CRITICAL EXPONENT GAP AND LEAFWISE DIMENSION

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Abstract. We show that for every nonarithmetic lattice $\Gamma < \mathrm{SL}_2(\mathbb{C})$ there is a gap $\varepsilon_\Gamma > 0$ such that for every $g \in \mathrm{SL}_2(\mathbb{C})$ the intersection $\mathrm{SL}_2(\mathbb{R}) \cap g\Gamma g^{-1}$ is either a lattice in $\mathrm{SL}_2(\mathbb{R})$ or has critical exponent $\delta(\mathrm{SL}_2(\mathbb{R}) \cap g\Gamma g^{-1}) \leq 1 - \varepsilon_\Gamma$.

1. Introduction

In his landmark work, Margulis [23] showed that there are no irreducible nonarithmetic lattices in higher-rank semisimple Lie groups (see Definition 2.1 of arithmetic lattice). However, there are nonarithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ and more generally $\mathrm{SO}(n,1)$ for $n \geq 2$. This paper focuses on nonarithmetic lattices in $G = \mathrm{SL}_2(\mathbb{C})$. There are several constructions for such lattices. One such construction is given by Gromov and Piatetski-Shapiro [19] as the fundamental group of a certain surgery of two arithmetic hyperbolic manifolds. Some other constructions can be found e.g. in [40, 33].

A recent result of Mohammadi and Margulis [25] and of Bader, Fisher, Miller and Strover [1] gives a geometric sufficient criterion for a lattice $\Gamma < G$ to be arithmetic, namely, if there are infinitely many totally geodesic surfaces in $\mathbb{H}^3/\Gamma$. This is equivalent to G/Γ having infinitely many periodic $\mathrm{SL}_2(\mathbb{R})$-orbits. The Bader, Fisher, Miller and Stover result is more general in that it deals with lattices in $SO(n,1)$ for any $n \geq 3$.

A notable distinction between arithmetic and nonarithmetic lattices is the following. Let $\mathrm{SL}_2(\mathbb{R}).x$ be an orbit for some $x \in G/\Gamma$. If Γ is arithmetic and $\mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(x)$ is Zariski dense in Γ, then a theorem of Borel and Harish-Chandra [3] says that $\mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(x)$ must be a lattice as well. In the nonarithmetic case, this is no longer true. One can quantify the "size" of a Zariski dense subgroup $\Lambda < \mathrm{SL}_2(\mathbb{R})$ by its critical exponent:

Definition 1.1 (Critical exponent)**.** Let $\Lambda < \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup. Define the *critical exponent* of Λ by

$$\delta(\Lambda) = \limsup_{R\to\infty} \frac{\log \#(B_{\mathrm{SL}_2(\mathbb{R})}(R) \cap \Lambda)}{R}.$$

Here $B_{\mathrm{SL}_2(\mathbb{R})}(R)$ is a ball in $\mathrm{SL}_2(\mathbb{R})$ around the identity, with respect to the natural metric which will be specified in Section 2. If Λ is a lattice, then $\delta(\Lambda) = 1$, and this is the maximal possible critical exponent.

The main result of this paper is the following theorem.

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Theorem 1.2 (Critical exponent gap)**.** *Let $\Gamma < G$ be a non-arithmetic lattice. For every $g \in G$ define*

$$\Gamma_g = \mathrm{SL}_2(\mathbb{R}) \cap g\Gamma g^{-1} = \mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(\pi_\Gamma(g)),$$

where $\pi_\Gamma(g)$ image of g in G/Γ, and consider the critical exponent $\delta(\Gamma_g)$. Then there is an $\varepsilon_\Gamma > 0$ such that for all $g \in G$ one of the following holds:

(1) $\delta(\Gamma_g) \leq 1 - \varepsilon_\Gamma$;
(2) Γ_g is a lattice.

To show that this ε_Γ cannot be chosen uniformly we prove the following.

Theorem 1.3. *For every $\varepsilon > 0$ there is a nonarithmetic lattice $\Gamma < G$ and $g \in G$ such that Γ_g is not a lattice but $\delta(\Gamma_g) > 1 - \varepsilon$.*

Remark 1.4. It seems likely that in the homogeneous space G/Γ we construct in Theorem 1.3 there are infinitely many orbits $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)$ so that $\delta(\Gamma_g) > 1 - \varepsilon$, but we do not know how to show it.

1.1. **Application to polynomial equidistribution.** We will relate Theorem 1.2 to a recent result of Lindenstrauss, Mohammadi, and Wang [22]. Let $\Gamma < G$ be a lattice. Let

$$\mathtt{u}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \mathtt{a}(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad \forall s, t \in \mathbb{R},$$

and let $x \in G/\Gamma$. Ratner's Equidistribution Theorem (See [31, 30, 32]) shows that $\mathtt{u}(s).x$ equidistributes in some homogeneous subspace of G/Γ. More formally, the sequence of measures

$$\mu_{T,x} = \frac{1}{T}\int_0^T \delta_{\mathtt{u}(s).x} \mathrm{d}s$$

converges to the Haar measure on a homogeneous subspace. Moreover, unless x lies in a $\mathtt{u}(s)$-invariant homogeneous subspace, $\mu_{T,x} \xrightarrow{T\to\infty} m_{G/\Gamma}$. Lindenstrauss, Mohammadi, and Wang [22] effectivized this claim whenever Γ is arithmetic. [22, Thm. 1.1] can be seen to be equivalent to the effectivization of Ratner's Equidistribution Theorem. Informally and inaccurately, it states that either $\mathtt{a}(t)\mu_{1,x}$ is $\exp(-\star t)$ close to the Haar measure $m_{G/\Gamma}$ or one of the following algebraic obstructions occurs:

- $\mathtt{u}(s)x$ is $\exp(-\star t)$ close to a periodic orbit $\mathrm{SL}_2(\mathbb{R}).x'$ of volume $\exp(-\star t)$ for all $t \geq 0$.
- x lies too deep in a cusp of G/Γ.

Lindenstrauss, Mohammadi and Wang also give a version of their theorem for nonarithmetic lattices in G ([22, Thm. 1.3]), but its statement is more complicated as it cites another, more complicated, type of obstruction, unrelated to periodic $\mathrm{SL}(2,\mathbb{R})$-orbits or cusp excursions, e.g. that the initial point is close to a point $\pi_\Gamma(g)$ for which Γ_g is Zariski dense but not a lattice. And indeed, this potentially is an obstruction: suppose that $x \in G/\Gamma$ has a stabilizer $\Lambda = \mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}\, x < \mathrm{SL}_2(\mathbb{R})$ with critical exponent $\delta(\Lambda)$, and suppose $\delta(\Lambda)$ is very close to 1 (its maximal value). This allows $\mathtt{a}(t)\mathtt{u}(s).x$ to return $\Theta(\exp(\delta(\Lambda)t))$ times to a ball $B_{\mathrm{SL}_2(\mathbb{R})}(1).x$ for $s \in [0,1]$. This gives an lower bound of $\exp((\delta(\Lambda) - 1)t)$ on the distance of $\mu_{T,x}$ and the Haar measure $m_{G/\Gamma}$.

Therefore, if one wants also for a nonarithmetic lattice a polynomial equidistribution theorem analogous to [22, Thm. 1.1], the first step is to bound $\delta(\Lambda)$, which is done by Theorem 1.2. In a follow-up to this paper, we will show the following polynomial unipotent equidistribution result.

Corollary 1.5 (Polynomial unipotent equidistribution in nonarithmetic lattices)**.** *For every $x_0 \in G/\Gamma$ and large enough R (depending only on X and the injectivity radius of x_0), for any $T \geq R^A$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(G/\Gamma)$,

$$\left|\int_0^1 \varphi(\mathtt{a}(\log T)\mathtt{u}(r))\mathrm{d}r - \int_{G/\Gamma} \varphi \mathrm{d}m_{G/\Gamma}\right| < S(\varphi)R^{-\kappa_1}.$$

where $S(\varphi)$ is a certain Sobolev norm.

(2) There exists $x_1 \in G/\Gamma$ such that the orbit $\mathrm{SL}_2(\mathbb{R})x_1$ is periodic and

$$d_{G/\Gamma}(x_0, x_1) < R^A(\log T)^A T^{-1}.$$

The constants A and κ_1 are positive and depend on X but not on x_0.

The Sobolev norm used here is the same as in [22]. See [25], [1] for a finiteness result of the periodic orbits in Option 2 in the above corollary.

Remark 1.6. The proof of Theorem 1.2 is via a limiting argument, invoking elements of [1] and uses Ratner's Measure Classification Theorem. Hence, Theorem 1.2 is not effective, which implies the same regarding the constants in Theorem 1.2 and Corollary 1.5. Similarly, the results of in [25] and [1] prove that for nonarithmetic Γ there are only finitely many $\mathrm{SL}_2(\mathbb{R})$ periodic orbits without any estimate on their number. In contrast, the constants in [22] are explicit in principle; cf. also [21, Thm. 1.4].

1.2. **Structure of the proof of the gap in critical exponent.** As mentioned above, the proof of Theorem 1.2 combines results from [1], with some extra ergodic theoretic arguments. In [1], the first step assumes to the contrary that there is a nonarithmetic lattice $\Gamma < G$ for which there are infinitely many periodic orbits $(\mathrm{SL}_2(\mathbb{R}).x_k)_{k=1}^\infty$ for $x_k = \pi_\Gamma(g_k) \in G/\Gamma$. Then, using Ratner's theorem (or more precisely a result of Mozes and Shah that relies on this theorem as well as the Dani-Margulis linearization method), the authors show that the sequence of Haar measures on these periodic orbits converges to the Haar measure, i.e.

$$m_{\mathrm{SL}_2(\mathbb{R}).x_k} \xrightarrow{k\to\infty} m_{G/\Gamma}.$$

In our case, $\Gamma_{g_k} = \mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(x_k)$ are not lattices, so the Haar measures on them are infinite. Instead, we will use for each k the Bowen-Margulis-Sullivan measure μ_k corresponding[1] to Γ_{g_k} on $\mathrm{SL}_2(\mathbb{R}).x_k$. It has an entropy $h_{\mu_k}(\mathtt{a}(1)) = \delta(\Gamma_{g_k})$. An $\mathtt{a}(1)$-invariant measure on G/Γ can have any entropy ≤ 2, so these entropies are certainly far from being maximal entropy. Thus we cannot show that such a measure is close to Haar using only the uniqueness of measure of maximal entropy on G/Γ (See e.g. [4], [39, §11]). However, all of the entropy of these μ_k comes "from the $\mathrm{SL}_2(\mathbb{R})$ direction". This intuition can be formalized to say that the $\mathtt{u}$-leafwise dimension (see Definition 3.7) of μ_k is almost everywhere $\delta(\Gamma_{g_k})$ which is close to the maximal value 1. This leads us to the ergodic component of the proof, Theorem 3.9 below. This theorem enables us to utilize this large dimension to show that any

[1]In fact, we use Bowen-Margulis-Sullivan measures corresponding to finitely generated subgroups of Γ_{g_k} so that the measure will be finite. We ignore this subtlety for the introduction.

weak-$*$ limit is $\mathrm{SL}_2(\mathbb{R})$-invariant, and is of interest by itself. However, there is yet work to be done to show that the limit is the Haar measure $m_{G/\Gamma}$, as there may be an escape of mass, or positive mass to $\mathrm{SL}_2(\mathbb{R})$-periodic orbit. To rule out these options we use linearization methods and Margulis functions.

1.3. **Structure of the construction of a lattice Γ with small gap.** As for Theorem 1.3, its proof can be divided to three main parts.

Part 1, construction of a homogeneous space: We implement a construction of a nonarithmetic lattice given by Gromov and Piatetski-Shapiro [19], who construct a nonarithmetic space G/Γ in the following way: Take two (carefully constructed) arithmetic spaces G/Γ_1 and G/Γ_2, and identify isomorphic codimension-1 submanifold $V_i \subseteq \mathbb{H}^3/\Gamma_i$. Cut $\mathbb{H}^3/\Gamma_i$ along V_i for each $i = 1, 2$ to obtain two hyperbolic threefolds with isomorphic boundaries. Finally, glue these manifolds along their boundaries to obtain a compact hyperbolic threefold of the form $\mathbb{H}^3/\Gamma$. The lattice Γ is the non-arithmetic manifold we look for.

Part 2, construction of an orbit: To construct the element g required by Theorem 1.3, we construct the orbit $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)$ as follows. Take a periodic $\mathrm{SL}_2(\mathbb{R})$-orbit $H.x_0$ in G/Γ_i for some $i = 1, 2$. Denote its projection to G/Γ_i by S_0. This is an immersed hyperbolic surface. Cut S_0 along V_i into finitely many pieces, and consider the image S_1 in $\mathbb{H}^3/\Gamma$ of one of these pieces. This yields an immersed hyperbolic surface $S_2 \cong \mathbb{H}^2/\Gamma_{g_2} \subset \mathbb{H}^3/\Gamma$ contains S_1. We show that by properly choosing S_0 and S_1 we can ensure that Γ_{g_2} is not a lattice.

Part 3, estimation of the critical exponent in the form of high Hausdorff dimension: The estimation of the critical exponent of Γ_{g_2} uses Sullivan [36, Thm. 1], which reduces the estimation of the desired critical exponent to an estimation of the Hausdorff dimension of the collection of geodesic in S_0 that originates form a point $p_0 \in S_0$ and do not intersect V_i. Viewing this problem in the universal cover of S_0, the inverse image of V_i is a union of geodesics. This reduces the question of giving a lower bound on the Hausdorff Dimension of set of rays from p_0 on S_0 avoiding V_i to the following two claims on an immersion $\iota_0 : \mathbb{H}^2 \to \mathbb{H}^3/\Gamma_i$.

- $\iota_0^{-1}(V_i)$ is composed of many hyperbolic lines in $\mathbb{H}^2$. Then there is a lower bound on the distances of these lines from one another,
- For every collection L of lines in $\mathbb{H}^2$ that are far from one another, and every point $p_0 \in \mathbb{H}^2$ not on any of these lines, the dimension of the set of geodesic rays from p_0 that do not hit any of the lines is large.

The first point follows from arithmetic considerations. The second can be reduced to an estimate of the dimension of a certain Cantor set.

1.4. **Structure of the paper.** In Section 2 we introduce several notations and conventions. Section 3 is divided into three parts: In Subsections 3.1 and 3.2 we recall a nonstandard definition for the leafwise measures and some of its properties. In Subsection 3.3 we introduce the leafwise Markov chain and complete the proof of Theorem 3.9.

In Section 4 we prove Theorem 1.2. The section is divided into three parts: Subsection 4.1 states several claims which will be of use in the next subsection. Subsection 4.2 follows the discussion at Subsection 1.2, and shows that a certain sequence of measures μ_k on G/Γ converges to the Haar measure. We prove $\mathrm{SL}_2(\mathbb{R})$-invariance, use Lemma 4.7 (whose proof is left to Section 5) to exclude escape of mass in the limit and any nontrivial homogeneous component, and finally use

Ratner's theorem [32] to conclude that the limit is the Haar measure. In Subsection 4.3 we conclude the proof of Theorem 1.2 by adapting the work of [1] to our setup. In Section 5 we use a linearization method and Margulis functions to prove Lemma 4.7. Section 6, which is independent of the rest of the paper, is dedicated to the proof of Theorem 1.3.

1.5. **Further research.** A natural question is to prove an effective version of Theorem 1.2.

Question 1.7. *Find an effective formula for an ε_Γ depending on the lattice $\Gamma < G$ such that there are only finitely many $\mathrm{SL}_2(\mathbb{R})$-orbits $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)$ such that $1-\varepsilon_\Gamma < \delta(\Gamma_g)$ but Γ_g is not a lattice. We expect ε_Γ to depend on the spectral gap of G/Γ, however, it may depend also on the arithmetic nature of Γ.*

The example given by Theorem 1.3 inspires us to formulate the following more optimistic question. We do not know if it helps to answer the previous one.

Question 1.8. *Let $\Gamma < G$ be a lattice. Is it true that there are only finitely many $\mathrm{SL}_2(\mathbb{R})$-orbits of points $x = \pi_\Gamma(g)$ in G/Γ with Γ_g Zariski dense in $\mathrm{SL}_2(\mathbb{R})$ for which there does not exist an arithmetic lattice $\Gamma_1 < G$ such that $\mathrm{SL}_2(\mathbb{R}).x$ lift bijectively to G/Λ where $\Lambda = \Gamma_1 \cap \Gamma$ and Λ is Zariski-dense in G? Can one find a finite collection ϖ of arithmetic lattices such that for every point $x = \pi_\Gamma(g)$ in G/Γ with Zariski dense Γ_g the orbit $\mathrm{SL}_2(\mathbb{R}).x$ lifts to G/Λ with $\Lambda = \Gamma_1 \cap \Gamma$ and $\Gamma_1 \in \varpi$, except perhaps for finitely many $\mathrm{SL}_2(\mathbb{R})$-orbits?*

One can also consider the analogous of Theorem 1.2 to nonarithmetic lattices in more general $\mathbb{R}$-rank 1 groups.

We now discuss possible extensions of Theorem 3.9 referred to above, and use similar notations. Let $B = \mathbb{R} \ltimes \mathbb{R}$ using the exponent action of $\mathbb{R}$ on $\mathbb{R}$.

Question 1.9. *Let $(\mu_k)_{k=1}^\infty$ be $\mathtt{a}(t)$-ergodic invariant probability measures on a locally compact second countable space X on which B acts continuously. Suppose that there is a weak-$*$ probability measure limit $\mu_\infty = \lim_{k\to\infty} \mu_k$ with ergodic decomposition $\int_X \mu_\infty^x \mathrm{d}\mu_\infty(x)$. Show that*

$$\int_X \dim^{\mathtt{u}} \mu_\infty^x \mathrm{d}\mu_\infty(x) \geq \limsup_{k\to\infty} \dim^{\mathtt{u}} \mu_k, \tag{1.1}$$

with the convention that $\dim^{\mathtt{u}} \mu_\infty^x = 1$ if μ_∞^x is not $\mathtt{u}$-free.

One can try to extend this to actions of more general semi-direct products.

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2. Notations

Definition 2.1 (Homogeneous dynamics notations)**.** Let $G = \mathrm{SL}_2(\mathbb{C})$ and $\Gamma < G$ a lattice. We say that Γ is *arithmetic* if there is an algebraic group $\mathbf{G}/\mathbb{Q}$ and a homomorphism with compact kernel $f : \mathbf{G}(\mathbb{R}) \to G$ whose image is open in G and Γ is commensurable to $f(\mathbf{G}(\mathbb{Z}))$. From now on we assume that Γ is nonarithmetic. Recall that $\mathtt{a}(t) = \mathrm{diag}(e^{t/2}, e^{-t/2})$ and $\mathtt{u}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ for all $t, s \in \mathbb{R}$, generates a subgroup $B < \mathrm{SL}_2(\mathbb{R})$. The group B is isomorphic to $\mathbb{R} \ltimes \mathbb{R}$, with the exponent action. Denote by $\pi_\Gamma : G \to G/\Gamma$ the standard projection.

Definition 2.2 (Metric notations)**.** For every metric space X, we will always denote its metric by d_X, and whenever there is a natural base point to the space we denote by $B_X(R)$ a ball of radius R around the base point.

Let d_G be the unique Riemannian metric on G that is right G-invariant and left $\mathrm{SU}(2)$-invariant, normalized so that

$$d_G\left(\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\right) = |t|,$$

for all $t \in \mathbb{R}$. This metric restricts to a Riemannian metric $d_{\mathrm{SL}_2(\mathbb{R})}$ on $\mathrm{SL}_2(\mathbb{R})$ and gives rise to the standard hyperbolic metrics $d_{\mathbb{H}^3}, d_{\mathbb{H}^2}$ on $\mathbb{H}^3 = \mathrm{SU}(2)\backslash G$ and $\mathbb{H}^2 = \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$, respectively. This makes $\mathbb{H}^3$ a right G-space and $\mathbb{H}^2$ a right $\mathrm{SL}_2(\mathbb{R})$-space. Since d_G is right invariant, it descends to a Riemannian metric $d_{G/\Gamma}$ on G/Γ.

Definition 2.3 (Measure notations)**.** For every measure space (X, μ) and a measurable function $f : X \to \mathbb{R}$, we define $\mu(f) = \int_X f \mathrm{d}\mu$ and $f \cdot \mu$ the measure $U \mapsto \int_U f \mathrm{d}\mu$ on X. For every two measurable spaces $(X, \mu), (Y, \nu)$, we denote by $\mu \times \nu$ the product measure on $X \times Y$.

Definition 2.4 (Law of a random variable)**.** Let (X, Σ) be a space together with a σ-algebra. Let μ be a probability measure on X. Whenever we think of X as a probability space, then any measurable function $y : X \to Z$ to any other space $(Z, \mathcal{B})$, is called a *random variable*. Denote by $\mathrm{Law}(y) = y_*\mu$ the pushforward probability measure on Z. For every two random variables $y_1 : X \to Z_1, y_2 : X \to Z_2$, measurable with respect to the σ-algebras $\mathcal{B}_1, \mathcal{B}_2$ on Z_1, Z_2 respectively, we define a random variable $\mathrm{Law}(y_1|y_2) : X \to \{\text{probability measures on } Z_1\}$ as follows. Let $\mathcal{B}_2' = y_2^{-1}\mathcal{B}_2$ be the σ-algebra of all the information on X given by y_2. Let $x \mapsto \mu^x_{\mathcal{B}_2'}$ be the conditional measure, and $\mathrm{Law}(y_1|y_2)(x) = (y_1)_*\mu^x_{\mathcal{B}_2'}$. Similarly, if $y_1, y_2, \ldots, y_n$ are random variables, then we denote $\mathrm{Law}(y_1|y_2, y_3, \ldots, y_n) = \mathrm{Law}(y_1|(y_2, y_3, \ldots, y_n))$, where $(y_2, y_3, \ldots, y_n)$ is the tuple random variable.

Definition 2.5 (Entropy notations)**.** For every $p_1, ..., p_k \in [0, 1]$ with $p_1 + p_2 + \cdots + p_k = 1$, denote

$$H(p_1, \ldots, p_k) = -\sum_{i=1}^k p_i \log p_i.$$

For every space X, a measure μ on X with countable support, denote

$$H(\mu) = -\sum_{p \in \mathrm{supp}\,(\mu)} \mu(\{p\}) \log \mu(p).$$

Removing the countable support assumption, let τ be a partition of X. Denote

$$H_\mu(\tau) = -\sum_{A \in \tau} \mu(A) \log \mu(A).$$

For every $x \in X$ denote by $[x]_\tau$ the unique element in τ containing x. Now, whenever we think of X as a probability space and function from X as random variables, let $y : X \to S$ be a random variable with a countable image. We denote

$$H(y) = H(y_*\mu) = H_\mu(\{y^{-1}(x) : x \in \mathrm{Im}(y)\}).$$

Let y_1, y_2 be two random variables, such that given y_2, the random variable y_1 has countably many options, that is, $\mathrm{Law}(y_1|y_2)$ is almost surely a measure with countable support. Then we denote $H(y_1|y_2) = \int_X H(\mathrm{Law}(y_1|y_2))\mathrm{d}\mu y_2$. Similarly, if $y_1, \dots, y_n$ are random variable, then we define

$$H(y_1|y_2, y_3, \dots, y_n) = H(y_1|(y_2, y_3, \dots, y_n)) = \int_X H(\mathrm{Law}(y_1|y_2, y_3, \dots, y_n))\mathrm{d}\mu y_2.$$

3. Leafwise measures

The purpose of this section is to prove Theorem 3.9 below. We will define the leafwise measures and leafwise dimension in Subsections 3.1 and 3.2, and recall some of their properties. At the end of Subsection 3.2 we state Theorem 3.9. In subsection 3.3 we introduce a different approach to the leafwise measures, and use it to prove Theorem 3.9.

3.1. The leafwise measures.

Let X be a locally compact second countable (LCSC) space, $\mathbb{R} \curvearrowright X$ be a continuous action via $\mathtt{u}(s) : X \to X$ for every $s \in \mathbb{R}$. Let μ be a measure on X, not neccessarily $\mathtt{u}$-invariant. We assume that μ is $\mathtt{u}$-*free*, that is, μ almost every point $x \in X$ is not fixed by $\mathtt{u}(s)$ for all $s \in \mathbb{R}$. We recall the following characterization for leafwise measures, which is equivalent to the one given in [12, §3].

Definition 3.1 (Anti-convolution of a function with a measure)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function with $\int_\mathbb{R} f(s)\mathrm{d}s = 1$. Denote

$$S_f\mu = \int_\mathbb{R} f(s)\mathtt{u}(-s).\mu\mathrm{d}s.$$

Theorem 3.2 (Fubini construction of Leafwise measures)**.** *There is a measurable map $y \mapsto \mu_y^{\mathtt{u}}$ which associates to every $y \in X$ a locally finite measure $\mu_y^{\mathtt{u}}$ on $\mathbb{R}$, which satisfies the following properties:*

(1) for every $s \in \mathbb{R}$ and $y \in X$ we have

$$\mu^{\mathtt{u}}_{\mathtt{u}(s).y} \propto T^s_*\mu^{\mathtt{u}}_y, \quad \text{where} \quad T^s : \mathbb{R} \to \mathbb{R}, \quad T^s(r) = r - s. \tag{3.1}$$

(2) Let

$$\omega = F_*(\mu \times m_\mathbb{R}), \quad \text{where} \quad F : X \times \mathbb{R} \to X \times \mathbb{R}, \quad F(x, s) = (\mathtt{u}(-s)x, s), \tag{3.2}$$

where $\mu \times m_\mathbb{R}$ is the product measure on $X \times \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative integrable function with $\int_\mathbb{R} f(s)\mathrm{d}s = 1$, and set $\tilde{f} : X \times \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{f}(x, s) = f(s)$. For $S_f\mu$-almost every $y \in X$, we have

$$0 < \mu^{\mathtt{u}}_y(f) < \infty \qquad \text{and} \tag{3.3}$$

$$\tilde{f} \cdot \omega = \int_X \delta_y \times \frac{f \cdot \mu^{\mathtt{u}}_y}{\mu^{\mathtt{u}}_y(f)}\mathrm{d}S_f\mu(y). \tag{3.4}$$

Here we use the notations regarding measures from §2.3, and $\pi_\mathbb{R} : X \times \mathbb{R} \to \mathbb{R}$ is the projection.

The map $y \mapsto \mu_y^{\mathtt{u}}$ is unique in the following sense. If $y \mapsto \mu_y^{\mathtt{u},1}$ and $y \mapsto \mu_y^{\mathtt{u},2}$ are maps satisfying the above conditions, then for μ-almost every $x \in X$ we have $\mu_y^{\mathtt{u},1} \propto \mu_y^{\mathtt{u},2}$.

Remark 3.3 (On Eq. (3.4)). A different way to write the left-hand side is

$$(f \circ \pi_{\mathbb{R}}) \cdot \omega = F_*(\mu \times (f \cdot m_{\mathbb{R}})),$$

where F is as in Eq. (3.2).

An alternative way to write the formula (3.4) is that for every compactly supported function and continuous function $g : X \times \mathbb{R} \to \mathbb{R}$

$$\int_{X\times\mathbb{R}} f(s)g(x,s)\mathrm{d}\omega(x,s) = \int_X \frac{1}{\mu_y^{\mathtt{u}}(f)} \int_{\mathbb{R}} g(y,s)f(s)\mathrm{d}\mu_y^{\mathtt{u}}(s)\mathrm{d}S_f\mu(y), \quad \forall g \in C_c(X\times\mathbb{R}). \tag{3.4′}$$

Corollary 3.4. *In the notations of Theorem 3.2, note that $F_*^{-1}((f\circ\pi_{\mathbb{R}})\cdot\omega) = (f\circ\pi_{\mathbb{R}})\cdot(\mu\times m_{\mathbb{R}})$. Applying this to Eq. (3.4) and projecting to X, we obtain*

$$\mu = \int_X \frac{(s \mapsto \mathtt{u}(s)y)_*(f\cdot\mu_y^{\mathtt{u}})}{\mu_y^{\mathtt{u}}(f)} \mathrm{d}S_f\mu(y). \tag{3.5}$$

Reduction of Theorem 3.2 to [12, §3]. We will describe the statement of [12, §3], restricted to our $\mathbb{R}$ action. Denote by $\mathcal{B}_X$ the borel σ-algebra of X. Consider the infinite measure $\mu \times m_{\mathbb{R}}$ on $X \times \mathbb{R}$. Define $\Psi : X\times\mathbb{R} \to X$ by $\Psi(x,s) = \mathtt{u}(-s).x$ and let $\mathcal{C} = \Psi^{-1}(\mathcal{B}_X)$. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be a positive continuous integrable function. Lift f_0 to a map $\tilde{f}_0 : X \times \mathbb{R} \to \mathbb{R}$ by $\tilde{f}_0(x,s) = f_0(s)$. The conditional measures of $\tilde{f}_0 \cdot (\mu \times m_{\mathbb{R}}) = \mu \times (f_0 \cdot m_{\mathbb{R}})$ with respect to the σ-algebra $\mathcal{C}$ are denoted $(\tilde{f}_0 \cdot (\mu\times m_{\mathbb{R}}))_y^{\mathcal{C}}$, for $y \in X \times \mathbb{R}$. This measure lies on the atom $[y]_{\mathcal{C}}$ of y which is of the form $\{(\mathtt{u}(s).x, s) : s \in \mathbb{R}\} = \Psi^{-1}(x)$ for $x = \Psi(y) \in X$, and $(\tilde{f}_0 \cdot (\mu\times m_{\mathbb{R}}))_y^{\mathcal{C}}$ depends only on the atom, that is, only on x, and is supported on this atom. Define $\mu_x^{\mathtt{u}}$ on $\mathbb{R}$ so that

$$(\tilde{f}_0 \cdot (\mu\times m_{\mathbb{R}}))_y^{\mathcal{C}} = \tilde{f}_0 \cdot a_*^x\mu_x^{\mathtt{u}}, \quad \text{where} \quad a^x : \mathbb{R} \to X\times\mathbb{R}, \quad a^x(s) = (\mathtt{u}(s).x, s). \tag{3.6}$$

Then [12, §3] ensures that Eq. (3.1) holds in a $\mathtt{u}$-invariant set $X' \subseteq X$ with $\mu(X') = 1$.

We will now deduce our formulation of the result. To ensure that Eq. (3.1) holds everywhere, we redefine $\mu_x^{\mathtt{u}} := 0$ for $x \notin X'$. This implies that Eq. (3.1) holds for all $x \in X$.

As to the second condition, Eq. (3.6) implies that $\tilde{f}_0 \cdot a_*^x\mu_x^{\mathtt{u}} = a_*^x(f_0\cdot\mu_x^{\mathtt{u}})$ is a probability measure for all $x \in X'$. That is, $\mu_x^{\mathtt{u}}(f_0) = 1$. In addition,

$$\begin{aligned}\tilde{f}_0 \cdot (\mu\times m_{\mathbb{R}}) &= \int_X (\tilde{f}_0 \cdot (\mu\times m_{\mathbb{R}}))_y^{\mathcal{C}} \mathrm{d}(\tilde{f}_0\cdot(\mu\times m_{\mathbb{R}}))(y) \\ &= \int_X \tilde{f}_0 \cdot a_*^x\mu_x^{\mathtt{u}} \mathrm{d}\Psi_*(\tilde{f}_0\cdot(\mu\times m_{\mathbb{R}}))(x).\end{aligned} \tag{3.7}$$

To simplify Eq. (3.7), first notice that

$$\Psi_*(\tilde{f}_0\cdot(\mu\times m_{\mathbb{R}})) = \Psi_*(\mu\times(f_0\cdot m_{\mathbb{R}})) = S_{f_0}\mu.$$

Second, we can multiply Eq. (3.7) by $\tilde{f}_0^{-1}$ and obtain

$$\mu \times m_{\mathbb{R}} = \int_X a_*^x \mu_x^{\mathtt{u}} \mathrm{d}S_{f_0}\mu(x). \tag{3.8}$$

Applying F_* to Eq. (3.8) we obtain

$$F_*(\mu \times m_{\mathbb{R}}) = \int_X \delta_x \times \frac{\mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f_0)} \mathrm{d}S_{f_0}\mu(x), \tag{3.9}$$

where the denominator could be added since it is almost surely the constant 1. This formula is equivalent to Eq. (3.4), for $f = f_0$, after multiplying with $\tilde{f}_0$. To obtain it for general nonnegative $f \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$, we multiply Eq. (3.8) with the function $\tilde{f} : X \times \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(x, s) = f(s)$:

$$F_*(\mu \times (f \cdot m_{\mathbb{R}})) = \tilde{f} \cdot F_*(\mu \times m_{\mathbb{R}}) = \int_X \delta_x \times \frac{f \cdot \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f_0)} \mathrm{d}S_{f_0}\mu(x). \tag{3.10}$$

From Eq. (3.10) we deduce that for $S_{f_0}\mu$-almost every $x \in X$ we have $f \cdot \mu_x^{\mathtt{u}}$ is a finite measure. Since $f \cdot \mu_x^{\mathtt{u}} = 0$ if and only if $\mu_x^{\mathtt{u}}(f) = 0$ we deduce that we may restrict the integral in the right-hand side of Eq. (3.10) to $X_0 = \{x \in X : \mu_x^{\mathtt{u}}(f) = 0\}$.

(3.11)

$$\begin{aligned} F_*(\mu \times (f \cdot m_{\mathbb{R}})) &= \int_{X_0} \delta_x \times \frac{f \cdot \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f_0)} \mathrm{d}S_{f_0}\mu(x) = \int_{X_0} \delta_x \times \frac{f \cdot \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f)} \frac{\mu_x^{\mathtt{u}}(f)}{\mu_x^{\mathtt{u}}(f_0)} \mathrm{d}S_{f_0}\mu(x) \\ &= \int_{X_0} \delta_x \times \frac{f \cdot \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f)} \mathrm{d}\mu_f(x) = \int_X \delta_x \times \frac{f \cdot \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f)} \mathrm{d}\mu_f(x), \end{aligned}$$

where

$$\mu_f = \int_X \delta_x \mathrm{d}\frac{\mu_x^{\mathtt{u}}(f)}{\mu_x^{\mathtt{u}}(f_0)} \mathrm{d}S_{f_0}\mu(x),$$

and last equality of Eq. (3.11) holds since μ_f is supported on X_0. To compute μ_f, project Eq. (3.11) to X. The projection of the right-hand side is μ_f. The projection of the left-hand side is $S_f\mu$, and hence $S_f\mu = \mu_f$. Therefore, Eq. (3.11) is equivalent to Eq. (3.4). Eq. (3.3) from the equality $S_f\mu = \mu_f$, the definition of μ_f, and the fact that $S_f\mu$ is a probability measure.

To show the uniqueness of the measures $\mu_x^{\mathtt{u}}$, note that we have established an equivalence between Eq. (3.4) applied to f_0 and

$$(\tilde{f}_0 \cdot (\mu \times m_{\mathbb{R}}))_y^{\mathcal{C}} = f_0 \cdot \frac{a_*^x \mu_x^{\mathtt{u}}}{\mu_x^{\mathtt{u}}(f_0)}.$$

for $\tilde{f}_0 \cdot (\mu \times m_{\mathbb{R}})$-almost every y and $x = \Psi(y)$. Since the conditional measures are uniquely defined almost everywhere, we deduce that $\mu_x^{\mathtt{u}}$ is uniquely defined S_{f_0} almost surely.

□

The following claims follow from the uniqueness of the characterization of Theorem 3.2.

Claim 3.5 (Equivariance of Leafwise measures)**.** *If $\alpha : X \to Y$ is an injective map of LCSC spaces, $\mathbb{R} \overset{\mathtt{u}'}{\curvearrowright} Y$, and*

$$\alpha(\mathtt{u}(s).x) = \mathtt{u}'(s).\alpha(x), \qquad \forall x \in X, s \in \mathbb{R},$$

then for μ-almost every $x \in X$,

$$\mu_x^{\mathtt{u}} \propto (\alpha_* \mu)^{\mathtt{u}'}_{\alpha(x)}. \tag{3.12}$$

Moreover, there is a set $X' \subseteq X$ that is $\mathtt{u}$ invariant and has $\mu(X') = 1$ such that Eq. (3.12) holds for all $x \in X'$.

Claim 3.6 (Rescaling of the action)**.** *Suppose that $\beta \neq 0$, and define the rescaled action $\mathbb{R} \overset{\mathtt{u}'}{\curvearrowright} X$ by $\mathtt{u}'(s).x = \mathtt{u}(\beta s).x$. Then μ-almost every for all $x \in X$ have*

$$\mu_x^{\mathtt{u}'} \propto (s \mapsto \beta^{-1} s)_* \mu_x^{\mathtt{u}}. \tag{3.13}$$

Moreover, there is a set $X' \subseteq X$ that is $\mathtt{u}$ invariant and has $\mu(X') = 1$ such that Eq. (3.13) holds for all $x \in X'$.

3.2. **Leafwise dimension.** We will discuss measures μ on an LCSC space X. We require that there is a measurable action $B \curvearrowright X$ where $B \cong \mathbb{R} \ltimes \mathbb{R}$ is defined as in Homogeneous Dynamics Notations 2.1. Our measures μ will be A-invariant and $\mathtt{u}$-free, and we will analyze their $\mathtt{u}$-leafwise measures.

Definition 3.7 (Leafwise dimension)**.** Let X be an LCSC space with a continuous action $B \curvearrowright X$. Let μ be an A-invariant $\mathtt{u}$-free probability measure on X. We say that μ has $\mathtt{u}$-*leafwise dimension* δ and write $\dim^{\mathtt{u}}(\mu) = \delta$ if for μ-almost-all x,

$$\lim_{t \to \infty} \frac{1}{t} \log \mu_x^{\mathtt{u}}([-e^{-t}, e^{-t}]) = -\delta. \tag{3.14}$$

If μ is ergodic, then $\dim^{\mathtt{u}}(\mu)$ exists. This existence is proved in the homogenous setting in [11, Thm 7.6(i)]. However, their proof works for our setting as well.

One can relate leafwise dimension to entropy. This will be useful in Section 3.

Theorem 3.8 (Relation of leafwise dimension to entropy)**.** *Let $\Lambda \subseteq \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup and μ an $\mathtt{a}$-invariant and ergodic probability measure on $\mathrm{SL}_2(\mathbb{R})/\Lambda$. Then*

$$h_\mu(\mathtt{a}(t)) = |t| \dim^{\mathtt{u}}(\mu).$$

This theorem is proved in [11, Thm 7.6 (ii)].

The main result of this section is the following:

Theorem 3.9. *Let $(\mu_k)_{k=1}^\infty$ be $\mathtt{a}(t)$-invariant and ergodic probability measures on an LCSC space X on which B acts continuously. We further assume that for every k the measure μ_k is $\mathtt{u}$-free. Suppose that the $\mathtt{u}$-leafwise dimensions*

$$\dim^{\mathtt{u}} \mu_k \xrightarrow{k \to \infty} 1. \tag{3.15}$$

Suppose that there is a weak-$*$ probability measure limit $\mu_\infty = \lim_{k\to\infty} (\mu_k)_{k=1}^\infty$. Then μ_∞ is $\mathtt{u}$-invariant.

3.3. **The leafwise Markov chain.** In this subsection, we will prove Theorem 3.9. To prove this theorem we need some macroscopic way to view the leafwise dimension, as opposed to Definition 3.7, which views it as a limit of a leafwise measure of very small intervals. To do this we introduce a Markov chain, somewhat similar to the one introduced in Furstenberg [17]. The properties of the Markov chain are summarized in the following lemma.

Lemma 3.10 (The leafwise Markov chain)**.** *Let X be an LCSC topological space, let $B \curvearrowright X$ be a continuous action and let μ be an A-invariant ergodic $\mathtt{u}$-free measure. Then there is a function $p : X \to [0,1]$ such that*

$$\int_X H(p(x), 1-p(x))\mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu(x) = \dim^u(\mu)\log 2, \tag{3.16}$$

$$\int_X \omega_x \mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu(x) = S_{\mathbb{1}_{[0,1)}}\mu, \tag{3.17}$$

where

$$\omega_x = p(x)\delta_{\mathtt{a}(\log 2).x} + (1-p(x))\delta_{\mathtt{u}(1)\mathtt{a}(\log 2).x}. \tag{3.18}$$

Fix an $\mathtt{a}$-invariant and ergodic $\mathtt{u}$-free measure μ on X.

Claim 3.11 (Almost no periods)**.** *In this setting,*

$$\mu\left(\{x \in X : \mathrm{stab}_U(x) \neq I\}\right) = 0.$$

The claim is fairly standard, see for instance [11, Lem. 7.12] in the homogeneous case.

Consider the dynamical system with space $X \times [0,1)$, measure $\nu_0 = \mu \times m_{[0,1)}$, and action $T_0(x,s) = (\mathtt{a}(\log(2))x, 2s \bmod 1)$, which preserves ν_0. Conjugate it by

$$F : X \times [0,1) \to X \times [0,1), \qquad F(x,s) = (\mathtt{u}(-s)x, s).$$

We obtain a dynamical system $(T, X \times [0,1), \nu)$, where

$$\nu = F_*\nu_0 \overset{(3.4)}{=} \int_X \delta_y \times \frac{\mu_y^{\mathtt{u}}|_{[0,1)}}{\mu_y^{\mathtt{u}}([0,1))}\mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu(y) \tag{3.19}$$

and $T(y,s) = (\mathtt{u}(b_1(s))\mathtt{a}(\log 2)x, 2s \bmod 1)$, where $b_1(s)$ is the 2^{-1} bit of the binary point in the binary expansion, $s = \sum_{i=1}^{\infty} 2^{-i}b_i(s)$, where $b_i(s) \in \{0,1\}$. Let $p_0 = (y_0, s)$ be a sample point in the probability space $(X \times [0,1), \nu)$. A different interpretation of Eq. (3.19) is the following almost sure equalities,

$$\mathrm{Law}(y_0) = S_{\mathbb{1}_{[0,1)}}\mu, \tag{3.20}$$

$$\mathrm{Law}(s|y_0) = \frac{\mu_y^{\mathtt{u}}|_{[0,1)}}{\mu_y^{\mathtt{u}}([0,1))}. \tag{3.21}$$

Denote by $\pi_X : X \times [0,1) \to X$ the projection. For every $n \geq 1$ let $p_n = T^n p_0$ and for every $n \geq 0$ denote $y_n = \pi_X(p_n)$. One can see that

$$y_n = \mathtt{a}(n\log 2)\mathtt{u}(s_n)y_0, \tag{3.22}$$

where $s_n = \sum_{i=1}^{n} 2^{-i}b_i(s)$. Let $X' = \{x \in X : \mu_x^{\mathtt{u}}([0,1)) > 0\}$. It has a full $S_{\mathbb{1}_{[0,1)}}\mu$-measure from Eq. (3.3). Define

$$p(x) = \begin{cases} \frac{\mu_x^{\mathtt{u}}([0,1/2))}{\mu_x^{\mathtt{u}}([0,1))}, & x \in X', \\ 0, & \text{otherwise}, \end{cases}$$

and let ω_x be as in Eq. (3.18).

Claim 3.12. *The stochastic process $(y_n)_{n=0}^{\infty}$ is a stationary Markov process, with the ν-almost always law*

$$\mathrm{Law}(y_n|y_0, y_1, \ldots, y_{n-1}) = \mathrm{Law}(y_n|y_{n-1}) = \omega_{y_{n-1}}. \tag{3.23}$$

Proof. It will be sufficient to show that the RHS and LHS of 3.23 coincide. Indeed, by Eq. (3.22) and Claim 3.11, $\mathrm{Law}(y_n|y_0, y_1, \dots, y_{n-1}) = \mathrm{Law}(y_n|s_{n-1}, y_0)$. Now,

(3.24)
$$\mathrm{Law}(s_n|s_{n-1}, y_0) \overset{(3.21)}{=} \begin{cases} s_{n-1}, & \text{with probability } \frac{\mu^{\mathtt{u}}_{y_0}([s_{n-1}, s_{n-1}+2^{-n}))}{\mu^{\mathtt{u}}_{y_0}([s_{n-1}, s_{n-1}+2^{-n+1}))}, \\ s_{n-1}+2^{-n}, & \text{with probability } \frac{\mu^{\mathtt{u}}_{y_0}([s_{n-1}+2^{-n}, s_{n-1}+2^{-n+1}))}{\mu^{\mathtt{u}}_{y_0}([s_{n-1}, s_{n-1}+2^{-n+1}))}. \end{cases}$$

We wish to apply Claims 3.5 and 3.6 to y_0. Since $u(s)y_0 \sim \mu$ we deduce that $y_0 \in X'$ almost surely, where X' is one of the sets in Claims 3.5 and 3.6, and hence these claim are applicable. Similarly, $y_n \in X'$ almost surely for all $n \geq 0$. Apply Claim 3.5 for the $\mathtt{a}((n-1)\log 2)$-action, which takes the $\mathtt{u}(s)$ action to $\mathtt{u}'(s) = \mathtt{u}(2^n s)$-action. Hence

$$\begin{aligned} \mu^{\mathtt{u}}_{y_{n-1}} &\overset{3.6}{\propto} (s \mapsto 2^{n-1}s)_* \mu^{\mathtt{u}'}_{y_{n-1}} \overset{(3.22)}{=} (s \mapsto 2^{n-1}s)_* \mu^{\mathtt{u}'}_{\mathtt{a}((n-1)\log 2)\mathtt{u}(s_{n-1})y_0} \\ &\overset{3.5}{\propto} (s \mapsto 2^{n-1}s)_* \mu^{\mathtt{u}}_{\mathtt{u}(s_{n-1})y_0} \overset{(3.1)}{\propto} (s \mapsto 2^{n-1}s)_* (s \mapsto s - s_{n-1})_* \mu^{\mathtt{u}}_{y_0} \\ &= (s \mapsto 2^{n-1}(s - s_{n-1}))_* \mu^{\mathtt{u}}_{y_0}. \end{aligned}$$

Thus Eq. (3.24) implies the desired. □

This shows Eq. (3.17).

Proof of Eq. (3.16). Let $C = \int_X H(p(x), 1-p(x))\mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu(x)$. Then $C = H(y_1|y_0)$. The Markov chain property implies that

$$C = H(y_n|y_{n-1}) = H(y_n|y_{n-1}, \dots, y_0) = H(b_n(s)|b_{n-1}(s), \dots, b_1(s), y_0).$$

Hence

(3.25)
$$H(b_n(s), b_{n-1}(s), \dots, b_1(s)|y_0) = \sum_{m=1}^{n} H(b_m(s)|b_{m-1}(s), \dots, b_1(s), y_0) = nC.$$

Denote by $\tau_n = \{[m2^{-n}, (m+1)2^{-n}) : m = 0, \dots, 2^n - 1\}$ the partition of $[0, 1)$. Rewriting Eq. (3.25) using Eq. (3.21), we obtain

$$\int_X H_{\frac{\mu^{\mathtt{u}}_{y_0}|_{[0,1)}}{\mu^{\mathtt{u}}_{y_0}([0,1))}}(\tau_n)\,\mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu(y_0) = nC. \tag{3.26}$$

It follows from Lemma 3.13 below and the Dominated Convergence Theorem that $C = \dim^{\mathtt{u}}(\mu)\log 2$. □

Lemma 3.13. *For μ-almost-all $x \in X$, and for all $t \in \mathbb{R}$ such that $\mu^{\mathtt{u}}_x([t, t+1]) \neq 0$, denote $M_{x,t} = (x \to x - t)_* \frac{\mu^{\mathtt{u}}_x|_{[t,t+1]}}{\mu^{\mathtt{u}}_x([t,t+1])}$. Then $\frac{1}{n} H(M_{x,y}, \tau_n) \xrightarrow{n\to\infty} \dim^{\mathtt{u}}(\mu)\log 2$.*

This lemma will follow from the following Lemma. Let

$$\tilde{\tau}_n = \{[m2^{-n}, (m+1)2^{-n}) : m \in \mathbb{Z}\},$$

denote the partition of $\mathbb{R}$. For every $s \in \mathbb{R}$ denote by $\tilde{\tau}_m(s)$ the unique partition element in $\tilde{\tau}_m$ containing s.

Claim 3.14. *Let ν be a locally finite measure on $\mathbb{R}$ satisfying that for ν-almost-all $s \in \mathbb{R}$,*

$$\frac{1}{t}\log \nu([s - e^t, s + e^t]) \xrightarrow{t\to\infty} \Delta. \tag{3.27}$$

Define the function $u_n : \mathbb{R} \to \mathbb{R}$ by $u_n(s) = -\frac{1}{n}\log \nu(\tilde{\tau}_n(s))$. Then $u_n \xrightarrow{m\to\infty} \Delta \log 2$ locally in $L^1(\mathbb{R},\nu)$. That is, for every finite interval $I \subseteq \mathbb{R}$,

$$\int_I |u_n(s) - \Delta\log 2|\, \mathrm{d}\nu(s) \xrightarrow{n\to\infty} 0.$$

Proof of Lemma 3.13 using Claim 3.14. Let

$$X_{good,0} = \left\{x \in X : -\frac{1}{t}\log \mu_x^{\mathtt{u}}([-e^{-t}, e^{-t}]) \xrightarrow{t\to\infty} \dim^{\mathtt{u}}(\mu)\right\}.$$

As mensioned in Definition 3.7, $\mu(X_{good,0}) = 1$. Let

$$X_{good,1} = \left\{x \in X : {\scriptstyle \text{for } \mu_x^{\mathtt{u}}\text{-almost all } s\in\mathbb{R} \atop \text{we have } \mathtt{u}(s)x\in X_{good,0}}\right\}.$$

This set is $\mathtt{u}$-invariant by Eq. (3.1). We will now show that $\mu(X_{good,1}) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive integrable function with $\int_{\mathbb{R}} f \mathrm{d}s = 1$.

$$\begin{aligned} 0 = \mu(X_{good,0}^c) &\overset{(3.5)}{=} \int_X \frac{\int_{\mathbb{R}} f(s_0)\mathbb{1}_{\mathtt{u}(s_0)y\in X_{good,0}^c}\mathrm{d}\mu_y^{\mathtt{u}}(s_0)}{\mu_y^{\mathtt{u}}(f)}\mathrm{d}S_f\mu(y) \\ &= \int_X\int_{\mathbb{R}} \frac{\int_{\mathbb{R}} f(s_0)\mathbb{1}_{\mathtt{u}(s_0)\mathtt{u}(s_1)x\in X_{good,0}^c}\mathrm{d}\mu_{\mathtt{u}(s_1)x}^{\mathtt{u}}(s_0)}{\mu_{\mathtt{u}(s_1)x}^{\mathtt{u}}(f)} f(-s_1)\mathrm{d}s_1\mathrm{d}\mu(x) \end{aligned}$$

Thus, the positivity of f implies that for $\mu \times m_{\mathbb{R}}$-almost all $x, s_1 \in X \times \mathbb{R}$,

$$\mu_{\mathtt{u}(s_1)x}^{\mathtt{u}}(\{s_0 \in \mathbb{R} : \mathtt{u}(s_0)\mathtt{u}(s_1)x \in X_{good,0}^c\}) = 0. \tag{3.28}$$

By Eq. (3.1), Eq. (3.28) is independent of s_1. Therefore, for μ-almost-all $x \in X$ we have Eq. (3.28) with $s_1 = 0$, which is equivalent to $x \in X_{good,1}$. Hence, $\mu(X_{good,1}) = 1$.

By Eq. (3.1), for every $x \in X_{good,1}$ the measure $\mu_x^{\mathtt{u}}$ satisfies the condition of Claim 3.14. This conclusion implies the desired result.

□

Proof of Claim 3.14. We will first show that it is enough to prove this claim under some simplifying assertions. It is sufficient to prove the convergence for intervals $I = (a, a+1)$ with $a \in 1/2\mathbb{Z}$. Then we may restrict ν to I, while preserving the property for ν-almost all $s \in I$. Translating ν, we may assume that ν is supported on $(0,1)$. Normalizing ν to a probability measure does not change the result as well. We may now prove that $u_n \xrightarrow{m\to\infty} \Delta\log 2$ in $L^1(\mathbb{R},\nu)$, under the assumption that ν is a probability measure on $(0,1)$.

Let $\varepsilon > 0$ be small numbers and n an integer going to ∞. By Eq. (3.27), for all sufficiently large n there is $S \subseteq (0,1)$ of volume $\nu(S) > 1-\varepsilon$ such that for all $s \in S$ and for all $t \geq n\log 2$,

$$\left|-\frac{1}{t}\log \nu([s - e^{-t}, s + e^{-t}]) - \Delta\right| < \varepsilon.$$

We will next show that $\int_0^1 (u_n - \Delta\log 2)^- \mathrm{d}\nu \xrightarrow{n\to\infty} 0$, where for every $x \in \mathbb{R}$ we have $x^+ = \max(x, 0)$ and $x^- = (-x)^+$, so that $x = x^+ - x^-$. Let

$$J_- = \{s \in [0,1) : u_n(s) < \Delta\log 2 - \varepsilon\}.$$

Then J_- is a union on τ_n elements. Note that $J_- \cap S = \emptyset$. Indeed, if $s \in J_- \cap S$, then we get the following contradiction:

$$\begin{aligned}\Delta \log 2 - \varepsilon > u_n(s) &= -\frac{1}{n} \log \nu(\tau_n(s)) \\ &\geq -\frac{1}{n} \log \nu([x + e^{-n\log 2}, x - e^{-n\log 2}]) \geq (\Delta - \varepsilon)\log 2.\end{aligned}$$

Consequently, $\nu(J_-) \leq 1 - \nu(S) < \varepsilon$. This implies that

$$\begin{aligned}\int_0^1 (u_n - \Delta \log 2)^- \mathrm{d}\nu &< \int_{J_-} (u_n - \Delta \log 2)^- \mathrm{d}\nu + \int_{J_-^c} (u_n - \Delta \log 2)^- \mathrm{d}\nu \\ &\overset{u_n \geq 0}{<} \Delta \nu(J_-) \log 2 + \varepsilon(1 - \nu(J_-)) \leq \varepsilon(\Delta \log 2 + 1 - \nu(J_-)).\end{aligned}$$

Taking $\varepsilon \to 0$ implies that $\int_0^1 (u_n - \Delta \log 2)^- \mathrm{d}\nu \xrightarrow{n\to\infty} 0$.

We will now show that $\int_0^1 (u_n - \Delta \log 2)^+ \mathrm{d}\nu \xrightarrow{n\to\infty} 0$. Define

$$J'_+ = \bigcup \{I \in \tau_n : \exists I' = I \pm 2^{-n} \text{ with } 3\varepsilon\nu(I') > \nu(I)\},$$

which satisfies

$$\nu(J'_+) = \sum_I \nu(I) < 3\varepsilon \sum_{I,I'} \nu(I') \leq 6\varepsilon, \tag{3.29}$$

where the sums are over I, I' as in the definition on J'_+. The rightmost inequality of Eq. (3.29) holds because each $I_0 \in \tau_n$ can appear as I' at most twice. Let

$$A = \{I \in \tau_n : I \subseteq J'_+ \cup S^c\}, \qquad J_+ = \bigcup_{I \in A} I.$$

We can estimate

$$\zeta := \nu(J_+) \leq \nu(J'_+) + \nu(S^c) \leq 6\varepsilon + \varepsilon.$$

Thus

$$\begin{aligned}\int_0^1 (u_n - \Delta \log 2)^+ \mathrm{d}\nu &= \int_{J_+} (u_n - \Delta \log 2)^+ \mathrm{d}\nu + \int_{J_+^c} (u_n - \Delta \log 2)^+ \mathrm{d}\nu \\ &= \sum_{I \in A} \nu(I) \left(-\frac{1}{n} \log \nu(I) - \Delta\right)^+ + \sum_{I \in \tau_n \setminus A} \nu(I) \left(-\frac{1}{n} \log \nu(I) - \Delta\right)^+. \end{aligned} \tag{3.30}$$

To bound the sum over $I \in A$

$$\begin{aligned}\sum_{I \in A} \nu(I) \left(-\frac{1}{n} \log \nu(I) - \Delta\right)^+ &\leq \frac{1}{n} \sum_{I \in A} -\nu(I) \log \nu(I) \\ &= -\zeta \log \zeta + \frac{\zeta}{n} \sum_{I \in A} -\frac{\nu(I)}{\zeta} \log \frac{\nu(I)}{\zeta} \leq -\zeta \log \zeta + \zeta \log 2.\end{aligned} \tag{3.31}$$

The last inequality holds since $(\frac{\nu(I)}{\zeta})_{I \in A}$ is a probability vector on at most 2^n and hence the maximal entropy it can have is $n \log 2$. For the other summand, let $I \in \tau_n \setminus A$. Since $I \notin A$ we deduce that $I \cap J'_+ = \emptyset$ and $I \cap S \neq \emptyset$. Thus there is

$s \in I \cap S$ with $s \notin J'_+$.

$$-\frac{1}{n}\log\nu(I) = -\frac{1}{n}\log\nu(\tau_n(s)) \overset{s\notin J'_+}{\leq} -\frac{1}{n}\log(\varepsilon\nu(\tau_n(s)\cup(\tau_n(s)-2^{-n})\cup(\tau_n(s)+2^{-n})))$$

(3.32)

$$\leq -\frac{1}{n}\log(\varepsilon\nu([s-2^{-n},s+2^{-n}])) \overset{s\in S}{\leq} -\frac{1}{n}\left(\log\varepsilon - n(\Delta+\varepsilon)\right) = \Delta+\varepsilon-\frac{\log\varepsilon}{n}$$

Combining Eqs. (3.30), (3.31), (3.32) we deduce that for all n sufficiently large

$$\int_0^1 (u_n - \Delta\log 2)^+ \mathrm{d}\nu \leq -7\varepsilon\log(7\varepsilon) + 7\varepsilon\log 2 + \varepsilon - \frac{\log\varepsilon}{n}.$$

Taking $\varepsilon \to 0$ implies that $\int_0^1 (u_n - \Delta\log 2)^+\mathrm{d}\nu \xrightarrow{n\to\infty} 0$. The desired follows. $\square$

Proof of Theorem 3.9. For every $k = 0, 1, \ldots$, let p^k, ν_x^k as in Lemma 3.10 constructed for μ^k. Let $f \in C_c(X)$ be a continuous compactly supported function. Then

$$\begin{aligned}
&S_{\mathbb{1}_{[0,1)}}\mu^k(f) - \frac{1}{2}\left(\mathtt{a}(\log 2).S_{\mathbb{1}_{[0,1)}}\mu^k\right)(f) - \frac{1}{2}\left(\mathtt{a}(\log 2).S_{\mathbb{1}_{[0,1)}}\mu^k\right)(f)\\
&\quad= \int_X \left(\left(p(x)-\frac{1}{2}\right) f(\mathtt{a}(\log 2).x) + \left(1-p(x)-\frac{1}{2}\right) f(\mathtt{u}(1)\mathtt{a}(\log 2).x)\right) \mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu^k(x)\\
&\quad\leq \|f\|_\infty \int_X \left|p(x)-\frac{1}{2}\right| \mathrm{d}\mu^k(x) \xrightarrow{k\to\infty} 0
\end{aligned}$$

The convergence follows from Eq. (3.16). Hence we get an equality of the weak-$*$ limits

$$S_{\mathbb{1}_{[0,1)}}\mu^\infty = \frac{1}{2}\mathtt{a}(\log 2).S_{\mathbb{1}_{[0,1)}}\mu^\infty + \frac{1}{2}\mathtt{a}(\log 2).S_{\mathbb{1}_{[0,1)}}\mu^\infty. \tag{3.33}$$

inductively applying (3.33) to itself, and using $\mathtt{a}(\log 2)\mathtt{u}(1) = \mathtt{u}(2)\mathtt{a}(\log 2)$, we get

$$S_{\mathbb{1}_{[0,1)}}\mu^\infty = \frac{1}{2^n}\sum_{i=0}^{2^n-1} \mathtt{u}(i)\mathtt{a}(n\log 2).S_{\mathbb{1}_{[0,1)}}\mu^\infty.$$

Hence $(\mathtt{u}(1)S_{\mathbb{1}_{[0,1)}}\mu^\infty - S_{\mathbb{1}_{[0,1)}}\mu^\infty)(f) \leq \frac{2}{2^n}\|f\|_\infty$ for every $f \in C_c(X)$. Taking $n \to \infty$ we deduce that $S_{\mathbb{1}_{[0,1)}}\mu^\infty$ is $\mathtt{u}(1)$ invariant, and hence $\mathtt{u}(k)$-invariant for every $k \in \mathbb{Z}$. Note that $\mathtt{a}(-n\log 2)S_{\mathbb{1}_{[0,1)}}\mu^\infty$ is $\mathtt{a}(-n\log 2)\mathtt{u}(k)\mathtt{a}(n\log 2) = \mathtt{u}(2^{-n}k)$-invariant, for every $k \in \mathbb{Z}$. On the other hand,

$$\begin{aligned}
\mathtt{a}(-n\log 2)S_{\mathbb{1}_{[0,1)}}\mu^\infty &= \int_0^1 \mathtt{a}(-n\log 2)\mathtt{u}(-s).\mu^\infty \mathrm{d}s = \int_0^1 \mathtt{u}(-2^{-n}s)\mathtt{a}(-n\log 2).\mu^\infty \mathrm{d}s\\
&= \int_0^1 \mathtt{u}(-2^{-n}s).\mu^\infty \mathrm{d}s \xrightarrow[n\to\infty]{\text{weak-}*} \mu^\infty.
\end{aligned}$$

where the last equality follows from the $\mathtt{a}$-invariance of μ^∞. Since μ^∞ is a weak-$*$ limit of measures with more and more invariance, we obtain that μ^∞ is invariant to $\mathtt{u}\left(\bigcup_{n=0}^\infty 2^{-n}\mathbb{Z}\right)$, and hence to the entire $\mathtt{u}$-action. $\square$

4. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2, conditioned on Lemma 4.7. The proof is an adaptation of [1], and is composed of four components, which we will enumerate from last to first. The last component is [1, Thm. 1.6], whose outcome contradicts the nonarithmeticity assumption we assumed. The third is a reduction of our problem to [1, Thm. 1.6]. We will do this similarly to [1, §3]. However, two new ergodic components will be needed for this part, namely, Theorem 3.9, and Lemma 4.7. This section will focus on the reduction to [1, Thm. 1.6].

4.1. Notation and preliminaries.

Theorem 4.1 (Properties of Bowen-Margulis-Sullivan measures)**.** *Let $\Lambda < \mathrm{SL}_2(\mathbb{R})$ be a finitely generated discrete group. Then there is an $\mathsf{a}(t)$-invariant probability measure of maximal entropy, μ_Λ on $\mathrm{SL}_2(\mathbb{R})/\Lambda$ with*

$$h_\mu(\mathsf{a}(t)) = |t|\delta(\Gamma).$$

See for instance [36]. To bound from below the critical exponent we will use the following claim:

Theorem 4.2 (Approximation of critical exponent)**.** *For every $0.5 > \varepsilon > 0$, there is $R_\varepsilon > 0$ such that the following holds. Let $\Lambda < \mathrm{SL}_2(\mathbb{R})$ be a discrete torsion-free subgroup such that $\delta(\Gamma) \le 1-\varepsilon$ for some $\varepsilon > 0$ and $\Lambda \cap B_{\mathrm{SL}_2(\mathbb{R})}(e^{-0.1\varepsilon R}) = \{I\}$ for some $R \ge R_\varepsilon$. Then*

$$\#\Lambda \cap B_{\mathrm{SL}_2(\mathbb{R})}(R) \le e^{(1-0.1\varepsilon)R}, \tag{4.1}$$

$$m_{\mathrm{SL}_2(\mathbb{R})}\left(\left\{h \in B_{\mathrm{SL}_2(\mathbb{R})}(R) : \#\{\gamma \in \Gamma : \gamma h \in B_{\mathrm{SL}_2(\mathbb{R})}(R)\} > e^{(1-0.1\varepsilon)R}\right\}\right) < e^{(1-\varepsilon/2)R}. \tag{4.2}$$

To use this, we need to replace Γ with a torsion-free subgroup of finite index. To do so we need two claims. The first is a special case of [29, Cor. 6.13], and the second will be proved in the appendix.

Claim 4.3. *Every finitely generated subgroup in a $\Lambda < G$ has a torsion-free subgroup of finite index.*

Claim 4.4. *For every two subgroups $\Lambda_1 < \Lambda_2 < \mathrm{SL}_2(\mathbb{R})$ with $[\Lambda_2 : \Lambda_1] < \infty$, we have $\delta(\Lambda_1) = \delta(\Lambda_2)$.*

4.2. **Beginning of the proof.** Recall that for every $g \in G$ we define $\Gamma_g = g\Gamma g^{-1} \cap \mathrm{SL}_2(\mathbb{R})$. We first use Claim 4.3 to replace Γ with a finite index subgroup which is also torsion-free. The critical exponents of the subgroups Γ_g do not change by Claim 4.4, hence we may assume from now on that Γ is torsion-free, and apply Theorem 4.2 to its subgroups.

The contrary Theorem 1.2 is the existence of a sequence $(g_k)_{k=1}^\infty \subset G$ such that $\delta(\Gamma_{g_k}) \xrightarrow{k\to\infty} 1$ but Γ_{g_k} is never a lattice. Note that if there is an element $g_0 \in G$ such that $\delta(\Gamma_{g_0}) = 1$ but Γ_g is not a lattice, the sequence $(g_k)_{k=1}^\infty$ may be the constant sequence g_0. We assume to the contrary that such a sequence exists.

Claim 4.5. *For every $k > k_0$ there is a finitely generate subgroup $\Gamma_k < \Gamma_{g_k}$ such that $\delta(\Gamma_k) \xrightarrow{k\to\infty} 1$. In addition, $\overline{\Gamma_k}^{\mathrm{zar}} = \mathrm{SL}_2(\mathbb{R})$.*

Proof. For each $k \geq 0$ by the definition of critical exponent, we can find arbitrarily large $R_k > 0$ so that

$$\#B_{\mathrm{SL}_2(\mathbb{R})}(R_k) \cap \Gamma_{g_k} > e^{R_k(\delta(\Gamma_k)-1/k)}.$$

Let $\Gamma_k = \left\langle B_{\mathrm{SL}_2(\mathbb{R})}(R_k) \cap \Gamma_{g_k} \right\rangle$. We now choose R_k sufficiently large to apply Theorem 4.2 and deduce that

$$\delta(\Gamma_k) > 1 - 10(1 - \delta(\Gamma_k)) - 10/k.$$

Thus $\delta(\Gamma_k) \xrightarrow{k\to\infty} 1$. The subgroups $(\Gamma_k)_{k=1}^{\infty}$ are finitely generated by their definition. For every $k > 0$ sufficiently large, $\delta(\Gamma_k) > 0$ and hence $\overline{\Gamma_k}^{\mathrm{zar}} = \mathrm{SL}_2(\mathbb{R})$. $\square$

Let μ'_k be the Bowen-Margulis-Sullivan probability measure on $\mathrm{SL}_2(\mathbb{R})/\Gamma_k$. It is $\mathtt{a}(t)$ invariant. Define $r_k : \mathrm{SL}_2(\mathbb{R})/\Gamma_k \to G/\Gamma$ as the map sending $\pi_{\Gamma_k}(x) \mapsto \pi_\Gamma(xg_k)$. Then define $\mu_k = (r_k)_*(\mu'_k)$. We wish to show that $\mu_k \xrightarrow{k\to\infty} m_{G/\Gamma}$. By Theorem 4.1, $h(\mu'_k) = \delta(\Gamma_k) \xrightarrow{k\to\infty} 1$. The following is a simple fact on the entropy of ergodic systems:

Claim 4.6. *Let $(X_1, \nu_1, T), (X_2, \nu_2, T_2)$ invertible ergodic systems and $f : X_1 \to X_2$ a factor map with countable fibers, such that $f_*\nu_1 = \nu_2$ and $f \circ T_1 = T_2 \circ f$. Then $h_{\nu_1}(T_1) = h_{\nu_2}(T_2)$.*

We will use it as follows: factor $r_k : \mathrm{SL}_2(\mathbb{R})/\Gamma_k \to G/\Gamma$ as

$$r_k : \mathrm{SL}_2(\mathbb{R})/\Gamma_k \xrightarrow{r'_k} \mathrm{SL}_2(\mathbb{R})/\Gamma_{g_k} \xrightarrow{r''_k} G/\Gamma$$

Note that r''_k is one-to-one and r'_k has countable fibers. Hence

$$1 \xleftarrow{k\to\infty} h(\mu'_k) \overset{4.6}{=} h((r'_k)_*\mu'_k) \overset{3.8}{=} \dim^{\mathtt{u}}((r'_k)_*\mu'_k) \overset{3.5}{=} \dim^{\mathtt{u}}((r''_k)_*(r'_k)_*\mu'_k) = \dim^{\mathtt{u}}(\mu_k).$$

We introduce now the geometric nondegeneracy lemma, which excludes degenerate limits to the sequence μ_k. We will prove it in Section 5.

Lemma 4.7 (Geometric nondegeneracy)**.** *Let μ_k be a sequence of $\mathtt{a}(t)$-invariant probability measures on G/Γ such that $\dim^{\mathtt{u}} \mu_k \xrightarrow{k\to\infty} 1$. Then μ_k has no escape of mass, that is,*

$$\sup_{\substack{K\subset G/\Gamma \\ \textit{compact}}} \liminf_{k\to\infty} \mu_k(K) = 1.$$

In addition, there is no nontrivial convergence to a periodic $\mathrm{SL}_2(\mathbb{R})$-orbit of G/Γ. In other words, let $g \in G$ such that Γ_g is a lattice. The $\mathrm{SL}_2(\mathbb{R})$-orbit $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g) \subset G/\Gamma$ is thus a closed set. Suppose that $\mu_k(\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)) = 0$ for every k. Then

$$\sup_{\substack{K\subset (G/\Gamma)\setminus \mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g) \\ \textit{compact}}} \liminf_{k\to\infty} \mu_k(K) = 1.$$

Consequently, restricting to a subsequence of k we may assume that $\mu_k \xrightarrow{k\to\infty} \mu_\infty$, where μ_∞ is a probability measure. By Theorem 3.9, μ_∞ is $\mathtt{u}$-invariant. By Theorem 3.9 applied for the negative time $\mathtt{a}(t)$ action, together with the transpose inverse action of B on G/Γ, we obtain that μ_∞ is $\mathtt{u}^t$-invariant. Hence μ_∞ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Let $\mu_\infty = \int_X \mu_\infty^x \mathrm{d}\mu_\infty(x)$ be the $\mathrm{SL}_2(\mathbb{R})$-ergodic decomposition. Then by Lemma 4.7, we deduce that $\mu_\infty(\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)) = 0$ for every periodic orbit $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)$. Thus $\mu_\infty^x(\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g)) = 0$ for μ_∞-almost every $x \in G/\Gamma$. By Ratner's Measure Classification Theorem [32], μ_∞^x is algebraic, i.e., there exists a connected intermediate group $\mathrm{SL}_2(\mathbb{R}) \leq L^x \leq G$ such that μ_∞^x is the Haar measure on a periodic L^x-orbit. However, we showed that $L^x \neq \mathrm{SL}_2(\mathbb{R})$ almost surely. One can see that there are no nontrivial connected intermediate subgroups between $\mathrm{SL}_2(\mathbb{R})$ and G, and hence μ_∞^x is the constant measure $m_{G/\Gamma}$, which implies that $\mu_\infty = m_{G/\Gamma}$.

4.3. **Completion of the proof using rigidity - lifting the measures to a projective bundle.** The weak-$*$ convergence $\mu_k \xrightarrow{k\to\infty} m_{G/\Gamma}$ is what we need to initiate the reduction to [1], with the measures μ_k supported on the dense orbits $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g_k)$, in place of the infinitely many periodic orbits. The proof goes the same, except for one point, where we need to prove some extra invariance. We will recite some details up to that point from [1, §2].

Definition 4.8 (Witness of the non-arithmeticity)**.** Let ℓ be the field generated by traces of the adjoint representations of Γ-elements. Here G is thought of as a real algebraic group, thus $\ell \subseteq \mathbb{R}$. The field ℓ is a number field ([34], [7], [28], [18]) contained in $\mathbb{R}$. The inclusion map is a real embedding $\sigma : \ell \to \mathbb{R}$. By [41], there is an ℓ-algebraic group $\mathbf{G}$ which is an ℓ-form of the image of G under the adjoint homomorphism such that $\mathrm{Ad}(\Gamma)$ lands in $\mathbf{G}(\ell)$. In other words, there is an isomorphism $\iota' : \mathrm{Ad}\, G \to \mathbf{G}(\mathbb{R})_0$ such that when we consider the composition $\iota = \iota' \circ \mathrm{Ad} : G \to \mathbf{G}(\mathbb{R})$ we have $\iota(\Gamma) \subseteq \mathbf{G}(\ell)$. Since Γ is nonarithmetic, there is a place $\nu \in S(\ell)$ such that

$$\iota_\nu = (\mathbf{G}(\ell) \to \mathbf{G}(\ell_\nu)) \circ \iota|_\Gamma : \Gamma \to \mathbf{G}(\ell_\nu)$$

does not factor continuously through $\Gamma \to G$, and the image $\iota_\nu(\Gamma)$ does not lie in a compact subgroup of $\mathbf{G}(\ell_\nu)$.

Claim 4.9 (Construction of invariant points)**.** *Up to restricting to a subsequence of k-s, there is an irreducible algebraic $\mathbf{G}(\ell_\nu)$-representation V independent of k and a point $P_k \in \mathbb{P}(V)$, for which the following holds.*

1. *Consider the action $\Gamma \curvearrowright \mathbb{P}(V)$ induced by the homomorphism ι_ν composed with the $\mathbf{G}(\ell_\nu)$ action on $\mathbb{P}(V)$. Then P_k is Γ_k^1 invariant, where $\Gamma_k^1 = g_k^{-1}\Gamma_k g_k \subseteq \Gamma$.*
2. *The representation V is in fact a representation of $\mathbf{H}(\ell_\nu)$, where $\mathbf{H}$ is the image of $\mathbf{G}$ under the adjoint representation. As such it is a faithful representation of $\mathbf{H}(\ell_\nu)$.*

The proof mimics one-to-one parts of [1, Prop. 3.4], however, we recite the main details.

Proof. For each $k = 1, 2, \ldots$, consider $\Gamma_{g_k}^1 = g_k^{-1}\Gamma_{g_k} g_k = \Gamma \cap g_k^{-1}\,\mathrm{SL}_2(\mathbb{R}) g_k \subseteq G$. Consider now the ℓ-Zariski closure of $\iota(\Gamma_{g_k}^1)$ in $\mathbf{G}$. This is an ℓ-algebraic subgroup $\mathbf{L}_k \subseteq \mathbf{G}$. The localization at σ is

$$\mathbf{L}_k(\mathbb{R}) = \iota\left(\overline{g_k^{-1}\Gamma_{g_k} g_k}^{\mathrm{zar}}\right) = \iota(g_k^{-1}\,\mathrm{SL}_2(\mathbb{R}) g_k).$$

Thus $\mathbf{L}_k$ is an $\mathbf{SL}_2$-form over ℓ, and in particular, is three dimensional. Since $\mathrm{SL}_2(\mathbb{R})$ is not a normal subgroup of G, we deduce that $\mathbf{L}_k$ is not a normal subgroup of $\mathbf{G}$. Let $\mathfrak{g} = \mathrm{Ad}(\mathbf{G})$ and $\mathfrak{l}_k = \mathrm{Ad}(\mathbf{L}_k)$ be the algebraic adjoint representations. The exterior product $\bigwedge^3 \mathfrak{g}(\ell_\nu)$ is an ℓ_ν-algebraic representations of $\mathbf{G}(\ell_\nu)$. The 3-dimensional subspace $\mathfrak{l}_k(\ell_\nu)$ of $\mathfrak{g}(\ell_\nu)$ induces a vector $p_k \in \bigwedge^3 \mathfrak{g}(\ell_\nu)$, well defined up to multiplication by scalar. Since $\mathbf{L}_k$ is not a normal subgroup of $\mathbf{G}$, it follows that p_k is not $\mathbf{G}(\ell_\nu)$-invariant. Thus, it projects nontrivially to some nontrivial irreducible $\mathbf{G}(\ell_\nu)$ sub-representation V of $\bigwedge^3 \mathfrak{g}(\ell_\nu)$. Denote this projection by π_V. Restricting to a subsequence we may assume that V is constant, that is, independent of k. The point $P_k = [\pi_V(p_k)] \in \mathbb{P}(V)$ is $\mathbf{L}_k(\ell_\nu)$ invariant. It is not $\mathbf{G}(\ell_\nu)$-invariant, as this would imply that V is one-dimensional, but $\mathbf{G}(\ell_\nu)$ is a semisimple group and has no nontrivial one-dimensional representations. Thus one observes the first point of the claim. The second follows from the construction, except for the faithfulness part. It follows in the same way as in [1, Prop. 3.4]. □

Consider the right action $G \times \mathbb{P}(V) \curvearrowleft \Gamma$ by $(g, P)\gamma = (g\gamma, \iota_\nu(\gamma^{-1})P)$. Consider the $\mathbb{P}(V)$-bundle $G \times \mathbb{P}(V)/\Gamma$, and the projection $\tilde{\pi} : G \times \mathbb{P}(V) \to G \times \mathbb{P}(V)/\Gamma$. Forgetting the $\mathbb{P}(V)$ coordinate yields a projection $\rho : G \times \mathbb{P}(V)/\Gamma \to G/\Gamma$. It has a left action by $\mathrm{SL}_2(\mathbb{R})$, acting only on the G coordinate. The following claim is analogous to the result of [1, Prop. 3.4] in our setting.

Claim 4.10. *There is an* $\mathrm{SL}_2(\mathbb{R})$*-invariant measure* $\tilde{\mu}$ *on* $\mathbb{P}(V) \times G/\Gamma$ *such that* $\rho_*\tilde{\mu} = m_{G/\Gamma}$.

Proof. Consider the point $\tilde{Q}_k = (g_k, P_k) \in G \times \mathbb{P}(V)$, $Q_k = \tilde{\pi}(\tilde{Q}_k) \in G \times \mathbb{P}(V)/\Gamma$, and the map $\tilde{r}'_k : \mathrm{SL}_2(\mathbb{R}) \to G \times \mathbb{P}(V)/\Gamma$ defined by $h \mapsto hQ_k$. This map is invariant to Γ_k from the right, indeed,

$$\begin{aligned} \tilde{r}'_k(h\gamma) &= h\gamma Q_k = \tilde{\pi}(h\gamma g_k, P_k) = \tilde{\pi}(hg_k g_k^{-1}\gamma g_k, P_k) \\ &= \tilde{\pi}(hg_k g_k^{-1}\gamma g_k, \iota_\nu(g_k^{-1}\gamma^{-1}g_k)P_k) = \tilde{\pi}(hg_k, P_k). \end{aligned}$$

The fourth equality follows from the fact that P_k is $g_k^{-1}\Gamma_k g_k$ invariant, and the last follows from the definition of the quotient map $\tilde{\pi}$, which is the quotient by the right Γ-action. Hence we may define $\tilde{r}_k : \mathrm{SL}_2(\mathbb{R})/\Gamma_k \to G \times \mathbb{P}(V)/\Gamma$ as the descent of $\tilde{r}'_k$. Thus in fact we factored $r_k : \mathrm{SL}_2(\mathbb{R})/\Gamma_k \to G/\Gamma$ as a composition of $\mathrm{SL}_2(\mathbb{R})$-maps, $r_k = \rho \circ \tilde{r}_k$. In a diagram,

$$\begin{tikzcd} \mathrm{SL}_2(\mathbb{R})/\Gamma_k \arrow[r, "\tilde{r}_k"] \arrow[dr, "r_k"] & G \times \mathbb{P}(V)/\Gamma \arrow[d, "\rho"] \\ & G/\Gamma \end{tikzcd}$$

Therefore, the measure $\mu_k = (r_k)_*\mu'_k$ on G/Γ lifts to probability measures $\tilde{\mu}_k = (\tilde{r}_k)_*\mu'_k$ satisfyies that $\rho_*\tilde{\mu}_k = \mu_k$. Similarly to μ_k, we obtain $\dim^{\mathtt{u}}(\tilde{\mu}_k) \xrightarrow{k\to\infty} 1$.

Since $(\mu_k)_{k=1}^\infty$ weak-$*$ converges to the probability measure $m_{G/\Gamma}$ and the fibers of ρ are compact, we deduce that there is no escape of mass in $(\tilde{\mu}_k)_k$, and hence we may restrict to a subsequence of k-s and assume that $\tilde{\mu}_k$ weak-$*$ converges to a measure $\tilde{\mu}$, satisfying that $\rho_*\tilde{\mu} = m_{G/\Gamma}$. By Theorem 3.9, the measure $\tilde{\mu}$ is $\mathtt{u}$-invariant. By Theorem 3.9 applied for the negative time $\mathtt{a}(t)$ action, together with the transpose inverse action of B on G/Γ, we obtain that $\tilde{\mu}$ is $\mathtt{u}^t$-invariant, and hence $\mathrm{SL}_2(\mathbb{R})$-invariant. □

The existence of such a measure $\tilde{\mu}$ is required in [1, Thm. 1.6]. The rest of the proof is now identical to the reduction of [1, Thms. 1.1 and 1.5.] to [1, Thm. 1.6]. [1, §3.4] implies the compatibility assumption of [1, Thm. 1.6], and hence its result holds, and shows that in fact $\iota_\nu : \Gamma \to \mathbf{G}(\ell_\nu)$ extends to a continuous homomorphism $G \to \mathbf{G}(\ell_\nu)$, which contradicts the assertion made in Definition 4.8. □

5. Proof of Lemma 4.7

The proof of Lemma 4.7 employs the linearization method. Linearization is a general technique, introduced by Dani and Margulis [10], and it uses representations to control the distance to homogeneous subvarieties.

In Subsection 5.1 we introduce the notion of $(\varepsilon; T_0, T_1)$-additive Margulis function. In Subsection 5.2 we prove Lemma 4.7, assuming a representational description of certain geometric notions, and prove them in Subsections 5.3, 5.4.

Remark 5.1. The results in this section are related to Mohammadi and Oh [26, Thm. 1.5]. In [26] the authors prove a separation result for closed $\mathrm{SL}_2(\mathbb{R})$-orbits in geometrically finite quotients of G. To do so, Mohammadi and Oh show that the Bowen-Margulis-Sullivan measure on one $\mathrm{SL}_2(\mathbb{R})$-orbit must be separated from the other $\mathrm{SL}_2(\mathbb{R})$-orbit. In this section, we also prove a separation result of measures and closed orbits. Although we allow our measures to be more general than Bowen-Margulis-Sullivan's measures, they are the ones in our application.

The proofs in this section and [26] share several similarities. First, the representation framework is similar. Second, both approaches use a Markov operator, though a different one. Third, we use a Margulis function similar to [26]; however, their Margulis function satisfies the Margulis inequality everywhere, while ours satisfies it only with high probability. The reason for this difference is how each paper effectivizes the high dimension of the leafwise measures $\mu_x^{\mathtt{u}}$ for the Bowen-Margulis-Sullivan measures μ.

Here we use Lemma 3.10. Mohammadi and Oh [26] use a different way to effectivize the dimension, by using a uniform bound $\frac{\mu_x^{\mathtt{u}}([-r,r])^{1/\delta'}}{r\mu_x^{\mathtt{u}}([-1,1])^{1/\delta'}} \le \mathtt{p}$ for μ-almost-all x and for all $r \in (0,2]$. Here δ' is either δ or $1-2(1-\delta)$. For our purposes this approach cannot be applied, since it requires a uniform bound for the $\mathtt{p}$ of all our μ_k, which we were not able to obtain.

5.1. $(\varepsilon; T_0, T_1)$-additive Margulis function.

In this section, we introduce the notion on $(\varepsilon; T_0, T_1)$-additive Margulis function and prove some results on the notion.

Definition 5.2 ($(\varepsilon; T_0, T_1)$-additive Margulis function)**.** Let (X, μ) be a measure space, $x \mapsto \nu_x$ a measurable map from X to measures on X such that $\mu = \int_X \nu_x \mathrm{d}\mu(x)$. In other words, $(X, x \mapsto \nu_x)$ is a Markov chain and μ is a stationary measure. A measurable function $\alpha : X \to [0, \infty)$ is called $(\varepsilon; T_0, T_1)$-*additive Margulis function* for some $T_1 > T_0 > 0$ large and $\varepsilon > 0$ if the following conditions hold:

M-a) For μ-almost all $x \in X$, and for ν_x-almost all $y \in X$, we have $\alpha(y) \in \alpha(x) + [-T_1, T_1]$.

M-b)

$$\mu\left(\left\{x \in X : T_1 \le \alpha(x) < T_0 + \int_X \alpha(y)\mathrm{d}\nu_x(y)\right\}\right) < \varepsilon. \tag{5.1}$$

Having equality to 0 in Eq. (5.1) is an additive version of the standard definition of a Margulis function.

Lemma 5.3. *In the setting of Definition 5.2,*

$$\mu(\{x \in X : \alpha(x) \geq t\}) \leq \frac{1}{\log\lfloor t/T_1 \rfloor - 1} + \frac{T_0 + T_1}{T_0}\varepsilon, \tag{5.2}$$

for all $t \geq 3T_1$.

Proof. For every interval $I \subseteq \mathbb{R}$ denote

$$A_I = \{x \in X : \alpha(x) \in I\},$$
$$B_I = \left\{ x \in A_I : \alpha(x) < T_0 + \int_X \alpha(y) \mathrm{d}\nu_x(y) \right\}.$$

For every $t_1 \geq T_1$, $t_2 > t_1 + 2T_1$, we use the stationarity of μ and Condition (M-a) to obtain

$$\begin{aligned}\int_{A_{[t_1+T_1, t_2-T_1]}} \alpha(y)\mathrm{d}\mu(y) &\leq \int_{A_{[t_1,t_2]}} \int_X \alpha(y)\mathrm{d}\nu_x(y)\mathrm{d}\mu(x) \\ &\leq \int_{A_{[t_1,t_2]}} (\alpha(x) - T_0)\mathrm{d}\mu(x) + \mu(B_{[t_1,t_2]}) \cdot (T_0 + T_1).\end{aligned}$$

Canceling common terms yields

$$T_0\mu(A_{[t_1,t_2]}) \leq (t_1 + T_1)\mu(A_{[t_1,t_1+T_1)}) + t_2\mu(A_{[t_2-T_1,t_2]}) + \varepsilon(T_0 + T_1). \tag{5.3}$$

Replacing t_2 by $t_2 + nT_1$ for some n yields different bounds

$$\begin{aligned}&T_0\mu(A_{[t_1,t_2+nT_1]}) \\ &\quad\leq (t_1 + T_1)\mu(A_{[t_1,t_1+T_1)}) + (t_2 + nT_1)\mu(A_{[t_2+(n-1)T_1,t_2+nT_1]}) + \varepsilon(T_0 + T_1).\end{aligned} \tag{5.4}$$

Now, note that

$$\liminf_{n\geq 0} (t_2 + nT_1)\mu(A_{[t_2+(n-1)T_1,t_2+nT_1]}) = 0.$$

Otherwise we have $\mu(A_{[t_2+nT_1,t_2+(n+1)T_1]}) > \frac{\delta}{t_2+(n+1)T_1}$ for some $\delta > 0$ and all n sufficiently large, but this is a divergent series. Taking liminf over n in Eq. (5.4) gives us

$$T_0\mu(A_{[t_1,\infty)}) \leq (t_1 + T_1)\mu(A_{[t_1,t_1+T_1)}) + \varepsilon(T_0 + T_1). \tag{5.5}$$

Let $n \geq 3$ and $n \geq m \geq 1$. Substitute $t_1 = mT_1$ to Eq. (5.5) and get

$$T_0\mu(A_{[nT_1,\infty)}) \leq T_0\mu(A_{[mT_1,\infty)}) \leq (m + 1)T_1\mu(A_{[mT_1,(m+1)T_1)}) + \varepsilon(T_0 + T_1). \tag{5.6}$$

Now, for every n consider

$$\delta_n = \min_{m=1}^{n} (m + 1)\mu(A_{[mT_1,(m+1)T_1)}).$$

We deduce that $\mu(A_{[mT_1,(m+1)T_1)}) \geq \frac{T_1\delta_n}{(m+1)T_0}$, and by additivity,

$$1 \geq \mu(A_{[T_1,(n+1)T_1)}) \geq \frac{T_1\delta_n}{T_0} \sum_{m=1}^{n} \frac{1}{m+1} \geq \frac{T_1\delta_n}{T_0}(\log n - 1).$$

Altogether, $\delta_n \leq \frac{T_0}{T_1(\log n - 1)}$. Plugging this to Eq. (5.6) yields

$$T_0\mu(A_{[nT_1,\infty)}) \leq \frac{T_0}{\log n - 1} + \varepsilon(T_0 + T_1).$$

Hence, for all $t \geq 3T_1$,

$$\mu(A_{[t,\infty)}) \leq \frac{1}{\log\lfloor t/T_1\rfloor - 1} + \varepsilon\frac{T_0+T_1}{T_0},$$

as desired. □

Remark 5.4. The summand $\frac{T_0+T_1}{T_0}\varepsilon$ in Eq. (5.2) is tight, and cannot be improved.

Claim 5.5. *In the setting of Definition 5.2, suppose that* $\alpha : X \to [0,\infty)$ *is an* $(\varepsilon; T_0, T_1)$*-additive Margulis function with* $T_0 > T_1/2$*, and* $\beta : X \to [0,\infty)$ *is a function satisfying Condition (M-a) for* T_1 *and*

$$\mu\left(\left\{x \in X : \alpha(x) + T_1 \leq \beta(x) < T_0 + \int_X \beta(y)\mathrm{d}\nu_x(y)\right\}\right) < \varepsilon. \tag{5.7}$$

Then $\gamma = \max(0, \alpha - 2T_1, \beta - 5T_1)$ *is a* $(2\varepsilon; 2T_0 - T_1, T_1)$*-additive Margulis function.*

Proof. One can easily see that γ satisfies Condition (M-a) for T_1. Let $T_0' = 2T_0 - T_1$. We claim that

$$X_{\gamma-\mathrm{bad}} = \left\{x \in X : T_1 \leq \gamma(x) < T_0' + \int_X \gamma(y)\mathrm{d}\nu_x(y)\right\},$$

is contained in the union of the sets $X_{\alpha-\mathrm{bad}}$ and $X_{\beta-\mathrm{bad}}$ estimated in Eqs. (5.1) and (5.7) respectively. Indeed, let $x \in X_{\gamma-\mathrm{bad}}$ be generic in the sense that it satisfies Condition (M-a) for α and β. We have

$$-T_0' < \int_X \gamma(y)\mathrm{d}\nu_x(y) - \gamma(x) = \int_X (\gamma(y) - (\gamma(x) - T_1))\mathrm{d}\nu_x(y) - T_1$$

Note that the integrand of the right-hand side is almost surely positive. Now denote $\alpha' = \alpha - 2T_1, \beta' = \beta - 5T_1$ and distinguish between the following three cases:

Case-a) $\gamma(x) = \alpha'(x)$ and $\beta'(x) \leq \alpha'(x) - 2T_1$.
Case-b) $\gamma(x) = \beta'(x)$ and $\alpha'(x) \leq \beta'(x) - 2T_1$.
Case-c) $\alpha'(x), \beta'(x) > \gamma(x) - 2T_1$.

In (Case-a), for $\nu_x(y)$-almost all y,

$$\alpha'(y) \geq \alpha'(x) - T_1 \geq \beta'(x) + T_1 \geq \beta'(y).$$

In addition, since $\gamma(x) = \alpha'(x) \geq T_1$, Condition (M-a) for α show that $\alpha'(y) \geq 0$, which implies that $\gamma(y) = \alpha'(y)$. Consequently,

$$\begin{aligned}
-T_0' &< \int_X (\gamma(y) - (\gamma(x) - T_1))\mathrm{d}\nu_x(y) - T_1 \\
&= \int_X (\alpha'(y) - (\alpha'(x) - T_1))\mathrm{d}\nu_x(y) - T_1 \\
&= \int_X (\alpha(y) - (\alpha(x) - T_1))\mathrm{d}\nu_x(y) - T_1
\end{aligned}$$

Since $\alpha(x) = \alpha'(x) + 2T_1 = \gamma(x) + 2T_1 \geq T_1$ we have $x \in X_{\alpha-\mathrm{bad}}$.

In (Case-b), for almost all y we have $\gamma(y) = \beta'(y)$ as in the previous case. The inequality $-T_0' < \int_X (\beta(y) - (\beta(x) - T_1))\mathrm{d}\nu_x(y) - T_1$ follows as well. Since

$$\alpha(x) = \alpha'(x) + 2T_1 \leq \gamma(x) = \beta'(x) = \beta(x) - 5T_1,$$

we deduce that $x \in X_{\beta-\mathrm{bad}}$.

In (Case-c),

$$\begin{aligned}-T_0' &< \int_X (\gamma(y) - (\gamma(x) - T_1))\mathrm{d}\nu_x(y) - T_1 \\ &\le \int_X \left((\alpha(y) - (\gamma(x) - T_1))^+ + (\beta(y) - (\gamma(x) - T_1))^+\right)\mathrm{d}\nu_x(y) - T_1 \\ &\le \int_X \left((\alpha(y) - (\alpha(x) - T_1)) + (\beta(y) - (\beta(x) - T_1))\right)\mathrm{d}\nu_x(y) - T_1,\end{aligned}$$

where $x^+ = \max(x, 0)$ for all $x \in \mathbb{R}$. Hence either

$$\int_X (\alpha(y) - (\alpha(x) - T_1))\nu_x(y) > \frac{T_1 - T_0'}{2} = T_1 - T_0 \tag{5.8}$$

or

$$\int_X (\beta(y) - (\beta(x) - T_1))\nu_x(y) > \frac{T_1 - T_0'}{2} = T_1 - T_0 \tag{5.9}$$

If Eq. (5.8) holds, $x \in X_{\alpha-\mathrm{bad}}$. Indeed, to show that $\alpha(x) \ge T_1$, we use

$$\alpha(x) = \alpha'(x) + 2T_1 \ge \gamma(x) \ge T_1.$$

If Eq. (5.9) holds, $x \in X_{\beta-\mathrm{bad}}$. Indeed, to show that $\beta(x) \ge T_1 + \alpha(x)$, we use

$$\beta(x) = \beta'(x) + 5T_1 \ge \gamma(x) + 3T_1 \ge \alpha'(x) + 3T_1 = \alpha(x) + T_1.$$

Consequently, $X_{\gamma-\mathrm{bad}} \subseteq X_{\alpha-\mathrm{bad}} \cup X_{\beta-\mathrm{bad}}$, and hence $\mu(X_{\gamma-\mathrm{bad}}) \le 2\varepsilon$. □

5.2. **Proof assuming representational description.** The proof of Lemma 4.7 will go as follows: given a closed $\mathrm{SL}_2(\mathbb{R})$-orbit S we will find a height function on its complement $(G/\Gamma) \setminus S$ which measures both how close a point is to S and how deep it is in the cusp. Then we show that it is an $(\varepsilon; T_0, T_1)$-additive Margulis function with respect to the leafwise Markov chain.

We will first describe representation-theoretic ways to view the cusps of G/Γ and with the periodic $\mathrm{SL}_2(\mathbb{R})$-periodic orbits in it. Note that the normalizer of $\mathrm{SL}_2(\mathbb{R})$ in G is an index 2 extension $\mathrm{SL}_2(\mathbb{R}) \lhd N(\mathrm{SL}_2(\mathbb{R})) < G$, and more explicitly,

$$N(\mathrm{SL}_2(\mathbb{R})) = \mathrm{SL}_2(\mathbb{R}) \sqcup \{ih : h \in M_{2\times 2}(\mathbb{R}) \text{ with } \det h = -1\}.$$

Any periodic orbit $S = \mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g_S)$ is contained in the periodic $N(\mathrm{SL}_2(\mathbb{R}))$-orbit $\bar{S} = N(\mathrm{SL}_2(\mathbb{R}))\pi_\Gamma(g_S)$. Let Cusps_Γ denote the set of cusps of G/Γ.

Lemma 5.6 (Description of the cusps of G/Γ using representations)**.** *There exists a 2 dimensional complex representation V of G equipped with a norm $\|-\|$, and a Γ-invariant subset $V_{\mathrm{Cusps}_\Gamma} \subset V$ such that for every $g \in G$ all the vectors in $g.V_{\mathrm{Cusps}_\Gamma} \cap \{v \in V : \|v\| < 1\}$ has the same length. This implies that the function $\alpha_{\mathrm{Cusps}_\Gamma} : G \to [0, \infty)$ defined by*

$$\alpha_{\mathrm{Cusps}_\Gamma} = \begin{cases} -\log \|g.v\|, & \text{if } \|g.v\| < 1 \text{ for some } v \in V_{\mathrm{Cusps}_\Gamma}, \\ 0, & \text{otherwise,} \end{cases}$$

is well-defined. Moreover, we claim that $\alpha_{\mathrm{Cusps}_\Gamma}$ is continuous, right Γ-invariant, and descends to a proper map $\alpha_{\mathrm{Cusps}_\Gamma}|_{G/\Gamma} : G/\Gamma \to [0, \infty)$, that is, $\alpha_{\mathrm{Cusps}_\Gamma}|_{G/\Gamma}^{-1}([0, T]) = \alpha_{\mathrm{Cusps}_\Gamma}^{-1}([0, T])/\Gamma$ is compact for every $T > 0$.

For the rest of the paper, we will use $\alpha_{\mathrm{Cusps}_\Gamma}$ both as a function on G and G/Γ.

Remark 5.7. For every $v \in V_{\mathrm{Cusps}_\Gamma}$ the set $\{\pi_\Gamma(g) : \|g.v\| < 1\}$ is a cusp neighborhood, and $\pi_\Gamma(g)$ gets deeper in the cups the smaller $\|g.v\|$ is.

Lemma 5.8 (Description of $N(\mathrm{SL}_2(\mathbb{R}))$-periodic orbits using representations)**.** *There exists a 4-dimensional real representation W equipped with norms $\|-\|$, such that the following happen: There is a vector $w_0 \in W$ such that such that $\mathrm{SL}_2(\mathbb{R}) = \mathrm{stab}_G(w_0)$, and $N(\mathrm{SL}_2(\mathbb{R})) = \mathrm{stab}_G(\{\pm w_0\})$. Let $W/w_0 = W/\mathbb{R}w_0 \cong w_0^\perp$ be the quotient space. It is an irreducible $\mathrm{SL}_2(\mathbb{R})$-representation (equivalent to Sym^2 of the standard representation). Let $\pi_{w_0} : W \to W/w_0$, be the standard projection. Let $\bar{S} = N(\mathrm{SL}_2(\mathbb{R}))\pi_\Gamma(g_{\bar{S}})$ be a periodic $N(\mathrm{SL}_2(\mathbb{R}))$-orbit, $w_{\bar{S}} = g_{\bar{S}}^{-1}.w_0$ be a vector which is stabilized by $g_{\bar{S}}^{-1}N(\mathrm{SL}_2(\mathbb{R}))g_{\bar{S}}$ up to sign and $W_{\bar{S}} = \Gamma.w_{\bar{S}}$. Define $\alpha_{\bar{S}} : G \to \mathbb{R} \sqcup \{\infty\}$ by*

$$\alpha_{\bar{S}}(g) = \max_{w \in W_{\bar{S}}} -\log \|\pi_{w_0}(g.w)\|. \tag{5.10}$$

Then

1. *$\alpha_{\bar{S}}$ is continuous and attains ∞ only on $\pi_\Gamma^{-1}(\bar{S})$.*
2. *There is $C_{\bar{S}} > 0$ such if $\alpha_{\bar{S}}(g) > 2\alpha_{\mathrm{Cusps}_\Gamma}(g) + C_{\bar{S}}$ for some $g \in G$ then for every $w \in W_{\bar{S}}$ exactly one of the following holds*
   - $-\log \|\pi_{w_0}(g.w)\| = \alpha_{\bar{S}}(g)$,
   - $-\log \|\pi_{w_0}(g.w)\| < 2\alpha_{\mathrm{Cusps}_\Gamma}(g) + C_{\bar{S}}$.

We postpone these lemmas' proofs to Subsections 5.3 and 5.4. Now that we have the height function, we recall that Lemma 3.10 gives us a the leafwise Markov chain on the space X with stationary measure $S_{\mathbb{1}_{[0,1)}}\mu_k$ and a transformation law $x \mapsto \nu_x^{(k)}$ given by

$$\nu_x^{(k)} = \begin{cases} \mathtt{a}(\log 2)x & \text{with probability } p^{(k)}(x), \\ \mathtt{u}(1)\mathtt{a}(\log 2)y & \text{with probability } 1 - p^{(k)}(x). \end{cases} \tag{5.11}$$

where $p^{(k)} : X \to [0,1]$ satisfies

$$\int_X H(p^{(k)}(x), 1 - p^{(k)}(x))\mathrm{d}S_{\mathbb{1}_{[0,1)}}\mu_k(x) = \dim^u(\mu_k)\log 2.$$

Fix a positive integer ℓ to be specified later.

Definition 5.9 (Iteration of the leafwise Markov chain $x \mapsto \nu_x^{(k)}$)**.** The ℓ iteration of the Markov chain $x \mapsto \nu_x^{(k)}$ is defined by

$$x_0 \mapsto \nu_{x_0}^{(k,\ell)} = \int_X \int_X \cdots \int_X \delta_{x_\ell} \mathrm{d}\nu_{x_{\ell-1}}^{(k)}(x_\ell)\mathrm{d}\nu_{x_{\ell-2}}^{(k)}(x_{\ell-1}) \ldots \mathrm{d}\nu_{x_2}^{(k)}(x_1)\mathrm{d}\nu_{x_1}^{(k)}(x_0).$$

In other words, given x_0 we sample x_1 via $\nu_{x_0}^{(k)}$, then sample x_2 via $\nu_{x_1}^{(k)}$, and so on, until we sample x_ℓ via $\nu_{x_{\ell-1}}^{(k)}$, and $\nu_{x_0}^{(k,\ell)} = \mathrm{Law}(x_\ell | x_0)$. Explicitly, $x_i = u(b_i)\mathtt{a}(\log 2)x_{i-1}$ for all $i = 1, \ldots, \ell$ where

$$b_i = \begin{cases} 0 & \text{with probability } p^{(k)}(x_{i-1}), \\ 1 & \text{with probability } 1 - p^{(k)}(x_{i-1}), \end{cases}$$

chosen independently of $b_1, \ldots, b_{i-1}$. Altogether,

$$x_\ell = u(b_\ell)\mathtt{a}(\log 2)u(b_{\ell-1})\mathtt{a}(\log 2) \cdots u(b_1)\mathtt{a}(\log 2)x_0 = u\left(\sum_{i=1}^{\ell} 2^{\ell-i}b_i\right)\mathtt{a}(n\log 2)x_0.$$

Let $b = \sum_{i=1}^{\ell} 2^{\ell-i} b_i$ and denote $p_j^{(k,\ell)}(x_0) := \mathbb{P}(b = j|x)$ for every $j = 0, 1, \dots, 2^\ell - 1$ so that

$$x_\ell = \Big\{ \mathtt{u}(j)\mathtt{a}(\ell \log 2)x_0 \text{ with probability } p_i^{(k,\ell)}(x_0) \text{ for each } i = 0, \dots, 2^{\ell-1}.$$

We have seen in Eq. (3.25) that

$$\begin{aligned}(5.12)\qquad &\int_X H\left(p_0^{(k,\ell)}(x_0), p_1^{(k,\ell)}(x_0), \dots, p_{2^\ell-1}^{(k,\ell)}(x_0)\right) \mathrm{d}(S_{\mathbb{1}_{[0,1)}}\mu_k)(x_0) \\ &\qquad = H(x_\ell|x_0) = \ell \dim^{\mathtt{u}} \mu_k \log 2.\end{aligned}$$

Since the map $q_1, \dots, q_n \mapsto H(q_1, \dots, q_n)$ obtain its maximal value only at $H(1/n, \dots, 1/n) = \log n$, we obtain the following observation.

Observation 5.10. For every $\delta > 0$ and $n > 0$ there is $\varepsilon > 0$ so that the following holds. Suppose that $\int_Z H(p_1(z), p_2(z), \dots, p_n(z))\mathrm{d}\nu \geq (1-\varepsilon)\log n$, where (Z, ν) is a probability space and $p_1, \dots, p_n : Z \to [0,1]$ has $p_1 + \cdots + p_n \equiv 1$. Then

$$(5.13)\qquad \nu(\{z \in Z : |p_i(z) - \frac{1}{n}| < \delta,\ \forall i = 1, \dots, n\}) > 1 - \delta.$$

Let $\delta > 0$ to be determined later. By Observation 5.10 and Eq. (5.12), for all k large enough as a function of ℓ and δ we have $S_{\mathbb{1}_{[0,1)}}\mu_k(X_{\text{good}}^{(k,\ell,\delta)}) > 1 - \delta$, where

$$X_{\text{good}}^{(k,\ell,\delta)} = \{y \in X : |p_i^{(k,\ell)}(y) - 2^{-\ell}| < \delta \text{ for all } i = 0, \dots, 2^\ell - 1\}.$$

We will now recall the following property of $\mathrm{SL}_2(\mathbb{R})$-representations.

Claim 5.11. *For every nontrivial irreducible real or complex representation W of $\mathrm{SL}_2(\mathbb{R})$ with highest weight n equipped with a norm $\|-\|$, there is $C_W > 0$ such that for every $m \geq 0$ and for every $w \in W \setminus \{0\}$,*

$$\frac{1}{2^m} \sum_{i=0}^{2^m-1} \log \|\mathtt{u}(i)\mathtt{a}(m \log 2).w\| - \log \|w\| \geq \frac{nm \log 2}{2} - C_W.$$

Proof. Note that there is $C_0 > 0$ such that for all $s \in [-1,1], w \in W$ we have $|\log \|\mathtt{u}(s).w\| - \log \|w\|| \leq C_0$. Let χ_W denote the maximal weight character on W. This is a character satisfying $\chi_W(\mathtt{a}(t).w) = e^{nt/2}\chi_W(w)$. Then

$$\begin{aligned}(5.14)\quad &\frac{1}{2^m} \sum_{i=0}^{2^m-1} \log \|\mathtt{u}(i)\mathtt{a}(m \log 2).w\| \geq \frac{1}{2^m} \int_0^{2^m} \log \|\mathtt{u}(s)\mathtt{a}(m \log 2).w\|\mathrm{d}s - C_0 \\ &\quad = \int_0^1 \log \|\mathtt{a}(m \log 2)\mathtt{u}(s).w\|\mathrm{d}s - C_0 \geq \int_0^1 \log |\chi_W(\mathtt{a}(m \log 2)\mathtt{u}(s).w)|\mathrm{d}s - C_0 \\ &\quad = \frac{mn}{2} \log 2 + \int_0^1 \log |\chi_W(\mathtt{u}(s).w)|\mathrm{d}s - C_0\end{aligned}$$

Now consider the function

$$f : W \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}, \qquad f(w) = \int_0^1 \log |\chi_W(\mathtt{u}(s).w)|\mathrm{d}s.$$

It satisfies $\forall \alpha \in \mathbb{R}^\times, f(\alpha w) = \log |\alpha| + f(w)$, hence is determined by its values on the unit sphere. One can see that it is continuous. We wish to show that it attains real values, (in contrast to $-\infty$). For that, we need to have that for every $w \neq 0$, the polynomial $s \mapsto \chi_W(\mathtt{u}(s).w)$ does not vanish. This is a standard result on SL_2-representations, which follows from their classification as homogeneous polynomials of degree n. Hence, f has a lower bound on the unit sphere, that is, for some $C_1 \in \mathbb{R}$, for all $w \in W$ with $\|w\| = 1$ we have $f(w) \geq -C_1$. Hence

$$\forall w \in W \setminus 0, \qquad f(w) \geq -C_1 + \log \|w\|.$$

Thus we can bound the right-hand side of (5.14) by

$$\frac{mn}{2}\log 2 + \int_0^1 \log |\chi_W(\mathtt{u}(s).w)| \mathrm{d}s - C_0 \geq \frac{mn}{2}\log 2 - C_1 + \log \|w\| - C_0.$$

The desired inequality follows for $C_W = C_0 + C_1$.

□

We now have a representation-theoretic tool to construct our $(\varepsilon; T_0, T_1)$-additive Margulis functions.

Claim 5.12. *Let $k, \ell \geq 1$ and consider the Markov chain $(G/\Gamma, S_{\mathbb{1}_{[0,1)}}\mu_k, \nu_y^{(k,\ell)})$. The functions $2\alpha_{\mathrm{Cusps}_\Gamma}$ and $\alpha_{\bar{S}}$ satisfy Condition (M-a) with $T_1 = \ell \log 2 + C_0$ respectively for some $C_0 > 0$.*

Proof. There is $C_0 > 0$ such that

- for all $v \in V \setminus \{0\}$ and $s \in [-1, 1]$ we have $\left|\log \frac{\|\mathtt{u}(s).v\|}{\|v\|}\right| < C_0/2$,
- for all $w \in W \setminus \{0\}$ and $s \in [-1, 1]$ we have $\left|\log \frac{\|\mathtt{u}(s).w\|}{\|w\|}\right| < C_0$.

Note that for all $t \in \mathbb{R}$,

$$\left|\log \frac{\|\mathtt{a}(t).v\|}{\|v\|}\right| \leq t/2 \qquad \text{for all} \qquad v \in V \setminus \{0\},$$
$$\left|\log \frac{\|\mathtt{a}(t).w\|}{\|w\|}\right| \leq t \qquad \text{for all} \qquad w \in W \setminus \{0\}.$$

The desired follows from the definition of the functions and the Markov chain. □

Claim 5.13. *In the setting of Claim 5.12, there exists δ_0 such that for all $\delta \in (0, \delta_0)$ the function $2\alpha_{\mathrm{Cusps}_\Gamma}$ is $(\delta; T_0, T_1)$-additive Margulis function, for*

$$T_1 = \ell \log 2 + C_0 + 1, T_0 = \ell \log 2 - 2C_V - 2^\ell \delta T_1,$$

provided that $T_0 > 0$ and $S_{\mathbb{1}_{[0,1)}}\mu_k(X_{\mathrm{good}}^{(k,\ell,\delta)}) > 1 - \delta$. Here C_0 is as in Claim 5.12 and C_V as in Claim 5.11.

Proof. Let $T_1 = \ell \log 2 + C_0 + 1$. Let $x = \pi_\Gamma(g) \in X_{\mathrm{good}}^{(k,\ell,\delta)}$ with $2\alpha_{\mathrm{Cusps}_\Gamma}(x) \geq T_1$. In particular, $\alpha_{\mathrm{Cusps}_\Gamma}(x) = -\log \|g.v\|$ for some $v \in V_{\mathrm{Cusps}_\Gamma}$. As in the proof of Claim 5.12, we deduce that for every $y = \mathtt{u}(i)\mathtt{a}(\ell \log 2)x \in \mathrm{supp}\,(\nu_x^{(k,\ell)})$ for $i = 0, 1, \ldots, 2^{\ell-1}$,

$$-2\log \|\mathtt{u}(i)\mathtt{a}(\ell \log 2)g.v\| > -2\log \|g.v\| - T_1 = 2\alpha_{\mathrm{Cusps}_\Gamma}(x) - T_1 \geq 0.$$

Thus, by its definition, $\alpha_{\mathrm{Cusps}_\Gamma}(y) = -\log\|\mathtt{u}(i)\mathtt{a}(\ell\log 2)g.v\|$. We estimate

$$\begin{aligned}
&\int_X 2\alpha_{\mathrm{Cusps}_\Gamma}(y)\mathrm{d}\nu_x^{(k,\ell)}(y)\\
&\quad= 2\alpha_{\mathrm{Cusps}_\Gamma}(x) + 2\sum_{i=0}^{2^\ell-1} p_i^{(k,\ell)}(x)(\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{u}(i)\mathtt{a}(\ell\log 2)y) - \alpha_{\mathrm{Cusps}_\Gamma}(x))\\
&\quad\overset{x\in X_{\mathrm{good}}^{(k,\ell,\delta)}}{\leq} 2\alpha_{\mathrm{Cusps}_\Gamma}(x) + 2\sum_{i=0}^{2^\ell-1} 2^{-\ell}(\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{u}(i)\mathtt{a}(\ell\log 2)y) - \alpha_{\mathrm{Cusps}_\Gamma}(x)) + 2^\ell\delta T_1\\
&\quad= 2\alpha_{\mathrm{Cusps}_\Gamma}(x) - 2\sum_{i=0}^{2^\ell-1} 2^{-\ell}\log\frac{\|\mathtt{u}(i)\mathtt{a}(\ell\log 2)g.v\|}{\|g.v\|} + 2^\ell\delta T_1\\
&\quad\overset{5.11}{\leq} 2\alpha_{\mathrm{Cusps}_\Gamma}(x) - \ell\log 2 + 2C_V + 2^\ell\delta T_1.
\end{aligned}$$

Consequently, $2\alpha_{\mathrm{Cusps}_\Gamma}$ is a $(\delta; T_0, T_1)$-additive Margulis function, with $T_0 = \ell\log 2 - 2C_V - 2^\ell\delta T_1$. □

Claim 5.14. *In the setting of Claim 5.12, there exists $\ell \geq 1$ large and $\delta_0 > 0$ small such that for all $\delta \in (0, \delta_0)$ the function* $\max(0, \alpha_{\mathrm{Cusps}_\Gamma} - 2T_1 - C_{\bar{S}}, \alpha_{\bar{S}} - 6T_1)$ *is a $(2\delta; T_0', T_1)$-additive Margulis function, for some $T_0' < T_1$, provided that $k \geq k_\delta$ for some k_δ depending on δ.*

Proof. Let $x = \pi_\Gamma(g) \in X_{\mathrm{good}}^{(k,\ell,\delta)}$ with $\alpha_{\bar{S}}(x) \geq 2T_1 + 2\alpha_{\mathrm{Cusps}_\Gamma}(x) + C_{\bar{S}}$. Then $\alpha_{\bar{S}}(x) = -\log\|\pi_{w_0}(g.w)\|$ for some $w \in W_{\bar{S}}$. As in the proof of Claim 5.12, we deduce that for every $y = \mathtt{u}(i)\mathtt{a}(\ell\log 2)x$ for $i = 0, 1, \ldots, 2^{\ell-1}$,

$$\begin{aligned}
-\log\|\pi_{w_0}(\mathtt{u}(i)\mathtt{a}(\ell\log 2)g.w)\| &> -\log\|\pi_{w_0}(g.w)\| - T_1 = \alpha_{\bar{S}}(x) - T_1\\
&\geq 2\alpha_{\mathrm{Cusps}_\Gamma}(x) + T_1 + C_{\bar{S}} > 2\alpha_{\mathrm{Cusps}_\Gamma}(y) + C_{\bar{S}}.
\end{aligned}$$

Hence, Lemma 5.8 point 2 implies that $\alpha_{\bar{S}}(y) = -\log\|\pi_{w_0}(\mathtt{u}(i)\mathtt{a}(\ell\log 2)g.w)\|$. As in the proof of Claim 5.14, we deduce that

$$\int_X \alpha_{\bar{S}}(y)\mathrm{d}\nu_x^{(k,\ell)}(y) \leq \alpha_{\bar{S}}(x) - \ell\log 2 + C_W + 2^\ell\delta T_1.$$

Let $T_0'' = \min(\ell\log 2 - C_W - 2^\ell\delta T_1, \ell\log 2 - 2C_V - 2^\ell\delta T_1)$. Let ℓ be sufficiently large so that

$$\min(\ell\log 2 - C_W, \ell\log 2 - 2C_V) > \frac{T_1}{2} = \frac{\ell\log 2 + C_0 + 1}{2}.$$

For $\delta > 0$ sufficiently small, $T_0'' > T_1/2$. Thus, applying Claim 5.5 for $\alpha = \alpha_{\mathrm{Cusps}}$, $\beta = \max(\alpha_{\bar{S}}(x) - T_1 - C_{\bar{S}}, 0)$, T_0'' and T_1 we deduce that $\max(0, \alpha_{\mathrm{Cusps}_\Gamma} - 2T_1 - C_{\bar{S}}, \alpha_{\bar{S}} - 6T_1)$ is a $(2\delta; 2T_0'' - T_1, T_1)$-additive Margulis function, as desired. □

Claim 5.14 and Lemma 5.3 imply that

$$\lim_{k\to\infty} \mu_k(\alpha_{\mathrm{Cusps}_\Gamma,\bar{S}}^{-1}([0,t])) \geq 1 - \frac{1}{\log\lfloor t/T_1\rfloor - 1}.$$

Since $\alpha_{\mathrm{Cusps}_\Gamma,\bar{S}}$ is continuous and proper on $(G/\Gamma)\setminus S$, and attains values in $[0,\infty)$, we deduce the result of 4.7. □

5.3. **Proof of Lemma 5.6.** We will prove a more general version of Lemma 5.6, which works also for $\mathrm{SL}_2(\mathbb{R})$, and later use it to understand periodic $\mathrm{SL}_2(\mathbb{R})$-orbits.

Definition 5.15 (General setting)**.** Let $F = \mathbb{R}$ or $\mathbb{C}$ and $H = \mathrm{SL}_2(F)$. $U_F = \{\mathtt{u}_F(s) : s \in F\}$ where $\mathtt{u}_F(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. Let $\mathtt{a}(t) = \mathrm{diag}(e^{t/2}, e^{-t/2})$ for $t \in \mathbb{R}$, and K_H be either $\mathrm{SO}(2)$ or $\mathrm{SU}(2)$ the maximal compact subgroup in H.

We will use this setting for the rest of the subsection.

Claim 5.16 (QR Decomposition)**.** *Any element $g \in H$ can be represented uniquely and continuously as $g = k\mathtt{a}(t)\mathtt{u}_F(s)$ for $t \in \mathbb{R}, k \in K_H$, and $s \in F$ where F, H, K_H, and $\mathtt{u}_F$ are as in Claim 5.19.*

Definition 5.17. Let $B_F^* = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(F) : |a| = 1 \right\}$. Let $M < B_F^*$ be a lattice, that is, a discrete subgroup such that B_F^*/M has finite volume. For every such lattice, the intersection $M \cap U_F$ is a lattice in U_F and hence of finite index in M.

Let $H_{\geq \tau} = \{k\mathtt{a}(-t)\mathtt{u}_F(s) : k \in K_H, t \geq \tau, s \in F\}$ for every $\tau \in \mathbb{R} \cup \{-\infty\}$. Then $H_{\geq \tau}$ is preserved by the right B_F^* action, and for every lattice $M < B_F^*$ denote $D_{M,\tau} = H_{\geq \tau}/M$. Explicit computations show that $D_{M,\tau}$ has finite volume via the Haar measure on H.

Claim 5.19 is a reformulation of [38, Prop. 5.11.1.] when we replace a homogeneous space H/Γ by the corresponding hyperbolic manifold $\mathbb{H}^k/\Gamma$, where $k = \begin{cases} 2, & \text{if } F = \mathbb{R}, \\ 3, & \text{if } F = \mathbb{C}. \end{cases}$ It states that lattice quotients H/Γ are composed of a compact part as well as disjoint sets of the form $D_{M,0}$, and will help us prove this both Lemmals 5.6 and 5.8.

Definition 5.18 (Quotient product)**.** For every two discrete subgroup $M_1, M_2 < H$ and an element $h_0 \in H$ such that $h_0^{-1} M_1 h_0 \subseteq M_2$, the map $h \mapsto hh_0$ descends to a map

$$x \mapsto x \bullet h_0 : H/M_1 \to H/M_2.$$

Sometimes we will use this notation to denote the restriction of $- \bullet h_0$ into subsets of H/M_1. To avoid confusion, whenever we use this notation, we will specify the source and target of $- \bullet h_0$.

Claim 5.19 (Siegel domains)**.** *Let $H = \mathrm{SL}_2(F)$, with $F = \mathbb{R}, \mathbb{C}$. Let Γ be a lattice in H. Then there is a finite set Cusps_Γ parameterizing the cusps of H/Γ, that satisfies the following properties:*

S-1) *For every $c \in \mathrm{Cusps}_\Gamma$ there is a chosen element $g_c \in H$.*
S-2) *For every $c \in \mathrm{Cusps}_\Gamma$ the group $M_c = B_F^* \cap g_c \Gamma g_c^{-1}$ is a lattice in B_F^* and satisfying $g_c^{-1} M_c g_c < \Gamma$.*

S-3) *Let $D_{c,\tau} = D_{M_c,\tau}$ for every $\tau \in \mathbb{R}$. The map*

$$- \bullet g_c : D_{c,-\infty} = H/M_c \to H/\Gamma, \tag{5.15}$$

restricts to a bijection on the image $(- \bullet g_c)|_{D_{c,0}} : D_{c,0} \xrightarrow{\sim} D_{c,0} \bullet g_c$, and the map $(- \bullet g_c)|_{D_{c,0}} : D_{c,0} \to H/\Gamma$ is proper.

S-4) *The images $D_{c,0} \bullet g_c$ for $c \in \mathrm{Cusps}_\Gamma$ are disjoint, and the complement $(H/\Gamma) \setminus \bigcup_{c \in \mathrm{Cusps}_\Gamma} D_{c,0} \bullet g_c$ is precompact.*

Definition 5.20. Consider the representation $V_F = F^2$. Denote its standard basis by e_1, e_2. It has the standard Euclidean norm $\| - \|$. Note that $H_{\geq \tau} = \{h \in H : \|he_1\| < e^{-\tau/2}\}$. Let $\alpha_{\mathrm{Siegel}} : H \to \mathbb{R}$ be $\alpha_{\mathrm{Siegel}}(h) = -\log \|he_1\|$ satisfying that whenever $h = k\mathtt{a}(-t)\mathtt{u}_F(s)$ is the QR-Decomposition of h, then $\alpha_{\mathrm{Siegel}}(h) = t/2$. This function is B_F^*-invariant from the right, hence descends to $\alpha_{\mathrm{Siegel}} : H/M \to \mathbb{R}$ for every lattice $M < B_F^*$.

Definition 5.21 (First definition of $\alpha_{\mathrm{Cusps}_\Gamma} : G \to [0,\infty)$)**.** Define $\alpha_{\mathrm{Cusps}_\Gamma} : G/\Gamma \to [0, \infty)$ by

$$\alpha_{\mathrm{Cusps}_\Gamma}(x) = \begin{cases} \alpha_{\mathrm{Siegel}}(z), & x = z \bullet g_c \text{ for some } c \in \mathrm{Cusps}_\Gamma, z \in D_{c,0}, \\ 0, & \text{otherwise.} \end{cases},$$

Here $- \bullet g_c$ is defined as in Eq. (5.15). We see that it is proper and continuous.

Definition 5.22. For any $c \in \mathrm{Cusps}_\Gamma$ consider the vector $v_c = g_c^{-1}.e_1 \in V_F$. Let $V_{\mathrm{Cusps}_\Gamma} = \bigcup_{c \in \mathrm{Cusps}_\Gamma} \Gamma.v_c \subseteq V_F$.

Corollary 5.23 (Reformulation of Lemma 5.6 in the general setting)**.** *The function $\alpha_{\mathrm{Cusps}_\Gamma} : H \to [0, \infty)$ satisfies*

$$\alpha_{\mathrm{Cusps}_\Gamma}(h) = \begin{cases} -\log \|h.v\|, & \text{if } \|h.v\| < 1 \text{ for some } v \in V_{\mathrm{Cusps}_\Gamma}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.16}$$

Proof. Suppose that $v = h\gamma.v_c = h\gamma g_c^{-1}.e_1 \in h.V_{\mathrm{Cusps}_\Gamma}$, satisfies that $\|v\| < 1$ for some $h \in H, c \in \mathrm{Cusps}_\Gamma$. Then $h\gamma g_c^{-1} \in H_{\geq 0}$, and hence $\pi_\Gamma(h) = \pi_{M_c}(h\gamma g_c^{-1}) \bullet g_c$ and $\alpha_{\mathrm{Cusps}_\Gamma}(h) = \alpha_{\mathrm{Siegel}}(h\gamma g_c^{-1}) = -\log \|v\|$. Here $- \bullet g_c$ is defined as in (5.15). This implies Eq. (5.16).

□

This claim proves Lemma 5.6 when using $F = \mathbb{C}$. We recall the following corollary, which is well known but follows easily from Corollary 5.23.

Definition 5.24. Let $B_F = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : b \in F, a \in F^\times \right\}$, and note that $B_F = \mathrm{stab}_H Fe_1$. Note that the restriction $\alpha_{\mathrm{Siegel}}|_{B_F} : B_F \to \mathbb{R}$ sends $\alpha_{\mathrm{Siegel}}\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = -\log a$ is a homomorphism.

Corollary 5.25. *Let $h \in H$. The following are equivalent:*

(1) The trajectory $\mathtt{a}(-t)\pi_\Gamma(h)$ diverges as $t \to \infty$.

(2) $h = h_0 g_c \gamma$ for $h_0 \in B_F, c \in \mathrm{Cusps}_\Gamma, \gamma \in \Gamma$.

Proof. The trajectory $\mathtt{a}(-t)\pi_\Gamma(h)$ diverges as $t \to \infty$, iff $\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{a}(-t)\pi_\Gamma(h)) \xrightarrow{t\to\infty} \infty$. Let $t_0 > 0$ be such that for all $t \geq t_0$ we have $\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{a}(-t)\pi_\Gamma(h)) > 0$. Let $v \in V_{\mathrm{Cusps}_\Gamma}$ satisfy that $\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{a}(-t_0)\pi_\Gamma(h)) = -\log \|\mathtt{a}(-t_0)h.v\|$. The two functions $f_1 : t \mapsto \alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{a}(-t)\pi_\Gamma(h))$ and $f_2 : t \mapsto -\log \|\mathtt{a}(-t)h.v\|$

- are continuous,
- coincide at $t = t_0$,
- satisfy that $f_1(t) > 0$ for all $t \geq t_0$, and
- for every $t \geq t_0$, if $f_2(t) > 0$, $f_1(t) = f_2(t)$.

We deduce that for all $t \geq t_0$ we have $f_1(t) = f_2(t)$, that is,

$$\alpha_{\mathrm{Cusps}_\Gamma}(\mathtt{a}(-t)\pi_\Gamma(h)) = -\log \|\mathtt{a}(-t)h.v\| \xrightarrow{t\to\infty} \infty.$$

This is equivalent to $\mathtt{a}(-t)h.v \xrightarrow{t\to\infty} 0$. Note that

$$\mathtt{a}(-t)h.v \xrightarrow{t\to\infty} 0 \iff h.v = h\gamma g_c^{-1}.e_1 \in Fe_1 \iff h\gamma g_c^{-1} \in B_F.$$

The desired equivalence follows. The desired follows. $\square$

5.4. **Proof of Lemma 5.8.** We now return to our original setting with $\Gamma < G$, as in Definition 2.1. Let $S = \mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g_S)$ be a periodic $\mathrm{SL}_2(\mathbb{R})$-orbit in G/Γ. Let $\Lambda = \mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}\,\pi_\Gamma(g_S) = g_S\Gamma g_S^{-1} \cap \mathrm{SL}_2(\mathbb{R})$, and $\pi_\Lambda : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})/\Lambda$ denote the standard projection. Since S is a periodic $\mathrm{SL}_2(\mathbb{R})$-orbit, it follows that Λ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. Let $\bar{S} = N(\mathrm{SL}_2(\mathbb{R}))\pi_\Gamma(g_S)$ be a periodic $N(\mathrm{SL}_2(\mathbb{R}))$-orbit.

Remark 5.26. The distinction between S and $\bar{S}$ bears no mathematical difficulties. If $\mathrm{stab}_{N(\mathrm{SL}_2(\mathbb{R}))}\,\pi_\Gamma(g_S) \subseteq \mathrm{SL}_2(\mathbb{R})$, then $\bar{S} = S \sqcup g_0 S$, where $g_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in N(\mathrm{SL}_2(\mathbb{R})) \setminus \mathrm{SL}_2(\mathbb{R})$. Otherwise $\bar{S} = S = g_0 S$.

Let

$$- \bullet g_S : \mathrm{SL}_2(\mathbb{R})/\Lambda \to G/\Gamma, \tag{5.17}$$

be defined as in Definition 5.18. In particular, it defines an isomorphism $- \bullet g_S : \mathrm{SL}_2(\mathbb{R})/\Lambda \xrightarrow{\sim} S$, which enables us to use Claim 5.19 for $F = \mathbb{R}$ to describe its cusps. We will need the following claim which describes how the cusps of S sit inside the cusps of G/Γ.

Claim 5.27 (How the cusps of S sit in the cusps of G/Γ)**.** *Let $c_\Lambda \in \mathrm{Cusps}_\Lambda$ be a cusp of $\mathrm{SL}_2(\mathbb{R})/\Lambda$. Then there are*

(1) a unique cusp $c_{\Gamma,c_\Lambda} = c_\Gamma \in \mathrm{Cusps}_\Gamma$;
(2) $h_{c_\Lambda} \in B_{\mathbb{C}}$;
(3) and $\gamma_{c_\Lambda} \in \Gamma$

such that

$$h_{c_\Lambda} g_{c_\Gamma} \gamma_{c_\Lambda} = g_{c_\Lambda} g_S, \tag{5.18}$$

$$M_{c_\Lambda} = h_{c_\Lambda} M_{c_\Gamma} h_{c_\Lambda}^{-1} \cap B_{\mathbb{R}}^*. \tag{5.19}$$

Proof. By Corollary 5.25 applied for $\mathrm{SL}_2(\mathbb{R})/\Lambda$, we deduce that $\mathtt{a}(-t)\pi_\Lambda(g_{c_\Lambda})$ diverges in $\mathrm{SL}_2(\mathbb{R})/\Lambda$ as $t \to \infty$. Thus $\iota(\mathtt{a}(-t)\pi_\Lambda(g_{c_\Lambda})) = \mathtt{a}(-t)\pi_\Gamma(g_{c_\Lambda} g_S)$ diverges in

G/Γ as $t \to \infty$. Corollary 5.25 applied for G/Γ now implies Eq. (5.18). To show Eq. (5.19), use the notation $a^b = b^{-1}ab$ for conjugation and note that

$$\begin{aligned} M_{c_\Lambda} &= B_{\mathbb{R}}^* \cap \Lambda^{g_{c_\Lambda}^{-1}} = B_{\mathbb{R}}^* \cap \Gamma^{g_S^{-1} g_{c_\Lambda}^{-1}} = B_{\mathbb{R}}^* \cap \Gamma^{g_{c_\Gamma}^{-1} h_{c_\Lambda}} \\ &= B_{\mathbb{R}}^* \cap \left(\Gamma^{g_{c_\Gamma}^{-1}} \cap B_{\mathbb{C}}^*\right)^{h_{c_\Lambda}} = B_{\mathbb{R}}^* \cap (M_{c_\Gamma})^{h_{c_\Lambda}}. \end{aligned}$$

□

Observation 5.28. Using (5.19) we may apply Definition 5.18 and define

$$z \mapsto z \bullet h_{c_\Lambda} : \mathrm{SL}_2(\mathbb{R})/M_{c_\Lambda} \to G/M_{c_\Gamma}. \tag{5.20}$$

Eq. (5.18) shows that for every $z \in D_{c_\Lambda,-\infty} = \mathrm{SL}_2(\mathbb{R})/M_{c_\Lambda}$ with $g \in \mathrm{SL}_2(\mathbb{R})$,

$$(z \bullet g_{c_\Lambda}) \bullet g_S = (z \bullet h_{c_\Lambda}) \bullet g_{c_\Gamma}. \tag{5.21}$$

Equivalently, the following diagram is commutative,

$$\begin{tikzcd} D_{c_\Lambda,-\infty} \arrow[r, "-\bullet h_{c_\Lambda}"] \arrow[d, "-\bullet g_{c_\Lambda}"] & D_{c_\Gamma,-\infty} \arrow[d, "-\bullet g_{c_\Gamma}"] \\ \mathrm{SL}_2(\mathbb{R})/\Lambda \arrow[r, "-\bullet g_S"] & G/\Gamma \end{tikzcd} \tag{5.21$'$}$$

A useful corollary is the following:

Corollary 5.29 (Only the cusps of S reach deep into the cusps of G/Γ)**.** *There is a constant $T_{\Gamma,S} \geq 0$ such that the following holds. For every point $x_0 \in S$ with $\alpha_{\mathrm{Cusps}_\Gamma}(x) > T_{\Gamma,S}$, is of the form*

$$\begin{aligned} x_0 &= (z_0 \bullet g_{c_{\Gamma,c_\Lambda}}) \bullet g_S \quad \textit{for} \quad z_0 \in D_{c_\Lambda,0}, \\ \alpha_{\mathrm{Cusps}_\Gamma}(x_0) &= \alpha_{\mathrm{Cusps}_\Lambda}(z_0) + \alpha_{\mathrm{Siegel}}(h_{c_\Lambda}). \end{aligned} \tag{5.22}$$

Proof. For every cusp $c_\Lambda \in \mathrm{Cusps}_\Lambda$, let $T_{c_\Lambda} = \max(-\alpha_{\mathrm{Siegel}}(h_{c_\Lambda}), 0) + 1$ and $c_\Gamma = c_{\Gamma,c_\Lambda}$. Let $z_1 \in D_{c_\Lambda,T_{c_\Lambda}}$, and let $x_1 = (z_1 \bullet g_{c_\Gamma}) \bullet g_S \overset{(5.21)}{=} (z_1 \bullet h_{c_\Lambda}) \bullet g_{c_\Gamma}$. Since

$$\alpha_{\mathrm{Siegel}}(z_1 \bullet h_{c_\Lambda}) = \alpha_{\mathrm{Siegel}}(z_1) + \alpha_{\mathrm{Siegel}}(h_{c_\Lambda}) \geq 0$$

we deduce that $z_1 \bullet h_{c_\Lambda} \in D_{c_\Gamma,0}$ and hence x_1 satisfy Eq. (5.22). Therefore, by Point (S-4), the set

$$S_{\mathrm{bad}} = \{x_0 \in S \text{ that does not satisfy Eq. (5.22)}\}$$

is precompact and we can take $T_{\Gamma,S} = \sup_{S_{\mathrm{bad}}} \alpha_{\mathrm{Cusps}_\Gamma}$.

□

Observation 5.30. Observing the definition of α_{Sigel}, we notice that it has no local maxima on $\mathrm{SL}_2(\mathbb{R})$. In view of Corollary 5.29 and Eq. (5.22), $\alpha_{\mathrm{Cusps}_\Gamma}$ has no local maxima on $S \cap \alpha_{\mathrm{Cusps}_\Gamma}^{-1}((T_{\Gamma,S}, \infty))$.

We introduce the representation W.

Claim 5.31. *There is a 4-dimensional irreducible real representation $W = \mathbb{R}^4$ of G and a vector $w_0 \in W$, such that $\mathrm{SL}_2(\mathbb{R}) = \mathrm{stab}_G(w_0)$ and $N(\mathrm{SL}_2(\mathbb{R})) = \mathrm{stab}_G(\{\pm w_0\})$. G preserves the real quadratic form $Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 - x_3^2 - x_4^2$ on W of type $(1,3)$, satisfying $Q(w_0) = -1$.*

Proof. We first introduce W as the space of Hermitian 2×2 matrices, on which G acts by $g.A = g^{-*}Ag^{-1}$. Here g^* refers to the complex conjugate of the transposed matrix, and $g^{-*} = (g^*)^{-1}$. The quadratic form det is preserved by the G action. Identify W with $\mathbb{R}^4$ using the basis

$$\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\right). \tag{5.23}$$

The remaining follows with $w_0 = \left(\begin{smallmatrix}0 & i\\ -i & 0\end{smallmatrix}\right)$. ◻

Recall that $w_{\bar S} = g_S^{-1}.w_0$ and $W_{\bar S} = \Gamma.w_{\bar S}$.

Claim 5.32. *The set $W_{\bar S}$ is discrete.*

Proof. Denote by $\pi_{\mathrm{SL}_2(\mathbb{R})} : G \to G/\mathrm{SL}_2(\mathbb{R})$ the natural projection. Note that

$$W_{\bar S} \subseteq R = \{w \in W : Q(w) = -1\} \overset{g.e_1 \mapsto \pi_{\mathrm{SL}_2(\mathbb{R})}(g)}{\cong} G/\mathrm{SL}_2(\mathbb{R}),$$

and R is closed in W. Hence to verify that $\Gamma.w_{\bar S}$ is discrete in W, it is sufficient to verify that $\pi_{\mathrm{SL}_2(\mathbb{R})}(\Gamma g_S^{-1})$ is discrete. Suppose that there is a sequence of points

$$\pi_{\mathrm{SL}_2(\mathbb{R})}(\gamma_i g_S^{-1}) \xrightarrow{i\to\infty} \gamma_\infty \pi_{\mathrm{SL}_2(\mathbb{R})}(g_0),$$

for $\gamma_1, \gamma_2, \ldots \in \Gamma$ and $g_0 \in G$. Hence there is a sequence of matrices $h_i \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\gamma_i g_S^{-1} h_i \xrightarrow{i\to\infty} g_0. \tag{5.24}$$

Inverting and projecting to G/Γ we deduce that

$$\pi_\Gamma(h_i^{-1} g_S) \xrightarrow{i\to\infty} \pi_\Gamma(g_0^{-1}).$$

Since $\pi_\Gamma(h_i^{-1}g_S) \in S$ and S is closed, also $\pi_\Gamma(g_0^{-1}) \in S$. Since S is a closed orbit, for some $\varepsilon > 0$ sufficiently small (depending on g_0), for every $p \in S$ with $d_{G/\Gamma}(\pi_\Gamma(g_0^{-1}), p) < \varepsilon$ we have $p = h.\pi_\Gamma(g_0^{-1})$ for some $h \in \mathrm{SL}_2(\mathbb{R})$ with $d_{\mathrm{SL}_2(\mathbb{R})}(h, I) = O(d_{G/\Gamma}(\pi_\Gamma(g_0^{-1}), p))$. Here $d_{\mathrm{SL}_2(\mathbb{R})}$ is the right invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$. It follows that for sufficiently large i, there is $h_i' \in H$ such that

$$\pi_\Gamma(h_i'^{-1} h_i^{-1} g_S) = \pi_\Gamma(g_0^{-1}), \tag{5.25}$$

and $d_{\mathrm{SL}_2(\mathbb{R})}(h_i', I) \xrightarrow{i\to\infty} 0$. Eq. (5.25) is equivalent to

$$g_S^{-1} h_i h_i' \in \Gamma g_0. \tag{5.26}$$

However, Eq. (5.24) and the size estimate for h_i' imply that

$$\gamma_i g_S^{-1} h_i h_i' = g_0. \tag{5.27}$$

for all i sufficiently large. Consequently, $\pi_{\mathrm{SL}_2(\mathbb{R})}(\gamma_i g_S^{-1}) = \pi_{\mathrm{SL}_2(\mathbb{R})}(g_0)$ for all i sufficiently large. This means that every converging sequence in $\pi_{\mathrm{SL}_2(\mathbb{R})}(\Gamma g_S^{-1})$ fixes on its limit, which implies that this set is discrete, as desired. ◻

We can now prove the remaining of Lemma 5.8. To construct $\alpha_{\bar S}$ we consider the quotient map and projection $\pi_{w_0} : W = \mathbb{R}^4 \to \mathbb{R}^3$ sending $(x_i)_{i=1}^4$ to $(x_i)_{i=1}^3$. Thus we consider the standard norm on $\mathbb{R}^3$ and Define $\alpha_{\bar S} : G \to \mathbb{R} \sqcup \{\infty\}$ by

$$\alpha_{\bar S}(g) = \sup_{w \in W_{\bar S}} -\log \|\pi_{w_0}(g.w)\|. \tag{5.28}$$

Claim 5.33. *The supremum in Eq.* (5.28) *is attained and the function* $\alpha_{\bar{S}}$ *is continuous.*

Proof. Let $R = \{w \in W : Q(w) = -1\}$ and $\ell : R \to \mathbb{R} \cup \{\infty\}$ be defined by $g(w) = -\log \|\pi_{w_0}(w)\|$. Note that $R_t := \ell^{-1}([t, \infty])$ is compact for all $t \in \mathbb{R}$. Since $W_{\bar{S}}$ is discrete, $g.W_{\bar{S}}$ is discrete as well for all $g \in G$, hence $R_t \cap g.W_{\bar{S}}$ in a finite set, for every $t \in \mathbb{R}$ and $g \in G$. Rewrite Eq. (5.28),

$$\alpha_{\bar{S}}(g) = \sup\{\ell(w) : w \in g.W_{\bar{S}}\} = \sup\{\ell(w) : w \in g.W_{\bar{S}}, \ell(w) \geq \alpha_{\bar{S}} - 1\} \tag{5.29}$$

$$= \sup\{\ell(w) : w \in g.W_{\bar{S}} \cap R_{\alpha_{\bar{S}}-1}\}. \tag{5.30}$$

The rightmost supremum in Eq. (5.29) ranges over a finite set and must be attained. Let $C \subseteq G$ be a compact subset. We will prove that $\alpha_{\bar{S}}$ is continuous on C. Then the infimum

$$z_C = \inf_{g \in C} \alpha_{\bar{S}}(g) \geq \inf_{g \in C} \ell(g.w_{\bar{S}}),$$

satisfies $z_C \in \mathbb{R} \cup \{\infty\}$ by the compactness of C. The set

$$W_{\bar{S},C} = W_{\bar{S}} \cap C^{-1}.R_{z_C},$$

is finite by the discreteness of $W_{\bar{S}}$. Then for all $g \in G$,

$$\begin{aligned}\alpha_{\bar{S}}(g) &= \sup\{\ell(w) : w \in g.W_{\bar{S}}, \ell(w) \geq z_C\} \\ &= \sup\{\ell(w) : w \in g.(W_{\bar{S}} \cap g^{-1}R_{Z_C})\} = \sup\{\ell(w) : w \in g.W_{\bar{S},C}\}.\end{aligned}$$

However, $W_{\bar{S},C}$ is finite, and hence $\alpha_{\bar{S}}$ is continuous on C. Since G is locally compact, this implies that $\alpha_{\bar{S}}$ is continuous everywhere. □

Proof of Claim 5.8 Point 1. Let $g \in G$. Note that $\alpha_{\bar{S}}(g) = \infty$ if and only if there exists $\gamma \in \Gamma$ such that $\pi_{w_0}(g\gamma g_S^{-1}.w_0) = 0$. Since $Q(g\gamma g_S^{-1}.w_0) = Q(w_0) = -1$ and $\pm w_0$ are the only vectors $w \in \ker \pi_{w_0}$ satisfying $Q(w) = -1$ we deduce that $g\gamma g_S^{-1}.w_0 = \pm w_0$, and hence $g\gamma g_S^{-1} \in N(\mathrm{SL}_2(\mathbb{R}))$. Altogether, we have the equivalence

$$\alpha_{\bar{S}}(g) = \infty \Leftrightarrow g \in N(\mathrm{SL}_2(\mathbb{R}))g_S\Gamma = \pi_\Gamma^{-1}(\bar{S}).$$

□

It remains to show the Point 2. We will use the following claims:

Claim 5.34. *There is a form* $\Psi : V \times W \to \mathbb{R}$ *that satisfies the following conditions*

1. Ψ *is* G*-invariant in the sense that* $\Psi(g.v, g.w) = \Psi(v, w)$ *for all* $v \in V$ *and* $w \in W$.
2. Ψ *is linear in* W *and Hermitian in* V.
3. *It satisfies that* $|\Psi(v, w_0)| = \inf_{g \in \mathrm{SL}_2(\mathbb{R})} \|g.v\|^2$.
4. *If the infimum is nonzero then it is attained.*

Proof. Since $V = \mathbb{C}^2$ and W is the space of hermitian matrices, we can define the form $\Psi(v, w) = v^*wv$, where v^* is the complex conjugate of v, thought of as a row vector. Writing $v = \binom{x}{y} \in \mathbb{C}^2$, algebraic manipulations show that

$$\Psi(v, w_0) = 2\mathrm{Im}(x\bar{y}) = \det(v, \bar{v})/i.$$

Here, by $\det(v, \bar{v})$ we refer to the determinant of the matrix whose columns are $v, \bar{v}$. Suppose that $\Psi(v, w_0) = 0$. Then $x\bar{y} \in \mathbb{R}$, which is equivalent to saying that v is proportional to a real vector. We know that $\mathrm{SL}_2(\mathbb{R})$ can shrink arbitrarily real vectors, which implies $\inf_{g\in \mathrm{SL}_2(\mathbb{R})}\|g.v\|^2=0$. Suppose now that $2\mathrm{Im}(x\bar{y})\neq 0$. Then the two vectors denote by $v_1=\frac{v+\bar{v}}{2}$ and $v_2=\frac{v-\bar{v}}{2i}$ are real. Hence,

$$\Psi(v,w_0)=\det(v,\bar{v})/i=-2\det(v_1,v_2).$$

Note that $\|v\|^2=\|v_1+iv_2\|^2=\|v_1\|^2+\|v_2\|^2$. One can verify that

$$|2\det(v_1,v_2)|\leq \|v_1\|^2+\|v_2\|^2 \tag{5.31}$$

for every pair of real vectors. Moreover, equality holds in Eq. (5.31) if and only if $v_1\perp v_2$ and $\|v_1\|=\|v_2\|$. Since whenever v_1 and v_2 are linearly independent, there is always $h\in \mathrm{SL}_2(\mathbb{R})$ such that $g.v_1\perp g.v_2$ and $\|g.v_1\|=\|g.v_2\|$, the desired result holds. □

Claim 5.35. *There is $\varepsilon=\varepsilon_{\Gamma,\bar{S}}=e^{-2T_{\Gamma,S}}>0$ such that for all $v\in V_{\mathrm{Cusps}_\Gamma}$ and $w\in W_{\bar{S}}$ we have that either $\Psi(v,w)=0$ or $|\Psi(v,w)|\geq \varepsilon_{\Gamma,\bar{S}}$.*

Proof. Let $v=\gamma_1 g_{c_\Gamma}^{-1}.e_1\in V_{\mathrm{Cusps}_\Gamma}, w=\gamma_2 g_S^{-1}.w_0\in W_{\bar{S}}$ be two vectors with $0\neq|\Psi(v,w)|<\varepsilon$. Let $g_0=g_S\gamma_2^{-1}$, be a matrix satisfying that $g_0.w=w_0$. We deduce that $g_0 v$ satisfy that $\Psi(v,w)=\Psi(g_0.v,w_0)$.

By Claim 5.34, there is $h_0\in \mathrm{SL}_2(\mathbb{R})$ such that $|\Psi(g_0.v,w_0)|=|\Psi(h_0g_0.v,w_0)|=\|h_0g_0.v\|^2$. Then $\alpha_{\mathrm{Cusps}_\Gamma}(h_0g_0)=-\log\|h_0g_0.v\|>T_{\Gamma,S}$. However, for every h_1 sufficiently small,

$$\alpha_{\mathrm{Cusps}_\Gamma}(h_1h_0g_0)=-\log\|h_1h_0g_0.v\|\leq -\frac{1}{2}\log|\Psi(h_0g_0.v,w_0)|=\alpha_{\mathrm{Cusps}_\Gamma}(h_0g_0),$$

which implies that $-\frac{1}{2}\log|\Psi(v,w)|$ is a local maximum of $\alpha_{\mathrm{Cusps}_\Gamma}$ at $\pi_\Gamma(h_0g_0)$ along the $\mathrm{SL}_2(\mathbb{R})$-orbit $\mathrm{SL}_2(\mathbb{R}).\pi_\Gamma(g_0)=S$. This contradicts Observation 5.30, and hence the equation $0\neq|\Psi(v,w)|<\varepsilon$.

□

Claim 5.36. *Consider the space*

$$W^{0+}=\{w:\Psi(e_1,w)=0\}=\left\{\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\right\}\subset W,$$

which contains w_0 and is the space of $\mathtt{a}(t)$ noncontracting elements in W. Then for all $k\in \mathrm{SU}(2), w\in W^{0+}, t\geq 0$,

$$\|\pi_{w_0}(k\mathtt{a}(t).w)\|\leq 4e^t\|\pi_{w_0}(k.w)\|.$$

Proof. Here we will use the matrix description of W, and will distinguish between the matrix multiplication denoted without a dot, and the group action on the representation w, denoted $(g,w)\mapsto g.w:G\times W\to W$. We will prove the claim with the Hilbert-Schmidt norm given by

$$\left\|\begin{pmatrix}a & b\\ \bar{b} & d\end{pmatrix}\right\|=\sqrt{a^2+2|b|^2+d^2}.$$

This norm is proportional to the one given by the basis Eq. (5.23). Let $w = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ and define $w_1 = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$. Then

$$\begin{aligned}
\|\pi_{w_0}(k\mathtt{a}(t).w)\| &\le \|\pi_{w_0}(k\mathtt{a}(t).w_1)\| + \|\pi_{w_0}(k\mathtt{a}(t).w_2)\| \\
&\le \|\pi_{w_0}(k.w_1)\| + e^t\|\pi_{w_0}(k.w_2)\| \le e^t(\|\pi_{w_0}(k.w_1)\| + \|\pi_{w_0}(k.w_2)\|) \\
&\le e^t(\|\pi_{w_0}(k.w)\| + 2\|\pi_{w_0}(k.w_2)\|) \le e^t(\|\pi_{w_0}(k.w)\| + 2\|k.w_2\|) \\
&= e^t(\|\pi_{w_0}(k.w)\| + 2|c|)
\end{aligned}$$

To interpret c in matrix terminology, note that

$$c = \operatorname{tr}(w) = \operatorname{tr}(kwk^{-1}) = \operatorname{tr}(k^{-*}wk^{-1}) = \operatorname{tr}(k.w) = \operatorname{tr}\pi_{w_0}(k.w).$$

Hence we may continue the estimate

$$\begin{aligned}
&= e^t(\|\pi_{w_0}(k.w)\| + 2|\operatorname{tr}\pi_{w_0}(k.w)|) \le e^t(\|\pi_{w_0}(k.w)\| + 2\sqrt{2}\|\pi_{w_0}(k.w)\|) \\
&= (1+2\sqrt{2})e^t\|\pi_{w_0}(k.w)\| \le 4e^t\|\pi_{w_0}(k.w)\|.
\end{aligned}$$

□

Proof of Lemma 5.8 Point 2. Let $T_0 > 0$ to be determined later, $\tilde{X}_{\mathrm{comp},1} \subseteq G$ be a compact set so that

$$X_{\mathrm{comp},1} = \pi_\Gamma(\tilde{X}_{\mathrm{comp},1}) = (G/\Gamma) \setminus \bigcup_{c_\Gamma \in \mathrm{Cusps}_\Gamma} D_{c_\Gamma,T_0} \bullet g_{c_\Gamma} = \alpha^{-1}_{\mathrm{Cusps}_\Gamma}([0,T_0/2]),$$

where $\alpha^{-1}_{\mathrm{Cusps}_\Gamma}$ is the preimage map of $\alpha_{\mathrm{Cusps}_\Gamma} : G/\Gamma \to [0,\infty)$, and $-\bullet g_{c_\Gamma}$ is defined as in Claim 5.19.

For every $g \in G$ denote by

$$\beta(g) = \inf\left\{ r > 0 : \substack{\text{there are } w_1,w_2 \in g.W_{\bar{S}} \text{ with } \{\pm w_1\} \neq \{\pm w_2\} \\ \text{such that } \|\pi_{w_0}(w_1)\|, \|\pi_{w_0}(w_2)\| < r} \right\}.$$

Note that $\beta(g) > 0$ for every $g \in G$. Indeed, otherwise there exist $w_1, w_2 \in g.W_{\bar{S}}$ such that $\|\pi_{w_0}(w_1)\| = \|\pi_{w_0}(w_2)\| = 0$ and $\{\pm w_1\} \neq \{\pm w_2\}$. But $\ker \pi_{w_0} \cap \{w \in W : Q(w) = -1\} = \{\pm w_0\}$, which contradicts this existence of two different such vectors. Denote by $\delta_0 = \min_{\tilde{X}_{\mathrm{comp},1}} \beta > 0$, and set $C_{\bar{S}} = -\log \delta_0 + 2\log 2 + 1$.

Suppose that $\alpha_{\bar{S}}(g) > 2\alpha_{\mathrm{Cusps}_\Gamma}(g) + C_{\bar{S}}$ for some $g \in G$. Since $\alpha_{\bar{S}}(g) > 0$ there is $\gamma_1 \in \Gamma$ such that $\alpha_{\bar{S}}(g) = -\log\|\pi_{w_0}(g\gamma_1.w_{\bar{S}})\|$. Suppose that $g\gamma_2.w_{\bar{S}}$ is another vector with $-\log\|\pi_{w_0}(g\gamma_2.w_{\bar{S}})\| \ge 2\alpha_{\mathrm{Cusps}_\Gamma}(g) + C_{\bar{S}}$, and $\{\pm g\gamma_1.w_{\bar{S}}\} \neq \{\pm g\gamma_2.w_{\bar{S}}\}$. Let $w_i = \gamma_i.w_{\bar{S}} \in W_{\bar{S}}$. By the definitions of $C_{\bar{S}}$ and δ_0 we deduce that $\beta(g) < \delta_0$, and hence $\pi_\Gamma(g) \notin X_{\mathrm{comp},1}$.

Remark 5.37. This shows that Claim 5.8 Point 2 holds provided that $\pi_\Gamma(g) \in X_{\mathrm{comp},1}$ for any $C'_{\bar{S}} > -\log\delta_0$, and in particular, for $C'_{\bar{S}} = -\log\delta_0 + 1$.

We deduce that for some $v \in V_{\mathrm{Cusps}_\Gamma}$ we have $\alpha_{\mathrm{Cusps}_\Gamma}(g) = -\log\|g.v\| > T_0/2$. To estimate $\Psi(v, w_i)$, note that there is a constant $C_0 > 0$ such that for all $v' \in V, w' \in W$ with $\|v'\|, \|w'\| \le 1$ we have $|\Psi(v', w')| \le C_0$ (direct computation leads to $C_0 = \sqrt{2}$). Then $|\Psi(v, w_i)| = |\Psi(g.v, g.w_i)| \le C_0\|g.v\|^2\|g.w_i\| \le C_0 e^{-T_0}\|g.w_i\|$ for $i = 1, 2$. To estimate $\|g.w_i\|$, note that there is a constant $C_1 > 0$ such if $w' \in W$ satisfies $\|\pi_{w_0} w'\| \le 1$ and $Q(w') = -1$ then $\|w'\| \le C_1$ (direct computation leads to $C_1 = \sqrt{3}$). This shows that $\|g.w_i\| \le C_1$, and hence $|\Psi(v, w_i)| \le C_0 C_1 e^{-T_0}$. Set

$T_0 = -\log(\varepsilon_{\Gamma,\bar{S}}/(C_0C_1))+1$ for $\varepsilon_{\Gamma,\bar{S}}$ as in Claim 5.35, and deduce that $\Psi(v,w_i)=0$ for $i=1,2$.

Suppose that

$$v = \gamma.v_{c_\Gamma} = \gamma g_{c_\Gamma}^{-1}.e_1, \tag{5.32}$$

for some $\gamma \in \Gamma$ and $c \in \mathrm{Cusps}_\Gamma$. QR-Decomposition 5.16 shows that one can represent

$$g\gamma g_{c_\Gamma}^{-1} = k\mathtt{a}(-t)\mathtt{u}_{\mathbb{C}}(s) \tag{5.33}$$

for some $k \in K_G, t \in \mathbb{R}, s \in \mathbb{C}$. Since $\|g\gamma g_{c_\Gamma}^{-1}.e_1\| = \|g.v\| < e^{-T_0/2}$ and $\|g\gamma g_{c_\Gamma}^{-1}.e_1\| = \|k\mathtt{a}(-t)\mathtt{u}_{\mathbb{C}}(s).e_1\| = e^{-t/2}$, it follows that $t > T_0$.

For every $\tau > 0$ write $g_\tau = k\mathtt{a}(-t+\tau)\mathtt{u}_{\mathbb{C}}(s)g_{c_\Gamma}\gamma^{-1} = k\mathtt{a}(\tau)k^{-1}g$. Let $\tau_0 = t - T_0 > 0$. Note that

$$\alpha_{\mathrm{Cusps}_\Gamma}(g_{\tau_0}) = -\log\|g_{\tau_0}.v\| = -\log\|k\mathtt{a}(-T_0)\mathtt{u}_{\mathbb{C}}(s).e_1\| = T_0.$$

Consequently, $\pi_\Gamma(g_{\tau_0}) \in X_{\mathrm{comp},1}$, and Remark 5.37 is applicable for g_{τ_0}. However, let us estimate $-\log\|g_{\tau_0}.w_i\|$.

Note that $\|g_{\tau_0}.w_i\| = \|k\mathtt{a}(\tau_0)k^{-1}g.w_i\|$. Denote $w_i' = k^{-1}g.w_i$ and notice that

$$0 = \Psi(k^{-1}g.v, w_i') \overset{(5.32)+(5.33)}{=} \Psi(k^{-1}k\mathtt{a}(t)\mathtt{u}_{\mathbb{C}}(s)g_{c_\Gamma}\gamma^{-1}\gamma g_{c_\Gamma}^{-1}.e_1, w_i') = \Psi(\mathtt{a}(t).e_1, w_i') = e^t\Psi(e_1, w_i').$$

Hence $\Psi(e_1, w_i') = 0$. Now we may apply Claim 5.36, and deduce that

$$\|\pi_{w_0}(g_{\tau_0}.w_i)\| = \|\pi_{w_0}(k\mathtt{a}(\tau_0).w_i')\| \overset{5.36}{\leq} 4e^{\tau_0}\|\pi_{w_0}(k.w_i')\| = 4e^{\tau_0}\|\pi_{w_0}(g.w_i)\|.$$

Therefore,

$$\begin{aligned} -\log\|\pi_{w_0}(g_{\tau_0}.w_i)\| &\geq -\log\|\pi_{w_0}(g.w_i)\| - 2\log 2 - \tau_0 \\ &\geq 2\alpha_{\mathrm{Cusps}_\Gamma}(g) + C_{\bar{S}} - 2\log 2 - \tau_0 \\ &= 2\alpha_{\mathrm{Cusps}_\Gamma}(g_{\tau_0}) - \log\delta_0 + 1. \end{aligned}$$

This contradicts Claim 5.8 Point 2 as shown in Remark 5.37 for $\pi_\Gamma(g_{\tau_0}) \in X_{\mathrm{comp},1}$. $\square$

6. Example with low ε_Γ

In this section, we Prove Theorem 1.3. The section is divided into five subsections. In Subsection 6.1 we construct a nonarithmetic lattice Γ such that G/Γ is glued from two homogeneous subspaces $G/\Gamma_1, G/\Gamma_2$. In Subsection 6.2 we construct an orbits $H.x$ in G/Γ, which comes from a piece of a periodic orbit in G/Γ_1. Then we reduce the problem of evaluating $\delta(\mathrm{stab}_H(x))$ into two independent problems. One arithmetic and one geometric. We then solve them in Subsections 6.3 and 6.4 respectively.

6.1. **Construction of a lattice.** In this subsection, we will construct a sublattice $\Gamma < G$ and show that it is nonarithmetic.

General setting. Let $Q\left((x_i)_{i=1}^4\right) = x_1^2 - x_2^2 - x_3^2 - x_4^2$ be a quadratic form. Let W^{op} be $\mathbb{R}^4$ thought of as row vectors, on which $\mathrm{SL}_4(\mathbb{R})$ acts from the right, and we consider Q as a quadratic form on W^{op}. Here we will use

$$G = \mathrm{SO}(3,1)^0 = \{g \in \mathrm{SL}_4(\mathbb{R}) : Q(w.g) = Q(w),\, \forall w \in W^{\mathrm{op}}\}^0,$$

which is isogenic to $\mathrm{SL}_2(\mathbb{C})$, via the action of $\mathrm{SL}_2(\mathbb{C})$ on W as in Claim 5.31. Recall that $\mathbb{H}^3$ is a right G-space, where here we identify

$$\mathbb{H}^3 = \{w \in W^{\mathrm{op}} : Q(w) = 1, w_1 > 0\}.$$

Let $p_0 = (1,0,0,0) \in \mathbb{H}^3$, and note that $K_G := \mathrm{stab}_G(p_0)$ is the maximal compact subgroup in G and is a copy of $\mathrm{SO}(3)$ embedded in G by the action on the last 3 coordinates. Let $H = \mathrm{SO}(2,1)^0$, embedded in $\mathrm{SO}(3,1)^0$ by action of the first 3 coordinates. The H action preserves the sign of the last coordinate, that is, it preserves

$$\mathbb{H}^2 = \{v \in \mathbb{H}^3 : v_4 = 0\} \subset \mathbb{H}^3,$$
$$(\mathbb{H}^3)^{\pm} = \{v \in \mathbb{H}^3 : \pm v_4 > 0\} \subset \mathbb{H}^3,$$

The maximal compact subgroup in H is $K_H = K_G \cap H$ and is isomorphic to $\mathrm{SO}(2)$ acting by rotations on the second and third coordinates of W^{op}.

The arithmetic components. Recall that $\Gamma(7) = \ker(\mathrm{SL}_4(\mathbb{Z}) \to \mathrm{SL}_4(\mathbb{Z}/7)) < \mathrm{SL}_4(\mathbb{Z})$ is a finite index torsion free subgroup. Let $A_1, A_2 > 0$ be big integers $\equiv 1 \mod 8$ such that A_1/A_2 is not a rational square. Let

$$Q_i = 7x_1^2 - x_2^2 - x_3^2 - A_i x_4^2 \in \mathbb{Z}[x_1,x_2,x_3,x_4] \qquad \text{for } i = 1,2,$$

be quadratic forms on W^{op}. Define

$$\mathrm{SO}(Q_i,\mathbb{Z}) = \{\gamma \in \mathrm{SL}_4(\mathbb{Z}) : Q_i(w.\gamma) = Q_i(w)\ \forall w \in W\}.$$

This is a subgroup of $\mathrm{SO}(Q_i,\mathbb{R})$, which is a lattice in it by Borel and Harish-Chandra's Theorem [3]. Let $\mathrm{SO}(Q_i,\mathbb{Z})' = \mathrm{SO}(Q_i,\mathbb{Z}) \cap \Gamma(7)$. This is a lattice in $\mathrm{SO}(Q_i,\mathbb{R})$ and torsion-free. Let $\Gamma_i = g_i^{-1}\,\mathrm{SO}(Q_i,\mathbb{Z})' g_i$ where $g_i = \mathrm{diag}(\sqrt{7},1,1,\sqrt{A_i})$. Then Γ_i is a torsion-free lattice in G. Note that, $\Gamma_3 = H \cap \Gamma_1 = H \cap \Gamma_2$ is a lattice in H similarly constructed from the quadratic form $Q_3 = 7x_1^2 - x_2^2 - x_3^2$.

Claim 6.1. *For $i = 1,2$ we have* $\mathrm{vol}(G/\Gamma_i) = \Omega(A_i^{1/2})$.

The proof relies on a certain arithmetic aspect of Γ_i and is given in Subsection 6.3.

Claim 6.2. *The lattices Γ_1, Γ_2 are cocompact in G and Γ_3 is cocompact in H.*

Proof. For a quadratic form Q in d variables, the lattice $\mathrm{SO}(Q,\mathbb{Z})$ is cocompact if and only if $Q(v) \neq 0$ for all $v \in \mathbb{Q}^d \setminus \{0\}$ (see [42, Prop. 5.3.4]). Hence it is sufficient to show that $Q_i(v) \neq 0$ for all $v \in \mathbb{Q}^4$. However, one can normalize v so that $v \in \mathbb{Z}^4$ and one of its coordinates is odd. Then there are no solutions modulo 8. $\square$

Construction of a hybrid manifold. Fix $i = 1, 2$ and consider the manifold $M_i = \mathbb{H}^3/\Gamma_i$ the submanifold $V = \mathbb{H}^2/\Gamma_3$, and the cover $\bar{M} = \mathbb{H}^3/\Gamma_3$ of M_i. These are indeed manifolds since Γ_i has no torsion elements. Let $\bar{\rho} : \mathbb{H}^3 \to \bar{M}$, $\rho_i : \mathbb{H}^3 \to M_i$, $\tau_i : \bar{M} \to M_i$ for each i and $\rho_3 : \mathbb{H}^2 \to V$ denote the standard projections. We think of V as a subset of $\bar{M}$.

$$\begin{tikzcd} & \mathbb{H}^3 \arrow[dl, "\rho_1"'] \arrow[d, "\bar{\rho}"] \arrow[dr, "\rho_2"] & \\ M_1 & \bar{M} \arrow[l, two heads, "\tau_1"] \arrow[r, two heads, "\tau_2"'] & M_2 \end{tikzcd}$$

Claim 6.3. *The projection $\tau_i : \bar{M} \to M_i$ restricts to an embedding on V.*

$$\begin{tikzcd} & V \arrow[dl, hook'] \arrow[d, hook] \arrow[dr, hook] & \\ M_1 & \bar{M} \arrow[l, two heads, "\tau_1"] \arrow[r, two heads, "\tau_2"'] & M_2 \end{tikzcd}$$

The proof relies on a certain arithmetic aspect of Γ_i and is given in Subsection 6.3. Denote by $V_i = \tau_i(V)$. By Claim 6.3 this is a submanifold.

We can now describe a new hyperbolic threefold R.

Definition 6.4 (A hybrid manifold)**.** Cut M_i along V_i. The resulting manifold M_i^{cut} is a hyperbolic threefold with a hyperbolic surface boundary composed of two isometric copies of V_i, namely, V_i^+, V_i^-. Near $V_i^{\pm}$, the manifold M_i^{cut} is locally isometric to $\mathbb{H}^2 \sqcup (\mathbb{H}^3)^{\pm}$. Glue M_1^{cut} to M_2^{cut} by gluing V_1^+ to V_2^- and V_1^- to V_2^+. The resulting manifold is an orientable compact hyperbolic threefold R.

For $i = 1, 2$ the embeddings $\chi_i : M_i^{\mathrm{cut}} \to R$, and the projections $\sigma_i : M_i^{\mathrm{cut}} \to M_i$.

Connectivity of R.

Theorem 6.5. *For each $i = 1, 2$, the manifold $M_i \setminus V_i$ is connected provided that A_i is sufficiently large.*

Proof. Assume to the contrary that $M_i \setminus V_i$ is not connected. This implies that $M_i = V_i \sqcup M_i^+ \sqcup M_i^-$, where $M_i^{\pm}$ are the different connected components of $M_i \setminus V_i$. We will estimate $\mathrm{vol}(M_i^{\pm})$. Since the matrix $g_{-1} = \mathrm{diag}(1, 1, 1, -1)$ normalize Γ_i, it acts on M_i. Since the g_{-1} action replaces the two sides of V in $\bar{M}$, it replaces the two sides of V_i in M_i. Thus $\mathrm{vol}(M_i^+) = \mathrm{vol}(M_i^-) = \frac{1}{2}\mathrm{vol}(M_i)$. By Claim 6.1, $\mathrm{vol}(M_i^{\pm}) = \frac{1}{2}\mathrm{vol}(M_i) = \Omega(A_i^{1/2})$. This implies that the Cheeger constant

$$h(M_i) := \inf_{S \subseteq M_i} \frac{\mathrm{vol}(\partial S)}{\min(\mathrm{vol}(S), \mathrm{vol}(M_i \setminus S))} \le \frac{\mathrm{vol}(V_i)}{\min(\mathrm{vol}(M_i^+), \mathrm{vol}(M_i^-))} = O(A_i^{-1/2}).$$

By Burger's inequality [6], we deduce that $\lambda_1(M_i) = O(h(M_i)^2 + h(M_i)) = O(A_i^{-1/2})$, where $\lambda_1(M_i)$ is the minimal nontrivial eigenvalue of minus the laplacian operator $-\Delta$ on M_i. By Property (τ) for congruence subgroups in arithmetic groups (See [35, 20, 5, 8]), there is an absolute constant λ_0 such that $\lambda_1(M_i) \ge \lambda_0$. This contradicts our previous estimate $\lambda_1(M_i) = O(A_i^{-1/2})$, as desired. □

We conclude that R is connected.

Remark 6.6 (Avoiding property (τ)). The use of property (τ) is the least elementary piece of the arguments in this section and can be avoided, as Theorem 6.5 is not necessary to the proof, and is only provided to give the reader a better picture of R and simplify the terminology.

Since R is a connected compact hyperbolic threefold, we deduce that $R \cong \mathbb{H}^3/\Gamma$ for some cocompact lattice $\Gamma < G$, which is our desired nonarithmetic lattice. Since A_1/A_2 is not a square we get the following theorem.

Theorem 6.7 ([19, §2.9]). *The lattice Γ is non-arithmetic.*

6.2. **Reduction of Theorem 1.3 into arithmetic and hyperbolic questions.** In this section, we will reduce the construction of an element g as in Theorem 1.3 to an arithmetic question.

Definition 6.8. For every complete hyperbolic manifold M and a point $p \in M$ denote by Ray_p the collection of geodesic rays $\gamma : [0, \infty) \to M$ originating from $p = \gamma(0)$. The derivative at 0 gives a metric isomorphism $\mathrm{Ray}_p \cong S^{\dim M - 1}$.

Claim 6.9. *Let $\Lambda < H$ be a subgroup and $U \subseteq \mathbb{H}^2/\Lambda$ be a precompact open subset. Then for every $p \in U$ we have*

$$H.\dim(\{\gamma \in \mathrm{Ray}_p : \gamma(t) \in U \ \forall t \geq 0\}) \leq \delta(\Lambda).$$

Proof. Let $\pi_\Lambda : \mathbb{H}^2 \to \mathbb{H}^2/\Lambda$ denote the standard projection. Let $U_0 \subseteq \mathbb{H}^2$ be precompact open set so that $\pi_\Lambda(U_0) = U$ and denote $\tilde{U} = \pi_\Lambda^{-1}(U) = \bigcup_{\lambda \in \Lambda} U_0.\lambda$. Let $\Lambda' = \langle \lambda \in \Lambda : U_0.\lambda \cap U_0 \neq \emptyset \rangle$. Since U_0 is precompact and Λ discrete, it follows that the set of generators we wrote to Λ' is finite and hence Λ' is geometrically finite. We will use Sullivan [36, Thm. 1] to give a lower bound on $\delta(\Lambda') \leq \delta(\Lambda)$.

Let $\tilde{p} \in U_0$ be a preimage of p and note that there is a bijection between rays Ray_p and $\mathrm{Ray}_{\tilde{p}}$ that gives an equality of the Hausdorff dimensions

$$H.\dim\left(\{\gamma \in \mathrm{Ray}_p : \gamma(t) \in U \ \forall t \geq 0\}\right) = H.\dim\left(\left\{\gamma \in \mathrm{Ray}_{\tilde{p}} : \gamma(t) \in \tilde{U} \ \forall t \geq 0\right\}\right).$$

Denote $X = \left\{\gamma \in \mathrm{Ray}_{\tilde{p}} : \gamma(t) \in \tilde{U} \ \forall t \geq 0\right\}$ and let $\gamma \in X$. We will show that $\lim_{t\to\infty} \gamma(t) \in \partial\mathbb{H}^2$ in fact lies in the limit set $D(\Lambda')$.

Since $(\{t \in [0, \infty) : \gamma(t) \in U_0.\lambda\})_{\lambda \in \Lambda}$ is an open cover of $[0, \infty)$ by bounded sets, there is a sequence $t_0 = 0 < t_1 < t_2 < \dots$ such that $\lim_{j\to\infty} t_j = \infty$ and a sequence $(\lambda_j)_{j=0}^\infty \subseteq \Lambda$ so that $\lambda_0 = I$ and for all $j = 0, 1, \dots$ and for all $t \in [t_j, t_{j+1}]$ we have $\gamma(t) \in U_0.\lambda_j$. Note that for all $j = 1, 2, \dots$ we have that $\gamma(t_j) \in U_0.\lambda_j \cap U_0.\lambda_{j-1}$. Hence $\lambda_{j-1}\lambda_j^{-1} \in \Lambda'$. By induction we deduce that $\lambda_j \in \Lambda'$ for all $j \geq 0$. This implies that

$$\lim_{t\to\infty} \gamma(t) = \lim_{j\to\infty} \tilde{p}.\lambda_j \in D(\Lambda').$$

Hence the limit embeds X in $D(\Lambda')$ which implies that

$$H.\dim(X) \leq H.\dim(D(\lambda')) \overset{[36,\text{ Thm. }1]}{=} \delta(\Lambda') \leq \delta(\Lambda).$$

$\square$

Direct computation shows that the normalizer $N(H)$ of H is given by

$$N(H) = H \cup g_0 H, \qquad g_0 = \mathrm{diag}(1, 1, -1, -1).$$

Observation 6.10 (The relation between H-orbits and immersed hyperbolic surfaces)**.** Let $\pi_{K_G} : G/\Gamma \to \mathbb{H}^3/\Gamma$ be the standard projection. Any H-orbit $H.\pi_\Gamma(g)$ in G/Γ is projected to an immersion $\iota_g : \mathbb{H}^2/\Gamma'_g \to \mathbb{H}^3/\Gamma$, where $\Gamma'_g = g\Gamma g^{-1} \cap N(H)$. The immersion is not necessarily bijective, however, the self-intersection is a countable union of geodesics, that is, the set $\{p \in \mathbb{H}^2/\Gamma_g : \iota^{-1}(\iota(p)) \neq \{p\}\}$ is a countable union of geodesics in $\mathbb{H}^2/\Gamma'_g$.

On the other hand, every immersion of open hyperbolic surface $\iota_U : U \to \mathbb{H}^3/\Gamma$ that satisfies that the self-intersection is a countable union of geodesics factors as $\iota_U = \iota_g \circ \iota_0$ for some $g \in G$, where $\iota_0 : U \to \mathbb{H}^2/\Gamma'_g$ is an isometric embedding.

Definition 6.11 (Semi-periodic immersed hyperbolic surface)**.** Let $H.x_0$ be a periodic orbit in G/Γ_1, where $x_0 = \pi_{\Gamma_1}(g)$. Assume that it is not the periodic orbits $\pi_{\Gamma_1}(H)$ or $\pi_{\Gamma_1}(Hg_0)$, whose projection to M_1 lands in V_1. Let $\iota_0 : \mathbb{H}^2/\Lambda \to M_1$ denote the corresponding immersion of hyperbolic surface, where $\Lambda = (\Gamma_1)'_g = g\Gamma_1 g_1^{-1} \cap N(H)$ is a lattice in $N(H)$. Then $\mathbb{H}^2/\Lambda$ is a finite volume compact space. Let U_1 be a connected component of $(\mathbb{H}^2/\Lambda) \setminus \iota_0^{-1}(V_1)$. Then ι_0 restricts to an embedding $\iota_0|_{U_1} : U_1 \to M_1^{\mathrm{cut}}$. Let $\iota_1 = \chi_1 \circ \iota_0|_{U_1} : U_1 \to \mathbb{H}^3/\Lambda$. By Observation 6.10 the immersion ι_1 factors as $\iota_1 = \iota_{g_2} \circ \iota_2$ for some $g_2 \in G$, $\iota_2 : U_1 \to \mathbb{H}^2/\Gamma'_{g_2}$. Then $\iota_{g_2} : \mathbb{H}^2/\Gamma'_{g_2} \to \mathbb{H}^3/\Gamma$ is a semi-periodic surface.

Reduction of Theorem 1.3 into two propositions. Let $g_0, \Lambda, U_1, \iota_1, g_2, \iota_2$ as in Definition 6.11. Since V_1 is a hyperbolic surface in M_1 that differs from $\iota_0(\mathbb{H}^2/\Lambda)$, we deduce that $\iota_0^{-1}(V_1)$ is a union of geodesics in H/Λ, and hence U_1 has finite diameter. Hence $\iota_2(U_1)$ is precompact in $\mathbb{H}^2/\Gamma'_{g_2}$. Let $\rho : \mathbb{H}^2/\Gamma_{g_2} \to \mathbb{H}^2/\Gamma'_{g_2}$ denote the standard projection. It is a proper covering map of index at most 2. Hence $\rho^{-1}(\iota_2(U_1))$ is precompact in $\mathbb{H}^2/\Gamma_{g_2}$. Let $p \in \rho^{-1}(\iota_2(U_1))$. It follows that

$$\begin{aligned}
\delta(\Gamma_{g_2}) &\overset{6.9}{\geq} H.\dim\left(\left\{\gamma \in \mathrm{Ray}_p : \gamma(t) \in \rho^{-1}(\iota_2(U_1))\ \forall t \geq 0\right\}\right) \\
&= H.\dim\left(\left\{\gamma \in \mathrm{Ray}_{\rho(p)} : \gamma(t) \in \iota_2(U_1)\ \forall t \geq 0\right\}\right) \\
&= H.\dim\left(\left\{\gamma \in \mathrm{Ray}_{p'} : \gamma(t) \in U_1\ \forall t \geq 0\right\}\right).
\end{aligned} \tag{6.1}$$

for $p' = \iota_2\rho(p) \in U_1 \subseteq \mathbb{H}^2/\Lambda$. Let $\pi_\Lambda : \mathbb{H}^2 \to \mathbb{H}^2/\Lambda$ denote the projection. We will express the right-hand side of Eq. (6.1) by this universal cover. Let $\tilde{p} \in \pi_\Lambda^{-1}(p')$. Since U_1 is the connected component of $(\mathbb{H}^2/\Lambda) \setminus \iota_0^{-1}(V_1)$, we can lift each geodesic ray to the universal cover $\mathbb{H}^2$ of U_1 get an equality

$$H.\dim\left(\left\{\gamma \in \mathrm{Ray}_{p'} : \gamma(t) \in U_1\ \forall t \geq 0\right\}\right) = H.\dim\left(\left\{\gamma \in \mathrm{Ray}_{\tilde{p}} : \gamma(t) \notin \mathcal{L}\ \forall t \geq 0\right\}\right), \tag{6.2}$$

where $\mathcal{L} = \pi_\Lambda^{-1}(\iota_0^{-1}(V_1))$ is a union of lines. we now introduce the following propositions on $\mathcal{L}$.

Proposition 6.12. *Let $\zeta : \mathbb{H}^2 \to M_1$ be a locally isometric immersion. Then the set $\mathcal{L}_\zeta = \zeta^{-1}(V_1)$ is a union of hyperbolic lines such that for every two geodesic lines $\ell_1 \neq \ell_2 \subseteq \mathcal{L}$ we have $d_{\mathbb{H}^2}(\ell_1, \ell_2) > \frac{1}{2}\log A_1 + O(1)$.*

Proposition 6.13. *Let $\mathcal{L} \subseteq \mathbb{H}^2$ be a union of lines so that for every two geodesic lines $\ell_1 \neq \ell_2 \subseteq \mathcal{L}$ we have $d_{\mathbb{H}^2}(\ell_1, \ell_2) > \mathcal{A}$. Then*

$$H.\dim\left(\left\{\gamma \in \mathrm{Ray}_p : \gamma(t) \notin \mathcal{L}\ \forall t \geq 0\right\}\right) > 1 - O(1/\mathcal{A}),$$

for every $p \in \mathbb{H}^2 \setminus \mathcal{L}$.

The combination of Propositions 6.12 and 6.13, together with Eqs. (6.1) and (6.2), show that $\delta(\Gamma_{g_2}) > 1 - O(1/\log A_1)$.

Leaving the proofs of these propositions to the next subsections, it is left to find $\iota_0, g_0, \Lambda, U_1, \iota_1, g_2, \iota_2$ as in Definition 6.11 so that Γ_{g_2} is not periodic. It follows from [16, Thm. 4.1] or [2, Prop. 12.1.], that if $\iota_0(\mathbb{H}^2/\Lambda)$ intersects V_1 non-orthogonally then Γ_{g_2} is not periodic. Such an immersion exists by the density of closed H-orbits in G/Γ_1.

□

6.3. **Behavior of the arithmetic space near the cutting plane.** In this section we prove Proposition 6.12, as well as claims 6.3 and 6.1. We begin the section by linearising the distance from a hyperbolic plane in $\mathbb{H}^3$.

Linearization of the distance from a hyperbolic plane.

Definition 6.14 (The representation W)**.** Let $W \cong \mathbb{R}^4$ denote the standard representation of $\mathrm{SL}_4(\mathbb{R})$ on which it acts from the left. Note that the quadratic form $Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 - x_3^2 - x_4^2$ is preserved by the G action (this time thought of as a quadratic form on W), similarly to the case with W^{op}.

Observation 6.15 (Identifying $\mathbb{H}^2$ in $\mathbb{H}^3$)**.** Let $\pi_{K_G} : G \to \mathbb{H}^3$ denote the standard projection. Note that $\mathrm{stab}_G(w_0) = H$ and

$$K_G.w_0 = \{(0, w_2, w_3, w_4)^t : w_2^2 + w_3^2 + w_4^2 = 1\} = \{w \in W : Q(w) = -1, \|w\| = 1\}.$$

Hence

$$\{g \in G : \|g.w_0\| = 1\} = \{g \in G : g.w_0 \in K_G.w_0\} = K_g H.$$

Hence $K_G H = \pi_{K_G}^{-1}(\mathbb{H}^2)$.

Definition 6.16 (Hyperbolic geometry relative to a hyperbolic plane)**.** Let $\varphi : \mathbb{H}^3 \to \mathbb{R}$ be the signed distance form $\mathbb{H}^2$, that is,

$$\varphi(p) = \begin{cases} d_{\mathbb{H}^3}(p, \mathbb{H}^2), & \text{if } p \in (\mathbb{H}^3)^+, \\ -d_{\mathbb{H}^3}(p, \mathbb{H}^2), & \text{if } p \in (\mathbb{H}^3)^-, \\ 0, & \text{if } p \in \mathbb{H}^2. \end{cases}$$

Note that φ is differentiable and the gradient is of fixed size 1.

Through every point $p \in \mathbb{H}^2$ passes a unique geodesic $\xi_p : \mathbb{R} \to \mathbb{H}^3$ with $\xi_p(0) = p$, which is orthogonal to $\mathbb{H}^2$ and oriented towards $(\mathbb{H}^3)^+$. These geodesics forms a foliation of $\mathbb{H}^3$, and satisfy $\varphi(\xi_p(t)) = t$ for all $p \in \mathbb{H}^2, t \in \mathbb{R}$. For every $h \in H$ we have that $\xi_{p.h} = \xi_p.h$. Recall $w_0 = (0, 0, 0, 1)^t \in W$, and define $\psi : G \to \mathbb{R}$ by $\psi(g) = (g.w_0)_1$. Since ψ is invariant from the left to K_H it descends to a map $\psi : \mathbb{H}^3 \to \mathbb{R}$.

Claim 6.17. *For every $p \in \mathbb{H}^3$ we have $\sinh(\varphi(p)) = \psi(p)$.*

Proof. Both functions $p \mapsto \psi(p)$ and $p \mapsto \sinh(\varphi(\pi_{K_G}(p)))$ are invariant from the right to H, which allows us to test this equality only on points of the form $\xi_{p_0}(t) = (\sinh(t), 0, 0, \cosh(t))$, on which the equality holds. □

Corollary 6.18. *Let $\mathbb{H}^2.g_1$ be a hyperbolic plane and $p_2 = \pi_{K_G}(g_2)$ be a point. Then $d_{\mathbb{H}^3}(p_2, \mathbb{H}^2.g_1) = \log \|g_2 g_1^{-1}.w_0\| + O(1)$.*

Proof. Using Claim 6.17 we deduce that

$$d_{\mathbb{H}^3}(p_2,\mathbb{H}^2.g_1) = d_{\mathbb{H}^3}(p_2.g_1^{-1},\mathbb{H}^2) = |\varphi(p_2.g_1^{-1})|$$
$$\overset{6.17}{=} |\sinh^{-1}(\psi(g_2g_1^{-1}))| = |\sinh^{-1}((g_2g_1^{-1}.w_0)_1)|.$$

The result follows from the fact that for every vector $w \in W$ with $Q(w) = -1$ we have

$$|\sinh^{-1}(w_1)| = \log\|w\| + O(1),$$

which is a direct computation. □

Denote by C_0 the implicit constant in Corollary 6.18 so that

$$\left|d_{\mathbb{H}^3}(p_2,\mathbb{H}^2.g_1) - \log\|g_2g_1^{-1}.w_0\|\right| \le C_0. \tag{6.3}$$

Finally, we prove the following claim

Claim 6.19. *Let $v \in W$ so that $Q(v) = -1$. Then there are $k, k' \in K_G$ and $t \in \mathbb{R}$ such that $k'\mathsf{a}(t)k.v = w_0$ and $\cosh(t) \le \|v\|$.*

Proof. Express $v = (v_1, v_2, v_3, v_4)^t$. For some $k \in K_G$, we have $k.v = (v_1, v_2', 0, 0)^t$ with $v_2' > 0$ and $v_1^2 - (v_2')^2 = -1$. This implies that for some $t' > 0$ we have $v_1 = \sinh t'$ and $v_2' = \cosh(t')$. In particular $\cosh(t') < \|v\|$ and $\mathsf{a}(-t')k.v = (0,1,0,0)^t$. The desired follows for $t = -t'$ and $k' = \left(\begin{smallmatrix} 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0\\ 0&1&0&0 \end{smallmatrix}\right)$. □

Arithmetic properties. Fix $i = 1, 2$ for the entire subsection. Consider the vector $w_0 = (0,0,0,1)^t \in W$ and the set $W_{\Gamma_i} = \Gamma_i.w_0 \subset W$.

Claim 6.20. *Let $v \in W_{\Gamma_i}$. Then either $v = w_0$, or $\|v\| \ge \sqrt{A_i/7}$.*

Proof. Let $W_{\mathbb{Z}} \cong \mathbb{Z}^4$ be the integer vectors in $W \cong \mathbb{R}^4$. By the definition of Γ_i we deduce that Γ_i preserves that lattice $W_{\mathbb{Z},i} = \sqrt{A_i}g_i^{-1}W_{\mathbb{Z}} \subset W$, where $g_i = \operatorname{diag}(\sqrt{7}, 1, 1, \sqrt{A_i})$.

Let $v \in W_{\Gamma_i} \setminus \{w_0\} \subseteq W_{\mathbb{Z},i}$. Note that $Q(v) = -1$. Assume to the contrary that $v \in \mathbb{R}w_0$. Then we must have $v = -w_0$. However, since Γ_i-s action on $W_{\mathbb{Z},i}$ descends to a trivial action on $W_{\mathbb{Z},i}/7W_{\mathbb{Z},i}$, we deduce that $W_{\Gamma_i} \cap -W_{\Gamma_i} = \emptyset$, which contradicts $v = -w_0$. Hence we have $v \notin \mathbb{R}w_0$. Let $j = 1, 2, 3$ be an index satisfying $(v)_j \ne 0$. Then since $v \in W_{\mathbb{Z},i}$ we must have $|(v)_j| \ge \begin{cases} \sqrt{A_i}, & \text{if } j = 2, 3, \\ \sqrt{A_i}/\sqrt{7}, & \text{if } j = 1. \end{cases}$. Hence $\|v\| \ge \sqrt{A_i/7}$. □

Claim 6.21. *There is $c > 0$ independent of A_i such that for all $g \in G$ there is at most one $z \in g.W_{\Gamma_i}$ such that $\|z\| < cA_i^{1/4}$.*

Proof. Suppose that $z \ne z' \in g.W_{\Gamma_i}$ and $\|z\|, \|z'\| < cA_i^{1/4}$. Assume that

$$z = g\gamma.w_0 \quad\text{and}\quad z' = g\gamma'.w_0. \tag{6.4}$$

Every $z \in g.W_{\Gamma_i}$ has $Q(z) = -1$. Applying Claim 6.19 to z, we get that there are $k, k' \in K_G$ and $t \in \mathbb{R}$ with $\cosh(t) \le \|z\| \le cA_i^{1/4}$ such that $k'\mathsf{a}(t)k.z = w_0$. In particular, $e^t < 2cA_i^{1/4}$. Substituting $z = g\gamma.w_0$ to the previous equality, we obtain $k'\mathsf{a}(t)kg\gamma.w_0 = w_0$. Using $H = \operatorname{stab}_G(w_0)$, we deduce that $k'\mathsf{a}(t)kg\gamma \in H$.

Since $\Gamma_i \cap H = \Gamma_3$ is cocompact in H (see Claim 6.2) and independent of A_i, there is $C > 0$ such that for every $h \in H$ there is $\gamma_3 \in H \cap \Gamma_i$ such that

$\|\gamma_3 h\|_{\mathrm{op}} < C$. Here we use the operator norm defined by the action on W. Applying this to $(k'\mathsf{a}(t)kg\gamma)^{-1} \in H$, we deduce that for some $\gamma_3 \in H \cap \Gamma_i$ we have $\|\gamma_3(k'\mathsf{a}(t)kg\gamma)^{-1}\|_{\mathrm{op}} < C$. Set

$$h = \gamma_3(k'\mathsf{a}(t)kg\gamma)^{-1}, \quad \text{satisfying} \quad \|h\|_{\mathrm{op}} < C. \tag{6.5}$$

Then $hk'\mathsf{a}(t)k.z = w_0$. Denote $v = hk'\mathsf{a}(t)k.z = w_0$ and $v' = hk'\mathsf{a}(t)k.z'$. We will now estimate v'. Note that

$$v' = hk'\mathsf{a}(t)k.z' \overset{(6.4)}{=} hk'\mathsf{a}(t)kg\gamma'.w_0 \overset{(6.5)}{=} \gamma_3\gamma^{-1}\gamma'.w_0 \in W_{\Gamma_i}.$$

By Claim 6.20, we obtain that $\|v'\| \geq \sqrt{A_i/7}$. On the other hand,

$$\|v'\| \leq \|z'\| \cdot \|k\mathsf{a}(t)k'h\|_{\mathrm{op}} \leq cA_i^{1/4}\|h\|_{\mathrm{op}}e^t \leq cA_i^{1/4} \cdot C \cdot 2cA_i^{1/4} \leq 2c^2CA_i^{1/2}.$$

Therefore, choosing c for which $c^2 < \frac{1}{2C\sqrt{7}}$, we obtain a contradiction to Claim 6.20 and the desired uniqueness follows. $\square$

Claim 6.22. *Let $\mathcal{A} = \frac{1}{4}\log A_i - C_0 + \log c = \Theta(\log A_i)$, wehre C_0 is as in Eq. (6.3) and c as in Claim 6.21. Define*

$$S = \{p \in \mathbb{H}^3 : d(p, \mathbb{H}^2) < \mathcal{A}\}.$$

Then for every $\gamma \in \Gamma_i$, either

(1) $\gamma \in \Gamma_3$ and then $S.\gamma = S$,
(2) or $\gamma \notin H$ and then $S.\gamma \cap S = \emptyset$.

Proof. If $\gamma \in H$ then the first option holds. If $\gamma \notin H$, assume to the contrary that $\pi_{K_G}(g) \in S.\gamma \cap S$. Then

$$\log\|g.w_0\| \overset{(6.3)}{\leq} d(\pi_{K_G}(g), \mathbb{H}^2) + C_0 < \mathcal{A} + C_0 = \frac{1}{4}\log A_i + \log c$$

Similarly,

$$\log\|g\gamma^{-1}.w_0\| \overset{(6.3)}{\leq} d(\pi_{K_G}(g), \mathbb{H}^2.\gamma) + C_0 < \mathcal{A} + C_0 = \frac{1}{4}\log A_i + \log c$$

Hence, by Claim 6.21 we deduce that $g\gamma^{-1}.w_0 = g.w_0$, which implies that $\gamma \in H$. This contradicts the assumption and completes the proof. $\square$

The following corollary is immediate.

Corollary 6.23. *For every two different hyperbolic planes $\mathbb{H}^2.\gamma, \mathbb{H}^2.\gamma'$ for $\gamma, \gamma' \in \Gamma_i$ we have $d_{\mathbb{H}^3}(\mathbb{H}^2.\gamma, \mathbb{H}^2.\gamma') \geq 2\mathcal{A}$.* $\square$

Corollary 6.24 (Strengthening of Claim 6.3)**.** *Recall the projection $\bar{\rho} : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_3 = \bar{M}$ and recall the standard projection $\tau_i : \bar{M} \to M_i$. Let $\bar{S} = \bar{\rho}(S)$. Then $\tau_i|_{\bar{S}}$ is one to one.*

Proof. Assume that for $p_1, p_2 \in \bar{S}$ we have $\tau_i(p_1) = \tau_i(p_2)$. Choose $\tilde{p}_j \in \bar{\rho}^{-1}(p_j)$ for $j = 1, 2$. Since $\tau_i \circ \bar{\rho} = \pi_{\Gamma_i}$ agrees on $\tilde{p}_1, \tilde{p}_2$ we deduce that for some $\gamma \in \Gamma_i$ we have $\tilde{p}_1 = \tilde{p}_2.\gamma$. Hence $\tilde{p}_1 \in S \cap S.\gamma$. Claim 6.22 implies that $\gamma \in \Gamma_3$, which in turn implies that $p_1 = p_2$. This proves the injectivity of τ_i on $\bar{S}$. $\square$

Proof of Claim 6.1. In view of Corollary 6.24 it is sufficient to show that $\mathrm{vol}(\bar{S}) = \Omega(A_i^{1/2})$.

Recall the map $\varphi : \mathbb{H}^3 \to \mathbb{R}$ form definition 6.16. Its gradient is of fixed size 1. This implies that for every set $\Omega \subseteq \mathbb{H}^3$,

$$\mathrm{Vol}(\Omega) = \int_{-\infty}^{\infty} \mathrm{Area}\left(\Omega \cap \varphi^{-1}(t)\right) \mathrm{d}t. \tag{6.6}$$

Recall the foliation $\{\xi_p : p \in \mathbb{H}^2\}$ of $\mathbb{H}^3$ form definition 6.16. This gives a parametrization $\mathbb{H}^2 \to \varphi^{-1}(t_0)$ for every $t_0 \in \mathbb{R}$ by $p \mapsto \xi_p(t_0)$. This parametrization can be seen to expand the Riemannian metric by $\cosh(t)$. Therefore, for every $\Omega \subseteq \mathbb{H}^2, t_0 \geq 1$,

(6.7)

$$\mathrm{vol}(\{\xi_p(t) : t \in [-t_0, t_0], p \in \Omega\}) = \mathrm{Area}(\Omega) \int_{-t_0}^{t_0} \cosh^2(t)\mathrm{d}t = \Theta(e^{2t_0}\mathrm{Area}(\Omega)).$$

The function φ is H invariant and hence descends to a function $\bar{\varphi} : \mathbb{H}^3/\Gamma_3 = \bar{M} \to \mathbb{R}$. For every $h \in H$, we have that $\xi_{p.h} = \xi_p.h$. Thus, the foliation $\xi_\bullet$ descends to $\bar{M}$ as follows: for every $q \in V = \mathbb{H}^2/\Gamma_3$ there is a geodesic $\bar{\xi}_q : \mathbb{R} \to \bar{M}$, and these geodesics form a foliation of $\bar{M}$. Choosing a fundamental domain $\Omega \subseteq \mathbb{H}^2$ to $V = \mathbb{H}^2/\Gamma_3$ we deduce from Eq. (6.7) that

$$\begin{aligned}\mathrm{vol}(\bar{S}) &= \mathrm{vol}(\{\bar{\xi}_q(t) : t \in (-\mathcal{A}, \mathcal{A}), q \in V\}) = \mathrm{vol}(\{\xi_p(t) : t \in (-\mathcal{A}, \mathcal{A}), p \in \Omega\}) \\ &= \Theta(e^{2\mathcal{A}}\mathrm{Area}(\Omega)) = \Theta(\sqrt{A}_i\mathrm{Area}(V)).\end{aligned}$$

Since $\mathrm{Area}(V)$ is fixed we deduce that $\mathrm{vol}(\bar{S}) = \Theta(\sqrt{A}_i)$. By Corollary 6.24 we obtain $\mathrm{vol}(M_i) = \Omega(\sqrt{A}_i)$. The equality $\mathrm{vol}(G/\Gamma_i) = \Omega(\mathrm{vol}(M_i))$ completes the proof. □

Proof of Proposition 6.12. Let $\zeta : \mathbb{H}^2 \to M_i$ be a locally isometric immersion. Recall the standard projection $\rho_i : \mathbb{H}^3 \to M_i$. Then ζ factors as $\zeta = \rho_i \circ \tilde{\zeta}$ for some isometric embedding $\tilde{\zeta} : \mathbb{H}^2 \to \mathbb{H}^3$.

Note that since $V_i = \rho_i(\mathbb{H}^2)$ we have

$$\rho_i^{-1}(V_i) = \rho_i^{-1}(\rho_i(\mathbb{H}^2)) = \bigcup_{\gamma \in \Gamma_i} \mathbb{H}^2.\gamma.$$

Hence

$$\zeta^{-1}(V_i) = \bigcup_{\gamma \in \Gamma_i} \tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma)$$

This is a representation of $\zeta^{-1}(V_i)$ as a union of lines. To complete the proof we need to show that for every γ, γ' for which $\tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma) \neq \tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma')$ we have

$$d_{\mathbb{H}^2}(\tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma), \tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma')) \geq 2\frac{1}{2}\log A_i + O(1).$$

However, since $\tilde{\zeta}^{-1}$ is an isometric embedding we obtain

$$d_{\mathbb{H}^2}(\tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma), \tilde{\zeta}^{-1}(\mathbb{H}^2.\gamma')) \geq d_{\mathbb{H}^3}(\mathbb{H}^2.\gamma, \mathbb{H}^2.\gamma') \overset{6.23}{\geq} 2\mathcal{A}.$$

□

6.4. **Proof of Proposition 6.13.** To prove Proposition 6.13, we will first rephrase it as a question on an estimate of the Hausdorff dimension of a certain Cantor set, and then bound it.

Reformulation of Proposition 6.13. Let $\mathcal{L} = \bigcup_{\ell \in L} \ell$ such that for every $\ell_1, \ell_2 \in L$ we have $d_{\mathbb{H}^2}(\ell_1, \ell_2) \geq \mathcal{A}$. We may assume without loss of generality that $\mathcal{A} \geq 10$. Let $p \in \mathbb{H}^2 \setminus \mathcal{L}$. Denote by U the connected component of $\mathbb{H}^2 \setminus \mathcal{L}$ containing p. Denote by $L' \subseteq L$ the collection of lines composing the boundary of U. Denote by $D(U) \subseteq \partial\mathbb{H}^2$ the limit set of U. Since U is convex,

$$D(U) = \{q \in \partial\mathbb{H}^2 : [p, q) \subseteq U\}.$$

Hence we have an equality of Hausdorff dimensions

$$\begin{aligned} H.\dim(\{\gamma \in \mathrm{Ray}_p : \gamma(t) \notin \mathcal{L}\ \forall t \geq 0\}) &= H.\dim(\{\gamma \in \mathrm{Ray}_p : \gamma(t) \in U\ \forall t \geq 0\}) \\ &= H.\dim(D(U)). \end{aligned}$$

For every geodesic line $\ell \subseteq \mathbb{H}^2 \setminus \{p\}$ denote by $x_\ell, y_\ell \in \partial\mathbb{H}^2$ the limit points of ℓ so that the ray $[p, x_\ell)$ can be rotated less then π degrees counterclockwise about p to obtain $[p, y_\ell)$. Denote by $I_\ell \subset \partial\mathbb{H}^2$ the open interval with the boundary points x_ℓ and y_ℓ, which lies on the other side of ℓ than p. Then $D(U) = \partial\mathbb{H}^2 \setminus \bigcup_{\ell \in L'} I_L$. The intervals I_L are disjoint.

Claim 6.25. *If ℓ_1, ℓ_2 are nonintersecting lines in $\mathbb{H}^2 \setminus \{p\}$ then*

$$\sinh(d(\ell_1, \ell_2)/2) = \sqrt{|[x_{\ell_1}, x_{\ell_2}; y_{\ell_2}, y_{\ell_1}]|}, \tag{6.8}$$

where $[a, b; c, d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ is the cross ratio on $\mathbb{P}^1_{\mathbb{R}} \cong \partial\mathbb{H}^2$.

Proof. The choice to labeling of the limit points of ℓ_1, ℓ_2 ensures that and $x_{\ell_1}, y_{\ell_1}, x_{\ell_2}, y_{\ell_2}$ are in this circular order on $\partial\mathbb{H}^2$. Up to an isometry, we may assume that

$$x_{\ell_1} = e^t, y_{\ell_1} = -e^t, x_{\ell_2} = -1, y_{\ell_2} = 1,$$

where $t = d_{\mathbb{H}^2}(\ell_1, \ell_2)$. Then Eq. (6.8) is a direct computation. □

Identify $\mathbb{H}^2$ with the Poincaré half-plane model in $\mathbb{C} \cup \{\infty\}$. Sample one $\ell_0 \in L'$. Up to an isometry we may assume that $x_{\ell_0} = 1, y_{\ell_0} = 0$ so that $|p - 1/2| < 1/2$ and $I_{\ell_0} = \mathbb{P}^1_{\mathbb{R}} \setminus [0, 1]$. Let $L'' = L' \setminus \{\ell_0\}$. Then $I_\ell = (x_\ell, y_\ell)$ for every $\ell \in L''$ and $D(U) = [0, 1] \setminus \bigcup_{\ell \in L}(x_\ell, y_\ell)$, where

- L-a) for all $\ell \in L''$ we have $x_\ell < y_\ell \in (0, 1)$;
- L-b) for all $\ell \in L''$ we have $\frac{x_\ell(1-y_\ell)}{x_\ell - y_\ell} \geq \sinh(\mathcal{A}/2)^2$;
- L-c) for all $\ell_1, \ell_2 \in L''$ we have $\frac{(x_{\ell_2} - y_{\ell_1})(y_{\ell_2} - x_{\ell_1})}{(y_{\ell_1} - x_{\ell_1})(y_{\ell_2} - x_{\ell_2})} \geq \sinh(\mathcal{A}/2)^2$.

Denote $\mathcal{A}' = \sinh(\mathcal{A}/2)^2 > 5000$.

Lower bound on the dimension of the Cantor set $D(U)$.

Observation 6.26. The function $a, b, c \mapsto \frac{b(a+b+c)}{ac}$ is monotone increasing in b and monotone decreasing in a, c whenever $a, b, c > 0$.

Definition 6.27 (A random variable in z in $D(U)$)**.** We construct a random sequence of decreasing intervals $[0, 1] = J_0 \supset J_1 \supset J_2 \supset \cdots$ such that J_{k+1} is one of the three thirds of J_k for every k. That is, if $J_k = [a_k, a_k + 3^{-k}]$, then $J_{k+1} = [a_{k+1}, a_{k+1} + 3^{-k-1}]$ for some $a_{k+1} \in \{a_k, a_k + 3^{-k-1}, a_k + 2 \cdot 3^{-k-1}\}$. We will show how to sample iteratively $J_1, J_2, J_3, \ldots$ so that so that for every $\ell \in L'', k \geq 0$ we have

$$|J_k \cap I_\ell| < 3^{-k-1}. \tag{6.9}$$

Note that Eq. (6.9) is satisfied for $J_0 = [0,1]$ as for every $\ell \in L''$ we have

$$|J_k \cap I_\ell| = x_\ell - y_\ell < \frac{x_\ell - y_\ell}{x_\ell(1-y_\ell)} \overset{(L-b)}{\leq} \frac{1}{\mathcal{A}'} \leq \frac{1}{3}.$$

Suppose that we have constructed J_k that satisfies Eq. (6.9). We say that J_k is a *regular interval* if for all $\ell \in L''$ we have $|J_k \cap I_\ell| < 3^{-k-2}$, and *irregular interval* otherwise. If J_k is a regular interval, we may choose each of the three thirds of J_k to be J_{k+1}. We sample J_{k+1} uniformly from these three thirds.

Claim 6.28. *If J_k is irregular, then the interval $\ell \in L''$ with $|J_k \cap I_\ell| \geq 3^{-k-2}$ is unique.*

Proof. Otherwise there are $\ell_1 \neq \ell_2 \in L''$ with $|J_k \cap I_{\ell_i}| \geq 3^{-k-2}$ for $i = 1, 2$. This implies that $y_{\ell_i} - x_{\ell_i} \geq 3^{-k-2}$. Assume without loss of generality $x_{\ell_2} > y_{\ell_1}$. Then since the two intervals intersect J_k we get that $x_{\ell_2} - y_{\ell_1} < 3^k$. Then

$$\mathcal{A}' \leq \frac{(x_{\ell_2} - y_{\ell_1})(y_{\ell_2} - x_{\ell_1})}{(y_{\ell_1} - x_{\ell_1})(y_{\ell_2} - x_{\ell_2})} \leq \frac{3^{-k} \cdot (2 \cdot 3^{-k-2} + 3^{-k})}{3^{-2(k+2)}} = 99$$

which is a contradiction. The last inequality follows from Observation 6.26 applied to $a = y_{\ell_1} - x_{\ell_1} \geq 3^{-k-2}$, $b = x_{\ell_2} - y_{\ell_1} \leq 3^{-k}$, $c = y_{\ell_2} - x_{\ell_2} \geq 3^{-k-2}$. □

Consequently, if J_k is an irregular interval, then there is a unique $\ell_k \in L''$ such that $|J_k \cap I_{\ell_k}| \geq 3^{-k-2}$. By Eq. (6.9) we obtain that $|J_k \cap I_{\ell_k}| \in [3^{-k-2}, 3^{-k-1})$. Hence at least one of the three thirds J of J_k satisfies $J \cap I_{\ell_k} = \emptyset$ and hence we choose J_{k+1} uniformly among these intervals. For every $\ell \in L''$, eihter $\ell = \ell_k$ and then $J_{k+1} \cap I_\ell = \emptyset$, or $\ell \neq \ell_k$, and then

$$|J_{k+1} \cap I_\ell| \leq |J_k \cap I_\ell| \leq 3^{-k-2}.$$

Hence J_{k+1} satisfies Eq. (6.9), as desired for the iterative process to continue. Let z be the unique element in $\bigcap_{k=0}^\infty J_k$.

Claim 6.29. *Sample z as in Definition 6.27. Then $z \in D(U)$.*

Proof. By its definition $z \in [0,1]$. Suppose that $z \in I_\ell$ for some $\ell \in L''$. Then since I_ℓ is open, for some k we have $J_k \subseteq I_\ell$. However, by Eq. (6.9) we have $|J_k \cap I_\ell| < 3^{-k-1} < 3^{-k} = |J_k|$. This contradicts $J_k \subseteq I_\ell$ and hence $z \in [0,1] \setminus \bigcup_{\ell \in L} I_\ell = D(U)$. □

Claim 6.30. *For every $J = [a/3^m, (a+1)/3^m]$ we have*

$$\mathbb{P}(J_m = J) < 3^{-(1-1/\mathcal{A}'')m+1},$$

where $\mathcal{A}'' = \log_3 \mathcal{A}' - 5 > 2$.

Proof. Let $F_m = \{k = 0, \dots, m-1 : J_k \text{ is an irregular interval}\}$. Let $k_1 < k_2 \in F_m$. Then $I_{\ell_{k_1}} \cap J_{k_1} \neq \emptyset$, $I_{\ell_{k_1}} \cap J_{k_1+1} = \emptyset$ and $I_{\ell_{k_2}} \cap J_{k_2} \neq \emptyset$. This implies that $\ell_{k_1} \neq \ell_{k_2}$. Note that $|I_{\ell_{k_1}}| \geq 3^{-k_1-2}, |I_{\ell_{k_2}}| \geq 3^{-k_2-2}$ and since both intervals intersect J_{k_1} we deduce that $d_{\mathbb{R}}(I_{\ell_{k_1}}, I_{\ell_{k_2}}) < 3^{-k_1}$. Applying Observation 6.26 and Point (L-c) to $I_{\ell_{k_1}}, I_{\ell_{k_2}}$ we deduce that

$$\mathcal{A}' \leq \frac{3^{-k_1}(3^{-k_1} + 3^{-k_1-2} + 3^{-k_2-2})}{3^{-k_1-2} \cdot 3^{-k_2-2}} \leq 3^5 \cdot 3^{k_2-k_1}.$$

Hence $k_2 - k_1 \geq \log_3 \mathcal{A}' - 5$. Therefore, $\#F_m < m/(\log_3 \mathcal{A}' - 5) + 1$. Note that when sampling J_{k+1} iteratively for $k = 0, ..., m-1$, if $k \notin F_m$ then J_{k+1} is sampled uniformly between the the three options. Hence the probability J_m was sampled is at most

$$\frac{1}{3^{m-\#F_m}} \le \frac{1}{3^{m-m/(\log_3 \mathcal{A}'-5)-1}}.$$

$\square$

Consequently, for every $J = [a/3^m, (a+1)/3^m]$ we have $\mathbb{P}(z \in J) \le 3^{-(1-1/\mathcal{A}'')m+1}$. This fact, together with Claim 6.29 and standard covering arguments shows that

$$H.\dim(D(U)) \ge 1 - 1/\mathcal{A}'' = 1 - O(1/\log \mathcal{A}') = 1 - O(1/\mathcal{A}).$$

This concludes the proof of Proposition 6.13.

Appendix A. Proofs of critical exponent-related results

Here we gather a few proofs that are fairly standard, and unrelated to the rest of the paper.

Proof of Theorem 4.2. Recall the Laplace-Beltrami operator Δ on $C^\infty(\Lambda\backslash\mathbb{H}^2)$. It is self-adjoint. The Elstrodt formula (see [13, 14, 15, 27, 37, 9]) relats the maximal eigenvalue $\lambda_0(\Delta)$ with the critical exponent as follows,

$$-\lambda_0(\Delta) = \begin{cases} \frac{1}{4}, & \text{if } \delta(\Lambda) \le \frac{1}{2} \\ \delta(\Lambda)(1-\delta(\Lambda)) & \text{if } \delta(\Lambda) \ge \frac{1}{2}. \end{cases} \tag{A.1}$$

We deduce that $-\lambda_0(\Delta) \ge \varepsilon(1-\varepsilon) \ge \varepsilon/2$.

Let x_0 be the $SO(2)$-invariant point in the hyperbolic plane $\mathbb{H}^2$, and $\pi^\Lambda : \mathbb{H}^2 \to \Lambda\backslash\mathbb{H}^2$ the quotient map. Note that π^Λ locally preserves the measure, hence defines a map $\pi^\Lambda_* : L^1(\mathbb{H}^2) \to L^1(\Lambda\backslash\mathbb{H}^2)$ by $(\pi^\Lambda_* f)(x) = \sum_{y\in(\pi^\Lambda)^{-1}(x)} f(y)$. One can see that π^Λ_* commutes with $e^{t\Delta}$.

To do so, let $f = \mathbb{1}(B_{\mathbb{H}^2}(1))$. This is an element of $L^2(\mathbb{H}^2)$ with $\|f\|^2 \asymp 1$. The computation of the Heat kernel [24] shows that

$$e^{t\Delta} f(x) \asymp \frac{\rho+1}{t^{3/2}} e^{-\frac{(t+\rho)^2}{4t}}, \qquad \text{where } \rho = d_{\mathbb{H}^2}(x, x_0). \tag{A.2}$$

Let $g = \pi^\Lambda_* f$, and estimate $\|g\|^2$.

$$\begin{aligned} \|g\|^2 &= \int_{\Lambda\backslash\mathbb{H}^2} g(p)^2 \mathrm{d}m_{\Lambda\backslash\mathbb{H}^2}(p) = \int_{B_{\mathbb{H}^2}(1)} g(\pi^\Lambda(q))\mathrm{d}m_{\mathbb{H}^2}(q) \\ &\ll \max_{q\in B_{\mathbb{H}^2}(1)} g(\pi^\Lambda(q)). \end{aligned} \tag{A.3}$$

For every $q \in B_{\mathbb{H}^2}(1)$,

$$\begin{aligned} g(\pi_\Lambda(q)) &= \#\{g \in \Lambda : gq \in B_{\mathbb{H}^2}(1)\} \\ &\le \#\Lambda \cap \{g \in \mathrm{SL}_2(\mathbb{R}) : gB_{\mathbb{H}^2}(1) \cap B_{\mathbb{H}^2}(1) \ne \emptyset\} = \#\Lambda \cap K, \end{aligned}$$

where K is a compact set. The assumption on the minimal element in Λ promises that $g(\pi_\Lambda(q)) \ll e^{0.3\varepsilon R}$, and hence $\|g\|^2 \ll e^{0.3\varepsilon R}$.

To prove Eq. (4.1), we will estimate $\langle g, e^{R\Delta} g\rangle$. On the one hand,

$$\langle g, e^{R\Delta} g\rangle \le \|g\|^2 e^{R\lambda_0(\Delta)} \le e^{0.3\varepsilon R} e^{R\lambda_0(\Delta)} \le e^{0.3\varepsilon R} e^{-R\varepsilon/2} = e^{-0.2\varepsilon R}.$$

On the other hand,

$$\langle g, e^{R\Delta} g\rangle = \langle \pi^\Lambda_* f, e^{R\Delta} g\rangle = \int_{B_{\mathbb{H}^2}(1)} e^{R\Delta} g(\pi^\Lambda(p))\mathrm{d}m_{\mathbb{H}^2}(p).$$

For any point $p \in B_{\mathbb{H}^2}(1)$ we have

$$\begin{aligned}(e^{R\Delta}g)(p) = (\pi_*^\Lambda e^{R\Delta} f)(p) = \sum_{\lambda\in\Lambda}(e^{R\Delta}f)(\lambda p) \geq \sum_{\lambda\in\Lambda\cap B_{\mathrm{SL}_2(\mathbb{R})}(R)}(e^{R\Delta}f)(\lambda p)\\ \asymp \sum_{\lambda\in\Lambda\cap B_{\mathrm{SL}_2(\mathbb{R})}(R)} \frac{d_{\mathrm{SL}_2(\mathbb{R})}(\lambda,e)+1}{R^{3/2}} e^{-\frac{(R+d_{\mathrm{SL}_2(\mathbb{R})}(\lambda,e))^2}{4R}}\\ \gg \sum_{\lambda\in\Lambda\cap B_{\mathrm{SL}_2(\mathbb{R})}(R)} \frac{1}{R^{3/2}}e^{-R} = \frac{\#\Lambda\cap B_{\mathrm{SL}_2(\mathbb{R})}(R)}{e^R R^{3/2}}\end{aligned}$$

Altogether, $e^{-0.2\varepsilon R} \gg \frac{\#\Lambda\cap B_{\mathrm{SL}_2(\mathbb{R})}(R)}{e^R R^{3/2}}$, which implies that $\#\Lambda \cap B_{\mathrm{SL}_2(\mathbb{R})}(R) \ll R^{3/2}e^{(1-0.2\varepsilon)R}$, and hence, for R sufficiently large as a function of ε we have Eq. (4.1).

To show Eq. (4.2), denote

$$\begin{aligned}X &= \left\{h \in B_{\mathrm{SL}_2(\mathbb{R})}(R) : \#\{\gamma\in\Gamma : \gamma h \in B_{\mathrm{SL}_2(\mathbb{R})}(R)\} > e^{(1-0.1\varepsilon)R}\right\},\\ X' &= \left\{q \in B_{\mathbb{H}^2}(R) : \#\{\gamma\in\Gamma : \gamma q \in B_{\mathbb{H}^2}(R)\} > e^{(1-0.1\varepsilon)R}\right\} \supseteq X.\end{aligned}$$

We will estimate $\|e^{t\Delta}g\|^2$. On the one hand,

$$\|e^{t\Delta}g\|^2 \leq e^{-2t\lambda_2(\Delta)}\|g\|^2 \ll e^{-\varepsilon R}e^{0.3\varepsilon R} = e^{-0.7\varepsilon R}.$$

On the other hand, similarly to Eq. (A.3),

(A.4)
$$\|e^{t\Delta}g\|^2 = \int_{\Lambda\backslash\mathbb{H}^2}(e^{t\Delta}g)(p)^2 \mathrm{d}m_{\Lambda\backslash\mathbb{H}^2}(p) = \int_{\mathbb{H}^2}\sum_{\lambda\in\Lambda}(e^{t\Delta}f)(p)(e^{t\Delta}f)(\lambda p)\mathrm{d}m_{\mathbb{H}^2}(p)$$

Using (A.2), we deduce that whenever $p \in B_{\mathrm{SL}_2(\mathbb{R})}(R)$, we have $(e^{t\Delta}f)(p) \gg \frac{1}{\sqrt{R}}e^{-R}$. Consequently, we can use for every $p \in X'$ the integrand in the LHS of Eq. (A.4) is roughly bounded from below by

$$\gg e^{(1-0.1\varepsilon)R}\cdot\left(\frac{1}{\sqrt{R}}e^{-R}\right)^2.$$

Hence, to estimate $\|e^{t\Delta}g\|^2$,

$$\|e^{t\Delta}g\|^2 \gg m_{\mathbb{H}^2}(X')\cdot e^{(1-0.1\varepsilon)R}\cdot\left(\frac{1}{\sqrt{R}}e^{-R}\right)^2 = \frac{m_{\mathbb{H}^2}(X')}{Re^{(1+0.1\varepsilon)R}}.$$

Altogether,

$$m_{\mathbb{H}^2}(X) \leq m_{\mathbb{H}^2}(X') \ll Re^{(1+0.1\varepsilon)R}\cdot e^{-0.7\varepsilon R} = Re^{(1-0.6\varepsilon)R},$$

which proves Eq. (4.2). □

Proof of Claim 4.4. Let $k = [\Lambda_2 : \Lambda_1]$. Let X be a system of representatives for the right cosets in Λ_2/Λ_1. The inequality $\delta(\Lambda_1) \leq \delta(\Lambda_2)$ is simple. As for the other inequality let $\varepsilon > 0$, let $R > 0$ be a big number such that $B_{\mathrm{SL}_2(\mathbb{R})}(R)\cap\Lambda_2 > e^{R(\delta(\Lambda_2)-\varepsilon/2)}$, and denote $r = d_{\mathrm{SL}_2(\mathbb{R})}(X, I)$. For every $\lambda \in B_{\mathrm{SL}_2(\mathbb{R})}(R)\cap\Lambda_2$, let $x_\lambda \in X$ such that $\lambda\Lambda_1 = x_\lambda\Lambda_1$. The collection $Y_R = \{x_\lambda^{-1}\lambda : \lambda \in B_{\mathrm{SL}_2(\mathbb{R})}(R)\cap\Lambda_2\} \subseteq \Lambda_1$

is of size $e^{R(\delta(\Lambda_2)-\varepsilon/2)}/k$, and lies in $B_{\mathrm{SL}_2(\mathbb{R})}(R+r)$. For R sufficiently big we have $ke^r < e^{R\varepsilon/2}$, and hence

$$\#Y_R \geq e^{R(\delta(\Lambda_2)-\varepsilon/2)}/k \geq e^{R(\delta(\Lambda_2)-\varepsilon/2)} \cdot e^{r-R\varepsilon/2} = e^{R(\delta(\Lambda_2)-\varepsilon)} \cdot e^r \geq e^{(R+r)(\delta(\Lambda_2)-\varepsilon)},$$

as desired. □

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