1 = \sqrt{\frac{m_0}{m_1}} r \cos\theta, \tag{4a}$$

$$x_2 = \sqrt{\frac{m_0}{m_2}} r \sin\theta, \tag{4b}$$

or, equivalently,

$$r = \sqrt{\frac{m_1 x_1^2 + m_2 x_2^2}{m_0}}, \tag{5a}$$

$$\theta = \arctan\left(\sqrt{\frac{m_2}{m_1}}\frac{x_2}{x_1}\right), \tag{5b}$$

where $0 \leq r < \infty$ and $0 \leq \theta \leq \pi/2$. Here $m_0(> 0)$ is an arbitrary mass scale introduced to assign the length dimension to the radius r. In this polar coordinate system, the general solution to the scaling law (3) can be written as

$$V(x_1, x_2) = \frac{\hbar^2}{2m_0}\frac{v(\theta)}{r^2}, \tag{6}$$

where $v(\theta)$ is some function of θ. Note that the solution (6) is in general not invariant under the translation $(x_1, x_2) \mapsto (x_1 + a, x_2 + a)$ and is different from the translation-invariant two-body non-contact interaction $V(x_1, x_2) \propto 1/(x_1 - x_2)^2$ discussed in the literature [7]. By using (6) and the identity $(1/m_1)\partial^2/\partial x_1^2 + (1/m_2)\partial^2/\partial x_2^2 = (1/m_0)r^{-1/2}(\partial^2/\partial r^2 + (1/r^2)\partial^2/\partial\theta^2)r^{1/2}$, we find

$$H = \frac{\hbar^2}{2m_0}r^{-1/2}\left(-\frac{\partial^2}{\partial r^2} + \frac{-\partial_\theta^2 + v(\theta) - 1/4}{r^2}\right)r^{1/2}. \tag{7}$$

The eigenvalue equation of this operator is easily solved by the method of separation of variables. In fact, by assuming the energy eigenfunction ψ is of the form,

$$\psi(x_1, x_2) = r^{-1/2}R(r)\Theta(\theta), \tag{8}$$

we immediately see that the eigenvalue equation $H\psi = E\psi$ is reduced to the following set of differential equations:

$$\left(-\frac{d^2}{d\theta^2} + v(\theta)\right)\Theta(\theta) = \lambda\Theta(\theta), \tag{9a}$$

$$\left(-\frac{d^2}{dr^2} + \frac{\lambda - 1/4}{r^2}\right)R(r) = \frac{2m_0 E}{\hbar^2}R(r). \tag{9b}$$

Notice that the energy eigenvalue E is solely determined by the one-body problem on the half line with the inverse-square potential (9b), which is known to break continuous scale invariance to discrete scale invariance when $\lambda < 0$ [8]. In this case, there arises a geometric sequence of negative energy eigenvalues $\{E_n\}$ that satisfy the scaling law $E_n/E_{n-1} = \exp(-2\pi/\sqrt{-\lambda})$. Hence, what we have to do is to solve the angular equation (9a) and to find out when the eigenvalue λ becomes negative. Let us next turn to this problem by using the most general scale-invariant two-body contact interactions consistent with the unitarity.

3 $U(2)$ family of scale-invariant two-body contact interactions

In the previous section, we have imposed the scaling law (2) and studied its consequence in the two-body problem on the half line. In this section, we further impose the contactness (3), which, in terms of the function $v(\theta)$, is described by

$$\mathrm{supp}(v) = \{\theta \in [0, \pi/2] : \theta = \theta_0\}. \tag{10}$$

Here θ_0 is the two-body coincidence point in the polar coordinate system (r, θ) defined through (5b) with $x_1 = x_2$:

$$\theta_0 = \arctan\left(\sqrt{\frac{m_2}{m_1}}\right). \tag{11}$$

Now, we wish to consider the two-body contact interaction as general as possible. One obvious way to do this is to use the most general function $v(\theta)$ with one-point support at the two-body coincidence point $\theta = \theta_0$. Another way to do this is to use the most general connection condition at $\theta = \theta_0$. Since the latter approach is much more well-established than the former, we shall use the connection-condition approach to the two-body contact interaction. Following [9], we shall focus on the most general connection condition consistent with the probability conservation, which, in our two-body problem, can be put into the following continuity condition for the probability current normal to the two-body coincidence line $x_1 = x_2$ in the two-body configuration space $\mathbb{R}_+^2$:

$$\boldsymbol{n} \cdot \boldsymbol{j}\big|_{x_1 - x_2 = 0_+} = \boldsymbol{n} \cdot \boldsymbol{j}\big|_{x_1 - x_2 = 0_-}. \tag{12}$$

Here $\boldsymbol{j} = (j_1, j_2)$ is the two-body probability current and $\boldsymbol{n} = (n_1, n_2)$ is a (unit) normal vector to the line $x_1 = x_2$. In the original Cartesian coordinate system, they are given by $j_a = (\hbar/(2im_a))(\overline{\psi}\partial\psi/\partial x_a - \overline{\partial\psi/\partial x_a}\psi)$ and $n_a = 1/\sqrt{2}$, where $a = 1, 2$ and overline ($\overline{}$) stands for the complex conjugate. In the following, we would like to obtain the most general connection condition for $\Theta(\theta)$ by solving (12).

First, in the polar coordinate system, the probability conservation (12) at the two-body coincidence point is equivalent to to the following condition:

$$\overline{\Theta(\theta_0+)}\Theta'(\theta_0+) - \overline{\Theta'(\theta_0+)}\Theta(\theta_0+) = \overline{\Theta(\theta_0-)}\Theta'(\theta_0-) - \overline{\Theta'(\theta_0-)}\Theta(\theta_0-), \tag{13}$$

where prime ($'$) indicates the derivative with respect to θ and $f(\theta_0\pm)$ stands for one-sided limits given by $f(\theta_0\pm) = \lim_{\epsilon\to 0_+} f(\theta_0 \pm \epsilon)$. Notice that (13) is quadratic equation with respect to Θ. However, it can be linearized and there exists a $U(2)$ family of linearized solutions. An important observation to see this is that (13) can be put into the form of inner product in two-dimensional complex vector space $\mathbb{C}^2$:

$$\begin{pmatrix} \Theta(\theta_0+) \\ \Theta(\theta_0-) \end{pmatrix}^{\dagger} \begin{pmatrix} \Theta'(\theta_0+) \\ -\Theta'(\theta_0-) \end{pmatrix} = \begin{pmatrix} \Theta'(\theta_0+) \\ -\Theta'(\theta_0-) \end{pmatrix}^{\dagger} \begin{pmatrix} \Theta(\theta_0+) \\ \Theta(\theta_0-) \end{pmatrix}, \tag{14}$$

where † stands for the hermitian conjugate. As shown in appendix A, the solution to this type of equation is parameterized by a 2×2 unitary matrix U and given by

$$(1 - U) \begin{pmatrix} \Theta(\theta_0+) \\ \Theta(\theta_0-) \end{pmatrix} + i(1 + U) \begin{pmatrix} \Theta'(\theta_0+) \\ -\Theta'(\theta_0-) \end{pmatrix} = 0, \quad U \in U(2). \tag{15}$$

This is the $U(2)$ family of connection conditions that describe all possible scale-invariant two-body contact interactions in the two-body problem on the half line.

Now, in order to solve the problem, we also have to specify the boundary conditions at $\theta = 0, \pi/2$. For simplicity, we will impose the Dirichlet boundary conditions:

$$\Theta(0) = \Theta(\pi/2) = 0. \tag{16}$$

What we have to do is then to solve the angular equation $-\Theta''(\theta) = \lambda\Theta(\theta)$ for $\theta \neq \theta_0$ under the conditions (15) and (16) and to determine the unitary matrix U that admits negative eigenvalue λ.

To this end, let us next parameterize the unitary matrix U. Let $e^{i\alpha_+}$ and $e^{i\alpha_-}$ be two eigenvalues of U, where $\alpha_\pm \in [-\pi, \pi)$ are real parameters. Then, the 2×2 unitary matrix U can be written as the following spectral decomposition:

$$U = e^{i\alpha_+} P_+ + e^{i\alpha_-} P_-, \tag{17}$$

where $P_\pm$ are projection operators that satisfy the completeness $P_+ + P_- = 1$, hermiticity $P_\pm^\dagger = P_\pm$, and orthonormality $P_a P_b = \delta_{ab} P_b$ ($a, b \in \{+, -\}$). Such 2×2 hermitian matrices are parameterized as follows:

$$P_\pm = \frac{1 \pm \boldsymbol{e} \cdot \boldsymbol{\sigma}}{2}, \tag{18}$$

where $\boldsymbol{e} = (e_1, e_2, e_3)$ is a real unit 3-vector satisfying $e_1^2 + e_2^2 + e_3^2 = 1$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-tuple of Pauli matrices. Eq. (17) is the most general parameterization of $U \in U(2)$, which contains four independent parameters $(\alpha_+, \alpha_-, \boldsymbol{e})$. What we want to do is to determine the parameter space of U that admits negative eigenvalue λ. Let us next do this by solving the angular equation $-\Theta''(\theta) = \lambda\Theta(\theta)$.

First, the general solution to the angular equation with the Dirichlet boundary conditions (16) can be written as the following linear combination of plane waves:

$$\Theta(\theta) = \begin{cases} A(\mathrm{e}^{i\sqrt{\lambda}(\theta_0-\theta)} - \mathrm{e}^{i\sqrt{\lambda}(\theta_0+\theta-\pi)}) & \text{for } \theta \in (\theta_0, \pi/2], \\ B(\mathrm{e}^{i\sqrt{\lambda}(\theta-\theta_0)} - \mathrm{e}^{-i\sqrt{\lambda}(\theta+\theta_0)}) & \text{for } \theta \in [0, \theta_0). \end{cases} \tag{19}$$

Here A and B are integration constants. Next, we impose the connection condition (15). Substituting (19) into (15) we get the following condition:

$$(1 - \mathcal{U}(\lambda)) \begin{pmatrix} A \\ B \end{pmatrix} = 0, \tag{20}$$

where

$$\mathcal{U}(\lambda) = \sum_{j=\pm} \mathrm{e}^{2i[\frac{\pi}{4}\sqrt{\lambda} + \arctan(\sqrt{\lambda}\cot(\alpha_j/2))]} P_j \, \mathrm{e}^{2i\sqrt{\lambda}(\frac{\pi}{4}-\theta_0)\sigma_3}. \tag{21}$$

Hence, in order to have nontrivial solution $(A, B) \neq (0, 0)$, the matrix $1 - \mathcal{U}(\lambda)$ must not have the inverse for any λ. Thus we get

$$\det(1 - \mathcal{U}(\lambda)) = 0. \tag{22}$$

This gives the quantization condition for the eigenvalue λ. This equation possesses negative solutions in a certain parameter region of U. Let us next see this by considering the simple case $m_1 = m_2$.

4 A simple example: mass-degenerate case

The quantization condition (22) becomes particularly simple in the mass-degenerate case $m_1 = m_2$, where the angle θ_0 becomes

$$\theta_0 = \arctan(1) = \frac{\pi}{4} \quad \text{for } m_1 = m_2. \tag{23}$$

In this case, the matrix (21) takes the following form:

$$\mathcal{U}(\lambda) = \sum_{j=\pm} \mathrm{e}^{2i[\frac{\pi}{4}\sqrt{\lambda} + \arctan(\sqrt{\lambda}\cot(\alpha_j/2))]} P_j. \tag{24}$$

It is clear from this expression that the unitary matrix $\mathcal{U}(\lambda)$ has the eigenvalues $\mathrm{e}^{2i[\frac{\pi}{4}\sqrt{\lambda} + \arctan(\sqrt{\lambda}\cot(\alpha_\pm/2))]}$, which must be 1 in order to fulfill the condition (22). Thus we find the following quantization condition:

$$\frac{\pi}{4}\sqrt{\lambda} + \arctan\left(\sqrt{\lambda}\cot\left(\frac{\alpha_\pm}{2}\right)\right) = n\pi, \quad (n : \text{integer}), \tag{25}$$

or, equivalently,

$$\alpha_\pm = -2\arctan\left(\sqrt{\lambda}\cot\left(\frac{\pi}{4}\sqrt{\lambda}\right)\right). \tag{26}$$

This equation is easily solved graphically and possesses infinitely many real roots $\lambda_{\pm,0} < \lambda_{\pm,1} < \lambda_{\pm,2} < \cdots$. As depicted in Fig. 1, the lowest root $\lambda_{\pm,0}$ becomes negative for $\alpha_\pm < \alpha_*$, where α_* is the critical value given by

$$\alpha_* = \lim_{\lambda\to 0}\left[-2\arctan\left(\sqrt{\lambda}\cot\left(\frac{\pi}{4}\sqrt{\lambda}\right)\right)\right] = -2\arctan\left(\frac{4}{\pi}\right). \tag{27}$$

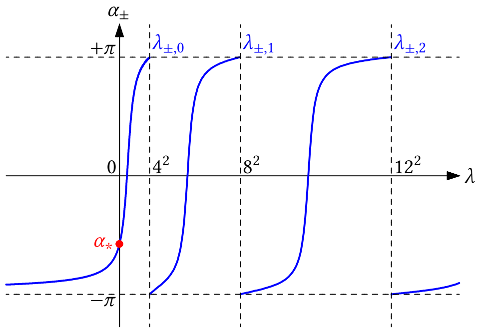

Figure 1: The $\alpha_\pm$-dependence of the eigenvalues $\lambda_{\pm,0} < \lambda_{\pm,1} < \lambda_{\pm,2} < \cdots$. The red dot represents the critical value $\alpha = \alpha_*$ below which the lowest eigenvalue $\lambda_{\pm,0}$ becomes negative.

Hence, continuous scale invariance is broken to discrete scale invariance for $\alpha_\pm < \alpha_*$. In consequence, there exist three distinct "phases" in this model. The first is the region $\{(\alpha_+, \alpha_-) : \alpha_+ < \alpha_* \ \& \ \alpha_- < \alpha_*\}$ in which there arise two distinct geometric series of two-body bound states in the $\lambda_{+,0}$- and $\lambda_{-,0}$-channels; the second is the region $\{(\alpha_+, \alpha_-) : \alpha_\pm < \alpha_* \ \& \ \alpha_\mp \geq \alpha_*\}$ in which there arises a single geometric series of two-body bound states in the $\lambda_{\pm,0}$-channel; and the third is the region $\{(\alpha_+, \alpha_-) : \alpha_+ \geq \alpha_* \ \& \ \alpha_+ \geq \alpha_*\}$ in which continuous scale invariance is unbroken and there is no two-body bound states. Notice that the parameter $\boldsymbol{e}$ has no effect in this phase structure.

5 Conclusion

In this paper, we have studied breakdown of continuous scale invariance to discrete invariance in two-body problems of nonidentical particles on the half line. We first discussed that, for nonidentical particles in one dimension, there exists the $U(2)$ family of scale-invariant two-body contact interactions consistent with the probability conservation. We then discussed the breakdown of continuous scale invariance under this $U(2)$ family of interactions. We showed that, in addition to the unbroken phase, there exist two additional phases in the scale-invariant two-body problem: one is the discrete scale invariant phase in which two distinct geometric series of two-body bound states appear, and the other is the discrete scale invariant phase in which a single geometric series of two-body bound states appear. This is in sharp contrast to the two-body problem of identical particles on the half line [6], where there arises at most a single geometric series of two-body bound states.

Throughout this paper we focused on two-body problems with boundary just for simplicity. Future studies should investigate a generalization to $n(\geq 3)$-body problems of nonidentical particles with or without boundary and reveal its phase structure. It is also interesting to study experimental realizations of the scenario discussed in this paper. Though currently unknown, the realization of the $U(2)$ family of two-body contact interactions in experiments will open up a new phase of many-body problems in one dimension.

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Appendix A Proof of (15)

Following the original work by Fülöp and Tsutsui [9], in this section we show that there exists a $U(2)$ family of solutions to the continuity condition (14). The essentially same derivation can also be found in [10].

To begin with, let us consider the following abstract problem. Let Ψ and Φ be some elements of an arbitrary Hilbert space $\mathscr{H}$ and subject to the following condition:

$$(\Psi, \Phi) = \overline{(\Phi, \Psi)}, \tag{28}$$

where $(*, *)$ stands for the inner product. We wish to find out all possible solutions to this equation. To solve this, we first note that any inner products can be written in terms of norms. In fact, the polarization identity gives (see, e.g., Eq. (1.11) of [11])

$$(\Psi, \Phi) = \frac{1}{4}\left(\|\Psi + \Phi\|^2 - \|\Psi - \Phi\|^2 + i\|\Psi + i\Phi\|^2 - i\|\Psi - i\Phi\|^2\right), \tag{29}$$

where $\| * \|$ stands for the norm. Note that $\overline{(\Phi, \Psi)} = \overline{(\Psi, \Phi)} = (1/4)(\|\Psi + \Phi\|^2 - \|\Psi - \Phi\|^2 - i\|\Psi + i\Phi\|^2 + i\|\Psi - i\Phi\|^2)$. Equating these expressions, one immediately sees that (28) is equivalent to the following condition:

$$\|\Psi + i\Phi\|^2 = \|\Psi - i\Phi\|^2. \tag{30}$$

This equation says that two distinct vectors $\Psi + i\Phi$ and $\Psi - i\Phi$ have the same norm. Hence there should exist a unitary U that satisfies the following relation:

$$\Psi + i\Phi = U(\Psi - i\Phi), \tag{31}$$

or, equivalently,

$$(1 - U)\Psi + i(1 + U)\Phi = 0. \tag{32}$$

This is the linearized relation that solves (28).

Now let us turn to the problem of solving the continuity condition (14). We first note that (14) is equivalent to (28) with the special choice $\mathscr{H} = \mathbb{C}^2$, $\Psi = {}^t(\Theta(\theta_0+), \Theta(\theta_0-))$, $\Phi = {}^t(\Theta'(\theta_0+), -\Theta'(\theta_0-))$, and $(\Psi, \Phi) = \Psi^\dagger\Phi$, where ^{t} stands for the transposition. Note also that U in this case is just a 2×2 unitary matrix. Hence there exists a $U(2)$ family of connection conditions given by (15). This completes the proof.

References

[1] V. Efimov, "Energy levels arising form the resonant two-body forces in a three-body system", Phys. Lett. B **33**, 563–564 (1970).

[2] P. Naidon and S. Endo, "Efimov physics: a review", Rept. Prog. Phys. **80**, 056001 (2017), arXiv:1610.09805 [quant-ph].

[3] T. Cheon and T. Shigehara, "Fermion boson duality of one-dimensional quantum particles with generalized contact interaction", Phys. Rev. Lett. **82**, 2536–2539 (1999), arXiv:quant-ph/9806041 [quant-ph].

[4] V. Caudrelier and N. Crampe, "Exact results for the one-dimensional many-body problem with contact interaction: Including a tunable impurity", Rev. Math. Phys. **19**, 349–370 (2007), arXiv:cond-mat/0501110 [cond-mat.other].

[5] N. Yonezawa and I. Tsutsui, "Inequivalent quantizations of the $N = 3$ Calogero model with scale and mirror-S_3 symmetry", J. Math. Phys. **47**, 012104 (2006), arXiv:hep-th/0510106 [hep-th].

[6] S. Ohya, "Efimov effect for two particles on a semi-infinite line", Am. J. Phys. **90**, 770 (2022), arXiv:2201.10869 [cond-mat.quant-gas].

[7] S. Moroz, J. P. D'Incao, and D. S. Petrov, "Generalized Efimov Effect in One Dimension", Phys. Rev. Lett. **115**, 180406 (2015), arXiv:1506.03856 [cond-mat.other].

[8] K. M. Case, “Singular Potentials”, Phys. Rev. **80**, 797–806 (1950).

[9] T. Fülöp and I. Tsutsui, “A free particle on a circle with point interaction”, Phys. Lett. A **264**, 366 (2000), arXiv:quant-ph/9910062 [quant-ph].

[10] G. Bonneau, J. Faraut, and G. Valent, “Self-adjoint extensions of operators and the teaching of quantum mechanics”, Am. J. Phys. **69**, 322 (2001), arXiv:quant-ph/0103153 [quant-ph].

[11] W. O. Amrein, *Hilbert Space Methods in Quantum Mechanics*, 1st (EFPL Press, Lausanne, 2009).