# Normal weak eigenstate thermalization 

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#### Abstract

Eigenstate thermalization has been shown to occur for few-body observables in a wide range of nonintegrable interacting models. For intensive observables that are sums of local operators, because of their polynomially vanishing Hilbert-Schmidt norm, weak eigenstate thermalization occurs in quadratic and integrable interacting systems. Here, we unveil a novel weak eigenstate thermalization phenomenon that occurs in quadratic models whose single-particle sector exhibits quantum chaos (quantum-chaotic quadratic models) and in integrable interacting models. In such models, we show that there are few-body observables with a nonvanishing Hilbert-Schmidt norm that are guarrantied to exhibit a polynomially vanishing variance of the diagonal matrix elements, a phenomenon we dub normal weak eigenstate thermalization. For quantum-chaotic quadratic Hamiltonians, we prove that normal weak eigenstate thermalization is a consequence of single-particle eigenstate thermalization, i.e., it can be viewed as a manifestation of quantum chaos at the single-particle level. We report numerical evidence of normal weak eigenstate thermalization for quantum-chaotic quadratic models such as the 3D Anderson model in the delocalized regime and the power-law random banded matrix model, as well as for the integrable interacting spin- $\frac{1}{2} \mathrm{XYZ}$ and XXZ models.


## I. INTRODUCTION

The eigenstate thermalization hypothesis (ETH) explains why thermalization in general occurs in isolated quantum systems [1-3]. The ETH can be written as an ansatz for the matrix elements of observables (few-body operators) $\hat{O}$ in the energy eigenstates $\{|\Omega\rangle\}[4,5]$ :

$$
\begin{equation*}
\langle\Omega| \hat{O}|\Gamma\rangle=O(\bar{E}) \delta_{\Omega \Gamma}+\frac{1}{\sqrt{e^{S(\bar{E})}}} f_{O}(\bar{E}, \omega) R_{\Omega \Gamma}^{O} \tag{1}
\end{equation*}
$$

where $\bar{E}=\left(E_{\Omega}+E_{\Gamma}\right) / 2$ is the average energy of the pair of eigenstates $|\Omega\rangle$ and $|\Gamma\rangle, \omega=E_{\Omega}-E_{\Gamma}$ is their energy difference, $O(\bar{E})$ and $f_{O}(\bar{E}, \omega)$ are smooth functions of their arguments, $S(\bar{E})$ is the thermodynamic entropy at energy $\bar{E}$, and $R_{\Omega \Gamma}^{O}$ are random numbers with zero mean and unit variance. Since $S(\bar{E})$ is an extensive quantity, it follows from this ansatz that the fluctuations of the diagonal matrix elements about their mean $O(\bar{E})$, as well as the magnitude of the off-diagonal matrix elements, decay exponentially with increasing the system size.

The ETH ansatz (1) describes the behavior of the matrix elements of few-body operators in the eigenstates of nonintegrable (quantum-chaotic) interacting models [57]. Specifically, eigenstate thermalization has been shown to occur in a wide range of quantum-chaotic Hamiltonians with short-range interactions [3, 8-34], and its extensions to describe higher-order correlations have been discussed [35-41]. On the other hand, eigenstate thermalization does not occur in quadratic nor in integrable interacting models [3, 8-10, 42, 43], in which the fluctuations of the diagonal matrix elements about their mean decays at most polynomially (as opposed to exponentially) with increasing the system size $[11,13,24,25,30,44-49]$, nor in many-body localized regimes [50-54], nor in systems with long-range interactions [55, 56]. Novel paradigms
in which the traditional interpretation of the ETH does not apply include quantum many-body scars [57-60] and Hilbert space fragmentation [61, 62].

For quadratic and integrable interacting Hamiltonians, it is understood that because of the lack of eigenstate thermalization, observables after quantum quenches do not thermalizate in general. Instead, their long-time averages are described by generalized Gibbs ensembles (GGEs) [42, 43, 63-79]. While the long-time averages of both quadratic and integrable interacting models are described by GGEs, there is a distinction between those models when it comes to the fluctuations of observables about their time averages. Equilibration occurs when those fluctuations become vanishingly small with increasing the system size. In quadratic models, the singleparticle eigenstates are often localized in a physically relevant single-particle basis, e.g., in quasimomentum space in the presence of discrete lattice translational invariance (single-particle states are Bloch states), or in position space in the presence of uncorrelated disorder (because of Anderson localization). Because of single-particle localization, in many-body sectors of localized quadratic models some observables equilibrate while others fail to do so [68, 71, 72, 75], see also Refs. [80-83]. On the other hand, in integrable interacting models (which are generally clean models in one dimension) few-body observables have been in general found to equilibrate $[8,9,76-78]$.

In recent years, we have explored the properties of a class of quadratic models that to some extent mirrors properties of integrable interacting models: those are quadratic models that exhibit quantum chaos and eigenstate thermalization at the single-particle level [84]. We refer to such models as quantum-chaotic quadratic (QCQ) models. In Ref. [85], one-body observables and their products were proved to always equilibrate in manybody sectors of QCQ models. Because there is an expo-
nentially large number (but a vanishing fraction in the thermodynamic limit) of many-body QCQ Hamiltonian eigenstates whose diagonal matrix elements are nonthermal, in general, observables after equilibration are described by a GGE rather than by a thermal ensemble [85].

In this work we use QCQ models to unveil a novel, and stricter, form of weak eigenstate thermalization. A form that, for few-body observables (as defined in Sec. II A), is only guarantied to arise in QCQ and (per our numerical results) in integrable interacting models. Previous works have argued that weak eigenstate thermalization is a phenomenon that occurs for intensive observables that are sums of local operators $[24,25,44,47]$. As discussed in Ref. [44], the fluctuations of the diagonal matrix elements of such operators in the energy eigenbasis vanish polynomially with increasing the system size, even in very simple quadratic models. This can be understood a being a consequence of that fact that the Hilbert-Schmidt norm of such operators also vanishes polynomially with increasing system size [25]. On the other hand, for QCQ and integrable interacting models, here we show that there is a stricter form of weak eigenstate thermalization, which always occurs for few-body operators with a nonvanishing Hilbert-Schmidt norm, namely, for normalized few-body operators. We dub it normal weak eigenstate thermalization as the fluctuations of the diagonal matrix elements vanish polynomially while the norm of the operators does not vanish. For QCQ models we prove for one-body observables, and argue for few-body observables, that normal weak eigenstate thermalization is guarantied to occur because eigenstate thermalization occurs in the singleparticle sector of the Hamiltonian. In contrast, we show that in localized quadratic models, normal weak eigenstate thermalization does not occur in general.

The presentation is organized as follows. In Sec. II, we introduce the few-body observables under investigation and their sums, and discuss the importance of the normalization. In Sec. III, we introduce the QCQ and localized quadratic Hamiltonians that are studied in this work. In Sec. IV, we establish the connection between single-particle eigenstate thermalization and normal weak eigenstate thermalization for one-body observables in QCQ models. We study numerically the scaling of the variances of the matrix elements of one-body observables in QCQ models in Sec. V A, and in localized quadratic models that do not exhibit quantum chaos in their single-particle sector in Sec. V B. The generalization to two-body and higher few-body observables is discussed in Sec. VI. Normal weak eigenstate thermalization in interacting integrable Hamiltonians is studied in Sec. VII. We conclude with a summary and discussion of our results in Sec. VIII.

## II. OBSERVABLES AND NORMALIZATION

We study one-body (two-point) and two-body (fourpoint) observables. General one-body observables can be
written as

$$
\begin{equation*}
\hat{o}_{\mathrm{I}}=\sum_{\alpha, \beta=1}^{V} o_{\alpha \beta}^{(\mathrm{I})} \hat{c}_{\alpha}^{\dagger} \hat{c}_{\beta} \tag{2}
\end{equation*}
$$

where $\hat{c}_{\alpha}^{\dagger}\left(\hat{c}_{\alpha}\right)$ creates (annihilates) a spinless fermion in a single-particle state $|\alpha\rangle$, while general two-body observables can be written as

$$
\begin{equation*}
\hat{o}_{\mathrm{II}}=\sum_{\alpha, \beta, \gamma, \delta=1}^{V} o_{\alpha \beta \gamma \delta}^{(\mathrm{II})} \hat{c}_{\alpha}^{\dagger} \hat{c}_{\beta}^{\dagger} \hat{c}_{\gamma} \hat{c}_{\delta} \tag{3}
\end{equation*}
$$

Our focus will be on specific forms of those observables to which we refer as one- and two-body observables, see Sec. II A, and sums of one- and two-body observables, see Sec. II B.

We study quadratic models with $V$ sites in onedimensional (1D) and three-dimensional (3D) lattices, where $V=L$ in 1D lattices and $V=L^{3}$ in 3D lattices, with $L$ being the linear number of sites. We also study integrable interacting chains, for which we refer to the number of lattice sites as $L$.

## A. One- and two-body observables

We call one-body observables the operators $\hat{o}_{\mathrm{I}}$ from Eq. (2) that, in the thermodynamic limit $(V \rightarrow \infty)$, can be expressed as finite sums of products of one creation and one annihilation operator in some single-particle basis. Formally, our one-body observables have rank $\mathbf{O}(1)$ in the single-particle Hilbert space, where the rank is the number of nondegenerate eigenvalues.

Specifically, we study the nearest neighbor hopping operator,

$$
\begin{equation*}
\hat{h}_{i}^{(1)}=\hat{c}_{i}^{\dagger} \hat{c}_{i+1}+\hat{c}_{i+1}^{\dagger} \hat{c}_{i} \tag{4}
\end{equation*}
$$

and the next-nearest neighbor hopping operator,

$$
\begin{equation*}
\hat{h}_{i}^{(2)}=\hat{c}_{i}^{\dagger} \hat{c}_{i+2}+\hat{c}_{i+2}^{\dagger} \hat{c}_{i} \tag{5}
\end{equation*}
$$

where we use $i$ to label the lattice sites. In 3D lattices, $i \equiv$ $\{x, y, z\}, i+1 \equiv\{x+1, y, z\}$, and $i+2 \equiv\{x+1, y+1, z\}$. We always compute these operators at a properly chosen site $i^{*}$, and report results for $\hat{h}_{1} \equiv \hat{h}_{i^{*}}^{(1)}$ and $\hat{h}_{2} \equiv \hat{h}_{i^{*}}^{(2)}$. For the models with periodic boundary conditions, the chosen site $i^{*}$ is not important. For the models with open boundary conditions, which are 1D models in this work, we take the site $i^{*}$ to be at the center of the chain.

The operators $\hat{h}_{1}$ and $\hat{h}_{2}$ are one-body operators that are local in position space. We also study a one-body operator that is nonlocal in position space, the occupation of the zero quasimomentum mode:

$$
\begin{equation*}
\hat{m}_{0}=\hat{f}_{0}^{\dagger} \hat{f}_{0} \tag{6}
\end{equation*}
$$

where $\hat{f}_{0}=\sum_{j=1}^{V} \hat{c}_{j} / \sqrt{V}$ in a lattice with $V$ sites.

We call two-body observables the operators $\hat{o}_{\text {II }}$ from Eq. (3) that can be expressed as finite sums of products of two creation and two annihilation operators in some single-particle basis. Specifically, we study the nearest neighbor density-density operator

$$
\begin{equation*}
\hat{o}_{i}^{(1)}=\hat{n}_{i} \hat{n}_{i+1} \tag{7}
\end{equation*}
$$

the next-nearest neighbor density-density operator,

$$
\begin{equation*}
\hat{o}_{i}^{(2)}=\hat{n}_{i} \hat{n}_{i+2} \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{n}_{i}=\hat{c}_{i}^{\dagger} \hat{c}_{i} \tag{9}
\end{equation*}
$$

and the correlated (pair) hopping,

$$
\begin{equation*}
\hat{o}_{i}^{(3)}=\hat{c}_{i}^{\dagger} \hat{c}_{i+3}^{\dagger} \hat{c}_{i+2} \hat{c}_{i+1}+\text { H.c. } \tag{10}
\end{equation*}
$$

Like for the one-body operators, we compute the previous two-body operators at a properly chosen site $i^{*}$, and report results for $\hat{o}_{1} \equiv \hat{o}_{i^{*}}^{(1)}, \hat{o}_{2} \equiv \hat{o}_{i^{*}}^{(2)}$, and $\hat{o}_{3} \equiv \hat{o}_{i^{*}}^{(3)}$. For models with periodic boundary conditions, the chosen site $i^{*}$ is not important. For our 1D models with open boundary conditions, we take the site $i^{*}$ to be at the center of the chain.

One can similarly define three-body and higher fewbody operators following this procedure. Those are the kind of operators that we have in mind whenever we mention few-body observables in this work.

## B. Sums of one- and two-body observables

We define sums of one-body (two-body) observables as the operators $\hat{o}_{\mathrm{I}}$ in Eq. (2) $\left[\hat{o}_{\mathrm{II}}\right.$ in Eq. (3)] that, in the thermodynamic limit $(V \rightarrow \infty)$, cannot be expressed as finite sums of products of two (four) creation and annihilation operators in some single-particle basis. In the case of one-body observables, such operators have rank $\mathbf{O}(V)$ in the single-particle Hilbert space.

Specifically, as sums of one-body observables, we study the sums of nearest and next-nearest neighbor hopping operators,

$$
\begin{equation*}
\hat{T}_{1}=\frac{1}{\sqrt{V}} \sum_{i=1}^{V} \hat{h}_{i}^{(1)}, \quad \hat{T}_{2}=\frac{1}{\sqrt{V}} \sum_{i=1}^{V} \hat{h}_{i}^{(2)} \tag{11}
\end{equation*}
$$

and, as sums of two-body observables, we study the sums of nearest and next-nearest neighbor density-density operators,

$$
\begin{equation*}
\hat{O}_{1}=\frac{1}{\sqrt{V}} \sum_{i=1}^{V} \hat{o}_{i}^{(1)}, \quad \hat{O}_{2}=\frac{1}{\sqrt{V}} \sum_{i=1}^{V} \hat{o}_{i}^{(2)} \tag{12}
\end{equation*}
$$

A common property of the sums of one- and two-body observables considered here is the prefactor $1 / \sqrt{V}$, which
ensures that the Hilbert-Schmidt norm of their traceless counterparts [defined in Eq. (16)] is $\mathbf{O}(1)$. We discuss the consequences of using the traditional $1 / V$ normalization, which ensures the sums are intensive operators, in Sec. II D.

One can similarly define sums of three-body and higher few-body operators following this procedure. Those are the kind of operators that we have in mind whenever we mention sums of few-body observables in this work.

## C. Variance of diagonal matrix elements

We are interested in how the fluctuations of the diagonal matrix elements of observables $O_{\Omega \Omega} \equiv\langle\Omega| \hat{O}|\Omega\rangle$ in the many-body energy eigenstates $|\Omega\rangle$ across the energy spectrum scale with the system size. Those fluctuations can be quantified by the variance

$$
\begin{equation*}
\sigma^{2}=\frac{1}{D} \sum_{\Omega=1}^{D}\left[O_{\Omega \Omega}-O_{\mathrm{mic}}\left(E_{\Omega}\right)\right]^{2} \tag{13}
\end{equation*}
$$

where $D$ is the dimension of the many-body Hilbert space and $O_{\text {mic }}\left(E_{\Omega}\right)$ is the expectation value of the observable in the microcanonical ensemble at energy $E_{\Omega}$. The latter can be computed as a running average [13, 19, 23, 25]

$$
\begin{equation*}
O_{\mathrm{mic}}\left(E_{\Omega}\right)=\frac{1}{\mathcal{N}} \sum_{E_{\Omega^{\prime}} \in \mathcal{M}} O_{\Omega^{\prime} \Omega^{\prime}} \tag{14}
\end{equation*}
$$

where $\mathcal{M}=\left\{E_{\Omega^{\prime}} ; E_{\Omega}-\Delta / 2 \leq E_{\Omega^{\prime}} \leq E_{\Omega}+\Delta / 2\right\}$ with a width $\Delta$ about $E_{\Omega}$, and $\mathcal{N}$ is the number of energy eigenstates in the microcanonical window.

## D. Normalized vs intensive sums of few-body observables

Modifying an operator by adding a constant neither changes its physical meaning nor the description of its matrix elements via the ETH ansatz [see Eq. (1)]. This constant can be trivially absorbed in the function $O(\bar{E})$. However, such a constant can change the Hilbert-Schmidt norm of the operator,

$$
\begin{equation*}
\|\hat{O}\|_{\mathrm{HS}}^{2}=\frac{1}{D} \operatorname{Tr}\left[\hat{O}^{2}\right] . \tag{15}
\end{equation*}
$$

In order to eliminate potentially ambiguous constants from our considerations, we normalize the traceless counterparts of the operators considered. In other words, we first project the operator $\hat{O}$ onto the subspace of traceless operators, $\hat{P} \hat{O}=\hat{O}-\operatorname{Tr}[\hat{O}] / D$, and then compute the Hilbert-Schmidt norm $\|\hat{P} \hat{O}\|_{\text {HS }}$. For simplicity, we
introduce the notation

$$
\begin{align*}
\|\hat{O}\|_{\mathrm{mb}}^{2} & \equiv\|\hat{P} \hat{O}\|_{\mathrm{HS}}^{2}=\left\|\hat{O}-\frac{1}{D} \operatorname{Tr}[\hat{O}]\right\|_{\mathrm{HS}}^{2} \\
& =\|\hat{O}\|_{\mathrm{HS}}^{2}-\left(\frac{1}{D} \operatorname{Tr}[\hat{O}]\right)^{2}  \tag{16}\\
& =\frac{1}{D} \sum_{\Omega=1}^{D}\langle\Omega| \hat{O}^{2}|\Omega\rangle-\left(\frac{1}{D} \sum_{\Omega=1}^{D}\langle\Omega| \hat{O}|\Omega\rangle\right)^{2}
\end{align*}
$$

where, in the last line, we explicitly wrote the trace in the basis $\{|\Omega\rangle\}$ of the many-body eigenstates of a Hamiltonian of interest. In the reminder of this paper, we refer to $\|\hat{O}\|_{\mathrm{mb}}$ as the many-body Hilbert-Schmidt norm of operator $\hat{O}$. Readers should keep in mind that we mean the Hilbert-Schmidt norm of the traceless version of $\hat{O}$, which is the one that is invariant under a shift of the operator $\hat{O}$ by an arbitrary constant.

The few-body observables defined in Sec. II A and their sums defined in Sec. II B have $\|\hat{O}\|_{\mathrm{mb}}^{2}=\mathbf{O}(1)$. In contrast, the commonly studied in the literature intensive sums of few-body observables $\hat{\mathcal{O}}$, in which, e.g., in Eqs. (11) and (21) one replaces $\sqrt{V} \rightarrow V$, the HilbertSchmidt norm vanishes in the thermodynamic limit,

$$
\begin{equation*}
\|\hat{\mathcal{O}}\|_{\mathrm{mb}} \propto \frac{1}{\sqrt{V}} \tag{17}
\end{equation*}
$$

A simple example of an intensive sum of few-body operators is the average site occupation $\hat{\bar{n}}=\frac{1}{V} \sum_{j} \hat{n}_{j}$, whose many-body (including all particle-number sectors) Hilbert-Schmidt norm is $\|\hat{\bar{n}}\|_{\mathrm{mb}}=1 /(2 \sqrt{V})$.

## E. Weak eigenstate thermalization vs normal weak eigenstate thermalization

We can now trace back weak eigenstate thermalization for intensive sums of few-body observables to the vanishing Hilbert-Schmidt norm of those observables. First, we rewrite Eq. (16) by introducing an identity operator in the first term, $\hat{O}^{2}=\hat{O} \hat{I} \hat{O}=\sum_{\Gamma} \hat{O}|\Gamma\rangle\langle\Gamma| \hat{O}$, from which it follows that

$$
\begin{equation*}
\|\hat{O}\|_{\mathrm{mb}}^{2}=\frac{1}{D} \sum_{\Omega, \Gamma=1}^{D}\left|O_{\Omega \Gamma}\right|^{2}-\left(\frac{1}{D} \sum_{\Omega=1}^{D} O_{\Omega \Omega}\right)^{2} \tag{18}
\end{equation*}
$$

Next, for simplicity, we consider structureless observables [i.e., $O_{\text {mic }}\left(E_{\Omega}\right)=$ const. in Eq. (14)]. In this case, we can rewrite Eq. (18) using Eq. (13) as

$$
\begin{equation*}
\|\hat{O}\|_{\mathrm{mb}}^{2}=\sigma^{2}+\frac{1}{D} \sum_{\Omega \neq \Gamma}\left|O_{\Omega \Gamma}\right|^{2} \tag{19}
\end{equation*}
$$

which shows that the variance is upper-bounded by the square of the Hilbert-Schmidt norm. Hence, by construction, the variance vanishes in the thermodynamic limit
for observables $\hat{\mathcal{O}}$ that are structureless intensive sums of few-body observables

$$
\begin{equation*}
\sigma^{2} \leq\|\hat{\mathcal{O}}\|_{\mathrm{mb}}^{2} \propto \frac{1}{V} \tag{20}
\end{equation*}
$$

and so they exhibit weak eigenstate thermalization.
From Eq. (20) it follows that, in translationally invariant models, local few-body observables also exhibit weak eigenstate thermalization. The diagonal matrix elements of local few-body observables and their corresponding intensive sums in position space are related via

$$
\begin{equation*}
\langle\Omega| \hat{\mathcal{O}}|\Omega\rangle=\frac{1}{V} \sum_{i=1}^{V}\langle\Omega| \hat{o}_{i}|\Omega\rangle \tag{21}
\end{equation*}
$$

Because of translational invariance, $\langle\Omega| \hat{o}_{i}|\Omega\rangle$ is independent of $i$, so $\langle\Omega| \hat{\mathcal{O}}|\Omega\rangle=\langle\Omega| \hat{o}_{i}|\Omega\rangle$, and we see that if $\hat{\mathcal{O}}$ exhibits weak eigenstate thermalization so must $\hat{o}_{i}$. In this work, the results for few-body observables and for their sums are independent of each other because none of the models considered exhibits translational invariance.

The previously discussed results open an important question, which is the focus of this work. Namely, are there models for which few-body observables such as those defined in Eqs. (7)-(10), and normalized sums of few-body observables such as those in Eqs. (11) and (21), are guarantied to exhibit polynomially vanishing variances? Namely, are there models for which normalized few-body observables always exhibit normal weak eigenstate thermalization?

## III. QUANTUM-CHAOTIC VS LOCALIZED QUADRATIC HAMILTONIANS

Central to the analytic derivations and numerical results reported in this work are quadratic Hamiltonians, i.e., Hamiltonians of the form

$$
\begin{equation*}
\hat{H}=\sum_{i, j} h_{i j} \hat{c}_{i}^{\dagger} \hat{c}_{j} \tag{22}
\end{equation*}
$$

where $h_{i j}$ are the Hamiltonian matrix elements in a single-particle basis. Such Hamiltonians can be brought to a diagonal form $\hat{H}=\sum_{\omega=1}^{V} \epsilon_{\omega} \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}$ via a canonical transformation of the fermionic creation and annihilation operators, where $\epsilon_{\omega}$ are the single-particle eigenenergies. A key property of quadratic Hamiltonians that we will use here is that the many-body eigenstates $|\Omega\rangle$ of $\hat{H}$ are Slater determinants (fermionic Gaussian states), i.e., they can be written as products of single-particle Hamiltonian eigenstates $|\omega\rangle$.

We report numerical results for two quadratic Hamiltonians that exhibit quantum chaos and eigenstate thermalization in their single-particle spectrum, despite the fact that their matrix elements $h_{i j}$ are not drawn from a fully random ensemble. Our Hamiltonians of interest
have a nontrivial structure, and this is the reason behind the fact that quantum chaos only occurs in some parameter regimes and not in others.

We consider the power-law random banded (PLRB) model,

$$
\begin{equation*}
\hat{H}=\sum_{i, j=1}^{V} \frac{\mu_{i, j}}{(1+|i-j| / b)^{a / 2}} \hat{c}_{i}^{\dagger} \hat{c}_{j}, \tag{23}
\end{equation*}
$$

where $\mu_{i, j}$ are random variables drawn from a normal distribution with zero mean and unit variance. We take open boundary conditions, and set $a=0.2$ and $b=0.1$ to ensure that the system exhibits single-particle quantum chaos [86-88].

We also consider the 3D Anderson model on a cubic lattice with $V$ sites:

$$
\begin{equation*}
\hat{H}=-\sum_{\langle i, j\rangle}\left(\hat{c}_{i}^{\dagger} \hat{c}_{j}+\text { H.c. }\right)+\sum_{i=1}^{V} \varepsilon_{i} \hat{n}_{i} \tag{24}
\end{equation*}
$$

where the first sum in Eq. (24) runs over nearest neighbor sites, and $\varepsilon_{i}$ are i.i.d. random variables drawn from a box distribution, $\varepsilon_{i} \in[-W / 2, W / 2]$. For this model we take periodic boundary conditions. For nonvanishing disorder strengths below the localization transition, $0<W<$ $W_{\mathrm{c}}$, where $W_{\mathrm{c}} \approx 16.5[89,90]$, the 3D Anderson model exhibits quantum chaos in the single-particle sector [91] and single-particle eigenstate thermalization [84], i.e., in this regime it is a QCQ model [84, 92].

We also study quadratic models that do not exhibit quantum chaos nor eigenstate thermalization in the single-particle sector. The single-particle energy eigenstates of such models are localized in position or quasimomentum space, so we refer to them as localized quadratic models [84]. We study two such models. The first one is the 3D Anderson model (24) above the localization transition, $W>W_{c}$ (we fix $W=30$ ). The second one is the 1D Aubry-André model

$$
\begin{equation*}
\hat{H}=-\sum_{i=1}^{V-1}\left(\hat{c}_{i}^{\dagger} \hat{c}_{i+1}+\text { H.c. }\right)+\lambda \sum_{i=1}^{V} \cos (2 \pi \eta i+\phi) \hat{n}_{i} \tag{25}
\end{equation*}
$$

where $\eta=(\sqrt{5}-1) / 2, \phi$ is randomly drawn from a box distribution, $\phi \in[0, \pi]$, and we take open boundary conditions. The single-particle eigenstates of this model are localized in quasimomentum space for $\lambda<2$, and in position space for $\lambda>2$ [93]. The former regime, being localized in a different basis than the 3D Anderson model at strong disorder, is the focus of our study here. We fix $\lambda=0.5$ in what follows.

In order to evaluate the variance $\sigma^{2}$ of the diagonal matrix elements across the energy spectrum of the models above, we proceed as follows. For each system size $V$, we first randomly select 200 many-body energy eigenstates at half filling, $n=N / V=1 / 2$, and calculate the corresponding diagonal matrix elements. We then repeat the calculation for 20 to 100 Hamiltonian realizations. Subsequently, we determine the minimal and maximal energies
in the simulations, divide the established energy interval into 100 bins, and compute the average diagonal matrix elements in each bin. Those averages are then plugged in Eq. (13) to calculate the variance $\sigma^{2}$ of the given set of matrix elements. [In Eq. (13), we replace $D$ by the corresponding number of matrix elements.]

## IV. NORMAL WEAK EIGENSTATE THERMALIZATION: ONE-BODY CASE

In this section we prove the occurrence of normal weak eigenstate thermalization for one-body observables in QCQ Hamiltonians. We will discuss the extension to two-body observables and beyond in Sec. VI.

## A. Single-particle eigenstate thermalization

We begin by noting that the normalization introduced in Eq. (16) is defined in the many-body Hilbert space, whose dimension $D$ increases exponentially with the number of lattice sites $V$. The normalization of observables is in general different in the single-particle Hilbert space, whose dimension is proportional to $V$. We sharpen that distinction next, as it is instrumental for the derivation of normal weak eigenstate thermalization in QCQ models.

The analog of the ETH ansatz from Eq. (1) was introduced for the matrix elements of one-body observables in the single-particle eigenstates $|\omega\rangle$ of QCQ Hamiltonians in Ref. [84]. For the diagonal matrix elements $o_{\omega \omega} \equiv\langle\omega| \hat{o}|\omega\rangle$, which are the ones needed in the derivations that follow, the single-particle eigenstate thermalization ansatz can be written as

$$
\begin{equation*}
\frac{o_{\omega \omega}}{\|\hat{o}\|_{\mathrm{sp}}}=o\left(\epsilon_{\omega}\right)+\frac{1}{\sqrt{\rho\left(\epsilon_{\omega}\right)}} \mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right) R_{\omega \omega}^{o} \tag{26}
\end{equation*}
$$

where $\rho\left(\epsilon_{\omega}\right)$ is the single-particle density of states, $o\left(\epsilon_{\omega}\right)$ and $\mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right)$ are smooth functions of the single-particle energies $\epsilon_{\omega}$, and $R_{\omega \omega}^{o}$ is a random number with zero mean and unit variance. In addition, $\|\hat{o}\|_{\text {sp }}$ is the HilbertSchmidt norm of the projected operator $\hat{p} \hat{o}=\hat{o}-\operatorname{Tr}[\hat{o}] / V$ in the single-particle Hilbert space, i.e.,

$$
\begin{align*}
\|\hat{o}\|_{\mathrm{sp}}^{2} & =\|\hat{p} \hat{o}\|_{\mathrm{HS}}^{2}=\left\|\hat{o}-\frac{1}{V} \operatorname{Tr}[\hat{o}]\right\|_{\mathrm{HS}}^{2} \\
& =\|\hat{o}\|_{\mathrm{HS}}^{2}-\left(\frac{1}{V} \operatorname{Tr}[\hat{o}]\right)^{2}  \tag{27}\\
& =\frac{1}{V} \sum_{\omega=1}^{V}\langle\omega| \hat{o}^{2}|\omega\rangle-\left(\frac{1}{V} \sum_{\omega=1}^{V}\langle\omega| \hat{o}|\omega\rangle\right)^{2} .
\end{align*}
$$

In what follows, we will call $\|\hat{o}\|_{\text {sp }}^{2}$ the single-particle Hilbert-Schmidt norm of the operator $\hat{o}$. Readers should keep in mind that we mean the single-particle HilbertSchmidt norm of its traceless counterpart $\hat{p} \hat{o}$.

Two important properties of one-body observables in the single-particle Hilbert space are: (i) One-body observables that are normalized in the many-body Hilbert space, i.e., for which $\|\hat{o}\|_{\mathrm{mb}}^{2}=\mathbf{O}(1)$, exhibit a vanishing Hilbert-Schmidt norm in the single-particle Hilbert space

$$
\begin{equation*}
\|\hat{o}\|_{\mathrm{sp}}^{2}=\mathbf{O}(1 / V) \tag{28}
\end{equation*}
$$

Consequently, $\sqrt{V} \hat{o}$ are the observables that are normalized in the single-particle Hilbert-space [84]. (ii) One-body observables that are normalized in the singleparticle Hilbert space have a structure function $o\left(\epsilon_{\omega}\right)$ in Eq. (26) that vanishes in the thermodynamic limit. We prove this property in Appendix A.

## B. Normal weak eigenstate thermalization

We are now ready to show that single-particle eigenstate thermalization implies normal weak eigenstate thermalization in many-body eigenstates. Using Eq. (2), we can write any one-body observable $\hat{o}$ in terms of the creation $\left\{\hat{f}_{\omega}^{\dagger}\right\}$ and annihilation $\left\{\hat{f}_{\omega}\right\}$ operators of spinless fermions in the single-particle energy eigenstates $\{|\omega\rangle\}$, and the corresponding matrix elements $o_{\omega \gamma}=\langle\omega| \hat{o}|\gamma\rangle$. We focus on one-body observables $\hat{o}$ that are normalized in the many-body Hilbert-space, namely, for which $\|\hat{o}\|_{\mathrm{mb}}^{2}=\mathbf{O}(1)$ [see Eq. (16)]. The diagonal matrix elements of $\hat{o}$ in many-body energy eigenstates, $o_{\Omega \Omega}=$ $\langle\Omega| \hat{o}|\Omega\rangle$, can therefore be written as

$$
\begin{equation*}
o_{\Omega \Omega}=\sum_{\omega, \gamma=1}^{V} o_{\omega \gamma}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\gamma}|\Omega\rangle=\sum_{\omega=1}^{V} o_{\omega \omega}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle . \tag{29}
\end{equation*}
$$

Plugging the single-particle eigenstate thermalization ansatz for $o_{\omega \omega}$ [see Eq. (26)] in Eq. (29) results in two contributions to $o_{\Omega \Omega}$,

$$
\begin{equation*}
o_{\Omega \Omega}=\chi_{1, \Omega}+\chi_{2, \Omega} \tag{30}
\end{equation*}
$$

of the form

$$
\begin{align*}
& \chi_{1, \Omega}=\|\hat{o}\|_{\mathrm{sp}} \sum_{\omega=1}^{V} o\left(\epsilon_{\omega}\right)\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle  \tag{31}\\
& \chi_{2, \Omega}=\|\hat{o}\|_{\mathrm{sp}} \sum_{\omega=1}^{V} \frac{1}{\sqrt{\rho\left(\epsilon_{\omega}\right)}} \mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right) R_{\omega \omega}^{o}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle . \tag{32}
\end{align*}
$$

The eigenstate fluctuations of $o_{\Omega \Omega}$ due to $\chi_{2, \Omega}$ were recently studied in Ref. [85]. The main result of that study is that $\chi_{2, \Omega}$ yields no structure in the many-body sector. In other words, the microcanonical average of $\chi_{2, \Omega}$ is energy independent. Using this result, we are able to calculate the variance of $\chi_{2, \Omega}$ analytically, as detailed in Appendix B1. We obtain

$$
\begin{equation*}
\operatorname{Var}\left[\chi_{2, \Omega}\right]=V\|\hat{o}\|_{\mathrm{sp}}^{2} n(1-n){\overline{A^{2}}}_{\mathrm{sp}} \tag{33}
\end{equation*}
$$

where $n$ is the average particle filling in the many-body Hilbert space, and

$$
\begin{equation*}
{\overline{A^{2}}}_{\mathrm{sp}} \equiv \frac{1}{V} \sum_{\omega=1}^{V} \frac{1}{\rho\left(\epsilon_{\omega}\right)}\left|\mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right)\right|^{2}\left|R_{\omega \omega}^{o}\right|^{2} \tag{34}
\end{equation*}
$$

The sum in Eq. (34) has $V$ terms that are $\mathbf{O}(1 / V)$, so it is $\mathbf{O}(1)$, which leads to $\overline{A^{2}}{ }_{\text {sp }} \propto 1 / V$ because of the prefactor of the sum. Recalling that $V\|\hat{o}\|_{\mathrm{sp}}^{2}=\mathbf{O}(1)$, see Eq. (28), we find that

$$
\begin{equation*}
\operatorname{Var}\left[\chi_{2, \Omega}\right] \propto \frac{1}{V} \tag{35}
\end{equation*}
$$

While $\operatorname{Var}\left[\chi_{2, \Omega}\right]$ vanishes polynomially with increasing $V$, we should stress that in Ref. [85] we proved that there is an exponentially large (in $V$ ) number of matrix elements $\chi_{2, \Omega}$ that are different from the average as $V \rightarrow \infty$.

Next, we study the contribution of $\chi_{1, \Omega}$ in Eq. (31). Following Ref. [25], we express the structure function $\mathcal{O}\left(\epsilon_{\omega}\right)$ as a polynomial,

$$
\begin{equation*}
o\left(\epsilon_{\omega}\right)=\sum_{i=0}^{V} p_{i}\left(\epsilon_{\omega}\right)^{i} \tag{36}
\end{equation*}
$$

where $\epsilon_{\omega}$ are the single-particle energy eigenvalues and $p_{i}$ are coefficients that characterize each observable, see Appendix A. In Appendix A, we show that for one-body observables the coefficients $p_{i}$ are at most $\mathbf{O}(1 / \sqrt{V})$, while for sums of one-body observables they can be $\mathbf{O}(1)$.

Using Eq. (36), one can write $\chi_{1, \Omega}$ as

$$
\begin{align*}
\chi_{1, \Omega}= & \|\hat{o}\|_{\mathrm{sp}} \sum_{i=0}^{V} p_{i} \sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{i}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle  \tag{37}\\
= & \|\hat{o}\|_{\mathrm{sp}} p_{0} N+\|\hat{o}\|_{\mathrm{sp}} p_{1} E_{\Omega} \\
& +\|\hat{o}\|_{\mathrm{sp}} \sum_{i=2}^{V} p_{i} \sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{i}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle . \tag{38}
\end{align*}
$$

The first and second terms on the r.h.s. of Eq. (38) have a simple interpretation: the first term is a constant shift and the second term introduces a linear $\left(\propto E_{\Omega}\right)$ structure in the diagonal matrix elements $o_{\Omega \Omega}$ in the many-body energy eigenstates. The third term in Eq. (38) introduces further structure, and it is the first term that introduces eigenstate to eigenstate fluctuations of $o_{\Omega \Omega}$. These fluctuations, as argued next, can result in a nonvanishing variance of the diagonal matrix elements of normalized sums of one-body observables.

To explore the possible behaviors of $\chi_{1, \Omega}$ numerically, we use the quadratic Sachdev-Ye-Kitaev (SYK2) model [84, 92, 94], in which the matrix elements in Eq. (22) are drawn from the Gaussian orthogonal ensemble, and we consider the following ( $p$-dependent) observables

$$
\begin{equation*}
\hat{\mathrm{u}}_{p}=\frac{1}{\sqrt{V}} \sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{p} \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega} \tag{39}
\end{equation*}
$$



FIG. 1. Diagonal matrix elements of the observables $\hat{\mathrm{u}}_{p}$ in Eq. (39) for (a),(b) $p=2$ and (c),(d) $p=3$ in (a),(c) singleparticle and (b),(d) many-body eigenstates of the SYK2 model with $V=4096$ sites. The calculations for the manybody eigenstates are carried out at half filling, i.e., for $N=$ $V / 2$. In (a),(c), we plot all the diagonal matrix elements for two realizations of the Hamiltonian. In (b),(d), we plot 200 diagonal matrix elements for 60 realizations of the Hamiltonian. The dark green points show bin averages of the light green points computed dividing the energy range shown in 100 identical bins. The red curves show least-square fits of the dark points to (a) a constant and (d) a linear function. The inset in (c) displays the variance $\sigma^{2}$ [see Eq. (13)] of $\hat{u}_{p}$ for $p=2$ (grey squares) and $p=3$ (red circles).

The matrix elements of $\hat{\mathrm{u}}_{p}$ in the single-particle energy eigenstates are smooth by construction, i.e., the fluctuating part of the single-particle eigenstate thermalization ansatz in Eq. (26) vanishes for any system size. This is illustrated for $\hat{u}_{2}$ and $\hat{u}_{3}$ in Figs. 1(a) and 1(c), respectively. Hence, the only contribution to the diagonal matrix elements of $\hat{u}_{p}$ in the many-body eigenstates comes from $\chi_{1, \Omega}$ in Eq. (38).

In Figs. 1(b) and 1(d), we show the matrix elements of $\hat{\mathrm{u}}_{2}$ and $\hat{\mathrm{u}}_{3}$ in the many-body eigenstates. In addition to the eigenstate to eigenstate fluctuations, there is no structure for $\hat{u}_{2}\left[\right.$ Fig. 1(b)], while there is structure for $\hat{\mathrm{u}}_{3}$ [Fig. 1(d)]. This shows that the structure in the diagonal matrix elements in the single-particle sector may or may not result in structure of the diagonal matrix elements in many-body eigenstates. We find similar results (not shown), in which there is no structure for $\mathrm{u}_{p}$ when $p$ is even, and there is a structure for $\mathrm{u}_{p}$ when $p$ is odd, for larger values of $p$.

With this knowledge in hand, we study the scaling of the variance of $\chi_{1, \Omega}$. We note that, because of the structure in $\chi_{1, \Omega}$, calculating the scaling of the variance of $\chi_{1, \Omega}$ is more challenging than calculating the one of $\chi_{2, \Omega}$. The reason is that we are interested in the fluctuations of the
matrix elements about the structure. Since we do not know the structure a priori, in what follows we provide an upper bound for the fluctuations by calculating the variance of the terms with $i \geq 2$ in Eq. (38),

$$
\begin{equation*}
\chi_{1, \Omega}^{(i)}=\|\hat{o}\|_{\mathrm{sp}} p_{i} \sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{i}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle \tag{40}
\end{equation*}
$$

We find that, see Appendix B 2,

$$
\begin{equation*}
\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]=V\|\hat{o}\|_{\mathrm{sp}}^{2} n(1-n) p_{i}^{2} \overline{\epsilon^{2 i}}{ }_{\mathrm{sp}}, \quad i \geq 2 \tag{41}
\end{equation*}
$$

where $\overline{\epsilon^{2 i}}{ }_{\text {sp }}=\mathbf{O}(1)$ is the mean value of the singleparticle energies $\left(\epsilon_{\omega}\right)^{2 i}$ in the single-particle sector, see Eq. (B7). Also, $V\|\hat{o}\|_{\mathrm{sp}}^{2}=\mathbf{O}(1)$, see Eq. (28).

Equation (41) shows that if $p_{i}=\mathbf{O}(1 / \sqrt{V})$, which is the case for one-body observables, $\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right] \propto 1 / V$. The latter scaling is the same as the one in Eq. (35). On the other hand, if $p_{i}=\mathbf{O}(1)$, which is the case for sums of one-body observables, $\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]=\mathbf{O}(1)$.

For even $i$ (with $i \geq 2$ ), for which there is no structure in the diagonal matrix elements $o_{\Omega \Omega}, \operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]$ in Eq. (41) gives the scaling of the fluctuations of the diagonal matrix elements $o_{\Omega \Omega}$. For odd $i$ (with $i \geq 3$ ), for which there is a structure in diagonal matrix elements $o_{\Omega \Omega}, \operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]$ only provides an upper bound for the scaling of the fluctuations of the diagonal matrix elements $o_{\Omega \Omega}$. Therefore, for (normalized) sums of one-body observables [for which $p_{i}=\mathbf{O}(1)$ ], Eq. (41) shows that normal weak eigenstate thermalization may or may not occur. In the inset of Fig. 1(c), we report our numerical results for $\sigma^{2}$ [see Eq. (13)] of the observables $\hat{\mathrm{u}}_{p}$ from Eq. (39) for $p=2$ (grey squares) and $p=3$ (red circles). The variances were calculated from 200 eigenstates and averaged over 60 Hamiltonian realizations. In agreement with our analytical calculations, we find that $\sigma^{2}$ is $\mathbf{O}(1)$ for those two observables, i.e., they do not exhibit normal weak eigenstate thermalization.

Summarizing our analysis in this section, we have proved that, for one-body observables, normal weak eigenstate thermalization always occurs in QCQ Hamiltonians. On the other hand, for sums of one-body observables, normal weak eigenstate thermalization may or may not occur in QCQ Hamiltonians.

## V. NUMERICAL RESULTS: ONE-BODY CASE

In this section we test our analytical predictions using numerical calculations to solve quadratic models that exhibit quantum chaos in the single-particle sector (QCQ models), and contrast the results to those obtained in models that do not exhibit single particle quantum chaos (localized quadratic models).

## A. Quantum-chaotic quadratic Hamiltonians

First, we calculate the variances of one-body observables and their sums in the PLRM model. Results for those observables are shown in Fig. 2. As predicted analytically, for the one-body observables $\hat{h}_{1}, \hat{h}_{2}$, and $\hat{m}_{0}$ [Fig. 2(a)], we find that $\sigma^{2} \propto 1 / V$, i.e., they exhibit normal weak eigenstate thermalization. For this model, because of the absence of structure for $\hat{T}_{1}$ and $\hat{T}_{2}$ in the single-particle sector, we find that those sums of fewbody observables also exhibit normal weak eigenstate thermalization $\left[\sigma^{2} \propto 1 / V\right.$, see Fig. 2(b)].

Next, we calculate the variances of one-body observables and their sums in the delocalized regime of the 3D Anderson model for $W=5$, i.e., in the quantum chaotic regime of the single-particle sector. The results are shown in Fig. 3. For the one-body observables, see Fig. 3(a), we find that $\hat{h}_{1}$ and $\hat{h}_{2}$ exhibit the normal weak eigenstate thermalization scaling, $\sigma^{2} \propto 1 / V$. The occupation of the zero quasimomentum mode $\hat{m}_{0}$, on the other hand, exhibits a slower decay. We expect this to be a finitesize effect that is a consequence of the proximity to the translationally invariant point at $W=0$. All singleparticle energy eigenstates at $W=0$ are localized in quasimomentum space (they are Bloch states). Full delocalization in quasimomentum space at $W=5$, so that $\sigma^{2} \propto 1 / V$ as predicted analytically, requires larger system sizes that those accessible to us here.

For the sums of one-body observables, see Fig. 3(b), we find that $\hat{T}_{1}$ and $\hat{T}_{2}$ do not exhibit normal weak eigenstate thermalization as their variances do not decrease with increasing $V$. This occurs because of the structure in the single-particle sector, which for $\hat{T}_{1}$ was studied in Ref. [84]. We note that $\sigma^{2}$ for $\hat{T}_{1}$ is much smaller than for $\hat{T}_{2}$. This is a consequence of the diagonal matrix elements of $\hat{T}_{1}$ in the single-particle sector being mostly proportional to the single-particle energies [i.e., $p_{1}>0$ in Eq. (38)]. The deviations from this linear behavior are the ones responsible for the breakdown of normal weak eigenstate thermalization. On the other hand, the diago-


FIG. 2. Variances $\sigma^{2}$, see Eq. (23), as functions of $V$ for the PLRB model with $a=0.2$ and $b=0.1$. (a) One-body observables, (b) sums of one-body observables. The straight lines show two-parameter fits $a V^{-b}$ to the results. We calculate $\sigma^{2}$ using 200 random eigenstates from each of 100 (20) Hamiltonian realizations for $V \leq 16^{3}\left(V>16^{3}\right)$.


FIG. 3. Same as Fig. 2 in the delocalized regime, $W=5$, of the 3D Anderson model.
nal matrix elements of $\hat{T}_{2}$ in the single-particle sector are mostly proportional to the square of the single-particle energies, which directly results in the breakdown of normal weak eigenstate thermalization.

We also study the distributions of the diagonal matrix elements of one-body observables. Note that, in Eq. (29), the matrix elements in the many-body eigenstates are written as weighted sums of the matrix elements in the


FIG. 4. Distributions of the diagonal matrix elements of (a),(d) $\hat{h}_{1}$, (b), (e) $\hat{h}_{2}$, and (c),(f) $\hat{m}_{0}$ in (a)-(c) the PRBM model with $\alpha=0.2$ and $\beta=0.1$, and (d)-(f) the 3D Anderson model with $W=5$. We consider $V=10648$ ( $V=22^{3}=10648$ ) lattice sites and 200 diagonal matrix elements for 20 realizations of the PRBM (3D Anderson) model. The lines show least-squares fits to a Gaussian function.
single-particle eigenstates. Hence, given that the singleparticle sector exhibits quantum chaos, for sufficiently large system sizes one expects the matrix elements in the many-body eigenstates to be normally distributed. This is the case, as in Ref. [48], for most of the observables under investigation for the system sizes considered.

Specifically, in Fig. 4 we plot the probability density functions (PDFs) of the diagonal matrix elements of onebody observables in the PLRB model (with $\alpha=0.2$ and $\beta=0.1$, left column) and in the 3D Anderson model (with $W=5$, right column), respectively. For all but one observable in one model, the distributions are close to Gaussian as advanced. The exception is $\hat{m}_{0}$ in the 3D Anderson model, see Fig. 4(e), which exhibits a bimodal distribution. The corresponding matrix elements are shown in Fig. 12 in Appendix D. The results for $\hat{m}_{0}$ make apparent that, in our finite systems, the matrix elements tend to accumulate at two values that depart from the mean. This is a remnant of what happens for $W=0$, for which the matrix elements of $\hat{m}_{0}$ are either 0 or 1 depending on whether the single-particle ground state is included or not in the many-body eigenstate. On the other hand, the PDF of the diagonal matrix elements of $\hat{m}_{0}$ in the PLRB model, see Fig. 4(e), is very close to Gaussian. We expect that, for sufficiently large systems (after full delocalization occurs in quasimomentum space), the matrix elements of $\hat{m}_{0}$ in the 3D Anderson model with $W=5$ will be normally distributed about the mean value as for all other observables shown.

## B. Localized quadratic Hamiltonians

We showed in previous sections that eigenstate thermalization in the single-particle sector results in normal weak eigenstate thermalization of one-body observables in many-body eigenstates of quadratic models. Our main conjecture for models that do not exhibit single-particle quantum chaos (because of single-particle localization) is that the lack of eigenstate thermalization in the single-


FIG. 5. Variances $\sigma^{2}$, see Eq. (23), as functions of $V$ for the 3D Anderson model with $W=30$. (a) One-body observables, (b) sums of one-body observables. The straight lines show two-parameter fits $a V^{-b}$ to the results. We calculate $\sigma^{2}$ using 200 random eigenstates from each of 100 (20) Hamiltonian realizations for $V \leq 16^{3}\left(V>16^{3}\right)$.


FIG. 6. Same as Fig. 5 for the Aubry-André model with $\lambda=0.5$, for which single-particle eigenstates are delocalizad in position space and localized in quasimomentum space.
particle sector results in no normal weak eigenstate thermalization of one-body observables, specifically, of onebody observables that are local in the basis in which localization occurs in the single-particle sector.

We first test this conjecture for the 3D Anderson model with $W=30$, see Fig. 5. For this model, in which singleparticle energy eigenstates are localized in position space, normal weak eigenstate thermalization does not occur for observables that are local in position space [84]. Figure 5(a) shows that the variance $\sigma^{2}$ of the diagonal matrix elements of $\hat{h}_{1}$ and $\hat{h}_{2}$ in the many-body energy eigenstates does not decay with $V$, i.e., they do not exhibit normal weak eigenstate thermalization. In contrast, the occupation of the zero quasimomentum mode $\hat{m}_{0}$ exhibits normal weak eigenstate thermalization $\left(\sigma^{2} \propto 1 / V\right)$. In Ref. [84], $\hat{m}_{0}$ was found to exhibit signatures of singleparticle eigenstate thermalization in the localized regime. Figure 5(b) shows that $\hat{T}_{1}$ and $\hat{T}_{2}$, which are sums of local one-body observables, do not exhibit normal weak eigenstate thermalization $\left[\sigma^{2}=\mathbf{O}(1)\right]$.

For the 1D Aubry-André model with $\lambda=0.5$, for which single-particle eigenstates are delocalizad in position space and localized in quasimomentum space, the scaling of the variances of one-body observables is opposite to that of the 3D Anderson model localized in position space. Specifically, Fig. 6(a) shows that the variance of both $\hat{h}_{1}$ and $\hat{h}_{2}$ scales as $\sigma^{2} \propto 1 / V$, while the variance of $\hat{m}_{0}$ does not decay with $V$. On the other hand, $\hat{T}_{1}$ and $\hat{T}_{2}$ (which are sums of local one-body observables) do not exhibit normal weak eigenstate thermalization in Fig. 6(b). The results in Figs. 5(b) and 6(b) show that the lack of normal weak eigenstate thermalization for sums of one-body observables in these models is independent of whether the observables are local or not in the basis in which the single-particle energy eigenstates are localized.

## VI. NORMAL WEAK EIGENSTATE THERMALIZATION: TWO-BODY CASE

In this section, we provide analytical arguments and report numerical results to argue that, like one-body observables, two-body observables always exhibit normal
weak eigenstate thermalization in QCQ models, i.e., a variance $\sigma^{2} \leq \mathbf{O}(1 / V)$. A detailed analytical calculation for observables that are products of occupation operators can be found in Appendix C.

## A. Analytical arguments

Let us consider a more general two-body observable $\hat{a}_{1} \hat{a}_{2} \hat{a}_{3} \hat{a}_{4}$, where $\hat{a}_{j}$ is either a fermionic creation or annihilation operator in an arbitrary single-particle state $|j\rangle$. The corresponding matrix elements in the manybody eigenstates can be written as $\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}=$ $\langle\Omega| \hat{a}_{1} \hat{a}_{2} \hat{a}_{3} \hat{a}_{4}|\Omega\rangle$. For quadratic models, the latter can be related to the expectation values of one-body observables using Wick's decomposition [95],

$$
\begin{equation*}
\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}=\sum_{P \in S_{4}}{ }^{\prime} \operatorname{sgn}(P)\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}, \tag{42}
\end{equation*}
$$

where $S_{4}$ is the permutation group of 4 elements and the sum is restricted (hence the prime) to the permutations with $P_{1}<P_{2}$ and $P_{3}<P_{4}$, as well as $P_{1}<P_{3}$. The first and second conditions maintain the alignment order, while the third condition excludes double counting.

To calculate the variance of $\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}$, we carry out two further simplifications. First, since $\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}$ in Eq. (42) is expressed as a sum of different contributions, one can use the covariance inequality (based on the Cauchy-Schwarz inequality) to upper bound the variance:

$$
\begin{equation*}
\sigma^{2}\left[\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}\right] \leq\left(\sum_{P \in S_{4}}{ }^{\prime} \sigma\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right]\right)^{2} \tag{43}
\end{equation*}
$$

Second, the variances of products of matrix elements of two different observables, which appear on the r.h.s. of the inequality in Eq. (43), can be expressed as

$$
\begin{align*}
& \sigma^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right] \\
& =\sigma^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\right] \sigma^{2}\left[\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right] \\
& \quad+\sigma^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\right] \mathbb{E}^{2}\left[\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right] \\
& \quad+\sigma^{2}\left[\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right] \mathbb{E}^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\right], \tag{44}
\end{align*}
$$

where we assumed a vanishing covariance between different matrix elements (independent matrix elements) and $\mathbb{E}[. .$.$] is the mean defined over the same set of states as the$ variance $\sigma^{2}$. Assuming that the one-body observables exhibit single-particle eigenstate thermalization, and using that (as we proved) the variances of one-body observables in such case scale as $\sigma^{2}=\mathbf{O}(1 / V)$, we find the scaling of the variance in Eq. (44) to be

$$
\begin{align*}
& \sigma^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right]=\mathbf{O}\left(1 / V^{2}\right)  \tag{45}\\
& +\mathbf{O}(1 / V) \mathbb{E}^{2}\left[\left(a_{P_{3}} a_{P_{4}}\right)_{\Omega \Omega}\right]+\mathbf{O}(1 / V) \mathbb{E}^{2}\left[\left(a_{P_{1}} a_{P_{2}}\right)_{\Omega \Omega}\right] .
\end{align*}
$$

We therefore see that when the means $\mathbb{E}[\ldots]$ vanish:

$$
\begin{equation*}
\sigma^{2}\left[\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}\right] \leq \mathbf{O}\left(1 / V^{2}\right) \tag{46}
\end{equation*}
$$

We show in Sec. VIB that this is what happens for $\hat{o}_{3}$ in Eq. (10). On the other hand, when the means $\mathbb{E}[. .$.$] do$ not vanish, which means that they are $\mathbf{O}(1)$, we get

$$
\begin{equation*}
\sigma^{2}\left[\left(a_{1} a_{2} a_{3} a_{4}\right)_{\Omega \Omega}\right] \leq \mathbf{O}(1 / V) \tag{47}
\end{equation*}
$$

We show in Sec. VIB that this is what happens for $\hat{o}_{1}$ and $\hat{o}_{2}$ from Eqs. (7) and (8), respectively. For the latter observables, we provide an explicit derivation of Eq. (47) in Appendix C.

No matter which of the two scalings applies to any given two-body observable, Eq. (46) or (47), we say that it exhibits normal weak eigenstate thermalization as the variance vanishes polynomially with increasing $V$.

The previous calculations can be extended to $p$-body observables with $p>2$, provided that $p=\mathbf{O}(1)$. When $p=\mathbf{O}(V)$, the upper bound for the variance from Eq. (43) may no longer decay polynomially with the system size. For sums of two-body observables, we provide numerical evidence that normal weak eigenstate thermalization occurs for the observables considered. Its analytical explanation is something we plan to explore in future works, together with what happens for sums of threeand higher few-body observables.

## B. Numerical results

Next, we show numerical results for two-body observables, and sums of two-body observables, in the 3D Anderson, the PLRB, and the 1D Aubry-André models.

In Fig. 7, we show results for the variances $\sigma^{2}$ in the (single-particle) quantum-chaotic regime of the PLRM


FIG. 7. Variances $\sigma^{2}$, see Eq. (23), as functions of $V$ for the (a),(b) PLRB model with $\alpha=0.2, \beta=0.1$ and (c),(d) 3D Anderson model with $W=5$. (a),(c) Two-body observables, (b),(d) sums of two-body observables. The straight lines show two-parameter fits $a V^{-b}$ to the results. We calculate $\sigma^{2}$ using 200 random eigenstates from each of 100 (20) Hamiltonian realizations for $V \leq 16^{3}\left(V>16^{3}\right)$.


FIG. 8. Distributions of diagonal matrix elements of (a),(d) $\hat{o}_{1},(\mathrm{~b}),(\mathrm{e}) \hat{o}_{2}$, and (c),(f) $\hat{o}_{3}$ in (a)-(c) the PRBM model with $\alpha=0.2$ and $\beta=0.1$, and (d)-(f) the 3D Anderson model with $W=5$. We consider $V=10648\left(V=22^{3}=10648\right)$ lattice sites and 200 diagonal matrix elements for 20 realizations of the PRBM (3D Anderson) model. The lines show least-squares fits to a Gaussian function.
model with $\alpha=0.2$ and $\beta=0.1$ [Figs. 7(a) and 7(b)] and of the 3D Anderson model with $W=5$ [Figs. 7(c) and $7(\mathrm{~d})]$. The results for two-body observables are shown in Figs. 7(a) and 7(c) and for sums of two-body observables in Figs. 7(b) and 7(d). All those observables exhibit normal weak eigenstate thermalization. However, the scaling of the variance depends on the observable. In Figs. $7(\mathrm{a})$ and $7(\mathrm{c})$, the variances of $\hat{o}_{1}$ and $\hat{o}_{2}$ scale as $\sigma^{2} \propto 1 / V$, while the variance of $\hat{o}_{3}$ scales as $\sigma^{2} \propto 1 / V^{2}$. Both scalings were argued in Sec. VI A to occur for twobody observables. Figures $7(\mathrm{~b})$ and $7(\mathrm{~d})$ show that the sums of two body observables $\hat{O}_{1}$ and $\hat{O}_{2}$, exhibit a scaling that is close to $\sigma^{2} \propto 1 / V^{\zeta}$ with $\zeta=1$ or 2 depending on the model. Why those scalings emerge is yet to be understood analytically.

Our results for two-body observables in QCQ models are consistent with the results obtained for the SYK2 model (which is also a QCQ model) in Ref. [96]. We note that in that study the variance scaled as $\sigma^{2} \propto 1 / V^{2}$. This is consistent with Eq. (47), and follows from the fact that the diagonal matrix elements of the one-body observables appearing in the Wick's decomposition have a vanishing


FIG. 9. Same as Fig. 7 for the (a),(b) 3D Anderson model with $W=30$, and for the (c),(d) 1D Aubry-André model with $\lambda=0.5$.
mean value in the many-body eigenstates.
In Fig. 8 we plot the probability density functions (PDFs) of the diagonal matrix elements in the PRBM model (with $\alpha=0.2$ and $\beta=0.1$, left column) and in the 3D Anderson model (with $W=5$, right column), respectively, for the two-body observables considered in Fig. 7. For all two-body observables in the two QCQ models the distributions are close to Gaussian.

In Fig. 9 we show results for the variances of the diagonal matrix elements of the same observables as in Fig. 7 but for localized quadratic Hamiltonians, i.e., for the 3D Anderson model with $W=30$ and for the 1D Aubry-André model with $\lambda=0.5$. The two-body observables $\hat{o}_{1}, \hat{o}_{2}$, and $\hat{o}_{3}$, which describe interactions and hoppings that are local in position space, exhibit variances in the localized regime of the 3D Anderson model that do not decay with $V$ [Fig. 9(a)]. On the other hand, these observables exhibit normal weak eigenstate thermalization in the 1D Aubry-André model with $\lambda=0.5$ [Fig. 9(c)], which is delocalized (localized) in position (quasimomentum) space. The results in Fig. 9(c) parallel what happens in the quantum-chaotic regimes in Figs. 7(a) and 7(c). Like the two-body observables, the variances of sums of two-body observables $\hat{O}_{1}$ and $\hat{O}_{2}$, do not exhibit weak eigenstate thermalization in the localized regime of the 3D Anderson model [Fig. 9(b)] and exhibit normal weak eigenstate thermalization in the 1D Aubry-André model with $\lambda=0.5$ [Fig. 9(d)].

Our results for the 1D Aubry-André model with $\lambda=$ 0.5 suggest that, for local two-body observables and their sums, single-particle localization in quasimomentum space results in very similar properties of the diagonal matrix elements as single-particle quantum chaos.


FIG. 10. Variances of the diagonal matrix elements of few-body observables for the (a) XYZ model, (b) XXZ model, and (c) noninteracting fermions with open boundary conditions. For (a),(b) we calculate the variance within $50 \%$ of the energy eigenstates about the mean energy from each symmetry sector, while for the noninteracting fermions in (c) we use the same method as for the other quadratic models. The solid lines show one parameter fits $a / L$ to the last four points of each corresponding data set. The dotted, dashed, and dashed-dotted lines show $L^{-2}, L^{-1}$, and $L^{0}$ functions, respectively, and are guides to the eye.

## VII. NORMAL WEAK EIGENSTATE THERMALIZATION: INTEGRABLE INTERACTING MODELS

In this section, we study the fluctuations of the diagonal matrix elements of various normalized observables in the eigenstates of paradigmatic integrable interacting spin- $\frac{1}{2}$ chains. We consider the spin- $\frac{1}{2}$ XYZ chain with open boundary conditions with the Hamiltonian

$$
\begin{equation*}
H_{\mathrm{XYZ}}=\sum_{\ell=1}^{L-1}(1+\eta) \hat{s}_{\ell}^{x} \hat{s}_{\ell+1}^{x}+(1-\eta) \hat{s}_{\ell}^{y} \hat{s}_{\ell+1}^{y}+\Delta \hat{s}_{\ell}^{z} \hat{s}_{\ell+1}^{z} \tag{48}
\end{equation*}
$$

where $\hat{s}_{\ell}^{\alpha}=\frac{1}{2} \sigma_{\ell}^{\alpha}$ are the spin- $\frac{1}{2}$ operators at site $\ell, \alpha=$ $x, y, z$, and $\sigma^{\alpha}$ are the Pauli matrices. We also consider the spin- $\frac{1}{2}$ XXZ chain with open boundary conditions with the Hamiltonian

$$
\begin{equation*}
H_{\mathrm{XXZ}}=\sum_{\ell=1}^{L-1} \hat{s}_{\ell}^{x} \hat{s}_{\ell+1}^{x}+\hat{s}_{\ell}^{y} \hat{s}_{\ell+1}^{y}+\Delta \hat{s}_{\ell}^{z} \hat{s}_{\ell+1}^{z} \tag{49}
\end{equation*}
$$

The latter model has $U(1)$ symmetry and is obtained from Eq. (48) by setting $\eta=0$. We focus on the zero magnetization sector of the spin- $\frac{1}{2} \mathrm{XXZ}$ chain.

We fix the parameters for the XYZ (XXZ) chain to be $\eta=\Delta=0.2(\Delta=0.55)$ as in Ref. [97]. We find the energy eigenstates and the corresponding matrix elements of observables carrying out exact diagonalization calculations for chains with even numbers of sites. To this end, we explicitly take into account all the symmetries relevant to each model. Specifically, for both models we account for the reflection and parity, $\mathcal{P}^{\alpha}=\prod_{\ell} \sigma_{\ell}^{\alpha}$ with $\alpha=x, y, z$, symmetries. For the XYZ chain with $L$ even, the three $\mathcal{P}^{\alpha}$ operators commute with each other. Because of the Clifford algebra of the Pauli matrices, one
can write any Pauli matrix as the product of the remaining two. We thus need to resolve only two of the parity symmetries (in addition to the reflection symmetry) [97]. For simplicity we choose $\mathcal{P}^{x}$ and $\mathcal{P}^{z}$. Therefore, the Hilbert space dimensions for the symmetry blocks considered for the XYZ model scale as $2^{L} / 8$. For the XXZ chain at zero magnetization, $\mathcal{P}^{z}$ acts trivially (it counts the number of up spins), so we only need to account for $\mathcal{P}^{x}$ (in addition to the reflection symmetry). Therefore, the Hilbert space dimensions for the symmetry blocks considered scale as $\binom{L}{L / 2} / 4$.

Our spin- $\frac{1}{2}$ models can be rewritten in terms of hardcore boson operators, and can be mapped onto interacting spinless fermions models using the Jordan-Wigner transformation [98]. Using the raising and lowering operators $\hat{s}_{\ell}^{ \pm}=\hat{s}_{\ell}^{x} \pm i \hat{s}_{\ell}^{y}$, which are equivalent to the creation and annihilation operators for hard-core bosons, and $\hat{s}_{\ell}^{z}$ operators, which are equivalent to site occupation operators (up to a $1 / 2$ offset), we define spin- $\frac{1}{2}$ observables that parallel the one- and two-body observables studied for the quadratic spinless fermion models. However, because of the need to carry out full exact diagonalization calculations for the spin chains, the system sizes that can be solved are much smaller than for quadratic models, so our results suffer from stronger finite size effects. Therefore, we focus on the nearest-neighbor one- and two-body operators at the center of the chains. They suffer from weaker finite-size effects than next-nearest neighbor and longer-range operators, and than observables that are sums of few-body operators.

We consider the (one-body) nearest-neighbor "hopping" operator

$$
\begin{equation*}
\hat{k}_{1}=\hat{s}_{L / 2}^{+} \hat{s}_{L / 2+1}^{-}+\hat{s}_{L / 2+1}^{+} \hat{s}_{L / 2}^{-} \tag{50}
\end{equation*}
$$

the (two-body) nearest-neighbor "density-density" operator

$$
\begin{equation*}
\hat{u}_{1}=\hat{s}_{L / 2}^{z} \hat{s}_{L / 2+1}^{z} \tag{51}
\end{equation*}
$$

and the equivalent of the (one-body) occupation of the zero quasimomentum mode

$$
\begin{equation*}
\hat{S}_{0}=\frac{1}{L} \sum_{\ell, \ell^{\prime}=1}^{L} \hat{s}_{\ell}^{+} \hat{s}_{\ell^{\prime}}^{-} \tag{52}
\end{equation*}
$$

The scaling of variances of the diagonal matrix elements of the observables in Eqs. (50)-(52) are shown in Fig. 10 for the XYZ [Fig. 10(a)] and the XXZ [Fig. 10(b)] models, together with the results for $\hat{h}_{1}, \hat{o}_{1}$, and $\hat{m}_{0}$ for spinless fermions with nearest neighbor hoppings in a chain with open boundary conditions [Fig. 10(c)]. Like for the Aubry-André model with $\lambda=0.5$ in Figs. 6(a) and 9 (c), in Fig. 10(c) one can see that $\sigma^{2}\left(\hat{h}_{1}\right) \propto 1 / L$, $\sigma^{2}\left(\hat{o}_{1}\right) \propto 1 / L$, and $\sigma^{2}\left(\hat{m}_{0}\right) \propto L^{0}$. In the spin chains [Figs. 10(a) and 10(b)], $\sigma^{2}\left(\hat{k}_{1}\right) \propto 1 / L, \sigma^{2}\left(\hat{u}_{1}\right) \propto 1 / L^{\alpha}$ with $\alpha \in[1,2]$, and $\sigma^{2}\left(\hat{S}_{0}\right) \propto 1 / L$. Therefore, because of the presence of interactions, in the clean spin chains $\hat{S}_{0}$ exhibits normal weak eigenstate thermalization. This is to be contrasted to the fact that the corresponding fermionic operator $\hat{m}_{0}$ does not exhibit normal weak eigenstate thermalization for spinless fermions with nearest neighbor hoppings in a chain with open boundary conditions. The latter is a consequence of single-particle localization in quasimomentum space.

Consistent with our results that normal weak eigenstate thermalization occurs for $\hat{S}_{0}$ in the XYZ and XXZ chains, in Ref. [48] it was shown that the variance of $\hat{m}_{0}$ is $\propto 1 / V$ in a translationally invariant model of hardcore bosons with nearest neighbor hoppings. Again, in such a clean model, interactions are responsible for the occurrence of normal weak eigenstate thermalization for $\hat{m}_{0}$, like single-particle eigenstate thermalization ensures that normal weak eigenstate thermalization occurs for one-body observables in QCQ models.

## VIII. SUMMARY AND DISCUSSION

We showed that single-particle eigenstate thermalization guaranties the occurrence of normal weak eigenstate thermalization of one- and two-body observables in QCQ models. We expect the same to apply to higher few-body observables. By normal weak eigenstate thermalization we mean that the variance of the diagonal matrix elements of normalized observables in many-body energy eigenstates decays polynomially with increasing system size. On the other hand, sums of one- and two-body observables may or may not exhibit normal weak eigenstate thermalization. For sums of one-body observables, we traced the possible behaviors of the variance to the structure of the operators in the single-particle sector. We presented numerical evidence supporting those results in


FIG. 11. Summary of our results for the fluctuations of the diagonal matrix elements of observables in the single-particle and many-body sectors of quantum-chaotic quadratic (QCQ) Hamiltonians. The upper row applies to the few-body (oneand two-body) observables introduced in Sec. II A, while lower row applies to the sums of few-body observables introduced in Sec. II B. Our main result is that eigenstate thermalization in the single-particle sector ensures normal weak eigenstate thermalization of one- and two-body observables in the many-body eigenstates. On the other hand, for sums of one-body observables the structure of the observable in the single-particle energy eigenstates may preclude normal weak eigenstate thermalization in the many-body eigenstates. The fluctuations of the diagonal matrix elements of few-body observables in many-body eigenstates of integrable interacting models were found to be qualitatively similar to those in QCQ models.
various QCQ models, and our results are summarized in Fig. 11. We also showed that normal weak eigenstate thermalization fails to occur for one- and two-body observables that are local in the space in which localization occurs in quadratic models that are not quantum chaotic.

We provided numerical evidence that normal weak eigenstate thermalization occurs for all the one- and two-body observables considered in integrable interacting models. Our findings suggest that in those models interactions effectively produce the same effect in the oneand two-body sectors of the many-body eigenstates (after tracing out all but one or two particles, respectively) that single-particle quantum chaos produces in QCQ models. This is remarkable in the light of the fundamental differences between the off-diagonal matrix elements (the ones responsible for the quantum dynamics [5]) of fewbody observables in integrable interacting and quadratic models. The off-diagonal matrix elements in many-body eigenstates of quadratic models are sparse [12, 48, 96] and the magnitude of the nonzero ones decays polynomially with the system size [48]. In integrable interacting models, on the other hand, the off-diagonal matrix elements are dense and their magnitude decays exponentially with increasing system size [24, 48].

Our results suggests that QCQ models can be used to gain analytical insights on the behavior of the diagonal matrix elements of few-body observables in interacting integrable models and, potentially, on that of few-
body observables that are nonlocal in the space in which localized quadratic models exhibit single-particle localization. Some important open questions still remain in the context of QCQ models. For example, for sums of two and higher few-body observables, it would be important to understand the conditions needed for normal weak eigenstate thermalization to occur. In the context of quantum dynamics, interesting open questions include: (i) are there classes of physically relevant initial states for which observables that exhibit normal weak eigenstate thermalization thermalize under noninteracting or integrable interacting dynamics and, (ii) for more general initial states, can one create nonthermal states after equilibration that are useful for quantum information storage and/or computing.

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## Appendix A: Absence of structure for one-body observables in single-particle eigenstates

We show that one-body observables in the singleparticle energy eigenstates do not have structure in the thermodynamic limit, i.e., that $o\left(\epsilon_{\omega}\right)$ in Eq. (26) is a constant. Specifically, we show that when writing

$$
\begin{equation*}
o\left(\epsilon_{\omega}\right)=\sum_{i=0}^{V} p_{i}\left(\epsilon_{\omega}\right)^{i} \tag{A1}
\end{equation*}
$$

where $\epsilon_{\omega}$ are the single-particle energies, the coefficients $p_{i}$ vanish for all $i \geq 1$.

Normalized one-body observables $\hat{o} /\|\hat{o}\|_{\text {sp }}$ can be written as [84]

$$
\begin{equation*}
\frac{\hat{o}}{\|\hat{o}\|_{\mathrm{sp}}}=\mathbf{O}(\sqrt{V}) \sum_{i=1}^{\mathbf{O}(1)} o_{i} \hat{f}_{i}^{\dagger} \hat{f}_{i} \tag{A2}
\end{equation*}
$$

where $\hat{f}_{i}^{\dagger}\left(\hat{f}_{i}\right)$ creates (annihilates) a spinless fermion in the $i^{\text {th }}$ single-particle eigenstate of $\hat{o}$, and $\sum_{i=1}^{\mathbf{O}(1)} o_{i}$ is at
most $\mathbf{O}(1)$. The diagonal matrix elements $o_{\omega \omega}=\langle\omega| \hat{o}|\omega\rangle$ in the single-particle energy eigenstates $|\omega\rangle$ are

$$
\begin{equation*}
\frac{o_{\omega \omega}}{\|\hat{o}\|_{\mathrm{sp}}}=\mathbf{O}(\sqrt{V}) \sum_{i=1}^{\mathbf{O}(1)} o_{i}|\langle\omega \mid i\rangle|^{2}, \tag{A3}
\end{equation*}
$$

where $\langle\omega| \hat{f}_{i}^{\dagger} \hat{f}_{i}|\omega\rangle=|\langle\omega \mid i\rangle|^{2}$.
Next, we assume that the structure in the singleparticle sector can be determined after coarse-graining all single-particle energy eigenstates into an arbitrary large but $\mathbf{O}(1)$ number of bins, each containing $\mathbf{O}(V)$ eigenstates, and averaging the diagonal matrix elements within each bin. Specifically, we define an energy window $\Delta$ that comprises $\mathcal{N}=\mathbf{O}(V)$ single-particle energy eigenstates from an energy bin with the mean energy $\bar{\epsilon}=\sum_{\omega \in \Delta} \epsilon_{\omega} / \mathcal{N}$. The average value of the observable at this energy is then

$$
\begin{equation*}
o(\bar{\epsilon})=\frac{1}{\mathbf{O}(V)} \sum_{\omega \in \Delta} \mathbf{O}(\sqrt{V}) \sum_{i=1}^{\mathbf{O}(1)} o_{i}|\langle\omega \mid i\rangle|^{2} . \tag{A4}
\end{equation*}
$$

Since $\sum_{\omega=1}^{V}|\langle\omega \mid i\rangle|^{2}=1$, we have that $\sum_{\omega \in \Delta}|\langle\omega \mid i\rangle|^{2}$ is at most $\mathbf{O}(1)$, and the structure function for traceless observables is upper bounded as

$$
\begin{equation*}
o(\bar{\epsilon}) \leq \mathbf{O}(1 / \sqrt{V}) \tag{A5}
\end{equation*}
$$

i.e., the coefficients $p_{i}$ in the polynomial expansion in Eq. (A1) are at most $\mathbf{O}(1 / \sqrt{V})$ and so they vanish in the thermodynamic limit.

For extensive sums of one-body observables, the equivalent of Eq. (A2) is [84]

$$
\begin{equation*}
\frac{\hat{o}}{\|\hat{o}\|_{\mathrm{sp}}}=\sum_{i=1}^{\mathbf{O}(V)} o_{i} \hat{f}_{i}^{\dagger} \hat{f}_{i} \tag{A6}
\end{equation*}
$$

Repeating the same steps as for one-body observables, one obtains that

$$
\begin{equation*}
o(\bar{\epsilon}) \leq \mathbf{O}(1) \tag{A7}
\end{equation*}
$$

i.e., the structure of sums of one-body observables in the single-particle sector may not vanish. Hence, there may exist $p_{i}$ (with $i \geq 1$ ) in the polynomial expansion in Eq. (A1) that are $\mathbf{O}(1)$.

## Appendix B: Variance of one-body observables

We calculate the variances of $\chi_{1, \Omega}$ and $\chi_{2, \Omega}$ in Eqs. (31) and (32), respectively. For simplicity, the variances $\operatorname{Var}(\ldots)$ and the means $\mathbb{E}[.$.$] are calculated over the entire$ many-body Hilbert space $D$ including all particle number sectors $N$. Since the numerical computations are carried out at a fixed particle filling $n=N / V=1 / 2$, at the end of each analytic calculation we mention how the results are modified when $n$ is fixed.

## 1. Variance of $\chi_{2, \Omega}$

First, we calculate the variance of $\chi_{2, \Omega}$ discussed in Sec. IV B. We rewrite $\chi_{2, \Omega}$ in Eq. (32) as

$$
\begin{equation*}
\chi_{2, \Omega}=\|\hat{o}\|_{\mathrm{sp}} \sum_{\omega=1}^{V} A_{\omega}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle \tag{B1}
\end{equation*}
$$

so that the mean of $\chi_{2, \Omega}$ can be expressed as

$$
\begin{equation*}
\mathbb{E}\left[\chi_{2, \Omega}\right]=V\|\hat{o}\|_{\mathrm{sp}} n \bar{A}_{\mathrm{sp}} \tag{B2}
\end{equation*}
$$

where $n$ is the average occupation of single-particle energy eigenstates in the many-body energy eigenstates, i.e., the average filling in the many-body Hilbert space. In Eqs. (B1) and (B2), we defined the quantity $A_{\omega}$ and its average $\bar{A}_{\text {sp }}$ in the single-particle Hilbert space as

$$
\begin{equation*}
A_{\omega}=\frac{\mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right)}{\sqrt{\rho\left(\epsilon_{\omega}\right)}} R_{\omega \omega}^{o}, \quad \bar{A}_{\mathrm{sp}}=\frac{1}{V} \sum_{\omega=1}^{V} \frac{\mathcal{F}_{o}\left(\epsilon_{\omega}, 0\right)}{\sqrt{\rho\left(\epsilon_{\omega}\right)}} R_{\omega \omega}^{o} \tag{B3}
\end{equation*}
$$

Since $\bar{A}_{\mathrm{sp}} \rightarrow 0$ for sufficiently large $V$, we have that $\mathbb{E}\left[\chi_{2, \Omega}\right] \rightarrow 0$. The variance can then be calculated as

$$
\begin{align*}
& \operatorname{Var}\left[\chi_{2, \Omega}\right]=\|\hat{o}\|_{\mathrm{sp}}^{2} \frac{1}{D} \sum_{\Omega=1}^{D}\left(\sum_{\omega=1}^{V} A_{\omega}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle\right)^{2} \\
& =\|\hat{o}\|_{\mathrm{sp}}^{2} \sum_{\omega=1}^{V}\left(A_{\omega}\right)^{2} \frac{1}{D} \sum_{\Omega=1}^{D}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle^{2}  \tag{B4}\\
& +\|\hat{o}\|_{\mathrm{sp}}^{2} \sum_{\omega \neq \omega^{\prime}=1}^{V} A_{\omega} A_{\omega^{\prime}} \frac{1}{D} \sum_{\Omega=1}^{D}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle\langle\Omega| \hat{f}_{\omega^{\prime}}^{\dagger} \hat{f}_{\omega^{\prime}}|\Omega\rangle
\end{align*}
$$

The averages over the many-body Hilbert space dimension $D$ in Eq. (B4) equal $n$ and $n^{2}$, respectively, and hence we obtain

$$
\begin{align*}
\operatorname{Var}\left[\chi_{2, \Omega}\right]= & \|\hat{o}\|_{\mathrm{sp}}^{2} n\left(\sum_{\omega=1}^{V}\left(A_{\omega}\right)^{2}\right) \\
& +\|\hat{o}\|_{\mathrm{sp}}^{2} n^{2}\left[\left(\sum_{\omega=1}^{V} A_{\omega}\right)^{2}-\sum_{\omega=1}^{V}\left(A_{\omega}\right)^{2}\right] \\
= & V\|\hat{o}\|_{\mathrm{sp}}^{2} n(1-n){\overline{A^{2}}}_{\mathrm{sp}}, \tag{B5}
\end{align*}
$$

which is the result reported in Eq. (35). This result is independent of whether the many-body Hilbert space, of dimension $D$, includes all particle-number sectors or it is limited to a fixed particle-number sector.

## 2. Variance of $\chi_{1, \Omega}$

Next, we calculate the variance of $\chi_{1, \Omega}^{(i)}$ in Eq. (40) for $i \geq 2$. (Recall that, for $i=0$ and 1 , the contributions to
$\chi_{1, \Omega}$ are smooth functions of the energy.) For $i \geq 2$, the mean of $\chi_{1, \Omega}^{(i)}$ reads

$$
\begin{equation*}
\mathbb{E}\left[\chi_{1, \Omega}^{(i)}\right]=V\|\hat{o}\|_{\mathrm{sp}} n p_{i} \bar{\epsilon}_{\mathrm{sp}}^{i} \tag{B6}
\end{equation*}
$$

where the mean of the single-particle energies to the $i$-th power is defined as

$$
\begin{equation*}
\overline{\epsilon_{\mathrm{sp}}^{i}}=\frac{1}{V} \sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{i} \tag{B7}
\end{equation*}
$$

We can write the variance of $\chi_{1, \Omega}^{(i)}$ as

$$
\begin{align*}
\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right] & =\|\hat{o}\|_{\mathrm{sp}}^{2} p_{i}^{2} \frac{1}{D} \sum_{\Omega=1}^{D}\left(\sum_{\omega=1}^{V}\left(\epsilon_{\omega}\right)^{i}\langle\Omega| \hat{f}_{\omega}^{\dagger} \hat{f}_{\omega}|\Omega\rangle\right)^{2} \\
& -\mathbb{E}^{2}\left[\chi_{1, \Omega}^{(i)}\right] \tag{B8}
\end{align*}
$$

The steps to simplify this expression are analogous to those in Eqs. (B4)-(B5) to obtain the variance of $\chi_{2, \Omega}$, and they yield

$$
\begin{equation*}
\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]=V\|\hat{o}\|_{\mathrm{sp}}^{2} n(1-n) p_{i}^{2} \overline{\epsilon_{\mathrm{sp}}^{2 i}} \tag{B9}
\end{equation*}
$$

which is the result reported in Eq. (41).
Equation (41) was obtained by carrying out the average over the many-body Hilbert space $D$ including all particle-number sectors. If one restricts the calculation to a sector with a fixed filling $n$ (we set $n=1 / 2$ in the numerical calculations), Eq. (B9) is modified to read

$$
\begin{equation*}
\operatorname{Var}\left[\chi_{1, \Omega}^{(i)}\right]=V\|\hat{o}\|_{\mathrm{sp}}^{2} n(1-n) p_{i}^{2}\left[\bar{\epsilon}^{2 i}{ }_{\mathrm{sp}}-\left(\bar{\epsilon}_{\mathrm{sp}}\right)^{2}\right] \tag{B10}
\end{equation*}
$$

The modification does not change the order of scaling of the variance with $V$, i.e., it does not affect whether normal weak eigenstate thermalization occurs or not. Note that the difference between Eqs. (B9) and (B10) arises from the fact that the particle-number variance at infinite temperature is different in the grand canonical ensemble compared to the canonical ensemble.

## Appendix C: Variance of products of site occupations

Building on the analysis in Sec. VI A, we detail our argument for the scaling of variances of two-body observables for the particular case in which they are products of site occupations, $\hat{n}_{i} \hat{n}_{j}$, with $i \neq j$. $\hat{o}_{1}$ and $\hat{o}_{2}$ in Eqs. (7) and (8), respectively, are examples of such observables.

The diagonal matrix elements of $\hat{n}_{i} \hat{n}_{j}$ can be expressed using the Wick decomposition [see Eq. (42)] as

$$
\begin{equation*}
\left(n_{i} n_{j}\right)_{\Omega \Omega}=\left(n_{i}\right)_{\Omega \Omega}\left(n_{j}\right)_{\Omega \Omega}-\left|\left(c_{i}^{\dagger} c_{j}\right)_{\Omega \Omega}\right|^{2} \tag{C1}
\end{equation*}
$$

where $\left(n_{i} n_{j}\right)_{\Omega \Omega}=\langle\Omega| \hat{n}_{i} \hat{n}_{j}|\Omega\rangle$, etc, and we assumed that $|\Omega\rangle$ is a Gaussian state with a fixed particle number (an
eigenstate of $\left.\hat{N}=\sum_{i=1}^{V} \hat{n}_{i}\right)$. The variance of $\left(n_{i} n_{j}\right)_{\Omega \Omega}$ then satisfies the inequality from Eq. (43),

$$
\begin{equation*}
\sigma^{2}\left[\left(n_{i} n_{j}\right)_{\Omega \Omega}\right] \leq\left(\sigma\left[\left(n_{i}\right)_{\Omega \Omega}\left(n_{j}\right)_{\Omega \Omega}\right]+\sigma\left[\left|\left(c_{i}^{\dagger} c_{j}\right)_{\Omega \Omega}\right|^{2}\right]\right)^{2} \tag{C2}
\end{equation*}
$$

The square of the first term in the parenthesis in Eq. (C2) can be expressed as

$$
\begin{align*}
& \sigma^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\left(n_{j}\right)_{\Omega \Omega}\right]=\sigma^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\right] \sigma^{2}\left[\left(n_{j}\right)_{\Omega \Omega}\right] \\
& \quad+\sigma^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\right] \mathbb{E}^{2}\left[\left(n_{j}\right)_{\Omega \Omega}\right]+\sigma^{2}\left[\left(n_{j}\right)_{\Omega \Omega}\right] \mathbb{E}^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\right] \tag{C3}
\end{align*}
$$

If both one-body observables $\hat{n}_{i}$ and $\hat{n}_{j}$ exhibit singleparticle eigenstate thermalization, we can estimate the scaling of this variance as

$$
\begin{align*}
\sigma^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\left(n_{j}\right)_{\Omega \Omega}\right] & =\mathbf{O}(1 / V) \mathbf{O}(1 / V)+\mathbf{O}(1 / V) \mathbb{E}^{2}\left[\left(n_{j}\right)_{\Omega \Omega}\right] \\
& +\mathbf{O}(1 / V) \mathbb{E}^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\right]=\mathbf{O}(1 / V) \tag{C4}
\end{align*}
$$

where we have used the fact that $\mathbb{E}^{2}\left[\left(n_{j}\right)_{\Omega \Omega}\right]$ and $\mathbb{E}^{2}\left[\left(n_{i}\right)_{\Omega \Omega}\right]$ are $\mathbf{O}(1)$.

In the second term in the parenthesis in Eq. (C2), assuming that the matrix element of the non-Hermitian operator $\left(c_{i}^{\dagger} c_{j}\right)_{\Omega \Omega}$ is real, which is the case for all quadratic Hamiltonians considered here, we replace it with a Hermitian operator, $\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)_{\Omega \Omega} / 2$. Therefore, the corresponding variance can be expressed as

$$
\begin{align*}
\sigma^{2}\left[\left|\left(c_{i}^{\dagger} c_{j}\right)_{\Omega \Omega}\right|^{2}\right] & \propto \sigma^{4}\left[\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)_{\Omega \Omega}\right] \\
& +2 \sigma^{2}\left[\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)_{\Omega \Omega}\right] \mathbb{E}^{2}\left[\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)_{\Omega \Omega}\right] \tag{C5}
\end{align*}
$$

Assuming again that the one-body observable $\hat{c}_{i}^{\dagger} \hat{c}_{j}+\hat{c}_{j}^{\dagger} \hat{c}_{i}$ exhibits the single-particle eigenstate thermalization, and exploiting that $\mathbb{E}\left[\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right)_{\Omega \Omega}\right]=0$, we arrive at

$$
\begin{equation*}
\sigma^{2}\left[\left|\left(c_{i}^{\dagger} c_{j}\right)_{\Omega \Omega}\right|^{2}\right]=\mathbf{O}\left(1 / V^{2}\right) \tag{C6}
\end{equation*}
$$

The above analysis shows that the upper bound of the variance of $\left(n_{i} n_{j}\right)_{\Omega \Omega}$ is set by Eq. (C4), and hence

$$
\begin{equation*}
\sigma^{2}\left[\left(n_{i} n_{j}\right)_{\Omega \Omega}\right] \leq \mathbf{O}(1 / V) \tag{C7}
\end{equation*}
$$

On the other hand, if for a given two-body observable all one-body operators appearing in the Wick decomposition have a vanishing mean of the diagonal matrix elements (the case for $\hat{c}_{i}^{\dagger} \hat{c}_{j}+\hat{c}_{j}^{\dagger} \hat{c}_{i}$ ), then the upper bound of the variance is $\mathbf{O}\left(1 / V^{2}\right)$.

## Appendix D: Additional results for matrix elements

The non-Gaussian PDF of the diagonal matrix elements of $\hat{m}_{0}$ for the 3D Anderson model in the singleparticle quantum-chaotic regime $(W=5)$ can be better understood directly looking at the behavior of the matrix elements. In Fig. 12, we plot randomly selected matrix elements for a system with $V=22^{3}$ sites. The matrix elements exhibit clustering at values that are different from the mean, which resembles the results in the translationally-invariant point $(W=0)$. At the latter point, the matrix elements only take two values $\pm 1$, depending on whether the single-particle ground state is occupied in a many-body energy eigenstate or not. We expect that, for sufficiently large systems, the matrix elements of $\hat{m}_{0}$ for the 3D Anderson model with $W=5$ will be normally distributed about the mean value.


FIG. 12. Diagonal matrix elements of $\hat{m}_{0}$ in the 3D Anderson model for $W=5$. The 3D lattice has $V=22^{3}=10648$ sites, and we report 200 diagonal matrix elements for 10 disorder realizations.
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