

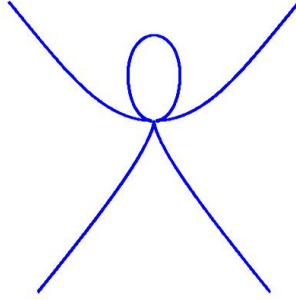
A remarkable eelliptic curve

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April 10, 2024

Abstract

We describe a system of plane algebraic curves defined over $\mathbb{Z}$, attached naturally to the exponential function. One of these is a remarkable curve of degree 6 that has genus equal to 1, and looks like



As the sextic curve has rational points, it is an elliptic curve and can be transformed over $\mathbb{Q}$ into the curve **1584.j1** of the LMFDB. One is left to wonder what the number 11, appearing in the factorisation $1584 = 2^4 \cdot 3^2 \cdot 11$, has to do with the exponential function.

1. The exponential function e^x is the first object we encounter in elementary analysis that transcends the algebraic geometrical world of algebraic functions. Being its own derivative and having no zeros or poles, it seems to lack clear geometrical features. A little later we learn to extend e^x into the complex domain and study the function e^z , $z = x + iy$, and discover that it has an imaginary period $2\pi i$, and is related, via

$$e^{x+iy} = e^x (\cos y + i \sin y),$$

to the functions $\sin$ and $\cos$, the circle and the rest of trigonometry and cyclotomy.

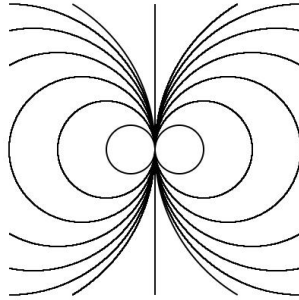
2. In the real domain, the function stretches between the limit value 0 for $x = -\infty$ and $+\infty$ for $x = +\infty$, thus showing that the point $z = \infty$ is an *essential singularity*. To

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study this special point, it is convenient to perform the inversion $z \mapsto 1/z$ and study $e^{1/z}$ in a neighbourhood of $z = 0$. Decomposing $z = x + iy$ in real and imaginary parts, the reciprocal transformation takes the form

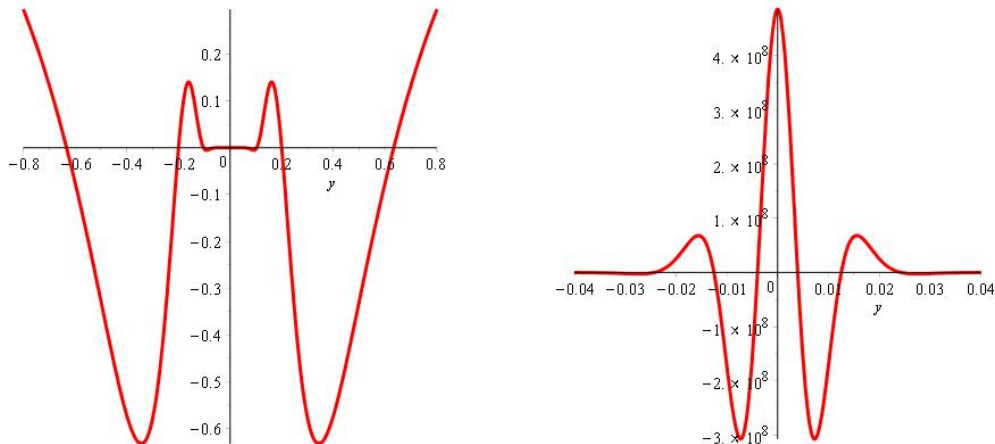
$$(x, y) \mapsto (x/(x^2 + y^2), -y/(x^2 + y^2)),$$

The level sets of $|e^z| = e^x$ are the vertical lines $x = \text{constant}$ and these are mapped to the system of circles tangent to line $x = 0$ and with center on the line $y = 0$, the level sets of $|e^{1/z}| = e^{x/(x^2+y^2)}$.



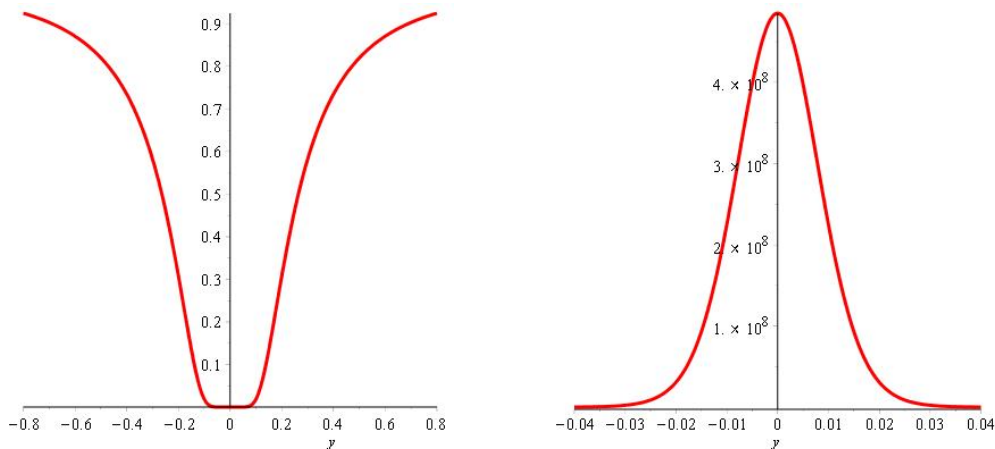
If we approach the origin $z = 0$ from right halfplane $x = \text{Re}(z) > 0$, the function $e^{1/z}$ is unbounded; coming from the left halfplane $\text{Re}(z) < 0$, the function approaches 0. The restriction to the imaginary axis $x = 0$ is $e^{1/(iy)}$ and has $\cos(1/y)$ as real part and exhibits the well-known wild oscillatory behaviour near 0.

3. To get a clearer view of what is happening, one can plot, for small $x = \text{constant}$, the function $\text{Re}(e^{1/(x+iy)}) = e^{x/(x^2+y^2)} \cos \frac{y}{x^2+y^2}$. For y large this function tends rapidly to its limiting value 1 so we look at a small interval around $y = 0$. For $x < 0$ one observes an oscillatory behaviour, whose amplitude suddenly drop to very small values, forming a *valley* around the origin. For $x > 0$ we have complementary behaviour: function is small well away from the origin, but develops oscillations that suddenly blow up when we approach the origin.



Plot of $\text{Re}(e^{1/(x+iy)})$ for $x = -0.05$ (left) and $x = 0.05$ (right).
(Note the difference in scaling on the axes.)

The graphs of $|e^{1/(x+iy)}| = e^{x/(x^2+y^2)}$ for $x = \text{constant}$ suggests the presence of two inflectional points that may serve to demarcate the location of the regions of there the function is very small.



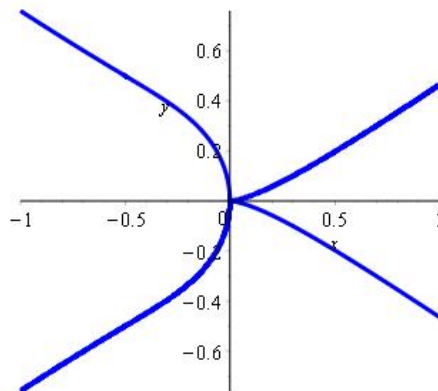
Plot of $|e^{1/(x+iy)}|$ for $x = -0.05$ (left) and $x = 0.05$ (right).
(Note the difference in scaling on the axes.)

These pictures illustrate the striking *difference in scale* for positive and corresponding negative values of x : the width of the valley for $x = -c < 0$ seems to be much bigger than the width of the corresponding peak for $x = c > 0$.

4. It is easy to calculate the location of these inflectional points by computing the second derivative of $e^{x/(x^2+y^2)}$ with respect to y . Its vanishing is determined by the polynomial factor $x F_2(x, y)$, where

$$F_2(x, y) := (x^2 - 3y^2)(x^2 + y^2) - 2xy^2,$$

which defines an irreducible quartic curve C_2 with a surprising singularity of type D_5 at the origin, consisting of a smooth branch and a transverse cuspidal branch.



The rational curve C_2 defined by $F_2 = 0$.

Hence we are dealing with a rational curve and it can be parametrised by the pencil of lines through 0, which intersect the curve C_2 in a unique further point; the

substitution $y = mx$ leads to

$$x = \frac{2m^2}{(1+m^2)(1-3m^2)}, \quad y = m \cdot x.$$

In the halfspace $x > 0$, C_2 exhibits its cuspidal branch, which is intersected by the line $x = c$ in two points, which span an interval of length

$$R_c = 2\sqrt{\frac{1}{2}}c^{3/2}\left(1 + \frac{c}{2} + \dots\right).$$

whereas in the halfspace $x < 0$ the curve C_2 is approximated near 0 by a parabola, and now the two points of intersection of C_2 with the line $x = c$ span an interval of length $L_c = 2\sqrt{\frac{2}{3}}(-c)^{1/2}\left(1 + \frac{c}{2} + \dots\right)$. The ratio between these intervals is

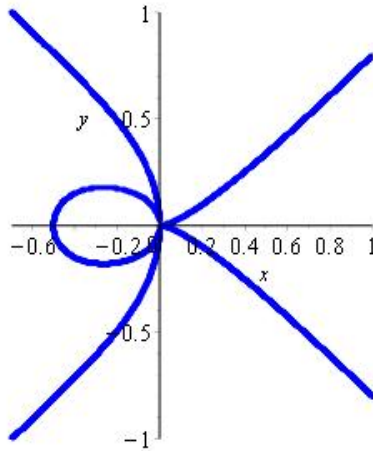
$$R_c/L_{-c} \sim \sqrt{\frac{3}{4}} \cdot c.$$

As a consequence, the central peak is indeed much narrower for small $x = c > 0$ than the corresponding central valley for $x = -c < 0$, explaining the feature observed in **3**.

5. Surprised by the beauty of the curve C_2 , one asks: why not take a look at higher derivatives of $e^{x/(x^2+y^2)}$? The third derivative factors out the irreducible polynomial

$$F_3(x, y) = 6(x^2 - y^2)(x^2 + y^2)^2 + 3x^5 - 6x^3y^2 - 9xy^4 - 2x^2y^2.$$

The curve defined by $F_3 = 0$ is a sextic C_3 , with a fourfold point at the origin, where we find two smooth and a cuspidal branch. One of the smooth branches closes up at the point $(x, y) = (-1/2, 0)$ to a loop lying in the left halfplane.



The curve C_3 defined by $F_3 = 0$.

In the affine x, y -plane, the curve has the origin as unique singular point. But we will look at the projective closure of this curve, lying in the projective closure of

the complexification $\mathbb{C}^2$ of $\mathbb{C}$ considered as $\mathbb{R}$ -vector space. As is visible from the equation, it passes doubly through the two circle points $(\pm i : 1 : 0)$ on the line at infinity. As we have two smooth and one cuspidal branch at the origin, and each pair of branches have mutual intersection multiplicity = 2, we obtain for the δ -invariant of C_3 at 0:

$$\delta(C_3, 0) = 1 + 0 + 0 + 2 + 2 + 2 = 7.$$

Together with the double points of C_3 at the circle points, we find by the *Riemann-Clebsch formula*:

$$\text{genus}(C_3) = \frac{(6-1)(5-2)}{2} - 7 - 1 - 1 = 1,$$

so C_3 is a curve of genus 1. As we have the rational point $(-1/2, 0)$ lying on C_3 , the curve is in fact an *elliptic curve defined over $\mathbb{Q}$* . We note the presence of the further rational points $(-1/6, \pm 1/6)$ on C_3 .

6. The pencil of lines through the origin intersect the curve C_3 in two further points, representing C_3 as double cover of the projective line of directions at 0. Setting $y = mx$ in $F_3(x, y)$, we obtain

$$F_3(x, mx) = -x^4(Ax^2 + Bx + C),$$

where

$$A = 6(m-1)(m+1)(m^2+1)^2, \quad B = 3(3m^2-1)(m^2+1), \quad C = 2m^2.$$

The x -coordinate of the two further intersection points of C_3 with the line $y = mx$ are the roots of $Ax^2 + Bx + C$. So the ramification points of the double cover are determined by the discriminant

$$D = B^2 - 4AC = 3(m^2+1)^2(11m^4 - 2m^2 + 3)$$

of the quadratic term. This shows that the curve C_3 is a double cover ramified over the four roots of the quartic

$$11m^4 - 2m^2 + 3.$$

From this we readily compute the j -invariant as

$$j = \frac{62500}{33} = \frac{2^2 \cdot 5^6}{3 \cdot 11},$$

which determines the elliptic curve as a complex curve. However, the above quartic has only complex roots, and it is not directly clear how to find the familiar Weierstrass normal form from here.

7. It is much more convenient to first perform the inversion $z \mapsto 1/z$, or $(x, y) \mapsto (x/(x^2 + y^2), -y/(x^2 + y^2))$ on our curve, so in a way 'undoing' the initial inversion to go from e^z to $e^{1/z}$. The corresponding substitution into F_3 leads to:

$$F_3\left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) = \frac{G_3(x, y)}{(x^2 + y^2)^4},$$

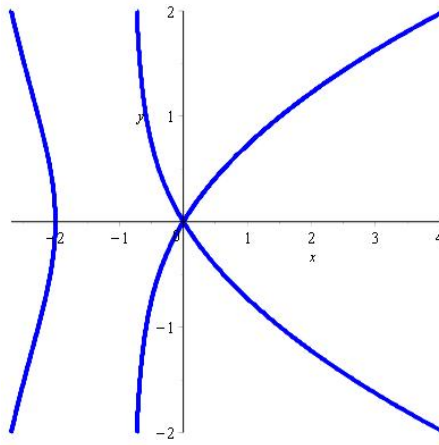
where

$$G_3 := -2x^2y^2 + 3x^3 - 9xy^2 + 6x^2 - 6y^2.$$

As the inversion is an involutory transformation, we also have

$$G_3 \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right) = \frac{F_3(x, y)}{(x^2 + y^2)^4}.$$

Indeed, from the standpoint of projective geometry, we are performing a *Cremona transformation* based at 0 and the two circle points. Any quadric through these three points (i.e. any linear combination of $xz, yz, x^2 + y^2$) intersects C_3 generally in $2 \cdot 6 - 4 - 2 - 2 = 4$ further points, so this transforms our sextic C_3 to a *quartic curve* Q , defined by vanishing of the above polynomial G_3 .



The quartic curve Q defined by $G_3 = 0$.

The (projective closure of) the curve Q has $(0 : 0 : 1)$ and $(0 : 1 : 0)$ as singular points, so the genus is indeed $3 - 1 - 1 = 1$. We note that the equation for Q is of a very special form:

$$0 = -(2x^2 + 9x + 6)y^2 + 3x^2(x + 2), \quad y^2 = 3x^2 \frac{x + 2}{2x^2 + 9x + 6}.$$

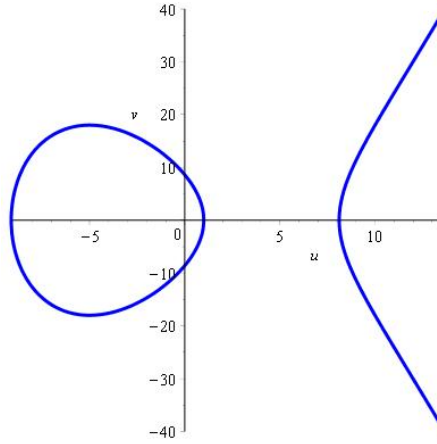
The introduction of the variable $y' := y(2x^2 + 9x + 6)/x$ converts this into the form

$$y'^2 = 3(x + 2)(2x^2 + 9x + 6) = 6x^3 + 39x^2 + 72x + 36.$$

Completing the cube and scaling $u = 6x + 13$, $v = 6y'$ brings the equation in *minimal Weierstrass form*

$$v^2 = u^3 - 75u + 74,$$

defining a standard elliptic curve E in the u, v -plane.



The elliptic curve E defined by $v^2 = u^3 - 75u + 74$.

Combining the transformations, we can express x and y in u and v :

$$x = \frac{u - 13}{6}, \quad y = \left(\frac{v}{2}\right) \frac{u - 13}{u^2 + u - 74},$$

and indeed

$$G_3\left(\frac{u - 13}{6}, \left(\frac{v}{2}\right) \frac{u - 13}{u^2 + u - 74}\right) = \frac{(u - 13)^2(u^3 - 75u + 74 - v^2)}{72(u^2 + u - 74)},$$

checking the E is mapped to Q . The composition

$$(u, v) \mapsto (x, y) \mapsto (x/(x^2 + y^2), -y/(x^2 + y^2))$$

is the sought for birational map from E to the sextic C_3 .

7. There are powerful software tools to handle algebraic curves and in particular elliptic curves.

MAPLE [5] has a nice package to deal with algebraic curves, their singularities, Puiseux expansions, their genera, etc. It has a built-in routine `Weierstrassform` that can be applied to find the Weierstrass normal form for the elliptic curve defined by the sextic C_3 . However, the algorithm produces a rather complicated transformation, whose geometric origin is obscure. The above simple algebra-geometric approach leads to a much simpler transformation.

PARI/GP [6] is a powerful software, pitched toward the computation of arithmetical invariants of (elliptic) curves defined over number fields. It can count points, handle modular forms and L -functions and much more. Convenient is the command `ellidentify`, which tells us that our curve E has the Cremona label **1584f1** [2], so the conductor of our curve is $1584 = 2^4 \cdot 3^2 \cdot 11$.

The LMFDB (L-functions and Modular Forms Data Base) [4] represents a huge extension of the earlier tables by J. Cremona [2]; it assigns to our curve the label **1584.j1** and presents further data attached to the curve in an attractive form. We

learn that the Mordell-Weil group of the curve is $\mathbb{Z} \oplus \mathbb{Z}/2$. The torsion point $(u, v) = (1, 0)$ of E corresponds to $(-2, 0)$ on Q and $(-1/2, 0)$ on C_3 . The generator of infinite order $(u, v) = (-5, 18)$ is mapped to $(-3, 3)$ on Q and to $(-1/6, 1/6)$ of C_3 . Furthermore, the curve E possesses 11 (!) points with integral coordinates, which map to special rational points on Q and C_3 . From LMFDB we also learn that our curve has the curve **264.c1** with equation $y^2 = x^3 + x^2 - 8x$ as minimal quadratic twist and that this curve has Mordell-Weil group $\mathbb{Z}/2$.

For general background information on algebraic curves, among the many books available, one of our favourites is [1]. For the arithmetic theory of elliptic curves [8] is indispensable.

8. Why stop at the third derivative? Write $\phi := \exp(x/(x^2 + y^2))$ and $\partial := \partial/\partial y$, so

$$\partial\phi = h\phi, \quad h := \partial\left(\frac{x}{x^2 + y^2}\right).$$

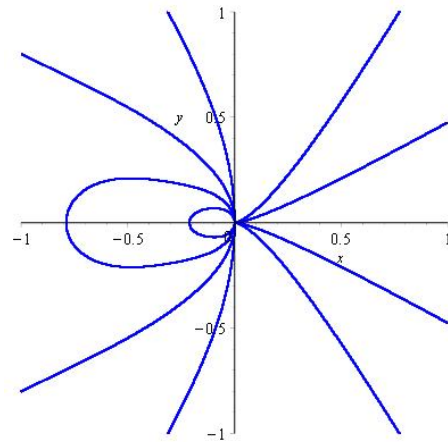
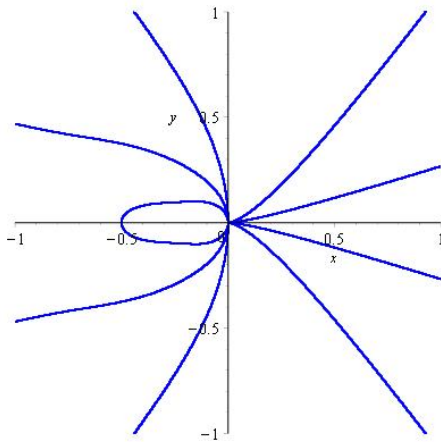
For the n -th derivative one then has

$$\partial^n \phi = h_n \phi, \quad h_{n+1} = \partial h_n + h \cdot h_n,$$

so that

$$h_n = \frac{f_n}{(x^2 + y^2)^{2n}},$$

where $f_n \in \mathbb{Z}[x, y]$ is a polynomial of degree $3n - 1$. For n odd f_n factorises as $f_n = xyF_n$, and for n even as $f_n = xF_n$, where $F_n \in \mathbb{Z}[x, y]$ is an irreducible polynomial (of degree $3n - 3$ (n odd) resp. $3n - 2$ (n even)). (Note $f_0 = 1$, $f_1 = -2xy$, so there are no curves C_0 or C_1 .)



The curve C_4 of genus 4 and C_5 of genus 7.

The curve C_n defined by $F_n = 0$ has a unique singularity at the origin, where it has $(n - 1)$ smooth local branches with common vertical tangent and $\lfloor \frac{n}{2} \rfloor$ cuspidal

branches with common horizontal tangent. Furthermore it has ordinary $(n - 1)$ -fold points at the circular points. The genus of C_n can be computed to be

$$\text{genus}(C_n) = \begin{cases} (n - 2)(3n - 4)/4 & \text{if } n \text{ is even} \\ (3n^2 + 1)/4 & \text{if } n \text{ is odd} \end{cases}$$

In the real domain, the curves C_n have an insect-like appearance. The $(n - 1)$ smooth branches are all in the halfspace $x \leq 0$; $\lfloor \frac{n-1}{2} \rfloor$ close up to 'heads', whereas the remaining branches form pairs of 'arms'; the cuspidal branches are in the halfspace $x \geq 0$ and form the 'legs'.

9. Of course, we do not need to restrict y derivatives only. It is simple to show that

$$P_{a,b} := (x^2 + y^2)^{2(a+b)} \left(\frac{\partial^{a+b}}{\partial x^a \partial y^b} e^{x/(x^2+y^2)} \right) e^{-x/(x^2+y^2)}$$

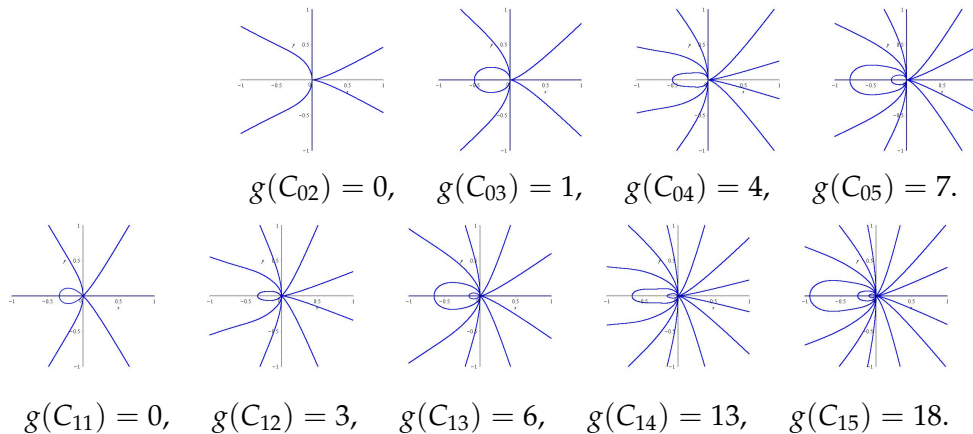
is a polynomial of degree $3(a + b) - 1$. For b an odd number, $P_{a,b}$ is divisible by y ; for $a = 0$ there is an additional factor x . Clearing out these factors, we obtain a polynomial $F_{a,b}$ that defines an irreducible curve $C_{a,b}$ and $C_n = C_{0,n}$. The curve $C_{a,b}$ has the origin as only singular point in the affine plane and furthermore ordinary $a + b$ -fold singularities at the two circle points. The genus is basically a quadratic function of a and b , to be more precise we have

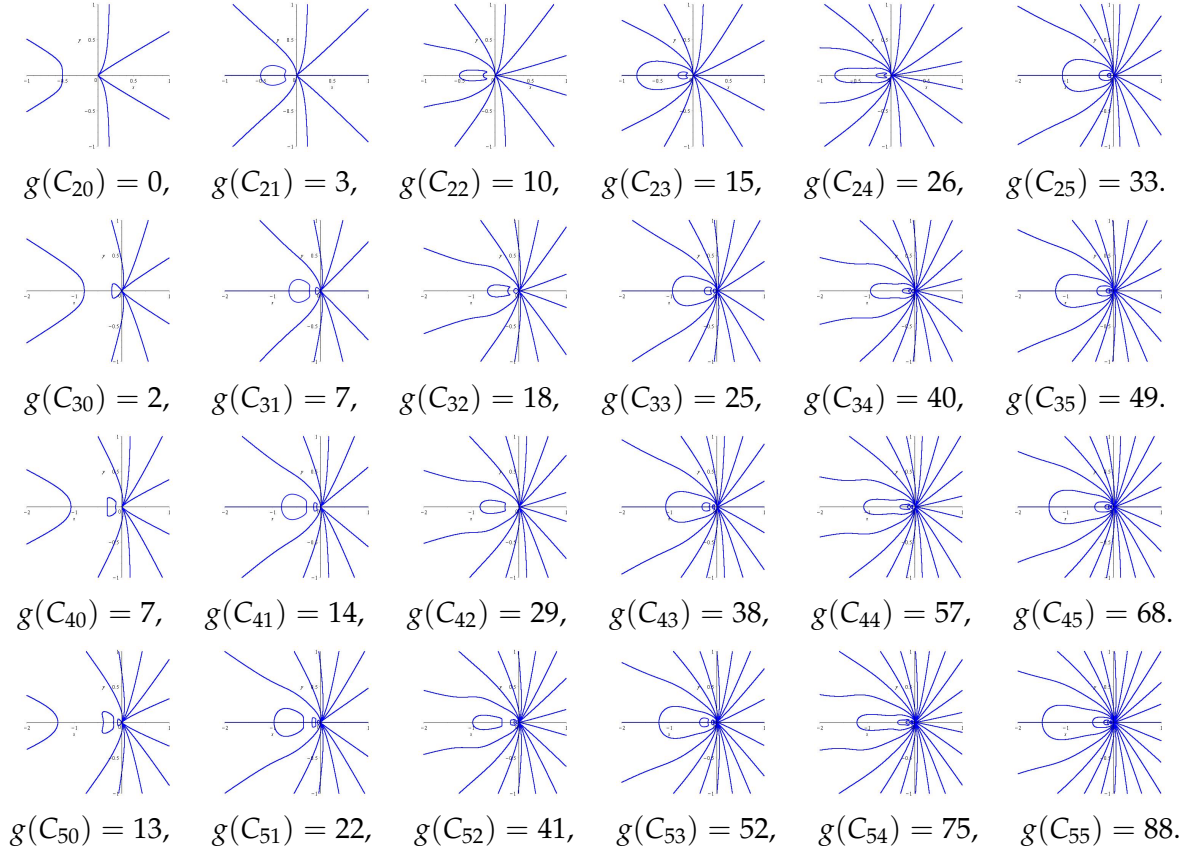
$$\text{genus}(C_{a,b}) = 9a^2/8 + 3ab + 3b^2/4 - 13a/4 - 5b/2 + \begin{cases} 2 & \text{if } a \equiv 0 \pmod{2}, \\ 13/8 & \text{if } a \equiv 0 \pmod{2}, b \equiv 0 \pmod{4}, \\ 9/8 & \text{if } a \equiv 0 \pmod{2}, b \equiv 2 \pmod{4}, \end{cases}$$

when b is even and

$$\text{genus}(C_{a,b}) = a^2 + 3ab + 3b^2/4 - 7a/2 - 3b + \begin{cases} 7/4 & \text{if } a \equiv 1 \pmod{2}, \\ 13/4 & \text{if } a = 0, \\ 9/4 & \text{if } a \equiv 0 \pmod{2}, a > 0, \end{cases}$$

when b is odd. Here a table of genera of the curves appearing for small $a, b \leq 5$.





10. It is somewhat of a mystery to me how the curve C_3 , with its rich specific arithmetic geometry, like its Galois representation of conductor $1584 = 2^4 3^2 11$, can arise without any specific choices from the exponential function, a basic function usually considered as belonging to “algebraic geometry over $\mathbb{F}_1$ ”. Of course there are many other algebraic geometric objects one can ‘naturally’ attach to the exponential function. For example, we can look at the Taylor polynomial

$$f_n(x) := 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n$$

of degree n of e^x . The location of the roots of $f_n(nx)$ and their convergence to the Szegő curve defined by $\{z \in \mathbb{C} \mid |ze^{1-z}| = 1\}$ is a well-known topic in classical analysis [7]. One can look at the sequence of hyperelliptic curves $y^2 = f_n(x)$, which for $n = 3, 4$ lead to two further ‘natural’ elliptic curves. But, in a sense I find hard to make precise, these curves look more artificial to me.

Rather it appears that the curves $C_{a,b}$ are natural *algebraic geometric satellites* of the exponential motive defined by $e^{1/z}$, and as such has some obvious generalisations. One may look at a similar system of curves for e^{1/z^2} . Here one also finds an elliptic curve, **576.c3**, with Mordel-Weil group $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In general, if $g, S \in K := \mathbb{Q}(x_1, x_2, \dots, x_n)$ are two rational functions, $\partial \in \text{Der}(K)$ a derivation of K , and $\phi := g \cdot e^S$, then repeated differentiation leads a sequence $g_1, g_2, g_3, \dots, g_n, \dots \in \mathbb{Q}(x_1, x_2, \dots, x_n)$ defined recursively by

$$g_1 := g, g_2 = \partial(g_1) + g_1 \cdot \partial S, \dots, g_{n+1} := \partial g_n + g_n \cdot \partial S, \dots$$

and which appear as prefactors of $\partial^n \phi = g_n \phi$. The classical motives attached to the g_n are in a similar way algebraic-geometrical satellites of the exponential motive with exponential periods $\int e^S \omega$, $\omega := g \cdot dx_1 dx_2 \dots dx_n$, [3].

Acknowledgement: I thank W. Zudilin for interest in the curve C_3 and many useful comments on this note and the suggestion to write it. Furthermore, I take the opportunity to thank the developers of MAPLE, PARI/GP, LMFDB which made this work possible.

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