

Floquet topological transitions in 2D Su-Schrieffer-Heeger model: interplay between time reversal symmetry breaking and dimerization

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We theoretically study the 2D Su-Schrieffer-Heeger model in the context of Floquet topological insulators (FTIs). FTIs are systems which undergo topological phase transitions, governed by Chern numbers, as a result of time reversal symmetry (TRS) breaking by a time periodic process. In our proposed model, the condition of TRS breaking is achieved by circularly polarized light irradiation. We analytically show that TRS breaking is forbidden in the absence of second order neighbors hopping. In the absence of light irradiation, we identify a symmetry-protected degeneracy and prove the appearance of a flat band along a specific direction in the momentum space. Furthermore, we employ a novel method to show that the four unit cell atoms, in the absence of irradiation, can be interpreted as conserved spin states. With the breaking of TRS via light irradiation, these spin states are no longer conserved, leading to the emergence of chiral edge states. We also show how the interplay between the TRS breaking and dimerization leads to a complex phase diagram. The validity of our findings is substantiated through Chern numbers, spectral properties, localization of chiral edge states and simulations of quantum Hall transport. Our model is suitable not only for condensed matter (materials), but also for cold gases trapped in optical lattices or electric circuits.

I. INTRODUCTION

The study of topological phases represents a highly topical branch of condensed matter physics that has attracted interest from researchers across various expertise areas. In the early 1980s, when the Quantum Hall Effect [1, 2] emerged as one of the most intriguing phenomena at the atomic scale, much of the scientific community endeavored to explain how charged particles could exhibit such exotic behavior (quantification of Hall resistance) under a magnetic field. In the years that followed, the study of topological phases transitioned from physical, insightful models and thought experiments [3] to a classification based on symmetry properties [4, 5]. Once the theoretical foundations were established, there was a surge in efforts to discover, synthesize, and characterize new topological materials. This led to a significant focus in the experimental research community on topological insulators (TIs) [6–13]. In more recent times, a notable historical landmark occurred with the Nobel Prize awarded to David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz for their “theoretical discoveries of topological phase transitions and topological phases of matter” [14].

Technically speaking, topological insulators (TIs) are materials that behave as common insulators in their bulk, while hosting conducting states at their surfaces or edges, depending on the dimensionality of the system [14, 15]. Given the robustness of the topological states against disorder [16, 17], these special materials are considered the building blocks of future technologies [18–23].

From a physical perspective, the Su-Schrieffer-Heeger (SSH) 1D atomic chain [24, 25] serves as the most basic model for understanding topological insulators. Developed to explain the insulating behavior of polyacetylene, its simplicity allows for analytical approaches and serves as a valuable starting point or analogy for studying TIs.

Moving to the 2D class, we must mention the emblematic TI, graphene [26–29], which was one of the first materials predicted to realize a topological phase transition. However,

among other lattice models such as Lieb [30, 31] and Kagome [32], the 2D SSH lattice structure [33–38] has made its way recently. It consists of a planar generalization of the SSH chain or, in other words, it may be seen as a number of reciprocally connected 1D SSH chains, giving rise to a square lattice. Beyond the conceptual model, there are numerous possibilities for materializing 2D SSH lattices, including condensed matter systems [39], ultra-cold atomic gases trapped in optical lattices [40, 41] or electric circuits [42, 43].

In the physics of TIs, the symmetries play the crucial role. The 2D TIs with time-reversal symmetry (TRS) are also known as quantum spin Hall insulators, and they are protected by the Z_2 invariant [44]. When the topological phase transition in a system is triggered by the breaking of the TRS, it is categorized as a *Chern insulator*. Over recent years, various methods have been proposed to induce such conditions [45]. What makes this type of TI interesting to study is the emergence of so-called *chiral edge states* in the energy gap of system. These states are strongly confined to the edges and conduct electricity without dissipation, thus sustaining the Quantum Hall Effect. The hallmark of a topological phase in a Chern insulator is its characterization by an invariant called the *Chern number*, from which it derives its name. During the transition between phases, this invariant takes on integer values of $\pm 1, \pm 2, \pm 3, \dots$, and is directly related to the quantization of the Hall resistance. In the absence of a phase transition, when the system is in a trivial (non-topological) phase, the Chern number remains zero. The most common method for breaking the TRS in such systems involves subjecting them to a magnetic field. However, in a groundbreaking paper, Haldane introduced an alternative approach based on imaginary hopping between second-order neighbors, employing the hexagonal (graphene) lattice as the foundational model [46], and demonstrated the emergence of a non-trivial phase.

More recently, graphene has been utilized as a platform, where it has been reported that circularly polarized light irradiation breaks TRS, realizing the Haldane model and thus inducing a topological phase [47–56]. Generally speaking,

materials that undergo a topological phase transition as a result of circularly polarized light driving are included in the class of Floquet topological insulators (FTIs), named after the framework used to model their physics [57–60]. Since a FTI in a topological phase has a non-zero Chern number, circularly polarized light triggers a phenomenon known as the ‘Quantum Anomalous Hall Effect’ [61]. It is termed anomalous because traditionally, the Hall effect is associated with the presence of a perpendicular magnetic field.

In this paper, we investigate the 2D SSH model from the perspective of FTIs. To achieve this aim, we introduce an interaction model using a tight-binding approach to elucidate the mechanism of topological phase transitions. Initially, we show analytically that TRS breaking is unreachable in the absence of diagonal hopping between lattice atomic sites. Subsequently, we explore the 2D SSH model in the absence of light irradiation and identify a symmetry-protected degeneracy that consistently maintains one of the three potential band gaps closed. This symmetry, incorporating the time-reversal operation, leads to the opening of the gap as circularly polarized light breaks the TRS. Furthermore, we employ a novel method to show that the four unit cell atoms, in the absence of irradiation, can be interpreted as conserved spin states throughout the Fourier space. However, with TRS broken by light irradiation, these spin states are no longer conserved, and this leads to the emergence of chiral edge states. Additionally, the system may undergo a phase transition due to dimerization, similarly to Peierls transition in SSH models [62]. In our model, we distinguish two dimerization cases, depending on the inter- and intra-cellular hopping parameters. We study the topological phases by comparing these two dimerization cases and highlight an intriguing interplay. That is, the system is topologically restricted to always maintaining one of its band gaps as trivial in one dimerization state, which closes inducing a topological phase transition, when passing into the other state. The phase diagram then gets very complex. We corroborate our results with an analysis of spectral features such as band gaps and the presence of chiral edge states, in addition to Chern numbers. We also explore the localization of topological states and show that they are confined to the boundaries. Finally, we verify the topological characteristics by simulating a 4-lead quantum Hall device.

The present paper is organized as follows: In Section II, we revisit the 1D SSH model, briefly describing its emergent topological properties, and introduce the dimerization concept. In fact, Section II serves as a benchmark for our research. In Section III, we elaborate the 2D SSH model and formulate the circularly polarized light interaction Hamiltonian within the Floquet formalism. As well, we analytically argue that the Floquet topological transitions are not allowed in the absence of diagonal direction hopping. In Section IV, we present the main results and give more physical insights about the Floquet topological phases and dimerization, discussing also their interplay. We support the findings using an ingenious interpretation of the 2D SSH lattice, which allows one to approach the unit cell atoms as spin states. In Section V, we discuss the implications of the light helicity reversal. In Section VI, we summarize our work, reiterating the main

results and present the conclusions.

II. SSH MODEL REVISITED

The SSH chain represents one of the most important topological models, originally introduced to describe the electronic properties of polyacetylene. It consists of an infinite bipartite atomic chain, as depicted in Fig. 1. The unit cell (gray shaded area) contains two atoms, indexed as A (blue) and B (red), respectively, separated from each other by lattice constant a . The topological properties of the SSH model are conferred by the existence of two distinct hopping parameters: γ_1 in each unit cell and γ_2 between the unit cells. Given its intrinsic simplicity, this system explains the fundamental mechanism of topological phase transitions in a straightforward manner as an interplay between γ_1 and γ_2 , as will be discussed.

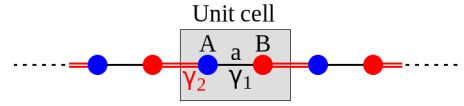


FIG. 1. SSH infinite chain. The unit cell (gray shaded area) contains two atoms indexed by A (blue) and, respectively, B (red). Inside the unit cell, the hopping parameter is γ_1 and, respectively, between unit cells, γ_2 . The lattice constant is a .

In Fourier space, the Hamiltonian of the system is given by:

$$H(k) = \begin{pmatrix} \varepsilon & \gamma_1 + \gamma_2 e^{-ika} \\ \gamma_1 + \gamma_2 e^{ika} & \varepsilon \end{pmatrix}, \quad (1)$$

where ε represents the on-site energy.

The topological properties of the SSH chain are governed by the Zak phase [63], which represents a Berry-like phase acquired by the wave function after one cyclic evolution through the Brillouin zone (BZ), defined as:

$$\varphi_{\text{Zak}} = i \int_{\text{BZ}} \langle u(k) | \partial_k | u(k) \rangle. \quad (2)$$

In the integral (2), which is performed over the entire BZ, $|u_n(k)\rangle$ represents the periodic component of the Bloch wave function $|\psi(k)\rangle = e^{ikx}|u(k)\rangle$.

The inversion symmetry property of the Hamiltonian (1), $\sigma_x H(k) \sigma_x = H(-k)$, leads to the quantization of the Zak phase as $\varphi_{\text{Zak}} = 0, \pm\pi$ [64–67]. Considering this effect, for $\varphi_{\text{Zak}} = 0$, the system lies in a trivial (non-topological) phase, and for $\varphi_{\text{Zak}} = \pm\pi$, the system undergoes a transition into a topological phase. Based on the Zak phase, the topological phase diagram is presented in Fig. 2(a), which depicts φ_{Zak} modulo 2π .

Now, we distinguish between two phases: the trivial phase for $|\gamma_2| < |\gamma_1|$ (black area) and the topological phase for $|\gamma_2| > |\gamma_1|$ (yellow area). Also, we generally associate the case of $|\gamma_2| < |\gamma_1|$ with one dimerization state and, respectively, $|\gamma_2| > |\gamma_1|$ with the other possible state. On the boundary of

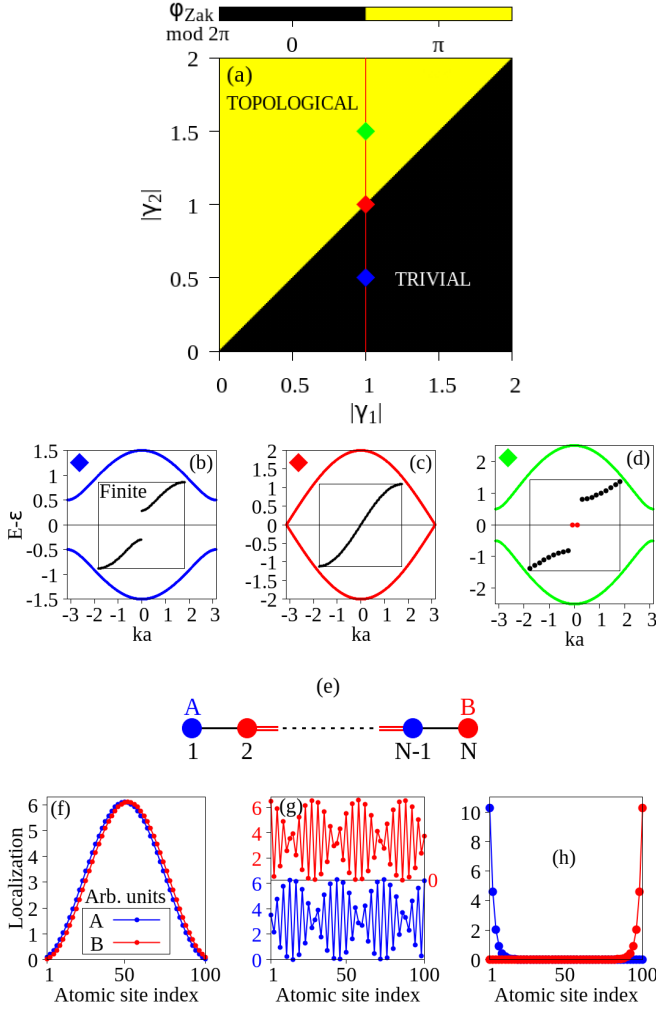


FIG. 2. Topological phases of the SSH chain. (a) Topological phase diagram based on Zak phase: $\varphi_{\text{Zak}} \bmod 2\pi$ vs. $|Y_1|$ and $|Y_2|$. The system lies in a topological phase whenever $|Y_2| > |Y_1|$. (b), (c), (d) Energy dispersion for the trivial phase [blue marker within (a), $|Y_2| = 0.5$], phase transition (red marker, $|Y_2| = |Y_1| = 1$), and topological phase (green marker, $|Y_2| = 1.5$). The reference level is set at the on-site energy ε . Insets show the eigenenergies for an SSH chain containing $N = 100$ atomic sites, as schematized in (e). In the case of the topological phase, two topological states arise inside the gap (red dots). (f), (g), (h) Localization of A and B states corresponding to (b), (c), and (d), respectively.

these two distinct phases, the Zak phase is not defined. In what follows, we present the energy dispersion for the trivial phase [blue marker within panel (a), $|Y_2| = 0.5$], the topological phase transition (red marker, $|Y_2| = 1$), and the topological phase (green marker, $|Y_2| = 1.5$) in panels (b), (c), and (d), respectively. The energy values are expressed with respect to the on-site energy ε as the reference level. The phase transition is fundamentally characterized by the energy band closing at $ka = \pm\pi$ when $|Y_1| = |Y_2|$. In this scenario, the off-diagonal coupling terms in the Hamiltonian (1) vanish, resulting in the band closing at the non-bonding combination of A and B or-

bitals, namely at the on-site energy ε . The insets depict the energy eigenvalues for a finite SSH chain containing $N = 100$ atomic sites, as schematized in Fig. 2(e). In the topological phase, two energy values emerge in the gap, representing the topological states (depicted as red dots). These states are protected by the chiral symmetry $\sigma_z H(k) \sigma_z = -H(k)$. To highlight this special property, using Eq. (55), we illustrate the localization within the chain for both A and B states in panels (f), (g), and (h), corresponding to the three cases presented in panels (b), (c), and (d), respectively. In the trivial phase, the atomic states are mainly localized in the middle of the chain. As the system evolves towards the topological phase, in the phase transition regime, the states become highly delocalized in the bulk of the atomic chain. Finally, in the topological phase, edge states emerge, with the A states confined to the left edge of the system and the B states localizing at the right edge.

The SSH chain topological transition governed by the Zak phase has its origin in the so-called dimerization of 1D atomic chains with one electron per atomic site and was explained by Rudolf Peierls in 1930. Nowadays, it is called *Peierls transition* or *Peierls distortion*. In such systems, the condition for a topological phase transition due to dimerization state switching is achieved when the intra-cellular hopping parameter equals the inter-cellular one. In Fourier space, the transition always takes place at the non-bonding energy point.

III. 2D SSH MODEL. FLOQUET TOPOLOGICAL TRANSITIONS

A. 2D SSH model

In this section, we briefly present the 2D SSH model. We study a square lattice system, as presented in Fig. 3 (a), with a unit cell (gray shaded area) containing four atoms [see Fig. 3 (b)]. The distance between the atoms is a . In our proposed model, we distinguish three hopping parameters: γ_1 inside each unit cell, γ_2 between the unit cells and γ_3 between the second order neighbor atoms (diagonal) only inside the unit cell. The role of γ_3 will become clear in the next section.

In Fourier space, the system is described by the following Hamiltonian:

$$H(k_x, k_y) = \begin{pmatrix} 0 & \gamma_1 + \gamma_2 e^{-ik_x} & \gamma_1 + \gamma_2 e^{-ik_y} & \gamma_3 \\ \gamma_1 + \gamma_2 e^{ik_x} & 0 & \gamma_3 & \gamma_1 + \gamma_2 e^{-ik_y} \\ \gamma_1 + \gamma_2 e^{ik_y} & \gamma_3 & 0 & \gamma_1 + \gamma_2 e^{-ik_x} \\ \gamma_3 & \gamma_1 + \gamma_2 e^{ik_y} & \gamma_1 + \gamma_2 e^{ik_x} & 0 \end{pmatrix}. \quad (3)$$

Here, for convenience, we have considered a zero on-site energy (diagonal elements) and set $a = 1$. The First BZ is depicted in Fig. 3(c) and the high symmetry points are $\Gamma : (0, 0)$, $X : (\pi, 0)$ and $M : (\pi, \pi)$.

For the case of $\gamma_3 = 0$, the 2D SSH model behaves similarly to the 1D SSH chain discussed in Section II. For instance, considering $\gamma_1 = -1$, we present in Fig. 4 the energy dispersion on M - X - Γ - M direction in BZ, for three fundamen-

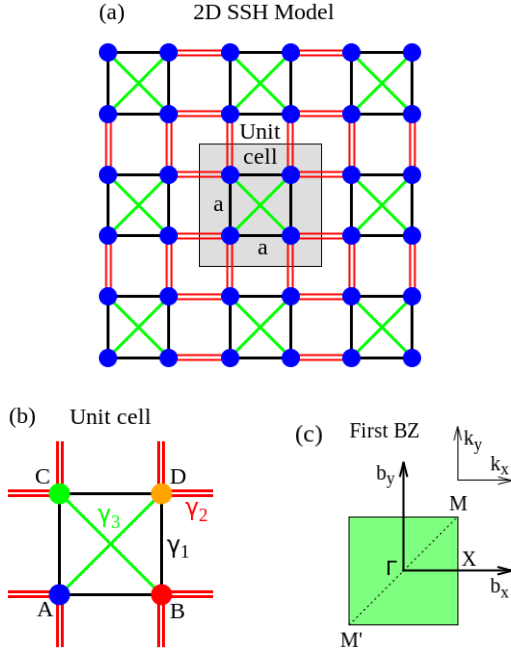


FIG. 3. 2D SSH model. (a) The 2D SSH square lattice in real space. The distance between the atoms is denoted by a . (b) The unit cell, containing four atoms indexed by A, B, C and D . Inside the unit cell, γ_1 represents the hopping parameter between the first order neighbors (horizontal and vertical) and, respectively, γ_3 , between the second order neighbors (diagonal hopping). (c) First BZ.

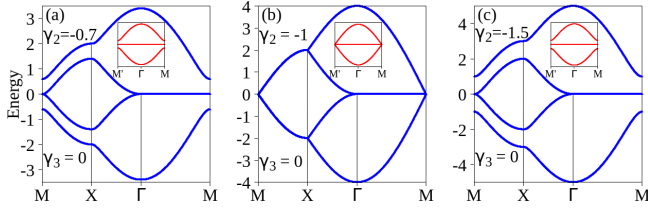


FIG. 4. Energy dispersion of a 2D SSH lattice on M - X - Γ - M direction, for $\gamma_1 = -1$ and $\gamma_3 = 0$. The inter-cellular hopping parameters are (a) $\gamma_2 = -0.7$ (trivial phase), (b) $\gamma_2 = -1$ (topological phase transition) and (c) $\gamma_2 = -1.5$ (topological phase). The insets show the dispersion on M' - Γ - M direction.

tal cases: trivial phase [panel (a), $\gamma_2 = -0.7$], phase transition [panel (b), $\gamma_2 = \gamma_1 = -1$] and topological phase [panel (c), $\gamma_2 = -1.5$]. The insets show the dispersion on M' - Γ - M direction. The gap closing is realized by the uppermost and lowermost bands in $M(M')$, while the inner bands close permanently on Γ and $M(M')$. At the phase transition, the gap closing is realized in X and $M(M')$. An other important property of the 2D SSH model is the occurrence of a completely flat band centered on the non-bonding energy level (0 in our present case), on the M' - Γ - M direction. This property will be carefully investigated in Section IV. An in-depth presentation may be found in Ref. [35].

B. TRS breaking: Floquet topological transition

In this section, we introduce an interaction model of 2D SSH system with a circularly polarized light beam. We consider the light ($\hbar\omega$) propagating perpendicularly on the lattice plane, as shown in Fig. 5. Our main purpose is to elucidate the mechanism of TRS breaking which actually represents the trigger of the Floquet topological transitions in the studied lattice model.

The physics of the light driven system is modeled using Peierls substitution. Generally speaking, within this approach, each hopping parameter contained by the Hamiltonian (3) is factorized by a time dependent phase term $e^{i\theta(t)}$, where

$$\theta(t) = \frac{-e}{h} \int_{\Gamma_{i \rightarrow j}} \mathbf{A}(t) \cdot d\mathbf{s}. \quad (4)$$

Here, the integral is performed along the path $\Gamma_{i \rightarrow j}$ which connects the neighbor atoms i and j , $\mathbf{A}(t)$ represents the light vector potential and $d\mathbf{s}$ parameterizes the integration path. The constant factors e and h represent the elementary charge and, respectively, the Planck constant.

Going further, we explicit Eq. (4) considering the following plane wave vector potential:

$$\mathbf{A}(t) = A_0 [\cos(\omega t)\mathbf{e}_x + \Lambda \sin(\omega t)\mathbf{e}_y], \quad (5)$$

where A_0 is a real constant amplitude, ω the light frequency, Λ the helicity quantum number, $\mathbf{e}_x$ and $\mathbf{e}_y$ the unit vectors along the x - and y -axis, respectively. Thus, we distinguish now four Peierls phases between the first order neighbors (horizontal and vertical hopping):

$$\theta_L(t) = \frac{eaA_0}{h} \cos(\omega t); \quad (6a)$$

$$\theta_R(t) = \frac{-eaA_0}{h} \cos(\omega t); \quad (6b)$$

$$\theta_D(t) = \Lambda \frac{eaA_0}{h} \sin(\omega t); \quad (6c)$$

$$\theta_U(t) = \Lambda \frac{-eaA_0}{h} \sin(\omega t). \quad (6d)$$

Eqs. (6a)–(6d) represent the phases acquired by the electron wave function corresponding to a left, right, down and, respectively, up hopping on a first order neighbor atomic site. For the case of second order neighbors hopping (on diagonally placed atomic sites), we introduce the following terms:

$$\theta_{LD}(t) = \frac{eaA_0}{h} [\cos(\omega t) + \Lambda \sin(\omega t)]; \quad (7a)$$

$$\theta_{RD}(t) = \frac{eaA_0}{h} [-\cos(\omega t) + \Lambda \sin(\omega t)]; \quad (7b)$$

$$\theta_{LU}(t) = \frac{eaA_0}{h} [\cos(\omega t) - \Lambda \sin(\omega t)]; \quad (7c)$$

$$\theta_{RU}(t) = \frac{eaA_0}{h} [-\cos(\omega t) - \Lambda \sin(\omega t)]. \quad (7d)$$

Analogously, Eqs. (7a)–(7d) represent the Peierls phases corresponding to a left-down, right-down, left-up and, respectively, right-up second order neighbors hopping.

Now, considering the matrix representation (3) and the

Peierls phases (6a)–(6d) and (7a)–(7d), we formulate the light driven 2D SSH time dependent Hamiltonian:

$$H(k_x, k_y; t) = \begin{pmatrix} 0 & \gamma_1 e^{i\theta_L(t)} + \gamma_2 e^{i\theta_R(t)} e^{ik_x} & \gamma_1 e^{i\theta_D(t)} + \gamma_2 e^{i\theta_U(t)} e^{ik_y} & \gamma_3 e^{i\zeta_{LD}(t)} \\ \gamma_1 e^{i\theta_R(t)} + \gamma_2 e^{i\theta_L(t)} e^{-ik_x} & 0 & \gamma_3 e^{i\zeta_{RD}(t)} & \gamma_1 e^{i\theta_D(t)} + \gamma_2 e^{i\theta_U(t)} e^{ik_y} \\ \gamma_1 e^{i\theta_U(t)} + \gamma_2 e^{i\theta_D(t)} e^{-ik_y} & \gamma_3 e^{i\zeta_{LU}(t)} & 0 & \gamma_1 e^{i\theta_L(t)} + \gamma_2 e^{i\theta_R(t)} e^{ik_x} \\ \gamma_3 e^{i\zeta_{RU}(t)} & \gamma_1 e^{i\theta_U(t)} + \gamma_2 e^{i\theta_D(t)} e^{-ik_y} & \gamma_1 e^{i\theta_R(t)} + \gamma_2 e^{i\theta_L(t)} e^{-ik_x} & 0 \end{pmatrix}. \quad (8)$$

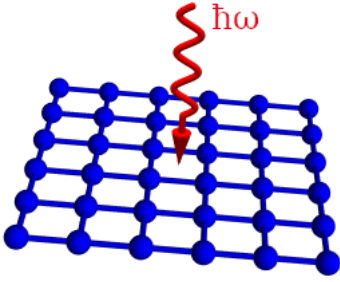


FIG. 5. 2D SSH square lattice under circularly polarized light irradiation ($\hbar\omega$). The light is propagating perpendicularly on the lattice plane.

Since the Hamiltonian (8) is periodic in time with the period $T = \frac{2\pi}{\omega}$, ω being the light frequency, we will make use of the Floquet formalism, described in what follows. The starting point is represented by the Floquet theorem [68, 69] which assures that a time periodic Hamiltonian

$$H(t + T) = H(t) \quad (9)$$

admits eigenfunctions of the form

$$\psi(t) = e^{\frac{-iWt}{\hbar}} \phi(t). \quad (10)$$

In Eq. (10), W is so-called *quasienergy* and $\phi(t)$, the *Floquet function*, having the same periodicity as the Hamiltonian (9):

$$\phi(t + T) = \phi(t). \quad (11)$$

Inserting the form (10) into the Schrödinger equation $H(t)\psi(t) = i\partial_t\psi(t)$, we end with the following eigenvalue equation for the *Floquet Hamiltonian* $H_F(t)$:

$$H_F(t)\phi(t) = W\phi(t); \quad (12)$$

$$H_F(t) = H(t) - i\hbar\partial_t. \quad (13)$$

Further, we exploit the time periodicity properties (9) and (11) to introduce the following Fourier representations for the

Hamiltonian and the Floquet function, respectively:

$$H(t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} H_n; \quad (14)$$

$$\phi(t) = \sum_{n=-\infty}^{\infty} e^{-in\omega t} \phi_n. \quad (15)$$

The corresponding inverse transformations are:

$$H_n = \frac{1}{T} \int_0^T e^{in\omega t} H(t) dt; \quad (16)$$

$$\phi_n = \frac{1}{T} \int_0^T e^{in\omega t} \phi(t) dt. \quad (17)$$

Finally, using the Eqs. (12)–(17), we obtain the *Floquet system of equations*:

$$\left(\sum_{m=-\infty}^{\infty} H_{m-n} + m\hbar\omega\delta_{mn} \right) \phi_m = W\phi_n; \quad (18)$$

$$n = -\infty, \dots, -1, 0, 1, \dots, \infty.$$

Note that the mathematical maneuver we have made to obtain the system (18) eliminated the time coordinate degree of freedom. Despite the system (18) contains an *infinite* number of *coupled* differential equations, we may formulate effective problems, truncating the system in a proper manner, as imposed by the energy scales. If the photon energy $\hbar\omega$ is large enough compared to the band width, the system is well described by the following high frequency Hamiltonian [70, 71], resulted from (18):

$$H_{\text{HF}}(k_x, k_y) = H_0(k_x, k_y) + H_{\text{INT}}(k_x, k_y); \quad (19)$$

$$H_{\text{INT}}(k_x, k_y) = \frac{1}{\hbar\omega} [H_{-1}(k_x, k_y), H_1(k_x, k_y)]. \quad (20)$$

The first term $H_0(k_x, k_y)$ within Eq. (20) represents the averaged Hamiltonian (8) over a period T . The second term $H_{\text{INT}}(k_x, k_y)$ describes the virtual interaction process of absorption-emission of one photon.

Now, let us evaluate $H_n(k_x, k_y)$, considering $\frac{ea}{\hbar} = 1$. Performing the integrals (16), we are led to the following substitutions:

$$e^{i\theta_L(t)} \rightarrow i^n J_n(A_0) e^{i\Lambda n\pi}, \quad (21a)$$

$$e^{i\theta_R(t)} \rightarrow i^n J_n(A_0); \quad (21b)$$

$$e^{i\theta_D(t)} \rightarrow i^n J_n(A_0) e^{i\Lambda n\pi/2}; \quad (21c)$$

$$e^{i\theta_U(t)} \rightarrow i^n J_n(A_0) e^{-i\Lambda n\pi/2}; \quad (21d)$$

$$e^{i\theta_{LD}(t)} \rightarrow i^n J_n(\sqrt{2}A_0) e^{-i\Lambda n3\pi/4}; \quad (21e)$$

$$e^{i\theta_{RD}(t)} \rightarrow i^n J_n(\sqrt{2}A_0) e^{-i\Lambda n\pi/4}; \quad (21f)$$

$$e^{i\theta_{LU}(t)} \rightarrow i^n J_n(\sqrt{2}A_0) e^{i\Lambda n3\pi/4}; \quad (21g)$$

$$e^{i\theta_{RU}(t)} \rightarrow i^n J_n(\sqrt{2}A_0) e^{i\Lambda n\pi/4}. \quad (21h)$$

Here, $J_n(x)$ denotes the n -th order Bessel function of the first kind.

Taking into account each harmonic order $n = 0, \pm 1$ and applying the substitutions (21a)–(21h), we find $H_0(k_x, k_y)$ and $H_{\text{INT}}(k_x, k_y)$, having the following form:

$$H_0(k_x, k_y) = J_0(A_0) \begin{pmatrix} 0 & \gamma_1 + \gamma_2 e^{-ik_x} & \gamma_1 + \gamma_2 e^{-ik_y} & \frac{J_0(\sqrt{2}A_0)}{J_0(A_0)} \gamma_3 \\ \gamma_1 + \gamma_2 e^{ik_x} & 0 & \frac{J_0(\sqrt{2}A_0)}{J_0(A_0)} \gamma_3 & \gamma_1 + \gamma_2 e^{-ik_y} \\ \gamma_1 + \gamma_2 e^{ik_y} & \frac{J_0(\sqrt{2}A_0)}{J_0(A_0)} \gamma_3 & 0 & \gamma_1 + \gamma_2 e^{-ik_x} \\ \frac{J_0(\sqrt{2}A_0)}{J_0(A_0)} \gamma_3 & \gamma_1 + \gamma_2 e^{ik_y} & \gamma_1 + \gamma_2 e^{ik_x} & 0 \end{pmatrix}; \quad (22)$$

$$H_{\text{INT}}(k_x, k_y) = i\gamma_3 \frac{2\sqrt{2}J_1(A_0)J_1(\sqrt{2}A_0)}{\hbar\omega} \begin{pmatrix} 0 & \gamma_1 - \gamma_2 \cos(k_y) & -\gamma_1 + \gamma_2 \cos(k_x) & 0 \\ -\gamma_1 + \gamma_2 \cos(k_y) & 0 & 0 & \gamma_1 - \gamma_2 \cos(k_x) \\ \gamma_1 - \gamma_2 \cos(k_x) & 0 & 0 & -\gamma_1 + \gamma_2 \cos(k_y) \\ 0 & -\gamma_1 + \gamma_2 \cos(k_x) & \gamma_1 - \gamma_2 \cos(k_y) & 0 \end{pmatrix}. \quad (23)$$

$H_0(k_x, k_y)$ respects the TRS: $H_0(k_x, k_y) = H_0^*(-k_x, -k_y)$. On the other hand, the interaction term $H_{\text{INT}}(k_x, k_y) \neq H_{\text{INT}}^*(-k_x, -k_y)$ breaks the TRS, therefore the virtual process of absorption-emission of photons is responsible for the topological phase transitions triggered by the light irradiation. However, obviously, this is valid only when $\gamma_3 \neq 0$. Hence, in our proposed model, *the presence of second order neighbors (diagonal) hopping is mandatory for Floquet topological transitions*. Contrastingly, the Hamiltonian remains time reversal symmetric and the 2D SSH system lies always in a trivial phase. In the absence of the diagonal hopping, the only effect of light irradiation will be the renormalization of the hopping parameters and, implicitly, of the spectrum band width.

IV. RESULTS AND DISCUSSION

In this Section, we present the main results and give more physical insights within the context of TRS breaking and TIs realm. Since in the absence of light irradiation, the 2D SSH model is endowed with some critical symmetries, we must start the analysis keeping the light off. By analogy with the SSH model described in Section II, we distinguish two dimerization cases: one for $\gamma_2 < \gamma_1$ and, respectively, the other one

for $\gamma_2 > \gamma_1$. Then, we investigate the effect of light driving separately for $\gamma_2 < \gamma_1$ and, thereafter, for $\gamma_2 > \gamma_1$. Our aim is to highlight the topological properties of the circularly polarized light driven 2D SSH model, comparing the two above discussed cases.

A. No light

First, in this Section, before discussing the Floquet topological transitions for some specific cases, we introduce the second order neighbors hopping, in the absence of light irradiation. In this case, in Γ and M within the Fourier space, the eigenvalues of Hamiltonian (3) read as follows:

$$E_1(\Gamma) = E_2(\Gamma) = -\gamma_3; \quad (24a)$$

$$E_3(\Gamma) = 2 - 2\gamma_2 + \gamma_3; \quad (24b)$$

$$E_4(\Gamma) = -2 + 2\gamma_2 + \gamma_3; \quad (24c)$$

$$E_1(M) = E_2(M) = -\gamma_3; \quad (24d)$$

$$E_3(M) = -2 - 2\gamma_2 + \gamma_3; \quad (24e)$$

$$E_4(M) = 2 + 2\gamma_2 + \gamma_3. \quad (24f)$$

Note that for both Γ and M , there exist two twofold degenerate energy levels with the value $-\gamma_3$, see Eqs. (24a) and

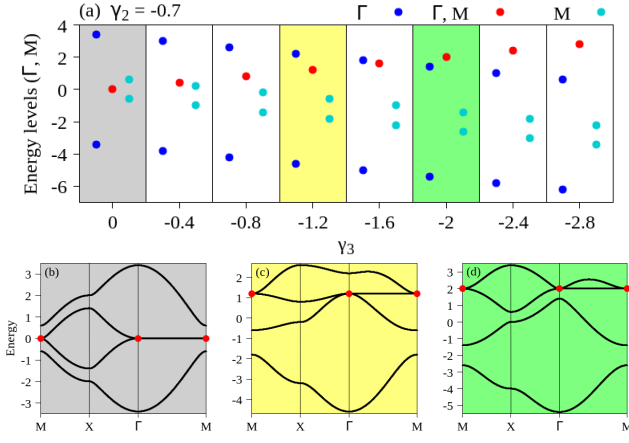


FIG. 6. (a) Energy values in Γ and M points vs. γ_3 , with $\gamma_2 = -0.7$: blue dots for Eqs. (24b) and (24c); cyan dots for Eqs. (24e) and (24f); red dots for Eqs. (24a) and (24d). The red dots represent the twofold degenerate energy levels where always two bands crosses. (b), (c), (d): Energy dispersion on M - X - Γ - M direction for: $\gamma_3 = 0$, $\gamma_3 = -1.2$ and $\gamma_3 = -2$, respectively (as indicated by the background colors). The red dots represent the degeneracy in Γ and M .

(24d). Consequently, two bands cross in these two high symmetry points at a value of $-\gamma_3$. In Fig. 6(a), we present some examples for a fixed $\gamma_2 = -0.7$ and varying discretely γ_3 . The blue dots correspond to Eq. (24b) and (24c), while the cyan ones, to (24e) and (24f), respectively. The red dots represent the twofold degenerate energy levels (24a) and (24d). In panels (b), (c) and (d), we show the energy dispersion for $\gamma_3 = 0$, $\gamma_3 = -1.2$ and $\gamma_3 = -2$, respectively. See also the background colors correspondence. For each case, the bands crossing is marked by the same red dot.

The twofold degeneracy in Γ and M points is a consequence of a so-called "hidden symmetry", as discussed in Refs. [72–74]. The main idea is that a (twofold) degeneracy is attributed to an antiunitary symmetry, expressed by an operator Υ with its square $\Upsilon^2 = -1$, satisfying $[\Upsilon, H] = 0$ in the degeneracy points.

In our problem, in Γ and M , the normalized eigenvectors of Hamiltonian (3) read:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}^T; \quad (25)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}^T; \quad (26)$$

$$|\psi_3\rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}^T; \quad (27)$$

$$|\psi_4\rangle = \frac{-1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T. \quad (28)$$

The degeneracy corresponds to Eqs. (25) and (26).

Further, extending the method presented in Ref. [74], we define the following antiunitary operator:

$$\Upsilon = (|\psi_1\rangle\langle\psi_2^*| - |\psi_2\rangle\langle\psi_1^*| + |\psi_3\rangle\langle\psi_4^*| - |\psi_4\rangle\langle\psi_3^*|) \mathcal{K}, \quad (29)$$

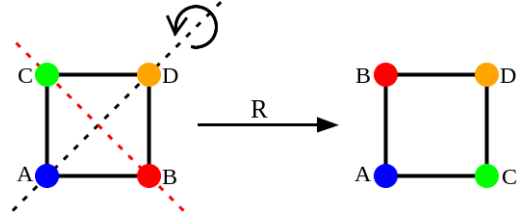


FIG. 7. Reflection of the unit cell with respect to its A - D diagonal direction (dotted line). The action of $\mathcal{R}$ interchanges the position of B and C atoms.

where $\mathcal{K}$ represents the complex conjugation operator. Substituting Eqs. (25)–(28) into Eq. (29), we find the operator in question in the following form:

$$\Upsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -i\sigma_y \mathcal{K} \otimes \sigma_z = \mathcal{T} \otimes \sigma_z, \quad (30)$$

where $\mathcal{T} = -i\sigma_y \mathcal{K}$ denotes the time reversal operator. Next, the following symmetries satisfied in the degeneracy points may be easily verified:

$$[\Upsilon, H(\Gamma)]|\psi_1\rangle = [\Upsilon, H(\Gamma)]|\psi_2\rangle = 0; \quad (31)$$

$$[\Upsilon, H(M)]|\psi_1\rangle = [\Upsilon, H(M)]|\psi_2\rangle = 0. \quad (32)$$

Finally, following the proof presented in Ref. [74], taking into account the antiunitarity of Υ and $\Upsilon^2 = -1$, one may straightforwardly argue that the symmetries (31) and (32) protect the twofold degeneracy in Γ and M .

Having elucidated the underlying physics of the twofold degeneracy in Γ and M , in what follows, we focus our attention on the flat band arising on Γ - M (M') direction, which is still present regardless of γ_3 magnitude. Also, this spectral property is given by an other symmetry, this time being the reflection of the unit cell with respect to its diagonal A - D or B - C axis, as shown in Fig. 7. The transformation discussed here is expressed by the operator

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (33)$$

acting on the state vector $|\psi\rangle = (A, B, C, D)^T$. One may easily prove that the Hamiltonian (3) obeys the following symmetry on Γ - M (M') direction ($k_x = k_y = k$):

$$[\mathcal{R}, H(k, k)] = 0. \quad (34)$$

To explain the occurrence of the flat band, we make use of a more subtle interpretation of the system, formulating a new effective problem. In this respect, taking into account the reflection symmetry discussed above, we rearrange the Hamil-

tonian (3) in the following block-form:

$$H(k_x, k_y) = \begin{pmatrix} h & \Gamma(k_x, k_y) \\ \Gamma^\dagger(k_x, k_y) & h \end{pmatrix}; \quad (35)$$

$$h = \begin{pmatrix} 0 & \gamma_3 \\ \gamma_3 & 0 \end{pmatrix}; \quad (36)$$

$$\Gamma(k_x, k_y) = \begin{pmatrix} \gamma_1 + \gamma_2 e^{-ik_x} & \gamma_1 + \gamma_2 e^{-ik_y} \\ \gamma_1 + \gamma_2 e^{ik_y} & \gamma_1 + \gamma_2 e^{ik_x} \end{pmatrix}. \quad (37)$$

Now, starting from the stationary Schrödinger eq. $H(k_x, k_y)\psi(k_x, k_y) = E(k_x, k_y)\psi(k_x, k_y)$, one may reduce the problem to the following effective Hamiltonian:

$$H_{\text{eff}}(k_x, k_y) = h + \Gamma(k_x, k_y) (E \cdot \mathbf{1}_2 - h)^{-1} \Gamma^\dagger(k_x, k_y). \quad (38)$$

In Eq. (38), the symbol $\mathbf{1}_2$ denotes the identity 2×2 matrix. The mathematical maneuver we have implied here is known as *decimation method*, see Refs. [53, 75–77].

As we have reformulated the problem, the Hamiltonian (38) may be also understood as describing the B and C states, taking into consideration the presence of the other two A and D atoms. Now, in order to evaluate the system on Γ - M (M') direction in Fourier space, we impose the condition $k_x = k_y = k$. After performing all the simplifications, Eq. (38) reduces to

$$H_{\text{eff}}(k, k) = \begin{pmatrix} \zeta & \zeta + \gamma_3 \\ \zeta + \gamma_3 & \zeta \end{pmatrix}; \quad (39)$$

$$\zeta = 2 \frac{E(\gamma_2^2 + 1) + \gamma_3 - 2\gamma_2(E + \gamma_3) \cos(k) + \gamma_2^2 \gamma_3 \cos(2k)}{(E - \gamma_3)(E + \gamma_3)}. \quad (40)$$

In the last step, to obtain the eigenvalues, we must solve the characteristic equation

$$\det(H_{\text{eff}}(k, k) - \lambda \cdot \mathbf{1}_2) = 0. \quad (41)$$

Straightforwardly, may be verified that Eq. (41) translates as

$$(\gamma_3 + \lambda)f(\lambda; E, k, \gamma_2, \gamma_3) = 0, \quad (42)$$

where f represents a function of λ and depends also on the other indicated parameters. It is not necessary to specify here its form. However, it is obvious that one eigenvalue of the effective Hamiltonian (39) is

$$\lambda = -\gamma_3. \quad (43)$$

Since it does not depend on k_x and k_y , Eq. (43) expresses exactly the flat band which arises on Γ - M (M') direction.

The decimation method we have involved, allows us to interpret the 2D SSH tetraatomic unit cell as an effective *bipartite* one, with a non-zero and $\mathbf{k}$ -dependent on-site energy. The effective model thus elaborated is depicted in Fig. 8(a). As can be observed here, the 2D SSH model is reproduced by the two-component effective unit cell (gray shaded area). The effective unit cell, shown in Fig. 8(b), contains two composite "atoms", indexed by $\tilde{A}$ and $\tilde{B}$, respectively. The "atom" $\tilde{A}$ is formed by the old ones A and D , while $\tilde{B}$, by old B and C .

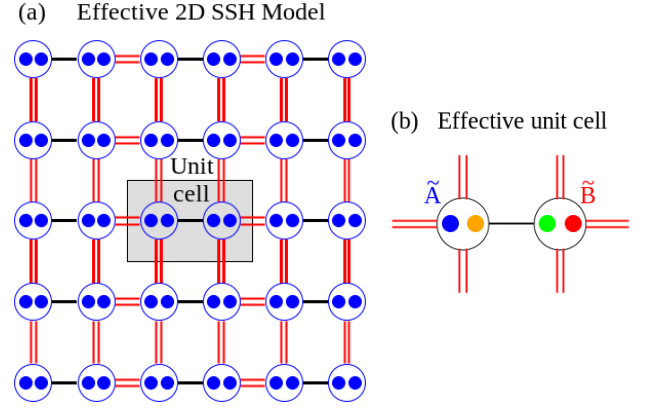


FIG. 8. Effective model. (a) The 2D SSH lattice is reproduced by a bipartite unit cell (gray shaded area). (b) The effective unit cell contains two composite "atoms" indexed by $\tilde{A}$ (formed by the old A and D atoms) and $\tilde{B}$ (formed by the old B and C).

Analyzing Eq. (40), we find that for $\gamma_1 = \gamma_2$ in Γ and M points of Fourier space [see Fig. 2(c)], the effective Hamiltonian (39) reduces respectively to

$$H_{\text{eff}}(\Gamma) = \begin{pmatrix} \frac{8}{E-\gamma_3} & \frac{8}{E-\gamma_3} + \gamma_3 \\ \frac{8}{E-\gamma_3} + \gamma_3 & \frac{8}{E-\gamma_3} \end{pmatrix}; \quad (44)$$

$$H_{\text{eff}}(M) = h = \begin{pmatrix} 0 & \gamma_3 \\ \gamma_3 & 0 \end{pmatrix}. \quad (45)$$

The corresponding eigenvalues read:

$$E_1(\Gamma) = \gamma_3 - 4; \quad E_2(\Gamma) = E_3(\Gamma) = -\gamma_3; \quad E_4(\Gamma) = \gamma_3 + 4; \quad (46)$$

$$E_1(M) = E_2(M) = \gamma_3; \quad E_3(M) = E_4(M) = -\gamma_3. \quad (47)$$

Since $\gamma_1 = \gamma_2$ is the condition for a topological phase transition due to dimerization state switching, we interpret Eqs. (46) and (47) as follows: $E_b = \gamma_3 - 4$ represents the bonding energy having the minimum value and $E_{nb} = \pm\gamma_3$, the non-bonding energy. In this case, all the gaps must close in Γ and M at E_{nb} values. See for instance Fig. 9 for $\gamma_3 = -1.2$, with $E_b = -5.2$ and $E_{nb} = \pm 1.2$.

B. Light irradiation: $\gamma_2 < \gamma_1$

Next, in this Section, we present the main results concerning the Floquet topological transitions within the 2D SSH model discussed in Section III, starting from the following hopping parameters: $\gamma_1 = -1$ and $\gamma_3 = -2$. The photon energy is fixed at $\hbar\omega = 4$ and the light helicity is $\Lambda = -1$. Throughout of the paper, the light vector potential amplitude A_0 , which gives the driving intensity, will be expressed in terms of $\hbar/(ea)$ constant. In this configuration, the circularly polarized light driven 2D SSH system is described by the Hamiltonian (19).

From the very beginning, we want to highlight two fundamental effects produced by the circularly polarized light.

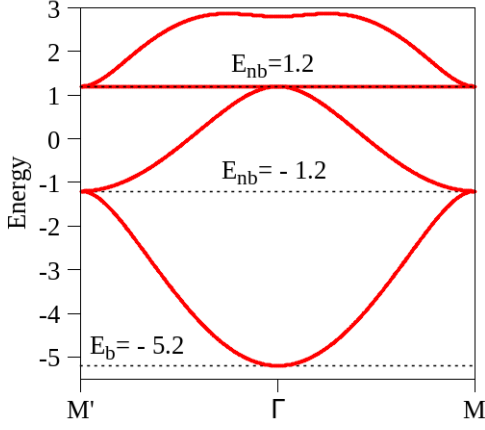


FIG. 9. Energy dispersion on $M'-\Gamma-M$ direction, for the case of $\gamma_1 = \gamma_2$ (topological phase transition condition). The gaps close in Γ and $M(M')$ at non-bonding energy $E_{nb} = \pm 1.2$. The bonding (minimum) energy is $E_b = -5.2$, in Γ .

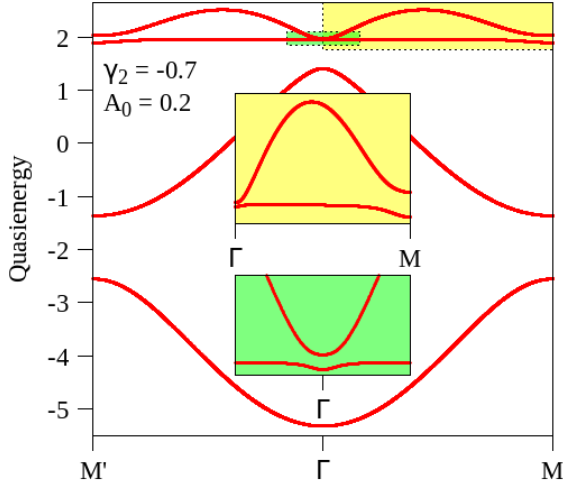


FIG. 10. Energy dispersion on $M'-\Gamma-M$ direction for the case of $\gamma_2 = -0.7$ and $A_0 = 0.2$. The energy band gap appearance and the flatness breaking of the third band are more clearly revealed in the insets which show a zoomed picture along $\Gamma-M$ direction and, respectively, in the vicinity of Γ .

Since the light irradiation breaks the TRS, as argued in Section III B, the symmetries (31) and (32) are no longer satisfied, given that Υ contains the time reversal operator $\mathcal{T}$. Hence, the two protected degeneracies will be lifted, giving rise to an energy band gap. Besides the TRS, the light irradiation breaks also the spatial symmetries of the system, including also (34), and, consequently, the flat band occurred in the absence of light begins to disperse. These two effects generated by circularly polarized light actually represent the precursors of the Floquet topological phase transitions. For instance, we show in Fig. 10 the quasienergy dispersion for the case of $\gamma_2 = -0.7$ and $A_0 = 0.2$. The two effects in question are more clearly

shown in the insets which present a zoomed picture along the $\Gamma-M$ direction and, respectively, in the vicinity of Γ . In this configuration, having three band gaps, the system is prepared to undergo Floquet topological transitions.

We begin our analysis in terms of topological invariants, namely *Chern numbers*. Since in our model, the circularly polarized light breaks the TRS, in the case of an infinite system, for each energy band there is assigned a possible non-zero Chern number (C_n), where n represents the band index, defined as follows:

$$C_n = \frac{1}{2\pi} \int_{\text{BZ}} \Omega_n(\mathbf{k}) \cdot d\mathbf{k}; \quad (48)$$

$$\Omega_n(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{A}(\mathbf{k}); \quad (49)$$

$$\mathbf{A}(\mathbf{k}) = i \langle u_n(\mathbf{k}) | \nabla_{\mathbf{k}} | u_n(\mathbf{k}) \rangle. \quad (50)$$

In Eq. (48), the integral is performed over the whole BZ and $\Omega_n(\mathbf{k})$ defined in Eq. (49), where $\nabla_{\mathbf{k}}$ denotes the nabla operator within Fourier space, represents the Berry curvature. Eq. (50) defines the Berry connection $\mathbf{A}(\mathbf{k})$ with $|u_n(\mathbf{k})\rangle$ being the periodic component of the Bloch wave function $|\psi(\mathbf{k})\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}|u_n(\mathbf{k})\rangle$.

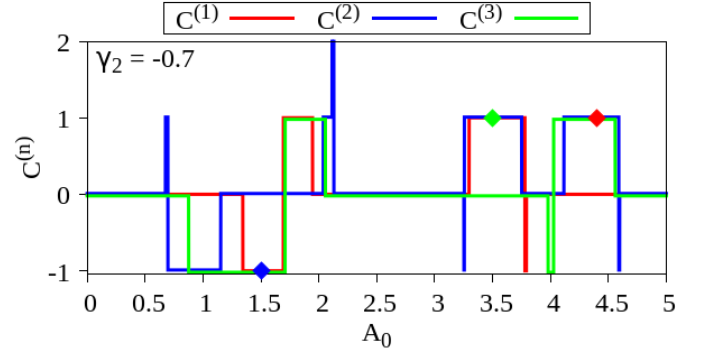


FIG. 11. Topological phase diagram. Summed Chern numbers $C^{(n)}$ for all of the three gaps within the quasienergy band structure vs. A_0 .

Whenever the Fermi level (FL) lies inside an energy gap (assumed to exist due to the presence of the light irradiation), one may describe the topological phase of the system summing the Chern numbers for all bands below the FL. Thus, the gap in which the FL lies is topologically described by the following invariant:

$$C^{(n)} = \sum_{n \in \text{occ}} C_n, \quad (51)$$

where the summation is performed over the all n occupied bands (below FL).

In Fig. 11, we present the topological phase diagram, considering all of the three quasienergy gaps, with respect to A_0 . As the A_0 value continuously varies, we observe the excitation of different topological phases, governed by an interesting restriction. That is, *there is impossible to induce topological phases for the whole three gaps at the same time*. In

other words, $C^{(1)} \neq 0$, $C^{(2)} \neq 0$ and $C^{(3)} \neq 0$ condition is not achievable simultaneously at a fixed A_0 . To explain this topological frustration, we make use again of the effective model discussed in Section IV A (Fig. 8).

First, since the effective unit cell contains only two atoms [see Fig. 8(b)], the second order neighbors hopping is no longer obvious. On the other hand, within the effective model, it becomes a new internal interaction process for "atoms" $\tilde{A}$ and $\tilde{B}$. Hence, A and D states become a new internal degree of freedom for atom $\tilde{A}$ and, analogously, B and C for $\tilde{B}$. Moreover, a deeper analysis reveals the following symmetry, in the absence of light:

$$[\sigma_x, H_{\text{eff}}(k, k)] = 0. \quad (52)$$

Therefore, within the atom $\tilde{A}$, A and D act as two well defined spin states. Obviously, the same is valid for B and C within $\tilde{B}$. Now, taking into consideration that generally the spin states are related by time reversal operation, the TRS breaking due to light irradiation will take place at "atomic" level (inside $\tilde{A}$ and $\tilde{B}$), between A and D and, respectively, between B and C . We anticipate now, that in a topological phase, where chiral edge states will arise, the mixing between A and D ; and

B and C states is topologically forbidden. In other words, in a finite configuration, A and D states will be localized at opposite edges, having also opposite momentum directions and, analogously, B and C . In Section IIIB, we have mathematically demonstrated that the presence of second order neighbors hopping is mandatory for TRS breaking, but with no much physical insight. Having in mind the interpretation of the second order neighbors hopping as an internal interaction process, we have now a deeper physical understanding concerning the TRS breaking in a 2D SSH system and its underlying mechanism.

Second, going outside the "atoms" $\tilde{A}$ and $\tilde{B}$, which are connected only by horizontal and vertical hopping, we do not expect any TRS breaking, regardless of light parameters such as intensity, photon energy and so on. However, even if there will not occur Floquet topological transitions, the system will undergo a topological phase transition due to dimerization state switching instead, governed by the interplay between γ_1 and γ_2 . Therefore, in the 2D SSH model, there will always be reserved an energy gap where the possible topological transition due to dimerization should take place. Actually, this is the reason why the simultaneous existence of three topological gaps is forbidden.

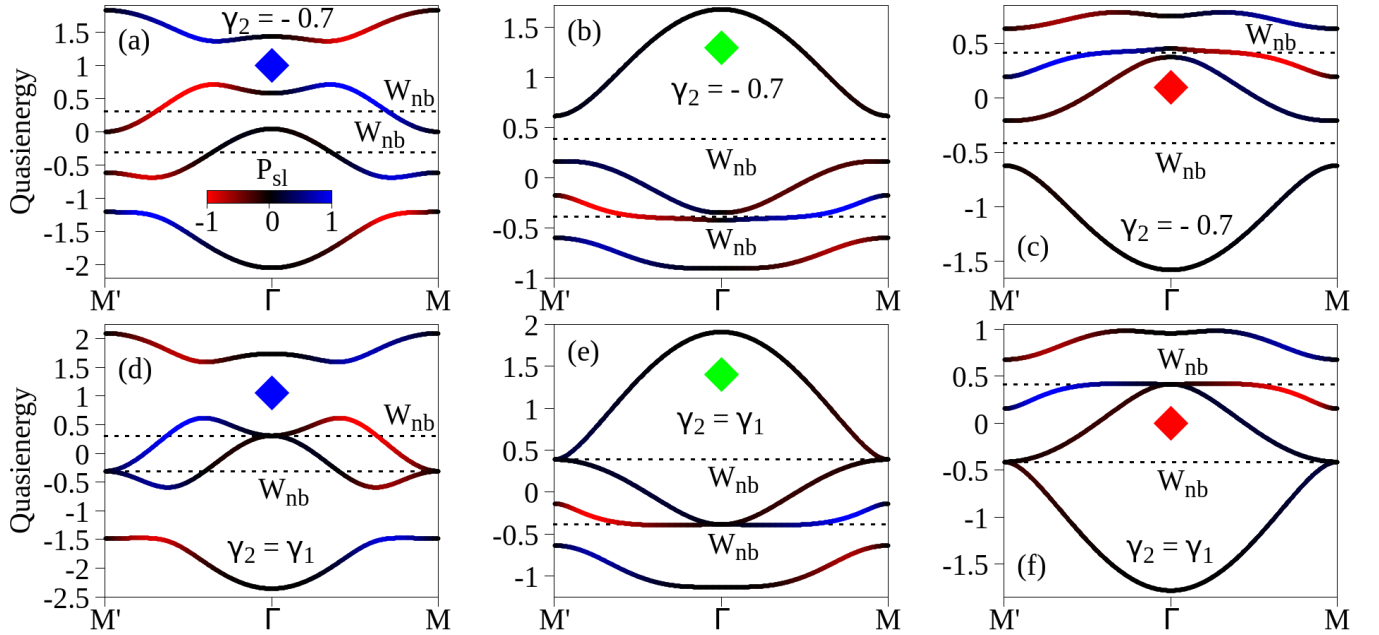


FIG. 12. Quasienergy dispersion on M' - Γ - M direction for different hopping parameters and A_0 magnitudes. (a), (b) and (c) show the topological and spectral properties of each phase indicated by blue, green and, respectively, red marker in Fig. 11. The intra-cellular hopping parameter is fixed at $\gamma_2 = -0.7$. (d), (e) and (f) illustrate the dimerization state switching scenarios (topological phase transition), achieved for $\gamma_2 = \gamma_1$, corresponding to (a), (b) and (c). The horizontal dashed lines represent the two non-bonding quasienergy values. The topological phases characterized by the Chern numbers in Fig. 11 reflect in the quasienergy dispersion and the bands polarization. (a) The first and the third gap reveal to be topological, according also to their invariants $C^{(1)} = C^{(3)} = -1$. The phase transition is realized inside the second (trivial) gap, when $\gamma_2 = \gamma_1$, as shown in (d) where the bands which bound the second gap intersect in Γ and $M(M')$, at the non-bonding energy value. (b) The first and second gap are topological, having their invariant $C^{(1)} = C^{(2)} = +1$. The third gap (trivial) is reserved for phase transition, as shown in (e). (c) The second and third gap are topological with $C^{(2)} = C^{(3)} = +1$ and the phase transition occurs inside the first (trivial) gap, as may be seen in (f).

The phase transition we are discussing about, will arise inside the gap which hosts the non-bonding quasienergy, that is

$$W_{\text{nb}} = \pm \gamma_3 J_0 (\sqrt{2} A_0). \quad (53)$$

Eq. (53) may be deduced by analogy with the case of light absence.

Having elucidated this topological property of the 2D SSH model, we go further and investigate the corresponding spectral properties, associated to three topological phases presented in Fig. 12 (blue, green and red markers).

First, we consider an infinite system. In order to describe the energy bands from a topological point of view, we divide the whole square structure in two sublattices, as a preliminary step. Since we have argued that the atoms A and D play the role of two different spin states and anticipated that they will acquire a well-defined chirality after the TRS breaking, we are lead to group A and B atoms in one sublattice, and, respectively, C and D in the other. Then, we introduce the sublattice polarization defined as

$$P_{\text{sl}} = \frac{(|\langle A|A\rangle|^2 + |\langle B|B\rangle|^2) - (|\langle C|C\rangle|^2 + |\langle D|D\rangle|^2)}{(|\langle A|A\rangle|^2 + |\langle B|B\rangle|^2 + |\langle C|C\rangle|^2 + |\langle D|D\rangle|^2)}. \quad (54)$$

$P_{\text{sl}} \in [-1, 1]$ expresses the sublattice population weight for each quasienergy band, at a given point in BZ. For instance, if $P_{\text{sl}} = 1$ at a given point $\mathbf{k}$ in BZ, in the n -th energy band, the contribution is given completely by A and B atoms. On the other hand, if $P_{\text{sl}} = -1$, there contribute C and D . At the middle of these two extreme situations, if $P_{\text{sl}} = 0$, the contribution is equal on one hand from A and B and on the other from C and D . In what follows, we identify the gaps as "first", "second" and "third" from the lowest to the highest quasienergy. The first investigated topological phase is for $A_0 = 1.5$, see blue marker in Fig. 11. The quasienergy dispersion is depicted in Fig. 12(a). The phase diagram suggests that in this regime, the first and the third gap lie in a topological phase. Specifically for a bipartite topological insulator with TRS, in its topological phase, there arises a band polarization (inversion) between the two bands which bound the topological gap. Indeed, the sublattice polarization function confirms the occurrence of this topological behavior also in our studied system, exactly for the first and the third gap. Thus, as the bands polarization indicates, in ribbon configuration, we anticipate the crossing of two pairs of chiral bands in $k = 0$ and $k = \pi$, in Fourier space. The second gap, trivial, is reserved for the topological phase transition. According to our effective bipartite unit cell interpretation (the effective model shown in Fig. 8), in the actual topological phase, one of the two non-bonding quasienergy levels $W_{\text{nb}} \approx \pm 0.3$, computed from Eq. (53), should be between the first and the third gap (see horizontal dashed lines). The bands crossing specific to the topological phase transition may be verified imposing $\gamma_2 = \gamma_1$. In Fig. 12(d), where we show the scenario of the dimerization state switching (topological phase transition), one may observe the gap closing in Γ and M , at W_{nb} .

Next, we analyze the phase indicated by the red marker in Fig. 11, where $A_0 = 3.5$. We expect to find the first and the

second gap in a topological phase. Fig. 12(b) confirms and, moreover, we see the specific localization of the non-bonding energy inside the third, trivial, gap. In Fig. 12(e), the SSH phase transition is shown. Interestingly, in this regime, the two non-bonding energy values does no longer lie in the same gap. Finally, in the phase corresponding to the red marker, we would have to find in topological phase the second and third gap and W_{nb} inside the first, trivial, gap. Fig. 12(c) shows this topological configuration. Also, the topological phase transition is illustrated in Fig. 12(f).

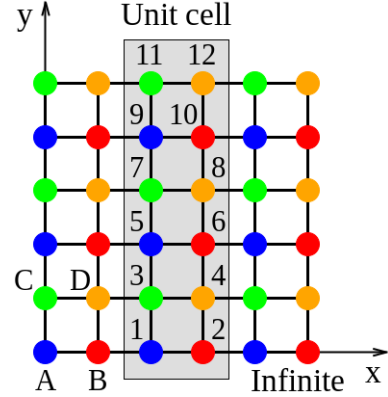


FIG. 13. 2D SSH ribbon. The system is kept infinite on x -direction (horizontal) and confined on y -direction (vertical). The unit cell (gray shaded area) contains the sequence $A, B, C, D, \dots, D$ atoms indexed by 1, 2, 3, 4, ..., 12.

Now, we investigate the topological properties of a 2D SSH ribbon under circularly polarized light irradiation, with the same parameters as above. We keep the system infinite on x -direction (horizontal), thus k_x is still a good quantum number and consider the lattice confined on y -direction. Since the periodic boundary condition becomes impossible to apply on y -direction, the old k_y good quantum number is replaced by an atomic index. For instance, in Fig. 13, we show a 2D SSH ribbon, where the unit cell (gray shaded area) contains the atoms sequence $A, B, C, D, \dots, D$ indexed by 1, 2, 3, ..., 12.

As predicted, in a ribbon, inside the topological gaps, there will arise pairs of chiral edge states. In the first case (blue marker), the chiral edge states occur inside the first and third gap, which have been identified in a topological phase, see Fig. 14(a). According to the value of the topological invariants $C^{(1)} = C^{(3)} = -1$, inside each gap will arise only one pair. The topological signature of the two quasienergy bands in question resides in their polarization. As one may observe, the positive direction momentum band is populated by C and D states, while the negative momentum direction band, by A and B . As well, since both invariants possess the same sign, the chirality of the topological states is the same for both gaps. Since besides the polarization, an other important topological signature is represented by the strong confinement of the topological states at the edges of the system, there will arise conduction channels with opposite direction, localized at the

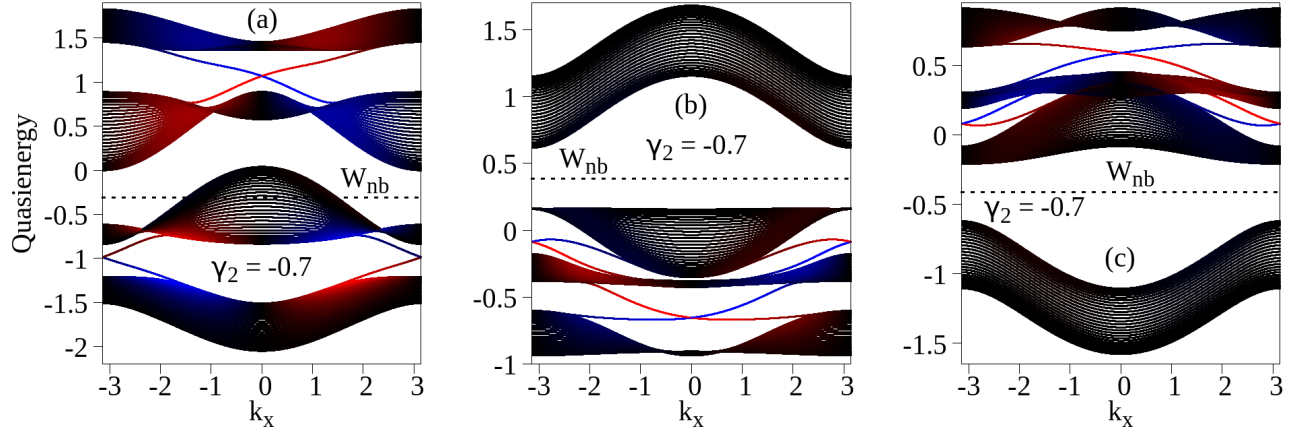


FIG. 14. 2D SSH ribbon quasienergy dispersion corresponding to: (a) blue, (b) green and, respectively, (c) red marker in Fig. 11. (a) The first and third gap are topological, according to their invariant $C^{(1)} = C^{(3)} = -1$. Inside each of the two topological gaps, there arise pairs of edge states. The positive momentum band is populated by C and D states, while the negative momentum direction band, by A and B . (b) The first and second gap are topological, having the invariants $C^{(1)} = C^{(2)} = +1$. In this case the chirality of the topological states reverses. (c) The second and third gap are topological, with $C^{(2)} = C^{(3)} = +1$. The chirality of the topological states is the same as in (b). The non-bonding energy lies inside the trivial gap for all the three cases.

systems extremities. This property will be investigated a little bit later. In Fig. 14(b), we show the energy dispersion corresponding to the second discussed topological phase (red marker). In this case, the phase is characterized by $C^{(1)} = C^{(2)} = +1$. Consequently, also in this regime, inside each topological gap (first and second) will arise only one pair of chiral bands. On the other hand, since the sign of the topological invariants is reversed, the chirality of the edge states is also reversed. Other way to say, at the edges of the system, the localization will not be affected, but, instead, the momentum direction will be inverted. However, this topological effect is obvious inside the topological gap, where one may observe that, unlike in the previous case, the positive momentum direction chiral band is populated by A and B states, while the negative momentum direction band, by C and D .

Finally, the third phase regime (red marker), see Fig. 14(c), characterized by $C^{(2)} = C^{(3)} = +1$, also confirms our expectations, according to the phase diagram and the analysis of the infinite system [Fig. 12(c)]. As well, in the ribbon configuration, the non-bonding energy lies inside the trivial gap.

For a further description of the Floquet topological phases of 2D SSH model, we intend to extend a little the above discussion. In order to highlight the fundamental characteristics of the found topological states in our system, we make use of the local density of states function (LDOS), defined as

$$\text{LDOS} = \frac{-1}{\pi} \text{Im}[G(i, i, E_F)], \quad (55)$$

where $G(i, i, E_F)$ represents the Green's function of the system Hamiltonian, i is the atomic index and E_F the value of FL.

We chose, for instance, the first analyzed phase (blue marker) and consider the FL at $E_F = 1.1$. The results are depicted in Fig. 15(a) for a ribbon with a unit cell containing a

number of 32 atoms. The distribution of the LDOS values, for each A, B, C and D state, highlights the strong confinement of the topological states at the ribbon edges. As well, this analysis confirms our prediction that the mixing between A and D states is topologically forbidden, as a consequence of TRS breaking inside the composite "atoms" $\tilde{A}$ and $\tilde{B}$ within the effective model represented in Fig. 8. The same is valid for B and C states. The inset sketches the formation of the conduction channels at the system extremities. According to Fig. 13 and the LDOS, at the bottom of the system, the conduction channel will transport A and B states, in the negative direction (from right to left). Analogously, at the top, the conduction channel will transport C and D states in the positive direction. Moreover, if we cut the ribbon on its infinite direction (finite sheet configuration), the chiral edge states will confine at the all four edges. In Fig. 15(b), we show the LDOS for a sheet formed by 40×20 atoms.

Finally, we simulate a Hall transport experiment, based on Landauer-Büttiker formalism. See Fig. 16(a) for the device design. We consider a 2D SSH finite lattice which has four leads attached, whose role is only to inject and collect charge. The leads are considered to be formed by a number of semi-infinite identical atomic chains which do not interact each other. For instance, we apply the potentials V_1 and V_2 on lead 1 and, respectively, lead 2 and measure the Hall resistance between lead 3 and lead 4. See Appendix A for a brief derivation of the Hall resistance R_H .

In Fig. 16, panels (b), (c) and (d), we present R_H in terms of h/e^2 constant, with respect to E_F , for the three topological phases analyzed above (blue, green and red markers in Fig. 11). The range of E_F is appropriately chosen to keep only the relevant results. The most obvious Hall Effect is for the first case [panel (b)], where two R_H plateaus are well defined

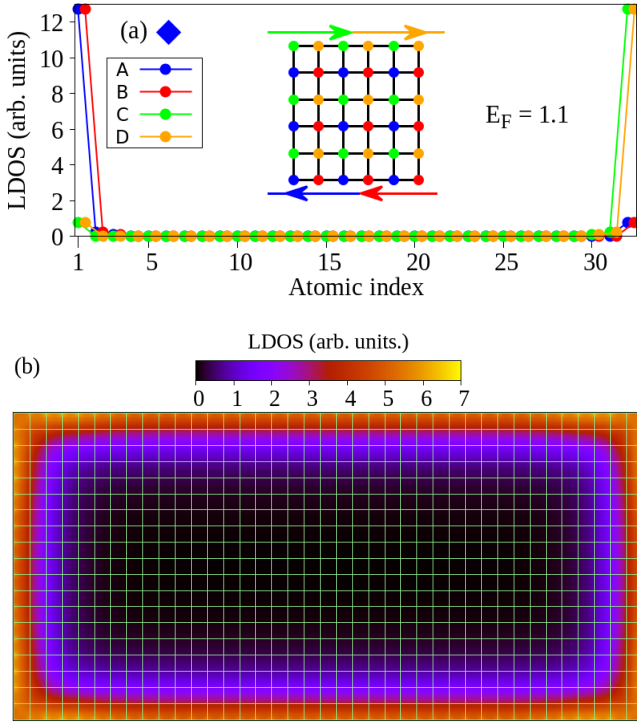


FIG. 15. LDOS for a (a) 2D SSH ribbon and, respectively, (b) for a 2D SSH sheet. The system is in the topological phase indicated by the blue marker in Fig. 11. (a) The FL is considered at $E_F = 1.1$, hence inside the third gap of the quasienergy dispersion shown in Fig. 14(a). At the bottom edge of the system, are localized *A* and *B* states, while at the top edge, *B* and *C*, respectively. The inset represents a sketch of the conduction channels formed at the edges of the ribbon. The upper channel transports *B* and *C* states in the positive direction, while the lower, *A* and *B* states in the negative direction. (b) The system contains 40×20 atomic sites.

at $R_H = -1$, corresponding to the first and the third gap of the quasienergy dispersion shown in Fig. 14(a) (topological gaps). Contrastingly, the next two cases [panels (c) and (d)] are not showing the intuited behavior, according to the Chern numbers in Fig. 11. As may be seen, only one gap has pure topological states [first gap in (b) and third in (c)], while inside the second one, the topological states are mixed with the bulk ones. Thus, the Hall Effect is quenched for any E_F value inside those gaps. However, the Quantum Hall Effect may be activated even in this situation, by inserting lattice defects in order to localize the bulk states, giving rise to a so-called *Topological Anderson Insulator* [78–80].

C. Light irradiation: $\gamma_2 > \gamma_1$

In this Section, we approach the case of $\gamma_2 > \gamma_1$. We consider that restricting ourselves to phase diagram, ribbon dispersion and Hall transport is enough to compare to the previous situation. The results presented here are obtained setting $\gamma_2 = -1.5$, keeping all the other parameters at the same value.

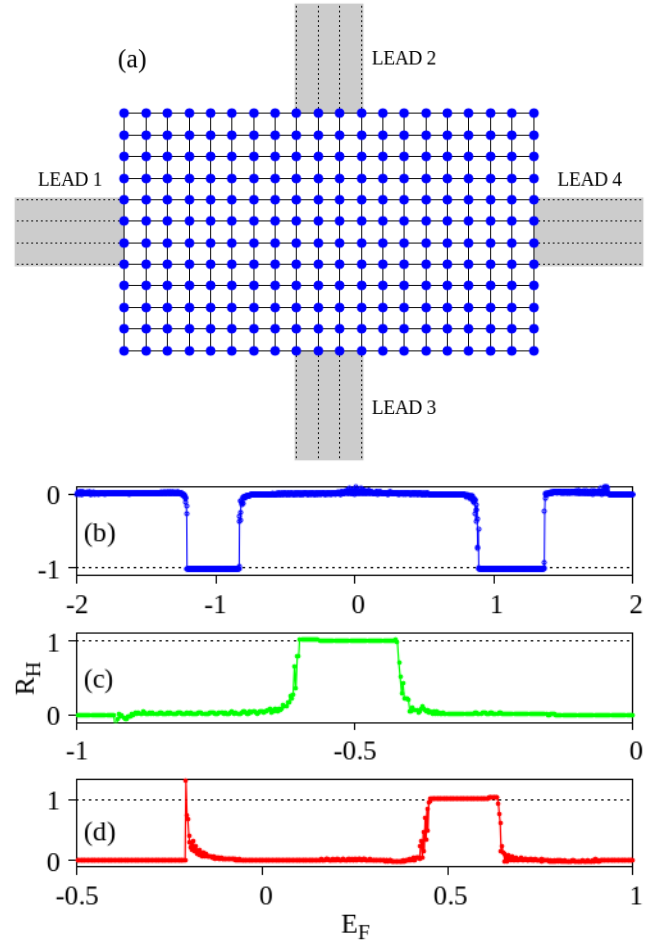


FIG. 16. Quantum Hall transport. (a) Hall device design. The 2D SSH lattice has four leads attached, formed by a number of non-interacting semi-infinite identical atomic chains, having the role only to inject and collect charge. (b), (c), (d) Quantum Hall transport simulation using Landauer-Büttiker formalism, for the case of blue, green and, respectively, red marker in Fig. 11. Only (b) shows the two intuited R_H plateaus, according to the corresponding Chern numbers of the two topological gaps. In (c) and (d), there is formed only one plateau, given that inside one of the two gaps, the edge states are mixed with the bulk ones.

Fig. 17(a) shows the phase diagram, which is obviously more intricate than the previous one (Fig. 11). Moreover, we may have several examples where the three invariants are simultaneously non-zero, which, as discussed, for $\gamma_2 < \gamma_1$, it is a forbidden configuration.

Fig. 17(b) illustrates the quasienergy dispersion for a ribbon, corresponding to the black marker in Fig. 17(a), namely for $A_0 = 3.4$. The spectrum is quite similar with Fig. 14(b) which is computed for $A_0 = 3.5$. However, in the present case, the third gap is topological, hosting two pairs of chiral bands. Comparing Figs. 14(b) and 17(b), we conclude that the dimerization state switching triggers a new topological phase governed by a Chern number $C^{(3)} = +2$. This new phase transition occurs inside the third gap [see also Fig. 12(e)]. Now,

we have simultaneously all the three gaps topological, even if the second one mixes edge with bulk states.

Finally, we simulate the Hall transport. As in the case of Fig. 16(c), the $R_H = +1$ plateau, corresponding to the first gap, is present and very well defined. As E_F rises, reaching the second gap, the Hall conduction is still absent as a consequence of edge and bulk states combination. Specific to our topological configuration, in the Hall resistance diagram, an $R_H = +\frac{1}{2}$ plateau arises, given the existence of the chiral edge states inside the third gap. This plateau represents the signature of the new topological phase induced as a result of dimerization state switching.

V. FINAL REMARKS: LIGHT HELICITY REVERSAL

Since the trigger of the Floquet topological phase transitions is the circularly polarized light, it is worth discussing also its helicity reversal case $\Lambda \rightarrow -\Lambda$. The clue resides in Eqs. (7a)–(7d). Thus, reversing the sign of Λ is similar to changing $\theta_{LD} \rightarrow \theta_{LU}$ and $\theta_{RD} \rightarrow \theta_{RU}$, or in other words, changing $y \rightarrow -y$. Hence, reflecting the whole system about its longitudinal axis, the particles chirality will be reversed, see Fig. 18 which is the counterpart of Fig. 14(a), for $\Lambda = +1$. In this case, A and B edge states will have a positive momentum, while C and D , a negative one. Recall Fig. 15(a). From a transport perspective, this is equivalent to reversing the conduction channels and, consequently, the Hall resistance and Chern numbers transform as $R_H \rightarrow -R_H$ and, respectively, $C^{(n)} \rightarrow -C^{(n)}$.

VI. SUMMARY AND CONCLUSIONS

In this paper, we investigated the topological properties of a 2D SSH system from the perspective of FTIs. TRS breaking was achieved through circularly polarized light irradiation. Additionally, besides the topological phases governed by Chern numbers, the system may also undergo a topological phase transition due to dimerization state switching. This transition is generally specific to SSH systems and is independent of light driving.

Our aim in this research was to fundamentally describe the interplay between TRS breaking and dimerization. We began our work by revisiting the well-known 1D SSH chain, highlighting its most important topological properties to serve as a benchmark for interpreting our results. Next, we introduced the 2D SSH lattice model, which allows for diagonal hopping between second-order neighbors. After formulating the 2D SSH Hamiltonian and utilizing Peierls substitution, we developed a model for interaction with circularly polarized light.

One significant finding was that TRS breaking, and implicitly, a topological phase transition, is not allowed in the absence of diagonal hopping. Moreover, this result was fully supported analytically.

Further, we performed a comprehensive numerical analysis to investigate the fundamental properties of the studied model. Initially, we examined the scenario without light,

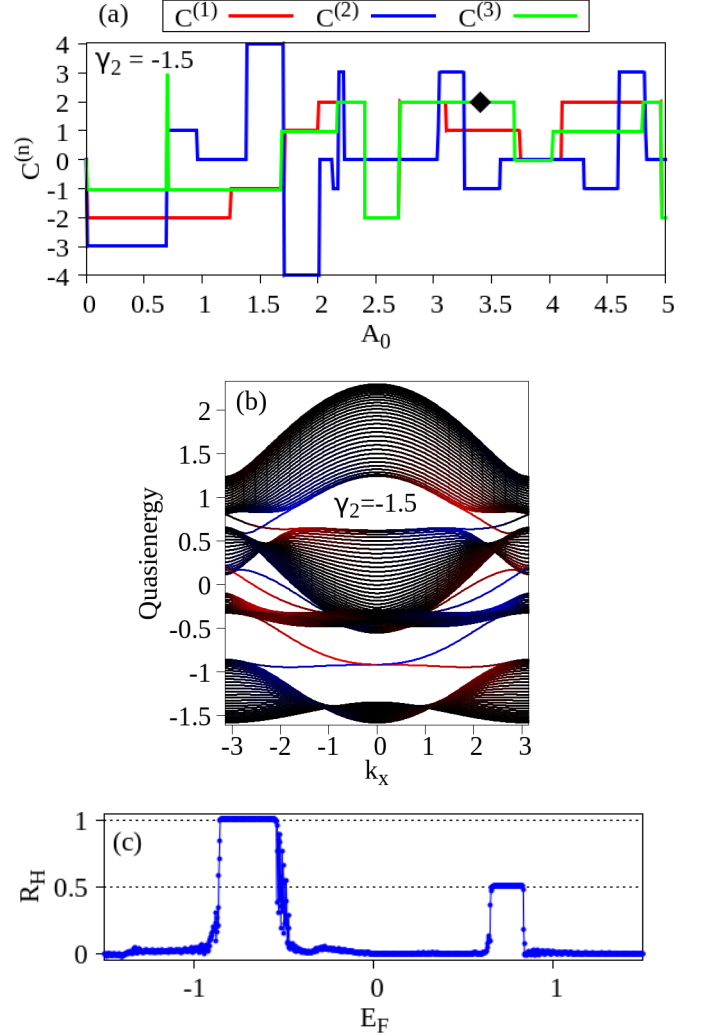


FIG. 17. Topological properties for the case of $\gamma_2 > \gamma_1$. (a) Topological phase diagram. Here, we have examples where the three invariants are simultaneously non-zero, situation forbidden for $\gamma_2 < \gamma_1$. (b) Ribbon quasienergy dispersion for the phase indicated by the black marker in (a), $A_0 = 3.4$. All the three gaps host chiral edge states, even though the second one combines edge with bulk states. (c) Hall transport simulation. The first $R_H = +1$ plateau corresponds to the first gap within (b). The second gap does not have an associated plateau, since there exists a combination of edge with bulk states. The $R_H = +\frac{1}{2}$ plateau is associated with the third gap and it represents the signature of the new topological phase induced as a result of dimerization state switching.

which served as our starting point. We showed that the degeneracy points at Γ and M are protected by the invariance under the action of the anti-unitary operator Υ (30) [Eqs. (31) and (32)], and that the reflection symmetry (34) guarantees the flat band appearance along the Γ - M direction.

Employing the decimation method, we proved that the lattice could be interpreted as being formed of 'composite' atoms, in which previous atoms are arranged in groups of two. The newly formulated model proved to be invariant under the

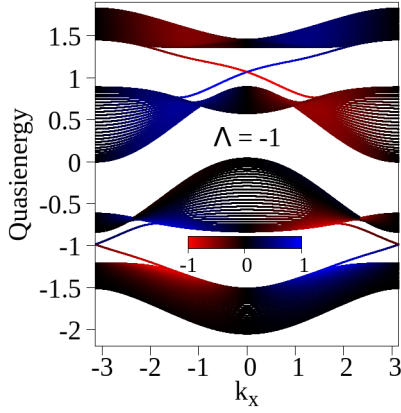


FIG. 18. The counterpart of Fig. 14(a), for $\Lambda = +1$. The particles chirality is reversed, as may be observed from the bands polarization: A and B states will have a positive momentum, while C and D , a negative one.

action of σ_x , allowing us to interpret the four atomic states (A - D and B - C) as conserved spin states, effectively serving as an internal degree of freedom for the two "composite" atoms of the effective unit cell. Given that the spin states are related by time reversal operation, we inferred that TRS breaking occurs between A and D states and similarly between B and C . Consequently, we anticipated that in a topological phase where chiral edge states arise, the mixing between $A(B)$ and $D(C)$ is topologically forbidden. In other words, one edge of the system will host A and B states, while the other, C and D , respectively.

Next, we introduced light irradiation to break the TRS and computed the topological phase diagram based on Chern numbers. We have thus found an interesting property. For $\gamma_2 < \gamma_1$, it is impossible to induce a phase in which all the three Chern numbers are non-zero and have interpreted this behavior in terms of the effective model of bipartite unit cell. By imposing $\gamma_2 = \gamma_1$, we demonstrated that a gap is always reserved for a topological phase transition due to the dimerization state switching, occurring at the non-bonding energy, as typically observed in SSH systems. Additionally, we highlighted the specific sublattice polarization of the topological bands in both infinite and ribbon configurations. Using the local density of states (LDOS) function, we confirmed the prediction that $A(B)$ and $D(C)$ states are separately localized at the system edges, giving rise to conduction channels with opposite chirality in a topological phase. Furthermore, we discussed the implications in terms of quantum Hall transport, particularly the formation of R_H plateaus.

Finally, we explored the case of $\gamma_2 > \gamma_1$. Here, we demonstrated that the phase diagram becomes more intricate and provided examples of simultaneous non-zero Chern numbers. For such a topological phase, we presented the quasienergy dispersion, and in the quantum Hall transport, a new $R_H = +\frac{1}{2}$ plateau represents the signature of the new topological phase

induced by the dimerization state switching. Moreover, we explored the scenario of light helicity reversal $\Lambda \rightarrow -\Lambda$, resulting in the switching of particle chirality.

The most significant findings from our research are that: (i) TRS breaking is not achievable in the absence of second-order neighbors (diagonal) hopping, and (ii) there exists an interplay between TRS breaking and dimerization, reflected in the values of the Chern numbers $C^{(n)}$.

VII. ACKNOWLEDGEMENTS

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Appendix A: Landauer-Büttiker formalism

If we consider a potential $V_1 - V_4$ applied between lead 1 and lead 4 and measure $V_2 - V_3$ between lead 2 and lead 3, the Hall resistance reads

$$R_H = \frac{V_2 - V_3}{I_1}, \quad (\text{A1})$$

where I_1 is the current which flows at terminal 1. Leads 2 and 3 are probes, thus $I_2 = I_3 = 0$. Setting $V_4 = 0$, the Ohm's law reads:

$$G \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ 0 \\ 0 \end{pmatrix}; \quad (\text{A2})$$

$$G = \begin{pmatrix} g_{12} + g_{13} + g_{14} & -g_{12} & -g_{13} \\ -g_{21} & g_{21} + g_{23} + g_{24} & -g_{23} \\ -g_{31} & -g_{32} & g_{31} + g_{32} + g_{34} \end{pmatrix}, \quad (\text{A3})$$

where G denotes the conductivity matrix with g_{ij} , the conductivity function between i and j leads. Eq. (A2) may be also translated as

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = R \begin{pmatrix} I_1 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{A4})$$

where $R = G^{-1}$ represents the resistance matrix. Finally, computing the inverse of (A3), the Hall resistance is defined as

$$R_H = \frac{g_{24}g_{31} - g_{21}g_{34}}{\det(G)} \left(\frac{h}{e^2} \right), \quad (\text{A5})$$

where h/e^2 is the resistance quantum.

Within the Landauer-Büttiker formalism, g_{ij} is numerically computed using the Green's functions method [81].

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