# Thermal Area Law in Long-Range Interacting Systems 

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#### Abstract

The area law of the bipartite information measure characterizes one of the most fundamental aspects of quantum many-body physics. In thermal equilibrium, the area law for the mutual information universally holds at arbitrary temperatures as long as the systems have short-range interactions. In systems with power-law decaying interactions, $r^{-\alpha}$ ( $r$ : distance), conditions for the thermal area law are elusive. In this work, we aim to clarify the optimal condition $\alpha>\alpha_{c}$ such that the thermal area law universally holds. A standard approach to considering the conditions is to focus on the magnitude of the boundary interaction between two subsystems. However, we find here that the thermal area law is more robust than this conventional argument suggests. We show the optimal threshold for the thermal area law by $\alpha_{c}=(D+1) / 2(D$ : the spatial dimension of the lattice), assuming a power-law decay of the clustering for the bipartite correlations. Remarkably, this condition encompasses even the thermodynamically unstable regimes $\alpha<D$. We verify this condition numerically, finding that it is qualitatively accurate for both integrable and non-integrable systems. Unconditional proof of the thermal area law is possible by developing the power-law clustering theorem for $\alpha>D$ above a threshold temperature. Furthermore, the numerical calculation for the logarithmic negativity shows that the same criterion $\alpha>(D+1) / 2$ applies to the thermal area law for quantum entanglement.


Introduction.- Quantum correlation and entanglement play pivotal roles in understanding quantum many-body systems from an information-theoretic standpoint. They help in identifying the exotic quantum phases $[1-3]$ and serve as crucial resources in quantum information processing [4]. One hallmark of quantum many-body systems is the area law for the ground state [5], which posits that the correlation and entanglement between two subsystems are constrained by the surface area of their interface. This principle has undergone extensive verification through both analytical and numerical approaches in a wide array of scenarios [6-16]. The area law not only elucidates the inherent complexities of quantum systems but also significantly enhances their simulatability using classical computers [17, 18].

The quantum Gibbs state describing thermal equilibrium obeys an analogous principle, known as the thermal area law, as long as the systems have short-range interactions [19-23]. This law is reflected by the behavior of mutual information, which contains both classical and quantum correlations, as illustrated by the following inequality:

$$
\begin{equation*}
\mathcal{I}(A: B) \leq 2 \beta\left\|H_{\partial A}\right\| \tag{1}
\end{equation*}
$$

where $\mathcal{I}$ represents the mutual information of the Gibbs state between two subsystems $A$ and $B$ at an inverse temperature $\beta$. Here, $\left\|H_{\partial A}\right\|$ denotes the operator norm of the boundary interaction Hamiltonian, defined as the sum of interactions whose supports intersect with the boundary $\partial A$ of $A$. In the short-range interacting case, the norm of these boundary interactions scales with the boundary area $|\partial A|$, thereby affirming the thermal area law. This law is not only relevant to mutual information

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FIG. 1. Schematic of the results. The thermal area law is satisfied for $\alpha>(D+1) / 2$ (indicated by a red region). The thermal area law is much more robust than the relation (1) suggests (a blue region). The key physics behind the area law is the power-law clustering of bipartite correlations.
but also extends to a wider range of bipartite information measures [20-22]. We also note that a recent study has further clarified that genuine quantum correlations cannot be as large as the system size for short-range interacting systems [24].

In contrast to short-range interactions, the general aspects of locality in long-range interacting systems remain less understood. Long-range interactions here refer to interactions that decay as $r^{-\alpha}$ with distance $r$ [25], which are ubiquitous in nature such as in atomic, molecular, and optical systems [26-31]. These interactions lead to non-trivial phenomena absent in shortrange systems [32-40]. Key unresolved issues include (i) identifying the critical threshold $\alpha_{\mathrm{c}}$ above which any systems have the thermal area law, (ii) understanding the underlying physics, and (iii) distinguishing between classical and quantum correlations. This paper addresses these questions as fundamental physics. In a practical aspect, this direction may lead to finding an efficiency-guaranteed algorithm for simulating the long-range quantum Gibbs state. Should the fundamental physics be governed solely by the magnitude of the boundary interaction, then the inequality (1) could already be considered optimal; that is, the condition of the thermal area law is given by $\alpha>D+1$ (or
$\alpha_{c}=D+1$ ), which is required to ensure $\left\|H_{\partial A}\right\| \propto|\partial A|$ as in [39]. Indeed, the threshold given by the finite boundary interaction recurs across various contexts in physics, indicating that physical properties undergo qualitative changes at this threshold in various manybody systems [39, 41-45].

However, in this work, we discover that the underlying physics of the thermal area law is influenced by a different characteristic: the clustering property, which is expected to remain robust in any noncritical phases. Note that the clustering property has been widely adopted as a basic and natural assumption in discussing various quantum properties such as entanglement area law [14], quantum Gibbs sampling [46, 47], the eigenstate thermalization [48, 49], etc. Consequently, we show that the thermal area law for mutual information possesses a greater resilience than traditionally anticipated. Utilizing the power-law clustering properties in bipartite correlations, we analytically find that the sufficient condition for the thermal area law in the mutual information is given $\alpha>(D+1) / 2$. Regarding the critical threshold $\alpha_{c}$ above which any systems satisfy the thermal area law, our numerical analysis concludes that $\alpha_{c}=(D+1) / 2$, since below $\alpha_{c}$ we provide explicit examples violating the area law. Remarkably, this condition encompasses $\alpha<D$, where the thermodynamic extensivity is violated [25]. In this regime, most analytical techniques break down, and universal behaviors remain largely unexplored. While the numerical investigation suggests that the clustering property robustly holds even for thermodynamic nonextensive regimes, we demonstrate that the power-law clustering theorem can be rigorously derived for $\alpha>D$ by identifying a suitable temperature regime. Consequently, the thermal area law unconditionally holds for $\alpha>D$. Our key results are depicted schematically in Fig. 1.

In addition, we investigate whether differences exist between quantum and classical correlations in the context of the thermal area law. By employing the logarithmic negativity proposed by Shapourian, Shiozaki, and Ryu for bilinear fermionic systems [50], we demonstrate that this criterion equally governs the thermal area law for quantum entanglement.

Setup. - We consider a quantum system with $N$ qudits or fermions on the $D$-dimensional lattice. Let $\Lambda$ be the set of all sites on the lattice of $N$ sites. We consider the Hamiltonian $H$ in the $k$-local form:

$$
\begin{equation*}
H=\sum_{Z:|Z| \leq k} h_{Z} \tag{2}
\end{equation*}
$$

where $Z$ is a set of interaction sites and $|Z|$ denotes the number of interacting sites (i.e., $|Z|$-body interaction). We consider the long-range interactions satisfying the following condition

$$
\begin{equation*}
J_{i, i^{\prime}}:=\sum_{Z: Z \ni\left\{i, i^{\prime}\right\}}\left\|h_{Z}\right\| \leq \frac{g}{\left(1+d_{i, i^{\prime}}\right)^{\alpha}} \tag{3}
\end{equation*}
$$

Here, $J_{i, i^{\prime}}$ is the maximum norm of the local Hamiltonian including the sites $i$ and $i^{\prime}$, and $g$ is some constant. The symbol $d_{i, i^{\prime}}$ stands for the Manhattan distance between the sites $i$ and $i^{\prime}$. The parameter $\alpha$ is the index of power-law decay in the interaction. We will provide examples of the Hamiltonian in (9) and (10) later.

Note that the Hamiltonian does not contain the Kac factor for the thermodynamically nonextensive regime $\alpha<D$, since such a factor does not exist in realistic situations [28, 51].

The quantum Gibbs state $\rho_{\beta}$ at an inverse temperature $\beta$ is defined as $\rho_{\beta}:=e^{-\beta H} / \operatorname{tr}\left(e^{-\beta H}\right)$. The mutual information of $\rho_{\beta}$ between two regimes $A$ and $B$ is defined as

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B):=S\left(\rho_{\beta}^{A}\right)+S\left(\rho_{\beta}^{B}\right)-S\left(\rho_{\beta}^{A B}\right) \tag{4}
\end{equation*}
$$

where $\rho_{\beta}^{X}$ is the reduced density matrix of $\rho_{\beta}$ on the subset $X \subseteq \Lambda$, and $S\left(\rho_{\beta}^{X}\right):=-\operatorname{tr}\left(\rho_{\beta}^{X} \log \rho_{\beta}^{X}\right)$ is the von Neumann entropy. We define the correlation function of two observables $O_{A}$ and $O_{B}$ as
$\operatorname{Cor}_{\rho_{\beta}}\left(O_{A}, O_{B}\right):=\operatorname{tr}\left(\rho_{\beta} O_{A} O_{B}\right)-\operatorname{tr}\left(\rho_{\beta} O_{A}\right) \cdot \operatorname{tr}\left(\rho_{\beta} O_{B}\right)$.

Main theorem. - We find that the thermal area law of the mutual information holds above some power-law threshold with the assumption of clustering.

Theorem 1: Let us assume that the correlation function of the quantum Gibbs state $\rho_{\beta}$ between two arbitrary operators $O_{i}$ and $O_{i^{\prime}}$ supported on the sites $i$ and $i^{\prime}$ satisfies the following power-law clustering property:

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{i}, O_{i^{\prime}}\right) \leq \frac{C}{d_{i, i^{\prime}}^{\alpha}}\left\|O_{i}\right\| \cdot\left\|O_{i^{\prime}}\right\|, \tag{6}
\end{equation*}
$$

where $C$ is an $\mathcal{O}(1)$ constant. Then, for the bipartition $A, B$ of $D$-dimensional lattice $\Lambda(A \cup B=\Lambda)$ and $\alpha>$ $(D+1) / 2$, the mutual information is upper bounded by

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \text { const. } \beta|\partial A| \tag{7}
\end{equation*}
$$

We postpone the proof to the end of the paper (the details are in the supplementary material (SM) [52]). Below, we focus on physically crucial points. The theorem ensures that the thermal area law holds for the regime $\alpha>(D+1) / 2$. This condition encompasses a broader scope compared to the condition found in previously established thermal area law (1). It is known that (e.g., Ref. [39]) the operator norm $\left\|H_{\partial A}\right\|$ in (1) is upper-bounded by the boundary area between $A$ and $B$ when $\alpha>D+1$ for generic interacting systems (the condition for the bilinear systems is $\alpha>D / 2+1$ ). Hence, the condition given in the theorem means that the thermal area law is much more robust than expected from a simple argument on the norm of the boundary interaction. Remarkably, the condition partially includes the thermodynamically nonextensive regime.

Power-law clustering and unconditional thermal area law. - Theorem 1 suggests that the crucial physics behind the thermal area law is not only from the magnitude of the boundary interaction but also from the clustering property. Note that the clustering property is expected to be one of the robust physical properties in non-critical thermal phases, rendering it an inherently natural assumption [14]. We here demonstrate that the power-law clustering property can be rigorously proven for $\alpha>D$ above a temperature threshold as shown in the following theorem 2 .

Theorem 2: Under the general Hamiltonian (2) with (3) for the regime $\alpha>D$, the following clustering property holds for the temperatures above $\beta_{c}^{-1}$ :

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right) \leq C\left\|O_{X}\right\| \cdot\left\|O_{Y}\right\| \frac{|X \| Y| e^{(|X|+|Y|) / k}}{d_{X, Y}^{\alpha}} \tag{8}
\end{equation*}
$$

where $d_{X, Y}=\min _{i \in X, j \in Y} d_{i, j}$ and $|X|$ and $|Y|$ are numbers of sites in the regions $X$ and $Y$, respectively. $C$ is an $\mathcal{O}(1)$ constant and the threshold temperature is given by $\beta_{c}=1 /(8 u g k)$ with $u=\sum_{j \in \Lambda} 1 /\left(1+d_{i, j}\right)^{\alpha}$.

The proof is based on the cluster expansion technique. The details in the proof are provided in the SM [52]. To the best of our knowledge, this is the first result to establish the power-law clustering theorem in quantum long-range interacting systems.

From this statement, the clustering property in Theorem 1 is not an assumption but rigorously holds above a threshold temperature as long as the thermodynamically extensive regime (i.e., $\alpha>D$ ) is considered. In particular, in 1D systems, the clustering property is believed to hold at arbitrary temperatures [53, 54]. Moreover, we stress that even in the thermodynamically nonextensive regime $(\alpha<D)$, numerical calculations for the two models below show the validity of the clustering property.

Numerical verification. - We numerically verify our theorems in both integrable and nonintegrable systems, establishing the bound's tightness.

As a typical example of integrable systems, we use the following long-range bilinear fermion system:

$$
\begin{equation*}
H=-\sum_{i, j \in \Lambda} \frac{t_{i, j}}{d_{i, j}^{\alpha}}\left(c_{i}^{\dagger} c_{j}+c_{j}^{\dagger} c_{i}\right) \tag{9}
\end{equation*}
$$

where $t_{i, j}$ is a hopping parameter of order $\mathcal{O}(1)$, and $c_{i}$ and $c_{i}^{\dagger}$ are the annihilation and creation operators of the spinless fermion at site $i$, respectively. We consider the Hamiltonian in the one dimension (1D) and the two dimension (2D).

We first verify the clustering property (6) in the Hamiltonian (9) [55, 56]. We analyze a 1D chain of $N=1000$ sites and a 2D square lattice of side length $N=40$, focusing on two-point correlation functions $\left\langle c_{i}^{\dagger} c_{i+r}\right\rangle$ in 1D and $\left\langle c_{i}^{\dagger} c_{i+r}\right\rangle$ in 2D, setting $\beta=2$. To consider general bilinear systems, we assign randomly chosen values $\in[0,1]$ to the hopping variable, and take the average over 1000 samples. In Figs. 2(a) and (b), we observe that $\left|\left\langle c_{i}^{\dagger} c_{i+r}\right\rangle\right| \times r^{\alpha}$ and $\left|\left\langle c_{i}^{\dagger} c_{\boldsymbol{i}+\boldsymbol{r}}\right\rangle\right| \times|\boldsymbol{r}|^{\alpha}$ saturate at large distances for any $\alpha$, thus verifying the clustering property in both dimensions. Note also that the clustering property holds even for $\alpha<D$. While we show here the averaged data over samples, even individual data show the clustering property (not shown). This indicates all correlation functions adhere to this property due to the Wick theorem.

After establishing the clustering property, we affirm the thermal area law is applicable for $\alpha>(D+1) / 2$, a sufficient condition whose optimality warrants examination. Let us investigate how tight the condition is, through the numerical calculation with the same model. The system is partitioned into subsystems $A$ and $B$, with $A$ including the first $N / 2$ sites and $B$ the rest in


FIG. 2. 1D and 2D long-range bilinear fermions. (a) The two-point correlation function $\left|\left\langle c_{i}^{\dagger} c_{i+r}\right\rangle\right|$ in the 1D chain of $N=1000$ sites between the $(i=250)$ th site and $(i+r)$-th site multiplied by $r^{\alpha}$ for different $\alpha$ 's. (b) The two-point correlation function $\left|\left\langle c_{i}^{\dagger} c_{i+r}\right\rangle\right|$ in the 2D square lattice with a side length $N=40$ between the site $\boldsymbol{i}=(10,10)$ and the site $\boldsymbol{i}+\boldsymbol{r}=(10+r, 10+r)$ multiplied by $|\boldsymbol{r}|^{\alpha}$ for different $\alpha$ 's. (c) The mutual information $\mathcal{I}_{\rho_{\beta}}(A: B)$ in the 1D chain of sites $N$ between half $(A)$ of the system and the other half $(B)$. (d) The mutual information $\mathcal{I}_{\rho_{\beta}}(A: B)$ in the 2D square lattice with a side length $N$ between half $(A)$ of the system and the other half $(B)$ divided by $N$. All the figures are the average over the 1000 samples of random variables $t_{i, j} \in[0,1]$ in Eq. (9) at $\beta=2$.

1D. For 2D, $A$ comprises the initial $(N \times N / 2)$ sites, and $B$ the remaining. The mutual information $\mathcal{I}_{\rho_{\beta}}(A: B)$ is calculated with the same parameters from the clustering property study. Results are presented in Figs. 2(c) for 1 D and (d) for 2D. The mutual information increases with the system size if $\alpha<1$ and it saturates to a constant value if $\alpha>1$ in 1D. In 2D, the mutual information divided by the boundary area $N$ grows for $\alpha<1.5$ and steadies for $\alpha>1.5$. These numerical results support the thermal area law holds for $\alpha>(D+1) / 2$, and hence the condition is optimal.

In general, bilinear systems can be special in several physical aspects not only for the non-equilibrium properties [57] but also for the static properties including the thermal entanglement [58-60]. However, the clustering is expected to be robust, regardless of the (non)integrability, and hence the thermal area law should also hold universally. We here check this universality using the specific nonintegrable system. Let us consider the one-dimensional long-range Heisenberg spin- $1 / 2$ chain of $N$ sites

$$
\begin{equation*}
H=\sum_{1 \leq i<j \leq N} \frac{1}{d_{i, j}^{\alpha}} \boldsymbol{S}_{i} \cdot \boldsymbol{S}_{j} \tag{10}
\end{equation*}
$$

with a spin-1/2 operator $\boldsymbol{S}_{i}=\left(S_{i}^{x}, S_{i}^{y}, S_{i}^{z}\right)$ at site $i$. Exploring the clustering property of $\left\langle\boldsymbol{S}_{i} \cdot \boldsymbol{S}_{i+r}\right\rangle$ with


FIG. 3. 1D long-range Heisenberg chain. (a) The two-point correlation function $\left|\left\langle\boldsymbol{S}_{i} \cdot \boldsymbol{S}_{i+r}\right\rangle\right|$ multiplied by $r^{\alpha}$ in the 1D chain with $N=200$ sites between $(i=50)$ th site and $(i+$ $r)$ th site at $\beta=2$ for different $\alpha$ 's. (b) The entanglement entropy Eq. (12) of the thermofield double state of the Gibbs state at $\beta=2$ for different $\alpha$ 's. We choose two subsystems as half of the system and the other half and change the size $N$ of the chain. Here, we use the XTRG algorithm and set bond dimensions (SU(2) multiplets) of 210 and 120 for (a) and (b), which correspond approximately 850 and 450 states, respectively.
$N=200$ and $\beta=2$, we employ the exponential tensor renormalization group (XTRG) algorithm, as detailed in references [61, 62], in conjunction with the QSpace tensor library [63, 64]. This approach enables us to construct the matrix product operator for the Gibbs state $\rho_{\beta}$. See [65] for details. Fig. 3(a) presents $\left|\left\langle\boldsymbol{S}_{i} \cdot \boldsymbol{S}_{i+r}\right\rangle\right| \times r^{\alpha}$, evidencing clustering by constant upper bounds. This observation, as well as the bilinear Hamiltonian case, strongly suggests that the clustering universally holds.

To verify the thermal area law, we compute the entanglement entropy $(\mathcal{E})$ for the thermofield double (TFD) state of Gibbs state $\rho_{\beta}$ as it upper bounds the mutual information [66]. We consider two identical copies $\mathcal{H}_{L}$ and $\mathcal{H}_{R}$ of the original Hilbert space $\mathcal{H}$. For the eigenvalues $\left\{E_{n}\right\}$ and eigenstates $\{|n\rangle\}$ of the Hamiltonian, the TFD state of the Gibbs state $\rho_{\beta}=\sum_{n} e^{-\beta E_{n}} / \operatorname{tr}\left(e^{-\beta H}\right)|n\rangle\langle n|$ is defined as

$$
\begin{equation*}
|\mathrm{TFD}\rangle:=\frac{1}{\sqrt{\operatorname{tr}\left(e^{-\beta H}\right)}} \sum_{n} e^{-\beta E_{n} / 2}|n\rangle_{L} \otimes|n\rangle_{R} \tag{11}
\end{equation*}
$$

Partitioning the system into halves $A$ and $B$, we define $\mathcal{E}$ as the von Neumann entropy of the reduced density matrix $\sigma_{A_{L}, A_{R}}=\operatorname{tr}_{B_{L}, B_{R}}(|\mathrm{TFD}\rangle\langle\mathrm{TFD}|)$ :

$$
\begin{equation*}
\mathcal{E}:=S\left(\sigma_{A_{L}, A_{R}}\right)=-\operatorname{tr}\left(\sigma_{A_{L}, A_{R}} \log \sigma_{A_{L}, A_{R}}\right) \tag{12}
\end{equation*}
$$

where $A_{L}$ and $A_{R}\left(B_{L}\right.$ and $\left.B_{R}\right)$ are the copies of the original subsystem $A(B)$. Then the mutual information is upper-bounded by $2 \mathcal{E}$, i.e. $\mathcal{I}_{\rho_{\beta}}(A: B) \leq 2 \mathcal{E}[22,67]$. We show the results of $\mathcal{E}$ in Fig. 3(b). $\mathcal{E}$ trends upward for $\alpha<1$ and stabilizes for $\alpha>1$, supporting the thermal area law's optimality condition. While this discussion focuses solely on Hamiltonian (10), it's worth noting that similar behaviors are numerically demonstrated even when the disorder is added into the Hamiltonian (see SM [52]).

Thermal area law in the quantum entanglement.The mutual information includes both classical and quantum correlations. We now extract purely quantum


FIG. 4. The SSR negativity $E_{\text {SSR }}$ of long-range bilinear fermions. (a) $E_{\text {SSR }}$ of the 1 D chain of $N$ sites and (b) $E_{\text {SSR }} / N$ of the 2D square lattice with a side length $N$ between one half of the system and the other half for different $\alpha$ 's. The figures are averaged over the 1000 samples of random variables $t_{i, j} \in[0,1]$ in Eq. (9) at $\beta=2$.
correlations from the Gibbs state to analyze the thermal area law condition. To this end, we investigate bilinear fermionic systems (9) using Shapourian, Shiozaki, and Ryu's (SSR) logarithmic negativity [50] to quantify quantum entanglement between subsystems $A$ and $B$ in a mixed state efficiently. The SSR logarithmic negativity is formulated through the partial time-reversal transform $R_{A}$, yielding:

$$
\begin{equation*}
E_{\mathrm{SSR}}(\rho):=\log \left\|\rho^{R_{A}}\right\|_{1} \tag{13}
\end{equation*}
$$

where $\|\cdot\|_{1}$ is the trace norm. Details on the SSR logarithmic negativity for bilinear systems are in the SM [52]. We calculate this for 1D and 2D Hamiltonian systems (9) with identical parameters and partitioning as mutual information, showing results in Figs. 4 (a) for 1 D and (b) for 2D. These figures demonstrate that the SSR logarithmic negativity shows the same behavior as the mutual information in Figs 2(c) and (d). Hence, the figures indicate that under the same conditions, namely $\alpha>(D+1) / 2$, quantum entanglement likewise follows the thermal area law.

Proof of main theorem. - We here provide the outline of the proof of the main theorem. For the sake of simplicity, we focus on the simple case where $h_{Z}$ is $h_{i, j}$ that acts only on two different points $i, j \in \Lambda$. The proof for the general case is in the SM [52]. Let us consider the total Hamiltonian $H=H_{A}+H_{B}+H_{\partial A}$, where $H_{A}, H_{B}$ are the Hamiltonian supported only on $A, B$, respectively. The part $H_{\partial A}$ is the interaction Hamiltonian between $A$ and $B$, given as $H_{\partial A}=\sum_{i \in A} \sum_{j \in B} h_{i, j}$ with $h_{i, j}=\sum_{s=1}^{d_{0}^{4}} h_{i}^{(s)} \otimes h_{j}^{(s)}$. Here, $d_{0}$ is the Hilbert dimension of the one site. Following Ref. [19], one has the following inequality from the Gibbs variational principle:

$$
\begin{equation*}
F(\rho)-F\left(\rho_{\beta}\right)=\beta^{-1} D\left(\rho \| \rho_{\beta}\right) \geq 0 \tag{14}
\end{equation*}
$$

where $F(\rho)$ is the nonequilibrium free energy defined as $F(\rho):=\operatorname{tr}(H \rho)+\beta^{-1} \operatorname{tr}[\rho \log (\rho)]$ and $D$ is the quantum relative entropy defined as $D(\rho \| \sigma)=\operatorname{tr}(\rho \log \rho)$ $\operatorname{tr}(\rho \log \sigma)$. Plugging $\rho=\rho_{\beta}^{A} \otimes \rho_{\beta}^{B}$ into this inequality,
one obtains the relation on the mutual information:

$$
\begin{align*}
\mathcal{I}_{\rho_{\beta}}(A: B) & \leq \beta \operatorname{tr}\left[\left(\rho_{\beta}^{A} \otimes \rho_{\beta}^{B}-\rho_{\beta}^{A B}\right) H_{\partial A}\right] \\
& =\beta \sum_{i \in A} \sum_{j \in B} \sum_{s} \operatorname{Cor}_{\rho_{\beta}}\left(h_{i}^{(s)}, h_{j}^{(s)}\right) . \tag{15}
\end{align*}
$$

The assumption of the clustering property leads to

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq C \sum_{i \in A} \sum_{j \in B} \sum_{s} \beta\left\|h_{i}^{(s)}\right\| \cdot\left\|h_{j}^{(s)}\right\| d_{i, j}^{-\alpha} . \tag{16}
\end{equation*}
$$

By the condition (3), we have

$$
\begin{equation*}
\max _{s}\left(\left\|h_{i}^{(s)}\right\|,\left\|h_{j}^{(s)}\right\|\right) \leq J_{i, j} \leq \frac{g}{d_{i, j}^{\alpha}} \tag{17}
\end{equation*}
$$

Combining Eq. (16) and Eq. (17), we finally obtain

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \sum_{i \in A} \sum_{j \in B} \frac{\beta d_{0}^{4} g C}{d_{i, j}^{2 \alpha}} . \tag{18}
\end{equation*}
$$

It can be proven that the summation of the distance $\sum_{i \in A} \sum_{j \in B} d_{i, j}^{-2 \alpha}$ is upper bounded by the boundary area when $2 \alpha>D+1$. Therefore, Ineq. (18) leads to the desired inequality (7).

Summary and outlook. - We consider the validity of the thermal area law in the systems with long-range interactions $r^{-\alpha}$. Under the assumption of the clustering property, we derive the critical threshold $\alpha_{\mathrm{c}}:=(D+$ 1) $/ 2$ above which any systems obey the thermal area law of the mutual information (See Fig.1). Remarkably, the regime covers thermodynamically nonextensive regime. This criteria potentially may allow for an efficient representation of quantum Gibbs states exhibiting a powerlaw decay up to $r^{-\alpha_{c}}$. Given this criterion, it is a crucial future problem to develop an efficiency-guaranteed algorithm for simulating long-range interacting systems at finite temperatures, as well as constructing tensor network states with polynomial bond dimensions $[68,69]$.
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## ACKNOWLEDGMENTS

D. K. thanks M. Kim for helpful discussions about QSpace. D. K. and T. K. acknowledge the Hakubi projects of RIKEN. T. K. was supported by Japan Science and Technology Agency Precursory Research for Embryonic Science and Technology (Grant No. JPMJPR2116). K. S. was supported by JSPS Grants-in-Aid for Scientific Research (No. JP19H05603, No. JP19H05791 and No. JP23H01099).

# Supplementary Material for "Thermal Area Law in Long-Range Interacting Systems" 

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## S.I. SETUP

We consider a quantum system, where we use $\Lambda$ to represent the set of sites. Given an arbitrary subset $X$ in $\Lambda, X \subseteq \Lambda$, we denote the cardinality of the set $X$, i.e., the number of sites contained in $X$, by $|X|$. For any two subsets $X, Y$ in $\Lambda, X, Y \subseteq \Lambda$, we define the distance between the two sets, $d_{X, Y}$, as the shortest path length of the graph that connects $X$ and $Y$. We remark that when $X$ intersects with $Y, X \cap Y \neq \emptyset$, the distance between the sets $X$ and $Y$ is zero: $d_{X, Y}=0$. We denote the complementary set of $X$ as $X^{c}$, namely $X^{c}:=\Lambda \backslash X$, and the surface subset of $X$ as $\partial X$, i.e. $\partial X:=\left\{i \in X \mid d_{i, X^{c}}=1\right\}$, respectively. Also, for an arbitrary operator $O$, we denote the support of $O$ by $\operatorname{Supp}(O)$. We often describe the support of the operator explicitly by adding an Index of lower right subscript such as $O_{X}$ with $\operatorname{Supp}\left(O_{X}\right)=X$.

We consider a $k$-local Hamiltonian $H$ with long-range interactions:

$$
\begin{equation*}
H=\sum_{Z:|Z| \leq k} h_{Z}, \tag{S.0}
\end{equation*}
$$

where $Z$ is a subset indicating the interacting sites, and $h_{Z}$ is the local Hamiltonian. Note that $|Z|$ is the number of the interacting sites, and hence $h_{Z}$ means $Z$-body interaction. We assume that the interaction form satisfies the following property

$$
\begin{equation*}
J_{i, i^{\prime}}:=\sum_{Z: Z \ni\left\{i, i^{\prime}\right\}}\left\|h_{Z}\right\| \leq \frac{g}{\left(1+d_{i, i^{\prime}}\right)^{\alpha}} \tag{S.1}
\end{equation*}
$$

Here, $J_{i, i^{\prime}}$ is the maximum operator norm of the local Hamiltonians containing the sites $i$ and $i^{\prime}$.
We consider the quantum Gibbs state $\rho_{\beta}$ with a fixed inverse temperature $\beta$ defined as follows:

$$
\begin{equation*}
\rho_{\beta}:=e^{-\beta H} / Z, \quad Z=\operatorname{tr}\left(e^{-\beta H}\right) . \tag{S.2}
\end{equation*}
$$

The mutual information of $\rho_{\beta}$ between two subsets $A$ and $B$ is defined as

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B):=S\left(\rho_{\beta}^{A}\right)+S\left(\rho_{\beta}^{B}\right)-S\left(\rho_{\beta}^{A B}\right), \tag{S.3}
\end{equation*}
$$

where $\rho_{\beta}^{X}$ is the density matrix of $\rho_{\beta}$ for the regime $X \in\{A, B, A B\}$ and $S\left(\rho_{\beta}^{X}\right):=-\operatorname{tr}\left(\rho_{\beta}^{X} \log \rho_{\beta}^{X}\right)$ is the von Neumann entropy. The standard correlation function of the density matrix $\rho$ between two observables $O_{A}$ and $O_{B}$ is defined as

$$
\begin{equation*}
\operatorname{Cor}_{\rho}\left(O_{A}, O_{B}\right):=\operatorname{tr}\left(\rho O_{A} O_{B}\right)-\operatorname{tr}\left(\rho O_{A}\right) \cdot \operatorname{tr}\left(\rho O_{B}\right) \tag{S.4}
\end{equation*}
$$

## S.II. PROOF OF MAIN THEOREM IN A GENERAL SETTING

Theorem 1. Let us assume that the correlation function of the quantum Gibbs state $\rho_{\beta}$ between two arbitrary operators $O_{X}$ and $O_{Y}$ supported on the subsets $X$ and $Y$ in $\Lambda$ satisfies the following power-law clustering property:

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right) \leq \frac{C}{d_{X, Y}^{\alpha}}\left\|O_{X}\right\| \cdot\left\|O_{Y}\right\| \tag{S.5}
\end{equation*}
$$

where $C$ is an $\mathcal{O}(1)$ constant. Then, for the bipartition $A, B$ of $D$-dimensional lattice $\Lambda(A \cup B=\Lambda)$ and $\alpha>$ $(D+1) / 2$, the mutual information is upper bounded by

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \beta \cdot C \cdot|\partial A| \cdot \text { const. } \tag{S.6}
\end{equation*}
$$

Proof. We partition the lattice $\Lambda$ into two subsets $A, B \subset \Lambda$ and write the Hamiltonian as $H=H_{A}+H_{B}+H_{\partial A}$. Here, $H_{A}, H_{B}$ are the Hamiltonian supported only on $A, B$, respectively, and $H_{\partial A}$ is the interaction between $A$ and $B$, i.e.,

$$
\begin{equation*}
H_{\partial A}=\sum_{\substack{Z: Z \cap A \neq \emptyset \\ Z \cap B \neq \emptyset}} h_{Z} . \tag{S.7}
\end{equation*}
$$

Using the non-negativity of the quantum relative entropy $D(\rho \| \sigma):=\operatorname{tr}(\rho(\log \rho-\log \sigma))$ yields

$$
\begin{equation*}
D\left(\rho \| \rho_{\beta}\right)=\beta \operatorname{tr}(\rho H)-S(\rho)+\log Z=\beta F_{\beta}(\rho)-\beta F_{\beta}\left(\rho_{\beta}\right) \geq 0 \tag{S.8}
\end{equation*}
$$

for an arbitrary density matrix $\rho$. Here, $F_{\beta}(\rho)$ is the free energy defined as

$$
\begin{equation*}
F_{\beta}(\rho):=\operatorname{tr}(\rho H)-\beta^{-1} S(\rho), \tag{S.9}
\end{equation*}
$$

and it becomes $F_{\beta}\left(\rho_{\beta}\right)=-\beta^{-1} \log Z$ at equilibrium. Substituting $\rho=\rho_{\beta}^{A} \otimes \rho_{\beta}^{B}$ and using $\mathcal{I}_{\rho_{\beta}}(A: B)=S\left(\rho_{\beta}^{A} \otimes\right.$ $\left.\rho_{\beta}^{B}\right)-S\left(\rho_{\beta}^{A B}\right)$ and $\operatorname{tr}\left(\left(H_{A}+H_{B}\right) \rho_{\beta}^{A} \otimes \rho_{\beta}^{B}\right)=\operatorname{tr}\left(\left(H_{A}+H_{B}\right) \rho_{\beta}^{A B}\right)$, we obtain

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \beta \cdot \operatorname{tr}\left(\left(\rho_{A} \otimes \rho_{B}-\rho_{A B}\right) H_{\partial A}\right) . \tag{S.10}
\end{equation*}
$$

For the $k$-local Hamiltonian, each local term $h_{Z}$ can be expanded by sum of the tensor products of two operators supported on $Z_{A}=Z \cap A$ and $Z_{B}=Z \cap B$ :

$$
\begin{equation*}
h_{Z}=\sum_{s=1}^{d_{0}^{2 k}} h_{Z_{A}}^{(s)} \otimes h_{Z_{B}}^{(s)} \tag{S.11}
\end{equation*}
$$

Equations (S.7), (S.10) and (S.11) lead to

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \sum_{\substack{Z: Z \cap A \neq \emptyset \\ Z \cap B \neq \emptyset}} \sum_{s=1}^{d_{0}^{2 k}} \beta \cdot \operatorname{tr}\left(\left(\rho_{A} \otimes \rho_{B}-\rho_{A B}\right) h_{Z_{A}}^{(s)} \otimes h_{Z_{B}}^{(s)}\right)=\sum_{\substack{Z: Z \cap A \neq \emptyset \\ Z \cap B \neq \emptyset}} \sum_{s=1}^{d_{0}^{2 k}} \beta \cdot \operatorname{Cor}_{\rho_{A B}}\left(h_{Z_{A}}^{(s)}, h_{Z_{B}}^{(s)}\right) . \tag{S.12}
\end{equation*}
$$

The assumption of the clustering property gives

$$
\begin{align*}
\mathcal{I}_{\rho_{\beta}}(A: B) & \leq \sum_{\substack{Z: Z \cap A \neq \emptyset \\
Z \cap B \neq \emptyset}} \sum_{\substack{=1}}^{d_{0}^{2 k}} \beta \cdot\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\| \frac{C}{d_{Z_{A}, Z_{B}}^{\alpha}}  \tag{S.13}\\
& \leq \sum_{i \in A} \sum_{j \in B} \sum_{\substack{: Z_{A} \ni i \\
Z_{B} \ni j}} \sum_{s=1}^{d_{0}^{2 k}} \beta \cdot\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\| \frac{C}{d_{i, j}^{\alpha}}  \tag{S.14}\\
& \leq \sum_{i \in A} \sum_{j \in B} \sum_{\substack{Z: Z_{A} \ni i \\
Z_{B} \ni j}} \beta \cdot d_{0}^{2 k} \max _{s}\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\| \frac{C}{d_{i, j}^{\alpha}} . \tag{S.15}
\end{align*}
$$

In the second inequality, we have used the fact that $d_{Z_{A}, Z_{B}}^{-\alpha}$ can be expressed as $d_{i, j}^{-\alpha}$ for some $i \in Z_{A}$ and $j \in Z_{B}$. In the third inequality, we has used $\sum_{s=1}^{d_{0}^{2 k}}\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\| \leq d_{0}^{2 k} \max _{s}\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\|$. By the condition Eq. (S.1) of the long-range interaction, we obtain

$$
\begin{equation*}
\sum_{\substack{Z: Z_{A} \ni i \\ Z_{B} \ni j}} \max _{s \in \mathbb{Z}_{\left[1, d_{0}^{2 k]}\right.}}\left\|h_{Z_{A}}^{(s)}\right\|\left\|h_{Z_{B}}^{(s)}\right\| \leq \sum_{Z: Z \ni\{i, j\}}\left\|h_{Z}\right\| \leq \frac{g}{d_{i, j}^{\alpha}} \tag{S.16}
\end{equation*}
$$

Combining Eq. (S.15) and Eq. (S.16),

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \sum_{i \in A} \sum_{j \in B} \frac{\beta d_{0}^{2 k} g C}{d_{i, j}^{2 \alpha}} \tag{S.17}
\end{equation*}
$$

It was proven that $\sum_{i \in A} \sum_{j \in B} d_{i, j}^{-2 \alpha}$ is upper bounded by the boundary area $|\partial A|$ when $2 \alpha>D+1$ [39]. With Eq. (S.17), we finally obtain

$$
\begin{equation*}
\mathcal{I}_{\rho_{\beta}}(A: B) \leq \beta \cdot C \cdot|\partial A| \cdot \text { const } \tag{S.18}
\end{equation*}
$$

for $2 \alpha>D+1$.

## S.III. POWER-LAW CLUSTERING THEOREM FOR $\alpha>D$

We consider the regime $\alpha>D$, so that the extensivity property holds as

$$
\begin{equation*}
\left\|\widehat{H_{L}}\right\| \leq g|L|, \quad \widehat{H_{L}}=\sum_{Z: Z \cap L \neq \emptyset} h_{Z}=H-H_{L^{\mathrm{c}}} \tag{S.19}
\end{equation*}
$$

for an arbitrary subset $L \subseteq \Lambda$. Then, the clustering theorem for the bipartite correlations is rigorously proven as
Theorem 2. The following relation holds for long-range interacting systems for the temperatures above $\beta_{c}^{-1}$ :

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right) \leq C\left\|O_{X}\right\| \cdot\left\|O_{Y}\right\| \frac{|X \| Y| e^{(|X|+|Y|) / k}}{d_{X, Y}^{\alpha}} \tag{S.20}
\end{equation*}
$$

where $C$ is an $\mathcal{O}(1)$ constant and the threshold temperature is given by $\beta_{c}=W(1 / u e) /(2 g k)$ with the Lambert $W$ function (i.e., the inverse function of $x e^{x}$ ) and $u=\sum_{j \in \Lambda} 1 /\left(1+d_{i, j}\right)^{\alpha}$. We can lower-bound $\beta_{c}$ as $\beta_{c} \geq 1 /(8 u g k)$ by using

$$
W(1 / u e) \leq 1 /(1+u e) \leq 1 /(4 u)
$$

where we use $u$ is always larger than 1 from the definition.

For the proof of the clustering theorem, we utilize the cluster expansion [69-71]. First of all, we describe the correlation function $\operatorname{Cor}_{\rho}\left(O_{X}, O_{Y}\right)$ can be written as

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right)=\frac{1}{\mathcal{Z}^{2}} \operatorname{tr}\left(e^{-\beta H^{(+)}} O_{X}^{(0)} O_{Y}^{(1)}\right) \tag{S.21}
\end{equation*}
$$

where we define the operators $O^{(+)}, O^{(0)}$ and $O^{(1)}$ as

$$
\begin{equation*}
O^{(+)}=O \otimes \hat{1}+\hat{1} \otimes O, \quad O^{(0)}=O \otimes \hat{1}, \quad O^{(1)}=O \otimes \hat{1}-\hat{1} \otimes O \tag{S.22}
\end{equation*}
$$

by considering a copy of the original Hilbert space. The quantity $\mathcal{Z}$ is a normalization factor $\mathcal{Z}^{2}=\operatorname{tr}\left(e^{-\beta H^{(+)}}\right)(=$ $\left[\operatorname{tr}\left(e^{-\beta H}\right)\right]^{2}$ in the original Hilbert space). It is simple to observe that the following lemma holds:

Lemma 1. For an arbitrary set of operators $\left\{O_{Z_{j}}\right\}_{j=1}^{m}$, the following relation holds:

$$
\begin{equation*}
\operatorname{tr}\left(O_{Z_{1}}^{(+)} O_{Z_{2}}^{(+)} \cdots O_{Z_{m}}^{(+)} O_{X}^{(0)} O_{Y}^{(1)}\right)=0 \tag{S.23}
\end{equation*}
$$

when the subsets $\left\{Z_{1}, Z_{2}, \ldots, Z_{m}\right\}$ cannot connect the subsets $X$ and $Y$ [see Fig. 5 (a)], i.e.,

$$
\begin{equation*}
\left(X \cup Z_{i_{1}} \cup Z_{i_{2}} \cup \cdots \cup Z_{i_{s}}\right) \cap\left(Y \cup Z_{i_{s+1}} \cup Z_{i_{s+2}} \cup \cdots \cup Z_{i_{m}}\right)=\emptyset \tag{S.24}
\end{equation*}
$$

with $\left\{i_{1}, i_{2}, \ldots, i_{m}\right\}=\{1,2, \ldots, m\}$.
Proof of Lemma 1. Without loss of generality, we let $\left\{i_{1}, i_{2}, \ldots, i_{s}\right\}=\{1,2, \ldots, s\}$ and $\{1,2, \ldots, s\} \backslash$ $\left\{i_{1}, i_{2}, \ldots, i_{m}\right\}=\{s+1, s+2, \ldots, m\}$. We then denote

$$
\begin{equation*}
O_{Z_{1}}^{(+)} O_{Z_{2}}^{(+)} \cdots O_{Z_{s}}^{(+)}=W_{1}, \quad O_{Z_{s+1}}^{(+)} O_{Z_{s+2}}^{(+)} \cdots O_{Z_{m}}^{(+)}=W_{2} \tag{S.25}
\end{equation*}
$$

with

$$
\begin{equation*}
\operatorname{Supp}\left(X \cup W_{1}\right) \cap \operatorname{Supp}\left(Y \cup W_{2}\right)=\emptyset \tag{S.26}
\end{equation*}
$$



FIG. 5. (a): Schematic picture of unconnected subsets. (b): Connected subsets. (c): Decomposition of the shortest subsets and the rest of the subsets.
where the second equation is equivalent to the condition (S.24). We then decompose as

$$
\begin{equation*}
\operatorname{tr}\left(O_{Z_{1}}^{(+)} O_{Z_{2}}^{(+)} \ldots O_{Z_{m}}^{(+)} O_{X}^{(0)} O_{Y}^{(1)}\right)=\operatorname{tr}\left(W_{1} O_{X}^{(0)}\right) \operatorname{tr}\left(W_{2} O_{Y}^{(1)}\right) \tag{S.27}
\end{equation*}
$$

Here, the operators $O_{Z_{i_{k}}}^{(+)}=O_{Z_{i_{k}}} \otimes \hat{1}+\hat{1} \otimes O_{Z_{i_{k}}}$ are symmetric for the exchange of the two Hilbert space while the operator $O_{Y}^{(1)}=O_{Y} \otimes \hat{1}-\hat{1} \otimes O_{Y}$ is antisymmetric, which yields $\operatorname{tr}\left(W_{2} O_{Y}^{(1)}\right)=0$. We thus prove the main equation (S.23).
$\qquad$ End of Proof of Lemma 1]
By using Lemma 1, the function $\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right)$ can be expanded as

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}=\frac{1}{\mathcal{Z}^{2}} \sum_{m=0}^{\infty} \sum_{Z_{1}, Z_{2}, \ldots, Z_{m}: \text { connected }} \frac{(-\beta)^{m}}{m!} \operatorname{tr}\left(h_{Z_{1}}^{(+)} h_{Z_{2}}^{(+)} \cdots h_{Z_{m}}^{(+)} O_{X}^{(0)} O_{Y}^{(1)}\right) \tag{S.28}
\end{equation*}
$$

where the summation $\sum_{Z_{1}, Z_{2}, \ldots, Z_{m} \text { :connected }}$ is taken over the collections $\left\{X, Z_{1}, Z_{2}, \ldots, Z_{m}, Y\right\}$ that connect $X$ and $Y$. See Fig. 5 (b). Note that $\left\{Z_{s}\right\}_{s=1}^{m}$ can be the same, e.g., $Z_{1}=Z_{2}$. Let us define the operator $\rho_{\mathrm{cl}}$ as

$$
\begin{equation*}
\rho_{\mathrm{cl}}=\frac{1}{\mathcal{Z}^{2}} \sum_{m=0}^{\infty} \sum_{Z_{1}, Z_{2}, \ldots, Z_{m}: \text { connected }} \frac{(-\beta)^{m}}{m!} h_{Z_{1}}^{(+)} h_{Z_{2}}^{(+)} \cdots h_{Z_{m}}^{(+)} \tag{S.29}
\end{equation*}
$$

Using the operator $\rho_{\mathrm{cl}}$, the correlation $\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right)$ has the following upper bound:

$$
\begin{equation*}
\operatorname{Cor}_{\rho_{\beta}}\left(O_{X}, O_{Y}\right) \leq 2\left\|O_{X}\right\| \cdot\left\|O_{Y}\right\| \cdot\left\|\rho_{\mathrm{cl}}\right\|_{1} \tag{S.30}
\end{equation*}
$$

To count the summation of $\sum_{Z_{1}, Z_{2}, \ldots, Z_{m} \text { :connected }}$, we first decompose as

$$
\begin{equation*}
\left\{Z_{1}, Z_{2}, \ldots, Z_{m}\right\}=w_{\mathrm{cl}} \oplus w_{\mathrm{cl}}^{\mathrm{c}}, \quad w_{\mathrm{cl}}^{\mathrm{c}}:=\left\{Z_{1}, Z_{2}, \ldots, Z_{m}\right\} \backslash w_{\mathrm{cl}} \tag{S.31}
\end{equation*}
$$

where $w_{\mathrm{cl}}$ is taken such that $w_{\mathrm{cl}} \oplus\{X, Y\}$ are connected to each other (See Fig. 5 (b)), in other words,

$$
\begin{equation*}
Z \cap Z^{\prime}=\emptyset \quad \text { for } \quad Z \in w_{\mathrm{cl}}, \quad Z^{\prime} \in w_{\mathrm{cl}}^{\mathrm{c}} . \tag{S.32}
\end{equation*}
$$

We define all the sets of $w_{\mathrm{cl}}$ and $w_{\mathrm{cl}}^{\mathrm{c}}$ as $\mathcal{G}_{\mathrm{cl}}$ and $\mathcal{G}_{\mathrm{cl}}^{\mathrm{c}}$, respectively. We also define the total support of the $w_{\mathrm{cl}}$ and $w_{\mathrm{cl}}^{\mathrm{c}}$ as $V_{w_{\mathrm{cl}}}$ and $V_{w_{\mathrm{cl}}^{\mathrm{c}}}$, respectively. We then obtain

$$
\begin{align*}
\rho_{\mathrm{cl}} & =\frac{1}{\mathcal{Z}^{2}} \sum_{m=0}^{\infty} \frac{(-\beta)^{m}}{m!} \sum_{s=0}^{m} \frac{m!}{(m-s)!s!} \sum_{\left\{Z_{1}, \cdots, Z_{s}\right\}=w_{\mathrm{cl}} \in \mathcal{G}_{\mathrm{cl}}} \tilde{h}\left(Z_{1}, \cdots, Z_{s}\right) \sum_{w_{\mathrm{cl}}^{\mathrm{c} \in \mathcal{G}_{\mathrm{cl}}^{\mathrm{c}}} \sum_{\substack{Z_{s+1}, \ldots, Z_{m} \\
\in w_{\mathrm{cl}}^{\mathrm{c}}}} h^{(+)}\left(Z_{s+1}\right) h^{(+)}\left(Z_{s+2}\right) \cdots h^{(+)}\left(Z_{m}\right)} \\
& =\frac{1}{\mathcal{Z}^{2}} \sum_{s=0}^{\infty} \frac{(-\beta)^{s}}{s!} \sum_{w_{\mathrm{cl}} \in \mathcal{G}_{\mathrm{cl}}} \tilde{h}\left(Z_{1}, \cdots, Z_{s}\right) e^{-\beta H_{V_{w_{\mathrm{cl}}}^{\mathrm{c}}}^{(+)}} \tag{S.33}
\end{align*}
$$

Here, we have defined

$$
\begin{equation*}
\tilde{h}\left(Z_{1}, \cdots, Z_{s}\right)=\sum_{P} h^{(+)}\left(Z_{P_{1}}\right) \cdots h^{(+)}\left(Z_{P_{s}}\right) \tag{S.34}
\end{equation*}
$$

where $P$ implies taking all combinations of different elements for $Z$. In particular, when $Z$ are all different, it means taking a permutation. From the expression (S.33), we obtain

$$
\begin{equation*}
\left\|\rho_{\mathrm{cl}}\right\|_{1} \leq \frac{1}{\mathcal{Z}^{2}} \sum_{s=0}^{\infty} \frac{\beta^{s}}{s!} \sum_{w_{\mathrm{cl}} \in \mathcal{G}_{\mathrm{cl}}}\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s}\right)\right\| \operatorname{tr}\left[e^{-\beta H_{V_{\mathrm{cl}}}^{(+)}}\right] \leq \sum_{s=0}^{\infty} \frac{\beta^{s}}{s!} \sum_{w_{\mathrm{cl}} \in \mathcal{G}_{\mathrm{cl}}} e^{2 \beta g\left|V_{w_{\mathrm{cl}}}\right|}\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s}\right)\right\| \tag{S.35}
\end{equation*}
$$

Here, we use the Golden-Thompson inequality as follows:

$$
\begin{align*}
\operatorname{tr}\left[e^{-\beta H_{V_{w 1}^{c}}^{(+)}}\right] & =\operatorname{tr}\left[e^{-\beta\left(H^{(+)}+H_{V_{w 1}^{c}}^{(+)}-H^{(+)}\right)}\right] \leq \operatorname{tr}\left[e^{-\beta H^{(+)}} e^{-\beta\left(H_{V_{w 1}^{c}}^{(+)}-H^{(+)}\right)}\right] \leq \mathcal{Z}^{2} e^{2 \beta\left\|H_{V_{w_{c l}^{\mathrm{c}}}^{\mathrm{c}}}-H\right\|} \\
& \leq \mathcal{Z}^{2} e^{2 \beta g\left|V_{w_{c l}}\right|} \leq \mathcal{Z}^{2} e^{2 \beta g k s} \tag{S.36}
\end{align*}
$$

where we use the condition (S.19) in the last inequality. From the $k$-locality of the Hamiltonian, we have $\left|V_{w_{\mathrm{cl}}}\right| \leq$ $k\left|w_{\mathrm{cl}}\right|=k s$.

We divide each element in the set $w_{\mathrm{cl}}$ into the two subsets $w_{\mathrm{cl}}^{\mathrm{s}}$ and $w_{\mathrm{cl}}^{\mathrm{r}}$, where $w_{\mathrm{cl}}^{\mathrm{s}}$ is a set representing the shortest path with the length $s^{\prime}$ and $w_{\mathrm{cl}}^{\mathrm{r}}$ is a set for the rest part with the length $\left(s-s^{\prime}\right)$. See Fig. 5 (c). We thus obtain

$$
\begin{align*}
\left\|\rho_{\mathrm{cl}}\right\|_{1} & \leq \sum_{s=0}^{\infty} \frac{\left(\beta e^{2 \beta g k}\right)^{s}}{s!} \sum_{s^{\prime}=1}^{s} \frac{s!}{s^{\prime}!\left(s-s^{\prime}\right)!} \sum_{\left\{Z_{1}, \cdots, Z_{s^{\prime}}\right\}=w_{\mathrm{cl}}^{\mathrm{s}}}\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s^{\prime}}\right)\right\|^{\prime} \sum_{\left\{Z_{s^{\prime}+1}, \cdots, Z_{s}\right\}=w_{\mathrm{cl}}^{\mathrm{r}}}\left\|\tilde{h}\left(Z_{s^{\prime}+1}, \cdots, Z_{s}\right)\right\|^{\prime} \\
& =\sum_{s=0}^{\infty}\left(\beta e^{2 \beta g k}\right)^{s} \sum_{s^{\prime}=1}^{s} \sum_{w_{\mathrm{cl}}^{\mathrm{s}}} \frac{\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s^{\prime}}\right)\right\|^{\prime}}{s^{\prime}!} \sum_{w_{\mathrm{cl}}^{\mathrm{r}}} \frac{\left\|\tilde{h}\left(Z_{s^{\prime}+1}, \cdots, Z_{s}\right)\right\|^{\prime}}{\left(s-s^{\prime}\right)!} \tag{S.37}
\end{align*}
$$

where

$$
\begin{equation*}
\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s^{\prime}}\right)\right\|^{\prime}:=\sum_{P}\left\|h_{Z_{P_{1}}}^{(+)}\right\| \cdots\left\|h_{Z_{P_{s^{\prime}}}}^{(+)}\right\| \leq 2^{s^{\prime}} \sum_{P}\left\|h_{Z_{P_{1}}}\right\| \cdots\left\|h_{Z_{P_{s^{\prime}}}}\right\| \tag{S.38}
\end{equation*}
$$

Similarly to (S.38), $\left\|\tilde{h}\left(Z_{s^{\prime}+1}, \cdots, Z_{s}\right)\right\|^{\prime}$ is defined. Below, we evaluate the terms on $w_{\mathrm{cl}}^{\mathrm{s}}$ and $w_{\mathrm{cl}}^{\mathrm{r}}$ in (S.37). To this end, the following lemma is useful.

Lemma 2. For $\alpha>D$, the quantity $J_{i, j}$ defined in (S.1) satisfies the following inequality

$$
\begin{equation*}
\sum_{j \in \Lambda} J_{i, j} J_{j, k} \leq \frac{g^{2} u}{\left(1+d_{i, k}\right)^{\alpha}} \tag{S.39}
\end{equation*}
$$

where $u=2^{\alpha} \sum_{j \in \Lambda}\left(1+d_{i, j}\right)^{-\alpha}$. The iterative use of (S.39) leads to $\left[\boldsymbol{J}^{\ell}\right]_{i, k} \leq g^{\ell} u^{\ell-1} /\left(1+d_{i, k}\right)^{\alpha}$.
Proof of Lemma 2.

$$
\begin{equation*}
\sum_{j \in \Lambda} J_{i, j} J_{j, k}=g^{2} \sum_{j \in \Lambda} \frac{1}{\left(1+d_{i, j}\right)^{\alpha}} \frac{1}{\left(1+d_{j, k}\right)^{\alpha}} \leq \frac{g^{2}}{\left(1+d_{i, k}\right)^{\alpha}} \sum_{j \in \Lambda} \frac{\left(2+d_{i, j}+d_{j, k}\right)^{\alpha}}{\left(1+d_{i, j}\right)^{\alpha}} \frac{1}{\left(1+d_{j, k}\right)^{\alpha}} \tag{S.40}
\end{equation*}
$$

Note $(x+y)^{\alpha} \leq 2^{\alpha-1}\left(x^{\alpha}+y^{\alpha}\right)$ for $\alpha>1$. Then we have

$$
\begin{equation*}
\sum_{j \in \Lambda} J_{i, j} J_{j, k} \leq \frac{g^{2} 2^{\alpha-1}}{\left(1+d_{i, k}\right)^{\alpha}} \sum_{j \in \Lambda} \frac{1}{\left(1+d_{i, j}\right)^{\alpha}}+\frac{1}{\left(1+d_{j, k}\right)^{\alpha}}=\frac{g^{2} u}{\left(1+d_{i, k}\right)^{\alpha}} \tag{S.41}
\end{equation*}
$$

The finiteness of $u$ is guaranteed by the condition $\alpha>D$.

Let us first consider the terms on $w_{\mathrm{cl}}^{\mathrm{s}}$ in (S.37).

$$
\begin{align*}
\sum_{w_{\mathrm{cl}}^{\mathrm{s}}} \frac{\left\|\tilde{h}\left(Z_{1}, \cdots, Z_{s^{\prime}}\right)\right\|^{\prime}}{s^{\prime}!} & \leq \frac{2^{s^{\prime}}}{s^{\prime}!} \sum_{i_{1} \in X} \sum_{i_{2} \in \Lambda} \sum_{Z_{1} \ni\left\{i_{1}, i_{2}\right\}} \sum_{i_{3} \in \Lambda} \sum_{Z_{2} \ni\left\{i_{2}, i_{3}\right\}} \cdots \sum_{i_{s^{\prime}} \in \Lambda} \sum_{i_{s^{\prime}+1} \in Y} \sum_{Z_{s^{\prime}} \ni\left\{i_{s^{\prime},}^{\prime}, s_{s^{\prime}+1}\right\}}\left\|h_{Z_{1}}\right\| \cdots\left\|h_{Z_{s^{\prime}}}\right\| \\
& =\frac{2^{s^{\prime}}}{s^{\prime}!} \sum_{i_{1} \in X} \sum_{i_{2} \in \Lambda} \cdots \sum_{i_{s^{\prime} \in \Lambda} \in \Lambda} \sum_{i_{s^{\prime}+1} \in Y} J_{i_{1}, i_{2}} \cdots J_{i_{s^{\prime},}, i_{s^{\prime}+1}} \leq \frac{u^{-1}(2 g u)^{s^{\prime}}}{s^{\prime}!}|X||Y| \frac{1}{\left(1+d_{X, Y}\right)^{\alpha}} \tag{S.42}
\end{align*}
$$

We next consider the terms of $w_{\mathrm{cl}}^{\mathrm{r}}$ in (S.37). We define $n:=s-s^{\prime}$ and $L:=X \cup Y \cup \omega_{\mathrm{cl}}^{\mathrm{s}}$ for a simplicity.

$$
\begin{align*}
& \frac{1}{n!} \sum_{w_{\mathrm{cl}}^{\mathrm{r}}}\left\|\tilde{h}\left(Z_{s^{\prime}+1} \cdots, Z_{s}\right)\right\|^{\prime} \leq \frac{2^{n}}{n!} \sum_{i_{1} \in L} \sum_{Z_{1} \ni i_{1}} \sum_{i_{2} \in L \cup Z_{1}} \sum_{Z_{2} \ni i_{2}} \sum_{i_{3} \in L \cup Z_{1} \cup Z_{2}} \sum_{Z_{3} \ni i_{3}} \cdots \sum_{i_{n} \in L \cup Z_{1} \cup \cdots \cup Z_{n-1}} \sum_{Z_{n} \ni i_{n}}\left\|h_{Z_{1}}\right\| \cdots\left\|h_{Z_{n}}\right\| \\
& \leq \frac{(2 g u)^{n}}{n!} \sum_{i_{1} \in L} \sum_{i_{2} \in L \cup Z_{1}} \sum_{i_{3} \in L \cup Z_{1} \cup Z_{2}} \cdots \sum_{i_{n} \in L \cup Z_{1} \cup \cdots \cup Z_{n-1}} \leq \frac{(2 g u)^{n}}{n!}|L|(|L|+k) \cdots(|L|+(n-1) k) \\
& \leq \frac{(2 g u)^{n}}{(n / e)^{n}}|L|(|L|+k) \cdots(|L|+(n-1) k) \leq(2 g u e k)^{n}\left(1+\frac{|L| / k}{n}\right)^{n} \\
& \leq(2 g u e k)^{n} e^{|L| / k} \leq(2 g u e k)^{n} e^{\left(|X|+|Y|+s^{\prime} k\right) / k} \tag{S.43}
\end{align*}
$$

where we have used $\sum_{Z \ni i}\left\|h_{Z}\right\|<g u,(n / e)^{n}<n!,(1+x / n)^{n}<e^{x}$ and $|L|<|X|+|Y|+s^{\prime} k$.
Finally we sum over $s^{\prime}$ to get the following expression

$$
\begin{align*}
\left\|\rho_{\mathrm{cl}}\right\|_{1} & \leq u^{-1}|X||Y| \frac{e^{(|X|+|Y|) / k}}{\left(1+d_{X, Y}\right)^{\alpha}} \sum_{s=0}^{\infty}\left(2 \beta g k e u e^{2 \beta g k}\right)^{s} \sum_{s^{\prime}=1}^{s} \frac{(1 / k)^{s^{\prime}}}{s^{\prime}!} \\
& \leq\left(e^{1 / k} / u\right)|X||Y| \frac{e^{(|X|+|Y|) / k}}{\left(1+d_{X, Y}\right)^{\alpha}} \frac{1}{1-2 \beta g k e u e^{2 \beta g k}} \tag{S.44}
\end{align*}
$$

Here, we use $\sum_{s^{\prime}=1}^{s}(1 / k)^{s^{\prime}} / s^{\prime}!<e^{1 / k}$. This completes the proof of the Theorem 2.

## S.IV. EXACT DIAGONALIZATION RESULTS FOR LONG-RANGE HEISENBERG CHAINS

Let's examine a more general 1D long-range Heisenberg chain described by the Hamiltonian

$$
\begin{equation*}
H=\sum_{1 \leq i<j \leq N} \frac{a_{i, j}}{d_{i, j}^{\alpha}} \boldsymbol{S}_{i} \cdot \boldsymbol{S}_{j} \tag{S.45}
\end{equation*}
$$

where $a_{i, j}$ is a parameter of order $\mathcal{O}(1), N$ is the number of sites, and $\boldsymbol{S}_{i}=\left(S_{i}^{x}, S_{i}^{y}, S_{i}^{z}\right)$ is a spin- $1 / 2$ operator at site $i$. To support our argument that the thermal area holds for $\alpha>(D+1) / 2$ and this criterion is optimal, we compute the mutual information $\mathcal{I}_{\rho_{\beta}}(A: B)$ between subsystems $A$ and $B$, with $A$ representing the first $N / 2$ sites and $B$ the remaining $N / 2$ sites, by using the exact diagonalization method. We randomly assign $a_{i, j}$ from the interval $[0,1]$, and then compute the average over 2000 samples. Fig. 6 shows that the mutual information increases with the system size when $\alpha<1$, while it does not exhibit an increase for $\alpha>1$.


FIG. 6. Exact diagonalization results: the averaged mutual information $\mathcal{I}_{\rho_{\beta}}(A: B)$ over 2000 samples of the long-range Heisenberg chain (S.45) with random numbers $a_{i, j}$ assigned from the interval $[0,1]$. Here $N$ is the number of sites, and $A$ and $B$ are the first $N / 2$ sites and the remaining $N / 2$ sites, respectively.

## S.V. NUMERICAL DETAIL OF SHAPOURIAN-SHIOZAKI-RYU NEGATIVITY

We consider $n$ spinless fermionic particles with the basis states $|0\rangle$ and $|1\rangle=c^{\dagger}|0\rangle$. The determinant of the $n \times n$ $\operatorname{matrix} \mathbf{X}=\left(x_{i, j}\right)$ is defined as

$$
\begin{equation*}
\operatorname{det} \mathbf{X}=\sum_{\sigma \in S_{n}} \operatorname{sgn}(\sigma) x_{1, \sigma(1)} \cdots x_{n, \sigma(n)} \tag{S.46}
\end{equation*}
$$

and the Pfaffian of the $2 n \times 2 n$ skew-symmetric matrix $\mathbf{Y}=\left(y_{i, j}\right)$, i.e. $\mathbf{Y}^{\mathrm{T}}=-\mathbf{Y}$, is defined as

$$
\begin{equation*}
\operatorname{Pf}[\mathbf{Y}]=\frac{1}{2^{n} n!} \sum_{\sigma \in S_{2 n}} \operatorname{sgn}(\sigma) y_{\sigma(1) \sigma(2)} \cdots y_{\sigma(2 n-1), \sigma(2 n)} \tag{S.47}
\end{equation*}
$$

Here, $S_{n}$ and $\operatorname{sgn}(\sigma)$ denotes the symmetric group and the sign of the permutation $\sigma$, respectively. For an arbitrary $2 n \times 2 n$ matrix $\mathbf{Z}$, we use the following expression

$$
\mathbf{Z}=\left(\begin{array}{ll}
{[\mathbf{Z}]^{(1,1)}} & {[\mathbf{Z}]^{(1,2)}}  \tag{S.48}\\
{[\mathbf{Z}]^{(2,1)}} & {[\mathbf{Z}]^{(2,2)}}
\end{array}\right)
$$

to describe the $n \times n$ block matrix $[\mathbf{Z}]^{(a, b)}$ with $a$-th row and $b$-th column $(a, b \in\{1,2\})$. We denote $\mathbb{1}_{n}$ as an $n \times n$ identity matrix.

We use the Grassmann variables $\left\{\xi_{j}, \bar{\xi}_{j}\right\}$ to describe the fermionic systems with the following notation:

$$
\begin{align*}
\xi & \equiv\left(\xi_{1}, \cdots, \xi_{n}\right)  \tag{S.49}\\
(\xi, \bar{\xi}) & \equiv\left(\xi_{1}, \cdots, \xi_{n}, \bar{\xi}_{1}, \cdots, \bar{\xi}_{n}\right), \tag{S.50}
\end{align*}
$$

with the differentials

$$
\begin{align*}
d \xi & \equiv d \xi_{n} \cdots d \xi_{1}  \tag{S.51}\\
d \bar{\xi} d \xi & \equiv d \bar{\xi}_{n} \cdots d \bar{\xi}_{1} d \xi_{n} \cdots d \xi_{1}  \tag{S.52}\\
d(\bar{\xi}, \xi) & \equiv d \bar{\xi}_{1} d \xi_{1} \cdots d \bar{\xi}_{n} d \xi_{n}=(-1)^{n(n-1) / 2} d \bar{\xi} d \xi \tag{S.53}
\end{align*}
$$

We define the state with the Grassmann variable as

$$
\begin{align*}
\left|\xi_{j}\right\rangle & \equiv|0\rangle_{j}-\xi_{j}|1\rangle_{j}  \tag{S.54}\\
\left|\bar{\xi}_{j}\right\rangle & \equiv\left\langle\left. 0\right|_{j}-\bar{\xi}_{j}\left\langle\left. 1\right|_{j},\right.\right.  \tag{S.55}\\
|\xi\rangle\langle\bar{\xi}| & =\left|\xi_{1}\right\rangle\left\langle\bar{\xi}_{1}\right| \otimes \cdots \otimes\left|\xi_{n}\right\rangle\left\langle\bar{\xi}_{n}\right| \tag{S.56}
\end{align*}
$$

The definition naturally leads to $\left\langle\bar{\xi}_{j} \mid \xi_{j}\right\rangle=e^{\bar{\xi}_{j} \xi_{j}}$. We also use the notation $(\chi, \bar{\chi})$ to describe another Grassmann variables.

In terms of the Grassmann variables, the trace of an operator $O$ is given by

$$
\begin{equation*}
\operatorname{Tr}[O]=\int\langle-\bar{\xi}| O|\xi\rangle e^{-\sum_{j} \bar{\xi}_{j} \xi_{j}} d(\bar{\xi}, \xi) \tag{S.57}
\end{equation*}
$$

We consider the following form of the density operator:

$$
\begin{equation*}
\rho=\frac{1}{Z_{\rho}} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2} \boldsymbol{\Gamma}(\xi, \bar{\xi})^{\mathrm{T}}+\sum_{j} \bar{\xi}_{j} \xi_{j}} \tag{S.58}
\end{equation*}
$$

Imposing $\operatorname{Tr}[\rho]=1$ with Eqs. (S.57) and (S.58), we have $Z_{\rho}=\operatorname{Pf}[\boldsymbol{\Gamma}]$. Therefore, the density matrix is written as

$$
\begin{equation*}
\rho=\frac{1}{\operatorname{Pf}[\boldsymbol{\Gamma}]} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2} \boldsymbol{\Gamma}(\xi, \bar{\xi})^{\mathrm{T}}+\sum_{j} \bar{\xi}_{j} \xi_{j}} \tag{S.59}
\end{equation*}
$$

Using Eq.(S.57), we can relate the two-point correlations functions of the fermionic creation and annihilation operators to the matrix $\boldsymbol{\Gamma}$ by

$$
\boldsymbol{\Gamma}^{-1}=\left(\begin{array}{cc}
{\left[\boldsymbol{\Gamma}^{-1}\right]^{(1,1)}} & {\left[\boldsymbol{\Gamma}^{-1}\right]^{(1,2)}}  \tag{S.60}\\
{\left[\boldsymbol{\Gamma}^{-1}\right]^{(2,1)}} & {\left[\boldsymbol{\Gamma}^{-1}\right]^{(2,2)}}
\end{array}\right)=\left(\begin{array}{cc}
\left\langle c_{j} c_{i}\right\rangle_{\rho} & -\left\langle c_{j}^{\dagger} c_{i}\right\rangle_{\rho} \\
\left\langle c_{i}^{\dagger} c_{j}\right\rangle_{\rho} & \left\langle c_{j}^{\dagger} c_{i}^{\dagger}\right\rangle_{\rho}
\end{array}\right) .
$$

In other words, we identify the density matrix with the covaraince matrix $\boldsymbol{\Gamma}$.
Now we move onto the Shapourian-Shiozaki-Ryu (SSR) negativity. We partition the total system $\Lambda$ into $A$ and $B$ with $|A|=n_{A}$ and $|B|=n_{B}$. To define the SSR negativity between $A$ and $B$, we need the partial time-reversal transformation $R_{A}$ on the subsystem $A$. The time-reversal transformation maps $|\xi\rangle\langle\bar{\xi}|$ to $|i \bar{\xi}\rangle\langle i \xi|$ in the coherent basis. Therefore, we have

$$
\begin{equation*}
\left(\left|\left\{\xi_{j}\right\}_{j \in A},\left\{\xi_{j}\right\}_{j \in B}\right\rangle\left\langle\left\{\bar{\chi}_{j}\right\}_{j \in A},\left\{\bar{\chi}_{j}\right\}_{j \in B}\right|\right)^{R_{A}}=\left|\left\{i \bar{\chi}_{j}\right\}_{j \in A},\left\{\xi_{j}\right\}_{j \in B}\right\rangle\left\langle\left\{i \xi_{j}\right\}_{j \in A},\left\{\bar{\chi}_{j}\right\}_{j \in B}\right| \tag{S.61}
\end{equation*}
$$

and therefore the density matrix after the partial time-reversal transformation becomes

$$
\begin{equation*}
\rho^{R_{A}}=\frac{1}{\operatorname{Pf}[\boldsymbol{\Gamma}]} \int d \bar{\xi} d \xi\left|i \bar{\xi}_{A}, \xi_{B}\right\rangle\left\langle i \xi_{A}, \bar{\xi}_{B}\right| e^{(\xi, \bar{\xi}) \frac{1}{2} \boldsymbol{\Gamma}(\xi, \bar{\xi})^{\mathrm{T}}+\sum_{j} \bar{\xi}_{j} \xi_{j}} \tag{S.62}
\end{equation*}
$$

Now, we rearrange the Grassmann variables. Let the vector $\left(\xi^{\prime}, \bar{\xi}^{\prime}\right) \equiv\left(\xi_{A}^{\prime}, \xi_{B}^{\prime}, \bar{\xi}_{A}^{\prime}, \bar{\xi}_{B}^{\prime}\right)=\left(-i \bar{\xi}_{A}, \xi_{B},-i \xi_{A}, \bar{\xi}_{B}\right)$, then we represent it by introducing the matrix $\mathbf{T}$ :

$$
\left(\xi^{\prime}, \bar{\xi}^{\prime}\right)^{\mathrm{T}}=\mathbf{T}(\xi, \bar{\xi})^{\mathrm{T}}, \quad \mathbf{T}=\left(\begin{array}{cccc}
0 & 0 & -i \mathbb{1}_{n_{A}} & 0  \tag{S.63}\\
0 & \mathbb{1}_{n_{B}} & 0 & 0 \\
-i \mathbb{1}_{n_{A}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{1}_{n_{B}}
\end{array}\right)
$$

Here, $\mathbb{1}_{n_{A}}\left(\mathbb{1}_{n_{B}}\right)$ is an $n_{A} \times n_{A}\left(n_{B} \times n_{B}\right)$ identity matrix. With this matrix, a partial transformed density matrix is rewritten as follows:

$$
\begin{equation*}
\rho^{R_{A}}=\frac{1}{\operatorname{Pf}[\boldsymbol{\Gamma}]} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2} \mathbf{S}^{\prime}(\xi, \bar{\xi})^{\mathrm{T}}} \tag{S.64}
\end{equation*}
$$

with a newly defined matrix

$$
\mathbf{S}^{\prime}:=\mathbf{T}\left[\boldsymbol{\Gamma}+\left(\begin{array}{cc}
0 & -\mathbb{1}_{n}  \tag{S.65}\\
\mathbb{1}_{n} & 0
\end{array}\right)\right] \mathbf{T} .
$$

The next step is to consider the product $\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}$. We introduce the matrix $\mathbf{S}^{\prime \prime}$ for the operator $\left(\rho^{R_{A}}\right)^{\dagger}$ :

$$
\mathbf{S}^{\prime \prime}=\left(\begin{array}{ll}
\left(\left[\mathbf{S}^{\prime}\right]^{(2,2)}\right)^{\dagger} & \left(\left[\mathbf{S}^{\prime}\right]^{(1,2)}\right)^{\dagger}  \tag{S.66}\\
\left(\left[\mathbf{S}^{\prime}\right]^{(2,1)}\right)^{\dagger} & \left.\left(\left[\mathbf{S}^{\prime}\right]\right]^{(1,1)}\right)^{\dagger}
\end{array}\right)
$$

This leads to

$$
\begin{equation*}
\left(\rho^{R_{A}}\right)^{\dagger}=\frac{1}{\operatorname{Pf}[\boldsymbol{\Gamma}]} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2}\left(\mathbf{S}^{\prime \prime}\right)(\xi, \bar{\xi})^{\mathrm{T}}} \tag{S.67}
\end{equation*}
$$

After a straightforward computation, we find the following expression for the product $\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}$ :

$$
\begin{equation*}
\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}=\frac{(-1)^{n^{2}} \operatorname{Pf}[\mathbf{B}]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2} \boldsymbol{\Gamma}^{\prime}(\xi, \bar{\xi})^{\mathrm{T}}+\sum_{j} \bar{\xi}_{j} \xi_{j}}, \tag{S.68}
\end{equation*}
$$

with introducing two matrices

$$
\begin{align*}
\mathbf{B} & =\left(\begin{array}{cc}
{\left[\mathbf{S}^{\prime}\right]^{(1,1)}} & -\mathbb{1}_{n} \\
\mathbb{1}_{n} & {\left[\mathbf{S}^{\prime \prime}\right]^{(2,2)}}
\end{array}\right),  \tag{S.69}\\
\boldsymbol{\Gamma}^{\prime} & =-\left(\begin{array}{cc}
0 & {\left[\mathbf{S}^{\prime \prime}\right]^{(1,2)}} \\
{\left[\mathbf{S}^{\prime}\right]^{(2,1)}} & 0
\end{array}\right) \mathbf{B}^{-1}\left(\begin{array}{cc}
0 & {\left[\mathbf{S}^{\prime}\right]^{(1,2)}} \\
{\left[\mathbf{S}^{\prime \prime \prime}\right]^{(2,1)}} & 0
\end{array}\right)+\left(\begin{array}{cc}
{\left[\mathbf{S}^{\prime \prime \prime}\right]^{(1,1)}} & 0 \\
0 & {\left[\mathbf{S}^{\prime}\right]^{(2,2)}}
\end{array}\right)-\left(\begin{array}{cc}
0 & -\mathbb{1}_{n} \\
\mathbb{1}_{n} & 0
\end{array}\right) . \tag{S.70}
\end{align*}
$$

Then the SSR negativity is

$$
\begin{equation*}
E_{\mathrm{SSR}}=\log \left\|\rho^{R_{A}}\right\|_{1}=\log \operatorname{Tr}\left[\sqrt{\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}}\right] \tag{S.71}
\end{equation*}
$$

## A. Numerical Procedure

Based on the above calculations, we present the following procedure to numerically calculate the SSR negativity.

1. We first compute the covariance matrix $\boldsymbol{\Gamma}$ from Eq. (S.60). Then we calculate $\mathbf{S}^{\prime}, \mathbf{S}^{\prime \prime}$, and $\mathbf{B}$ from Eqs. (S.65), (S.66), and (S.69) to construct $\Gamma^{\prime}$ by Eq. (S.70).
2. Note the relation

$$
\begin{align*}
\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}} & =\frac{(-1)^{n^{2}} \operatorname{Pf}[\mathbf{B}] \operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}} \rho^{\prime \prime}  \tag{S.72}\\
\rho^{\prime \prime} & =\frac{1}{\operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]} \int d \bar{\xi} d \xi|\xi\rangle\langle\bar{\xi}| e^{(\xi, \bar{\xi}) \frac{1}{2} \boldsymbol{\Gamma}^{\prime}(\xi, \bar{\xi})^{\mathrm{T}}+\sum_{j} \bar{\xi}_{j} \xi_{j}} . \tag{S.73}
\end{align*}
$$

From $\boldsymbol{\Gamma}^{\prime}$, using Eq. (S.60) again, we define the covariance matrix $\mathbf{C}$ of $\rho^{\prime \prime}$ :

$$
\mathbf{C}=\left(\begin{array}{ll}
\left\langle c_{i}^{\dagger} c_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle c_{i}^{\dagger} c_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}}  \tag{S.74}\\
\left\langle c_{i} c_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle c_{i} c_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}}
\end{array}\right)=\left(\begin{array}{cc}
{\left[\left(\boldsymbol{\Gamma}^{\prime}\right)^{-1}\right]^{(2,1)}} & \left(\left[\left(\boldsymbol{\Gamma}^{\prime}\right)^{-1}\right]^{(2,2)}\right)^{\mathrm{T}} \\
\left(\left[\left(\boldsymbol{\Gamma}^{\prime}\right)^{-1}\right]^{(1,1)}\right)^{\mathrm{T}} & \mathbb{1}_{n}+\left[\left(\boldsymbol{\Gamma}^{\prime}\right)^{-1}\right]^{(1,2)}
\end{array}\right) .
$$

3. We choose the unitary transformation $U$ to define another fermionic annihilation operators $\mathbf{d}=\left(d_{1}, \cdots, d_{n}\right)$ :

$$
\begin{equation*}
\binom{\mathbf{d}^{\dagger}}{\mathbf{d}}=U\binom{\mathbf{c}^{\dagger}}{\mathbf{c}} \tag{S.75}
\end{equation*}
$$

so that it diagonalizes $\rho^{\prime \prime}$ as

$$
\begin{equation*}
\rho^{\prime \prime}=\frac{e^{-\sum_{i=1}^{n} \epsilon_{i}^{\prime \prime} d_{i}^{\dagger} d_{i}}}{Z^{\prime \prime}} \tag{S.76}
\end{equation*}
$$

where $Z^{\prime \prime}=\operatorname{Tr}\left[\rho^{\prime \prime}\right]=\prod_{i=1}^{n}\left(1+e^{-\epsilon_{i}^{\prime \prime}}\right)$. Equation. (S.76) yields

$$
\left(\begin{array}{ll}
\left\langle d_{i}^{\dagger} d_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle d_{i}^{\dagger} d_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}}  \tag{S.77}\\
\left\langle d_{i} d_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle d_{i} d_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}}
\end{array}\right)=\left(\begin{array}{cc}
\frac{1}{e_{i}^{\epsilon_{i}^{\prime \prime}}+1} \delta_{i, j} & 0 \\
0 & \frac{e^{\epsilon_{i}^{\prime \prime}}}{e_{i}^{\epsilon_{i}^{\prime \prime}}+1} \delta_{i, j}
\end{array}\right)=U\left(\begin{array}{ll}
\left\langle c_{i}^{\dagger} c_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle c_{i}^{\dagger} c_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}} \\
\left\langle c_{i} c_{j}\right\rangle_{\rho^{\prime \prime}} & \left\langle c_{i} c_{j}^{\dagger}\right\rangle_{\rho^{\prime \prime}}
\end{array}\right) U^{\dagger} .
$$

It means that we can obtain $\epsilon_{i}^{\prime \prime}$ by diagonalizing the covariance matrix $\mathbf{C}$ :

$$
\begin{align*}
\mathbf{C} & =U^{\dagger} \operatorname{diag}\left(\left\langle d_{i}^{\dagger} d_{i}\right\rangle_{\rho^{\prime \prime}},\left\langle d_{i} d_{i}^{\dagger}\right\rangle_{\rho^{\prime \prime}}\right) U  \tag{S.78}\\
\left\langle d_{i}^{\dagger} d_{i}\right\rangle_{\rho^{\prime \prime}} & =\frac{1}{e^{\epsilon_{i}^{\prime \prime}}+1} \tag{S.79}
\end{align*}
$$

4. From the following identity

$$
\begin{equation*}
\operatorname{Tr}\left[\sqrt{\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}}\right]=\sqrt{\frac{(-1)^{n^{2}} \operatorname{Pf}[\mathbf{B}] \operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}}} \operatorname{Tr} \sqrt{\rho^{\prime \prime}}=\sqrt{\frac{\operatorname{Pf}[\mathbf{B}] \operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}}} \prod_{i=1}^{n} \frac{1+e^{-\frac{\epsilon_{i}^{\prime \prime}}{2}}}{\sqrt{1+e^{-\epsilon_{i}^{\prime \prime}}}}, \tag{S.80}
\end{equation*}
$$

we get the following expression of the SSR negativity with computable quantities:

$$
\begin{align*}
E_{\mathrm{SSR}} & =\log \operatorname{Tr}\left[\sqrt{\left(\rho^{R_{A}}\right)^{\dagger} \rho^{R_{A}}}\right]  \tag{S.81}\\
& =\frac{1}{2} \log \left[\frac{(-1)^{n^{2}} \operatorname{Pf}[\mathbf{B}] \operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}}\right]+\sum_{i=1}^{n}\left[\log \left(1+e^{-\frac{\epsilon_{i}^{\prime \prime}}{2}}\right)-\frac{1}{2} \log \left(1+e^{-\epsilon_{i}^{\prime \prime}}\right)\right]  \tag{S.82}\\
& =\frac{1}{2} \log \left[\frac{(-1)^{n^{2}} \operatorname{Pf}[\mathbf{B}] \operatorname{Pf}\left[\boldsymbol{\Gamma}^{\prime}\right]}{\operatorname{Pf}[\boldsymbol{\Gamma}]^{2}}\right]+\sum_{i=1}^{n} \log \left(\sqrt{\left\langle d_{i}^{\dagger} d_{i}\right\rangle_{\rho^{\prime \prime}}}+\sqrt{\left\langle d_{i} d_{i}^{\dagger}\right\rangle_{\rho^{\prime \prime}}}\right) . \tag{S.83}
\end{align*}
$$


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