#### UPPER BOUNDS OF PRIME DIVISORS OF FRIEND OF 10

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ABSTRACT. In this paper we propose necessary upper bounds for second, third and fourth smallest prime divisors of friend of 10 based on the number of distinct prime divisors of it.

#### 1. Introduction

For any positive integer n,  $I(n) = \frac{\sigma(n)}{n}$  is called the abundancy index of n where  $\sigma(n)$  is sum of divisors of n. A positive integer n is said to be a friendly number if there exist a positive integer m other than n having same abundancy index as n i.e., I(n) = I(m) and in that case we say m is a friend of n. If n has no friends then we say n is a solitary number. 10 is the smallest positive integer whose classification in terms of solitary and friendly is unknown. J. Ward [6] proved that if 10 has a friend then it is an odd square having at least six distinct prime divisors with 5 being the least one and there exists a prime divisor congruent to 1 modulo 3. In [8] it has been proven that the friendly number of 10 necessarily has at least seven distinct prime divisors and has two primes p and q not necessarily distinct such that  $p \equiv 1 \pmod 6$  and  $q \equiv 1 \pmod 10$ , also it has prime divisor r such that  $2a + 1 \equiv 0 \pmod f$  where f is the least odd positive integer greater than 1 satisfying  $5^f \equiv 1 \pmod r$ , provided  $5^{2a} \parallel n$ ,  $a \in \mathbb{Z}^+$  (where n is the friendly number of 10). In this paper we prove the following,

**Theorem 1.1.** Let n be a friend of 10 and  $q_2$  be the second smallest prime divisor of n. Then necessarily

$$7 \leq q_2 < \lceil \frac{7\omega(n)}{3} \rceil \left\{ \log \left( \lceil \frac{7\omega(n)}{3} \rceil \right) + \log \log \left( \lceil \frac{7\omega(n)}{3} \rceil \right) \right\}.$$

**Theorem 1.2.** Let n be a friend of 10 and  $q_3$  be the third smallest prime divisor of n. Then necessarily

$$11 \leq q_3 < \lceil \frac{180\omega(n)}{41} \rceil \left\{ \log \left( \lceil \frac{180\omega(n)}{41} \rceil \right) + \log \log \left( \lceil \frac{180\omega(n)}{41} \rceil \right) \right\}.$$

**Theorem 1.3.** Let n be a friend of 10 and  $q_4$  be the fourth smallest prime divisor of n. Then necessarily

$$13 \leq q_4 < \lceil \frac{390\omega(n)}{47} \rceil \left\{ \log \left( \lceil \frac{390\omega(n)}{47} \rceil \right) + \log \log \left( \lceil \frac{390\omega(n)}{47} \rceil \right) \right\}.$$

Where  $\omega(n) \in \mathbb{Z}^+$  is the number of distinct prime divisors of n and  $\lceil . \rceil$  is the ceiling function.

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#### 2

### 2. Preliminaries

In this section, we note some elementary properties of the abundancy index and some of useful properties of ceiling function.

# Properties of Abundancy Index:

- 1. I(n) is weakly multiplicative.
- 2. Let a, n be two positive integers and a > 1. Then I(an) > I(n).
- 3. Let  $p_1, p_2, p_3, ..., p_m$  be m distinct primes and  $a_1, a_2, a_3, ..., a_m$  are positive integers then

$$I\bigg(\prod_{i=1}^m p_i^{a_i}\bigg) = \prod_{i=1}^m \bigg(\sum_{j=0}^{a_i} p_i^{-j}\bigg) = \prod_{i=1}^m \frac{p_i^{a_i+1}-1}{p_i^{a_i}(p_i-1)}.$$

4. If  $p_1,...,p_m$  are distinct primes,  $q_1,...,q_m$  are distinct primes such that  $p_i \leq q_i$  for all  $1 \leq i \leq m$ . If  $t_1,t_2,...,t_m$  are positive integers then

$$I\left(\prod_{i=1}^{m} p_i^{t_i}\right) \ge I\left(\prod_{i=1}^{m} q_i^{t_i}\right).$$

5. If  $n = \prod_{i=1}^{m} p_i^{a_i}$ , then  $I(n) < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}$ .

# Properties of Ceiling Function [.]:

For  $n \in \mathbb{Z}^+$  and  $x \in \mathbb{R}^+$ 

i. 
$$\lceil x + n \rceil = \lceil x \rceil + n$$

ii. 
$$\lceil x \rceil - |x| = 1$$
 if  $x \notin \mathbb{Z}^+$ 

iii. 
$$\lceil x \rceil - \lfloor x \rfloor = 0$$
 if  $x \in \mathbb{Z}^+$ 

iv. 
$$\{x\} = x - |x|$$

where |.| is floor function and {.} is fractional part function.

# 3. Proofs of the main theorems

**Proposition 3.1.** Let a function  $\psi:(1,\infty)\to\mathbb{R}$  be defined by  $\psi(x)=\frac{x}{x-1}$  then  $\psi$  is a strictly decreasing function of x in  $(1,\infty)$ .

*Proof.* Let  $\psi:(1,\infty)\to\mathbb{R}$  be defined by  $\psi(x)=\frac{x}{x-1}$ . Since  $\psi(x)$  is a rational function and  $x\neq 1, \psi(x)$  is differentiable. Therefore

$$\psi'(x) = \frac{-1}{(x-1)^2} < 0 \; ; \; for \; all \; x \in (1, \infty)$$

This proves that  $\psi$  is a strictly decreasing function of x in  $(1, \infty)$ .

**Proposition 3.2.** Let a function  $\Omega : [\alpha, \infty) \to \mathbb{R}$  be defined by  $\Omega(x) = \frac{ax-b}{cx-d}$   $(a, b, c, d, \alpha \in \mathbb{Z}^+)$  such that  $bc > ad, \alpha > \frac{d}{c}$  then  $\Omega$  is a strictly increasing function of x in  $[\alpha, \infty)$ .

*Proof.* Let  $\Omega: [\alpha, \infty) \to \mathbb{R}$  be defined by  $\Omega(x) = \frac{ax-b}{cx-d} (a, b, c, d, \alpha \in \mathbb{Z}^+)$  where bc > ad,  $\alpha > \frac{d}{c}$ . Since  $\Omega(x)$  is a rational function and  $x \neq \frac{d}{c}$ ,  $\Omega(x)$  is differentiable. Therefore

$$\Omega'(x) = \frac{bc - ad}{(cx - d)^2} > 0 \; ; \; for \; all \; x \in [\alpha, \infty)$$

This proves that  $\Omega$  is a strictly increasing function of x in  $[\alpha, \infty)$ .

**Remark 3.3.** If  $p_n$  is the  $n^{th}$  prime number then  $p_n > n$  for each  $n \in \mathbb{Z}^+$ . Using Proposition 3.1 we can say that  $\frac{p_n}{p_n-1} < \frac{n}{n-1}$  for each  $n \in \mathbb{Z}^+$ .

**Lemma 3.4.** ([1]) If  $p_n$  is  $n^{th}$  prime number then

$$p_n < n(\log n + \log \log n)$$

for  $n \geq 6$ .

3.1. **Proof of Theorem 1.1.** Let  $n=5^{2a_1}\prod_{2\leq i\leq \omega(n)}q_i^{2a_i}(q_1=5)$  be a friend of 10, where  $5< q_2< q_3...< q_{\omega(n)}$  are prime numbers and  $2a_i$  are positive integers. Now we will show that  $q_2$  must be strictly less than  $p_{\lceil\frac{7\omega(n)}{3}\rceil}$ , where  $p_{\lceil\frac{7\omega(n)}{3}\rceil}$  is the  $\lceil\frac{7\omega(n)}{3}\rceil^{th}$  prime number. Which will immediately prove this theorem by Lemma 3.4. If possible suppose that  $q_2\geq p_{\lceil\frac{7\omega(n)}{3}\rceil}$ . Then by Property (4) and Property (5) we have

$$I(n) \leq I\bigg(5^{2a_1} \prod_{2 \leq i \leq \omega(n)} p_{\lceil \frac{7\omega(n)}{3} \rceil + i - 2}^{2a_i}\bigg) < \frac{5}{4} \prod_{2 \leq i \leq \omega(n)} \frac{p_{\lceil \frac{7\omega(n)}{3} \rceil + i - 2}}{p_{\lceil \frac{7\omega(n)}{3} \rceil + i - 2} - 1}$$

Using Remark 3.3,

$$I(n) < \frac{5}{4} \prod_{2 < i < \omega(n)} \frac{\left\lceil \frac{7\omega(n)}{3} \right\rceil + i - 2}{\left\lceil \frac{7\omega(n)}{3} \right\rceil + i - 3} = \frac{5}{4} \frac{\left\lceil \frac{7\omega(n)}{3} \right\rceil + \omega(n) - 2}{\left\lceil \frac{7\omega(n)}{3} \right\rceil - 1}$$

Now we will show that for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\lceil \frac{7\omega(n)}{3} \rceil + \omega(n) - 2}{\lceil \frac{7\omega(n)}{2} \rceil - 1} < \frac{10}{7}$$

Using Property i, we obtain

$$\frac{\lceil \frac{7\omega(n)}{3} \rceil + \omega(n) - 2}{\lceil \frac{7\omega(n)}{3} \rceil - 1} = \frac{\lceil 2\omega(n) + \frac{\omega(n)}{3} \rceil + \omega(n) - 2}{\lceil 2\omega(n) + \frac{\omega(n)}{3} \rceil - 1} = \frac{3\omega(n) - 2 + \lceil \frac{\omega(n)}{3} \rceil}{2\omega(n) - 1 + \lceil \frac{\omega(n)}{3} \rceil}$$

Now, either  $3 \mid \omega(n)$  or  $3 \nmid \omega(n)$  therefore considering two cases:

If  $3 \nmid \omega(n)$  then  $\frac{\omega(n)}{3} \not\in \mathbb{Z}^+$  therefore Using Property ii and Property iv we get

$$\begin{split} \frac{3\omega(n)-2+\lceil\frac{\omega(n)}{3}\rceil}{2\omega(n)-1+\lceil\frac{\omega(n)}{3}\rceil} &= \frac{3\omega(n)-2+1+\frac{\omega(n)}{3}-\{\frac{\omega(n)}{3}\}}{2\omega(n)-1+1+\frac{\omega(n)}{3}-\{\frac{\omega(n)}{3}\}}\\ &= \frac{10\omega(n)-3-3\{\frac{\omega(n)}{3}\}}{7\omega(n)-3\{\frac{\omega(n)}{2}\}} \end{split}$$

Since

$$1 \le 3\{\frac{\omega(n)}{3}\} \le 2$$

We obtain

(1) 
$$10\omega(n) - 5 \le 10\omega(n) - 3 - 3\{\frac{\omega(n)}{3}\} \le 10\omega(n) - 4$$

and

(2) 
$$7\omega(n) - 2 \le 7\omega(n) - 3\{\frac{\omega(n)}{3}\} \le 7\omega(n) - 1$$

Using 1 and 2 we finally have

$$\frac{3\omega(n) - 2 + \lceil \frac{\omega(n)}{3} \rceil}{2\omega(n) - 1 + \lceil \frac{\omega(n)}{2} \rceil} \le \frac{10\omega(n) - 4}{7\omega(n) - 2}$$

Now define  $f:[1,\infty)\to\mathbb{R}$  by  $f(t)=\frac{10t-4}{7t-2}$ . Then f is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}f(t)=\frac{10}{7}$  we have  $f(t)<\frac{10}{7}$  for all  $t\in[1,\infty)$ . In particular for  $t=\omega(n)$  we have

$$\frac{10\omega(n) - 4}{7\omega(n) - 2} < \frac{10}{7}$$

Which immediately implies that

$$\frac{\lceil \frac{7\omega(n)}{3} \rceil + \omega(n) - 2}{\lceil \frac{7\omega(n)}{3} \rceil - 1} < \frac{10}{7}$$

Now if  $3 \mid \omega(n)$  then  $\frac{\omega(n)}{3} \in \mathbb{Z}^+$ . Therefore

$$\frac{3\omega(n)-2+\lceil\frac{\omega(n)}{3}\rceil}{2\omega(n)-1+\lceil\frac{\omega(n)}{3}\rceil}=\frac{3\omega(n)-2+\frac{\omega(n)}{3}}{2\omega(n)-1+\frac{\omega(n)}{3}}=\frac{10\omega(n)-6}{7\omega(n)-3}$$

Now define  $g:[1,\infty)\to\mathbb{R}$  by  $g(t)=\frac{10t-6}{7t-3}$ . Then g is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}g(t)=\frac{10}{7}$  we have  $g(t)<\frac{10}{7}$  for all  $t\in[1,\infty)$ . In particular for  $t=\omega(n)$  we have

$$\frac{10\omega(n) - 6}{7\omega(n) - 3} < \frac{10}{7}$$

which immediately implies that

$$\frac{\left\lceil \frac{7\omega(n)}{3} \right\rceil + \omega(n) - 2}{\left\lceil \frac{7\omega(n)}{3} \right\rceil - 1} < \frac{10}{7}$$

Therefore for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\left\lceil \frac{7\omega(n)}{3} \right\rceil + \omega(n) - 2}{\left\lceil \frac{7\omega(n)}{3} \right\rceil - 1} < \frac{10}{7}$$

Which shows that

$$I(n) < \frac{5}{4} \frac{\left\lceil \frac{7\omega(n)}{3} \right\rceil + \omega(n) - 2}{\left\lceil \frac{7\omega(n)}{3} \right\rceil - 1} < \frac{25}{14} < \frac{9}{5}$$

Therefore for  $q_2 \ge p_{\lceil \frac{7\omega(n)}{3} \rceil}$ , n can not be a friend of 10. Hence necessarily  $q_2 < p_{\lceil \frac{7\omega(n)}{3} \rceil}$  and this proves the theorem.

3.2. **Proof of Theorem 1.2.** Let  $n=5^{2a_1}\prod_{2\leq i\leq \omega(n)}q_i^{2a_i}(q_1=5)$  be a friend of 10, where  $5< q_2< q_3...< q_{\omega(n)}$  are prime numbers and  $2a_i$  are positive integers. Now we will show that  $q_3$  must be strictly less than  $p_{\lceil \frac{180\omega(n)}{41} \rceil}$  where  $p_{\lceil \frac{180\omega(n)}{41} \rceil}$  is the  $\lceil \frac{180\omega(n)}{41} \rceil^{th}$  prime number. Which

will immediately prove this theorem by Lemma 3.4. If possible suppose that  $q_3 \geq p_{\lceil \frac{180\omega(n)}{41} \rceil}$ . Then by Property (4) and Property (5) we have

$$I(n) \leq I \left(5^{2a_1}7^{2a_2} \prod_{3 \leq i \leq \omega(n)} p_{\lceil \frac{180\omega(n)}{41} \rceil + i - 3}^{2a_i} \right) < \frac{5}{4} \cdot \frac{7}{6} \prod_{3 \leq i \leq \omega(n)} \frac{p_{\lceil \frac{180\omega(n)}{41} \rceil + i - 3}}{p_{\lceil \frac{180\omega(n)}{41} \rceil + i - 3} - 1}$$

Using Remark 3.3,

$$I(n) < \frac{5}{4} \cdot \frac{7}{6} \prod_{3 \le i \le \omega(n)} \frac{ \lceil \frac{180\omega(n)}{41} \rceil + i - 3}{ \lceil \frac{180\omega(n)}{41} \rceil + i - 3 - 1} = \frac{5}{4} \cdot \frac{7}{6} \frac{ \lceil \frac{180\omega(n)}{41} \rceil + \omega(n) - 3}{ \lceil \frac{180\omega(n)}{41} \rceil - 1}$$

Now we will show that for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} < \frac{221}{180}$$

Using Property i, we obtain

$$\frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} = \frac{5\omega(n) - 3 + \left\lceil \frac{16\omega(n)}{41} \right\rceil}{4\omega(n) - 1 + \left\lceil \frac{16\omega(n)}{41} \right\rceil}$$

Now, either 41 |  $\omega(n)$  or 41 |  $\omega(n)$  therefore considering two cases :

If  $41 \nmid \omega(n)$  then  $\frac{\omega(n)}{41} \not\in \mathbb{Z}^+$  therefore using Property ii and Property iv we get

$$\frac{5\omega(n) - 3 + \lceil \frac{16\omega(n)}{41} \rceil}{4\omega(n) - 1 + \lceil \frac{16\omega(n)}{41} \rceil} = \frac{5\omega(n) - 2 + \frac{16\omega(n)}{41} - \{\frac{16\omega(n)}{41}\}}{4\omega(n) + \frac{16\omega(n)}{41} - \{\frac{16\omega(n)}{41}\}}$$
$$= \frac{221\omega(n) - 82 - 41\{\frac{16\omega(n)}{41}\}}{180\omega(n) - 41\{\frac{16\omega(n)}{41}\}}$$

Since

$$1 \le 41\{\frac{16\omega(n)}{41}\} \le 40$$

We obtain

(3) 
$$221\omega(n) - 122 \le 221\omega(n) - 82 - 41\left\{\frac{16\omega(n)}{41}\right\} \le 221\omega(n) - 83$$

and

(4) 
$$180\omega(n) - 40 \le 180\omega(n) - 41\left\{\frac{16\omega(n)}{41}\right\} \le 180\omega(n) - 1$$

Using 3 and 4 we finally have

$$\frac{221\omega(n) - 82 - 41\{\frac{16\omega(n)}{41}\}}{180\omega(n) - 41\{\frac{16\omega(n)}{41}\}} \le \frac{221\omega(n) - 83}{180\omega(n) - 40}$$

Now define  $f:[1,\infty)\to\mathbb{R}$  by  $f(t)=\frac{221t-83}{180t-40}$ . Then f is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}f(t)=\frac{221}{180}$  we have  $f(t)<\frac{221}{180}$  for all  $t\in[1,\infty)$ . In

particular for  $t = \omega(n)$  we have

$$\frac{221\omega(n) - 83}{180\omega(n) - 40} < \frac{221}{180}$$

Which immediately implies that

$$\frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} < \frac{221}{180}$$

Now if  $41 \mid \omega(n)$  then  $\frac{\omega(n)}{41} \in \mathbb{Z}^+$ . Therefore

$$\frac{5\omega(n) - 3 + \left\lceil \frac{16\omega(n)}{41} \right\rceil}{4\omega(n) - 1 + \left\lceil \frac{16\omega(n)}{41} \right\rceil} = \frac{5\omega(n) - 3 + \frac{16\omega(n)}{41}}{4\omega(n) - 1 + \frac{16\omega(n)}{41}} = \frac{221\omega(n) - 123}{180\omega(n) - 41}$$

Now define  $g:[1,\infty)\to\mathbb{R}$  by  $g(t)=\frac{221t-123}{180t-41}$ . Then g is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}g(t)=\frac{221}{180}$  we have  $g(t)<\frac{221}{180}$  for all  $t\in[1,\infty)$ . In particular for  $t=\omega(n)$  we have

$$\frac{221\omega(n) - 123}{180\omega(n) - 41} < \frac{221}{180}$$

which immediately implies that

$$\frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} < \frac{221}{180}$$

Therefore for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} < \frac{221}{180}$$

Which shows that

$$I(n) < \frac{5}{4} \cdot \frac{7}{6} \frac{\left\lceil \frac{180\omega(n)}{41} \right\rceil + \omega(n) - 3}{\left\lceil \frac{180\omega(n)}{41} \right\rceil - 1} < \frac{1547}{864} < \frac{9}{5}$$

Therefore for  $q_3 \ge p_{\lceil \frac{180\omega(n)}{41} \rceil}$ , n can not be a friend of 10. Hence necessarily  $q_3 < p_{\lceil \frac{180\omega(n)}{41} \rceil}$  and this proves the theorem.

3.3. **Proof of Theorem 1.3.** Let  $n=5^{2a_1}\prod_{2\leq i\leq \omega(n)}q_i^{2a_i}(q_1=5)$  be a friend of 10, where  $5< q_2< q_3...< q_{\omega(n)}$  are prime numbers and  $2a_i$  are positive integers. Now we will show that  $q_4$  must be strictly less than  $p_{\lceil\frac{390\omega(n)}{47}\rceil}$  where  $p_{\lceil\frac{390\omega(n)}{47}\rceil}$  is the  $\lceil\frac{390\omega(n)}{47}\rceil^{th}$  prime number. Which will immediately prove this theorem by Lemma 3.4. If possible suppose that  $q_4\geq p_{\lceil\frac{390\omega(n)}{47}\rceil}$ . Then by Property (4) and Property (5) we have

$$I(n) \leq I\left(5^{2a_1}7^{2a_2}11^{2a_3}\prod_{\substack{4\leq i\leq \omega(n)}}p_{\lceil\frac{390\omega(n)}{47}\rceil+i-4}^{2a_i}\right) < \frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\prod_{\substack{4\leq i\leq \omega(n)}}\frac{p_{\lceil\frac{390\omega(n)}{47}\rceil+i-4}}{p_{\lceil\frac{390\omega(n)}{47}\rceil+i-4}-1}$$

Using Remark 3.3,

$$\begin{split} I(n) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \prod_{4 \leq i \leq \omega(n)} \frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + i - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil + i - 5} \\ = \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} \end{split}$$

Now we will show that for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} < \frac{437}{390}$$

Using Property i, we obtain

$$\frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} = \frac{9\omega(n) - 4 + \left\lceil \frac{14\omega(n)}{47} \right\rceil}{8\omega(n) - 1 + \left\lceil \frac{14\omega(n)}{47} \right\rceil}$$

Now, either 47 |  $\omega(n)$  or 47 |  $\omega(n)$  therefore considering two cases:

If  $47 \nmid \omega(n)$  then  $\frac{\omega(n)}{47} \not\in \mathbb{Z}^+$  therefore using Property ii and Property iv we get

$$\frac{9\omega(n) - 4 + \lceil \frac{14\omega(n)}{47} \rceil}{8\omega(n) - 1 + \lceil \frac{14\omega(n)}{47} \rceil} = \frac{9\omega(n) - 3 + \frac{14\omega(n)}{47} - \{\frac{14\omega(n)}{47}\}}{8\omega(n) + \frac{14\omega(n)}{47} - \{\frac{14\omega(n)}{47}\}}$$
$$= \frac{437\omega(n) - 141 - 47\{\frac{14\omega(n)}{47}\}}{390\omega(n) - 47\{\frac{14\omega(n)}{47}\}}$$

Since

$$1 \le 47\{\frac{14\omega(n)}{47}\} \le 46$$

We obtain

(5) 
$$437\omega(n) - 187 \le 437\omega(n) - 141 - 47\left\{\frac{14\omega(n)}{47}\right\} \le 437\omega(n) - 142$$

and

(6) 
$$390\omega(n) - 46 \le 390\omega(n) - 47\left\{\frac{14\omega(n)}{47}\right\} \le 390\omega(n) - 1$$

Using 5 and 6 we finally have

$$\frac{437\omega(n) - 141 - 47\{\frac{14\omega(n)}{47}\}}{390\omega(n) - 47\{\frac{14\omega(n)}{47}\}} \le \frac{437\omega(n) - 142}{390\omega(n) - 46}$$

Now define  $f:[1,\infty)\to\mathbb{R}$  by  $f(t)=\frac{437t-142}{390t-46}$ . Then f is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}f(t)=\frac{437}{390}$  we have  $f(t)<\frac{437}{390}$  for all  $t\in[1,\infty)$ . In particular for  $t=\omega(n)$  we have

$$\frac{437\omega(n) - 142}{390\omega(n) - 46} < \frac{437}{390}$$

Which immediately implies that

$$\frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} < \frac{437}{390}$$

Now if  $47 \mid \omega(n)$  then  $\frac{\omega(n)}{47} \in \mathbb{Z}^+$ . Therefore

$$\frac{9\omega(n) - 4 + \lceil \frac{14\omega(n)}{47} \rceil}{8\omega(n) - 1 + \lceil \frac{14\omega(n)}{47} \rceil} = \frac{9\omega(n) - 4 + \frac{14\omega(n)}{47}}{8\omega(n) - 1 + \frac{14\omega(n)}{47}} = \frac{437\omega(n) - 188}{390\omega(n) - 47}$$

Now define  $g:[1,\infty)\to\mathbb{R}$  by  $g(t)=\frac{437t-188}{390t-47}$ . Then g is strictly increasing function of t in  $[1,\infty)$  by Proposition 3.2. Since  $\lim_{t\to\infty}g(t)=\frac{437}{390}$  we have  $g(t)<\frac{437}{390}$  for all  $t\in[1,\infty)$ . In particular for  $t=\omega(n)$  we have

$$\frac{437\omega(n)-188}{390\omega(n)-47}<\frac{437}{390}$$

Which immediately implies that

$$\frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} < \frac{437}{390}$$

Therefore for any  $\omega(n) \in \mathbb{Z}^+$  we have

$$\frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} < \frac{437}{390}$$

Which shows that

$$I(n) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \frac{\left\lceil \frac{390\omega(n)}{47} \right\rceil + \omega(n) - 4}{\left\lceil \frac{390\omega(n)}{47} \right\rceil - 1} < \frac{33649}{18720} < \frac{9}{5}$$

Therefore for  $q_4 \geq p_{\lceil \frac{390\omega(n)}{47} \rceil}$ , n can not be a friend of 10. Hence necessarily  $q_4 < p_{\lceil \frac{390\omega(n)}{47} \rceil}$  and this proves the theorem.

# 4. Conclusion

We use an elementary approach to obtain bounds for the second, third and fourth smallest prime divisors of the friendly number of 10 although we are unable to produce upper bounds for other prime divisors based on the number of distinct prime divisors of it. Moreover we highly believe that the largest prime divisor of friend of 10 has no upper bounds. However 10 is not the only number whose status is unknown; in fact the status of 14, 15, 20 and many others are active topics for Research. It may be proving whether 10 has a friend or not is as much as difficult as finding an odd perfect number. Computer search shows that if 10 has a friend then its smallest friend will be at least  $10^{30}$ .

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